University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2012**
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA
Exam Type: A

General Comments:

1. Most of this exam consisted of completely routine questions. In particular, questions 4, 5, 7 and 8 were almost identical to questions on previous exams.

2. The only question that could be considered non-routine is question 9. However, there are many correct ways to do both parts (a) and (b), and many people got them. But most students had no idea how to tackle part (b). Some students wrote things like $B = 3I = I$ or

$$\vec{u}\vec{u}^T + \vec{v}\vec{v}^T + (\vec{u} \times \vec{v})(\vec{u} \times \vec{v})^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}!$$

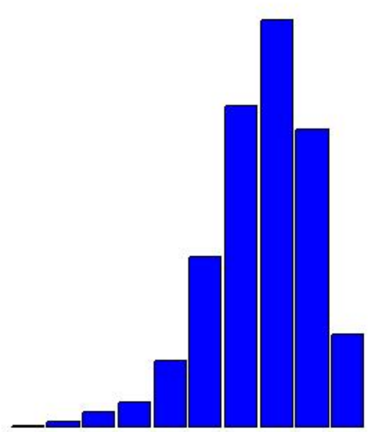
   Others wrote things like $\vec{u}\,\vec{u}^T = \|\vec{u}\|^2$. In fact, $\vec{u}\,\vec{u}^T$ is a 3×3 matrix.

3. The range on every question was zero to perfect. Students did surprisingly well on the True and False, but surprisingly poorly on the first question.

4. There were three perfect papers.

Breakdown of Results: 971 students wrote this exam. The marks ranged from 9% to 100%, and the average was 70.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	6.6%
A	27.8%	80-89%	21.2%
B	29.0%	70-79%	29.0%
C	23.0%	60-69%	23.0%
D	12.2%	50-59%	12.2%
F	8.0%	40-49%	4.7%
		30-39%	1.8%
		20-29%	1.0%
		10-19%	0.4%
		0-9%	0.1%



1. [avg: 5.3/10] Find the following:

(a) [5 marks] a vector equation for the line of intersection common to the two planes with equations $x + y - z = 5$ and $2x + y + 2z = 2$.

Solution: solve the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 2 & 1 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 0 & 1 & -4 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & -3 \\ 0 & 1 & -4 & 8 \end{array}\right]$$

Let $z = t$ be a parameter, then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 - 3t \\ 8 + 4t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix},$$

which is the vector equation of a line.

(b) [5 marks] the shortest distance between the two parallel lines $\mathbb{L}_1$ and $\mathbb{L}_2$

$$\mathbb{L}_1 : \left\{ \begin{array}{ccccc} x & = & 1 & + & t \\ y & = & 0 & - & t \\ z & = & 1 & + & t \end{array} \right. ; \quad \mathbb{L}_2 : \left\{ \begin{array}{ccccc} x & = & 2 & - & s \\ y & = & 3 & + & s \\ z & = & 1 & - & s \end{array} \right.$$

where s and t are parameters.

Solution: let the shortest distance be D. Let P be the point on $\mathbb{L}_1$ with coordinates $(1, 0, 1)$; let Q be the point on $\mathbb{L}_2$ with coordinates $(2, 3, 1)$; let

$$\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

be a direction vector for both lines. Then $D^2 + \left\| \mathrm{proj}_{\vec{n}} \overrightarrow{PQ} \right\|^2 = \left\| \overrightarrow{PQ} \right\|^2$, so

$$D^2 = \left\| \overrightarrow{PQ} \right\|^2 - \left\| \frac{\overrightarrow{PQ} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right\|^2 = \left\| \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\|^2 - \left\| -\frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\|^2 = 10 - \frac{4}{3} = \frac{26}{3};$$

and

$$D = \sqrt{\frac{26}{3}}.$$

2. [avg: 7.9/10]

(a) [5 marks] Find the area of the triangle ΔPQR passing through the three points

$$P(1,3,1),\ \ Q(2,2,2),\ \ R(0,2,0).$$

Solution: let the area be A. Then

$$A = \frac{1}{2}\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\| = \frac{1}{2}\left\|\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right\| = \frac{1}{2}\left\|\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}\right\| = \frac{\sqrt{8}}{2} = \sqrt{2}.$$

(b) [5 marks] Let $U = \left\{\begin{bmatrix} x & y & z \end{bmatrix}^T \mid xyz = 0\right\}$.

1. Is U non-empty?

   Solution: No. The vector $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ is in U since $(0)(0)(0) = 0$.

2. Is U closed under scalar multiplication?

   Solution: Yes. If the vector $X = \begin{bmatrix} x & y & z \end{bmatrix}^T$ is in U, so is tX for any scalar t, since $(tx)(ty)(tz) = t^3xyz = t^3(0) = 0$.

3. Is U closed under vector addition?

   Solution: No. The vectors $X = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $Y = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ are in U but the vector $X + Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is not in U, since $(1)(1)(1) = 1 \neq 0$.

4. Is U a subspace of $\mathbb{R}^3$?

   Solution: No, because U is not closed under addition.

3. [2 marks for each part; avg: 7.9/10] Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) If $\lambda = 0$ is an eigenvalue of the $n \times n$ matrix A then A is not invertible.

True or **False**

True: let a corresponding eigenvector be X; so X is non-zero and $AX = 0X = O$. Thus the homogeneous system has a non-zero solution, which is equivalent to A being non-invertible. OR: $\det(0I - A) = 0 \Leftrightarrow \det(A) = 0$, so A is not invertible.

(b) $\dim\left(\operatorname{im}\begin{bmatrix} 1 & 1 & 1 & -2 \\ 3 & 6 & 1 & 1 \\ 5 & 8 & 3 & -3 \end{bmatrix}\right) = 3$ **True** or **False**

False: $\dim(\operatorname{im}(A)) = r$, where r is the rank of A. The matrix

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 3 & 6 & 1 & 1 \\ 5 & 8 & 3 & -3 \end{bmatrix}$$

has rank 2 because it only has 2 independent rows: $R_3 = R_2 + 2R_1$.

(c) If U is a subspace of $\mathbb{R}^7$ and $\dim U = 4$ then $\dim U^\perp = 3$. **True** or **False**

True: $7 = \dim U + \dim U^\perp \Rightarrow \dim U^\perp = 7 - 4 = 3$.

(d) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$. **True** or **False**

False: in the spanning set, the first vector is the sum of the next two. So the question reduces to, Are there scalars a and b such that

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}?$$

This system is obviously inconsistent, since $1 + 1/3 \neq 2$.

(e) If $\{X_1, X_2, X_3\}$ is linearly independent, then so is $\{X_2, X_3\}$. **True** or **False**

True: $sX_2 + tX_3 = O \Rightarrow 0X_1 + sX_2 + tX_3 = O \Rightarrow s = t = 0$ since $\{X_1, X_2, X_3\}$ is linearly independent.

4. [10 marks; avg: 8.5/10] Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & -1 & 2 \\ 2 & -1 & 3 & -2 & 2 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is $r = 3$, the number of leading 1's in R.

U	$\dim U$	description of basis	vectors in basis
row$A$	$r = 3$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$
		three independent rows of A	$\{R_1, R_2, R_4\}$ $\{R_1, R_3, R_4\}$ $\{R_2, R_3, R_4\}$ NB: $R_3 = R_2 + 2R_1$
col A	$r = 3$	three independent columns of A	$\{C_1, C_2, C_5\}$ NB: $C_3 = 2C_1 + C_2; C_4 = -C_1$
		(*not* three independent columns of R)	(col$R \neq$ col$A$)
null A	$5 - r = 2$	two basic solutions to $RX = 0$	$\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

5. [avg: 8.5/12] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+5y \\ 2x+y \end{bmatrix}.$$

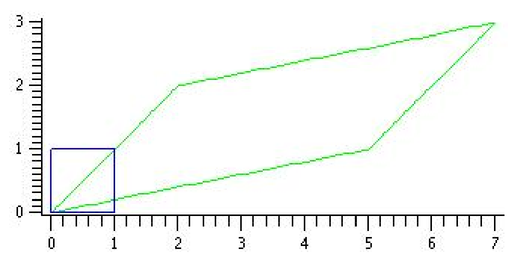
(a) [6 marks] Draw the image under T of the unit square, and calculate its area.

Solution: the image of the unit square is the parallelogram determined by

$$T(\vec{\imath}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ and } T(\vec{\jmath}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

In the diagram to the right, the unit square is in blue, and the image of the unit square is in green. Its area is

$$\left|\det\begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\right| = |2-10| = 8.$$



(b) [6 marks] Find the formula for $T \circ R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ if R is a rotation of $\pi/6$ clockwise around the origin.

Solution: the matrix of R is

$$\begin{bmatrix} \cos(-\pi/6) & -\sin(-\pi/6) \\ \sin(-\pi/6) & \cos(-\pi/6) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} = B$$

and the matrix of T is

$$\begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix} = A;$$

so

$$\begin{aligned} T \circ R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = AB\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} 2\sqrt{3}-5 & 2+5\sqrt{3} \\ 2\sqrt{3}-1 & 2+\sqrt{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} (2\sqrt{3}-5)\,x + (2+5\sqrt{3})\,y \\ (2\sqrt{3}-1)\,x + (2+\sqrt{3})\,y \end{bmatrix}. \end{aligned}$$

6. [avg: 8.8/12] Let $Q = \dfrac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix}$.

(a) [4 marks] Find the eigenvalues of Q.

Solution:

$$\det(\lambda I - Q) = \det \begin{bmatrix} \lambda - \left(\frac{-8}{10}\right) & -\frac{6}{10} \\ -\frac{6}{10} & \lambda - \frac{8}{10} \end{bmatrix} = \lambda^2 - \left(\frac{64}{100}\right) - \left(\frac{36}{100}\right) = \lambda^2 - 1.$$

So the eigenvalues of Q are $\lambda_1 = 1$ and $\lambda_2 = -1$.

(6) [8 marks] Find a basis for each eigenspace[1] of Q and plot the eigenspaces of Q in the plane, indicating which eigenspace corresponds to which eigenvalue of Q.

Solution:

$$E_1(Q) = \text{null}(I - Q) = \text{null}\begin{bmatrix} 1 + \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & 1 - \frac{4}{5} \end{bmatrix} = \text{null}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix};$$

so could take

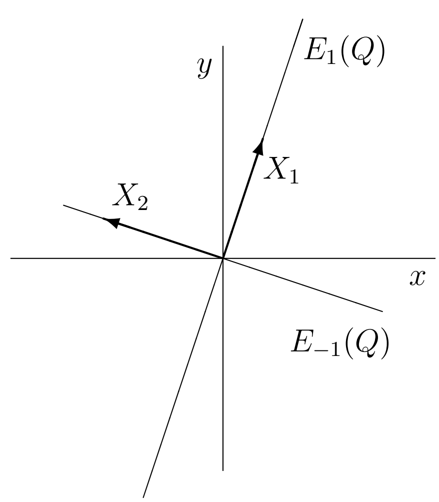
$$X_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

$$E_{-1}(Q) = \text{null}(-I - Q) = \text{null}\begin{bmatrix} -1 + \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -1 - \frac{4}{5} \end{bmatrix} = \text{null}\begin{bmatrix} -1 & -3 \\ -3 & -9 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix};$$

so could take

$$X_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Plot of Eignespaces:



Note: Q is the matrix of a reflection in the line $y = 3x$, so $E_1(Q)$ *is* the axis of reflection, and $E_{-1}(Q)$ must be the line orthogonal to the axis of reflection.

[1] Recall: if A is an $n \times n$ matrix, the eigenspace of A corresponding to λ is $E_\lambda(A) = \{X \in \mathbb{R}^n \mid AX = \lambda X\}$.

7. [avg: 10.4/12]

Let $U = \text{span}\left\{X_1 = [\ 0\ \ 1\ \ -1\ \ 0\ ]^T, X_2 = [\ 1\ \ 0\ \ 0\ \ -1\ ]^T, X_3 = [\ 1\ \ -1\ \ 0\ \ 0\ ]^T\right\}$; let $X = \left[\ 1\ \ 1\ \ 0\ \ 1\ \right]^T$. Find:

(a) [6 marks] an orthogonal basis of U.

Solution: since $X_1 \cdot X_2 = 0$ already, you only need to use the Gram-Schmidt algorithm to find F_3. Take $F_1 = X_1,\ F_2 = X_2$, and

$$F_3 = X_3 - \frac{X_3 \cdot X_1}{\|X_1\|^2} X_1 - \frac{X_3 \cdot X_2}{\|X_2\|^2} X_2 = X_3 + \frac{1}{2} X_1 - \frac{1}{2} X_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Optional: clear fractions and take

$$F_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Either way $\{F_1, F_2, F_3\}$ is an orthogonal basis of U.

(b) [6 marks] $\text{proj}_U(X)$.

Solution: using $\{F_1, F_2, F_3\}$ with fractions cleared.

$$\begin{aligned} \text{proj}_U X &= \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 + \frac{X \cdot F_3}{\|F_3\|^2} F_3 \\ &= \frac{1}{2} F_1 + (0) F_2 + \frac{1}{4} F_3 \\ &= \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}. \end{aligned}$$

Cross-check/Alternate Solution: $U^\perp = \text{span}\{Y\}$ with $Y = [\ 1\ \ 1\ \ 1\ \ 1\ ]^T$. Then

$$\text{proj}_U X = X - \text{proj}_{U^\perp}(X) = X - \frac{X \cdot Y}{\|Y\|^2} Y = X - \frac{3}{4} Y = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}.$$

8. [12 marks; avg: 9.8/12] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 2 & 0 \\ -1 & 0 & \lambda - 1 \end{bmatrix} = (\lambda - 2) \det \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{bmatrix} \\ &= (\lambda - 2)\left((\lambda - 1)^2 - 1\right) = (\lambda - 2)(\lambda^2 - 2\lambda) \\ &= \lambda\,(\lambda - 2)^2 \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2$.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_0(A) = \text{null} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$E_2(A) = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

OR: use either $E_0(A) = (E_2(A))^\perp$ or $E_2(A) = (E_0(A))^\perp$ to simplify calculations. That is, $E_2(A)$ is the plane with equation $x = z$ and $E_0(A)$ is the line normal to the plane.

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

9. [avg: 3.3/12] Suppose $\vec{u}$ and $\vec{v}$ are two orthogonal unit vectors in $\mathbb{R}^3$.

Let $A = \left[\ \vec{u} \mid \vec{v} \mid \vec{u}\times\vec{v}\ \right]$ and let $B = \vec{u}\,\vec{u}^T + \vec{v}\,\vec{v}^T + (\vec{u}\times\vec{v})\,(\vec{u}\times\vec{v})^T$.

(a) [6 marks] Explain why A must be an orthogonal matrix.

Solution 1: show that the columns of A form an orthonormal basis of $\mathbb{R}^3$.
It is given that $\|\vec{u}\| = \|\vec{v}\| = 1$ and $\vec{u}\cdot\vec{v} = 0$; and $\vec{u}\times v$ is always orthogonal to both $\vec{u}$ and $\vec{v}$. So need only show that $\vec{u}\times\vec{v}$ is also a unit vector:

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin(\pi/2) = (1)(1)(1) = 1.$$

Solution 2: show that $A^TA = I$: $A^TA = \left[\ \vec{u} \mid \vec{v} \mid \vec{u}\times\vec{v}\ \right]^T \left[\ \vec{u} \mid \vec{v} \mid \vec{u}\times\vec{v}\ \right] =$

$$\begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ (\vec{u}\times\vec{v})^T \end{bmatrix} \left[\ \vec{u} \mid \vec{v} \mid \vec{u}\times\vec{v}\ \right] = \begin{bmatrix} \vec{u}^T\vec{u} & \vec{u}^T\vec{v} & \vec{u}^T(\vec{u}\times\vec{v}) \\ \vec{v}^T\vec{u} & \vec{v}^T\vec{v} & \vec{v}^T(\vec{u}\times\vec{v}) \\ (\vec{u}\times\vec{v})^T\vec{u} & (\vec{u}\times\vec{v})^T\vec{v} & (\vec{u}\times\vec{v})^T(\vec{u}\times\vec{v}) \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v} & \vec{u}\cdot(\vec{u}\times\vec{v}) \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} & \vec{v}\cdot(\vec{u}\times\vec{v}) \\ (\vec{u}\times\vec{v})\cdot\vec{u} & (\vec{u}\times\vec{v})\cdot\vec{v} & (\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v}) \end{bmatrix} = \begin{bmatrix} \|\vec{u}\|^2 & 0 & 0 \\ 0 & \|\vec{v}\|^2 & 0 \\ 0 & 0 & \|\vec{u}\times\vec{v}\|^2 \end{bmatrix} = I.$$

(b) [6 marks] Explain why B must be I, the 3×3 identity matrix.

Solution 1: To save space, let $\vec{w} = \vec{u}\times\vec{v}$. By part (a), $AA^T = I$, so

$$I = AA^T = \left[\ \vec{u} \mid \vec{v} \mid \vec{w}\ \right] \begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ \vec{w}^T \end{bmatrix} = \vec{u}\,\vec{u}^T + \vec{v}\,\vec{v}^T + \vec{w}\,\vec{w}^T = B.$$

Solution 2: $\{\vec{u},\vec{v},\vec{w}\}$ is an orthonormal basis. Apply the Expansion Theorem, for $\vec{x}\in\mathbb{R}^3$: $\vec{x} = (\vec{x}\cdot\vec{u})\vec{u} + (\vec{x}\cdot\vec{v})\vec{v} + (\vec{x}\cdot\vec{w})\vec{w} = \vec{u}\vec{u}^T\vec{x} + \vec{v}\vec{v}^T\vec{x} + \vec{w}\vec{w}^T\vec{x} = B\vec{x} \Rightarrow B = I$.

Solution 3: show $\vec{u},\vec{v},\vec{w}$ are all eigenvectors of B with eigenvalue $\lambda = 1$.

$$\begin{aligned} B\vec{u} &= (\vec{u}\vec{u}^T + \vec{v}\vec{v}^T + \vec{w}\vec{w}^T)\vec{u} = \vec{u}\vec{u}^T\vec{u} + \vec{v}\vec{v}^T\vec{u} + \vec{w}\vec{w}^T\vec{u} \\ &= \vec{u}(\vec{u}\cdot\vec{u}) + \vec{v}(\vec{v}\cdot\vec{u}) + \vec{w}(\vec{w}\cdot\vec{u}) = \vec{u}(1) + \vec{v}(0) + \vec{w}(0) = \vec{u}; \\ B\vec{v} &= (\vec{u}\vec{u}^T + \vec{v}\vec{v}^T + \vec{w}\vec{w}^T)\vec{v} = \vec{u}\vec{u}^T\vec{v} + \vec{v}\vec{v}^T\vec{v} + \vec{w}\vec{w}^T\vec{v} \\ &= \vec{u}(\vec{u}\cdot\vec{v}) + \vec{v}(\vec{v}\cdot\vec{v}) + \vec{w}(\vec{w}\cdot\vec{v}) = \vec{u}(0) + \vec{v}(1) + \vec{w}(0) = \vec{v}; \\ B\vec{w} &= (\vec{u}\vec{u}^T + \vec{v}\vec{v}^T + \vec{w}\vec{w}^T)\vec{w} = \vec{u}\vec{u}^T\vec{w} + \vec{v}\vec{v}^T\vec{w} + \vec{w}\vec{w}^T\vec{w} \\ &= \vec{u}(\vec{u}\cdot\vec{w}) + \vec{v}(\vec{v}\cdot\vec{w}) + \vec{w}(\vec{w}\cdot\vec{w}) = \vec{u}(0) + \vec{v}(0) + \vec{w}(1) = \vec{w}. \end{aligned}$$

Thus the matrix $A = \left[\ \vec{u} \mid \vec{v} \mid \vec{w}\ \right]$ orthogonally diagonalizes B and

$$I = \operatorname{diag}(1,1,1) = A^TBA \Leftrightarrow B = AIA^T = I.$$