Supplementary Materials

Uncover the relation between consensus and topology of directed network: the minimum node in-degree of directed cycles

1 Part I: The derivation details of Eq. (9)

The characteristic equation of the new Laplacian with ℓ disjoint directed cycles is as follows

$$
\begin{aligned}
&\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\kappa=1}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(\lambda-d_{\kappa i}-1)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(\lambda-h_{\kappa j})^{c_{\kappa j}}\right)\\
&+\sum_{a=1}^{\ell}(-1)^{n+n-\sum\limits_{i=1}^{q_{a1}}w_{ai}-\sum\limits_{j=1}^{q_{a2}}c_{aj}+1}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\substack{\kappa=1\\ \kappa\neq a}}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(\lambda-d_{\kappa i}-1)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(\lambda-h_{\kappa j})^{c_{\kappa j}}\right)\\
&+\sum_{a=1}^{\ell-1}\sum_{b=a+1}^{\ell}(-1)^{n+n-\sum\limits_{i=1}^{q_{a1}}w_{ai}-\sum\limits_{j=1}^{q_{a2}}c_{aj}-\sum\limits_{i=1}^{q_{b1}}w_{bi}-\sum\limits_{j=1}^{q_{b2}}c_{bj}+2}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\substack{\kappa=1\\ \kappa\neq a,b}}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(\lambda-d_{\kappa i}-1)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(\lambda-h_{\kappa j})^{c_{\kappa j}}\right)\\
&+\ldots\\
&+(-1)^{n+\sum\limits_{k=1}^{q_3}e_k+\ell}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\\
&=0,
\end{aligned}
$$

which is a seemingly complex equation. Simply put, the first term on the left side of the equation considers only the nodes of the entire network, the second term for one cycle from these ℓ cycles, the third term for two cycles, and goes down in sequence, the last term for ℓ cycles. Then writing $\lambda-d_{\kappa i}-1$ and $\lambda-h_{\kappa j}$ as $-(d_{\kappa i}+1-\lambda)$ and $-(h_{\kappa j}-\lambda)$, we find that the power terms of -1 can be removed. Therefore, we have

$$
\begin{aligned}
&\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\kappa=1}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(\lambda-d_{\kappa i}-1)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(\lambda-h_{\kappa j})^{c_{\kappa j}}\right)\\
&-\sum_{a=1}^{\ell}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\substack{\kappa=1\\ \kappa\neq a}}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(\lambda-d_{\kappa i}-1)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(\lambda-h_{\kappa j})^{c_{\kappa j}}\right)\\
&+\sum_{a=1}^{\ell-1}\sum_{b=a+1}^{\ell}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\substack{\kappa=1\\ \kappa\neq a,b}}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(\lambda-d_{\kappa i}-1)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(\lambda-h_{\kappa j})^{c_{\kappa j}}\right)\\
&+\ldots\\
&+(-1)^{\ell}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\\
&=0,
\end{aligned}
$$

where the positive and negative signs of each item are exactly staggered. Through further organization, the characteristic equation is as follows

$$\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\prod_{\kappa=1}^{\ell}\left(\prod_{i=1}^{q_{\kappa 1}}(d_{\kappa i}+1-\lambda)^{w_{\kappa i}}\prod_{j=1}^{q_{\kappa 2}}(h_{\kappa j}-\lambda)^{c_{\kappa j}}-1\right)=0.$$

That is, Eq. (9).

2 Part II: The derivation details of Eq. (12)

Since we consider ℓ_1 fully intersected directed cycles and each directed cycle contains p_1 nodes and the corresponding nodes have the same in-degree, the characteristic equation of Laplacian is as follows

$$\prod_{i=1}^{q_1}(\lambda-d_i-1)^{w_i}\prod_{j=1}^{q_2}(\lambda-h_j)^{c_j}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}+\ell_1(-1)^{2n-p_1+1}\frac{\prod_{i=1}^{q_1}(\lambda-d_i-1)^{w_i}\prod_{j=1}^{q_2}(\lambda-h_j)^{c_j}\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}}{\prod_{i=1}^{q_1{}'}(\lambda-d_i{}'-1)^{w_i{}'}\prod_{j=1}^{q_2{}'}(\lambda-h_j{}')^{c_j{}'}}=0,$$

where w_i and c_j represent the number of nodes for all cycles with in-degrees d_i+1 and h_j, respectively; $w_i{}'$ and $c_j{}'$ represent the number of nodes on a cycle with in-degrees $d_i{}'+1$ and $h_j{}'$, respectively, satisfying $\sum_{i=1}^{q_1{}'}w_i'+\sum_{j=1}^{q_2{}'}c_j'=p_1$; e_k is the number of nodes which are not on any directed cycle, having in-degrees g_k.

Writing $\lambda-d_i-1$, $\lambda-h_j$, $\lambda-d_i{}'-1$, $\lambda-h_j{}'$ as $-(d_i+1-\lambda)$, $-(h_j-\lambda)$, $-(d_i{}'+1-\lambda)$, $-(h_j{}'-\lambda)$, respectively, one has

$$\prod_{i=1}^{q_1}(d_i+1-\lambda)^{w_i}\prod_{j=1}^{q_2}(h_j-\lambda)^{c_j}\prod_{k=1}^{q_3}(g_k-\lambda)^{e_k}-\ell_1\frac{\prod_{i=1}^{q_1}(d_i+1-\lambda)^{w_i}\prod_{j=1}^{q_2}(h_j-\lambda)^{c_j}\prod_{k=1}^{q_3}(g_k-\lambda)^{e_k}}{\prod_{i=1}^{q_1{}'}(d_i{}'+1-\lambda)^{w_i{}'}\prod_{j=1}^{q_2{}'}(h_j{}'-\lambda)^{c_j{}'}}=0.$$

Extracting $\prod_{k=1}^{q_3}(g_k-\lambda)^{e_k}$ and defining

$$Q(\lambda)=\frac{\prod_{i=1}^{q_1}(d_i+1-\lambda)^{w_i}\prod_{j=1}^{q_2}(h_j-\lambda)^{c_j}}{\prod_{i=1}^{q_1{}'}(d_i{}'+1-\lambda)^{w_i{}'}\prod_{j=1}^{q_2{}'}(h_j{}'-\lambda)^{c_j{}'}},$$

we have

$$\prod_{k=1}^{q_3}(g_k-\lambda)^{e_k}\left[\prod_{i=1}^{q_1}(d_i+1-\lambda)^{w_i}\prod_{j=1}^{q_2}(h_j-\lambda)^{c_j}-\ell_1Q(\lambda)\right]=0,$$

or

$$\prod_{k=1}^{q_3}(\lambda-g_k)^{e_k}\left[\prod_{i=1}^{q_1}(d_i+1-\lambda)^{w_i}\prod_{j=1}^{q_2}(h_j-\lambda)^{c_j}-\ell_1Q(\lambda)\right]=0.$$

That is, Eq. (12).

3 Part III: Schematic diagram of directed chain network

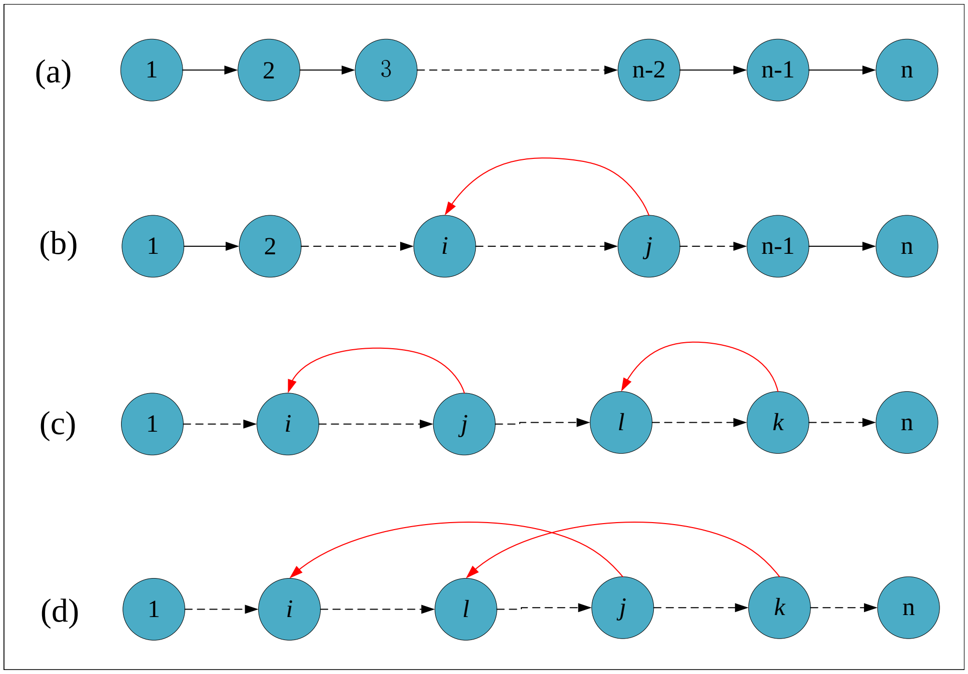

Figure A : Schematic diagram of adding edges to a directed chain network corresponding to Examples 1 to 3.