UNIVERSITY OF TORONTO, FACULTY OF APPLIED SCIENCE AND ENGINEERING

MAT187H1S - Calculus II

Final Exam - April 24, 2015

EXAMINERS: D. BURBULLA, S. COHEN, B. GALVÃO-SOUSA, P. MILGRAM, D. PANCHENKO, M. PAWLIUK, L.-P. THIBAULT

Time allotted: 150 minutes

No Aids permitted

Total marks: 100

Full Name: ______________________________________
Last First

Student ID: ______________________________________

Email: ______________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages and a detached **formula sheet**. Make sure you have all of them.
- You can use pages 13–14 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 13–14.

GOOD LUCK!

PART I. No explanation is necessary. **(20 Marks)**

1. **(2 marks)** Compute the following integral.

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}}\,dx = \rule{8cm}{0.4pt}.$$

2. **(2 marks)** When **calculating the integral**

$$\int \frac{x^4+2}{x(x-3)^2(1+x^2)}\,dx,$$

we obtain the sum of the following terms (**circle all that apply**):

(a) $A\ln|x|$

(b) $\dfrac{B}{x}$

(c) C

(d) $D\ln|x-3|$

(e) $\dfrac{E}{x-3}$

(f) $\dfrac{F}{(x-3)^2}$

(g) $G\ln(1+x^2)$

(h) $\dfrac{H}{1+x^2}$

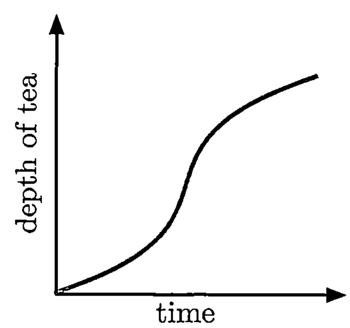
(i) $\dfrac{Ix}{1+x^2}$

(j) $J\arctan(1+x^2)$

(k) $K\arctan(x)$

(l) $L\,x\arctan(x)$

3. **(2 marks)** Tea is poured in a mug at a constant rate (constant volume per unit time). The graph on the right shows the depth of tea in the mug as a function of time.



Which of the mugs below was used? (**circle one option**).

4. **(2 marks)** Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. How many terms do we need to add to guarantee that the error of the approximation is smaller than $\frac{1}{2016}$?

$p =$ ________________________________

5. **(2 marks)** Approximate $f(\theta) = \sin\left(\frac{\theta^2}{3}\right)$ with a 6^{th}-order Taylor polynomial centred at $\theta = 0$.

$p_6(\theta) =$ ________________________________.

For **questions 6–9**, consider the power series:

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{4n+5}}(x-1)^n$$

6. **(2 marks)** The radius of convergence is

$R =$ ________________________________.

7. **(2 marks)** The interval of convergence is

$x \in$ ________________________________.

8. **(2 marks)** $f^{(2016)}(1) =$ ________________________________.

9. **(2 marks)** Consider the initial-value problem

$$\begin{cases} u'(x) = f(x) \\ u(1) = -2 \end{cases}$$

where $f(x)$ is defined above. Then, the Taylor series for the solution is

$u(x) =$ ________________________________.

10. **(2 marks)** (Hard!) Write a **power series** that converges for $x \in [7, 9.5)$.

____

Continued...

PART II. Answer the following questions. **Justify** your answers.

11. Air pressure $p(y)$ at altitude y (in metres) is the force per unit area exerted by the weight of the air above, so **(20 Marks)**

$$p(y) = g \int_{y}^{30\,000} f(z)\,dz,$$

where $f(y)$ = density of air at altitude y, which is related to the atmospheric pressure:

$$p(y) = Rf(y)T(y),$$

where R is a constant and $T(y)$ is the temperature (in K) at altitude y.

(a) **(5 marks)** The temperature is a linear function of the altitude. Assume that the altitude is measured from sea-level, the temperature at sea-level is 300K and at 5000m is 250K. Find an expression for $T(y)$.

$T(y) =$

(b) **(5 marks)** Show that $p(y)$ satisfies the differential equation

$$p'(y) = -\frac{g}{R}\frac{p(y)}{T(y)}.$$

(c) **(8 marks)** Assume that the air pressure at sea level is 10^5 Pa.

Find a formula for $p(y)$ (it can depend on R and g).

$p(y) =$

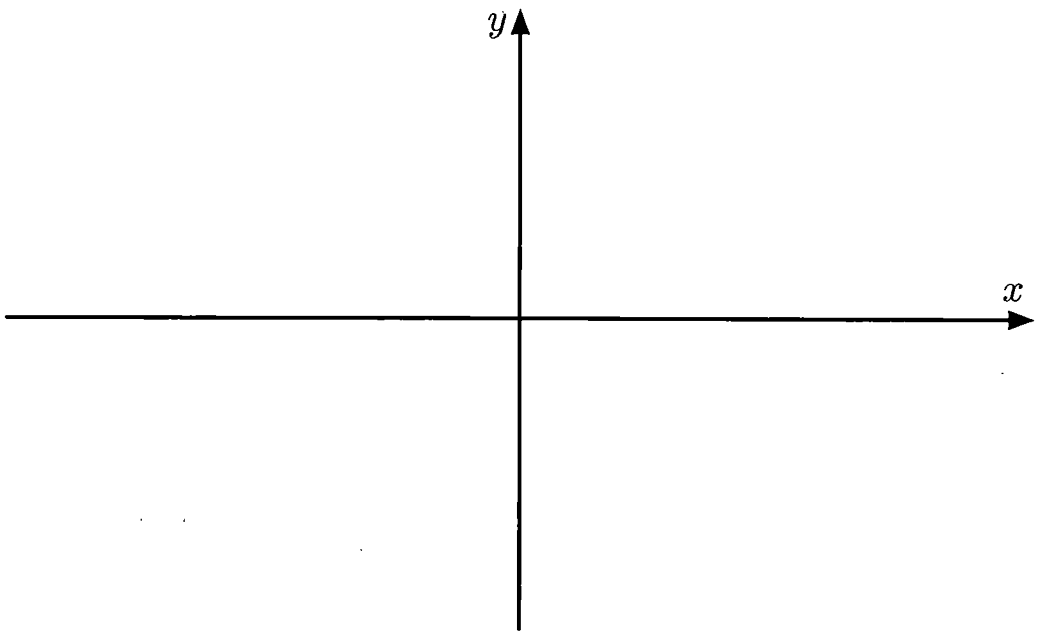
(d) **(2 marks)** The speed of sound v_s satisfies $v_s(y) = \sqrt{\dfrac{p(y)}{f(y)}}$.

Show that the speed of sound decreases with the altitude.

12. Consider the polar curves **(20 Marks)**

$$r = f(\theta) = \cos(\theta) \qquad \text{and} \qquad r = g(\theta) = \sqrt{3}\sin(\theta).$$

(a) **(5 marks)** Sketch both curves $r = f(\theta)$ and $r = g(\theta)$.



(b) **(5 marks)** These two polar curves intersect at two points P and Q. What are the two points of intersection?

$$P = \Big(\underline{\qquad\qquad\qquad}\ ,\ \underline{\qquad\qquad\qquad}\Big)_{x,y}$$

$$Q = \Big(\underline{\qquad\qquad\qquad}\ ,\ \underline{\qquad\qquad\qquad}\Big)_{x,y}$$

Continued...

(c) **(10 marks)** Calculate the area of the region inside both $r = f(\theta)$ and $r = g(\theta)$.

Area =

13. Consider a particle moving with position **(20 Marks)**

$$\vec{r}(t) = \Big(2\cos(t) \quad , \quad -\cos(t) + \sqrt{3}\sin(t) \quad , \quad -\cos(t) - \sqrt{3}\sin(t)\Big) \qquad \text{for all } t \geqslant 0.$$

(a) **(5 marks)** Show that this particle is always at the same distance from the origin.

(b) **(5 marks)** Show that the acceleration vector is perpendicular to the velocity vector.

(c) **(5 marks)** Calculate the curvature of the path of the particle.

$\kappa =$

(d) **(5 marks)** Find the binormal vector $\vec{B}(t)$. Does this particle moves on a plane?

(bonus) **(2 marks)** Based on **(a)** and **(d)**, what is the curve described by this particle?

(You don't need to have solved **(a)** or **(d)** to be able to answer)

14. After the MAT187 test, the good luck engineering panda arrived home and had a sore arm. It was hard-work giving all that candy and waving good luck to all the students. So he decided to use a spring mechanism for next time! **(20 Marks)**

The panda knows that a spring satisfies the differential equation

$$u''(t) + ku(t) = 0,$$

where $u(t)$ is the displacement from equilibrium (in cm) at time t (in seconds) and k is a constant.

(a) **(5 marks)** The panda initially compresses the spring 4 cm and then lets it go (without initial speed). Find a formula for $u(t)$.

(b) **(5 marks)** The spring oscillates back and forth. What is the period?

(c) **(5 marks)** The panda needs to deliver candy once every 10 seconds. For that he needs a spring with a specific constant k. Find k.

(d) **(5 marks)** The panda could only find an old spring, which satisfies

$$u''(t) + 2u'(t) + \frac{6}{5}u(t) = 0.$$

Find the general solution $u(t)$ that starts with no displacement from equilibrium, but with initial velocity v_0.

(bonus) **(2 marks)** Estimate how long it takes for the amplitude of the oscillation $u(t)$ to decay to $\dfrac{|v_0|}{10\sqrt{5}}$.

Continued...

USE THIS PAGE FOR ROUGH WORK.

Continued...

USE THIS PAGE FOR ROUGH WORK.

The end.