University of Toronto
Faculty of Applied Sciences and Engineering

# MAT187 - Summer 2025

Lecture

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We will start 10 minutes past the hour. Use this time to make a new friend.

# Separable Differential Equations

A seperable ODE is a first-order ODE of the form

$$y' = f(y)\,g(x)$$ $\Leftarrow$ can seperate the "y-stuff" and the "x-stuff"

ex//

$y' = xy$ $\Leftarrow$ seperable $f(y) = y$, $g(x) = x$

$y' = x^2 y$ $\Leftarrow$ seperable $f(y) = y$, $g(x) = x^2$

$y' = (x^2 + e^x)(y + \sin(y))$ $\Leftarrow$ seperable

$y' = x + y$ $\Leftarrow$ non-seperable

## How to solve?

formal derivation

$$y' = f(y)\,g(x)$$

$$\frac{y'}{f(y)} = g(x) \quad \Leftarrow \text{integrate both sides w.r.t. to } x$$

$$\int \frac{y'}{f(y)}\,dx = \int g(x)\,dx \qquad y = y(x)$$

$$dy = y'\,dx$$

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

ex//

informal derivation

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = x\,dx$$

$$\int \frac{dy}{y} = \int x\,dx \Rightarrow \ln|y| = \frac{1}{2}x^2 + C \Rightarrow \boxed{y = e^{\frac{1}{2}x^2 + C} = Ce^{\frac{1}{2}x^2}}$$

$y' = \frac{x^2+1}{2y}$ and $y(3) = -5$

$$\frac{dy}{dx} = \frac{x^2+1}{2y}$$

$$2y\,dy = (x^2+1)\,dx$$

$$\int 2y\,dy = \int (x^2+1)\,dx$$

$$y^2 = \frac{1}{3}x^3 + x + C$$

$$\boxed{y = \pm\sqrt{\frac{1}{3}x^3 + x + C}}$$

depends on IVP

general solution

$$y(x=3) = -5$$

$$\Rightarrow (-5)^2 = \frac{1}{3}(3)^3 + 3 + C$$

$$25 = 12 + C$$

$$\boxed{C = 13}$$

$$\boxed{y = -\sqrt{\frac{1}{3}x^3 + x + 13}}$$

negative root b/c of IVP

$y' = \frac{2x}{y^2+1}$ and $y(0) = 1$

→ seperable ODE

$$\frac{dy}{dx} = \frac{2x}{y^2+1}$$

$$\int (y^2+1)\,dy = \int 2x\,dx$$

$$\boxed{\frac{1}{3}y^3 + y = x^2 + C}$$

implicit general solution

$\overset{\text{IVP}}{\Longrightarrow}$

$$y(x=0) = 1$$

$$\frac{1}{3}(1)^3 + (1) = 0^2 + C$$

$$\boxed{C = \frac{4}{3}}$$

$$\Rightarrow \boxed{\frac{1}{3}y^3 + y = x^2 + \frac{4}{3}}$$

implicit solution to IVP

$$(1+x)y' = (x+2)(y-1)$$

General Solution?

# Integrating Factors

→ the method of using integrating factors works to solve any linear first-order ODE:

$$\boxed{y' + g(x)y = h(x)}$$

→ Method: multiply ODE by some unknown function $p(x)$ and use product rule

$$y' + g(x)y = h(x) \quad \Leftarrow \text{multiply by } p(x)$$

$$\underbrace{p(x)y' + g(x)p(x)y}_{\substack{\text{write this using} \\ \text{product rule as} \\ (p(x)y)' = p(x)y' + p'(x)y}} = h(x)p(x)$$

$\Rightarrow$

$$p'(x)y = g(x)p(x)y$$

$$p'(x) = p(x)g(x)$$

$$\frac{p'(x)}{p(x)} = g(x)$$

$$(\ln(p(x)))' = g(x)$$

$$\ln(p(x)) = \int g(x)\,dx$$

$$p(x) = \exp\left(\int g(x)\,dx\right)$$

→ with integrating factor

$$(p(x)\,y)' = h(x)\,p(x) \quad \Leftarrow \text{ for appropriate choice of } p(x)$$

$$\boxed{p(x) = \exp\left(\int g(x)\,dx\right)}$$

$$\int (p(x)\,y)'\,dx = \int h(x)\,p(x)\,dx$$

$$p(x)\,y = \int h(x)\,p(x)\,dx$$

$$\boxed{y = \frac{1}{p(x)} \int h(x)\,p(x)\,dx}$$

$y' + y = e^x$ and $y(0) = 2$

→ 1st order linear ⇒ use integrating factor

$$(*)\quad \underbrace{P(x)y' + P(x)y}_{\text{express as}} = P(x)e^x$$

$$(P(x)y)' = \underbrace{P(x)y' + P'(x)y}_{(**)} \Rightarrow$$

$$\overbrace{P(x)y' + P(x)y}^{(*)} = \overbrace{P(x)y' + P'(x)y}^{(**)}$$

$$P'(x) = P(x)$$

$$P(x) = e^x$$

$$e^x y' + e^x y = e^x(e^x)$$

$$(e^x y)' = e^{2x}$$

$$e^x y = \int e^{2x}\,dx$$

$$e^x y = \frac{1}{2}e^{2x} + C$$

$$\boxed{y = \frac{1}{2}e^x + \frac{C}{e^x}}$$

General sol'n

$\xRightarrow{\text{IVP}}$

$$y(x=0) = 2$$

$$2 = \frac{1}{2}e^0 + \frac{C}{e^0}$$

$$C = \frac{3}{2}$$

$$\boxed{y = \frac{1}{2}e^x + \frac{3}{2e^x}}$$

$y' + \frac{2y}{x} = \sin(x)$ and $y(\pi/2) = 0$

→ first-order linear (use integrating factors)

using formula

$$\begin{cases} P(x) = \exp\left(\int g(x)\,dx\right) = \exp\left(\int \frac{2}{x}\,dx\right) = e^{2\ln(x)} = x^2 \\ y(x) = \frac{1}{P(x)} \int P(x)h(x)\,dx = \frac{1}{x^2} \int x^2 \sin(x)\,dx \end{cases}$$

→ full derivation method

$$\underbrace{P(x)y' + \frac{2}{x}P(x)y}_{(yP(x))'} = P(x)\sin(x)$$

$$\Rightarrow P'(x) = \frac{2}{x}P(x)$$

$$\frac{P'}{P} = \frac{2}{x} \Rightarrow \ln(P) = 2\ln(x)$$

$$P = x^2$$

$$x^2y' + 2xy = x^2\sin(x)$$

$$(x^2y)' = x^2\sin(x)$$

by parts

use I.V.

$$y = \frac{1}{x^2}\int x^2\sin(x)\,dx = \frac{2}{x^2}\cos(x) + \frac{2}{x}\sin(x) + \cos(x) + C$$