University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION, DECEMBER, 2013
Duration: 2 and 1/2 hours
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA
Exam Type: A

SURNAME: (as on your T-card) ____________________

YOUR FULL NAME: ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

Examiners:
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Calculators Permitted:
Casio FX-991 or Sharp EL-520.

INSTRUCTIONS: Attempt all questions.
Present your solutions in the space provided. You must show your work and give full explanations to get full marks. Partial credit can be obtained for partially correct work, but NO credit may be given if your work is poorly presented, difficult to decipher, uses incorrect mathematical notation, or makes no sense.
Use the backs of the sheets if you need more space. Do not tear any pages from this exam. Make sure your exam contains 10 pages.

MARKS: All nine questions are worth 10 marks.

TOTAL MARKS: 90

QUESTION	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
Q9	
TOTAL	

1. The parts of this question are unrelated.

(a) [5 marks] Calculate $\det \begin{bmatrix} 2 & 0 & -1 & 5 \\ -1 & 3 & 0 & 1 \\ 5 & 4 & 2 & 3 \\ 1 & 6 & 0 & 4 \end{bmatrix}$.

(b) [5 marks] Find a basis for $S^{\perp}$ if $S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix} \right\}$.

2.(a) [5 marks] Find all values of c for which $\det \begin{bmatrix} 1 & c & c \\ 1 & -1 & 1 \\ c & c & 1 \end{bmatrix} = 0$.

2.(b) [5 marks] Find all values of c for which the system of equations

$$\begin{array}{rcrcrcl} x_1 & + & cx_2 & + & cx_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 4 \\ cx_1 & + & cx_2 & + & x_3 & = & 2 \end{array}$$

has *(i)* no solution, *(ii)* a unique solution, *(iii)* infinitely many solutions.

3. [10 marks; 2 marks each.] Indicate if the following statements are True or False, and give a *brief* explanation why. Circle your choice.

(a) If λ is an eigenvalue of the matrix A then λ^2 is an eigenvalue of the matrix A^2.

True **False**

(b) If A is an $n \times n$ diagonalizable matrix then $A + 3I_n$ is also diagonalizable.

True **False**

(c) If the rows of the 5×7 matrix A are linearly independent then $\text{null}(A) = \{\mathbf{0}\}$.

True **False**

(d) If $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is a linear transformation and T is one-to-one, then T is onto.

True **False**

(e) If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal column vectors in $\mathbb{R}^n$ then $\mathbf{y}\,\mathbf{x}^T = 0$.

True **False**

4. Find the solution to the system of linear differential equations

$$\begin{aligned} y_1' &= 3y_1 - 2y_2 \\ y_2' &= -4y_1 + y_2 \end{aligned}$$

where y_1, y_2 are functions of t, and $y_1(0) = 4, y_2(0) = -1$.

5. The reduced echelon form of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [1 mark] The rank of A is ________.

(b) [2 marks] $\dim(\text{row}(A)) =$ ________ and a basis for the row space of A is:

(c) [3 marks] $\dim(\text{col}(A)) =$ ________ and a basis for the column space of A is:

(d) [4 marks] $\dim(\text{null}(A)) =$ ________ and a basis for the null space of A is:

6. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y \\ 2x + y \end{bmatrix}.$$

(a) [5 marks] Draw the image of the unit square under T, and calculate its area.

(b) [5 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

7. Suppose $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \; T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; T\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) = -\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Find a 3×3 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x}$ in $\mathbb{R}^3$.

8. Let $S = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Find $\operatorname{proj}_S(\mathbf{x})$.

9. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$