Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 9

1. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & c & 5 \\ 2 & 4 & -4 \end{bmatrix}$.

(a) Find $\det(A)$.

(b) For what values of c is A invertible?

(c) For the values of c such that A is invertible, find $\det(A^{-1})$.

Solution:

(a) We compute $\det(A)$ by cofactor expansion along the second row:

$$\det(A) = c \cdot \det \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = -2c + 10$$

For the final equality we used the (hopefully!) familiar formula $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

(b) We recall one of the most important properties of the determinant: A is invertible if and only if $\det(A) \neq 0$. In our case, it follows from part **(a)** that A is invertible iff $-2c + 10 \neq 0$. In other words, A is invertible iff $c \neq 5$.

(c) Whenever A is invertible (i.e. when $c \neq 5$), we have

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-2c + 10}$$

2. By row-reducing to upper-triangular form, evaluate

$$\det \begin{bmatrix} 3 & 2 & 1 & 4 & -1 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 & 1 \\ -3 & 4 & 1 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Solution: Using Theorem 3 from Section 5.1 along with Theorems 1, 2, and 4 from Section 5.2 in the textbook, we perform elementary row operations:

$$\det\begin{bmatrix} 3 & 2 & 1 & 4 & -1 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 & 1 \\ -3 & 4 & 1 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 & 1 \\ -3 & 4 & 1 & 6 & 7 \\ 3 & 2 & 1 & 4 & -1 \end{bmatrix} \qquad \text{(switch } R_1 \text{ and } R_5\text{)}$$

$$= -\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -6 & -12 & -18 & -24 \\ 0 & -3 & -4 & -5 & -9 \\ 0 & 10 & 10 & 18 & 22 \\ 0 & -4 & -8 & -8 & -16 \end{bmatrix} \qquad \text{(adding multiples of } R_1 \text{ to the others)}$$

$$= -(-6)(-4)\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -3 & -4 & -5 & -9 \\ 0 & 10 & 10 & 18 & 22 \\ 0 & 1 & 2 & 2 & 4 \end{bmatrix} \qquad \text{(pulling common factors out of rows)}$$

$$= -24\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & -10 & -12 & -18 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \qquad \text{(adding multiples of } R_2 \text{ to those below it)}$$

$$= 24\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -10 & -12 & -18 \end{bmatrix} \qquad \text{(switch } R_4 \text{ and } R_5\text{)}$$

$$= 24\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 8 & -3 \end{bmatrix} \qquad \text{(add } 5R_3 \text{ to } R_5\text{)}$$

$$= 24\det\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \qquad \text{(add } 8R_4 \text{ to } R_5\text{)}$$

This gives $24 \times 6 = 144$, since the determinant of an upper-triangular matrix is the product of its diagonal entries.

3. Think of $U = (u_1\, u_2\, \ldots u_n), V = (v_1\, v_2\, \ldots v_n)$, and $A_i = (a_{i1}\, a_{i2}\, \ldots a_{in})$ as $1 \times n$ rows, $i = 2, \ldots, n$.

(a) Show that

$$\det \begin{bmatrix} U+V \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \det \begin{bmatrix} U \\ A_2 \\ \vdots \\ A_n \end{bmatrix} + \det \begin{bmatrix} V \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

This is known as the additive property of the determinant. Note that similar equalities hold for other rows too. If you're interested, please see "A Linearity Property of the Determinant Function" below.

(b) Suppose that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -3$. Find $\det \begin{bmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{bmatrix}$.

(c) True or False: $A = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 & 3 & 9 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \\ 2 & 2 & 18 & 22 \end{bmatrix}$ have the same determinant.

Hint: You can answer this **without** computing the determinants of A and B.

Solution:

(a) Denote these matrices by

$$X = \begin{bmatrix} U+V \\ A_2 \\ \vdots \\ A_n \end{bmatrix}, \quad Y = \begin{bmatrix} U \\ A_2 \\ \vdots \\ A_n \end{bmatrix}, \quad Z = \begin{bmatrix} V \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

We will do a cofactor expansion of these determinants along the first row. Observe that the cofactors along the first row are all equal:

$$c_{1,j}(X) = c_{1,j}(Y) = c_{1,j}(Z), \qquad \text{for all } 1 \leq j \leq n$$

Indeed, for each of these cofactors we remove row 1 and column j. Row 1 is the only place where X, Y, Z differ, so we get the same result in all cases. Then

$$\begin{aligned} \det(X) &= (u_1+v_1)c_{1,1}(X) + (u_2+v_2)c_{1,2}(X) + \ldots + (u_n+v_n)c_{1,n}(X) \\ &= (u_1c_{1,1}(X) + u_2c_{1,2}(X) + \ldots u_nc_{1,n}(X)) + (v_1c_{1,1}(X) + v_2c_{1,2}(X) + \ldots + v_nc_{1,n}(X)) \\ &= (u_1c_{1,1}(Y) + u_2c_{1,2}(Y) + \ldots u_nc_{1,n}(Y)) + (v_1c_{1,1}(Z) + v_2c_{1,2}(Z) + \ldots + v_nc_{1,n}(Z)) \\ &= \det(Y) + \det(Z) \end{aligned}$$

as was claimed.

(b) Performing row operations:

$$\begin{aligned} \det \begin{bmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{bmatrix} &= 2\det \begin{bmatrix} a & b & c \\ 3d-a & 3e-b & 3f-c \\ 4g+3a & 4h+3b & 4i+3c \end{bmatrix} \\ &= 2\det \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{bmatrix} \\ &= 2 \cdot 3 \cdot 4 \cdot \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\ &= -72 \end{aligned}$$

(c) For the first row of A, we have $\begin{bmatrix} 2 & 4 & 2 & 6 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 3 & 6 & 3 & 9 \end{bmatrix}$. Therefore we have

$$\det(A) = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{bmatrix} = \frac{2}{3}\det\begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 3 & 27 & 33 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \end{bmatrix}$$

Switching R_2 and R_3, and then R_3 and R_4, this is equal to

$$-\frac{2}{3}\det\begin{bmatrix} 2 & 4 & 2 & 6 \\ 2 & 1 & 5 & 2 \\ 3 & 3 & 27 & 33 \\ 6 & 1 & -3 & 3 \end{bmatrix} = \frac{2}{3}\det\begin{bmatrix} 2 & 4 & 2 & 6 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 3 \\ 3 & 3 & 27 & 33 \end{bmatrix}$$

Bringing the constant $\frac{2}{3}$ into R_4, this is equal to

$$\det\begin{bmatrix} 2 & 4 & 2 & 6 \\ 2 & 1 & 5 & 2 \\ 6 & 1 & -3 & 2 \\ 2 & 2 & 18 & 2 \end{bmatrix} = \det(B)$$

So, it is true that $\det(A) = \det(B)$.

4. Let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \mid \det\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} = 0 \right\}$.

(a) Show that W is a subspace of $\mathbb{R}^4$.

(b) Find a basis for W and determine $\dim(W)$.

Solution:

(a) We provide two methods for solving this problem.

Method 1: Expand the determinant in the definition of W, e.g. using cofactor expansion along the first row. Then we (eventually!) get the linear equation $-x_1 + x_3 = 0$. W is the set of solutions to this equation, so it is a subspace of $\mathbb{R}^4$.

Method 2: We apply the subspace test. Suppose that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ are both in W. From problem **3 (a)**, we know that

$$\det\begin{bmatrix} x_1+y_1 & x_2+y_2 & x_3+y_3 & x_4+y_4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} = \det\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} + \det\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix}$$

Since $\mathbf{x}, \mathbf{y} \in W$, the two determinants on the right are zero. Hence $\mathbf{x} + \mathbf{y} \in W$.

Similarly, for $\mathbf{x} \in W$ and $c \in \mathbb{R}$, we have

$$\det \begin{bmatrix} cx_1 & cx_2 & cx_3 & cx_4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} = c \cdot \det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix}$$

This is zero since $\mathbf{x} \in W$, so we see that $c\mathbf{x} \in W$. It follows that W is a subspace.

(b) We provide two methods for solving this problem.

Method 1: We saw above that W is the set of solutions to the equation $-x_1 + x_3 = 0$. The parametric form for the solutions to this equation gives us, in the usual way, a basis for W

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

so that $\dim(W) = 3$.

Method 2: We have

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} = 0, \quad \det \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} = 0, \quad \det \begin{bmatrix} 2 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \end{bmatrix} = 0$$

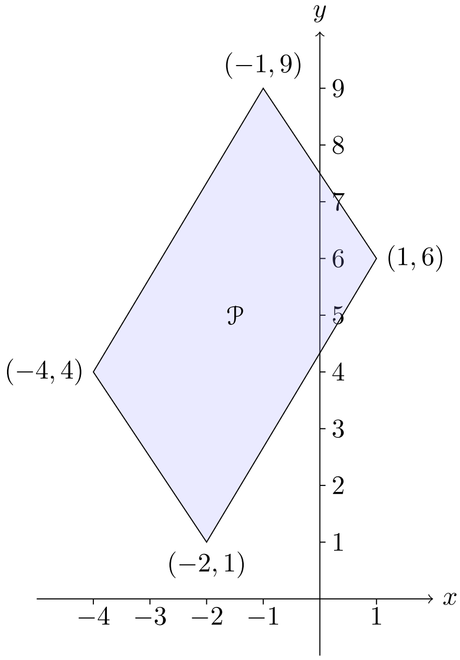
since in each case the matrix has two identical rows. This tells us that the set

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} \right\}$$

is a subset of W. This set is linearly independent (check this!). Since W is three dimensional,$\mathcal{S}$ must also span W. Hence $\mathcal{S}$ is a basis for W.

5. Let P be the parallelogram with vertices $(-2,1), (-4,4), (1,6)$, and $(-1,9)$. Sketch P and compute the area of P using determinants.

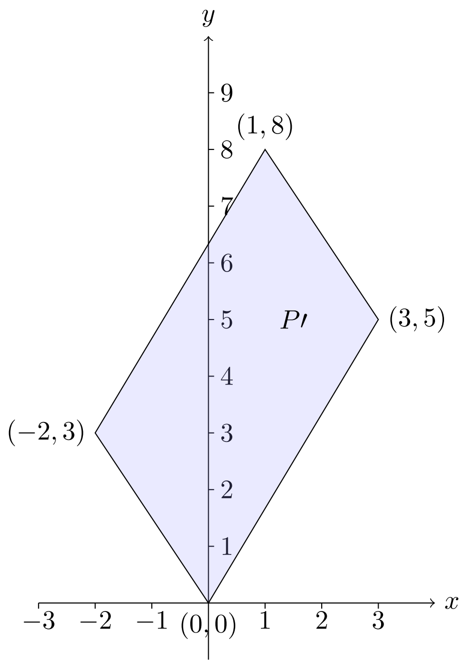
Solution:



Method 1

The area of P is the absolute value of the determinant of the matrix formed by row vectors representing the parallelogram's sides. In this case, one side is $v_1 = \begin{bmatrix} 1-(-2) \\ 6-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, and the other side is $v_2 = \begin{bmatrix} -4-(-2) \\ 4-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, so our matrix has rows equal to these vectors:
$A = \begin{bmatrix} 3 & 5 \\ -2 & 3 \end{bmatrix}$.
Then $\text{area}(P) = |\det(A)| = |\det \begin{bmatrix} 3 & 5 \\ -2 & 3 \end{bmatrix}| = |3 \cdot 3 - (-2) \cdot 5| = 19$

Method 2



First we shift P to the origin by translating all points by $(2, -1)$. This is clearly an area-preserving transformation, and we get the parallelogram $P\prime$ above.

Let T by the linear transformation that maps the unit square to $P\prime$. Let A be the matrix satisfying $T(\mathbf{x}) = A\mathbf{x}$. Then $\text{area}(P\prime) = |\det(A)|$.

T maps the unit square as follows: $\begin{cases} (0,0) \mapsto (0,0) \\ (1,0) \mapsto (3,5) \\ (0,1) \mapsto (-2,3) \\ (1,1) \mapsto (1,8) \end{cases}$

$\Rightarrow A = \begin{bmatrix} T\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) & T\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 3 \end{bmatrix}$

$\Rightarrow \text{area}(P) = \text{area}(P\prime) = |\det(A)| = |\det \begin{bmatrix} 3 & -2 \\ 5 & 3 \end{bmatrix}| = |3 \cdot 3 - (-2) \cdot 5| = 19$

A Linearity Property of the Determinant Function

For an $n \times n$ matrix A you can think of $\det(A)$ as a function of the n rows of A. More precisely, if all the rows of A except one are fixed, then $\det(A)$ is a *linear transformation* of that one row/vector variable.

Suppose the 1st row of A is allowed to vary but the remaining rows are held fixed. Say,

$$A = \begin{bmatrix} U \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

Define T: $\mathbb{R}^n \to \mathbb{R}$ by

$$T(U) = \det \begin{bmatrix} U \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

Then,

$$T(cU) = cT(U), \text{ for all scalars } c \text{ and all } U \in \mathbb{R}^n \text{ (as a row vector not a column vector)} \tag{1}$$

and

$$T(U+V) = T(U) + T(V), \text{ for all } U, V \in \mathbb{R}^n \text{ (again, as row vectors not column vectors)} \tag{2}$$

In other words, T is a linear transformation! (1) is Theorem 4, on page 267 of the textbook, and (2) is what you're asked to prove in question 3(a). Notice there is nothing special about allowing the first row to vary and holding all other rows fixed, we could just as easily let the j-th row vary and hold all the others rows fixed. Note that same result is true for the columns of A - i.e. we may replace the word "row" with "column" above.