UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Quiz 6

Date: Jun 18, 2025
Duration: 50 minutes

Course: MAT 187 (Calculus II)
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Instructions

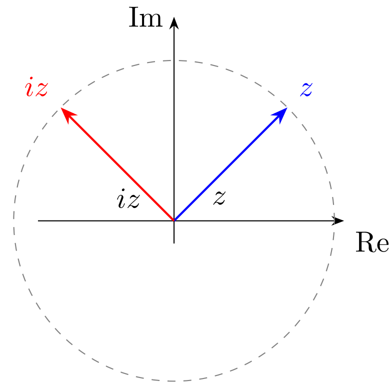
- This is a Type A assessment and **does not** allow any external aids.
- Read all instructions carefully and **justify all your answers.** No points will be awarded for a correct answer without justification.
- Read each question carefully. **No clarification or content related questions will be answered.**

You may use the following space for scratch work or to continue your solutions if you run out of room. If you do so, please clearly indicate in the original question that part of your solution appears here.

1. (2 points) Explain what multiplying a complex number by i does geometrically in the complex plane. Include a sketch to support your explanation.

Solution:

Multiplying a complex number by i rotates it 90° counter-clockwise around the origin.



2. (3 points) Consider the driven oscillator described by the equation

$$\ddot{x} + \omega^2 x = \cos(\omega t).$$

This system experiences resonance. Explain in plain words (i.e., without using math) what resonance means in this context, and why it occurs in this particular case.

Solution: Resonance is a phenomenon that occurs when an external force drives a system at the same frequency as the system's natural frequency. In this situation, each push from the external force adds energy at just the right time to reinforce the system's motion.

In this particular case, the system's natural frequency is ω, and the forcing term also oscillates with frequency ω. Because they match, the system absorbs energy efficiently and the motion builds over time. This leads to a steadily increasing amplitude — a hallmark of resonance — even though the force itself is not getting stronger.

3. (5 points) Find the general solution to the ODE.

$$y''' - 5y'' + 8y' - 4y = 0$$

Hint: $(r-2)^2(r-1)$

Solution: We begin by solving the characteristic equation associated with the differential equation:

$$r^3 - 5r^2 + 8r - 4 = 0$$

We can verify that the characteristic polynomial:

$$r^3 - 5r^2 + 8r - 4 = (r-2)^2(r-1)$$

This gives us the characteristic roots:

$$r = 2 \quad \text{(multiplicity 2)}, \quad r = 1 \quad \text{(multiplicity 1)}$$

From these roots, the general solution to the ODE is:

$$\boxed{y(t) = C_1 e^{2t} + C_2 t e^{2t} + C_3 e^t}$$

where $C_1, C_2, C_3 \in \mathbb{R}$ are arbitrary constants.

4. (10 points) Solve the initial value problem.

$$y'' - 3y' + 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$$

Solution: **Step 1: Solve the homogeneous equation.**

The associated homogeneous equation is:

$$y'' - 3y' + 2y = 0$$

The characteristic equation is:

$$r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$$

So the general solution to the homogeneous equation is:

$$y_h(t) = C_1 e^t + C_2 e^{2t}$$

Step 2: Find a particular solution to the non-homogeneous equation.

Since the forcing term is a polynomial $2t$, we guess a particular solution of the form:

$$y_p(t) = At + B$$

Then:

$$y_p' = A, \quad y_p'' = 0$$

Substitute into the differential equation:

$$0 - 3A + 2(At + B) = 2t \Rightarrow 2At + 2B - 3A = 2t$$

Match coefficients:

$$2A = 2 \Rightarrow A = 1, \quad 2B - 3A = 0 \Rightarrow 2B = 3 \Rightarrow B = \frac{3}{2}$$

So a particular solution is:

$$y_p(t) = t + \frac{3}{2}$$

Step 3: General solution.

The general solution is:

$$y(t) = C_1 e^t + C_2 e^{2t} + t + \frac{3}{2}$$

Step 4: Apply initial conditions.

Use $y(0) = 0$:

$$C_1 + C_2 + \frac{3}{2} = 0 \Rightarrow C_1 + C_2 = -\frac{3}{2} \quad (1)$$

Compute the derivative:

$$y'(t) = C_1 e^t + 2C_2 e^{2t} + 1$$

Apply $y'(0) = 1$:

$$C_1 + 2C_2 + 1 = 1 \Rightarrow C_1 + 2C_2 = 0 \quad (2)$$

Solve (1) and (2): From (2): $C_1 = -2C_2$

Substitute into (1):

$$-2C_2 + C_2 = -\frac{3}{2} \Rightarrow -C_2 = -\frac{3}{2} \Rightarrow C_2 = \frac{3}{2}, \quad C_1 = -3$$

Final Answer:

$$\boxed{y(t) = -3e^t + \frac{3}{2}e^{2t} + t + \frac{3}{2}}$$

5. A mass is attached to a spring, stretched, and released at time $t = 0$. The motion of the system is described by the function:

$$x(t) = 10\cos(2t).$$

(a) (5 points) Determine the initial value problem (i.e., the differential equation and initial conditions) that governs the motion of this system.

Solution: We recognize this as the solution to an undamped harmonic oscillator, which has the general form:

$$\ddot{x} + \omega^2 x = 0,$$

where ω is the natural frequency of the system. In this case, $\omega = 2$, so the equation becomes:

$$\ddot{x} + 4x = 0.$$

To find the initial conditions, we evaluate $x(t)$ and $\dot{x}(t)$ at $t = 0$:

$$x(0) = 10, \quad \dot{x}(0) = -20\sin(0) = 0.$$

Final answer:

$$\boxed{\begin{cases} \ddot{x} + 4x = 0 \\ x(0) = 10, \quad \dot{x}(0) = 0 \end{cases}}$$

(b) (5 points) Now suppose the spring-mass system is placed in an environment where it experiences a damping force $2\dot{x}$ that opposes the velocity. The mass is again stretched and released with the same initial conditions as in part (a).

Sketch the position as a function of time and provide a written justification for your sketch.

Solution: The equation governing the motion is:

$$\ddot{x} + 2\dot{x} + 4x = 0.$$

This is a second-order linear differential equation with constant coefficients. The characteristic equation is:

$$r^2 + 2r + 4 = 0 \quad \Rightarrow \quad r = -1 \pm i\sqrt{3}.$$

Since the roots are complex with a negative real part, the system is **underdamped**. The solution exhibits oscillatory motion with an amplitude that decays exponentially over time.