University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 10

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We will start 10 minutes past the hour. Use this time to make a new friend.

Scenario 1: Population Growth with Resource Limits

Context: A population of deer is introduced into a large fenced park. Initially, the population grows rapidly, but the environment has limited resources (space, food). When the population is small, it grows approximately proportionally to its size. The growth rate slows down as population $P(t)$ approaches a maximum population of 500 (the carrying capacity).

Derive an ODE that describes the population growth.

Which variable to define the ODE with? $P(t)$

Logistic equation:

$$\boxed{P'(t) = \underbrace{rP}_{\text{proportional to size}} \left(1 - \frac{P}{500}\right)}$$

→ need to make $P'(t) = 0$ when $P = 500$

→ slow down $P'(t)$ as $P \to 500$

function that $= 0$ at $P = 500$?

$$\left(1 - \frac{P}{500}\right)$$

any other options?

$(500 - P)$ equivalent

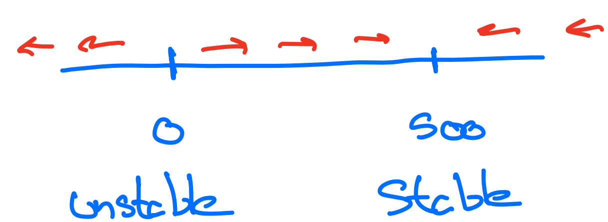
$\left(1 - \frac{P}{500}\right)^2$ is correct but simpler solution preferred

Scenario 1: Population Growth with Resource Limits

Context: A population of deer is introduced into a large fenced park. Initially, the population grows rapidly, but the environment has limited resources (space, food).
What are the equilibrium values of P? What do these mean biologically?

$$\dot{P} = rP\left(1-\frac{P}{500}\right)$$

equilibrium $\quad 0 = P' = rP\left(1-\frac{P}{500}\right) \Rightarrow P=0, \; P=500$



Biologically

→ at zero deer
→ add few deers ⇒ population grows away from $P=0$

→ at 500 deer
→ add few deer ⇒ deer die until back at carrying capacity

→ remove deer ⇒ grow until carrying capacity

Variant: Introducing Constant Harvesting

Extension: Suppose park rangers harvest a fixed number of deer per year for population control. **How does this change your model? Determine the new equilibrium values and interpret them. Under what harvesting rate will the population collapse?**

$$P'(t) = rP\left(1-\frac{P}{500}\right) - \underbrace{h}_{\text{harvest rate}}$$

⇐ limits range of validity
i.e. not valid at $P=0$

→ carrying capacity $P=500$ no longer equilibrium

→ new equilibrium

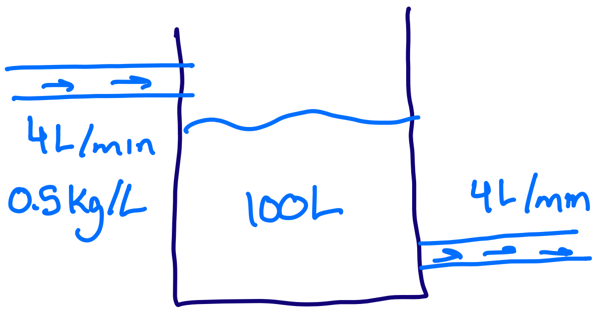
$$0 = rP\left(1-\frac{P}{500}\right) - h$$

→ equilibrium should be lower

→ when is collapse? when equilibrium is only at $P=0$

Scenario 2: Mixing Tank with Salt Solution

Context: A tank initially contains 100 L of pure water. A salt solution with concentration 0.5 kg/L flows in at 4 L/min. The tank is well-mixed, and the outflow is also 4 L/min. **Derive a differential equation for $A(t)$. Find the equilibrium value and interpret it.**



→ possible variables:

→ concentration (kg/L)

→ amount of salt (kg) ⇐ better option

change in salt = salt in − salt out

→ let $A(t)$ = amount of salt in tank at time t

$$\underbrace{\frac{dA}{dt}}_{\text{kg/min}} = \text{salt in} - \text{salt out}$$

$$= \underbrace{\left(0.5\ \text{kg/L}\right)\left(4\ \text{L/min}\right)}_{\text{kg/min}} - \underbrace{\left(\frac{A(t)}{100L}\right)\left(4L/\text{min}\right)}_{\text{kg/min}}$$

$$\Rightarrow \frac{dA}{dt} = -\frac{1}{25} A(t) + 2 \qquad \underbrace{A(0) = 0}_{\text{initial condition}}$$

→ equalibrium

$$0 = \frac{dA}{dt} = -\frac{1}{25} A + 2$$

$$\boxed{A = 50 \text{ Kg}}$$

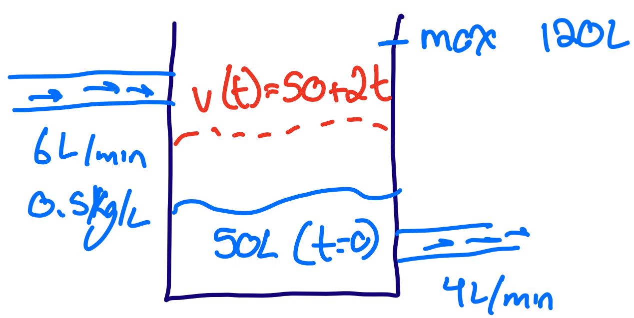
interpretation: equilibrium when salt concentration = concentration of inflow $= 0.5$ Kg/L

$$\Rightarrow \text{amount of salt} = (0.5 \text{ Kg/L})(100 \text{ L}) = 50 \text{ Kg}$$

Variant: Tank Fills and Overflows

Extension: Now the inflow rate is 6 L/min and the outflow is 4 L/min. The tank starts with 50 L of water and can hold up to 120 L before it overflows.

Derive a model for $A(t)$ before overflow starts. What might happen once the tank reaches capacity?



$$A(t) = \text{salt in tank at time } t$$

$$\frac{dA}{dt} = \text{salt in} - \text{salt out}$$

$$= \left(6\,L/\text{min}\right)\left(0.5\,\text{kg}/L\right) - \frac{A(t)}{\text{vol}(t)}\left(4\,L/\text{min}\right)$$

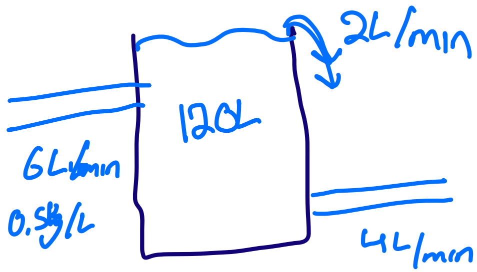
$$= 3\,\text{kg}/\text{min} - \frac{A(t)}{(50+2t)\,L}\left(4\,L/\text{min}\right)$$

$$\boxed{\frac{dA}{dt} = -\frac{4A}{50+2t} + 3}$$

→ valid until overflow

$$\boxed{t < 35 \text{ mins}}$$

→ after $t = 3s$



⇐ identical to previous problem but with 6L/min in and 6L/min

Scenario 3: iPhone Adoption Over Time

Context: A new iPhone model is released. Early adopters quickly buy the phone, but over time the pool of new customers shrinks. The rate at which new people adopt the phone slows down. Assume the total market size is 10 million people and the adoption rate is proportional to the number of people who haven't bought it yet

Derive a differential equation for $N(t)$. What is the equilibrium, and what does it represent? How does the adoption curve behave over time?

→ candidates for variables:

→ people who haven't bought iphone
→ people who have bought iphone

} roughly equivalent in difficulty

$N(t)$ = People who have bought in millions

$$\boxed{\frac{dN}{dt} = \text{adoption rate} = k\,(10 - N(t))}$$

$$\boxed{k > 0}$$

→ equilibrium

$$0 = \frac{dN}{dt} = k(10 - N(t))$$

$$\Rightarrow \boxed{N = 10}$$ ⇐ no change when everyone has the phone

→ adoption curve:

→ quick at first

→ slow at end

Variant: Influence from Existing Users

Extension: Suppose people are more likely to buy the phone if their friends already have it. Now adoption depends both on how many haven't bought it and how many already have.

Derive the updated differential equation. How might the adoption rate change early vs. late in the product cycle? Is this model reasonable?

$$\frac{dN}{dt} = k \underbrace{(10-N)}_{\text{available pool}} N$$

word of mouth

Other options?

$$\frac{dN}{dt} = k(10-N)N^2$$

$$\frac{dN}{dt} = k(10-N)N^3$$

correct but simplest model preferred

→ adoption rate:

→ slow at first (low word of mouth)

→ gets faster

→ gets slow once $N \to 10$ (all people have adopted)

Is this reasonable? Any other factors?

→ broken phones?

→ new technology?

→ outdated phone?