

University of Toronto  
Faculty of Applied Sciences and Engineering

**MAT187 - Summer 2025**

Lecture 12

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We will start 10 minutes past the hour. Use this time to make  
a new friend.

## Second order linear ODEs :

$$y'' + f(x)y' + g(x)y = h(x)$$

A second order linear ODE is called homogeneous if  $h(x)=0$   
→ a second order ODE has 2 unknown parameters (solution space is 2D)

IVP:  $y(0) = y_0$       } 2 degrees of  
 $y'(0) = v_0$       freedom

For homogeneous linear ODEs, if  $y_1(x)$  and  $y_2(x)$  are solutions then only linear combination  $y(x) = c_1 y_1(x) + c_2 y_2(x)$  is also a solution.

Pf: → plug into ODE:  $y = c_1 y_1 + c_2 y_2$

$$y'' + f(x)y' + g(x)y = 0$$

$$(c_1 y_1 + c_2 y_2)'' + f(x)(c_1 y_1 + c_2 y_2)' + g(x)(c_1 y_1 + c_2 y_2) = ? 0$$

$$c_1 y_1'' + c_2 y_2'' + f(x)(c_1 y_1' + c_2 y_2') + g(x)(c_1 y_1 + c_2 y_2) \stackrel{?}{=} 0$$

$$c_1 \underbrace{(y_1'' + f(x)y_1' + g(x)y_1)}_{=0 \text{ b/c } y_1 \text{ is a solution}} + c_2 \underbrace{(y_2'' + f(x)y_2' + g(x)y_2)}_{=0} \stackrel{?}{=} 0$$

$$c_1 0 + c_2 0 \stackrel{\checkmark}{=} 0$$

$\therefore y = c_1 y_1 + c_2 y_2$   
is a solution to  
SDE

If  $y_1$  &  $y_2$  are two linearly independent solutions to homogeneous second-order ODE, then

$y(x) = c_1 y_1(x) + c_2 y_2(x)$  is the general solution

$y_1$  &  $y_2$  are linearly independent if  $y_1(x) \neq c y_2(x)$  for any  $c$  in the whole interval of interest

$$y'' - 5y' + 6y = 0 \quad \Leftarrow \text{constant coeff.}$$

→ guess  $y(t) = e^{rt}$

$$(e^{rt})'' - 5(e^{rt})' + 6(e^{rt}) = 0$$

$$\cancel{r^2} e^{rt} - 5\cancel{r} e^{rt} + 6e^{rt} = 0$$

$$r^2 - 5r + 6 = 0$$

$\Leftarrow$  polynomial in  $r$  (called  
polynomial for the characteristic  
ODE)

$$(r-3)(r-2) = 0$$

$$\Rightarrow r=3 \quad r=2$$

$$y_1(t) = e^{3t}$$

$$y_2(t) = e^{2t}$$

linearly independent

$$y(x) = C_1 e^{3t} + C_2 e^{2t}$$

General  
solution

Given any homogeneous ODE with constant coeff.

$$0 = a_0 y + a_1 y' + a_2 y'' + \dots + a_n y^{(n)}$$

→ guess solution  $y = e^{rt}$

$$0 = a_0 (e^{rt}) + a_1 (e^{rt})' + \dots + a_n (e^{rt})^{(n)}$$

$$0 = a_0 e^{rt} + a_1 r e^{rt} + \dots + a_n r^n e^{rt}$$

$$0 = a_0 + a_1 r + a_2 r^2 + \dots + a_n r^n$$

← called the characteristic polynomial

→ solutions:  $y(t) = e^{r_1 t}$        $\Leftarrow r_1, r_2$  are roots

$$y(t) = e^{r_2 t}$$

⋮

of char. polynomial

$$y'' - 3y' + 2y = 0, \quad y(0) = 1 \text{ and } y'(0) = 0$$

$\Rightarrow$  Char. polynomial

$$0 = r^2 - 3r + 2$$

$$= (r-1)(r-2)$$

$\Rightarrow r=1, r=2$  are sol'n

$\Rightarrow$  general sol'n of ODE

$$y(t) = C_1 e^t + C_2 e^{2t}$$

$$\Rightarrow y'(t) = C_1 e^t + C_2 2e^{2t}$$

$\Rightarrow$  apply I.E.

$$1 = y(0) = C_1 e^0 + C_2 e^{2 \cdot 0} = C_1 + C_2$$

$$0 = y'(0) = C_1 + 2C_2$$

$$\boxed{\begin{aligned} C_1 + C_2 &= 1 \\ C_1 + 2C_2 &= 0 \end{aligned}}$$

$\Rightarrow$

$$\begin{aligned} C_1 &= 2 \\ C_2 &= -1 \end{aligned}$$

$$\boxed{y(t) = 2e^t - e^{2t}}$$

$$y'' - 4y' + 4y = 0$$

→ char. poly.

$$0 = r^2 - 4r + 4$$

$$= (r-2)^2$$

$$\Rightarrow \boxed{r=2}$$

$$y(t) = C_1 e^{2t} + C_2 t e^{2t} = \underbrace{(C_1 + C_2 t) e^{2t}}_{\text{one degree of freedom}}$$

| → need to find another linearly independent sol'n

$$\rightarrow \text{guess } y(t) = t e^{2t}$$

$$(t e^{2t})'' - 4(t e^{2t})' + 4(t e^{2t}) = 0$$

$$(e^{2t} + 2t e^{2t})' - 4(e^{2t} + 2t e^{2t}) + 4(t e^{2t}) = 0$$

$$\cancel{2e^{2t} + 2e^{2t} + 4te^{2t}} - \cancel{4e^{2t} - 8te^{2t} + 4te^{2t}} = 0$$

= 0 So sol'n to

BDE

$$y_1(t) = e^{2t}$$

$$y_2(t) = t e^{2t}$$

linearly independent

$$\Rightarrow \boxed{y(t) = C_1 e^{2t} + C_2 t e^{2t}}$$

↪ general solution

Given a constan coeff. homogenous ODE

$$0 = a_0 y + a_1 y' + \dots + a_n y^{(n)}$$

$\Rightarrow$  char poly

$$0 = a_0 + a_1 r^1 + \dots + a_n r^n$$

$$0 = (r - r_1)^m \underbrace{\dots}_{\text{other roots}}$$

$\Leftarrow$  If char poly has  $r_1$  as repeated root then the solutions are

m solutions from root  $r_1$  which was repeated m times

$$\left\{ \begin{array}{l} y_1(t) = e^{r_1 t} \\ y_2(t) = t e^{r_1 t} \\ y_3(t) = t^2 e^{r_1 t} \\ \vdots \\ y_m = t^{(m-1)} e^{r_1 t} \end{array} \right\}$$

not solutions if  $r_1$  is not a repeated root

remaining solution from other roots

$$\left\{ \begin{array}{l} \vdots \\ \vdots \end{array} \right.$$

$$y'' + 2y' + y = 0, \quad y(0) = 3 \text{ and } y'(0) = -5$$

$\Rightarrow$  Char. Poly

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$r = -1$  repeated root

$$y_1(t) = e^{-t}$$

$$y_2 = te^{-t}$$

because of  
repeated root

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

$$\Rightarrow y'(t) = -C_1 e^{-t} + C_2 (e^{-t} - t e^{-t})$$

$\Rightarrow$  apply I.V.

$$3 = y(0) = C_1$$

$$\Rightarrow \begin{aligned} C_1 &= 3 \\ C_2 &= -2 \end{aligned}$$

$$-5 = y'(0) = -C_1 + C_2$$

$$y(t) = 3e^{-t} - 2te^{-t}$$

$$y'' + 2y' + 5y = 0$$

$\Rightarrow$  char. poly

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$r = -1 \pm \frac{\sqrt{-16}}{2}$$

$$r = -1 \pm 2\sqrt{-1}$$

$$\boxed{r = -1 \pm 2i}$$

$$y_1 = e^{(-1+2i)t}$$

$$y_2 = e^{(-1-2i)t}$$

$\Leftarrow$  not red root  
let  $i = \sqrt{-1}$

} what does this mean?  
 $\rightarrow$  next lecture