University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER, 2010

Duration: 2 and 1/2 hours

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA

Exam Type: A

Examiners:
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SURNAME: (as on your T-card) ____________________

YOUR FULL NAME: ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

Calculators Permitted: Casio 260, Sharp 520 or TI 30.

INSTRUCTIONS: Attempt all questions. Present your solutions in the space provided. Use the backs of the sheets if you need more space. Do not tear any pages from this exam. Make sure your exam contains 10 pages.

MARKS: Question 1 is worth 24 marks; 6 marks for each part.

Question 2 is worth 10 marks; 2 marks for each part.

Questions 3, 4 and 5 are each worth 10 marks.

Questions 6, 7 and 8 are each worth 12 marks.

TOTAL MARKS: 100

QUESTION	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
TOTAL	

1. Find the following:

   (a) the characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   (b) all values of a for which the following system of equations has

$$\begin{cases} x + y + z = 1 \\ -x + ay = 3 \\ 6y + az = 8 \end{cases}$$

   infinitely many solutions.

(c) a linear combination of the three vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ equal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(d) a basis for $U^{\perp}$ if $U = \text{span}\left\{[\begin{array}{ccccc} 1 & 0 & 1 & 2 & 1 \end{array}]^T, [\begin{array}{ccccc} 1 & 0 & 2 & 1 & 1 \end{array}]^T, [\begin{array}{ccccc} 1 & 1 & 2 & 1 & 1 \end{array}]^T\right\}$.

2. Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) $\dim\left(\operatorname{im}\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 3 & 3 & 0 \\ 3 & 1 & 7 & 3 \\ 2 & 1 & 5 & 2 \end{bmatrix}\right) = 2$ **True** or **False**

(b) If U is a subspace of $\mathbb{R}^5$ and $\dim U = 2$ then $\dim U^{\perp} = 2$. **True** or **False**

(c) If A is a diagonalizable matrix then so is A^T. **True** or **False**

(d) If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 6x - 5y \\ 3x - 4y \end{bmatrix}$ then the area of the image of the unit square is 9.

True or **False**

(e) If T_1 is a rotation of 180° and T_2 is a reflection in the line $y = 2x$, then $T_1 \circ T_2$ is a projection on the line $y = -2x$. **True** or **False**

3. Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & -1 \\ 2 & -1 & 4 & 3 & 0 \\ 4 & 1 & 10 & 9 & -2 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

4. Find the least squares approximating line for the data points

$$(-2,0),(0,1),(1,1),(1,2),(2,3).$$

5. Find the general solution to the following system of differential equations:

$$\begin{aligned} f_1'(x) &= & & & & f_3(x) \\ f_2'(x) &= & f_1(x) & - & f_2(x) & + f_3(x) \\ f_3'(x) &= & 4f_1(x) & & & \end{aligned}$$

6. Let $U = \text{span}\left\{\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T, \begin{bmatrix} 2 & 0 & -1 & -1 \end{bmatrix}^T\right\}$;

   let $X = \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T$. Find:

   (a) [6 marks] an orthogonal basis of U.

   (b) [6 marks] $\text{proj}_U(X)$.

7. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

8.(a) [3 marks] Let $A = \begin{bmatrix} 1 \\ m \end{bmatrix}$. Show that $P = A\,(A^T A)^{-1} A^T$ is the matrix of a projection.

8.(b) [9 marks] Let U be a subspace of $\mathbb{R}^n$ with basis $X_1, X_2, \ldots, X_k$; let A be the $n \times k$ matrix with columns $X_1, X_2, \ldots, X_k$. Let $P_A = A\,(A^T A)^{-1} A^T$.

(i) [3 marks] Show each of the following:

$P_A^2 = P_A$

$P_A^T = P_A$

$P_A\, A = A$

(ii) [6 marks] Show that $\operatorname{im}(P_A) = U$ and $\operatorname{null}(P_A) = U^{\perp}$.