Practise Problem Set 7

MAT 187 - Summer 2025

These questions are meant for your own practice for the exam and are not to be handed in. Some of these questions, or problems similar to these, may appear on the quizzes or exams. Therefore, solutions to these problems will not be posted but you may, of course, ask about these questions during office hours, or on Piazza.

Suggestions on how to complete these problems:

- Solution writing is a skill like any other, which must be practiced as you study. After you write down your rough solutions, take the time to write a clear readable solution that blends sentences and mathematical symbols. This will help you to retain, reinforce, and better understand the concepts.
- After you complete a practice problem, reflect on it. What course material did you use to solve the problem? What was challenging about it? What were the main ideas, techniques, and strategies that you used to solve the problems? What mistakes did you make at the first attempt and how can you prevent these mistakes on a Term Test? What advice would you give to another student who is struggling with this problem?
- Discussing course content with your classmates is encouraged and a mathematically healthy practice. Work together, share ideas, explain concepts to each other, compare your solutions, and ask each other questions. Teaching someone else will help you develop a deeper level of understanding. However, it's also important that reserve some time for self-study and self-assessment to help ensure you can solve problems on your own without relying on others.

1. Convert these rectangular coordinates to polar coordinates.
   (a) $(x, y) = (0, 1)$
   (b) $(x, y) = (2, -3)$
   (c) $(x, y) = (-4, 1)$
2. Convert these polar coordinates to rectangular coordinates.
   (a) $(r, \theta) = (1, 0)$
   (b) $(r, \theta) = (2, \pi)$
   (c) $(r, \theta) = (3, -\pi/3)$
   (d) $(r, \theta) = (-2, 3\pi)$
3. Graph the following polar curves.

(a) $r = \sin^2(\theta/2)$

(b) $r = 2 - 2\sin(\theta)$

4. Convert the following polar curves to Cartesian and simplify. Describe the curves in terms of the parameters involved.

   (a) $r^2 - 2r(a\cos(\theta) + b\sin(\theta)) = R^2 - a^2 - b^2$

   (b) $r = \frac{b}{\sin(\theta) - m\cos(\theta)}$

5. Compute the area enclosed by the following polar curves:

   (a) $r(\theta) = 1 + \cos\theta$, for $\theta \in [0, 2\pi]$

   (b) $r(\theta) = \sin(3\theta)$, for one petal

   (c) $r(\theta) = 2$, for $\theta \in [0, \pi]$

6. Compute the arc length of the following polar curves (feel free to use a compute software to solve the final integral):

   (a) $r(\theta) = \theta$, for $\theta \in [0, 2\pi]$

   (b) $r(\theta) = 1 + \sin\theta$, for $\theta \in [0, \pi]$

7. Compute the arc length of the following parametric curves:

   (a) $x(t) = t^2,\ y(t) = t^3$, for $t \in [0, 1]$

   (b) $x(t) = \cos t,\ y(t) = \sin t$, for $t \in [0, \pi]$

   (c) $x(t) = e^t,\ y(t) = e^{-t}$, for $t \in [0, 1]$

8. This question is about tracing out curves in $\mathbb{R}^2$ using parametric equations.

   (a) Find parametric equations $x(t)$ and $y(t)$ whose graph $\{(x(t), y(t)) : t \in [0, 1]\} \subseteq \mathbb{R}^2$ is a circle with radius 1 that is traced out **counter-clockwise** over a time interval of 1 second.

   (b) Find parametric equations $x(t)$ and $y(t)$ whose graph $\{(x(t), y(t)) : t \in [0, 1]\} \subseteq \mathbb{R}^2$ is a circle with radius 1 that is traced out **clockwise** over a time interval of 1 second.

   (c) Find parametric equations $x(t)$ and $y(t)$ whose graph $\{(x(t), y(t)) : t \in [0, 1]\} \subseteq \mathbb{R}^2$ is a circle with radius 1 that is traced out **clockwise** over a time interval of 1 second but where the speed the circle is traced out starts at 0 and ends at 2.

   (d) Find parametric equations $x(t)$ and $y(t)$ whose graph $\{(x(t), y(t)) : t \in [0, 1]\} \subseteq \mathbb{R}^2$ is a **square** with lower left corner at $(0, 0)$ and upper right corner at $(1, 1)$.
   Hint: you may use piecewise functions.

   (e) Find parametric equations $x(t)$ and $y(t)$ whose graph $\{(x(t), y(t)) : t \in [0, 1]\} \subseteq \mathbb{R}^2$ is an ∞ symbol.

9. Consider the parametric equations $x(t) = t^3$ and $y(t) = t^2$. Let G be the graph of $(x(t), y(t))$ for $t \in \mathbb{R}$.

   (a) Sketch G.

   (b) Find a **function** f so that the graph of $y = f(x)$ is the same as G.

   (c) Does the graph G have a tangent line at every point? Explain.

(d) Are $x(t)$ and $y(t)$ differentiable at every point?

(e) If $p(t)$ and $q(t)$ are differentiable parametric functions, does the graph of $(p(t), q(t))$ need to have a tangent line at every point?

10. Consider the parametric equations $x(t) = t$ and $y(t) = \frac{1}{1+t^2}$ and the graph $G = \{(x(t), y(t)) : t \in \mathbb{R}\}$. The vector $(x'(t), y'(t))$ is always a *tangent* vector to G at the point $(x(t), y(t))$, provided $(x'(t), y'(t)) \neq \vec{0}$. The *speed* at which the graph G is being traced out at the point $(x(t), y(t))$ is given by $\|(x'(t), y'(t))\|$.

(a) Find the maximum and minimum speeds at which G is traced out. (You may use a calculator/computer to solve any equations.)

(b) Find new parametric equations whose graph is the same as G but which trace out G in a total of 2 units of time. Without computing, what is the maximum speed at which your new parametric equations trace out G?

11. Consider the parametric equations $x(t) = \sin(7\pi t)$ and $y(t) = \cos(5\pi t)$ for $t \in [0, \pi/8]$ and their graph $G \subseteq \mathbb{R}^2$.

(a) Find the tangent line to G at the point $(x(\pi/12), y(\pi/12))$. Express your answer in $y = mx + b$ form.

(b) Use a computer/Desmos to estimate: what is the closest that G comes to the point $(-1, -1)$?

12. Consider the polar equation $r(\theta) = \frac{1}{2} + \sin\theta$

(a) Find parametric equations $x(\theta)$ and $y(\theta)$ so that the graph of (x, y) is the same as the graph of $r(\theta)$.

(b) Set up an integral to find the area inside the graph of $r(\theta)$ for $\theta \in [-\pi/6, 7\pi/6]$. Then, find the area.

(c) Set up an integral to find the area inside the graph of $r(\theta)$ for $\theta \in [7\pi/6, 11\pi/6]$. Then, find the area.

(d) Use Desmos to graph $r(\theta)$ for $\theta \in [-\pi/6, 11\pi/6]$. Set up an integral to compute the area in the larger of the two enclosed regions.

(e) Why doesn't $\displaystyle\int_{-\pi/6}^{11\pi/6} \tfrac{1}{2}\Big(\sin(\theta) + \tfrac{1}{2}\Big)^2 \, d\theta$ result in the correct area for the larger of the two enclosed regions? Explain.

(f) A friend is trying to find the area of the circle enclosed by the polar equation $r(\theta) = \sin\theta$. The compute

$$\int_0^{2\pi} \tfrac{1}{2}\sin^2\theta \, d\theta = \frac{\pi}{2}.$$

However your friend knows that the area of the circle is actually $\frac{\pi}{4}$. Explain to your friend why they got the wrong answer and how to avoid such a mistake in the future.