MAT187 - Calculus II - Winter 2019

Term Test 1 - February 5, 2019

Time allotted: 70 minutes. Aids permitted: None.

Total marks: 50

Instructions:

- Do not write on the QR code at the top of the pages.
- This test contains 12 pages and a detached booklet for multiple-choice questions and formula sheet. DO NOT DETACH ANY PAGES.
- You can use pages 10–11 to complete questions (**mark clearly** which questions you are answering).
- Calculators, cellphones, or any other electronic devices are not allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.
- DO NOT START the test until instructed to do so.

GOOD LUCK!

LONG ANSWER PART

7. Consider the integral **(13 marks)**

$$\int_0^{\sqrt{\frac{\pi}{2}}} \sin(x^2)\, dx.$$

This integral can't be calculated using elementary functions, so we need to approximate it.

(a) (5 marks) If you want to make sure your approximation is less than the actual integral (that it underestimates the integral), which method would you use to approximate this integral? Justify your answer.

Hint. What is the sign of the derivative of the integrand?

(b) **(5 marks)** Your boss doesn't want to know about your reasoning for **(a)** and she decides that you must use the Right-hand rule. She also wants the error to be smaller than $\frac{1}{100}$. How many intervals should you use? Justify your answer.

Hint 1. There is only one solution of $2z^2\tan(z^2) = 1$ in $\left[0, \sqrt{\frac{\pi}{2}}\right]$, whose values you can assume are $z_1 = 0.8$ and $\cos(z_1^2) = 0.8$.

OR

Hint 2. Note that $\sqrt{\frac{\pi}{2}} \leqslant 1.3$.

(c) **(3 marks)** Consider a function $f(x)$ which is continuous in the interval $[0, 4]$ satisfying $f(0) = f(4) + 8$ and consider the integral $\displaystyle\int_0^4 f(x)\,dx$.

Calculate $L_{128} - R_{128}$.

8. It's the 1960's and NASA is trying to find out how much work it will take to get a spaceship to escape the gravitational pull of Earth. This means that the spaceship should be able to keep going forever, getting further and further away from Earth. **(13 marks)**

The formula for work is $W = F\,d$, where F is the force acting on the object and d is the displacement. The gravitational pull of Earth has magnitude $\dfrac{GMm}{r^2}$, where G is the universal gravitational constant, M is the mass of the Earth, m is the mass of the spaceship, and r is the distance from the centre of the Earth. You can denote the radius of the Earth as R.

(a) **(8 marks)** Find an integral formula for the work it takes to bring a spaceship from the surface of the Earth to a distance h from the surface.

Explain every step and define all your variables and constants.

(You can continue your answer to **(a)** on the next page)

(b) **(1 mark)** Find an integral formula for the work it takes for a spaceship starting from the surface of the Earth to go on forever, getting further and further away from Earth. Justify your answer.

(c) **(4 marks)** Is the amount of work you found in **(b)** finite or infinite? Justify your answer.

9. Consider two functions $f(t)$ and $g(t)$ that satisfy the properties: **(12 marks)**

(P_1) $f^2(t) + g^2(t) = 1$ for all t;

(P_2) $f'(t) = g(t)$ for all t;

(P_3) $g'(t) = \dfrac{1}{tg(t)}$ for all $t \neq 0$.

We want to study the integral $\displaystyle\int \sqrt{1-x^2}\,dx$.

(a) (4 marks) As a first step, use the substitution $x = f(t)$ to show that

$$\int \sqrt{1-x^2}\,dx = \int g^2(t)\,dt.$$

Hint. Use properties (P_1) and (P_2).

(b) (4 marks) Now, integrate by parts to calculate the integral $\int \sqrt{1-x^2}\,dx$.

(c) (4 marks) This question is independent of the others.

Find a function $g(t)$ that satisfies property (P_3). Ignore the properties (P_1) and (P_2).

USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

If you wish to have this page marked, make sure to refer to it in your original solution.

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