University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 18

Instructor: Arman Pannu

We will start 10 minutes past the hour. Use this time to make a new friend.

We Value Your Feedback!

Course Evaluations Are Open!

- Your feedback is **anonymous** and makes a real difference.
- It helps us improve the course for future students.
- It helps instructors reflect and grow in their teaching.

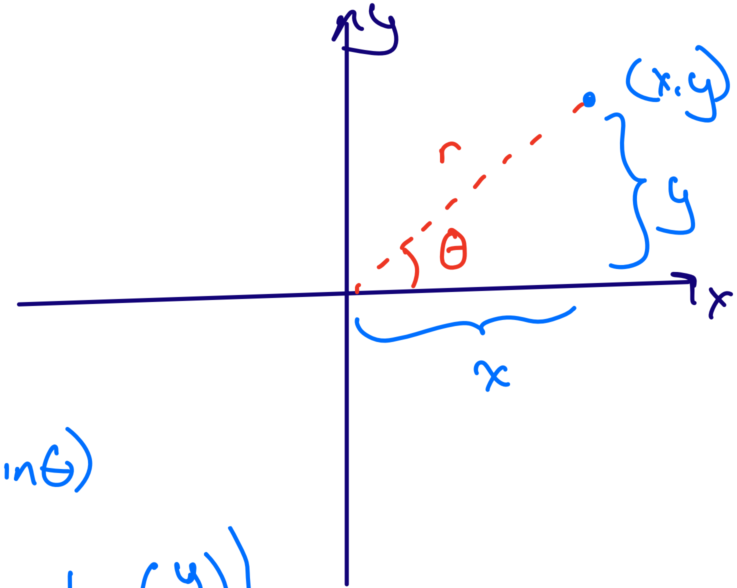
What to Comment On:

- Course structure - was it clear and well-organized?
- Teaching style - what worked (or didn't)?
- Assessments - were they fair and helpful?
- Resources - were they accessible and useful?
- Suggestions - what could be improved for next time?

Bonus: Consider leaving a review on platforms like `RateMyProfessors.com` *to help other students too!*

Polar Coordinates

Polar coordinates describe a Point in $\mathbb{R}^2$ with distance from origin r and angle with x-axis θ



$$(r,\theta) \longmapsto (x,y) = (r\cos\theta, r\sin\theta)$$

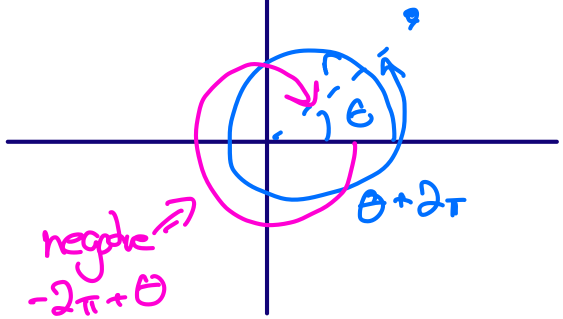
$$(x,y) \longmapsto (r,\theta) = \Big(\sqrt{x^2+y^2},\ \underbrace{\arctan\left(\tfrac{y}{x}\right)}_{\text{true in 1st \& 2nd quadrant}}\Big)$$

→ up to 2π factor

Remarks

→ we only need $r \geq 0$ and $\theta \in [0, 2\pi)$ to describe any Point uniquely

→ if we allow (r,θ) outside this range then there is redundancy

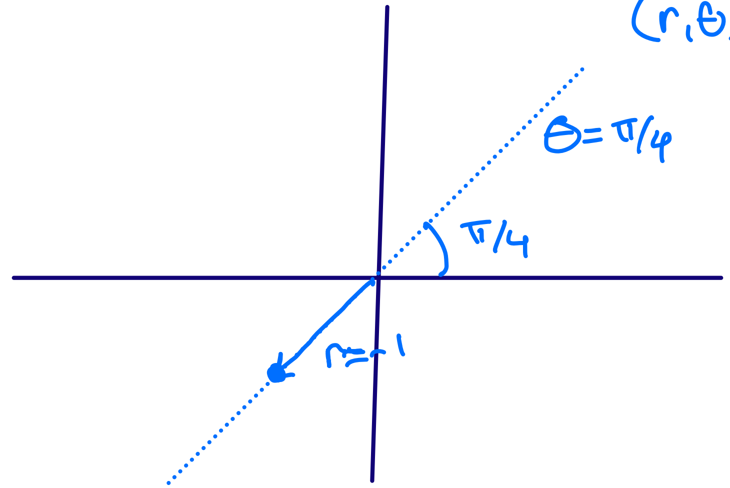


(r,θ) same pt.

or $(r, \theta+2\pi)$

and $(r, -2\pi+\theta)$

$(r,\theta) = (-1, \pi/4)$



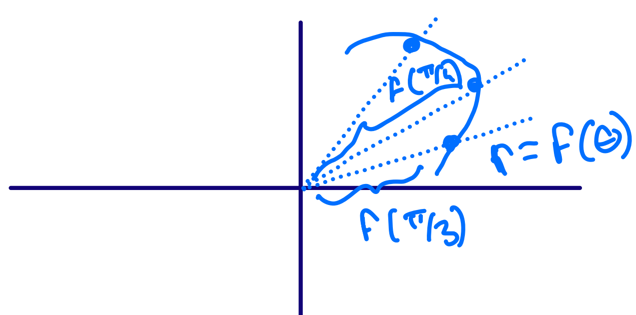
$(r,\theta) = (-1, \pi/4)$ is same pt.

as $(r,\theta) = (1, 5\pi/4)$

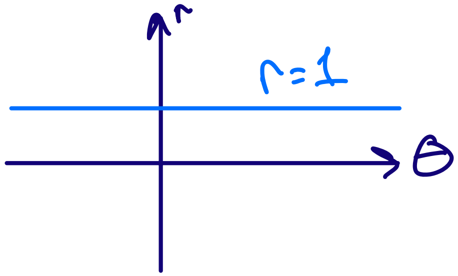
Polar Curves

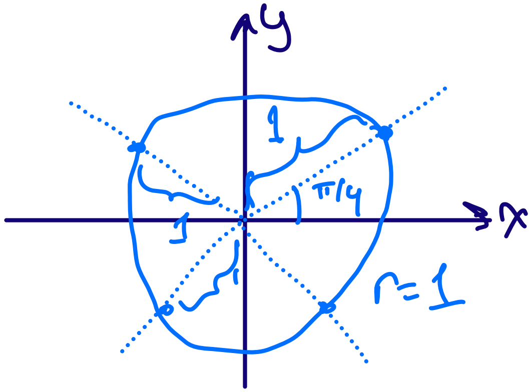
→ a function $y = f(x)$ gives a relation between x & y

→ similarly we can have a relation between θ, r via $r = f(\theta)$

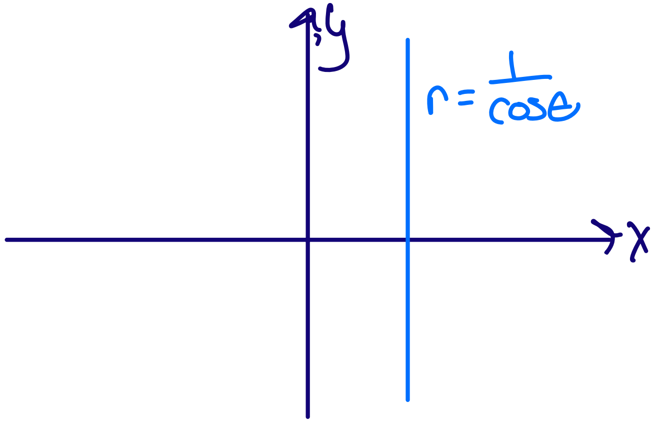


ex 11 $\quad r = f(\theta) = 1 \quad \Leftarrow$ constant



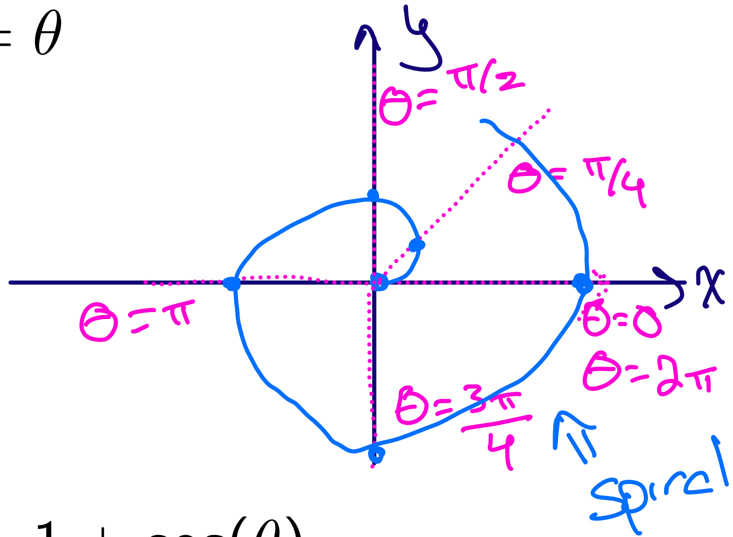


$$r = \frac{1}{\cos\theta} \Rightarrow \underbrace{r\cos\theta}_{x} = 1 \Rightarrow x = 1$$



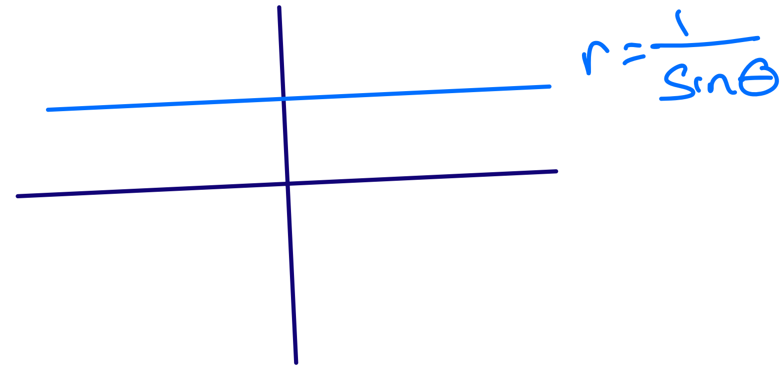
Sketch the following curves.

$r = \theta$

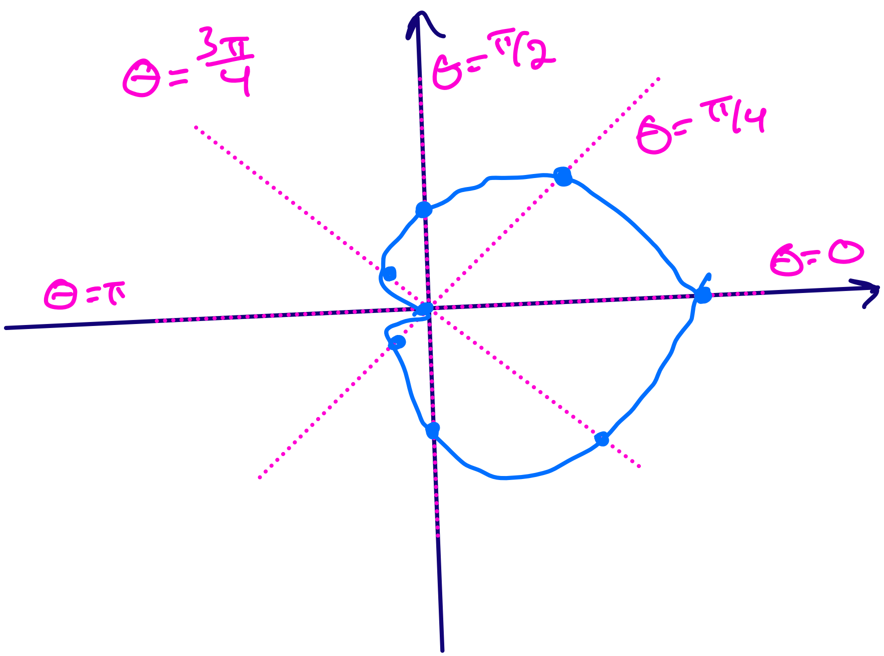


$r = \frac{1}{\sin(\theta)}$ $\Rightarrow$ $r\sin\theta = 1$

$y = 1$



$r = 1 + \cos(\theta)$



⇐ cardioid

Arclength

→ recall arclength for a parametric equation

$$\text{arclength} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

→ given polar curve $r = f(\theta)$

→ resulting graph is parametric curve

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = \underbrace{f(\theta)\sin\theta}_{\text{Parameterization with parameter } \theta}$$

$$\Rightarrow x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$$
$$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\Rightarrow \quad x'(\theta)^2 + y'(\theta)^2 = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2$$

$$= \underline{f'(\theta)^2\cos^2\theta} + \underline{f(\theta)^2\sin^2\theta} - \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta}$$

$$+ \underline{f'(\theta)^2\sin^2\theta} + \underline{f(\theta)^2\cos^2\theta} + \cancel{2f'(\theta)f(\theta)\cos\theta\sin\theta}$$

$$= f'(\theta)^2 + f(\theta)^2$$

arclength of polar curve $= \displaystyle\int_{\theta_1}^{\theta_2} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$ where $r = f(\theta)$

$\rightarrow$ notation $\displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2}\, d\theta$

Find the arclength of the following polar curves:

$$\underbrace{r = 1 \qquad 0 \leq \theta \leq 2\pi}_{\text{Full-circle}}$$

$$\text{arclength} = \int_0^{2\pi} \sqrt{r^2 + r'^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1)^2 + (0)^2}\, d\theta = \underbrace{2\pi}_{\text{Perimeter of circle}}$$

$r = \sin(\theta) \qquad 0 \leq \theta \leq 2\pi$

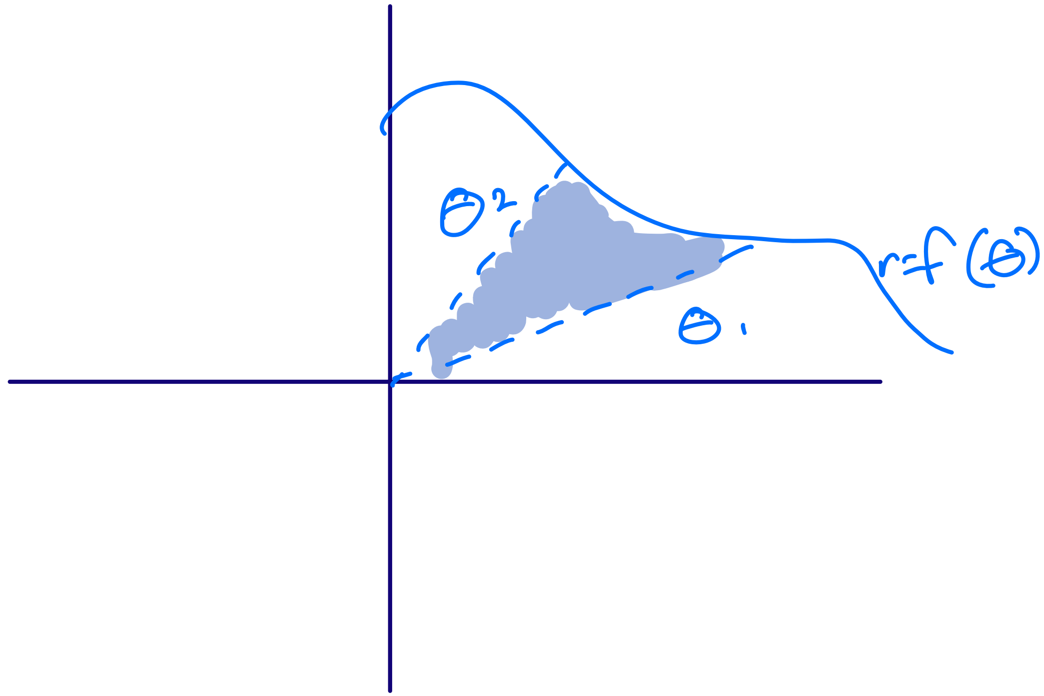
$$\text{arclength} = \int_0^{2\pi} \sqrt{r^2 + r'^2}\, d\theta$$

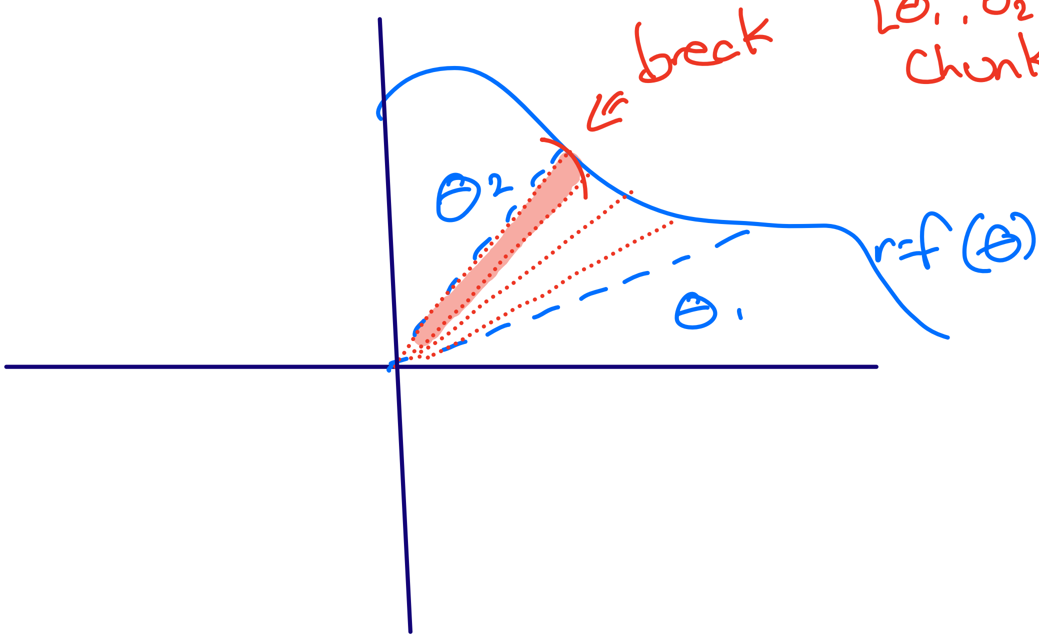
$$= \int_0^{2\pi} \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1}\, d\theta = 2\pi$$

Area

area enclosed by polar curve?

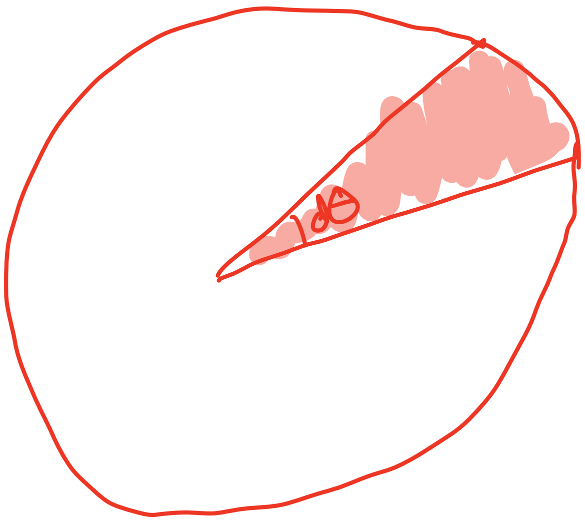




break $[\theta_1, \theta_2]$ into chunks of size $d\theta$

→ assume r is constant on each $d\theta$ slice

→ resultant slice is a sector of a circle

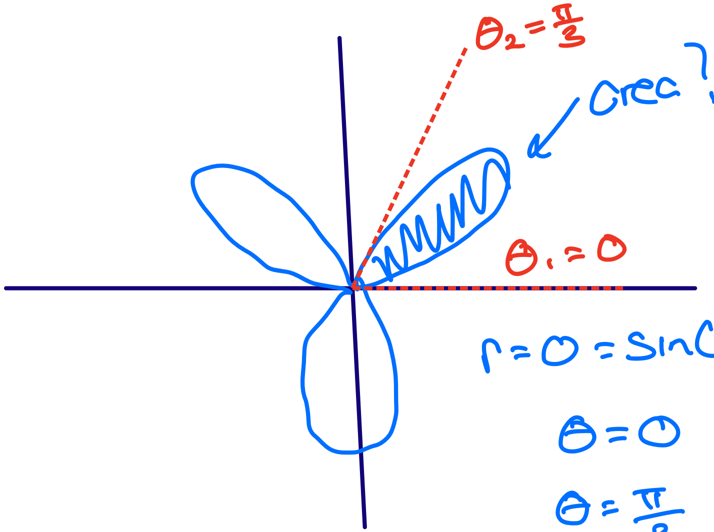


$$\text{area} = \underbrace{\frac{d\theta}{2\pi}}_{\text{percentage of circle}} \underbrace{\pi r^2}_{\text{area of full circle}} = \frac{1}{2} r^2 \, d\theta$$

$$\text{total area} = \int \text{sectors}$$

$$\boxed{= \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta}$$

Find the area enclosed in the first petal of the following curve:
$r = \sin(3\theta)$



$r = 0 = \sin(3\theta)$

$\theta = 0$

$\theta = \frac{\pi}{3}$

$$\text{area} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \frac{1}{2} \sin^2(3\theta) \, d\theta$$

$$= \int_{0}^{\pi/3} \frac{1}{2} \sin^2(3\theta) \, d\theta$$

$\vdots$