UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Quiz 3

Date: May 28, 2025
Duration: 50 minutes

Course: MAT 187 (Calculus II)
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Instructions

- This is a Type A assessment and **does not** allow any external aids.
- Read all instructions carefully and **justify all your answers.** No points will be awarded for a correct answer without justification.
- Read each question carefully. **No clarification or content related questions will be answered.**

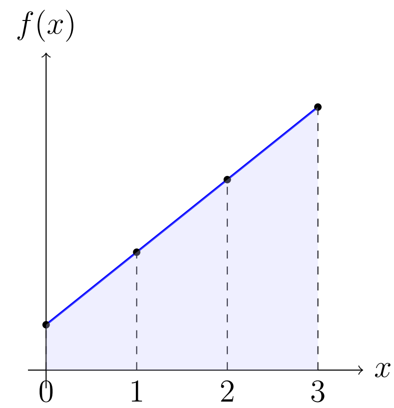
You may use the following space for scratch work or to continue your solutions if you run out of room. If you do so, please clearly indicate in the original question that part of your solution appears here.

1. (3 points) Give an example of a non-constant function for which the trapezoidal rule gives the exact value of the definite integral over an interval. Explain why this happens. Use a sketch to justify your answer.

Solution:
The trapezoidal rule gives the exact value of a definite integral for any linear function:

$$f(x) = mx + b$$

on any interval $[a, b]$, regardless of how many subintervals are used.
This is because the trapezoidal rule approximates the area under a curve by connecting points with straight line segments. For a linear function, this exactly matches the graph. Since the graph is composed of straight lines, each trapezoid perfectly fills the area under the curve with no error.



2. (3 points) A student approximates the integral $\int_0^1 f(x)\,dx$ using both the midpoint rule and the trapezoidal rule with the same number of subintervals and finds:

$$M_n = 1.83 \quad \text{and} \quad T_n = 2.17$$

Assuming no inflection points on the interval $[0, 1]$, what can you conclude about the concavity of $f(x)$? Justify your answer.

Solution: The trapezoidal rule overestimates the integral if the function is concave up, and the midpoint rule underestimates it in that case.

Here, $M_n < T_n$, which suggests:

$$\text{Midpoint estimate } < \text{Actual value } < \text{Trapezoidal estimate}$$

Since there is no inflection points in the interval where the concavity changes, this implies the function is **concave up** on the interval.

3. (3 points) The second-degree Taylor polynomial for a function f, centered at $x = 1$, is given by: $P_2(x) = 5 + 2(x-1) - 3(x-1)^2$. Based on this polynomial, describe the local behaviour of $f(x)$ near $x = 1$. Is the function increasing or decreasing? Is it concave up or concave down? Justify your reasoning.

Solution: The second-degree Taylor polynomial is:

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = 5 + 2(x-1) - 3(x-1)^2$$

So we can read off the derivatives:

- $f'(1) = 2$: the function is **increasing** at $x = 1$.
- $\frac{f''(1)}{2} = -3 \Rightarrow f''(1) = -6$: the function is **concave down** near $x = 1$.

Conclusion: Near $x = 1$, the function is increasing and concave down.

4. (6 points) Suppose you want to approximate $\int_0^2 \ln(x+1)\,dx$ using the midpoint rule with an error less than $\frac{1}{100}$. How many subintervals n are needed to guarantee this accuracy?

Hint: Use the midpoint rule error bound:

$$|E_M| \leq \frac{(b-a)^3}{24n^2} \cdot \max_{a\leq x\leq b} |f''(x)|$$

Solution: We are given $f(x) = \ln(x+1)$. Compute the second derivative:

$$f'(x) = \frac{1}{x+1}, \quad f''(x) = -\frac{1}{(x+1)^2}$$

So on $[0,2]$, $|f''(x)| = \frac{1}{(x+1)^2}$, which is largest when $x = 0$:

$$\max |f''(x)| = \frac{1}{1^2} = 1$$

Now apply the midpoint error bound:

$$|E_M| \leq \frac{(2)^3}{24n^2} \cdot 1 = \frac{8}{24n^2} = \frac{1}{3n^2}$$

We want:

$$\frac{1}{3n^2} < \frac{1}{100} \quad \Rightarrow \quad 3n^2 > 100 \quad \Rightarrow \quad n^2 > \frac{100}{3} \approx 33.3$$

Since $5^2 = 25 < 33.3$ and $6^2 = 36 > 33.3$, the smallest integer n is:

$$\boxed{n = 6}$$

5. The table below records the values of a function $C(t)$ over time. The exact formula for $C(t)$ is unknown.

t (hr)	0	1	2	3	4
$C(t)$ (units)	10	8	5	2.5	1

(a) (3 points) Use Simpson's Rule to estimate the value of $\int_0^4 C(t)\,dt$.

Solution: We use Simpson's Rule with 4 subintervals (evenly spaced):

$$h = \frac{4-0}{4} = 1$$

Simpson's Rule formula:

$$\int_0^4 C(t)\,dt \approx \frac{h}{3}\left[C(0) + 4C(1) + 2C(2) + 4C(3) + C(4)\right]$$

$$= \frac{1}{3}[10 + 4(8) + 2(5) + 4(2.5) + 1] = \frac{1}{3}(10 + 32 + 10 + 10 + 1) = \frac{63}{3} = \boxed{21}$$

(b) (3 points) Based on the data, explain why Simpson's Rule is a good choice here compared to other basic numerical methods.

Solution: The function values decrease steadily and appear to follow a smooth, nonlinear trend, possibly exponential decay. Since Simpson's Rule fits a quadratic polynomial across each pair of subintervals, it generally gives more accurate results for smooth, curved data than trapezoidal or midpoint rules.

Because the data is evenly spaced and relatively smooth, Simpson's Rule is well-suited here. It captures the curvature of the function more accurately than linear approximations would. However, if the actual function changes more sharply between points or is not well-approximated by quadratics, some error may still occur.

6. In optics, the Fresnel integral

$$S(x) = \int_0^x \sin(t^2)\,dt$$

arises when modeling light intensity near the edge of a diffraction pattern. This integral has no elementary antiderivative, so we must rely on approximations.

(a) (5 points) Use the sixth-degree Taylor polynomial for $\sin(t^2)$ centered at $t = 0$ to approximate $S(1) = \int_0^1 \sin(t^2)\,dt$. Leave your answer as an exact simplified fraction.

Solution:
We begin with the well-known Taylor polynomial for $\sin(x)$ at $x = 0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Now, substitute $x = t^2$ into the expansion to get a Taylor polynomial for $\sin(t^2)$:

$$\sin(t^2) = t^2 - \frac{t^6}{6} + \frac{t^{10}}{120} - \cdots$$

To construct a sixth-degree Taylor polynomial, we keep terms up to t^6:

$$\sin(t^2) \approx t^2 - \frac{t^6}{6}$$

Now we use this approximation in the integral:

$$\int_0^1 \sin(t^2)\,dt \approx \int_0^1 \left(t^2 - \frac{t^6}{6}\right)\,dt = \left[\frac{t^3}{3} - \frac{t^7}{42}\right]_0^1 = \frac{1}{3} - \frac{1}{42} = \frac{14-1}{42} = \boxed{\frac{13}{42}}$$

(b) (4 points) It can be shown that on the interval $[0, 1]$:

$$\left|\frac{d^7}{dt^7}\sin(t^2)\right| \leq 300$$

Use Taylor's Remainder Theorem to bound the error in your approximation.

Solution:
From Taylor's Remainder Theorem:

$$|R_6(1)| \leq \int_0^1 \frac{M}{7!}t^7 dt = \frac{M}{7!}\cdot\frac{1}{8} \quad \text{where } M = 300$$

$$\Rightarrow \text{Error bound} = \boxed{\frac{300}{8!}}$$

So the approximation $\frac{13}{42}$ is accurate to within $\frac{300}{8!}$.