UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2015

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

EXAMINERS: D. BURBULLA, S. COHEN, K. LEUNG, S. LIU,
Y. LOIZIDES, F. PARSCH, B. SCHACHTER

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Full Name: ______________________________

UTor email: ______________________________ @mail.utoronto.ca

Signature: ______________________________

Instructions:

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 12 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of eight questions, some with many parts. Attempt all of them. Each question is worth 10 marks. Marks for parts of a question are indicated in the question. **Total Marks: 80**
- PRESENT YOUR SOLUTIONS IN THE SPACE PROVIDED. You can use pages 10, 11 and 12 for rough work. If you want anything on pages 10, 11 or 12 to be marked you must indicate in the relevant previous question that the solution continues on page 10, 11 or 12.

1. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}$. Show $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal set, and write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

2. Let $A = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbf{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue.

3. An $n \times n$ matrix A is called **idempotent** if $A^2 = A$.

   (a) [2 marks] Show that the only invertible idempotent matrix is the identity matrix.

   (b) [2 marks] Show that if A is idempotent then so is $I - A$, where I is the $n \times n$ identity matrix.

   (c) [3 marks] Let $\mathbf{v}$ be an eigenvector of an idempotent matrix A, with corresponding eigenvalue λ. Show that $\lambda = 0$ or $\lambda = 1$.

   (d) [3 marks] Show that if A is an idempotent matrix then it is diagonalizable. Hint: what can you say about the dimension of the eigenspaces of A?

4. Find *all* linear transformations $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that

$$\ker(T) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ and } \operatorname{range}(T) = \operatorname{span}\left\{ \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \right\}.$$

Continued...

5. Find the solution to the system of linear differential equations $\begin{cases} y_1' = y_1 + 3y_2 \\ y_2' = 2y_1 + 2y_2 \end{cases}$, where y_1, y_2 are functions of t, and $y_1(0) = 0,\ y_2(0) = 5$.

Continued...

6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$.

Continued...

7. Let $S = \operatorname{span}\left\{[\begin{matrix} 1 & -1 & 0 & 0 \end{matrix}]^T, [\begin{matrix} 1 & 0 & 1 & 0 \end{matrix}]^T, [\begin{matrix} 0 & 1 & 1 & 1 \end{matrix}]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 2 & 0 & 3 & 1 \end{bmatrix}^T$. Find $\operatorname{proj}_S(\mathbf{x})$.

8. Let $A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$.

(a) [4 marks] Show that if $a + b + c = 0$, then A is not invertible.

(b) [6 marks] Show that if $a+b+c = \pm 1$ and $a^2+b^2+c^2 = 1$, then A is orthogonal. Hint: $(\pm 1)^2 = 1$.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

Continued...

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.