UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, May 2006

First Year

MAT188H1 S – LINEAR ALGEBRA

Calculator Type: 3

Exam Type: A

Examiner – H. Witteman

Instructions: Write your solutions in the examination booklets required.
Please make sure to number the questions correctly.

1. **[10 marks]**

For $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ -2 & -2 & 5 \\ 2 & 4 & -3 \end{bmatrix}$

(a) **[4 marks]** Find $\mathbf{A}^{-1}$ or explain why it does not exist.

(b) **[2 marks]** Solve **AX=B** for $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) **[2 marks]** Are the columns of **A** linearly independent? Why or why not?

(d) **[2 marks]** Do the columns of **A** form a basis for $\mathbf{R}^3$? Why or why not?

2. **[2 marks]**

If **A** is a 4x4 matrix such that det(**A**) = 3, and matrix **B** is obtained from **A** by switching rows 2 and 4 and then multiplying the entire matrix by 2, what is det(**B**)?

3. **[4 marks]**

For $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & k & k \\ 1 & k & 0 & k \\ 1 & k & k & 0 \end{bmatrix}$, where k is a constant:

(a) **[3 marks]** What is det(**A**)?

(b) **[1 mark]** For what values of k is **A** invertible?

4. **[8 marks]**

Given $\mathbf{A} = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$, find $\mathbf{A}^{75}$.

5. **[5 marks]**

Given a matrix transformation **T** such that $\mathbf{T}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-2} \end{bmatrix}$ and $\mathbf{T}\begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{5} \end{bmatrix}$:

(a) **[4 marks]** Find the matrix of **T**.

(b) **[1 mark]** State whether **T** is linear, and explain briefly why or why not.

6. **[6 marks]**

Find the equation of the plane in $\mathbf{R}^3$ that passes through the point P(2,0,3) and is parallel to the plane containing points A(1,1,-5), B(0,1,-2) and C(1,0,6).

7. **[8 marks]**

Find the shortest distance between the point P(-1,2,0) and the solution to the system:

$$\begin{aligned} x - y + 3z &= 1 \\ -x + 3y - z &= 3 \\ y + z &= 2 \end{aligned}$$

8. **[10 marks]**

Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & -2 & 0 & 1 \\ 2 & -2 & 1 & 0 & -7 & 0 & 0 \\ -3 & 3 & -1 & 0 & 9 & 1 & -2 \\ -2 & 2 & -1 & 0 & 7 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 4 & -4 \end{bmatrix}$

with $\mathbf{RREF(A)} = \begin{bmatrix} 1 & -1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$:

(a) **[2 marks]** Write the general solution to **AX=0**.

(b) **[4 marks]** Give a basis for:
 (i) the rowspace of **A**
 (ii) the columnspace of **A**
 (iii) the nullspace of **A**
 (iv) the image of **A**

(c) **[2 marks]** Give the dimension of:

(i) the rowspace of **A**
(ii) the columnspace of **A**
(iii) the nullspace of **A**
(iv) the image of **A**

(d) **[2 marks]** Complete the statements:

(i) the rowspace of **A** is a subspace of ____
(ii) the columnspace of **A** is a subspace of ____
(iii) the nullspace of **A** is a subspace of ____
(iv) the image of **A** is a subspace of ____

9. **[8 marks]**

Find the best approximation to a solution for the inconsistent system:

$$2x_1 + x_2 = 1$$
$$x_1 - x_2 = 2$$
$$x_1 + 5x_2 = 3$$

10. **[9 marks]**

Given the data set

x	-3	-1	0	2	4
y	2	1	-1	2	3

:

(a) **[3 marks]** Write the form of the interpolating polynomial for the data and give the system of equations you would use to solve for the coefficients of the polynomial. You may write the system in equation form or matrix form. **Do not solve the system.**

(b) **[4 marks]** Write the form of the least squares approximating quadratic for the data and give the corresponding matrices and vectors **M**, **Z** and **Y** that you would use in the matrix equation $(\mathbf{M}^T\mathbf{M})\mathbf{Z}=\mathbf{M}^T\mathbf{Y}$ to solve for the coefficients of the quadratic. **Do not solve the system.**

(c) **[2 marks]** Explain briefly why least squares approximating polynomials are used more often in practice than interpolating polynomials.

11. **[12 marks]**

Given $\mathbf{U} = span\{(-1,1,1,1), (1,-1,2,0)\}$, $\mathbf{X} = (1, 1, 3, 1)$:

(a) **[4 marks]** Show that **U** is a subspace of $\mathbf{R}^4$.

(b) **[1 mark]** What is the dimension of **U**?

(c) **[2 marks]** Find $\mathbf{U}^{\perp}$.

(d) **[5 marks]** Find $proj_U X$ and $proj_{U^{\perp}} X$.

12. **[6 marks]**

Given a linearly independent set of vectors $\{\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}\}$ in $\mathbf{R}^n$ and a set $\{\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y_3}\}$, where $\mathbf{Y_1} = \mathbf{X_1} + \mathbf{X_2}$, $\mathbf{Y_2} = \mathbf{X_2} + \mathbf{X_3}$, and $\mathbf{Y_3} = \mathbf{X_1} + \mathbf{X_3}$, is the set $\{\mathbf{Y_1}, \mathbf{Y_2}, \mathbf{Y_3}\}$ linearly independent?

13. **[12 marks, 2 for each part]** For each of the following statements, state whether it is true or false. If it is true, show or explain why. If it false, explain why or give a counterexample.

(a) If two (nonparallel) planes in $\mathbf{R}^3$ intersect in a line, then that line is parallel to the normals of both planes.

(b) Any plane in $\mathbf{R}^3$ that passes through the origin is a subspace of $\mathbf{R}^3$.

(c) $\mathbf{R}^2$ is a subspace of $\mathbf{R}^3$.

(d) It is possible for a 7x5 matrix to have a nullspace of dimension 6.

(e) If U is a subspace of $\mathbf{R}^n$ then $\mathbf{U}^{\perp}$ is a subspace of $\mathbf{R}^n$.

(f) It is possible for a homogeneous system to have no solution.

14. **BONUS QUESTION [3 bonus marks]**

Consider **M(2,2)**, the vector space of 2x2 matrices. Show that the subset of 2x2 diagonal matrices is a subspace of **M(2,2)** and give a basis for that subspace.

Note: If your solution to this question is correct, you will be awarded 3 bonus marks. **It is not necessary to attempt this question.**

1	10
2	2
3	4
4	8
5	5
6	6
7	8
8	10
9	8
10	9
11	12
12	6
13	12
	100
bonus	5