University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 8

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We will start 10 minutes past the hour. Use this time to make a new friend.

Power Series

A power series can be thought of as an infinite degree Polynomial

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

ex//

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad \boxed{a_n = 1}$$

$$\sum_{n=0}^{\infty} r^n x^n = 1 + rx + r^2x^2 + \ldots \qquad \boxed{a_n = r^n}$$

$$\sum_{n=0}^{\infty} n x^n = 0 + x + 2x^2 + 3x^3 + \ldots \qquad \boxed{a_n = n}$$

Does the power series converge for any x?

$\rightarrow$ $x=0$, always converges

ex// $\sum_{n=0}^{\infty} x^n$ where else does this converge?

→ evaluate at $x=r$ $\Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ $|r|<1$

$\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$ and is <u>equal</u> to the function $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

→ for $|x| \geq 1$, power series diverges

A power series centered at a is of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \ldots$$

Radius of Convergence

→ apply ratio test to power series

$$\sum_{n=0}^{\infty} \underbrace{a_n (x-a)^n}_{c_n} = \sum_{n=0}^{\infty} c_n$$

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n (x-a)^n} \right|$$

$$= |x-a| \underbrace{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}_{1/R}$$

→ R may be finite, infinite or zero

$$L = |x-a| \, 1/R$$

→ by ratio test, series converges if $L = |x-a| \frac{1}{R} < 1 \Rightarrow \boxed{|x-a| < R}$

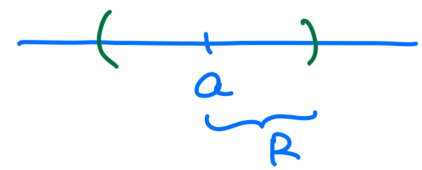
diverges if $L = |x-a| \frac{1}{R} > 1 \Rightarrow \boxed{|x-a| > R}$

inconclusive if $L = 1 \Rightarrow \boxed{x = a \pm R}$

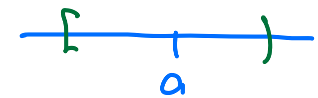
Given a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ there exists an R (which may be infinite or zero) s.t. the power series converges on either

depends on convergence at end-points

$(a-R, a+R)$



$[a-R, a+R)$



$(a-R, a+R]$

$[a-R, a+R]$

→ if $R=0$ then convergence on $[a-0, a+0] = \{a\}$ single point

→ if $R=\infty$ then convergence on $(a-\infty, a+\infty) = \mathbb{R}$ all numbers

The value R is called radius of convergence

$\sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!}}_{c_n}$ ⇐ center $a=0$

Ratio test ⇒ $L = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n\to\infty} \frac{1}{n+1}$

$L = |x| \cdot 0 = 0 < 1$ for all x

∴ radius of convergence is infinite

⇒ convergences on all of $\mathbb{R}$

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ ⇐ center $a=0$

Ratio test ⇒ $L = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = |x| \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right| = |x| \cdot 1$

$L = |x| \cdot 1 < 1 \Rightarrow \boxed{|x| < 1}$

∴ radius of convergence $\boxed{R=1}$

→ check end-points $x=1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series)

$x=-1 \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges (AST)

$\boxed{\text{Interval of convergence } [-1, 1]}$

Properties of Power Series

Suppose you have power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$ } common interval of convergence I

then ① $f(x)+g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ will converge on I

② $f(x)\,g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$ $\Leftarrow$ distributive multiplication

$$= a_0 b_0 + (a_0 b_1 + b_0 a_1) x + (a_0 b_2 + b_0 a_2 + 2a_1 b_1) x^2 + \ldots$$

③ $f(x^n) = \sum_{n=0}^{\infty} a_n (x^n)^n$ converges whenever $x^n \in$ interval of convergence

④ $f(cx) = \sum_{n=0}^{\infty} a_n (cx)^n$ for $cx \in$ interval of convergence

Integration and Differentiation

If $$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

① $$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

② $$\int f(x)\,dx = \sum_{n=0}^{\infty} \int a_n x^n\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} + C$$

} radius of convergence will be the same for new same

→ you can differentiate / integrate term-term

→ these also apply to power series not centered at $a=0$

ex// $$\frac{1}{1-2x} = \frac{1}{1-y} = \sum_{n=0}^{\infty} y^n = \sum_{n=0}^{\infty} 2^n x^n$$ ⇐ radius of convergence is $|2x| < 1$

$$\boxed{|x| < 1/2}$$

$y = 2x$

geometric series

coeff $a_n = 2^n$

Find a power series for:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\frac{1}{1-(\text{banana})} = \sum_{n=0}^{\infty} (\text{banana})^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$\Leftarrow$ convergence for

$|-x^2| < 1$

$|x| < 1$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$\Leftarrow$ convergence for $|x| < 1$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$$

Taylor Series

Given a function $f(x)$, the taylor series centered at a is defined as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

→ the partial sums of taylor series are the Taylor Polynomials

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

→ taylor's remainder thm:

$$\text{error} = |f(x) - P_n(x)| = \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

→ if f is "well-behaved" then as $n \to \infty$, error $\to 0$ (atleast locally around a)

$\Rightarrow$ taylor series $\sum \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$ around a

→ note there are functions for which Taylor series $\neq f(x)$, called non-analytic functions

→ most functions in engineering are analytic :)

Uniqueness Thm

If $f(x) = \sum a_n (x-a)^n$ around a then the power series is the Taylor series $\Rightarrow a_n = \frac{f^{(n)}(a)}{n!}$

ex/ $\arctan(x) = 1x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots$

$$1 \leftarrow \frac{f'(0)}{1!}, \quad \frac{1}{3} \leftarrow \frac{f'''(0)}{3!}, \quad \frac{1}{5} \leftarrow \frac{f^{(5)}(0)}{5!}$$

ex/ Taylor Series

$f(x) = e^x \Rightarrow f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(x) = e^x \qquad = 1$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n}$$

(converges everywhere)

Find the Taylor series for:

$f(x) = \cos(x)$ $\quad f(0) = 1$

$f'(x) = -\sin(x)$ $\quad f'(0) = 0$

$f''(x) = -\cos(x)$ $\quad -1$

$f'''(x) = \sin(x)$ $\quad 0$

$f^{(4)}(x) = \cos(x)$ $\quad 1$

$f^{(5)}(x) = -\sin(x)$ $\quad 0$

$\downarrow$ $\quad -1$

repeat $\quad 0$

$\vdots$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 + \ldots$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \ldots$$

$$\boxed{\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

$\rightarrow$ likewise

$$\boxed{\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots}$$

$\cos(\sqrt{x})$

$\rightarrow$ use series for $\cos(x)$

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$

Radius of Convergence and Domain

$$f(x) = \underbrace{\sum \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

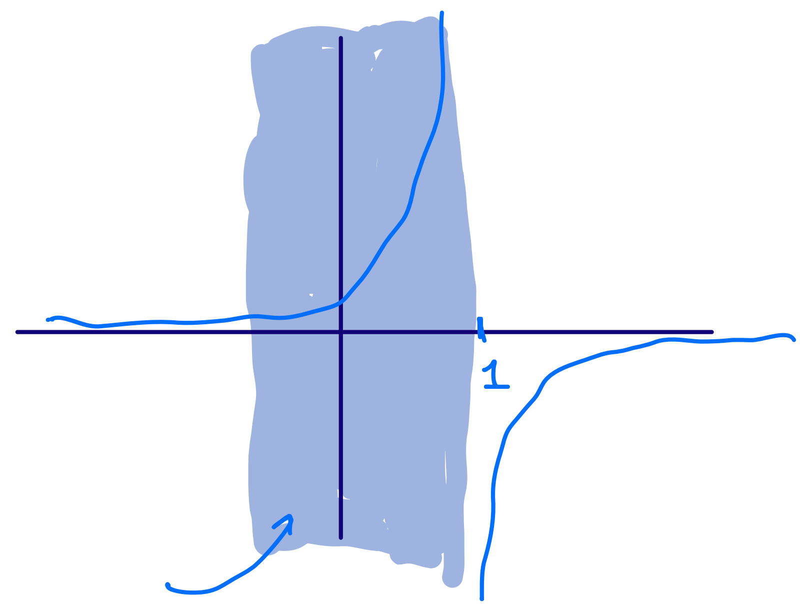
→ convergence of this series doesn't need to equal domain $f(x)$

ex //

$$\underbrace{\frac{1}{1-x}} = \underbrace{\sum_{n=0}^{\infty} x^n}$$

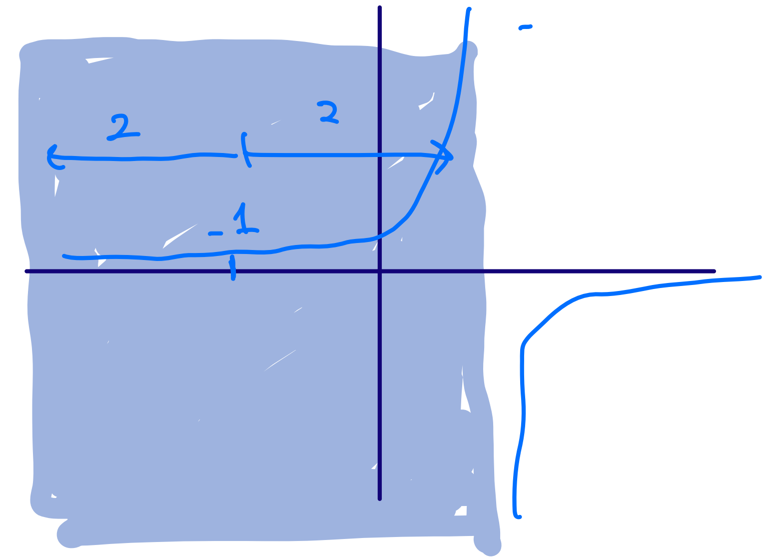
domain: $x \neq 1$

convergence only $|x|<1$



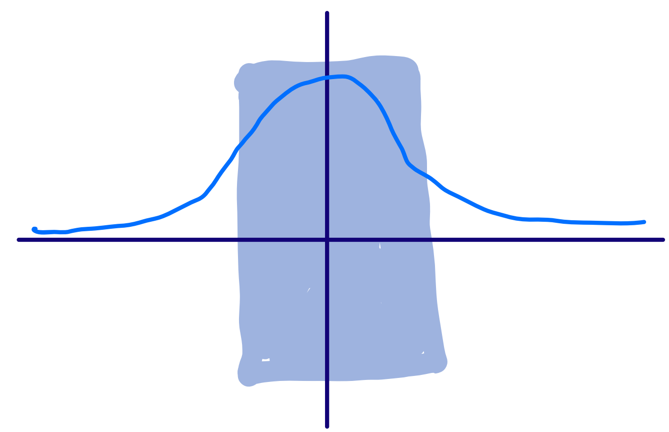
region of convergence for Taylor series centered at zero ex:

$$\frac{1}{1-x} = \underbrace{\sum_{n=0}^{\infty} \cdots (x+1)^n}_{\text{Taylor series centered at } -1}$$



ex:

$$\underbrace{\frac{1}{1+x^2}}_{\text{defined everywhere}} = \underbrace{\sum_{n=1}^{\infty} (-1)^n x^{2n}}_{\text{radius of convergence } |x|<1}$$



ex:

$$\underbrace{\cos(\sqrt{x})}_{\text{domain } x>0} = \underbrace{\sum (-1)^n \frac{x^n}{(2n)!}}_{\text{converges everywhere}}$$

$\Leftarrow$ sometimes series converges at more points than the function