UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Quiz 2

Date: May 21, 2025
Duration: 50 minutes

Course: MAT 187 (Calculus II)
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Instructions

- This is a Type A assessment and **does not** allow any external aids.
- Read all instructions carefully and **justify all your answers.** No points will be awarded for a correct answer without justification.
- Read each question carefully. **No clarification or content related questions will be answered.**

You may use the following space for scratch work or to continue your solutions if you run out of room. If you do so, please clearly indicate in the original question that part of your solution appears here.

1. (4 points) Consider the improper integral:

$$\int_{-\infty}^{\infty} \frac{1}{(x+1)}\,dx$$

Define what it means for this improper integral to **converge or diverge**. *You do not need to evaluate the integral—just explain the definition.*

Solution: This integrand has **vertical asymptotes** at $x=-1$, so the function is **not continuous on the interval** $(-\infty,\infty)$. Additionally, the interval of integration is infinite.

To evaluate the convergence of this integral, we must break it into parts at the discontinuities and use limits. The integral is defined as:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{(x+1)}\,dx = \lim_{a\to-\infty}\int_{a}^{-2} \frac{1}{(x+1)}\,dx + \lim_{b\to 1^-}\int_{-2}^{b} \frac{1}{(x+1)}\,dx \\ + \lim_{c\to 1^+}\int_{c}^{0} \frac{1}{(x+1)}\,dx + \lim_{d\to\infty}\int_{0}^{d} \frac{1}{(x+1)}\,dx\end{aligned}$$

The **integral converges** only if **all four limits exist and are finite**. If **any one** of these limits does not exist or is infinite, the integral is said to **diverge**. *Note that we do not need to split the integrals at $x=-2$ and $x=0$, any points above and below $x=-1$ will work.*

2. (3 points) **True or False:** If $f(x)\le g(x)$ for all $x\ge 1$, and $\int_1^\infty g(x)\,dx$ converges, then it implies that $\int_1^\infty f(x)\,dx$ also converges. **Justify your answer.**

Solution: False. The statement is not always true without an additional assumption.

The **comparison test** requires that both functions be **nonnegative** on the interval. If $f(x)\le g(x)$, but $f(x)$ takes on **negative values**, the conclusion about convergence may not hold.

Counterexample: Let $g(x)=\frac{1}{x^2}$, which is positive and

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$

converges.

Now let $f(x)=-1$, which satisfies $f(x)\le g(x)$ for all $x\ge 1$, but

$$\int_1^\infty f(x)\,dx = \int_1^\infty (-1)\,dx$$

diverges to $-\infty$. So, the original statement is false.

3. (3 points) Consider the improper integral:

$$\int_1^\infty \frac{1}{x^p}\,dx$$

For which values of $p>0$ does this integral converge? (*No justification necessary.*) Explain in words why the integral converges for some values of p and diverges for others, even though $\frac{1}{x^p}\to 0$ as $x\to\infty$ for all $p>0$.

Solution: The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if $p>1$ and diverges if $p\le 1$. This is often referred to as the p-test.

Even though $\frac{1}{x^p}\to 0$ as $x\to\infty$ for all $p>0$, the rate at which the function approaches zero determines whether the total area under the curve is finite.

When $p\le 1$, the function decays too slowly — the "tail" of the function is thick enough that the area accumulates without bound, causing the integral to diverge.

When $p>1$, the function decays quickly enough that the area under the curve from 1 to ∞ is finite, and so the integral converges.

4. (6 points) Evaluate the integral or show it diverges:

$$\int_0^\infty xe^{-x}\,dx$$

Solution: This is an improper integral due to the infinite upper limit. We evaluate it as a limit:

$$\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$

We compute the definite integral using integration by parts: Let

$$u = x \quad \Rightarrow \quad du = dx, \quad dv = e^{-x}dx \quad \Rightarrow \quad v = -e^{-x}$$

Then:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C$$

Now apply the limits:

$$\begin{aligned}\int_0^t xe^{-x}\,dx &= \left[-(x+1)e^{-x}\right]_0^t \\ &= -(t+1)e^{-t} + (0+1)e^0 \\ &= -(t+1)e^{-t} + 1\end{aligned}$$

Now take the limit as $t \to \infty$:

$$\lim_{t\to\infty}\left[-(t+1)e^{-t} + 1\right] = 1$$

Final Answer: The integral converges and equals $\boxed{1}$.

5. (6 points) Determine whether the following improper integral converges or diverges. Justify your answer using the **comparison test**:

$$\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$$

Solution: As $x \to 0^+$, $\sqrt{x} \gg x^2$, so:

$$\frac{1}{\sqrt{x}+x^2} \sim \frac{1}{\sqrt{x}}$$

Thus we attempt to compare the integrand with $\frac{1}{\sqrt{x}}$ to show convergence.

Firstly note that $0 \le \frac{1}{\sqrt{x}+x^2}$, so we can apply the comparison test.

For $0 < x < 1$, we have:

$$\sqrt{x} + x^2 > \sqrt{x} \quad \Rightarrow \quad 0 \le \frac{1}{\sqrt{x}+x^2} < \frac{1}{\sqrt{x}}$$

Now,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \text{ converges by the } p\text{-test}$$

Therefore by the comparison test, $\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$ converges.

6. (8 points) According to Coulomb's law, the magnitude of the electrostatic force between two point charges q_1 and q_2 separated by a distance r is given by:

$$F(r) = \frac{kq_1q_2}{r^2}$$

where k is a positive constant.

The work required to move a particle from position $x = r_1$ to $x = r_2$ with a force $F(x)$ is given by:

$$\text{Work} = \int_{r_1}^{r_2} F(x)\,dx$$

Assuming both charges have the same sign (i.e., they repel each other), explain whether it is physically possible to bring the two charges to a distance of zero.

Solution: If the two charges have the same sign, they repel each other. To bring them closer together, an external agent must apply a force opposing this repulsion. The amount of work required is given by:

$$\text{Work} = \int_{r_1}^{r_2} \frac{kq_1q_2}{x^2}\,dx$$

Since q_1 and q_2 are both positive (or both negative), $kq_1q_2 > 0$, so the force is repulsive and points away from the other charge. To bring the charges together, we must integrate from $r_1 > 0$ to $r_2 = 0$, which gives:

$$\text{Work} = \int_{r_1}^{0} \frac{kq_1q_2}{x^2}\,dx$$

This is an improper integral because the integrand has a vertical asymptote at $x = 0$ and by the p-test, this integral diverges.

Therefore, the work required to bring the charges to a distance of zero is infinite. This means that, in practice, it is physically impossible to bring two like charges all the way to $r = 0$, because it would require an infinite amount of energy.