Faculty of Applied Science & Engineering, University of Toronto

MAT188H1F - Linear Algebra

Fall 2016

Tutorial Problems 4

1. Given a system of the linear equations, the last column of the augmented matrix of the system is called the **constant matrix** of the system, and the matrix obtained from the augmented matrix by deleting the last column is called the **coefficient matrix** of the system (cf. page 69 of the textbook). We say the coefficient matrix is **nonsingular** if there exists a unique solution to to the system, regardless of the value of the constant matrix.

(a) Is the coefficient matrix of the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= a \\ x_1 + 3x_2 + 4x_3 &= b \\ 2x_1 + 4x_2 + x_3 &= c \end{aligned}$$

nonsingular? Explain your answer.

(b) Suppose a system of three equations in three unknowns has a nonsingular coefficient matrix. What would be the rank of the coefficient matrix? Describe the reduced row-echelon form of the coefficient matrix as accurately as possible.

(c) For what value(s) of c is the coefficient matrix of the system

$$\begin{aligned} x_1 + x_2 + cx_3 &= 1 \\ x_1 + cx_2 + cx_3 &= 1 \\ cx_1 + x_2 + x_3 &= 1 \end{aligned}$$

nonsingular? Find the solution in terms of c.

(d) Can a system of two equations in three unknowns have a nonsingular coefficient matrix? Explain your answer in terms of the reduced row-echelon form of the coefficient matrix.

(e) Can a system of four equations in three unknowns have a nonsingular coefficient matrix? Explain your answer in terms of the reduced row-echelon form of the coefficient matrix.

(f) Can a system of m equations in n unknowns where $m \neq n$ have a nonsingular coefficient matrix? Explain your answer in terms of the reduced row-echelon form of the coefficient matrix.

Solution:

(a) The augmented matrix, and its row-echelon form (check!), are

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 3 & 4 & b \\ 2 & 4 & 1 & c \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 3 & b-a \\ 0 & 0 & 1 & 2a-c \end{array}\right]$$

Since every variable is a leading variable, there is a unique solution for any choice of a, b, c. This means that the coefficient matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ is nonsingular.

(b) Consider the reduced row-echelon form of such a system. Since there is a unique solution, there must be exactly 3 leading variables, all appearing in the (row-reduced) coefficient matrix.

We took the augmented matrix to this form by elementary row operations. If we look only at the coefficient matrix, we can perform the same row operations to it (i.e. forget about the constant matrix). The result will be the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the RREF for the coefficient matrix, and its rank is 3.

(c) Perform row operations to the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 1 & c & c & 1 \\ c & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2^{new}=R_2-R1} \left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 0 & c-1 & 0 & 0 \\ c & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3^{new}=R_3-cR1} \left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 0 & c-1 & 0 & 0 \\ 0 & 1-c & 1-c^2 & 1-c \end{array}\right]$$

$$\xrightarrow{R_3^{new}=R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & c & 1 \\ 0 & c-1 & 0 & 0 \\ 0 & 0 & 1-c^2 & 1-c \end{array}\right]$$

So, if $c \neq 1, -1$ there is a unique solution for *any* choice of constant matrix (check this!). So the coefficient matrix is nonsingular if and only if $c \neq 1, -1$.

(d) For a system of two equations in three unknowns, the coefficient matrix is 2 by 3. Such a matrix has rank 1 or 2, and its RREF looks roughly like (note: the leading ones can appear in other columns)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

If the system is consistent then there is at least one free variable, and so there are infinitely many solutions. If the system is inconsistent (this can only happen in the rank 1 case), then there are no solutions. Overall, there is no way for us to have a unique solution for all constant matrices: the coefficient matrix is singular.

(e) For a system of four equations in three unknowns, the coefficient matrix is 4 by 3. This implies that the coefficient matrix has rank 1, 2 or 3, and looks roughly like

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We can deal with the first two cases using the same argument as in part **(d)**. For the rank 3 case, we will argue that for *some* choice of constant matrix the system is inconsistent: we can arrange so that the reduced row-echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Namely, to put the coefficient matrix into its reduced row-echelon form, we performed a series of elementary row operations. Elementary row operations are *invertible*: they can be undone. Now, apply these inverse operations to the augmented matrix written above. The result is the augmented matrix for a system as desired.

(f) No, the coefficient matrix can be nonsingular only if $m = n$. If we have $m < n$, then the proof from part **(d)** generalizes to this case. If $m > n$, then we can generalize the proof from part **(e)**.

2. Consider the systems

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 &= d_3 \end{aligned} \qquad \begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 &= d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= d_2 \\ a_3x_1 + b_3x_2 + c_3x_3 &= d_3 + 1 \end{aligned}$$

The system on the left, we'll call system (i); the one on the right, system (ii). Decide if the following statements are true or false:

(a) If (i) has a unique solution, then so does (ii).

(b) If the solution set of (i) is a line, then the same is true for (ii).

(c) If (i) is inconsistent, then so is (ii).

Solution:

(a) True. By question 1(b), (i) has a unique solution iff rank $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 3$ iff (ii) has a unique solution.

(b) False. A counter-example is:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 2x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 2x_3 &= 1 \end{aligned}$$

Then (i) has the line $t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as its solution set, but (ii) has no solution.

(c) False. A counter-example is:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 2x_3 &= -1 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + 2x_3 &= 0 \end{aligned}$$

Then (i) has no solution but (ii) does.

3 (a) Let $\{\mathbf{x}, \mathbf{y}\}$ be linearly independent set of vectors in $\mathbb{R}^n$. Let $\mathbf{u} = 3\mathbf{x} - 2\mathbf{y}$, $\mathbf{v} = \mathbf{x} + \mathbf{y}$. Show that the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent in $\mathbb{R}^n$.

3 (b) Let $\{\mathbf{x}, \mathbf{y}\}$ be linearly independent set of vectors in $\mathbb{R}^n$. Let $\mathbf{u} = a\mathbf{x} + b\mathbf{y}$, $\mathbf{v} = c\mathbf{x} + d\mathbf{y}$. Show that the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent in $\mathbb{R}^n$ if and only if the set $\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right\}$ is linearly independent in $\mathbb{R}^2$.

3 (c) Create an exercise similar to part (a), using different coefficients for $\mathbf{x}$ and $\mathbf{y}$, so that the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent.

Solution:

3 (a) We have to show that if

$$\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = \mathbf{0}$$

then $\lambda_1 = \lambda_2 = 0$.

Now, $\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = \mathbf{0}$ is equivalent to $\lambda_1(3\mathbf{x} - 2\mathbf{y}) + \lambda_2(\mathbf{x} + \mathbf{y}) = \mathbf{0}$ which is equivalent to $(3\lambda_1 + \lambda_2)\mathbf{x} + (-2\lambda_1 + \lambda_2)\mathbf{y} = \mathbf{0}$. And since $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, we have that

$$\begin{aligned} 3\lambda_1 + \lambda_2 &= 0 \\ -2\lambda_1 + \lambda_2 &= 0 \end{aligned}$$

which has only the trivial solution $\lambda_1 = \lambda_2 = 0$, as required.

3 (b) First observe that

$$\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = \lambda_1(a\mathbf{x} + b\mathbf{y}) + \lambda_2(c\mathbf{x} + d\mathbf{y}) = (\lambda_1 a + \lambda_2 c)\mathbf{x} + (\lambda_1 b + \lambda_2 d)\mathbf{y}. \qquad (*)$$

Now, $\{\mathbf{u}, \mathbf{v}\}$ is linearly **dependent** in $\mathbb{R}^n$ if and only if we have

$$\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = \mathbf{0}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$ not both zero. Using $(*)$, this condition is **equivalent** to having

$$(\lambda_1 a + \lambda_2 c)\mathbf{x} + (\lambda_1 b + \lambda_2 d)\mathbf{y} = \mathbf{0}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$ not both zero. And using the fact that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, this last condition is **equivalent** to having

$$\begin{aligned} \lambda_1 a + \lambda_2 c &= 0 \\ \lambda_1 b + \lambda_2 d &= 0 \end{aligned}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$ not both zero. We can rewrite these last two equations as

$$\lambda_1 \begin{bmatrix} a \\ b \end{bmatrix} + \lambda_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In summary, what we've observed is that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent in $\mathbb{R}^n$ if and only if

$$\lambda_1 \begin{bmatrix} a \\ b \end{bmatrix} + \lambda_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$ not both zero. This last bit is the same as saying $\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\}$ is linearly dependent in $\mathbb{R}^2$.

Thus, we've proved that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent in $\mathbb{R}^n$ if and only if $\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\}$ is linearly dependent in $\mathbb{R}^2$. Therefore, $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent in $\mathbb{R}^n$ if and only if $\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\}$ is linearly independent in $\mathbb{R}^2$, as desired.

3 (c) Many choices here. For example, take $\mathbf{u} = \mathbf{x} - 2\mathbf{y}$, $\mathbf{v} = 2\mathbf{x} - 4\mathbf{y}$.

4. Let $S = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} \right\}$.

(a) Is $\mathbb{R}^3 = \operatorname{span}(S)$? If not, find a vector in $\mathbb{R}^3$ that cannot be written as a linear combination of the vectors in S.

(b) Let S be the set in part (a). Is the set of solutions to the homogeneous system $11x_1 + 5x_2 + 2x_3 = 0$ equal to the span of S?

Solution:

(a) We check if any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ lies in $\operatorname{span}(S)$ which amounts to checking that the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x \\ -3 & -4 & -5 & y \\ 2 & -1 & 7 & z \end{array}\right]$$

is consistent for choices of x, y, z. We row reduce:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & x \\ -3 & -4 & -5 & y \\ 2 & -1 & 7 & z \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 0 & 2 & -2 & y+3x \\ 0 & -5 & 5 & z-2x \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & x \\ 0 & 2 & -2 & y+3x \\ 0 & 0 & 0 & 2z+11x+5y \end{array}\right].$$

The system is inconsistent if $2z + 11x + 5y \neq 0$ so $\operatorname{span}(S) \neq \mathbb{R}^3$. For instance, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in $\operatorname{span}(S)$.

(b) Yes they are the same. In part (a) we saw that the augmented matrix was consistent if and only if $2z + 11x + 5y = 0$ which is the equation in question.