NAME (Given then Family):

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University of Toronto
Faculty of Applied Science & Engineering

Winter 2024/2025
110 Minutes

MAT187H1S Universal Makeup Test Solutions

Exam Reminders:

- Fill out your name, UTORid, and email address at the top of this page.
- Do not begin writing the exam until instructed to do so.
- As a student, you help create a fair and inclusive writing environment; unauthorized aids are prohibited and using one may result in you being charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. These devices may *not* be left in your pockets.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator.
- In the event of a fire alarm, do not check cell phones or other electronic devices unless authorized to do so.

Special Instructions:

- Write legibly and darkly.
- For questions with a boxed area, ensure your answer is completely within the box.
- Fill in your bubbles completely.

Good: ● A ○ B Bad: ◐ A ⊙ B ⊗ C

Scoring

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	1	2	1	2	4	1	1	6	3	21
Score:										

Formula Sheet

The following formulae are provided for your use during the exam.

Simpson's Rule

If p is a quadratic polynomial,

$$\int_a^b p(x)\,\mathrm{d}x = \frac{b-a}{6}\left(p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right).$$

Taylor's Remainder Theorem

Let T_n be the nth Taylor approximation of function f centered at a. Further, assume f has at least $n+1$ continuous derivatives. Then

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.

Geometric Sum

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

1. (1 point) Which of the following express the point $(x, y) = (2, 2) \in \mathbb{R}^2$ in polar coordinates? Mark all that apply.

- ○ $\sqrt{8}e^{\frac{\pi}{4}i}$
- ● $(\sqrt{8}, \frac{\pi}{4})$
- ○ $(-\sqrt{8}, \frac{\pi}{4})$
- ● $(-\sqrt{8}, \frac{5\pi}{4})$
- ○ $-\sqrt{8}e^{\frac{5\pi}{4}i}$
- ○ $(2, 2)$

2. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers and define the number $A = \sum_{i=3}^{5} a_i$.

(a) (1 point) Express A as a sum starting from 4.

$$A = \sum_{i=4}^{6} a_{i-1}$$

(b) (1 point) Let $B = \sum_{i=0}^{3} a_{2i}$. Express $B - A$ as an explicit sum (i.e., without using Σ-notation or "$\cdots$" notation).

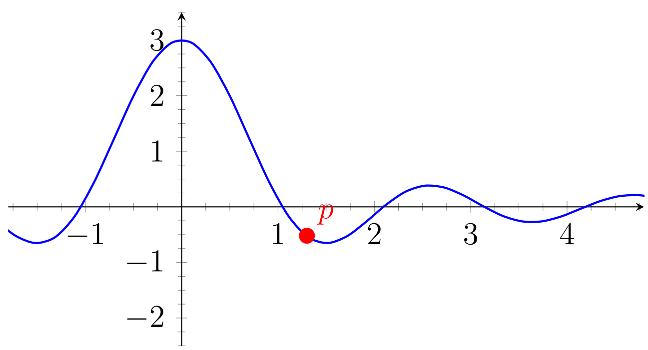
$$B - A = a_0 + a_2 - a_3 - a_5 + a_6$$

Scratch Work:

3. (1 point) Let $I = \int_a^b f(x)\,\mathrm{d}x$. Explain in words why using Simpson's Rule to estimate I is likely to be more accurate than using the Trapezoid rule to estimate I.

Since Simpson's rule is based on Quadratic interpolation rather than the Linear interpolation used for the Trapezoid rule, the interpolation used with Simpson's Rule will more closely match the function f compared to the linear interpolations used with the Trapezoid rule. Since the function is more closely approximated, the integral will be more closely approximated.

4. The graph of an unknown function f is given below. Marked on the graph is the point $p = (1.3, f(1.3))$.



(a) (1 point) Let T_1 be the first Taylor approximation to f centered at 1.3. The graph of T_1 intersects the graph of f at p and at one other point $q = (q_x, q_y)$. Give your best estimate for the value of q_x. Note: your estimate should be within at least 0.5 of the true answer.

$q_x \quad \approx \quad \boxed{-0.5}$

(b) (1 point) Let $P(x) = a(x-1.3)^2 + b(x-1.3) + c$ be the second Taylor approximation to f centered at 1.3. What can you say about a? Select the best answer.

● $a > 0$ ○ $a = 0$
○ $a < 0$ ○ Not enough information to determine

5. The function f has a Taylor series centered at 0 of

$$T(x) = \sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} x^n.$$

(a) (1 point) Provide a well-written definition of the statement "the series $T(3)$ converges". *Do not determine whether $T(3)$ converges. Only write the definition of convergence.*

The series $T(3) = \displaystyle\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} 3^n$ converges if the limit of the partial sums $S_N = \displaystyle\sum_{n=0}^{N} \frac{(n+1)^2}{n!} 3^n$ exists and is finite. That is, $\lim_{N\to\infty} S_N$ exists.

(b) (1 point) Let L be the 3[rd] Taylor approximation to f centered at 0. Write an expression for L

$$L(x) = \boxed{1 + 2^2 x + \frac{3^2}{2!}x^2 + \frac{4^2}{3!}x^3 = 1 + 4x + \frac{9}{2}x^2 + \frac{8}{3}x^3 = \sum_{n=0}^{3} \frac{(n+1)^2}{n!} x^n}$$

(c) (2 points) Let F be the Taylor series for $\int_0^x f(t)\,dt$ centered at 0. Express F using Σ-notation.

$$F(x) = \boxed{\sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^{n+1} = \sum_{n=0}^{\infty} \frac{(n+1)^2}{(n+1)!} x^{n+1} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} x^n}$$

Scratch work:

6. (1 point) In this question, you are asked to draw, if possible, the graph of a function f subject to various restrictions. If no such function exists, you are to mark the option "No such f exists".

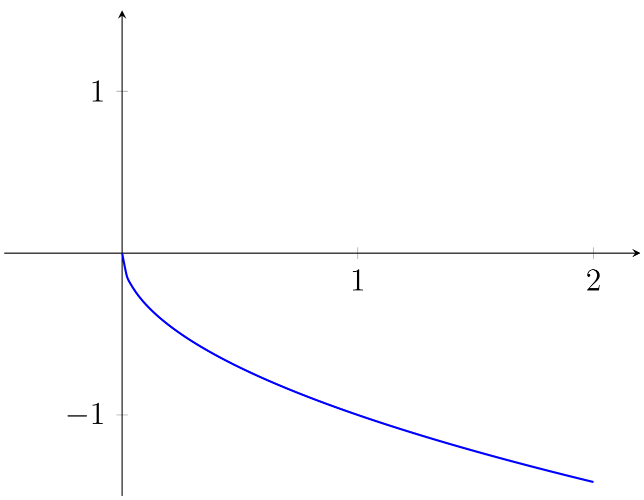
   Note: if you feel like your drawing is unclear, you may add a description in words below your graph.

   f satisfies that

   - f is continuous;
   - f is defined on its entire domain $[0, 2]$;
   - $f(0) = 0$; and
   - the improper integral $\displaystyle\int_0^2 \frac{1}{f(x)}\mathrm{d}x$ **converges** and is **negative**.

   Note: you are ***drawing*** *f but* ***integrating*** *$1/f$.*

   ◯ Such an f exists ◯ No such f exists



7. (1 point) The function f is a polynomial with roots 4, -1, $6+i$, and $6-i$. Define $h(x) = \frac{x}{f(x)}$. Let T be the Taylor series for h **centered at 1**.

   What is the radius of convergence of T?

   $R = \boxed{2}$

8. Pam, an astronomer, wants to determine the mass of a planet of radius 10^6 metres. Using spectroscopy, Pam has determined the chemical makeup of the planet and used this information to estimate the radial density of the planet (i.e., the density r metres from its centre). The estimate is given by

$$\tilde{\rho}(r) = \frac{1}{r^2(r+4)}$$

in units of kilograms per metre cubed.

In the following questions, you may use the fact that the mass of a sphere with radial density $\rho(r)$ and radius R is given by the formula

$$M = \int_0^R 4\pi r^2 \cdot \rho(r)\,\mathrm{d}r.$$

(a) (2 points) Based on Pam's calculations, she has determined that $|\tilde{\rho}(r) - \rho(r)| \leq \frac{1}{100}$, where $\rho(r)$ is the true radial density of the planet.

Using this information, set up **an integral** that gives the best possible upper bound for the mass, M, of the planet. *Do not evaluate your integral.*

$$M \leq \boxed{\int_0^{10^6} 4\pi r^2 \cdot \left(\frac{1}{r^2(r+4)} + \frac{1}{100}\right) \mathrm{d}r = \int_0^{10^6} \frac{4\pi}{r+4} + \frac{\pi r^2}{25}\,\mathrm{d}r}$$

Find an upper bound for M. *There is no need to simplify, but all integrals must be evaluated.*

$$M \leq \boxed{\ln(10^6+4)\cdot 4\pi - \ln(4)\cdot 4\pi + \frac{\pi 10^{18}}{75} = \pi\left(4\ln(10^6+4) - 4\ln(4) + \frac{10^{18}}{75}\right)}$$

Scratch work:

(b) After reading Pam's paper on planet mass, her graduate student, Alice, suggests a correction. Alice claims that the uncertainty in the radial density of the planet is greater near its core than at the outer edges.

Pam and Alice have the following exchange:

> *Alice*: A better bound to use would be $|\tilde{\rho}(r) - \rho(r)| \leq \frac{1}{r^3(r+4)}$, where $\rho(r)$ is the true density of the planet at radius r.
> *Pam:* But at $r = 0$, that bound is invalid.
> *Alice:* If we use improper integrals, we can arrive at a bound even if some of the functions are undefined at a particular point.

i. (1 point) Write down **a single** (improper) integral that uses *Alice*'s idea to get an upper bound for the mass, M, of the planet.

$$M \quad \leq \quad \boxed{\int_0^{10^6} 4\pi r^2 \left(\frac{1}{r^2(r+4)} + \frac{1}{r^3(r+4)} \right) \mathrm{d}r \quad = \quad \int_0^{10^6} \frac{4\pi}{r+4} + \frac{4\pi}{r(r+4)} \,\mathrm{d}r}$$

ii. (1 point) Express the (improper) integral above as limit(s) of proper integrals.

$$\lim_{k \to 0^+} \int_k^{10^6} \frac{4\pi}{r+4} + \frac{4\pi}{r(r+4)} \,\mathrm{d}r$$

iii. (2 points) Is Alice correct? By using an improper integral, will Alice get a valid (i.e., real number) bound on the mass of the planet? Provide a well-written justification.

○ **Will** provide a valid bound ● **Will not** provide a valid bound

Justification:

To evaluate the improper integral from part ii, we first compute

$$\int \frac{4\pi}{r+4} + \frac{4\pi}{r(r+4)}\,\mathrm{d}r$$

$$= 4\pi \ln(r+4) + \pi \ln r - \pi \ln(r+4) + C$$

$$= 3\pi \ln(r+4) + \pi \ln r + C$$

Now, using this antiderivative to compute the improper integral, we see

$$\lim_{k\to 0^+} \left(3\pi \ln(r+4) + \pi \ln r + C\right)\Big|_k^{10^6}$$

$$= 3\pi \ln(10^6+4) + \pi \ln(10^6) - 3\pi \ln 4 - \pi \ln k = \infty,$$

and so the improper integral diverges. This means that the bound coming from Alice's suggestion is not helpful and is not a real-number.

Scratch work:

9. The *moment about the y-axis* of a function q on the interval $[a,b]$, sometimes written $\mathbb{M}(q)$, is $\displaystyle\int_a^b x\cdot q(x)\,\mathrm{d}x$.

(a) (1 point) Find the *moment about the y-axis* of the sine function on the interval $[0,\pi]$. *No need to simplify, however all integrals must be evaluated.*

$\mathbb{M}(\sin) \quad = \quad \boxed{\pi}$

(b) (2 points) Find the *moment about the y-axis* of the function $h(x) = e^{-x^4}$ on the interval $[-2,2]$. **Simplify your answer and provide a well-written justification.**

Note 1: You must simplify your final answer.

Note 2: You must show your work.

Note 3: Even though h does not have an anti-derivative in terms of elementary functions, you have enough information to answer this question.

$\mathbb{M}(h) \quad = \quad \boxed{0}$

Justification:

Method 1:

Let $g(x) = xh(x) = xe^{-x^4}$. Notice that $g(-x) = -xe^{-(-x)^4} = -xe^{-x^4} = -g(x)$ and so g is an odd function. Therefore, $\displaystyle\int_{-2}^{2} g(x)\,\mathrm{d}x = 0$, since the domain of integration is symmetric about the origin.

Method 2:

Proceed with integration by substitution. Let $u = x^2$ and so $\mathrm{d}u = 2x\mathrm{d}x$. Now,

$$\int_{x=-2}^{x=2} xe^{-x^4}\,\mathrm{d}x = \int_{x=-2}^{x=2} \tfrac{1}{2}e^{-u^2}\,\mathrm{d}u.$$

Since $u = x^2$, when $x = -2$ we have $u = 4$ and when $x = 2$, we have $u = 4$. Therefore

$$\int_{x=-2}^{x=2} xe^{-x^4}\,\mathrm{d}x = \int_{u=4}^{u=4} \tfrac{1}{2}e^{-u^2}\,\mathrm{d}u.$$

Since this is an integral from 4 to 4, its value is zero.

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If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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