UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2004
First Year - CHE, CIV, IND, LME, MEC, MMS

MAT188H1F – LINEAR ALGEBRA

Exam Type: A

SURNAME ______________________________

GIVEN NAME ______________________________

STUDENT NO. ______________________________

SIGNATURE ______________________________

Examiners

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INSTRUCTIONS:

Non-programmable calculators permitted.

Answer all questions.

Present your solutions in the space provided; use the back of the **preceding** page if more space is required.

TOTAL MARKS: 100

The value for each question is shown in parentheses after the question number.

MARKER'S REPORT	
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
Q9	
Q10	
TOTAL	

1. [10 marks] Let $A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -3 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{pmatrix}$.

(a) [3 marks] Find the reduced row-echelon form of A^T.

(b) [2 marks] Find the basic solutions of the homogeneous system $A^T X = O$, where X is a 4×1 matrix.

(c) [2 marks] Let B be the matrix whose columns are the basic solutions of $A^T X = O$. Compute $B^T A$.

(d) [3 marks] Circle the following formulas, if any, that are illustrated by the above computations.

(i) $\left(\text{null}(A^T)\right)^\perp = \text{col}(A)$ (ii) $(\text{row}(A))^\perp = \text{null}(A)$ (iii) $\text{null}(A^T) = (\text{col}(A))^\perp$

2. [10 marks] The parts of this question are unrelated.

(a) [4 marks] Show that the matrix $A = \begin{pmatrix} 1 & -a & b \\ a & 1 & 2 \\ -b & 0 & 1 \end{pmatrix}$ is invertible for any real values of a and b.

(b) [6 marks] B is a 2×2 matrix such that

$$E_2(B) = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \text{ and } E_5(B) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Find B.

3. [10 marks; one mark for each part] Let A be an $n \times n$ matrix. For each statement below, decide if it is equivalent to the statement

A is invertible.

If it is, circle Yes at the right; if it isn't, circle No.

(a) $\lambda = 0$ is an eigenvalue of A. Yes No

(b) A^7 is invertible. Yes No

(c) The rank of A is n. Yes No

(d) AA^T is invertible. Yes No

(e) A is a product of elementary matrices. Yes No

(f) $\operatorname{col}(A) = \mathbf{R}^n$ Yes No

(g) $\operatorname{null}(A) = \mathbf{R}^n$ Yes No

(h) A is similar to the identity matrix I_n Yes No

(i) O is not in the row space of A Yes No

(j) For any vector B in $\mathbf{R}^n$, the equation $AX = B$ has a solution. Yes No

4. [10 marks] Indicate whether each of the following statements is True or False. (Circle your choice.) It is not necessary to justify your choice, and there is no penalty for an incorrect answer.

(a) True or False: 1 is the minimum distance between the two planes with equations $x + y + z = 1$ and $x + y + z = 0$.

(b) True or False: The line of intersection of the two planes with equations $x+y+z = 3$ and $2x-3y+4z = 6$ is parallel to the vector $[\ 7 \quad -2 \quad -5\ ]^T$.

(c) True or False: If λ is an eigenvalue of the $n \times n$ matrix A, then $\lambda^2 + 4$ is an eigenvalue of the matrix $A^2 + 4I_n$.

(d) True or False: An LU-factorization of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T$$

(e) True or False: If E and F are both $n \times n$ elementary matrices, then $(EF)^{-1} = E^{-1}F^{-1}$.

(f) True or False: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 12 \end{pmatrix} \right\}$ is an independent set in $\mathbf{R}^3$.

(g) True or False: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ -1 \\ 12 \end{pmatrix} \right\}$ is a spanning set for $\mathbf{R}^3$.

(h) True or False: Every symmetric matrix is diagonalizable.

(i) True or False: If A is an $m \times n$ matrix with rank equal to n and $A = QR$ is a QR-factorization of A, then $A^T A = R^T R$.

(j) True or False: If A is an $n \times n$ matrix such that $A^3 - A^2 + A - 2I_n = O$, then $(A^2 - A + I_n)^{-1} = 2A$.

5. [10 marks] Given that the reduced row-echelon form of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 2 & 3 & 4 & -1 \\ 2 & 2 & 6 & 4 & 2 \\ 3 & 4 & 11 & 8 & 4 \end{pmatrix} \text{ is } R = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

6. [10 marks] Solve the system of differential equations

$$\begin{aligned} f_1'(x) &= -f_1(x) + 5f_2(x) \\ f_2'(x) &= f_1(x) + 3f_2(x) \end{aligned}$$

for f_1 and f_2 as functions of x, given the initial conditions $f_1(0) = 1$ and $f_2(0) = -1$.

7. [10 marks] Let $U = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & -1 & 2 & 3 \end{bmatrix}^T \right\}$; let $X = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$. Find $\operatorname{proj}_U(X)$.

8. [10 marks] Find the least squares approximating line for the data points

$$(1,1),(2,2),(3,2),(4,3).$$

9. [10 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

You may assume that the eigenvalues of A are $\lambda = 0$ and $\lambda = 3$.

10. [10 marks; 5 marks for each part.] The parts of this question are unrelated.

(a) Suppose $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is a linear transformation such that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Find

$$T\begin{pmatrix} x \\ y \end{pmatrix} \text{ and } T^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) Let X and Y be any two vectors in $\mathbf{R}^n$. Prove that

X and Y are orthogonal if and only if $\|X+Y\|^2 = \|X-Y\|^2$.