

University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 9

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We will start 10 minutes past the hour. Use this time to make
a new friend.

Differential Equations

Def'n: A differential equation is an equation that involves a variable y and its derivatives

$$f\left(\frac{dy}{dx}, y, x\right) = 0, \quad f(y'', y', y, x) = 0, \dots$$

$\uparrow \uparrow$
 x -derivatives

Use: When derivative of a quantity depends on the quantity itself

ex/ ① Population (more population = more babies = higher derivative)

② Chemical concentrations (radioactive decay)

③ : more next time

Given differential equation, the order is the largest derivative that appears in equation

ex/ $y' + yx^2 + x = 0$ \Leftarrow 1st order

$y'' + y'y + x^2 = 0$ \Leftarrow 2nd order

A linear ODE is one where all derivatives appear linear

ex// $y'' + 2y' + y + x^2 = 0$ \Leftarrow linear

$$y'' + 2(y')^2 + y = 0 \quad \Leftarrow \text{non-linear}$$

$$y'' + 2y'y = 0 \quad \Leftarrow \text{non-linear}$$

$$y'' + x^2y' + (2x^4)y = 0 \quad \Leftarrow \text{linear}$$

A differential equation is autonomous if it doesn't depend on the independent variable (x or t or...)

ex// $y'' + y' = 0 \quad \Leftarrow \text{autonomous}$

$$y' + x = 0 \quad \Leftarrow \text{not autonomous}$$

$$y'' + xy = 0$$

$$\frac{dy}{dx} + 2y = \cos(x)$$

→ linear

→ non-autonomous

→ 1st order

$$y'' + y \cdot \frac{dy}{dx} = 0$$

→ 2nd order

→ non linear

→ autonomous

$$\frac{dy}{dx} = y(1 - y) = y - y^2$$

→ 1st order

→ autonomous

→ non-linear

Solving Differential Equations - Guess and Check

ex/ $y' = y$

→ function equal to its own derivative?

$$y = e^x ? \quad (e^x)' \checkmark = (e^x)$$

$$y = 0 ? \quad (0)' \checkmark = 0$$

$$y = 3e^x \quad (3e^x)' \checkmark = 3e^x$$

ex/ $y' = 3y$

→ not e^x but what about e^{cx} ?

$$y' = 3y \quad \Leftarrow \text{plug-in} \quad y = e^{cx}$$

$$ce^{cx} = 3e^{cx}$$

$$c = 3 \quad \therefore y = e^{3x} ?$$

$$y'' = 3y$$

exponential? try $e^x \Rightarrow$ ~~$y = e^x \neq 3y = 3e^x$~~

$$\text{try } e^{cx} \Rightarrow y'' = c^2 e^{cx} \stackrel{?}{=} 3y = 3e^{cx}$$

$$\Rightarrow c^2 = 3 \quad \text{this works}$$

$$\text{Sol'n} \Rightarrow y = e^{\sqrt{3}x}$$

and

$$y = e^{-\sqrt{3}x}$$

$$\text{Check: } y = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x} \quad C_1, C_2 \in \mathbb{R}$$

$y'' = -3y$ is a solution cosine?

$$\text{try } y = \cos(cx) \Rightarrow y'' = -c^2 \cos(cx) \stackrel{?}{=} -3y = -3 \cos(cx)$$

$$\Rightarrow -c^2 = 3 \Rightarrow c = \pm \sqrt{3}$$

$$\therefore y = \cos(\sqrt{3}x)$$

also

$$y = \sin(\sqrt{3}x)$$

← check these

also

$$y = C_1 \sin(\sqrt{3}x) + C_2 \cos(\sqrt{3}x)$$

Solutions to ODE

A particular solution to an ODE is any $y(x)$ that satisfies the ODE

The general solution is the family of all possible solutions to an ODE

$$\text{ex/1 } y' = y \quad \text{General solution } y = Ce^x \quad C \in \mathbb{R}$$

$$y'' = -3y \quad \text{General sol'n } y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x) \quad C_1, C_2 \in \mathbb{R}$$

$$y' = x \Rightarrow y = \frac{1}{2}x^2 + C \quad C \in \mathbb{R}$$

In-general: \rightarrow 1st order ODE has 1D solution space

i.e. one free constant

\rightarrow 2nd order ODE has 2D solution space

i.e. two free constants

An initial value problem (IVP) is an ODE along with an initial condition

1st order $\begin{cases} f(y, y, x) = 0 \\ y(x=c) = y_0 \end{cases}$ \Leftarrow specifies the choice of constant c

2nd order $\begin{cases} f(y'', y', y, x) = 0 \\ y(x=c) = y_0 \\ y'(x=c) = v_0 \end{cases}$ \Leftarrow specify constants c_1, c_2

ex:

$$\begin{cases} y' = -y \\ y(t=0) = y_0 \end{cases}$$
 \Leftarrow radioactive decay
 \Leftarrow starting amount

general soln: $y(t) = c e^{-t}$, plug in $y(0) = c e^{0} = y_0$
 $c = y_0$

$$\therefore \boxed{y(t) = y_0 e^{-t}}$$

Analyzing Differential Equations

→ we can learn a lot about a system without even solving for an ODE

Given 1st order autonomous ODE $y' = F(y)$, an equilibrium point are the points where $y' = 0$

→ equilibrium = no change in the system

e.g. deer population reproduction + carrying capacity

equilibrium. ① zero deer

Points: ② population at capacity

An equilibrium point is:

① stable if $F' < 0$ at equilibrium point

② unstable if $F' > 0$ " "

③ semistable if F is local min or max

- stable = if you move away from equilibrium, system eventually returns back to some equilibrium
- unstable = moves away from equilibrium when perturbed
- semi-stable = returns back when perturbed in one direction and moves away in other

$$\frac{dy}{dx} = y(2-y)(y+3).$$

Classify each equilibrium solution as **stable** or **unstable**.

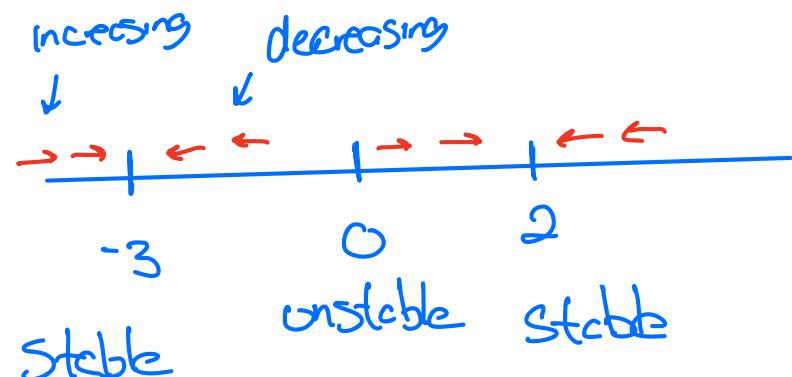
Determine the behaviour between the equilibrium Points

→ equilibrium $0 = \frac{dy}{dx} = y(2-y)(y+3) \Rightarrow y=0, y=2, y=-3$

Sign Chart

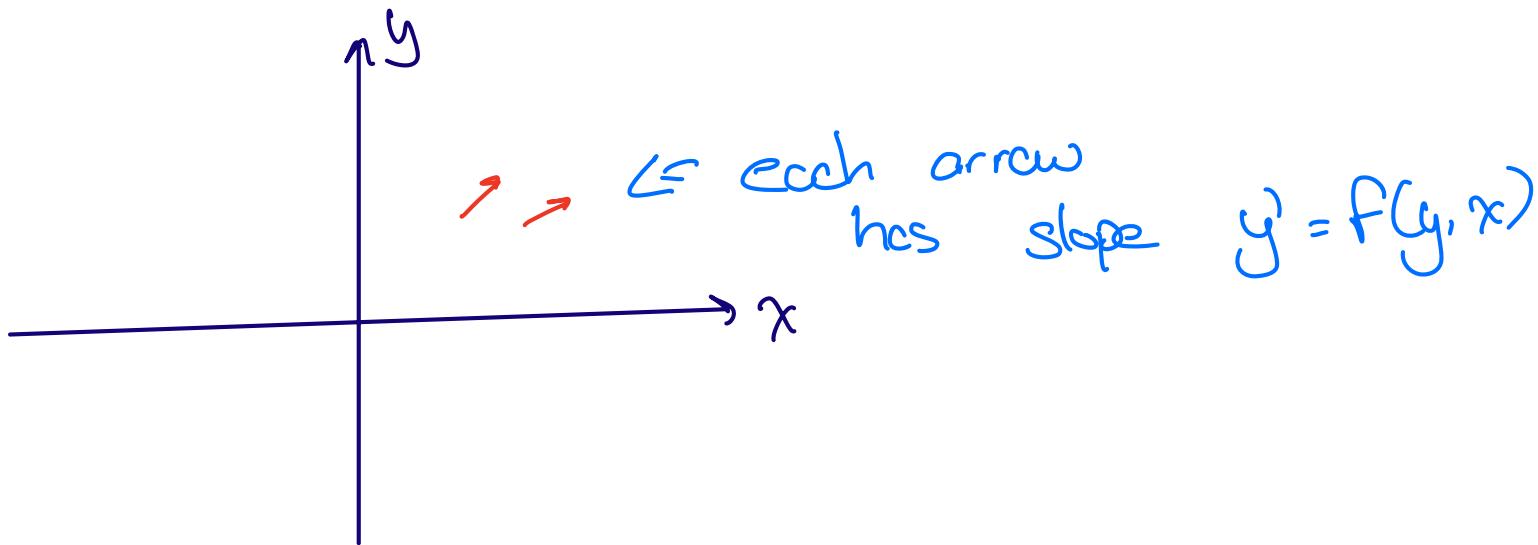
	$y < -3$	$-3 < y < 0$	$0 < y < 2$	$2 < y$
y	(-)	(-)	(+)	(+)
$2-y$	(+)	(+)	(+)	(-)
$y+3$	(-)	(+)	(+)	(+)
$\frac{dy}{dx}$	(+)	(-)	(+)	(-)

↑ ↑ ↑ ↑
 growing decreasing increasing decreasing
 $y < -3$

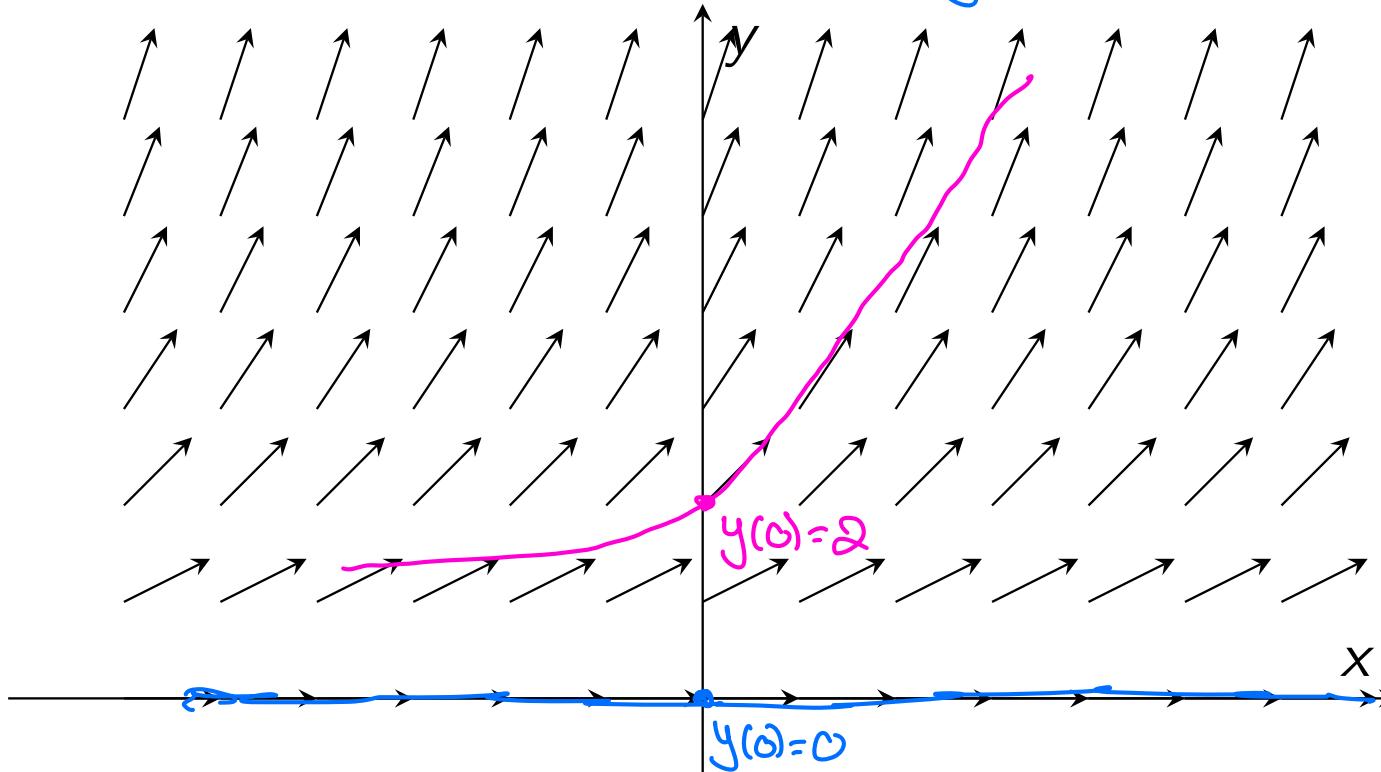


Direction Fields

→ visualization of $y' = f(y, x)$

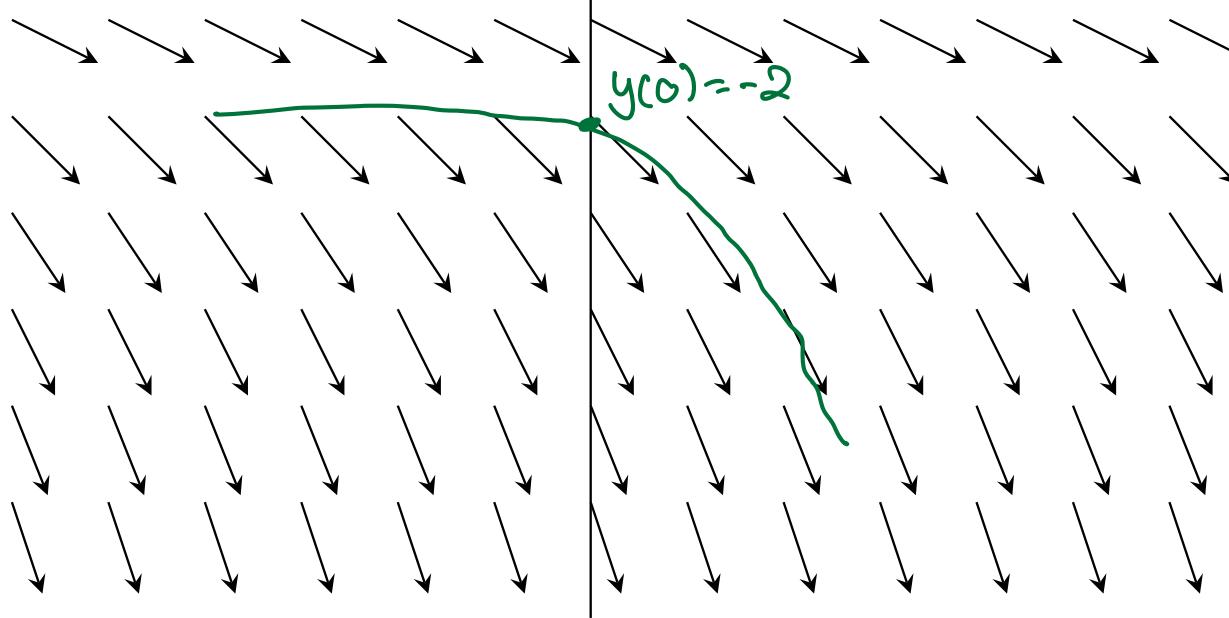


Direction Field for $\frac{dy}{dx} = y = F(x,y)$

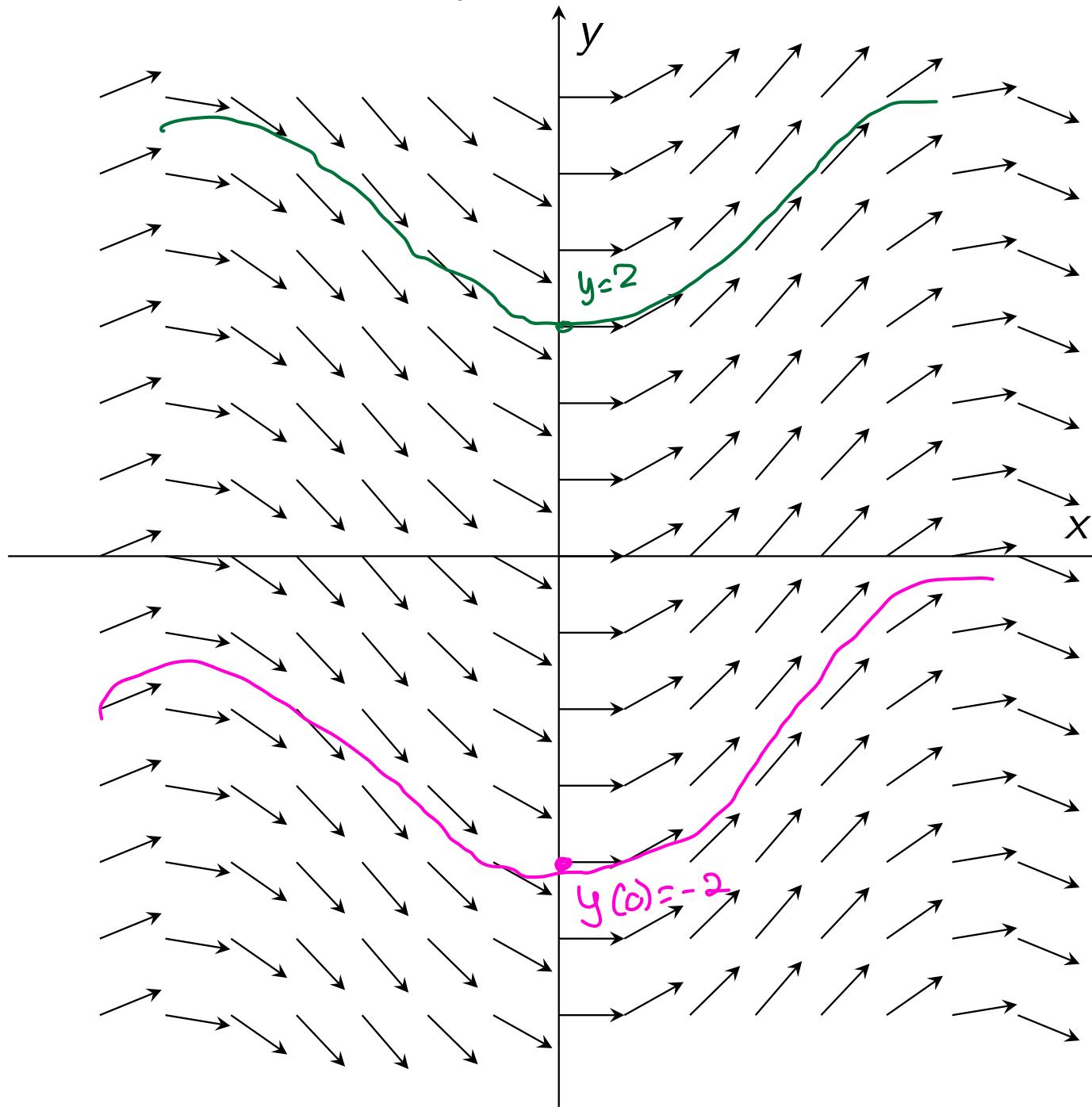


→ autonomous ODE
(arrows don't change with x)

→ zero is equilibrium solution



Direction Field for $\frac{dy}{dx} = \sin(x)$



→ non-autonomous

→ no equilibrium

Direction Field for $\frac{dy}{dx} = y(1 - y) + x$

