University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER, 2011

Duration: 2 and 1/2 hours

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA

Exam Type: A

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SURNAME: (as on your T-card) ______________________

YOUR FULL NAME: ______________________

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

Calculators Permitted: Casio 260, Sharp 520 or TI 30.

INSTRUCTIONS: Attempt all questions. Present your solutions in the space provided. Use the backs of the sheets if you need more space. Do not tear any pages from this exam. Make sure your exam contains 10 pages.

MARKS: Question 1 is worth 24 marks; 6 marks for each part.

Question 2 is worth 10 marks; 2 marks for each part.

Questions 3, 4 and 5 are each worth 10 marks.

Questions 6, 7 and 8 are each worth 12 marks.

TOTAL MARKS: 100

QUESTION	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
TOTAL	

1. Find the following:

(a) the values of a for which the matrix $A = \begin{bmatrix} 1 & a & 0 \\ 2 & 0 & a \\ a & -1 & 1 \end{bmatrix}$ is not invertible.

(b) the minimum distance between the skew lines $\mathbb{L}_1$ and $\mathbb{L}_2$

$$\mathbb{L}_1 : \begin{cases} x = 1 + t \\ y = 0 - t \\ z = 1 + 3t \end{cases} ; \quad \mathbb{L}_2 : \begin{cases} x = 2 - s \\ y = 3 - s \\ z = 1 + s \end{cases}$$

where s and t are parameters.

(c) a basis for $U^{\perp}$ if

$$U = \text{span}\left\{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 2 & -3 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T\right\}.$$

(d) the scalar equation of the plane passing through the three points

$$P(-2,3,5),\ Q(2,2,1),\ R(2,0,0).$$

2. Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) An $n \times n$ matrix A is not invertible if and only if $\text{col}(A)$ contains the zero vector.

True or **False**

(b) $\dim\left(\text{im}\begin{bmatrix} 5 & 1 & -1 & 2 \\ 3 & 6 & 1 & 1 \\ 2 & -5 & -2 & 1 \end{bmatrix}\right) = 2$

True or **False**

(c) If U is a subspace of $\mathbb{R}^6$ and $\dim U = 2$ then $\dim U^{\perp} = 4$.

True or **False**

(d) If $\lambda \neq 0$ is an eigenvalue of A, then $\dfrac{\det(A)}{\lambda}$ is an eigenvalue of $\text{adj}(A)$.

True or **False**

(e) If the eigenvalues of A are all real, then A is symmetric.

True or **False**

3. Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ -1 & 4 & 3 & -1 & -1 \\ 1 & -1 & 0 & 1 & 2 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

4. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x - 4y \\ 2x + y \end{bmatrix}.$$

(a) [5 marks] Draw the image under T of the unit square, and calculate its area.

(b) [5 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

5. Let $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) [5 marks] Find the eigenvalues of R.

(b) [5 marks] Show geometrically that R has eigenvalues in $\mathbb{R}$ and eigenvectors in $\mathbb{R}^2$ only if θ is an integral multiple of π. What are the eigenvalues?

6. Let $U = \text{span}\left\{[\ 0\ \ 1\ \ -1\ \ 0\ ]^T, [\ 2\ \ 0\ \ 0\ \ -1\ ]^T, [\ 1\ \ 1\ \ 0\ \ -1\ ]^T\right\}$;

let $X = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$. Find:

(a) [6 marks] an orthogonal basis of U.

(b) [6 marks] $\text{proj}_U(X)$.

7. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T A P$, if

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

8. Let $X_1 = [\ 1 \ \ 3 \ \ 1 \ \ 0\ ]^T, X_2 = [\ 2 \ \ 4 \ \ 1 \ \ -1\ ]^T, X_3 = [\ 1 \ \ 5 \ \ 0 \ \ 2\ ]^T$.

(a) [6 marks] Show that $S = \{X_1, X_2, X_3\}$ is linearly independent.

(b) [6 marks] Show that $U = \{X \in \mathbb{R}^4 \mid \det[\ X_1 \mid X_2 \mid X_3 \mid X\ ] = 0\}$ is a subspace of $\mathbb{R}^4$ and find a basis for U.