Practise Problem Set 6

MAT 187 - Summer 2025

These questions are meant for your own practice for quiz 6 and are not to be handed in. Some of these questions, or problems similar to these, may appear on the quizzes or exams. Therefore, solutions to these problems will not be posted but you may, of course, ask about these questions during office hours, or on Piazza.

Suggestions on how to complete these problems:

- Solution writing is a skill like any other, which must be practiced as you study. After you write down your rough solutions, take the time to write a clear readable solution that blends sentences and mathematical symbols. This will help you to retain, reinforce, and better understand the concepts.
- After you complete a practice problem, reflect on it. What course material did you use to solve the problem? What was challenging about it? What were the main ideas, techniques, and strategies that you used to solve the problems? What mistakes did you make at the first attempt and how can you prevent these mistakes on a Term Test? What advice would you give to another student who is struggling with this problem?
- Discussing course content with your classmates is encouraged and a mathematically healthy practice. Work together, share ideas, explain concepts to each other, compare your solutions, and ask each other questions. Teaching someone else will help you develop a deeper level of understanding. However, it's also important that reserve some time for self-study and self-assessment to help ensure you can solve problems on your own without relying on others.

1. Simplify the following expressions:
   (a) $(3+4i)-(1-2i)$
   (b) $(2+3i)(1-i)$
   (c) $\frac{5+2i}{1-i}$
   (d) $\overline{-3+5i}$ (Complex conjugate)
   (e) $|4-3i|$ (Modulus)

2. Convert each complex number to polar form (exponential form):
   (a) $1+i$
   (b) $-2+2i$
   (c) $-1-\sqrt{3}i$

3. Convert the following complex numbers to rectangular form $a+bi$:
   (a) $3e^{i\pi/4}$
   (b) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
   (c) $4e^{-i\pi/2}$

4. (a) The function $T_1(z) = w_1 z$ acts geometrically by halving the modulus of each complex number z. What is w_1?
   (b) The transformation $T_2(z) = w_2 z$ acts geometrically by rotating each complex number z by angle $\frac{\pi}{3}$. What is w_2?

5. Solve the following homogeneous linear differential equations with constant coefficients. Find the general solution in each case. You may use a computational tool to factor the characteristic polynomial if needed.
   (a) $y'' - 5y' + 6y = 0$
   (b) $y'' + 2y' + 5y = 0$
   (c) $y'' + 4y = 0$
   (d) $y'' + 6y' + 9y = 0$
   (e) $y''' - 6y'' + 11y' - 6y = 0$
   (f) $y''' - 3y'' + 3y' - y = 0$
   (g) $y^{(4)} - 2y'' + y = 0$
   (h) $y^{(4)} + 8y^{(3)} + 24y'' + 32y' + 16y = 0$

6. Solve the following second-order nonhomogeneous differential equations using the method of undetermined coefficients. Find the general solution in each case.
   (a) $y'' - 3y' + 2y = e^t$
   (b) $y'' - 3y' + 2y = \cos t$
   (c) $y'' - 3y' + 2y = \cos t + e^t$ *Hint: If you did the previous 2 parts, this is really easy.*
   (d) $y'' - 2y' + y = te^t$
   (e) $y'' - 2y' + y = 7$
   (f) $y'' - y = t^2 + t$

7. Find the period and amplitude of the solution to the initial value problem

$$\begin{cases} y'' + 9y = 0 \\ y(0) = 0 \\ y'(0) = 5. \end{cases}$$

8. A mass sitting on a frictionless table is attached to a wall by an ideal spring. The mass is pulled 5 cm out from its natural resting length and let go. It travels with a period of 4 seconds.
   (a) Find an ordinary differential equation of the form $ay'' + by' + cy = 0$ with corresponding initial-values to model the motion of the mass.
   (b) What would be changed in the equation from (a) if the mass was instead pushed 3 cm toward the wall and is still approaching it at 1 cm/s when let go?

(c) What would be changed in the equation from (a) if the mass was doubled?

9. Let $y(t)$ denote the displacement of a spring from its equilibrium position. Compute the displacement of the spring when $t = 3\pi$ seconds, where $y(t)$ is modelled by the initial value problem

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 1 \\ y'(0) = 2, \end{cases}$$

and $g(t)$ is the piecewise function $g(t) = \begin{cases} -\sin(t), & 0 \le t \le \frac{\pi}{2} \\ \cos(2t), & t > \frac{\pi}{2}. \end{cases}$

10. In this question, you will explore some of the ideas behind "guessing" the form of a particular solution of a non-homogeneous ODE. Make sure to carefully explain your reasoning in your solution.

Consider a mass-spring system with mass m, drag constant μ and Hooke's constant k with external forcing being applied to the system given by $f(t) = e^{\alpha t}$, where $\alpha \in \mathbb{R}$. Let $P(r)$ denote the characteristic polynomial of the ODE that models this system. In this problem, we search for particular solutions to this ODE of the form $y_p(t) = A(t)e^{\alpha t}$, where $A(t)$ is some unknown function.

(a) Find the ODE that models this system. Show that if $y_p(t) = A(t)e^{\alpha t}$ is a particular solution to this ODE, then $A(t)$ must solve the differential equation

$$\frac{P''(\alpha)}{2} A''(t) + P'(\alpha)A'(t) + P(\alpha)A(t) = 1.$$

(b) Suppose $P(\alpha) = 0$ and $P'(\alpha) \neq 0$. Find a particular solution to the ODE that models this spring-mass system and a single solution to the corresponding homogeneous system.

(c) Suppose $P(\alpha) = P'(\alpha) = 0$. Find the general solution of the ODE that models this spring-mass system.

11. In this question we explore how solutions to a forced undamped spring without resonance can be used to obtain the resonating solution through taking a limit. Make sure to carefully explain your reasoning in your solution.

Consider the initial value problem for an undamped spring-mass system with oscillatory external forcing,

$$\begin{cases} y'' + 9y = \cos(\omega t) \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

(a) Let $y_\omega(t)$ denote the solution to the initial value problem for $\omega \neq 3$. Compute $y_\omega(t)$.

(b) Compute the limit $\lim_{\omega \to 3} y_\omega(t)$. Make sure you explain your steps. Call this limit– if exists– $y_3(t)$. Show that $y_3(t)$ solves the initial value problem

$$\begin{cases} y'' + 9y = \cos(3t) \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$