Tutorial Problems 10

1. Three of the four vectors $\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{x_3} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{x_4} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ are eigenvectors of

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 3 & -4 & 0 \\ 3 & -1 & -3 \end{bmatrix}.$$

(a) What are the eigenvalues of A?

Solution: Notice that $A\mathbf{x_1} = -\mathbf{x_1}$, $A\mathbf{x_2} = -2\mathbf{x_2}$, and $A\mathbf{x_4} = -3\mathbf{x_4}$, so that $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_4}$ are the eigenvectors of A with corresponding eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$, and $\lambda_3 = -3$ respectively.

(b) Find $A^{188}\mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$. **Hint**: Write $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$ as a linear combination of the eigenvectors of A.

Solution: Notice that $\mathbf{b} = 3\mathbf{x_1} + 2\mathbf{x_2} - \mathbf{x_4}$. Therefore, $A^{188}\mathbf{b} = A^{188}(3\mathbf{x_1} + 2\mathbf{x_2} - \mathbf{x_4}) = 3A^{188}\mathbf{x_1} + 2A^{188}\mathbf{x_2} - A^{188}\mathbf{x_4}$. Now, from part (a), we know that $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_4}$ are eigenvectors of A. Notice that if $A\mathbf{v} = \lambda\mathbf{v}$, then $A^k\mathbf{v} = A^{k-1}(A\mathbf{v}) = \lambda A^{k-1}\mathbf{v} = \ldots = \lambda^k\mathbf{v}$. Thus, $A^{188}\mathbf{b} = 3\lambda_1^{188}\mathbf{x_1} + 2\lambda_2^{188}\mathbf{x_2} - \lambda_3^{188}\mathbf{x_4} = 3\mathbf{x_1} + 2^{189}\mathbf{x_2} - 3^{188}\mathbf{x_4}$.

2. Let $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

(a) Find all eigenvalues of A and a basis for each eigenspace.

(b) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ and use this to find A^{-1}.

Solution:

(a) Computing the characteristic polynomial of A, we get (check this!)

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -1 & 2 \\ 1 & \lambda - 2 & -1 \\ 0 & -1 & \lambda + 1 \end{bmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 1)$$

Therefore the eigenvalues of A are $\lambda = -1, 1, 2$.

For $\lambda = -1$, the eigenspace E_{-1} is the null space

$$E_{-1} = \text{Null}(-I - A) = \text{Null} \begin{bmatrix} -2 & -1 & 2 \\ 1 & -3 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

The RREF for this matrix is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the above null space has basis (from the parametric form of the solutions to this system) given by

$$E_{-1} = \text{Null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Similary, for the other eigenvalues we get

$$E_1 = \text{span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}, \quad E_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Thus we have found all eigenvalues and a basis for each eigenspace.

(b) To find D and P such that $P^{-1}AP = D$, we take the diagonal matrix D consisting of the eigenvalues of A, and the matrix P of corresponding eigenvectors:

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and find that } P^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} & \frac{7}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Then $D = P^{-1}AP$ or equivalently $A = PDP^{-1}$. Inverting both sides, we get that

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

Substituting the above matrices, we find that

$$A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{-1} & 0 & 0 \\ 0 & \frac{1}{1} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} & \frac{7}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 & -5/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -3/2 \end{bmatrix}$$

3. Find a 2×2 matrix A that has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Solution: Let $P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, so that $P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$. Also, let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. It follows that $A = PDP^{-1} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

4. Find conditions on $a, b, c,$ and d such that the matrix

$$A = \begin{bmatrix} 2 & 1 & a & b \\ 0 & 3 & -1 & c \\ 0 & 0 & 2 & d \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is diagonalizable.

Solution: First let's compute the characteristic polynomial:

$$\det(A - \lambda I) = \det \begin{bmatrix} \lambda - 2 & -1 & -a & -b \\ 0 & \lambda - 3 & 1 & -c \\ 0 & 0 & \lambda - 2 & -d \\ 0 & 0 & 0 & \lambda - 3 \end{bmatrix} = (\lambda - 2)^2 (\lambda - 3)^2$$

Note that this is easy because the matrix is upper-triangular, so its determinant is the product of the diagonal entries!

We see that A has eigenvalues $\lambda = 2, 3$ and that both have multiplicity 2. Therefore A is diagonalizable if and only if

$$\dim E_2 = 2 \quad \text{and} \quad \dim E_3 = 2$$

We will seek the conditions on a, b, c, d that determine when these equalities hold.

For $\lambda = 2$, we have

$$E_2 = \text{Null}(2I - A) = \text{Null} \begin{bmatrix} 0 & -1 & -a & -b \\ 0 & -1 & 1 & -c \\ 0 & 0 & 0 & -d \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

This matrix can be reduced (check this!) to the form $B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & a+1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Observe that $\text{rank}(B) = $

3 when $a + 1 \neq 0$, and that $\text{rank}(B) = 2$ when $a + 1 = 0$. By the Rank-Nullity Theorem,

$$\dim \text{Null}(B) + \text{rank}(B) = \mathbb{R}^4 = 4$$

Therefore, the $\dim E_2 = \dim \text{Null}(B) = 1$ when $a + 1 \neq 0$, and $\dim E_2 = 2$ when $a + 1 = 0$. We want the second case!

For $\lambda = 3$, we have

$$E_3 = \text{Null}(3I - A) = \text{Null} \begin{bmatrix} 1 & -1 & -a & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix can be reduced to the form $C = \begin{bmatrix} 1 & -1 & -a & -b \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & d-c \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This time we see that $\text{rank}(C) = 3$

when $d - c \neq 0$, and that $\text{rank}(C) = 2$ when $d - c = 0$. Applying the Rank-Nullity Theorem to C we get that $\dim E_3 = \dim \text{Null}(C) = 1$ when $d - c \neq 0$, and $\dim E_3 = 2$ when $d - c = 0$. Again we want the second case!

In conclusion, A is diagonalizable

$$\iff \quad \dim E_2 = 2 \quad \text{and} \quad \dim E_3 = 2$$
$$\iff \quad a + 1 = 0 \quad \text{and} \quad d - c = 0$$

In other words, A is diagonalizable if and only if $a = -1$ and $c = d$.