Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 3

1 (a) Do the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ intersect? If they do, find the point of intersection.

Solution: Suppose that the two lines intersect, then there is a point with coordinates (x, y, z) that lies on both lines, i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \tag{1}$$

or equivalently,

$$s \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \tag{2}$$

for some $s, t \in \mathbb{R}$. Notice that the above system is satisfied for $s = t = -1$. Therefore, the point $(2, 3, 0)$ lies on both lines, i.e. the lines intersect. Another way to solve this is to row-reduce the augmented matrix of (2), into:

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 3 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \tag{3}$$

which shows that (2) is always consistent and thus has a unique solution.

1 (b) Consider the non-parallel lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. Find the shortest distance between the two lines and find the points on the lines that are closest together.

Solution: Let $P(1+s, -1+s, s)$ and $Q(2+3t, -1+t, 3)$ be the points on the lines corresponding to the shortest distance. Then, the vector from P to Q is denoted as $\mathbf{PQ} = \mathbf{Q} - \mathbf{P} = (1 + 3t - s, t - s, 3 - s)$. Because P and Q represent the shortest distance between the two lines, then $\mathbf{PQ}$ is orthogonal to both lines, i.e. $\mathbf{PQ} \cdot \mathbf{u_1} = 0$ and $\mathbf{PQ} \cdot \mathbf{u_2} = 0$, where $\mathbf{u_1}(1, 1, 1)$ and $\mathbf{u_2}(3, 1, 0)$ are the direction vectors of the two lines. Thus, we can write:

$$\mathbf{PQ} \cdot \mathbf{u_1} = 0 \quad \Longleftrightarrow \quad 4t - 4s + 4 = 0 \tag{4}$$

$$\mathbf{PQ} \cdot \mathbf{u_2} = 0 \quad \Longleftrightarrow \quad 10t - 4s + 3 = 0 \tag{5}$$

Solving the system of equations (4) and (5), we get $t = 1/6$ and $s = 7/6$, so that $\mathbf{PQ} = (1/3, -1, 11/6)$, $P = (13/6, 1/6, 7/6)$ and $Q = (5/2, 5/6, 3)$, and the shortest distance is $|\mathbf{PQ}| = \sqrt{1/9 + 1^2 + 121/36} = \sqrt{161/36}$

1 (c) Show that the shortest distance from a point Q to the line through the point P with direction vector $\mathbf{d}$ is

$$\frac{||\vec{PQ} \times \mathbf{d}||}{||\mathbf{d}||}$$

Solution: Recall that $||\vec{PQ} \times \mathbf{d}||$ is the area of the paralellogram with base length $\|\mathbf{d}\|$. In general, the area of a paralellogram is base $\times$ height. Therefore, $||\vec{PQ} \times \mathbf{d}||$ = base $\times$ height = $\|d\| \times$ height, or equivalently, height = $\frac{||\vec{PQ} \times \mathbf{d}||}{||\mathbf{d}||}$, which is the shortest distance from point Q to the line.

2. Consider the system of linear equations

$$\begin{aligned} x_1 - 3x_2 &= 2 \\ -2x_1 + 6x_2 &= -4 \end{aligned}$$

(a) Give two different particular solutions to the given system (your choice). Write your solutions as (x_1, x_2) and (y_1, y_2). i.e. Think of them as points common to both lines. Show that $\dfrac{(x_1, x_2) + (y_1, y_2)}{2}$ is a solution to the system but $(x_1, x_2) + (y_1, y_2)$ is not.

Solution: Let $(x_1, x_2) = (5, 1)$ and $(y_1, y_2) = (2, 0)$, and notice that both of them satisfy the above system of equations. $\frac{(x_1,x_2)+(y_1,y_2)}{2} = (7/2, 1/2)$ satisfies both equations, but $(x_1, x_2) + (y_1, y_2) = (7, 1)$ does not.

(b) Show that if (x_1, x_2) and (y_1, y_2) are two solutions to the given system, then $a(x_1, x_2) + b(y_1, y_2)$ is also a solution if and only if $a + b = 1$.

Solution: Substitute $a(x_1, x_2) + b(y_1, y_2)$ in the first equation to get:

$$\begin{aligned} & ax_1 + by_1 - 3ax_2 - 3by_2 = 2 && (6) \\ \iff\ & a(x_1 - 3x_2) + b(y_1 - 3y_2) = 2 && (7) \\ \iff\ & 2a + 2b = 2 && (8) \\ \iff\ & 2(a + b) = 2 && (9) \\ \iff\ & (a + b) = 1 && (10) \end{aligned}$$

where we used the fact that (x_1, x_2) and (y_1, y_2) satisfy the system of equations, so that $x_1 - 3x_2 = 2$ and $y_1 - 3y_2 = 2$.

3. Consider the system of linear equations

$$\begin{aligned} ax_1 + bx_2 &= c \\ bx_1 + cx_2 &= d \end{aligned}$$

where $a \neq 0$. Find conditions on the remaining constants b, c, d such that the system has a unique solution; infinitely many solutions; or no solutions.

Solution: Reducing the above system we get:

$$\left[\begin{array}{cc|c} 1 & \frac{a}{b} & \frac{c}{a} \\ 0 & c - \frac{b^2}{a} & d - b\frac{c}{a} \end{array}\right] \qquad (11)$$

Note that we may divide by a since we have assumed that $a \neq 0$.

(i) If $c - b\frac{b}{a} \neq 0$, then there is a unique solution.

(ii) If $c - b\frac{b}{a} = 0$, then there are two further possibilities: if $d - b\frac{c}{a} = 0$ then there are infinitely many solutions, while if $d - b\frac{c}{a} \neq 0$ there are no solutions at all.

4. Suppose a linear system of equations in the variables x_1, x_2, x_3 has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 1 & c & 1 & 1 \\ c & 1 & 1 & -2 \end{array}\right]$$

For what values of c is the system (i) inconsistent, (ii) consistent with a unique solution, and (iii) consistent with infinitely many solutions?

Solution: The operations $R_3 - cR_1 and R_2 - R_1$ give

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 0 & c-1 & 1-c & 0 \\ 0 & 1-c & 1-c^2 & -c-2 \end{array}\right]$$

Now $R_3 + R_2$ gives

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 0 & c-1 & 1-c & 0 \\ 0 & 0 & 2-c-c^2 & -c-2 \end{array}\right]$$

If $c \neq 1$ and $c \neq -2$. Then, $c-1 \neq 0$ and $2-c-c^2 \neq 0$. This allows us to perform the operations $\frac{1}{c-1}R_2$ and $\frac{1}{2-c^2-c}R_2$ which gives

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{-c-2}{2-c^2-c} \end{array}\right]$$

so that the system has a unique solution.

If $c = 1$ then we have,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

which gives us no solution since the last row gives us the equation $0 = -3$.

Finally, if $c = -2$ we have,

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which after $-R_2/3$ gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which gives us infinitely many solutions (Column 3 has no leading co-efficient, i.e. x_3 can be set as a parameter).

Thus we have $c = 1$ gives no solution, $c = -2$ gives us infinitely many solutions to the system of equations. And for all other values of c we have a unique solution to the system.