

# Practise Problem Set 3

MAT 187 - Summer 2025

These questions are meant for your own practice for quiz 3 and are not to be handed in. Some of these questions, or problems similar to these, may appear on the quizzes or exams. Therefore, solutions to these problems will not be posted but you may, of course, ask about these questions during office hours, or on Piazza.

## Suggestions on how to complete these problems:

- Solution writing is a skill like any other, which must be practiced as you study. After you write down your rough solutions, take the time to write a clear readable solution that blends sentences and mathematical symbols. This will help you to retain, reinforce, and better understand the concepts.
- After you complete a practice problem, reflect on it. What course material did you use to solve the problem? What was challenging about it? What were the main ideas, techniques, and strategies that you used to solve the problems? What mistakes did you make at the first attempt and how can you prevent these mistakes on a Term Test? What advice would you give to another student who is struggling with this problem?
- Discussing course content with your classmates is encouraged and a mathematically healthy practice. Work together, share ideas, explain concepts to each other, compare your solutions, and ask each other questions. Teaching someone else will help you develop a deeper level of understanding. However, it's also important that reserve some time for self-study and self-assessment to help ensure you can solve problems on your own without relying on others.

1. Use the trapezoidal rule with  $n = 4$  subintervals to approximate

$$\int_0^2 \sqrt{1+x^3} dx.$$

- (a) Compute the approximation.  
(b) Based on concavity, is this an overestimate or underestimate? Explain.
2. Let  $f(x) = \frac{1}{x}$ . Consider the integral  $\int_1^5 \frac{1}{x} dx$ .
  - Estimate the integral using the trapezoidal rule with  $n = 2$  subintervals.
  - Estimate the integral using the midpoint rule with  $n = 2$  subintervals.
  - Which approximation is more accurate, and why?
  - What happens as you increase the
3. You are given function values but not a formula:

$x$	0	1	2	3	4
$f(x)$	1	0.5	0.2	0.1	0.05

- (a) Estimate  $\int_0^4 f(x) dx$  using the trapezoidal rule.  
 (b) Estimate  $\int_0^4 f(x) dx$  using the Simpson's rule.  
 (c) Based on the data, can you say which rule is more likely to be accurate?

4. Let  $f(x) = \arctan x$ .

- (a) Find the second-degree Taylor polynomial  $P_2(x)$  centered at  $a = 0$ .  
 (b) Use your result to estimate  $\arctan(0.2)$ .  
 (c) Predict whether your estimate is an overestimate or underestimate. Justify briefly.  
 (d) Find a bound for the error to your estimate using Taylor's remainder theorem.

5. Let  $f(x) = \ln(1 + x)$

- (a) Compute the 4th Taylor Polynomial for  $f$  centered at  $a = 0$   
 (b) Use it to approximate  $\ln(1.5)$ .  
 (c) Compare your result to a calculator value and find the absolute error.  
 (d) Find a bound for the error to your estimate using Taylor's remainder theorem.  
 (e) Will increasing to degree 6 improve the approximation? Explain without computing.

6. Let  $f(t) = e^{-t^2}$ .

- (a) Find the second-degree Taylor polynomial centered at  $t = 0$ .  
 (b) Use it to approximate  $f(0.3)$ .  
 (c) Find a bound for the error to your estimate using Taylor's remainder theorem.  
 (d) Why might a polynomial approximation of this function be useful in physics or engineering?

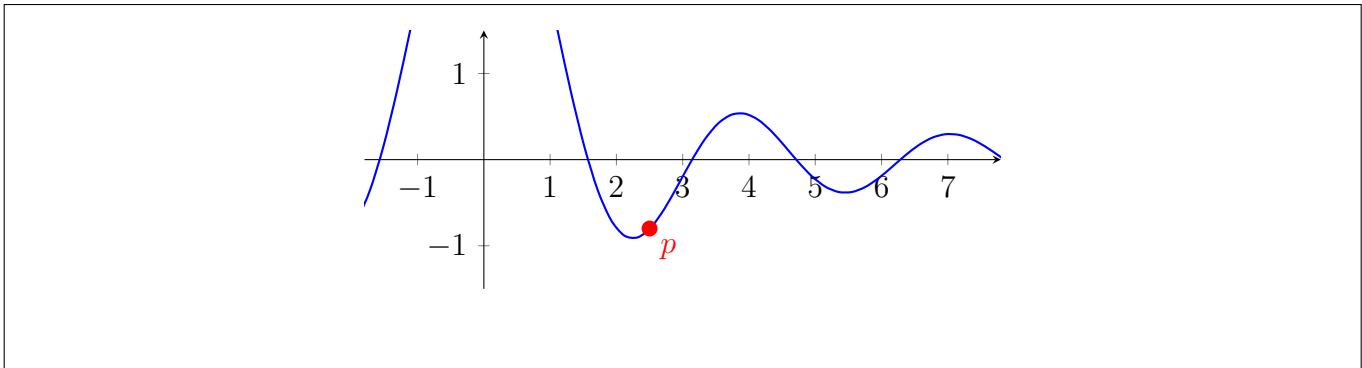
7. For each of the following, sketch a function  $f$  that has the required properties.

- (a) Left-endpoint approximations give the *exact* value of  $\int_0^1 f(x) dx$ .  
 (b) Left-endpoint approximations give an *under estimate* of the value of  $\int_0^1 f(x) dx$ .  
 (c) Left-endpoint approximations give an *over estimate* of the value of  $\int_0^1 f(x) dx$ .  
 (d) Trapezoidal approximations give the *exact* value of  $\int_0^1 f(x) dx$ .  
 (e) Trapezoidal approximations give an *under estimate* of the value of  $\int_0^1 f(x) dx$ .  
 (f) Trapezoidal approximations give an *over estimate* of the value of  $\int_0^1 f(x) dx$ .  
 (g) Midpoint approximations give an *over estimate* of the value of  $\int_0^1 f(x) dx$ .

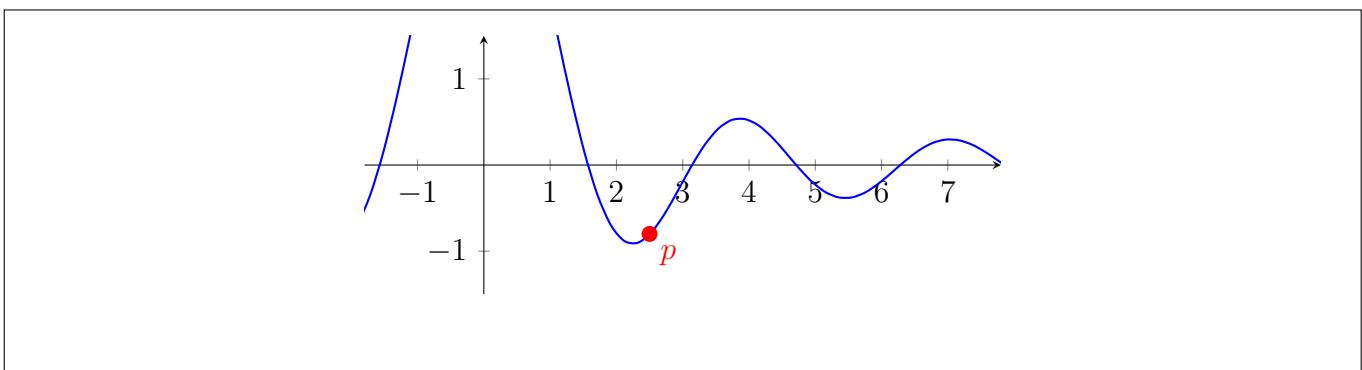
8. For each part of this question you are given the graph of an unknown function  $f$ . On this graph is marked the point  $p$ . For each part, sketch the requested polynomial.

Note: your sketch only needs to be approximate, but it should demonstrate the important behaviour of the requested polynomial.

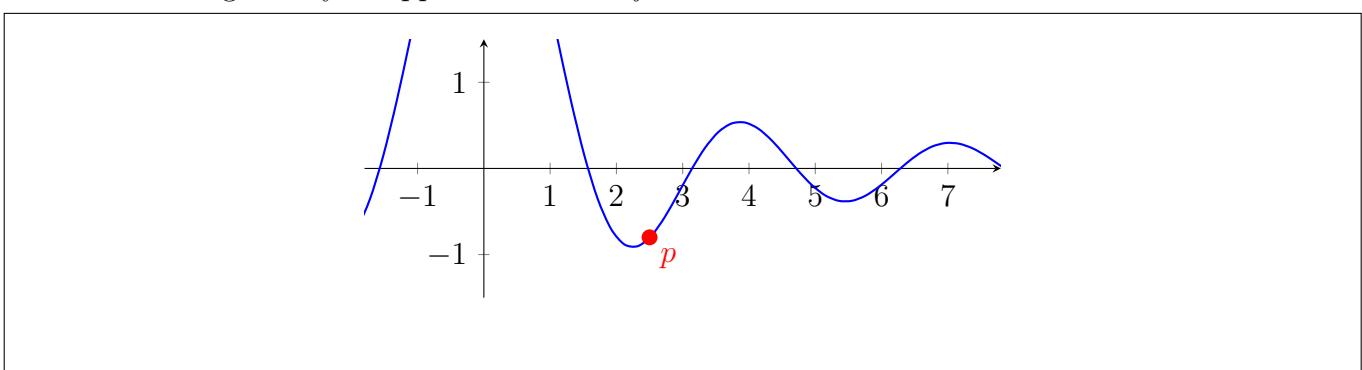
- (a) Sketch a 0th degree Taylor approximation to  $f$  centered at 2.5.



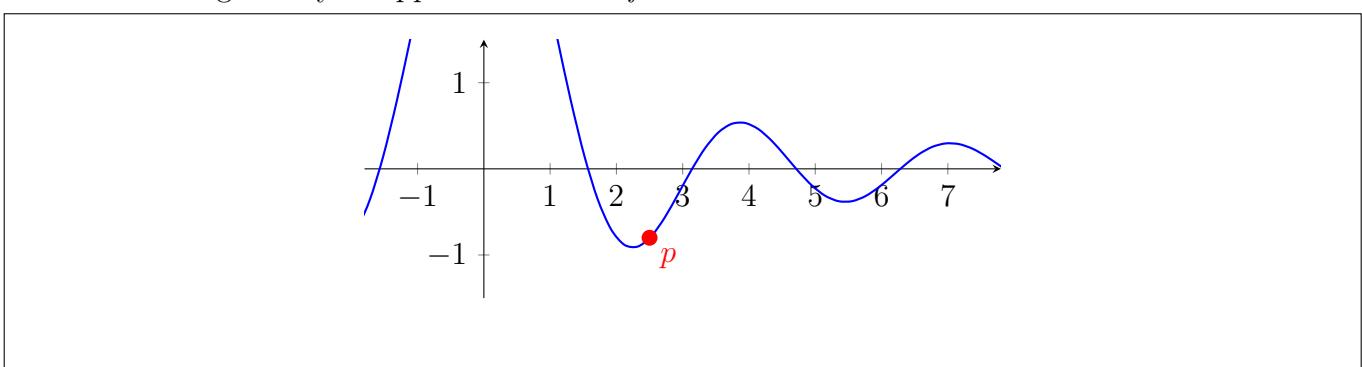
- (b) Sketch a 1st degree Taylor approximation to  $f$  centered at 2.5.



- (c) Sketch a 2nd degree Taylor approximation to  $f$  centered at 2.5.



- (d) Sketch a 3rd degree Taylor approximation to  $f$  centered at 2.5.



9. Let  $f(x) = \sin(x)$  and let  $I = \int_0^\pi f(x) dx$ . By using antiderivatives, we know that  $I = 2$ , exactly.

For this question, you may use a computer/calculator to evaluate any sums.

- (a) Let  $L_n$  be an approximation of  $I$  based on *left* endpoints where the interval  $[0, \pi]$  is divided into  $n$  regions.
  - i. Write down a formula (using summation notation) for  $L_n$ .
  - ii. What is the smallest  $n$  so that  $L_n$  is within 0.01 of the correct answer?
- (b) Let  $M_n$  be an approximation of  $I$  based on the midpoint rule where the interval  $[0, \pi]$  is divided into  $n$  regions.
  - i. Write down a formula (using summation notation) for  $M_n$ .
  - ii. What is the smallest  $n$  so that  $M_n$  is within 0.01 of the correct answer?
- (c) Let  $T_n$  be an approximation of  $I$  based on the trapezoid rule where the interval  $[0, \pi]$  is divided into  $n$  regions.
  - i. Write down a formula (using summation notation) for  $T_n$ .
  - ii. What is the smallest  $n$  so that  $T_n$  is within 0.01 of the correct answer?
- (d) Let  $S_n$  be an approximation of  $I$  based on Simpson's rule where the interval  $[0, \pi]$  is divided into  $n$  regions.
  - i. Write down a formula (using summation notation) for  $S_n$ .
  - ii. What is the smallest  $n$  so that  $S_n$  is within 0.01 of the correct answer?

## Bonus:

10. Simpson's Rule can be easily derived based off of the following fact: if  $p$  is a quadratic polynomial, then  $I = \int_{x_0}^{x_1} p(x) dx$  can be computed as a linear combination of  $p(x_0)$ ,  $p(x_1)$ , and  $p(\frac{x_0+x_1}{2})$ . That is

$$I = ap(x_0) + bp\left(\frac{x_0 + x_1}{2}\right) + cp(x_1)$$

where  $a, b, c$  are constants that do not depend on  $p$  nor on  $x_0$  or  $x_1$ .

- (a) Let  $p(x) = x^2$  and compute  $\int_{-2}^0 p(x) dx$ ,  $\int_{-1}^1 p(x) dx$ , and  $\int_0^2 p(x) dx$ .
- (b) For each of the previous integrals, what are the values of  $x_0$ ,  $x_1$ , and  $\frac{x_0+x_1}{2}$ ? Use these values to compute  $p(x_0)$ ,  $p(x_1)$ , and  $p(\frac{x_0+x_1}{2})$ .
- (c) Use what you've learned from the previous parts to solve for  $a, b, c$  in the expression  $I = ap(x_0) + bp\left(\frac{x_0+x_1}{2}\right) + cp(x_1)$ .
- (d) Let  $f(x) = \frac{1}{1+x^2}$  and let  $p$  be a quadratic interpolating polynomial through  $(-1, f(-1))$ ,  $(0, f(0))$ , and  $(1, f(1))$ . Use your knowledge from the previous parts to find  $\int_{-1}^1 p(x) dx$  without finding  $p$ .
- (e) (Bonus) The integral of a cubic can be computed as a linear combination of the value of that cubic at four equally spaced points. Come up with a "Super" Simpson's rule based on using cubic interpolation instead of quadratic interpolation.

11. Taylor polynomials tend to be accurate in a symmetric region about the point where they are centered. That is, if  $T_n$  is an  $n$ th Taylor approximation of  $f$  centered at  $a$ , we would expect  $T$  to be accurate in some region  $(a - \varepsilon, a + \varepsilon)$ . Here  $\varepsilon$  would depend on  $n$ ,  $a$ , and  $f$ .

For this question, you may use the following Desmos link to graph Taylor approximations:

<https://www.desmos.com/calculator/jdv3z3trvu>

However, *be warned* for Taylor polynomials above degree 9 or so, Desmos suffers from significant rounding errors.

- (a) For this part we will be approximating the function  $f(x) = 1/x$ . Let  $T_n$  be the  $n$ th Taylor approximation to  $f$  centered at  $a$ .
  - i. When  $a = 2$ , for which  $\varepsilon$  is  $T_8$  a good approximation to  $f$  on the interval  $(a - \varepsilon, a + \varepsilon)$ ? (Hint: you will need to use your judgement to decide what a “good” approximation is.)
  - ii. When  $a = 1$ , for which  $\varepsilon$  is  $T_8$  a good approximation to  $f$  on the interval  $(a - \varepsilon, a + \varepsilon)$ ?
  - iii. When  $a = 1/2$ , for which  $\varepsilon$  is  $T_8$  a good approximation to  $f$  on the interval  $(a - \varepsilon, a + \varepsilon)$ ?
  - iv. Why does  $\varepsilon$  keep getting smaller as  $a$  gets smaller?
  - v. Based on your conjecture, find a formula for an upper bound for  $\varepsilon$  based on  $a$ .
- (b) Repeat the previous part using  $f(x) = \sin x$ . Does the region of “good” approximation,  $(a - \varepsilon, a + \varepsilon)$ , change as  $a$  changes?
- (c) For this part, we will use the function  $f(x) = \frac{1}{1+x^2}$ .

Use the following Desmos link: <https://www.desmos.com/calculator/opdydtzixw>

- i. When  $a = 4$ , for which  $\varepsilon$  is  $T_{20}$  a good approximation to  $f$  on the interval  $(a - \varepsilon, a + \varepsilon)$ ?
- ii. When  $a = 3$ , for which  $\varepsilon$  is  $T_{20}$  a good approximation to  $f$  on the interval  $(a - \varepsilon, a + \varepsilon)$ ?
- iii. When  $a = 2$ , for which  $\varepsilon$  is  $T_{20}$  a good approximation to  $f$  on the interval  $(a - \varepsilon, a + \varepsilon)$ ?
- iv. Find all (complex) vertical asymptotes of the function  $f$ .
- v. Why did  $\varepsilon$  shrink in the previous parts?
- vi. Based on your conjecture, find a formula for an upper bound for  $\varepsilon$  based on  $a$ . (Hint: this part will involve doing some geometry in the complex plane.)

12. Consider the function  $f(x) = \frac{1}{1-x}$  on  $(-\infty, 1)$ . Let  $P_n$  denote the  $n^{th}$  Taylor polynomial of  $f$  centred at  $a = 0$ .

1. Use the Taylor Remainder Theorem to find an upper bound for the error at  $x = \frac{1}{2}$ .
  2. The geometric sum formula states  $\sum_{n=0}^k \alpha^n = \frac{\alpha^{k+1}-1}{\alpha-1}$ . Use the geometric sum formula to find a closed-form expression for  $P_n(x)$  (i.e., an expression that does not use summation notation or any “...”s).
- Use your formula to find an exact expression for the error at  $x = \frac{1}{2}$  and describe how the error decreases as we increase  $n$ .
3. Use Desmos to graph  $f$  and  $P_n$  for  $n = 1$  to  $n = 50$ . Based on your visual evidence, does  $P_n$  give a good approximation for  $f$  in  $(-\infty, 1)$ ? If not, what is the interval over which  $P_n$  is a reasonable approximation of  $f$ ?
  4. Taylor’s remainder theorem states that  $|f(x) - P_n(x)| = \frac{f^{n+1}(c(x))x^{n+1}}{(n+1)!}$ , where  $c(x)$  takes on a mysterious value that depends on  $x$  and is somewhere between  $x$  and  $a$ . Normally, we cannot compute  $c(x)$ , but in this example we can. Find the function  $c(x)$ .