University of Toronto, Faculty of Applied Science and Engineering

MAT187H1S - Calculus II

Final Exam - April 24, 2015

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Duration: 150 minutes. No Aids permitted.

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages and a detached **formula sheet**. Make sure you have all of them.
- You can use pages 13–14 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 13–14.

GOOD LUCK!

PART I. No explanation is necessary. **(20 Marks)**

For **questions 1-4**, consider the differential equation for $y(t)$:

$$y' + ty + t = 0.$$

1. The equilibrium solution is (circle one option)

(a) $y(t) = -1$ **(c)** $y(t) = 1$

(b) $y(t) = 0$ **(d)** There is no equilibrium solution

2. If a solution $y(t)$ has a horizontal asymptote, then what is it?

$y =$ -1 .

3. Consider the solution $y(t)$ which satisfies $y(0) = 1$. Complete the blanks:

$y'(0) =$ 0 $[y' = -t(y+1)]$,

$y''(0) =$ -2 $[y'' = -(y+1) - t\,y']$.

4. (Harder!) Consider the solution $y(t)$ which satisfies $y(0) = 1$. What is the first value $T > 0$ for which $y(t) < 0$ for all $t > T$?

$T =$ $\sqrt{2\ln 2}$. $\left(y = -1 + 2e^{-t^2/2}\right)$

5. Consider two particles with positions $r_1(t) = \big(\cos(t), \frac{t}{2}\big)$ and $r_2(t) = \big(1 + t^2, \pi e^t\big)$.

(a) These particles collide and the paths intersect.

(b) The particles collide, but the paths don't intersect.

(c) The particles don't collide, but the paths intersect.

(d) The particles don't collide and the paths don't intersect.

Continued...

For **questions 6–9**, please consider the following power series

$$f(x) = \sum_{k=0}^{\infty} \frac{k+1}{4^k} x^k.$$

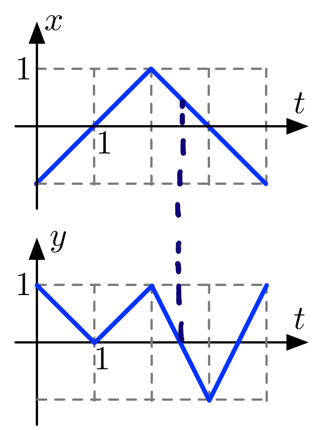
6. The radius of convergence for this series is $R =$ 4.

7. $\int f(x)\,dx =$ $\dfrac{x}{1-x/4} + C$ $\left[x \sum_{k=0}^{\infty} \left(\frac{x}{4}\right)^k + C \right]$ (not as a power series).

8. $f(x) =$ $\dfrac{1}{(1-x/4)^2} = \dfrac{16}{(4-x)^2}$ (not as a power series).

9. $\sum_{k=0}^{\infty} (-1)^k \frac{(k+1)}{2^k} =$ $f(-2) = \frac{4}{9}$.

10. The position of a particle is $\vec{r}(t) = \big(x(t), y(t)\big)$ as given in the figure on the right.



(a) The slope of the path of the particle at the point $\left(\frac{1}{2}, 0\right)$ is

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-2}{-1} = 2$ $[t = 2.5]$.

(b) At this point, the particle is moving to the

left / **right** and **up** / **down** .

x decreasing y decreasing

(circle the correct options)

PART II. Answer the following questions. **Justify** your answers.

11. Consider the curve $\vec{r}(t) = \left(-\frac{t^3}{3}, t^2, 1-2t\right)$. **(20 Marks)**

(a) (10 marks) Find the maximum curvature.

(**Hint.** Use the formula with the cross product)

$$k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\vec{v} = (-t^2, 2t, -2) \qquad \vec{a} = (-2t, 2, 0)$$

$$|\vec{v}| = \sqrt{t^4 + 4t^2 + 4} = t^2 + 2$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -t^2 & 2t & -2 \\ -2t & 2 & 0 \end{vmatrix} = 4\vec{i} + 4t\vec{j} + 2t^2\vec{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{16 + 16t^2 + 4t^4} = 2\sqrt{4 + 4t^2 + t^4} = 2(t^2+2)$$

$$\Rightarrow \boxed{k = \frac{2(t^2+2)}{(t^2+2)^3} = \frac{2}{(t^2+2)^2}}$$

max k is attained for min t^2+2 which happens for $t=0$.

So $\boxed{\max k = \frac{2}{2^2} = \frac{1}{2}.}$

(b) (6 marks) Show that there is no minimum curvature.

$$\kappa = \frac{2}{(2+t^2)^2} > 0$$

and $\lim_{t\to\infty} \kappa = 0$

$\Rightarrow$ There is no minimum, since it would be 0, but $\kappa \neq 0$ for any t.

(c) (4 marks) Consider the particles with positions $\vec{r_1}(t) = \vec{r}(t)$ and $\vec{r_2}(t) = \vec{r}(e^3 - \pi t)$. Will these particles collide? If so, when?

These 2 particles are following the same path $\vec{r}(t)$ in opposite directions, so $\boxed{\text{they will collide.}}$

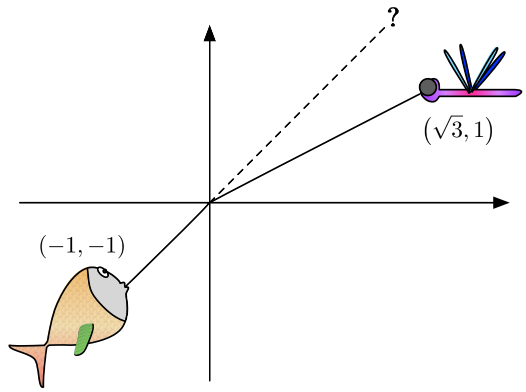
This will happen when $\vec{r_1}(t) = \vec{r_2}(t) \Leftrightarrow t = e^3 - \pi t \Leftrightarrow \boxed{t = \frac{e^3}{1+\pi}}$

Continued...

12. Archer fish hunt by spitting a jet of water at a nearby flying insect. **(20 Marks)**

(a) (5 marks) Assume that the fish's position, as shown, is $(-1,-1)$ and the insect's position is $(\sqrt{3},1)$ and the water surface is at $y=0$.

When the fish sees the insect, water refraction changes the angle of the light as in the figure on the right, causing the fish to misjudge the insect's position.



The fish sees the path to the insect as a straight line. What does the fish think is the position of the insect?

The misjudged position of the insect is

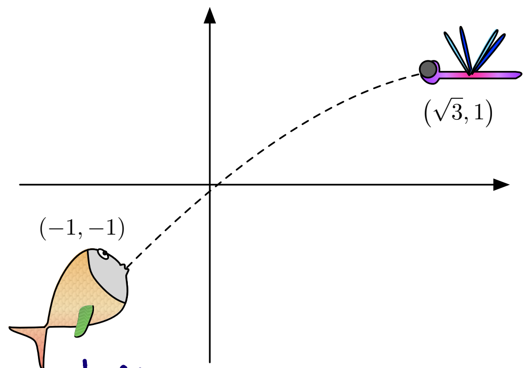
- in polar coordinates : $\theta = \frac{\pi}{4}$ and $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

  $(r,\theta) = (2, \pi/4)$

- in rectangular coordinates : $(x,y) = (\sqrt{2}, \sqrt{2}) = \left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$

Continued...

(b) (10 marks) The fish hunts the insect by spitting a jet of water in the direction of the dotted line. Ignoring the effects of water and air resistance, but considering gravity, $\vec{a} = (0, -g)$, how fast should the fish spit the water to hit the insect? Assume all distances given are in cm and the insect is not moving.



The position of the jet of water $\vec{r}(t)$ satisfies:

$$\begin{cases} \vec{r}''(t) = (0, -g) \\ \vec{r}(0) = (-1, -1) \\ \vec{r}'(0) = (u_0, u_0) \end{cases}$$

(it starts from the fish's mouth)

($u_0 = v_0$ because the fish spits the water in the direction of the dotted line: the perceived direction of the insect: $\theta = \pi/4$)

$\Rightarrow \vec{r}(t) = \left(-1 + u_0 t, -1 + u_0 t - \frac{g}{2}t^2\right)$

we want to hit the insect, so $\vec{r}(T) = (\sqrt{3}, 1)$:

$$\begin{cases} -1 + u_0 T = \sqrt{3} \\ -1 + u_0 T - \frac{g}{2}T^2 = 1 \end{cases} \Leftrightarrow \begin{cases} -1 + u_0 T = \sqrt{3} \\ -\frac{g}{2}T^2 = 1 - \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} u_0 = \frac{1+\sqrt{3}}{T} \\ T = \frac{2}{g}(\sqrt{3} - 1) \end{cases}$$

The fish should spit the water with initial velocity $\boxed{\vec{r}'(0) = (u_0, u_0) \text{ with } u_0 = \frac{1+\sqrt{3}}{\sqrt{3}-1}\frac{g}{2}}$

This gives a speed $|\vec{r}'(0)| = \sqrt{2}\, u_0 = \boxed{\frac{1+\sqrt{3}}{\sqrt{3}-1}\frac{g}{\sqrt{2}}}$

(c) (5 marks) How much time does the insect have to move out of the way?

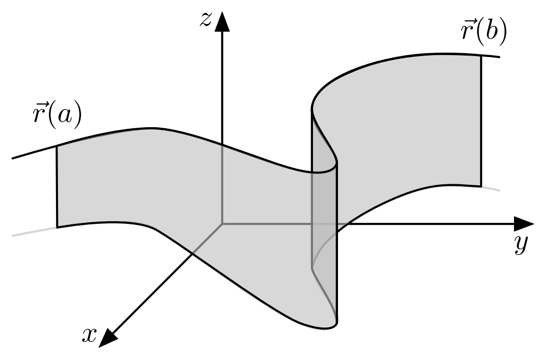
The insect has $T = \frac{2}{g}(\sqrt{3} - 1)$ seconds to move out of the way.

13. Consider a vector-valued function **(20 Marks)**

$$\vec{r}(t) = \big(x(t), y(t), z(t)\big),$$

for $t \in [a, b]$. The graph of this function forms a curve.

Find a formula for the area between the curve and the $xy-$plane (as indicated in the figure).



Hint. To make the exercise easier, you can follow the steps:

(a) Let $a < t_0 < t_1 < b$. Sketch the rectangle with coordinates

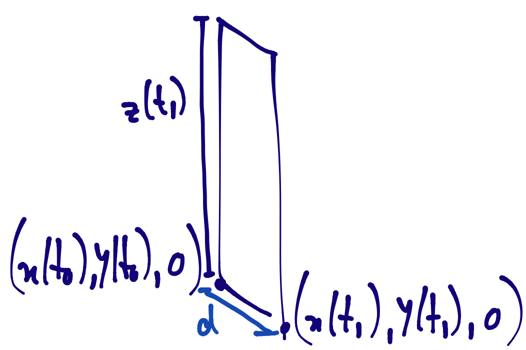
$P_0 = \big(x(t_0), y(t_0), 0\big)$, $P_1 = \big(x(t_0), y(t_0), z(t_1)\big)$, $P_2 = \big(x(t_1), y(t_1), z(t_1)\big)$, $P_3 = \big(x(t_1), y(t_1), 0\big)$

(b) What is the area of the rectangle from **(a)**?

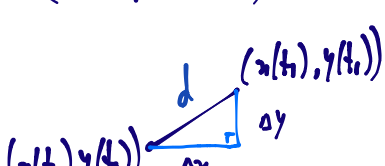
(c) Approximate the area (between the curve and the $xy-$plane) with the area of n thin rectangles.

(d) Find a formula for the exact area between the curve and the $xy-$plane by taking the limit as $n \to \infty$.

(a)



(b) $A = z(t_1) \cdot d$



$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\big(x(t_1) - x(t_0)\big)^2 + \big(y(t_1) - y(t_0)\big)^2}$$

$$\text{Area} = z(t_1)\sqrt{\big(x(t_1) - x(t_0)\big)^2 + \big(y(t_1) - y(t_0)\big)^2}$$

Continued...

(Use this page to continue question 13)

(c) Split the interval $[a,b]$ in $\underline{n}$ subintervals

$$a = t_0 < t_1 < \dots < t_n = b$$

with $\begin{cases} t_i = a + i\,\Delta t \\ \Delta t = \dfrac{b-a}{n} \end{cases}$

Then we can approximate the area of the "curtain" by adding the area of $\underline{n}$ narrow (vertical) rectangles with vertices

$$(x(t_{i-1}), y(t_{i-1}), 0),\ (x(t_{i-1}), y(t_{i-1}), z(t_i)),\ (x(t_i), y(t_i), z(t_i)),\ (x(t_i), y(t_i), 0)$$

Each of these rectangles is of the same form as in parts (a) and (b), so the area is $A_i = z(t_i) \cdot \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

We add these areas to obtain an approximation of the exact area:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} z(t_i)\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} z(t_i)\sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\,\Delta t \quad \text{(Riemann Sum)}$$

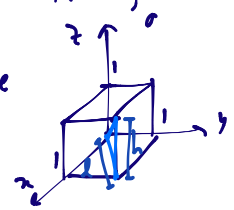
(d) Take the limit as $n \to \infty$ to deduce the formula for the area:

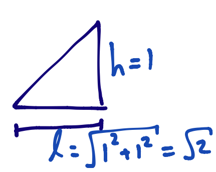
$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} z(t_i)\sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\,\Delta t = \int_a^b z(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt}$$

Bonus. Test your formula. Use your formula to find the area for the function **(4 Marks)**
$\vec{r}(t) = (t, t, t)$ for $t \in [0, 1]$ and confirm that it matches the formula for the area of a triangle.

The formula above gives $A = \int_0^1 t\sqrt{1^2 + 1^2}\,dt = \left[\sqrt{2}\,\frac{t^2}{2}\right]_0^1 = \frac{\sqrt{2}}{2}$.

The area forms a triangle





$\Rightarrow$ The area is $A = \dfrac{l \cdot h}{2} = \dfrac{\sqrt{2}}{2}$

They are the same, which confirms the formula.

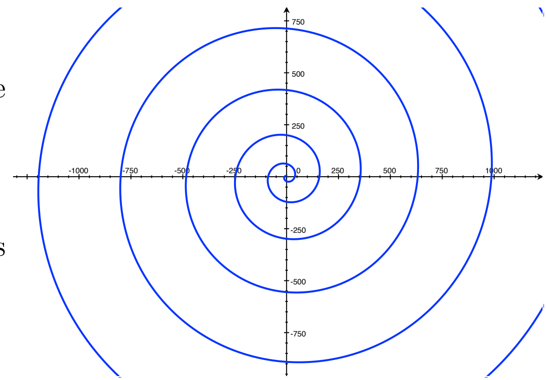
Continued...

14. In this question we will investigate the shells of some land snails. **(20 Marks)**

(a) (8 marks) A certain snail's shell has the shape of the spiral

$$r = \theta^2,$$

where the magnitude of the angle θ in radians equals the number of days since the snail was hatched.



If the variable r is in mm, how long will it take for the shell's spiral to be $\dfrac{19}{3}$ mm long?

We need to find the angle θ (also number of days) when the length of the spiral is $\frac{19}{3}$.

$$\text{length} = \int_0^T \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta = \int_0^T \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^T \theta \sqrt{4+\theta^2}\, d\theta = \frac{1}{2}\int_4^{4+T^2} \sqrt{u}\, du$$

$$\boxed{\begin{aligned} u &= 4+\theta^2 \\ du &= 2\theta\, d\theta \end{aligned}}$$

$$= \left[\frac{1}{2}\cdot\frac{2}{3} u^{3/2}\right]_4^{4+T^2} = \frac{1}{3}\left[(4+T^2)^{3/2} - 4^{3/2}\right]$$

So $\text{length} = \frac{19}{3} \iff (4+T^2)^{3/2} - 8 = 19 \iff (4+T^2)^{3/2} = 27$

$\iff 4+T^2 = (27)^{2/3} = 9 \iff T^2 = 5 \iff \boxed{T = \sqrt{5} \text{ days}}$

Continued...

(b) (8 marks) A different snail's shell also forms a spiral. Each night, since it's humid, the length grows by $\dfrac{1}{2n}$ cm, but during the day, since it's dry the length decreases by $\dfrac{1}{2n+1}$ cm, where n is the number of days since it hatched.

If the snail lives forever, how long will the shell's spiral be?

(**Hint.** If you use $\sum$ notation, it should remind you of a Taylor series, with x set equal to -1)

[The snail's growth cannot start with $n=0$, otherwise we would have $\frac{1}{2\cdot 0}$]

length(n) = length of the shell's spiral at the end of day n.

length$(1) = \frac{1}{2} - \frac{1}{3}$, length$(2) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

length$(3) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}$

If this happens forever, the shell will have length $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k}$.

As stated in the hint, this series resembles the Taylor serie $\ln(1-x) = -\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$ with $x=-1$

We then have $\ln(1-(-1)) = -\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \sum_{k=2}^{\infty} \frac{(-1)^k}{k}$

So the length of the snail's shell will be $\boxed{\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k} = 1 - \ln(2) \text{ cm}}$ long.

Continued...

(c) (4 marks) For the same snail as in **(b)**, how many days will it take for the shell's spiral to be within $\frac{1}{100}$ cm of the limiting size?

At the end of N days (night + day), the length is

$$\text{length}(N) = \sum_{k=2}^{2N+1} \frac{(-1)^k}{k}$$

We want to find N such that the remainder of this alternating series is smaller than $\frac{1}{100}$.

$$|R_{2N+1}| = \frac{1}{2N+2} \leq \frac{1}{100} \iff 2N+2 \geq 100$$

$$\iff N \geq \frac{98}{2} = 49$$

At the end of 49 days (nights and days), the snail's shell will be within $\frac{1}{100}$ cm of its limiting size.

Bonus. Excluding this bonus question, what is your mark on this exam? __________ ±2 points **(3 marks)**