University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 4

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We will start 10 minutes past the hour. Use this time to make a new friend.

Numerical Integration

Numerical integration is technique for approximating $\int_a^b f(x)\,dx$ without finding anti-derivative

How? → $\int_a^b f(x)\,dx$ is area under curve

→ use elementary shapes to approximate that area

Why? → anti-derivative might be difficult to determine

→ anti-derivative might not exist in terms of elementary functions ex// $\int e^{-x^2}\,dx$

$$\mathrm{erf}(x) = F(x) = \int_0^x e^{-t^2}\,dt$$

ex//

→ some functions are defined as an integral

$$\ln(x) = \int_1^x \frac{1}{x}\,dx$$

→ we may not know the integrand but only certain values exist

$t=0$ $\quad$ $t=1$ $\quad$ $t=3$ $\quad \Leftarrow$ approximate distance from sampled velocity

$v=1$ $\quad$ $v=2$ $\quad$ $v=6$

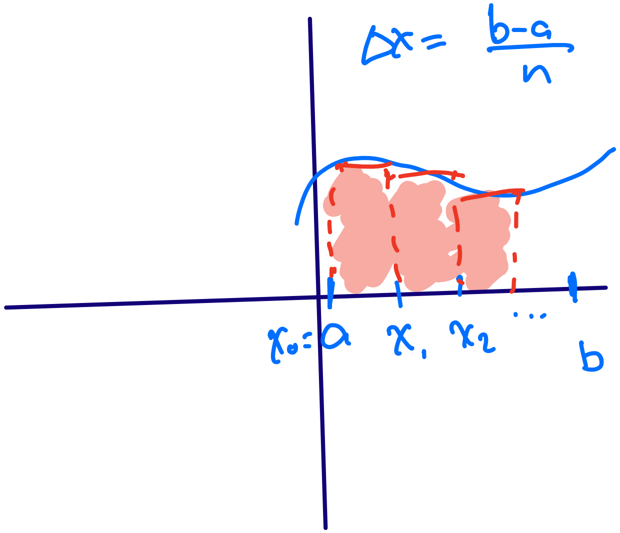
Recall Riemann Sums

→ partition $[a,b] = \{x_0, x_1, \dots, x_n\}$ $\Leftarrow$ equally spaced

$x_i = a + \frac{b-a}{n}$

$\Delta x = \frac{b-a}{n}$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \int_a^b f(x)\,dx$$



x_i^* some element in $[x_{i-1}, x_i]$

→ use Riemann sums as an approximation with finite n

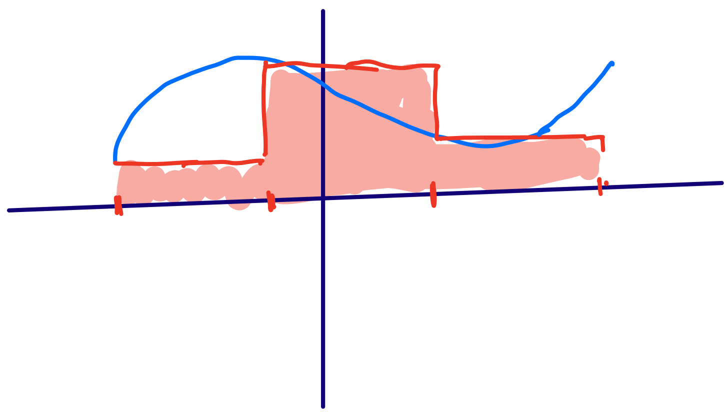
→ approximate each interval with constant function (0^{th} degree poly)

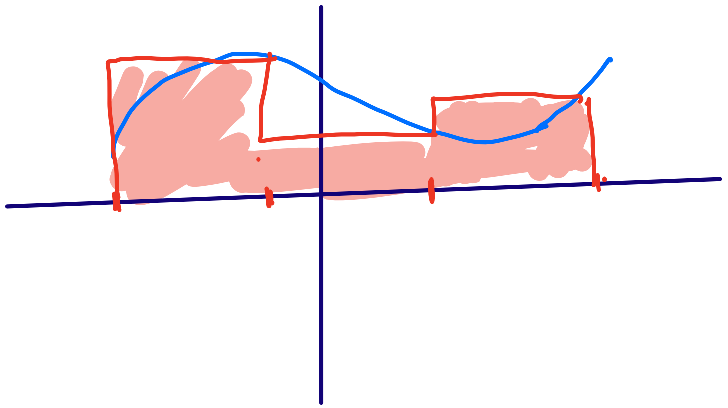
Left and Right Endpoint

→ choice of $x_i^* \in [x_{i-1}, x_i]$

$x_i^* = x_{i-1}$ (left end-point)

$x_i^* = x_i$ right endpoint





→ left-endpoint is overestimate for decreasing function

underestimate for increasing ..

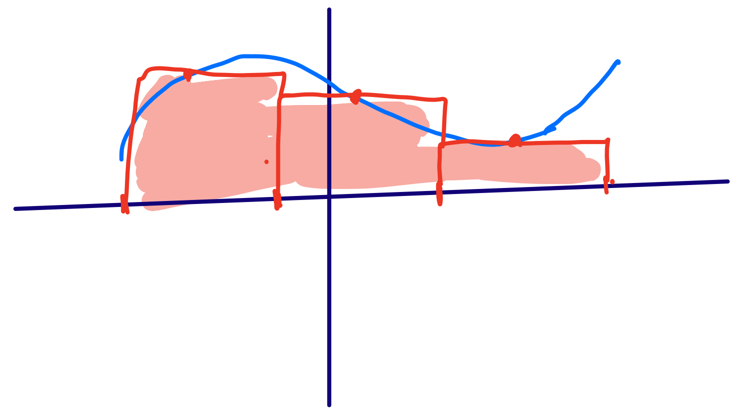
→ right-endpoint reverse

Midpoint Rule

→ take x_i^* to be middle point of $[x_{i-1}, x_i]$

$$x_i^* = \frac{x_i - x_{i-1}}{2}$$

→ mid-point in general more accurate



Approximate $\int_0^1 x^2 dx$ using midpoint rule.

$n=4$ (4 portions)

$\Leftarrow f(x)=x^2$

$$\int_0^1 x^2 dx \approx \sum_{i=0}^{4} f(x_i^*)\,\Delta x$$

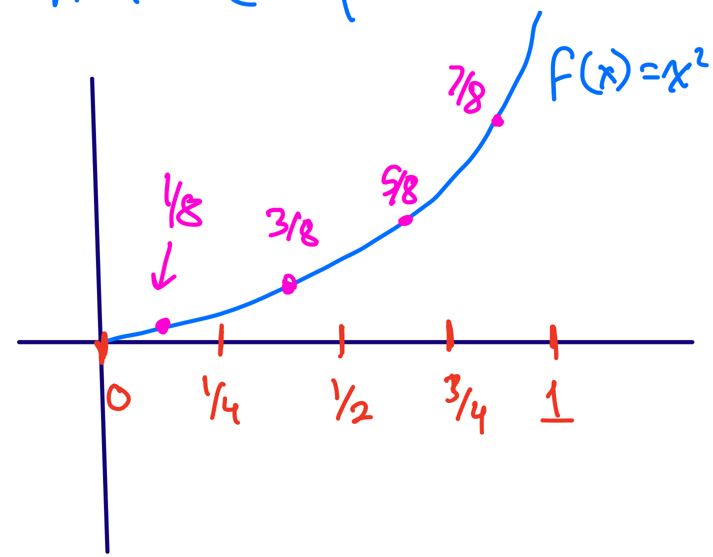
$$= \frac{1}{4}\left(f\left(\tfrac{1}{8}\right)+f\left(\tfrac{3}{8}\right)+f\left(\tfrac{5}{8}\right)+f\left(\tfrac{7}{8}\right)\right)$$

$$= \frac{1}{4}\left(\frac{1}{64}+\frac{9}{64}+\frac{25}{64}+\frac{49}{64}\right)$$

$$= \frac{1}{4}\left(\frac{84}{64}\right)$$

$$= \frac{1}{2}\left(\frac{42}{64}\right)$$

$$= \frac{21}{64}$$



$\rightarrow$ compare

$$\int_0^1 x^2 dx = \frac{1}{3}$$

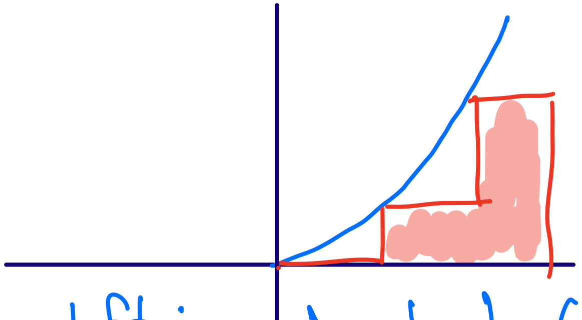
$$\left|\frac{1}{3}-\frac{21}{64}\right| = \frac{1}{192}$$

$\uparrow$

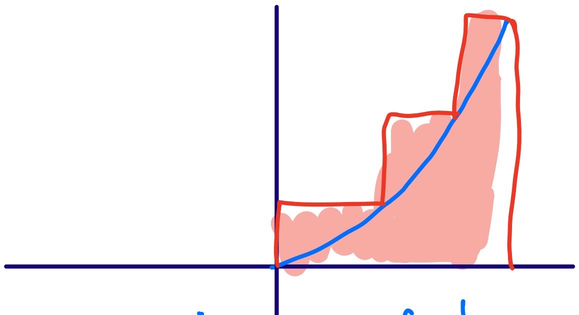
error

For each of the following, which rule is an underestimate, overestimate, undetermined?

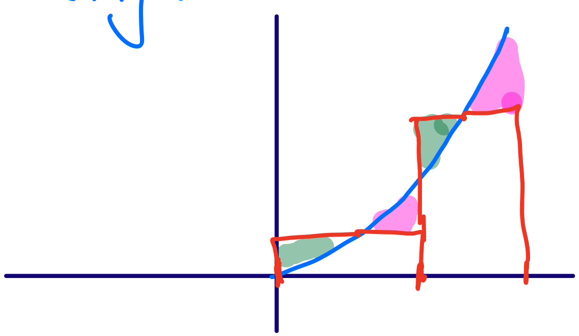
$\int_0^1 x^2 dx$



→ left: underestimate (increasing)

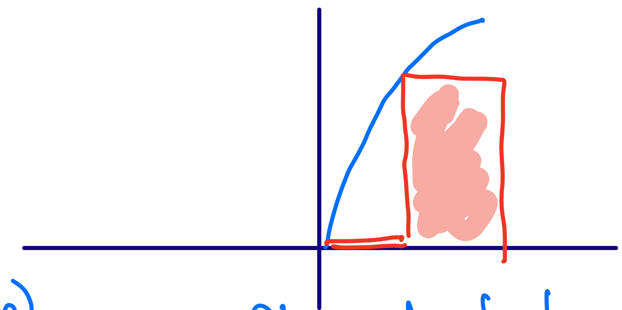


→ right: overestimate

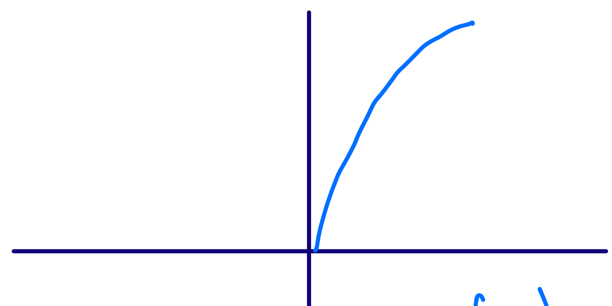


→ midpoint: underestimate

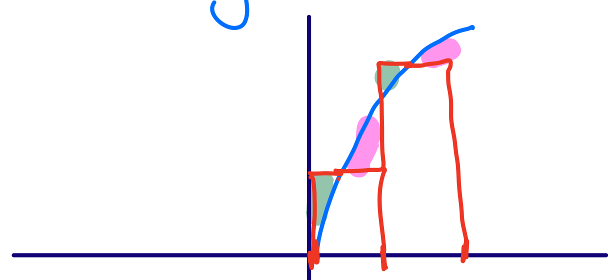
$\int_0^{\frac{\pi}{2}} \sin(x) dx$



→ left: underestimate

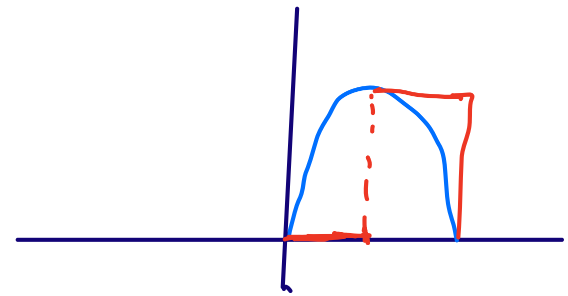


→ right: overestimate

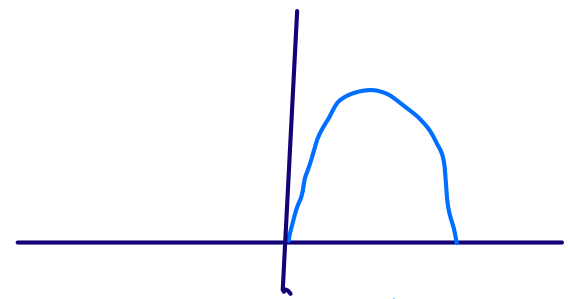


→ midpoint: overestimate

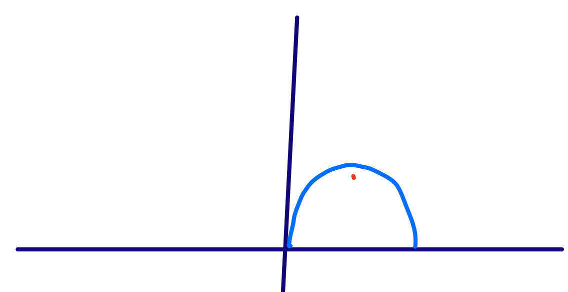
$\int_0^{\pi} \sin(x) dx$



undermined



undermined



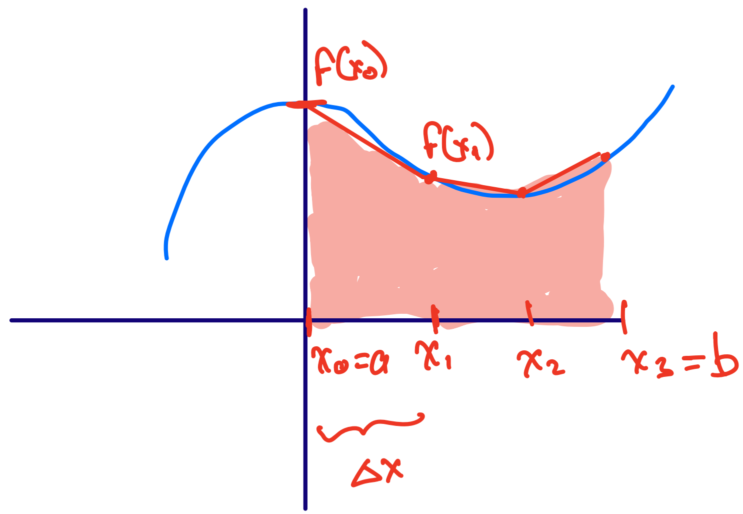
midpoint: overestimate

General rule for mid-point rule

→ in general if f is concave up then underestimate

→ if concave down then overestimate

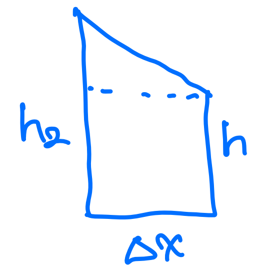
Trapezoid Rule



$$\Delta x = \frac{b-a}{n}$$

→ approximating each interval with a <u>linear</u> function (1st degree polynomial)

→ area of a trapezoid



area = (rect) + (triangle)

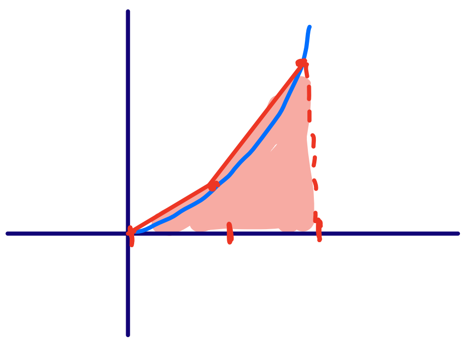
$$\boxed{= \frac{1}{2}(h_1 + h_2)\Delta x}$$

$$\text{approx} = \frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x + \frac{1}{2}\left(f(x_1) + f(x_2)\right)\Delta x + \dots \frac{1}{2}\left(f(x_{n-1}) + f(x_n)\right)\Delta x$$

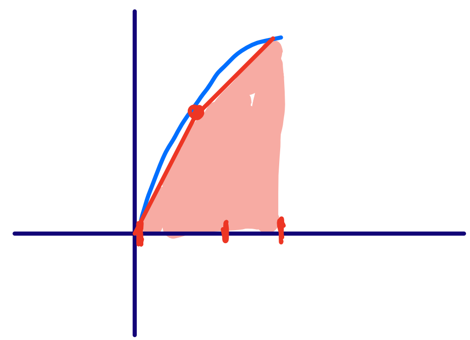
$$= \Delta x\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right.$$

For each of the following, the trapezoid rule is an underestimate, overestimate, undetermined?

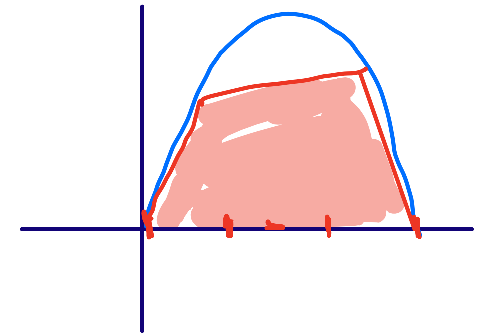
$\int_0^1 x^2 dx$ $\qquad$ $\int_0^{\frac{\pi}{2}} \sin(x)dx$ $\qquad$ $\int_0^{\pi} \sin(x)dx$



overestimate



underestimate



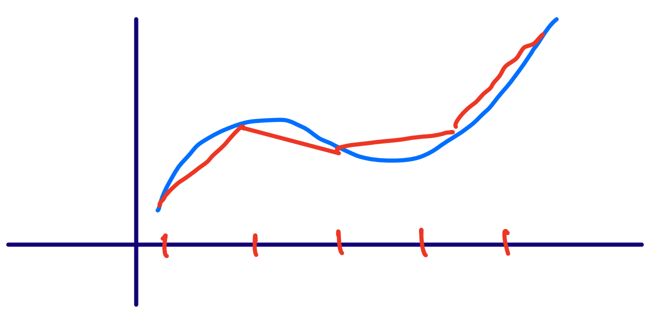
under

General rule for trapezoid rule

→ concave up ⇒ overestimate

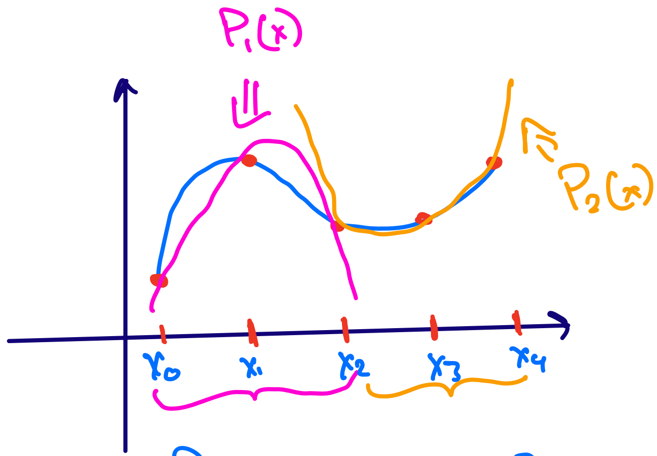
→ concave down ⇒ underestimate

} opposite of mid-point rule



⇐ undetermined b/c concavity changes

Simpson's Rule



→ in general given 3 points $(x_0, f(x_0))$ $(x_1, f(x_1))$ $(x_2, f(x_2))$ there is a quadratic that goes through those 3 points

→ for every 3 adjacent points in partition, find a quadratic that fits through 3 points

$$\rightarrow \int f(x)\,dx \approx \int_{x_0}^{x_2} P_1(x)\,dx + \int_{x_2}^{x_4} P_2(x)\,dx + \underbrace{\cdots}_{\text{if more than 4 intervals}}$$

→ note Simpson's requires <u>even</u> # of intervals

→ useful formula: if $P(x)$ is a quadric fit between 3 <u>equally</u> spaced points $(x_0, f(x_0))$ $(x_1, f(x_1))$ $(x_2, f(x_2))$ then

$$\int_{x_0}^{x_2} P(x)\,dx = \frac{\Delta x}{6}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big) \Leftarrow \Delta x = x_1 - x_0 = x_2 - x_1$$

→ we don't need to figure out P_1, P_2

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{6}\Big(f(x_0)+4f(x_1)+f(x_2)\Big) + \frac{\Delta x}{6}\Big(f(x_2)+4f(x_3)+f(x_4)\Big)$$

$$= \frac{\Delta x}{6}\Big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\Big)$$

General Simpson's rule:

$$\int_a^b f(x)\,dx = \frac{\Delta x}{6}\Big(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\dots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big)$$

coeff : 1 4 2 4 2 4 2 4 ... 2 4 2 4 1

→ in general simpson's rule is more accurate than midpoint / trapezoid

Use Simpson's rule to approximate $\int_0^2 \frac{1}{1+x^2} dx$

Error Bounds

Midpoint rule:

$$\text{error} \le \frac{M(b-a)^3}{24n^2}$$

$\Leftarrow$ M is an upper bound

$\Leftarrow$ for $\underbrace{|f''(x)|}_{2^{nd}\text{ derivative}}$ for $a \le x \le b$

Trapezoid rule:

$$\text{error} \le \frac{M(b-a)^3}{12n^2}$$

Simpson's rule:

$$\text{error} \le \frac{M(b-a)^5}{180n^4}$$

$\Leftarrow$ M is an upper bound

for $\underbrace{|f^{(4)}(x)|}_{4^{th}\text{ derivative}}$ for $a \le x \le b$

What number of partitions do we need to approximate $\int_1^2 \frac{1}{x}\,dx = \ln(2)$ using midpoint and Simpson's rule. to an error of most 0.01

Midpoint rule

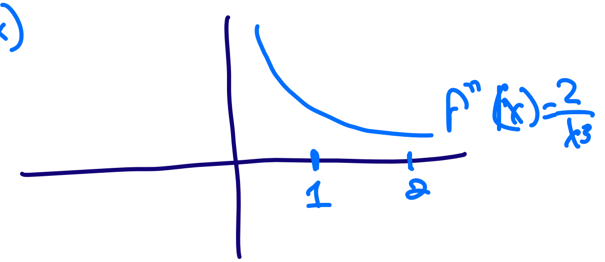
$\rightarrow$ error $\leq \dfrac{M(b-a)^3}{24n^2} = 0.01$

$M =$ bound on $f''(x)$

$f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$\boxed{f''(x) = \frac{2}{x^3}}$



$\rightarrow$ at $x=1 \Rightarrow |f''(x)| = 2$

$x = 2 \Rightarrow |f''(x)| = \frac{1}{4}$

$\rightarrow \frac{2}{x^3}$ decreasing so max value for $1 \leq x \leq 2$ is at $x=1$

$\boxed{M = 2}$

$$\frac{2(2-1)^3}{24n^2} = 0.01$$

$$n^2 = \frac{2(2-1)^3}{24(0.01)}$$

$$n = 2.88$$

$\therefore$ we want $n \geq 3$ to guarantee error < 0.01

Simpson's Rule

$\rightarrow$ exercise