

University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 9

Instructor: Arman Pannu

We will start 10 minutes past the hour. Use this time to make a new friend.

Differential Equations

Def'n: A differential equation is an equation that involves a variable y and its derivatives

$$f\left(\frac{dy}{dx}, y, x\right) = 0, \quad f(y'', y', y, x) = 0, \dots$$

↑ ↑
x-derivatives

Use: When derivative of a quantity depends on the quantity itself

ex: ① Population (more population = more babies = higher derivative)

② Chemical concentrations (or radioactive decay)

③ ∴ more next time

Given differential equation, the order is the largest derivative that appears in equation

ex: $y' + yx^2 + x = 0$ \Leftarrow 1st order

$y'' + y'y + x^2 = 0$ \Leftarrow 2nd order

A linear ODE is one where all derivatives appear linearly

ex|| $y'' + 2y' + y + x^2 = 0$ \Leftarrow linear

$y'' + 2(y')^2 + y = 0$ \Leftarrow non-linear

$y'' + 2y'y = 0$ \Leftarrow non-linear

$y'' + x^2 y' + (2x^4)y = 0$ \Leftarrow linear

A differential equation is autonomous if it doesn't depend on the independent variable (x or t or ...)

ex|| $y'' + y' = 0$ \Leftarrow autonomous

$y' + x = 0$ \Leftarrow not autonomous

$y'' + xy = 0$ \Leftarrow

$$\frac{dy}{dx} + 2y = \cos(x)$$

→ linear

→ non-autonomous

→ 1st order

$$y'' + y \cdot \frac{dy}{dx} = 0$$

→ 2nd order

→ non linear

→ autonomous

$$\frac{dy}{dx} = y(1 - y) = y - y^2$$

→ 1st order

→ autonomous

→ non-linear

Solving Differential Equations - Guess and Check

ex I $y' = y$

→ function equal to its own derivative?

$$y = e^x? \quad (e^x)' = \checkmark (e^x)$$

$$y = 0? \quad (0)' = \checkmark 0$$

$$y = 3e^x \quad (3e^x)' = \checkmark 3e^x$$

ex II $y' = 3y$

→ not e^x but what about e^{cx} ?

$$y' = 3y \quad \Leftarrow \text{plug-in } y' = e^{cx}$$

$$\cancel{c}e^{cx} = 3\cancel{e^{cx}}$$

$$c = 3$$

$$\therefore y = e^{3x}?$$

$$y'' = 3y$$

exponential? try $e^x \Rightarrow y'' = e^x \neq 3y = 3e^x$

$$\text{try } e^{cx} \Rightarrow y'' = c^2 e^{cx} \stackrel{?}{=} 3y = 3e^{cx}$$

$$\Rightarrow c^2 = 3 \quad \text{this works}$$

$$\text{sol'n} \Rightarrow \boxed{y = e^{\sqrt{3}x}} \quad \text{and} \quad \boxed{y = e^{-\sqrt{3}x}}$$

$$\text{check: } y = C_1 e^{\sqrt{3}x} + C_2 e^{-\sqrt{3}x} \quad C_1, C_2 \in \mathbb{R}$$

$$y'' = -3y$$

is a solution

cosine?

$$\text{try } y = \cos(cx) \Rightarrow y'' = -c^2 \cos(cx) \stackrel{?}{=} -3y = -3\cos(cx)$$

$$\Rightarrow -c^2 = 3 \Rightarrow c = \pm\sqrt{3}$$

$$\therefore \boxed{y = \cos(\sqrt{3}x)}$$

also

$$\boxed{y = \sin(\sqrt{3}x)} \quad \leftarrow \begin{array}{l} \text{check} \\ \text{these} \end{array}$$

also

$$\boxed{y = C_1 \sin(\sqrt{3}x) + C_2 \cos(\sqrt{3}x)}$$

Solutions to ODE

A particular solution to an ODE is any $y(x)$ that satisfies the ODE

The general solution is the family of all possible solutions to an ODE

ex// $y' = y$ General solution $y = Ce^x$ $C \in \mathbb{R}$

$y'' = -3y$ General sol'n $y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)$
 $C_1, C_2 \in \mathbb{R}$

$y' = x \Rightarrow y = \frac{1}{2}x^2 + C$ $C \in \mathbb{R}$

In-general: \rightarrow 1st order ODE has 1D solution space
i.e. one free constant

\rightarrow 2nd order ODE has 2D solution space
i.e. two free constants

An initial value problem (IVP) is an ODE along with an initial condition

$$\text{1st order} \begin{cases} F(y, y', x) = 0 \\ y(x=c) = y_0 \end{cases} \quad \Leftarrow \text{specifies the choice of constant } c$$

$$\text{2nd order} \begin{cases} F(y'', y', y, x) = 0 \\ y(x=c) = y_0 \\ y'(x=c) = v_0 \end{cases} \quad \begin{array}{l} \Leftarrow \text{specify constants} \\ \Leftarrow c_1, c_2 \end{array}$$

ex 11

$$\begin{cases} y' = -y & \Leftarrow \text{radioactive decay} \\ y(t=0) = y_0 & \Leftarrow \text{starting amount} \end{cases}$$

general sol'n: $y(t) = ce^{-t}$, plug-in $y(0) = ce^0 = y_0$
 $c = y_0$

$$\therefore \boxed{y(t) = y_0 e^{-t}}$$

Analyzing Differential Equations

→ we can learn a lot about a system without even solving for an ODE

Given 1st order autonomous ODE $y' = F(y)$, an equilibrium point are the points where $y' = 0$

→ equilibrium = no change in the system

ex: deer population reproduction + carrying capacity

equilibrium. ① zero deer

Points: ② population at capacity

An equilibrium point is:

① stable if $F' < 0$ at equilibrium point

② unstable if $F' > 0$ " " "

③ semistable if F is local min or max

→ stable = if you move away from equilibrium, system eventually returns back to same equilibrium

→ unstable = moves away from equilibrium when perturbed

→ semi-stable = returns back when perturbed in one direction and moves away in other

$$\frac{dy}{dx} = y(2 - y)(y + 3).$$

Classify each equilibrium solution as **stable** or **unstable**.

Determine the behaviour between the equilibrium Points

→ equilibrium $0 = \frac{dy}{dx} = y(2 - y)(y + 3) \Rightarrow y = 0, y = 2, y = -3$

Sign Chart

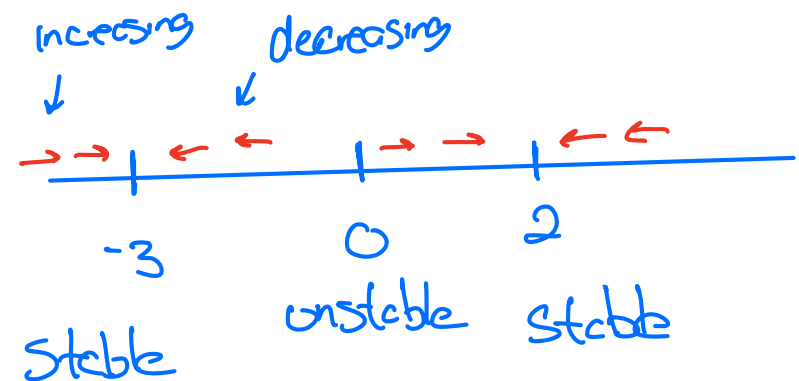
	$y < -3$	$-3 < y < 0$	$0 < y < 2$	$2 < y$
y	(-)	(-)	(+)	(+)
$2 - y$	(+)	(+)	(+)	(-)
$y + 3$	(-)	(+)	(+)	(+)
$\frac{dy}{dx}$	(+)	(-)	(+)	(-)

↑
growing
 $y < -3$

↑
decreasing

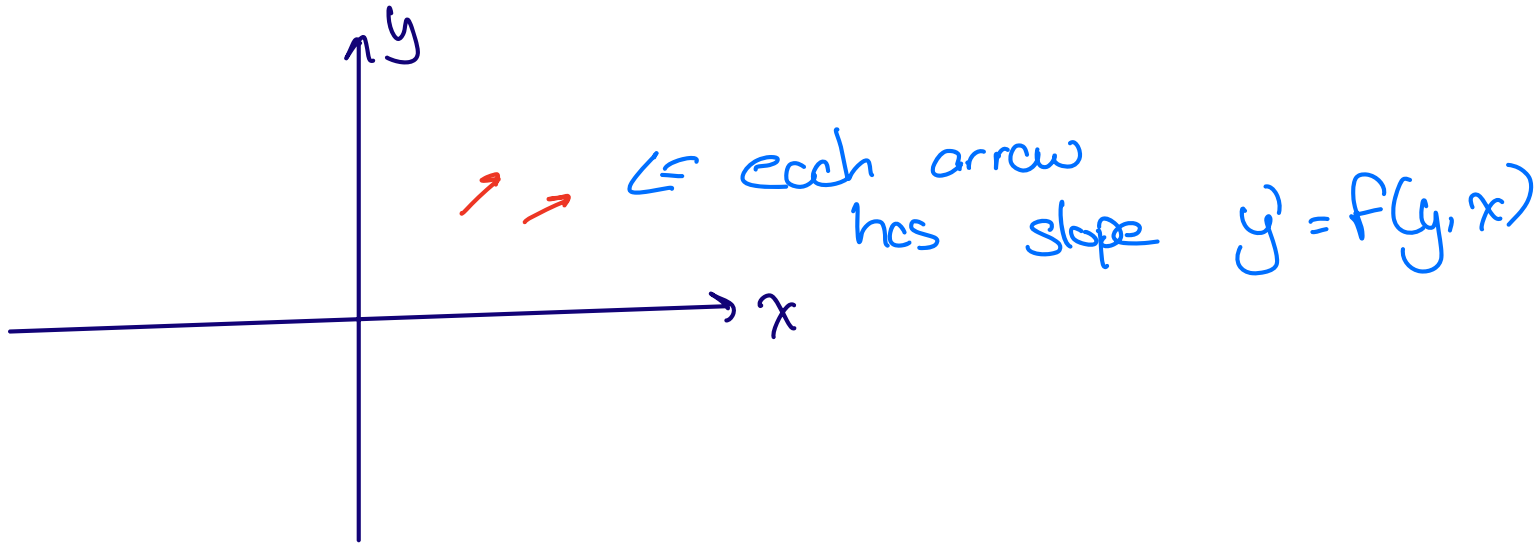
↑
increasing

↑
decreasing

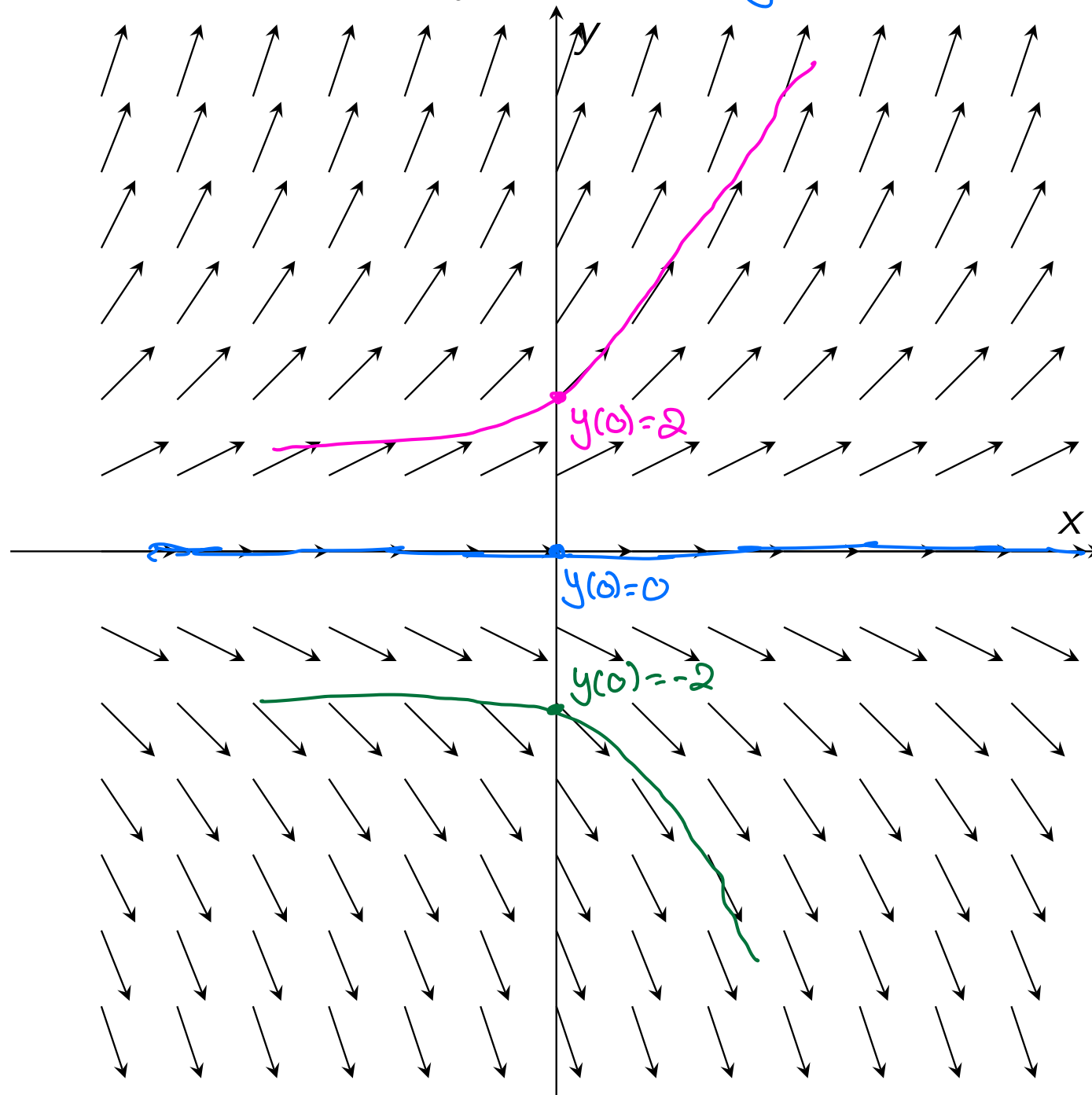


Direction Fields

→ visualization of $y' = f(y, x)$



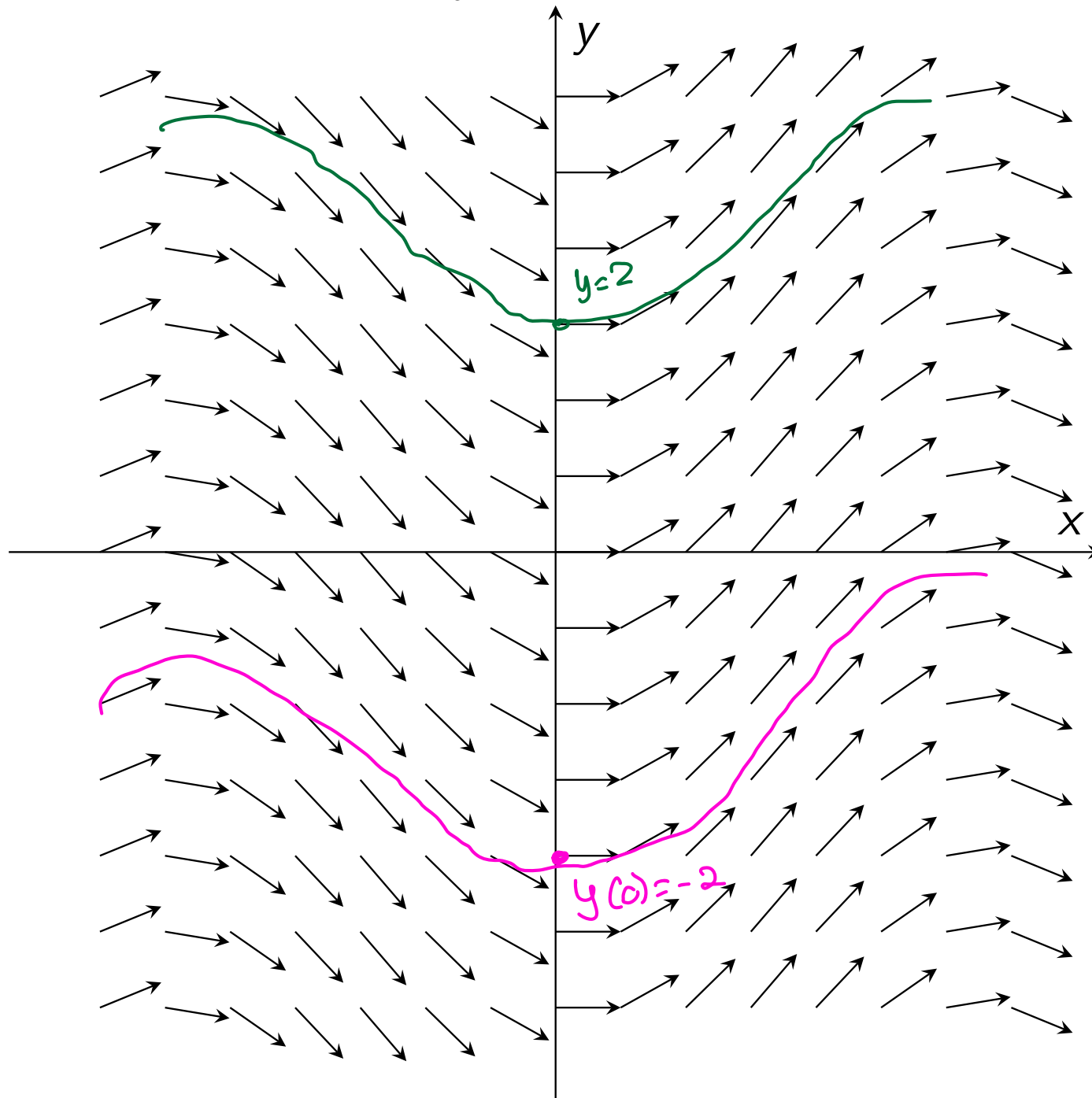
Direction Field for $\frac{dy}{dx} = y = F(x,y)$



→ autonomous ODE
(arrows don't
change with
 x)

→ zero is
equilibrium
solution

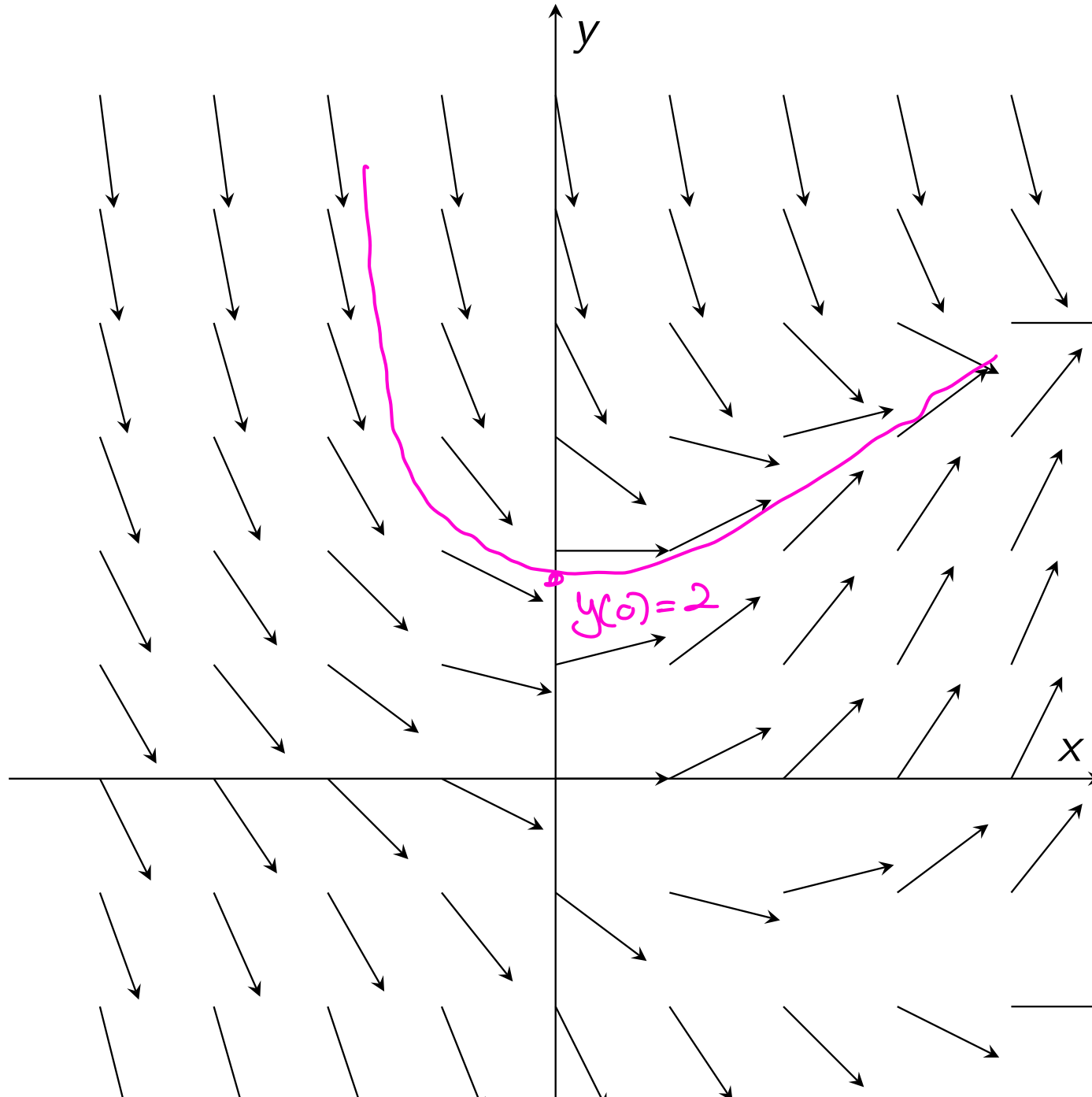
Direction Field for $\frac{dy}{dx} = \sin(x)$



→ non-autonomous

→ no equilibrium

Direction Field for $\frac{dy}{dx} = y(1 - y) + x$



→ non-autonomous

$y(0)=2$