UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 2015

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

EXAMINER: D. BURBULLA

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Full Name: ______________________________

Last First

Student Number: ______________________________

Signature: ______________________________

Instructions:

- ONLY THE **FRONT PAGES** WILL BE SCANNED. THE BACK PAGES WILL NOT BE SEEN BY THE EXAMINERS.
- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 10 pages (including this cover page). Make sure you have all of them. Do not tear any pages from this exam.
- You can use the back of the pages and page 10 for rough work.
- This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

PART I : No explanation is necessary.

1. Big Theorem, Final Exam Version: Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix}$$

be the matrix with the vectors in $\mathcal{A}$ as its columns, and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Decide if the following statements are equivalent to the statement, "A is invertible." Circle Yes if the statement is equivalent to "A is invertible," and No if it isn't.

Note: +1 for each correct choice; −1 for each incorrect choice; and 0 for each part left blank.

(a) $\det(A) \neq 0$. Yes No

(b) A is diagonalizable. Yes No

(c) The reduced echelon form of A is I, the $n \times n$ identity matrix. Yes No

(d) T is one-to-one. Yes No

(e) $\mathbf{0}$ is not in $\text{row}(A)$. Yes No

(f) $\text{range}(T) = \mathbb{R}^n$. Yes No

(g) $\text{row}(A) = \text{span}(\mathcal{A})$. Yes No

(h) $\dim(\text{col}(A)) + \dim(\text{null}(A)) = n$. Yes No

(i) $A = A^T$. Yes No

(j) $\lambda = 0$ is not an eigenvalue of A. Yes No

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

2. Find the following:

(a) [2 marks] $\dim(S^{\perp})$, if S is a subspace of $\mathbb{R}^7$ and $\dim(S) = 3$.

(b) [2 marks] $\det(-3A^TB^2)$, if A and B are 3×3 matrices with $\det(A) = 1$ and $\det(B) = 2$.

(c) [2 marks] $\text{proj}_{\mathbf{a}}\mathbf{u}$, if $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$.

(d) [2 marks] $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}^{-1}$.

(e) [2 marks] $\det \begin{bmatrix} 4 & 3 & 10 & 6 \\ 13 & 5 & e & 10 \\ -9 & 4 & \pi & 8 \\ 14 & -1 & \sqrt{2} & -2 \end{bmatrix}$.

3. For any real numbers a, b, c, let $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

(a) [6 marks] Determine for which values of a, b, c, if any, A and B are invertible.

(b) [4 marks] If either A or B is invertible, find its inverse.

4. Suppose A is an $n \times n$ invertible matrix such that $A^2 = 5A$.

   (a) [4 marks] What are the possible eigenvalues of A?

   (b) [2 marks] What must A be if it is invertible?

   (c) [4 marks] Suppose that A is diagonalizable. Find the rank and nullity of A in terms of the multiplicities of its eigenvalues.

5. Find the solution to the system of linear differential equations

$$\begin{aligned} y_1' &= -\frac{3}{2}y_1 + \frac{1}{2}y_2 \\ y_2' &= y_1 - y_2 \end{aligned}$$

where y_1, y_2 are functions of t, and $y_1(0) = 5,\ y_2(0) = 4$.

6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

7. Let $S = \text{span}\left\{[\ 1\ \ 1\ \ 0\ \ 1\ ]^T, [\ 0\ \ 1\ \ 1\ \ 1\ ]^T, [\ 1\ \ 0\ \ 1\ \ 1\ ]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 1 & 5 & 3 & 1 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

8. The parts of this question are unrelated. Each part is worth 5 marks.

(a) Suppose $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are four vectors in $\mathbb{R}^4$ and $\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4\ ] = 5$. Find the value of $\det[\ \mathbf{a}_1 \quad 3\mathbf{a}_2 + 6\mathbf{a}_4 \quad \mathbf{a}_3 \quad 2\mathbf{a}_4 - 8\mathbf{a}_2\ ]$.

(b) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be three linearly independent vectors in $\mathbb{R}^4$. Define $T : \mathbb{R}^4 \longrightarrow \mathbb{R}$ by $T(\mathbf{x}) = \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{x}\ ]$. Show that T is a linear transformation and find range(T).

This page is for rough work; it will only be marked if you indicate you want it marked.