Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 1 Solutions

1. The language of sets is ubiquitous in mathematics and is something we will work with often in this course. Loosely, a set is a collection of objects called members, or elements, of the set. We will use the following shorthand notation to indicate certain relationships between sets.

- $\emptyset$ denotes the empty set. i.e. The set with no members.
- $a \in A$ means a is a member of the set A. It's typical that sets are designated by upper case letters and members of that set by the corresponding lower case letter.
- $A = B$ means the set A is equal to the set B. i.e. A and B have exactly the same members and nothing more.
- $A \subseteq B$ means A is a subset of B. i.e. Every element of A is an element of B. It may be that A and B are equal.
- $A \subset B$ means A is a proper subset of B. i.e. A is a subset of B but not equal to B. You may also see the notation $A \subsetneq B$ to denote A is a proper subset of B.

There are two ways we may define a set: we may list the elements of the set, such as $A = \{1, 2, 3\}$, or specify them by a rule, such as $A = \{x \mid x \in \mathbb{Z} \text{ and } x > -1\}$ (you read this as the set of x such that x is an integer and x is greater than -1).

Definition: Let A and B be sets.

The *intersection* of A and B is defined to be the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

The *union* of A and B is defined to be the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (this is an inclusive or which means that $x \in A$ or $x \in B$ or both).

The *difference* of A and B is defined to be the set $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

(a) Let $A = \{x \mid x \in \mathbb{R} \text{ and } x^2 < 3\}$ and $B = \{x \mid x \in \mathbb{Z} \text{ and } x > -1\}$. Determine $A \cap B$, $B \setminus A$, $\mathbb{Z} \setminus B$, and $\mathbb{R} \cap A$.

Solution:

$$\begin{aligned}
A \cap B &= \{x \mid x \in A \text{ and } x \in B\} \\
&= \{x \mid x \in \mathbb{R} \text{ and } x \in \mathbb{Z} \text{ and } x^2 < 3 \text{ and } x > -1\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } -\sqrt{3} < x < \sqrt{3} \text{ and } x > -1\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } -1.73\cdots < x < 1.73\cdots \text{ and } x > -1\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } -1 < x < 1.73\cdots\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } 0 \leq x \leq 1\} \\
&= \{0, 1\}
\end{aligned}$$

$$B \setminus A = \{x \mid x \in B \text{ and } x \notin A\}$$

Notice that, for any x, $x \notin A$ is equivalent to $x \in \mathbb{R}$ and $x^2 \geq 3$. Therefore,

$$\begin{aligned}
B \setminus A &= \{x \mid x \in \mathbb{Z} \text{ and } x > -1 \text{ and } x \in \mathbb{R} \text{ and } x^2 \geq 3\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x > -1 \text{ and } x \geq \sqrt{3}\} \cup \{x \mid x \in \mathbb{Z} \text{ and } x > -1 \text{ and } x \leq -\sqrt{3}\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x \geq 2\} \cup \phi \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x \geq 2\} \\
&= \{2, 3, 4, \ldots\}
\end{aligned}$$

$$\begin{aligned}
\mathbb{Z} \setminus B &= \{x \mid x \in \mathbb{Z} \text{ and } x \notin B\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x \in \mathbb{Z} \text{ and } x \leq -1\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x \leq -1\} \\
&= \{-1, -2, -3, \ldots\}
\end{aligned}$$

$$\begin{aligned}
\mathbb{R} \cap A &= \{x \mid x \in \mathbb{R} \text{ and } x \in A\} \\
&= \{x \mid x \in \mathbb{R} \text{ and } x \in \mathbb{R} \text{ and } x^2 < 3\} \\
&= \{x \mid x \in \mathbb{R} \text{ and } -\sqrt{3} < x < \sqrt{3}\}
\end{aligned}$$

(b) Let $C = \{x \mid x \in \mathbb{Z} \text{ and } x^2 > 4\}$ and $D = \{x \mid x \in \mathbb{Z} \text{ and } x > -1\}$. Determine $C \cup D$, $D \setminus C$, $D \cap \emptyset$, $\mathbb{R} \cup D$.

Solution:

$$\begin{aligned}
C \cup D &= \{x \mid x \in \mathbb{Z} \text{ and } x^2 > 4 \text{ and } x > -1\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x > 2 \text{ and } x > -1\} \cup \{x \mid x \in \mathbb{Z} \text{ and } x < -2 \text{ and } x > -1\} \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x > 2\} \cup \phi \\
&= \{x \mid x \in \mathbb{Z} \text{ and } x > 2\} \\
&= \{3, 4, 5, \ldots\}
\end{aligned}$$

$$\begin{aligned}
D \setminus C &= \{x \mid x \in \mathbb{Z} \text{ and } -1 < x \leq 2\} \\
&= \{0, 1, 2\}
\end{aligned}$$

$$D \cap \phi = \phi$$

$$\begin{aligned}
\mathbb{R} \cup D &= \{x \mid x \in \mathbb{R} \text{ and } x \in \mathbb{Z} \text{ and } x > -1\} \\
&= \{0, 1, 2, \ldots\}
\end{aligned}$$

(c) Let $E = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2\}$ and $F = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 1\}$. Determine $E \cup F$, $E \cap F$, $E \setminus F$, and $F \setminus E$.

Solution:

$$E \cup F = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2 \text{ or } x_1^2 + x_2^2 \geq 1\}$$

$$E \cup F = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2 \text{ and } x_1^2 + x_2^2 \geq 1\}$$

$$E \setminus F = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > x_2 \text{ and } x_1^2 + x_2^2 < 1\}$$

$$F \setminus E = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq x_2 \text{ and } x_1^2 + x_2^2 \geq 1\}$$

2. For each of the following subsets of $\mathbb{R}^3$, determine whether they are a subspace of $\mathbb{R}^3$. Give reasons why or why not in each case.

(a) $S_a = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{Z}\}$

Solution: Notice that $(0, 0, 0) \in S_a$, so that S_a is non-empty. Let $\mathbf{x} = (x_1, x_2, x_3) = (1, 1, 1) \in S_a$. Notice that S_a is not closed under scalar multiplication, due to the following. Let $t = 1.5$, $t\mathbf{x} = (1.5, 1.5, 1.5) \notin S_a$. Therefore, S_a is not a subspace.

(b) $S_b = \{(x_1, x_2, x_3) \mid x_1^2 = x_2^3\}$

Solution: Notice that $(0, 0, 0) \in S_b$, because $0^2 = 0^3$, so that S_b is non-empty. Let $\mathbf{x} = (x_1, x_2, x_3) = (1, 1, 1) \in S_b$, because $1^2 = 1^3$. Notice that S_b is not closed under scalar multiplication, due to the following. Let $t = 1.5$, $t\mathbf{x} = (1.5, 1.5, 1.5) \notin S_b$, because $1.5^2 \neq 1.5^3$. Therefore, S_b is not a subspace.

(c) $S_c = \{(x_1, x_2, x_3) \mid x_1 - 3x_2 + 4x_3 = 0 \text{ and } x_1 = x_2\}$

Solution: Notice that $(0, 0, 0) \in S_c$, because $0^2 = 0^3$, so that S_c is non-empty. Let $\mathbf{x}, \mathbf{y} \in S_c$. Then they must satisfy the condition of the set, so $x_1 - 3x_2 + 4x_3 = 0$, $x_1 = x_2$, $y_1 - 3y_2 + 4y_3 = 0$, $y_1 = y_2$.

To show that S_c is closed under addition, we must show that $\mathbf{x} + \mathbf{y} = (x_1 + y_2, x_2 + y_2, x_3 + y_3)$ satisfies the condition of S_c. Notice that $x_1 + y_1 - 3(x_2 + y_2) + 4(x_3 + y_3) = 0$ and $x_1 + y_1 = x_2 + y_2$. Hence, $\mathbf{x} + \mathbf{y} \in S_c$.

Similarly, for any $t \in \mathbb{R}$, we have $t\mathbf{x} = (tx_1, tx_2, tx_3)$, so that $tx_1 - 3t_x 2 + 4tx_3 = 0$ and $tx_1 = tx_2$, i.e. $t\mathbf{x} \in S_c$. Therefore, S_c is closed under scalar multiplication and S_c is a subspace of $\mathbb{R}^3$.

(d) $S_d = \{(x_1, x_2, x_3) \mid \sin(x_1) = x_3\}$

Solution: Notice that $(0, 0, 0) \in S_d$, because $\sin(0) = 0$, so that S_d is non-empty. Let $\mathbf{x} = (x_1, x_2, x_3) = (\pi/2, 0, 1) \in S_d$, because $\sin(\pi/2) = 1$. Notice that S_d is not closed under scalar multiplication, due to the following. Let $t = 2$, $t\mathbf{x} = (\pi, 0, 2) \notin S_d$, because $\sin(\pi) = 0 \neq 2$. Therefore, S_d is not a subspace.

(e) $S_e = \{(x_1, x_2, x_3) \mid (x_1 + x_2 + x_3)^2 = 0\}$

Solution: Notice that $(0, 0, 0) \in S_e$, because $(0 + 0 + 0)^2 = 0$, so that S_e is non-empty. Let $\mathbf{x}, \mathbf{y} \in S_e$. Then they must satisfy the condition of the set, so $(x_1 + x_2 + x_3)^2 = 0$, or equivalently $x_1 + x_2 + x_3 = 0$, and $(y_1 + y_2 + y_3)^2 = 0$, or equivalently $y_1 + y_2 + y_3 = 0$.

To show that S_e is closed under addition, we must show that $\mathbf{x} + \mathbf{y} = (x_1 + y_2, x_2 + y_2, x_3 + y_3)$ satisfies the condition of S_e. Notice that $(x_1 + y_2 + x_2 + y_2 + x_3 + y_3)^2 = 0$. Hence, $\mathbf{x} + \mathbf{y} \in S_e$.

Similarly, for any $t \in \mathbb{R}$, we have $t\mathbf{x} = (tx_1, tx_2, tx_3)$, so that $(tx_1 + tx_2 + tx_3)^2 = 0$, i.e. $t\mathbf{x} \in S_e$. Therefore, S_e is closed under scalar multiplication and S_e is a subspace of $\mathbb{R}^3$.

3 (a) Show that $\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\} = \operatorname{span}\left\{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\end{bmatrix}\right\}$.

Solution: Denote by $S_1 = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$ and $S_2 = \operatorname{span}\left\{\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\end{bmatrix}\right\}$.

By definition of span, for any $x \in S_1$:

$$\mathbf{x} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_3 \\ t_2 + t_3 \\ t_2 + t_3 \end{bmatrix}$$

By definition of span, for any $x \in S_2$:

$$\mathbf{y} = a_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_1 - a_2 \\ 2a_1 - a_2 \end{bmatrix}$$

For any $t_1, t_2, t_3 \in \mathbb{R}$, we can let $a_1 = t_1 + 2t_3$ and $a_2 = 2t_2 - t_2 + 3t_3$, so that

$$\mathbf{y} = \begin{bmatrix} t_1 + 2t_3 \\ t_2 + t_3 \\ t_2 + t_3 \end{bmatrix}$$

Therefore, $S_1 = S_2$.

3 (b) Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Show that $\text{span}\{\mathbf{u} + \mathbf{w}, \mathbf{v} - \mathbf{w}\} \subseteq \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Are these spans are actually equal?

Solution: Denote by $S_1 = \text{span}\{\mathbf{u}+\mathbf{w}, \mathbf{v}-\mathbf{w}\}$ and $S_2 = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, so that $S_1 = \text{span}\left\{ \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} \right\}$ and $S_2 = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$.

By definition of span, for any $x \in S_1$:

$$\mathbf{x} = t_1 \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4t_1 - 2t_2 \\ 0 \\ 4t_1 - 2t_2 \end{bmatrix} = (4t_1 - 2t_2) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where $t = 4t_1 - 2t_2 \in \mathbb{R}$.

By definition of span, for any $x \in S_2$:

$$\mathbf{y} = a_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2a_1 + 2a_3 \\ -a_1 + a_2 + a_3 \\ a_1 + a_2 + 3a_3 \end{bmatrix}$$

Thus, for any t, let $a_1 = 0$, $a_2 = -t/2$, $a_3 = t/2$. Notice that

$$\mathbf{y} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

so that $\mathbf{x} = \mathbf{y} \in \mathbf{S_2}$. It follows that $S_1 \subseteq S_2$.

In the following, we will show that $S_1 \neq S_2$ by showing that there exists $\mathbf{y} \in \mathbf{S_2}$ such that $\mathbf{y} \notin \mathbf{S_1}$. Let $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$. Notice that $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \mathbf{S_1}$. Therefore, $S_1 \subset S_2$.

4. Let $W = \{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ be a set of four vectors in $\mathbb{R}^n$. Suppose that $\mathbf{x} = \mathbf{y} - 5\mathbf{z} + \mathbf{w}$ and $\mathbf{z} = \mathbf{x} - \mathbf{y}$.

(a) Is the set $\{\mathbf{x}, \mathbf{y}, \mathbf{w}\}$ linearly independent? How about the set $\{\mathbf{x}, \mathbf{z}, \mathbf{w}\}$? Explain your answers.

Solution:

$$\begin{aligned}
&\mathbf{x} = \mathbf{y} - \mathbf{5z} + \mathbf{w} \\
&\mathbf{x} = \mathbf{y} - \mathbf{5}(\mathbf{x} - \mathbf{y}) + \mathbf{w} \\
&6\mathbf{x} - \mathbf{6y} - \mathbf{w} = \mathbf{0}
\end{aligned}$$

Therefore, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{w}\}$ is, by definition, linearly dependent.

$$\begin{aligned}
&\mathbf{x} = \mathbf{y} - \mathbf{5z} + \mathbf{w} \\
&\mathbf{x} = (\mathbf{x} - \mathbf{z}) - \mathbf{5z} + \mathbf{w} \\
&0\mathbf{x} + \mathbf{6z} - \mathbf{w} = \mathbf{0}
\end{aligned}$$

Therefore, the set $\{\mathbf{x}, \mathbf{z}, \mathbf{w}\}$ is, by definition, linearly dependent.

(b) Can you conclude anything about the linear independence or dependence of the set $\{\mathbf{x}, \mathbf{z}\}$?

Solution: No. It depends on the vectors $\mathbf{y}$ and $\mathbf{w}$. Let's take some examples from $\mathbb{R}^3$. Notice that if $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$, then we have $\mathbf{x} = \mathbf{y} - \mathbf{5z} + \mathbf{w} = -\mathbf{5z}$, it follows that $\mathbf{x} + \mathbf{5z} = \mathbf{0}$ and so $\mathbf{x}$ and $\mathbf{z}$ are linearly dependent. However, if we let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix}$, we can check that $\mathbf{x} = \mathbf{y} - \mathbf{5z} + \mathbf{w}$ and $\mathbf{z} = \mathbf{x} - \mathbf{y}$. However, the system $t_1\mathbf{x} + t_2\mathbf{z} = \mathbf{0}$ only has the trivial solution $t_1 = 0$ and $t_2 = 0$ (equivalently, that $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ are not multiples of one another). Therefore, $\mathbf{x}$ and $\mathbf{z}$ will be linearly independent in this case.

5. Let A and B be subspaces of $\mathbb{R}^3$.

(a) Show that $A \cap B$ is a subspace of $\mathbb{R}^3$.

Solution: Because A and B are subspaces of $\mathbb{R}^3$, then $(0,0,0) \in A$ and $(0,0,0) \in B$, so that $(0,0,0) \in A \cap B = S_a$ and S_a is non-empty. Let $\mathbf{x}, \mathbf{y} \in S_a$, then $\mathbf{x}, \mathbf{y} \in A$ and $\mathbf{x}, \mathbf{y} \in B$. But A is a subspace of $\mathbb{R}^3$, so that A is closed under addition, i.e. $\mathbf{x} + \mathbf{y} \in A$. Similarly, B is a subspace of $\mathbb{R}^3$, so that B is closed under addition, i.e. $\mathbf{x} + \mathbf{y} \in B$. Therefore, $\mathbf{x} + \mathbf{y} \in A \cap B = S_a$. It follows that S_a is closed under addition.

Furthermore, A is closed under scalar multiplication, i.e. $t\mathbf{x} \in A$, for any $t \in \mathbb{R}$, and B is closed under scalar multiplication, i.e. $t\mathbf{x} \in B$, for any $t \in \mathbb{R}$. Therefore, $t\mathbf{x} \in A \cap B = S_a$, for any $t \in \mathbb{R}$, so that S_a is closed under scalar multiplication. Therefore, S_a is a subspace of $\mathbb{R}^3$.

(b) Find an example of subspaces A and B to show $A \cup B$ is not necessarily a subspace of $\mathbb{R}^3$.

Solution: Let $A = \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}$ and $B = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}$, which are clearly non-empty sets. Let $\mathbf{x}, \mathbf{z} \in A$, and notice that $\mathbf{x} + \mathbf{z} \in A$ and $t\mathbf{x} \in A$, for all $t \in \mathbb{R}$. Thus, A is a subspace of $\mathbb{R}^3$. Similarly, we can show that B is a subspace of $\mathbb{R}^3$.

Let $\mathbf{x} \in A$ and $\mathbf{y} \in B$. Notice that $\mathbf{x} = (x_1, 0, 0)$ and $\mathbf{y} = (0, y_2, 0)$. It follows that $\mathbf{x} + \mathbf{y} = (x_1, y_2, 0)$ which doesn't belong to either A or B. Therefore, $A \cup B$ is not closed under addition and $A \cup B$ is not a subspace.

(c) Is $A \setminus B$ ever a subspace of $\mathbb{R}^3$? Why or why not?

Solution: Notice that $A \setminus B = A \cap \overline{B}$, where $\overline{B}$ is the complement of the set B, i.e. $\overline{B} = \{x \mid x \notin B\}$. Because B is a subspace, then B contains the origin $(0, 0, 0)$, which means that $\overline{B}$ does not contain the origin. Thus, $A \setminus B = A \cap \overline{B}$ does not contain the origin. It follows that $A \setminus B$ is not a subspace.

Notice that if B is not a subspace, say for example $B = \phi$, then $A \setminus B = A \cap \overline{B} = A \cap \mathbb{R}^3 = A$ may be a subspace. An example of this is $A = \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}$, which we have proved in 5 (b) to be a subspace.