University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 17

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We will start 10 minutes past the hour. Use this time to make a new friend.

We Value Your Feedback!

Course Evaluations Are Open!

- Your feedback is **anonymous** and makes a real difference.
- It helps us improve the course for future students.
- It helps instructors reflect and grow in their teaching.

What to Comment On:

- Course structure - was it clear and well-organized?
- Teaching style - what worked (or didn't)?
- Assessments - were they fair and helpful?
- Resources - were they accessible and useful?
- Suggestions - what could be improved for next time?

Bonus: Consider leaving a review on platforms like `RateMyProfessors.com` *to help other students too!*

Parametric Curves

A parametric curve is a function from $\mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R} \to \mathbb{R}^3$

$$t \longmapsto (x(t), y(t))$$

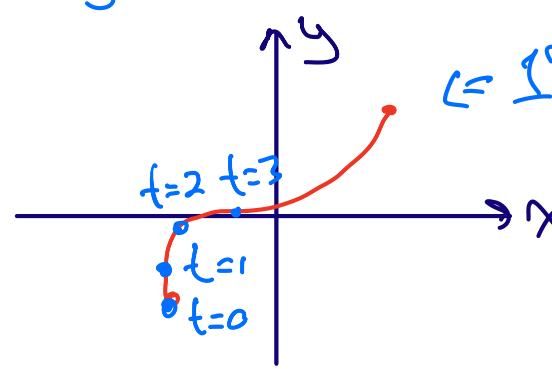
or $$t \longmapsto (x(t), y(t), z(t))$$

<u>Uses</u> ① represent path of an object in 2D or 3D

$$\underbrace{t}_{\text{time}} \longmapsto \underbrace{(x(t), y(t))}_{\text{Position at time } t}$$

② represent a 1-dimensional object embedded in 2D or 3D space

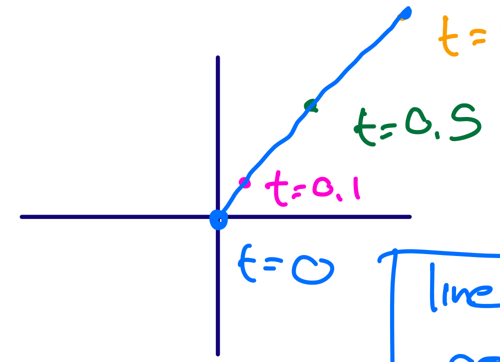
$$\underbrace{t}_{\text{coordinate}} \longmapsto (x(t), y(t))$$



$\Leftarrow$ 1D string in 2D space

ex // ①

$$\gamma(t) = (\underbrace{t}_{x(t)}, \underbrace{t}_{y(t)}) \qquad 0 \le t \le 1$$



$\boxed{\text{line segment}}$

Plot some Points:

$t=0 \longmapsto \gamma(0)=(0,0)$

$t=0.1 \longmapsto \gamma(0.1)=(0.1, 0.1)$

$t=0.5 \longmapsto \quad = (0.5, 0.5)$

$t=1 \longmapsto \quad = (1,1)$

Eliminate the parameter:

$$x(t)=t$$
$$y(t)=t$$

$\Rightarrow \boxed{x=y}$ ← lose timing information

⇑ straight line with slope 1

→ note that multiple functions $\gamma(t)$ can represent the same curve ex// $\tilde{\gamma}(t) = (2t, 2t)$ $\quad 0 \le t \le \frac{1}{2}$

$\tilde{\tilde{\gamma}}(t) = (t^2, t^2)$ $\quad 0 \le t \le 1$

line segments from (0,0) to (1,1) but varying speed

② $\gamma(t) = (\underbrace{\cos(t)}_{x(t)}, \underbrace{\sin(t)}_{y(t)})$ $\quad 0 \le t \le \pi$

$$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$$

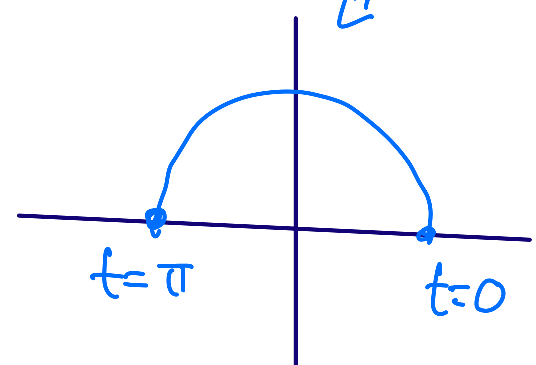
$\boxed{x^2 + y^2 = 1}$ $\Leftarrow$ circle

half circle

→ bounds of parameterization

$t = 0 \mapsto \gamma(0) = (1, 0)$

$t = \pi \mapsto \gamma(\pi) = (-1, 0)$



Sketch the following curves:

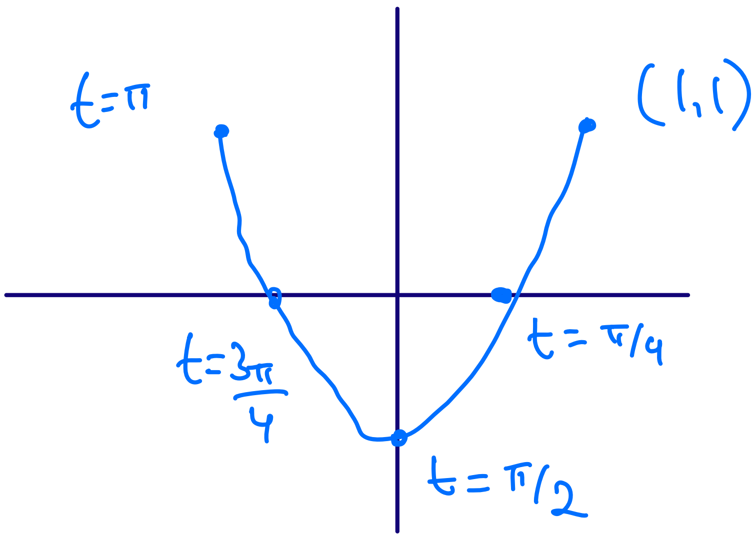
$(x, y) = (\cos t, \cos(2t))$ $\quad 0 \le t \le \pi$

$t = 0 \mapsto (1, 1)$

$\to x(t), y(t)$ both decrease

$t = \pi/4 \mapsto \left(\frac{1}{\sqrt{2}}, 0\right)$

$t = \pi/2 \mapsto (0, -1)$



$(x, y) = (2t + 1, t^2)$ $\quad 0 \le t \le 2$

$x(t) = 2t+1 \Rightarrow t = \frac{x-1}{2}$

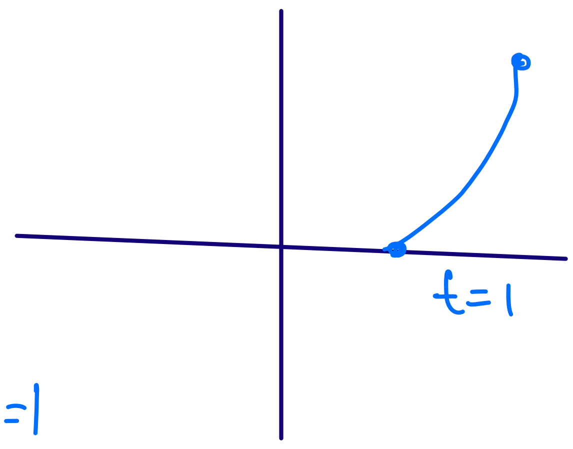
$y(t) = t^2$

$\Rightarrow y = \left(\frac{x-1}{2}\right)^2$

$y = \frac{1}{4}(x-1)^2 \Leftarrow$ Parabola edge at $x=1$

$t=0 \mapsto (1, 0)$

$t=2 \mapsto (5, 4)$



Derivatives of Parametric Curves

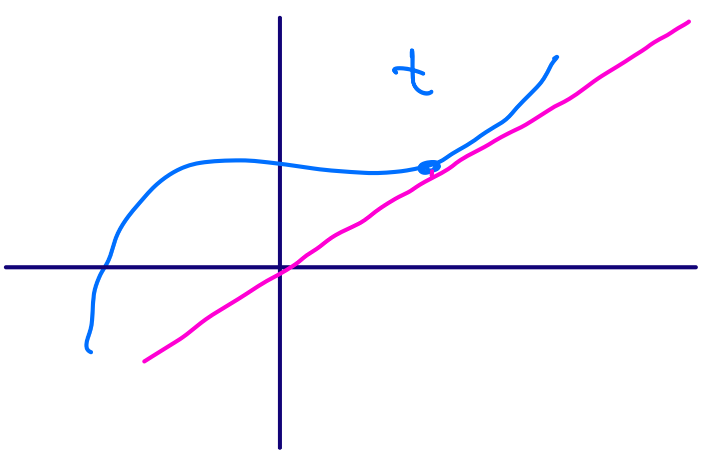
$\gamma(t) = (x(t), y(t))$ (also possibly $z(t)$)

The velocity of $\gamma(t)$ at time t is the component by component derivative w.r.t to t

$$\gamma'(t) = (x'(t), y'(t))$$

$\to$ the speed of γ is the norm of velocity $\|\gamma'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$

The tangent line to γ at time t is a line with slope $\dfrac{dy/dt}{dx/dt}$ and passing through $\gamma(t)$



Determine the velocity vector and tangent line for each curve at $t = 0$.

$(x, y) = (t^2 + 2t, t^3 + t)$

velocity vector

$(x'(t), y'(t)) = (2t+2, 3t^2+1)$

$t=0 \Rightarrow (x'(0), y'(0)) = (2, 1)$

Tangent line

slope $= \frac{y'(0)}{x'(0)} = \frac{1}{2}$

$\rightarrow$ passes through $t=0$, $(x_0, y_0) = (0,0)$

$y - y_0 = m(x - x_0)$ $\Leftarrow$ line passing through (x_0, y_0)

$\boxed{y = \frac{1}{2}x}$

$(x, y) = (\cos(t) + t, \sin(t) - t^2)$

$(x'(t), y'(t)) = (-\sin(t)+1, \cos(t) - 2t)$

$t=0 \Rightarrow \boxed{(x'(0), y'(0)) = (1, 1)}$

Tangent Line

slope $= \frac{y'(0)}{x'(0)} = \frac{1}{1} = 1$

$\rightarrow$ passes throug $t=0$ $(x_0, y_0) = (1, 0)$

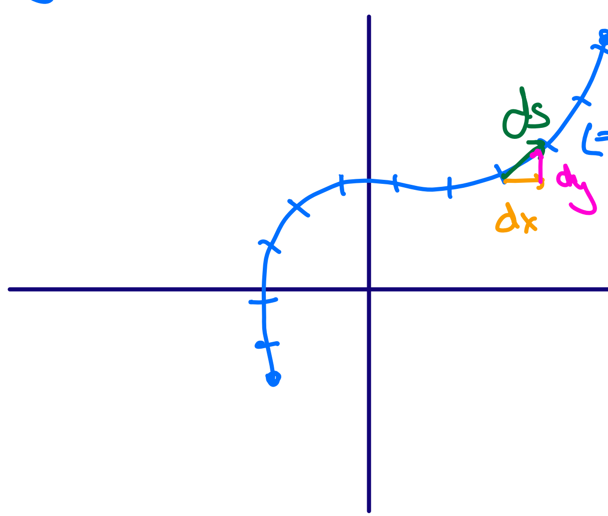
$y - y_0 = m(x - x_0)$

$y = 1(x-1)$

$\boxed{y = x - 1}$

Arclength

→ given curve $\gamma(t) = (x(t), y(t))$, $a \le t \le b$. How to find its length?



⇐ assume ds is straight line

→ divide curve into pieces of size ds

$$\text{total arclength} = \int ds$$

→ compute ds

$$ds = \sqrt{dx^2 + dy^2}$$

x is function of t

y is function of t

$$\Rightarrow \quad dx = x'(t)\,dt$$

$$dy = y'(t)\,dt$$

$$ds = \sqrt{(x'(t)\,dt)^2 + (y'(t)\,dt)^2}$$

$$= \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Conclusion

$$\text{arclength} = \int_{t=a}^{t=b} \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Determine the arc length of $(x, y) = (t^2, \frac{t^3}{3})$, $0 \leq t \leq 2$

$$\text{arclength} = \int_{t=a}^{t=b} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$x(t) = t^2 \Rightarrow x'(t) = 2t$$

$$y(t) = \frac{t^3}{3} \Rightarrow y'(t) = t^2$$

$$= \int_0^2 \sqrt{(2t)^2 + (t^2)^2}\, dt$$

$$= \int_0^2 t\sqrt{4+t^2}\, dt$$

$$u = 4 + t^2 \qquad du = 2t\,dt \Rightarrow \begin{array}{l} t=0 \Rightarrow u=4 \\ t=2 \Rightarrow u=8 \end{array}$$

$$= \frac{1}{2}\int_{u=4}^{u=8} \sqrt{u}\, du$$

$\vdots$