University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

**FINAL EXAMINATIONS, APRIL 2006**
First Year - CIV, CHE, IND, LME, MEC, MMS

# MAT 187H1S - CALCULUS II

Exam Type: A

**Examiners**

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D. Burbulla

E. Lawes

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SURNAME _____

GIVEN NAME _____

STUDENT NO. _____

SIGNATURE _____

**Calculators Permitted:** Casio 260, Sharp 520 or Texas Instrument 30

**INSTRUCTIONS:**

Attempt all questions.

Questions 1 through 6 are Multiple Choice; circle the single correct choice for each question. Each correct choice is worth 4 marks.

Questions 7 and 8 are each worth 6 marks.

Question 9 consists of three parts; each part is worth 4 marks.

Questions 10 through 13 are long questions for which you must show your work. Each long question is worth 13 marks.

TOTAL MARKS: 100

Use the backs of the pages if you need more space.

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| **TOTAL** | |

1. What is the fourth degree Taylor polynomial of the function $f(x) = \sin(x)$ at $a = 0$?

   (a) $x + \dfrac{x^3}{6}$

   (b) $x - \dfrac{x^3}{6}$

   (c) $1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$

   (d) $1 + \dfrac{x^2}{2} - \dfrac{x^4}{24}$

2. Suppose the half-life of a certain radioactive substance is 32 minutes. How long will it take for 75% of an initial amount of this substance to decay?

   (a) 16 min

   (b) 32 min

   (c) 48 min

   (d) 64 min

3. What is the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n + 5^n}(x-1)^n$?

   (a) $\dfrac{5}{2}$

   (b) $\dfrac{2}{5}$

   (c) $\dfrac{3}{2}$

   (d) $\dfrac{2}{3}$

4. What is the slope of the tangent line to the polar graph of the polar equation $r = e^{\theta}$ at the point $(x, y) = (0, e^{(\pi/2)})$?

   (a) 0

   (b) $-1$

   (c) 1

   (d) $-e^{(\pi/2)}$

5. The area of the region inside the cardioid with equation $r = 2 - 2\cos\theta$ but outside the circle with equation $r = 1$ is given by

   (a) $\int_0^{\pi/3} [1 - 2\cos\theta]\, d\theta$

   (b) $\int_0^{\pi/3} [(2 - 2\cos\theta)^2 - 1]\, d\theta$

   (c) $\int_{\pi/3}^{\pi} [1 - 2\cos\theta]\, d\theta$

   (d) $\int_{\pi/3}^{\pi} [(2 - 2\cos\theta)^2 - 1]\, d\theta$

6. What is the area of the region bounded by the curve with parametric equations $x = t^2$ and $y = t^3 - t$, for $-1 \leq t \leq 1$?

   (a) $\frac{2}{15}$

   (b) $\frac{4}{15}$

   (c) $\frac{8}{15}$

   (d) $\frac{11}{15}$

7. [6 marks] Approximate $\int_0^{1/2} \frac{1}{\sqrt{1+x^6}}\,dx$ correctly to within $10^{-4}$, and explain why your approximation *is* correct to within $10^{-4}$.

8. [6 marks] Find all positive values of $a$ for which the initial value problem

$$y'' + ay = 0, y(0) = 0, y(1) = 0$$

has non-trivial (that is, not identically zero) solutions.

9. [4 marks each] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\sum_{n=0}^{\infty} \frac{\sin^2 n}{n^4+1}$

◯ Converges ◯ Diverges

by _____

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

◯ Converges ◯ Diverges

by _____

(c) $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$

◯ Converges ◯ Diverges

by _____

10. [13 marks] Find and classify all the critical points of $f(x, y) = x^3 - 2y^2 - 2y^4 + 3x^2y$.

11. If $x$ is the amount of salt disolved in a saline solution of volume $V$, at time $t$, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where $c_i$ is the concentration of salt in a solution entering the mixing tank at rate $r_i$, and $r_0$ is the rate at which the well-mixed solution is leaving the tank.

A tank initially contains 10 liters of pure water. Saltwater containing 10 grams of salt per liter enters the tank at 1 liter per min and the (perfectly mixed) solution leaves the tank at 2 liters per min.

(a) [2 marks] How many minutes will it take until the tank is empty?

(b) [8 marks] Find the amount of salt (in grams) in the tank after t min.

(c) [3 marks] What is the maximum amount of salt in the tank, at any one time?

12. [13 marks] Torricelli's Law states that

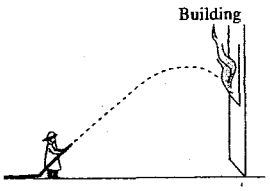
$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

where $y$ is the depth of a fluid in a tank at time $t$, $A(y)$ is the cross-sectional area of the tank at height $y$ above the exit hole, $a$ is the cross-sectional area of the exit hole, and $g = 32$ ft/sec$^2$ is the acceleration due to gravity.

The shape of a water tank is obtained by revolving the curve $y = x^{4/3}$ around the $y$-axis (units on the coordinate axes are in feet). A plug at the bottom is removed at 12 noon, when the water depth in the tank is 12 ft. At 1 PM the water depth is 6 ft. When will the tank be empty?

13. Water issues from the nozzle of a fire hose with speed $S$ meters per sec.

(a) [6 marks] Suppose the hose is held at ground level, $D$ meters from a wall, and aimed at angle $\alpha$ to the horizontal. Write down parametric equations for the trajectory of the water, with $x$ and $y$ in terms of time $t$. Assume the origin $(x, y) = (0, 0)$ is chosen to be the base of the wall.




(b) [7 marks] Show that the maximum height attainable by the water on the wall is given by

$$\frac{S^4 - g^2 D^2}{2gS^2},$$

where $g$ is the acceleration due to gravity.