MAT186 Calculus I
Term Test 2

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Instructions:

1. This test contains a total of 12 pages, and a total of 39 marks available.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this test, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. $\lim_{x\to 1} \frac{(x-1)-\ln x}{(x-1)^2} =$ _____? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ 0
- ● $\frac{1}{2}$
- ○ 1
- ○ DNE ($\to \infty$)
- ○ DNE ($\to -\infty$)

We observe that the limit has the indeterminate form $\frac{0}{0}$ and apply L'Hôpital's Rule. The result,

$$\lim_{x\to 1} \frac{1-\frac{1}{x}}{2(x-1)}$$

still has the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's Rule a second time and find that

$$\lim_{x\to 1} \frac{\frac{1}{x^2}}{2} = \frac{1}{2}$$

2. Let $f(x) = 10^{\log_2(x)}$. Which of the limits below represents $f'(4)$? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $\lim_{h\to 0} \frac{100(10^{\log_2(h)}) - 100}{h}$.
- ○ $\lim_{h\to 0} \frac{100^{\log_2(h)} - 100}{h}$.
- ○ $\lim_{h\to 0} \frac{10^{\log_2(4h)} - 100}{h}$.
- ● $\lim_{h\to 0} \frac{10^{\log_2(4+h)} - 100}{h}$.

By definition,

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\ &= \lim_{h\to 0} \frac{10^{\log_2(4+h)} - 10^{\log_2(4)}}{h} \end{aligned}$$

Note that $10^{\log_2(4)} = 10^2$.

Multiple Choice: No justification is required. Only your final answer will be graded.

Use the following to answer Question 3 and Question 4.

The table below gives the values for two differentiable functions $f(x)$ and $g(x)$, along with their derivatives $f'(x)$ and $g'(x)$ at the points $x = -2, -1, 0, 1, 2$.

x	-2	-1	0	1	2
$f(x)$	-5	-3	-2	-1	2
$f'(x)$	2	3	0	3	-4
$g(x)$	6	4	0	-1	-2
$g'(x)$	-7	-5	-3	-1	-1

3. Let $p(x) = f(x)g(x)$. Then $p'(2) =$ _____? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ -4.
- ○ -2.
- ○ 4.
- ● 6.
- ○ 8.

Using the product rule,

$$p'(2) = f'(2)g(2) + f(2)g'(2) = (-4)(-2) + (2)(-1) = 6.$$

4. Let $h(x) = f(g(x))$. Then $h'(2) =$ _____? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ -4.
- ○ -3.
- ● -2.
- ○ 2.
- ○ 4.

Using the chian rule,

$$h'(2) = f'(g(2))g'(2) = f'(-2)g'(2) = (2)(-1) = -2.$$

Multiple Choice: No justification is required. Only your final answer will be graded.

5. It is generally understood that, given a sound source (e.g. a stereo speaker), each time we double of the number of speakers, the sound intensity increases by 3 decibels (dB).

Let $f(x)$ be the number of speakers, and let x be the sound intensity (in dB). Which of the following statements is the best interpretation of $(f^{-1})'(15) = 0.2$? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ If we have 15 speakers, then adding another speaker will increase the sound intensity by 0.2 dB.

○ The sound intensity with 15 speakers is approximately 0.2 times the sound intensity with 16 speakers.

○ With 15 speakers, the rate at which the sound intensity is increasing is 0.2 speakers per dB.

● The sound intensity with 13 speakers is approximately 0.4 dB less than the sound intensity with 15 speakers.

○ If we increase the number of speakers from 15 to 16, then the sound intensity would be approximately 0.2 dB greater than the sound intensity with one speaker.

The first option is incorrect as the sound intensity will increase by *approximately* 0.2 dB, not *exactly* 0.2 dB as stated. The second option expresses the derivative as a multiplicative factor which is incorrect, as well as reversing the roles of 15 and 16 speakers in our interpretation. The third option assigns incorrect units to the derivative of the inverse (should be dB/speaker). The final option incorrectly references the relationship between $(f^{-1})'(15), (f^{-1})'(16)$ and the sound intensity with one speaker. Therefore, the correct choice is the fourth statement. This statement correctly expresses the approximate linear relationship between the loudness at 15 speakers, and a nearby point (13 speakers).

6. Suppose that $f(x)$ is differentiable everywhere, and that $f'(x) > 0$ for all x. Which of the statements below are true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

○ $(f^{-1})'(x) = \dfrac{1}{f(x)}$.

● $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

● $(f^{-1})'(x) > 0$.

● $(f^{-1})'(f(x)) > 0$.

● If $f'(x)$ is increasing, then $(f^{-1})'(f(x))$ is decreasing.

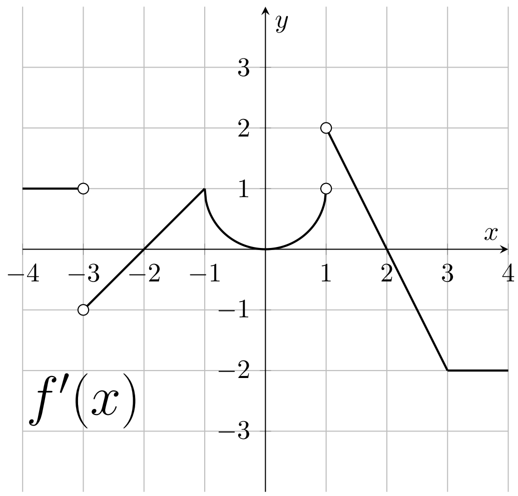
We are given $f(x)$ is differentiable and striclty increasing (invertible) over its whole domain. Therefore, $f(f^{-1}(x)) = x$ for every x, and by chain rule

$$f'(f^{-1}(x))(f^{-1})'(x) = 1$$

which rearranges to the second option. The remaining three options follow from this expression.

Multiple Choice: No justification is required. Only your final answer will be graded.

7. Suppose that $f(x)$ is a continuous function on $[-4, 4]$. Below is the graph of $y = f'(x)$.



Use this graph to answer parts (a)-(e) below. For each part, fill in all the options that apply (unfilled ○ filled ●).

(a) $f(x)$ has a critical point at $x =$ _____? [1 mark]

● -3. ● -2. ○ -1. ● 0. ● 1. ● 2. ○ 3.

(b) $f'(x)$ has a critical point at $x =$ _____? [1 mark]

○ -3. ○ -2. ● -1. ● 0. ○ 1. ○ 2. ● 3.

(c) $f(x)$ has a local minimum at $x =$ _____? [1 mark]

○ -3. ● -2. ○ -1. ○ 0. ○ 1. ○ 2. ○ 3.

(d) $f(x)$ has a local maximum at $x =$ _____? [1 mark]

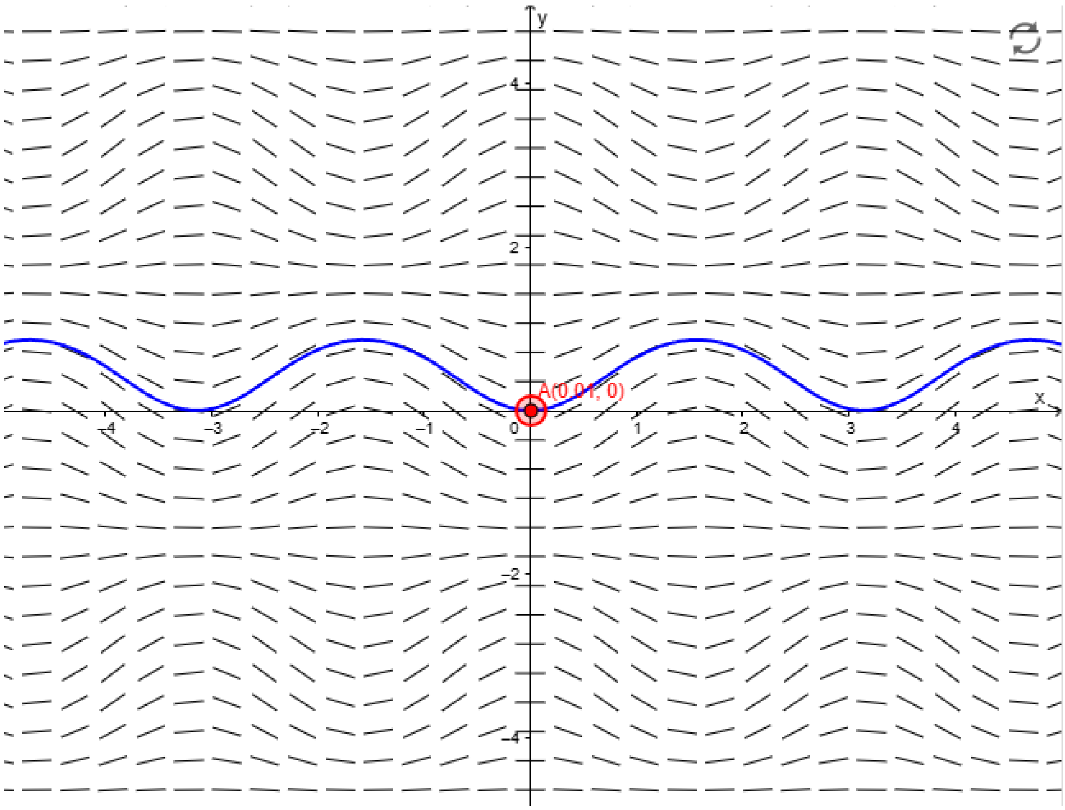
● -3. ○ -2. ○ -1. ○ 0. ○ 1. ● 2. ○ 3.

(e) $f(x)$ has a point of inflection at $x =$ _____? [1 mark]

○ -3. ○ -2. ● -1. ● 0. ● 1. ○ 2. ○ 3.

Graphing, Multiple Choice: No justification is required. Only your final answer will be graded.

The graph of a slope field for a differential equation $\frac{dy}{dx} = f(x, y)$ is pictured below. Use this slope field to answer Question 8 and Question 9.



8. On the slope field, carefully sketch the solution for $x \in (-5, 5)$ that passes through $(0, 0)$. [2 marks]

See above.

9. The given slope field is the slope field of which differential equation below? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ $\frac{dy}{dx} = \cos y \cos 2x$.

● $\frac{dy}{dx} = \cos y \sin 2x$.

○ $\frac{dy}{dx} = \sin y \sin 2x$.

○ $\frac{dy}{dx} = \sin y \cos 2x$.

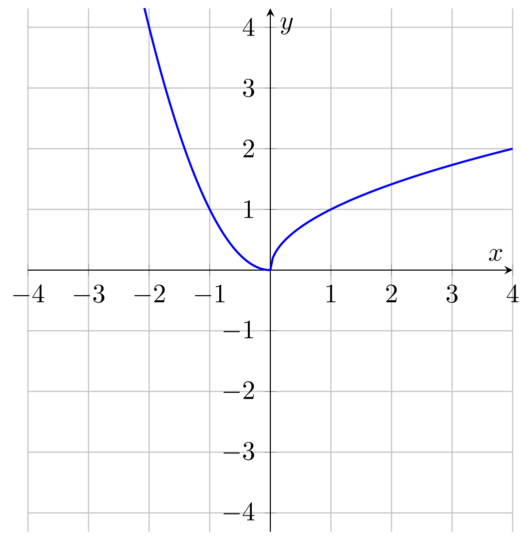
Observing the slope field, we note that $dy/dx = 0$ at $y \approx \pi/2$. In addition, we observe that $dy/dx = 0$ at $x = \pm\pi$, for all y. The only option which satisfies these criteria is

$$\frac{dy}{dx} = \cos y \sin 2x.$$

Graphing: No justification is required. Only your final answer will be graded.

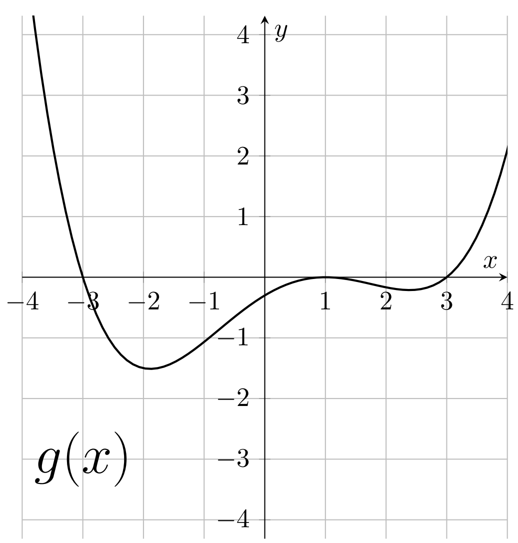
10. Carefully sketch the graph of a function $f(x)$ on the axes below that satisfies all the given properties. [2.5 marks]

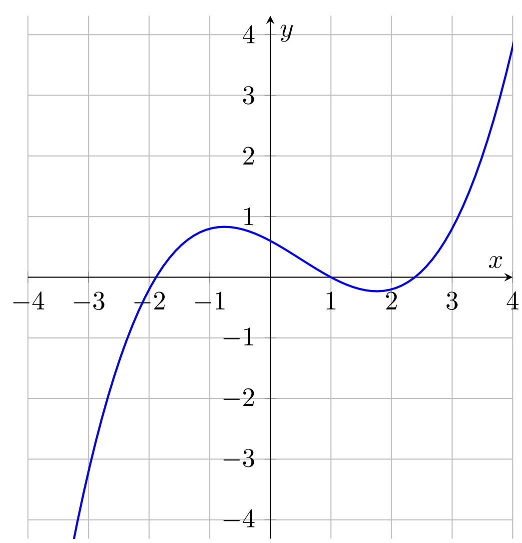
- $f(x)$ is continuous on $(-4, 4)$.
- $f'(x)$ is negative and increasing on $(-4, 0)$.
- $f'(x)$ is positive and decreasing on $(0, 4)$.
- $\lim_{h\to 0^-} \dfrac{f(h)-f(0)}{h} = 0$.
- $\lim_{h\to 0^+} \dfrac{f(h)-f(0)}{h} \to \infty$.



To satisfy these conditions, for $x < 0$ we choose a function that is decreasing ($f'(x)$ is negative) and concave up ($f'(x)$ is increasing). Conversely, for $x > 0$ we choose a function which is increasing ($f'(x)$ is positive) and concave down ($f'(x)$ is decreasing). In addition, we require that the derivative of the function as it approaches $x = 0$ from the left is 0, while the derivative from the right approaches ∞ (i.e., tends towards a vertical line). Finally, both sides of the function must meet at the same y-value when $x = 0$ to maintain continuity.

11. A function $g(x)$ on $(-4, 4)$ is pictured below on the left. Carefully sketch the graph of $g'(x)$ on $(-4, 4)$ on the given axes on the right. [2.5 marks]



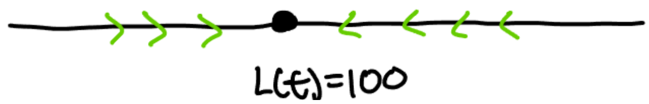


Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

12. Consider the differential equation $\dfrac{dL}{dt} = \dfrac{1}{2}(100 - L)$.

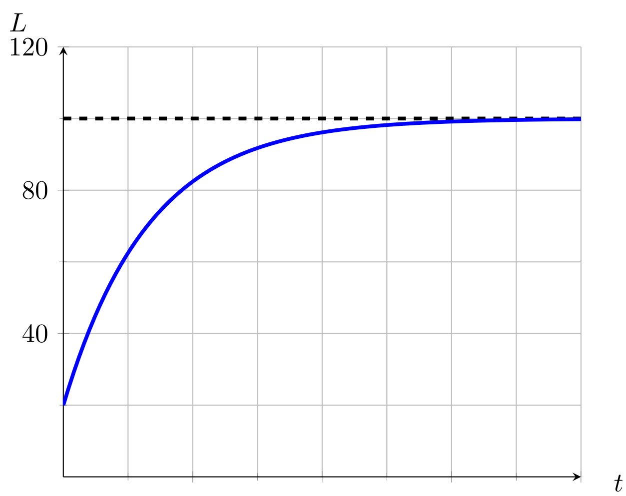
(a) Sketch the phase line for this differential equation and classify the equilibrium point(s) on your sketch. [2 marks]

Solving $\frac{dL}{dt} = 0$, we find a single equilibrium point at $L(t) = 100$. Since $dL/dt > 0$ for $L < 100$, and $dL/dt < 0$ for $L > 100$, the equilibrium is a stable equilibrium.



(b) The given differential equation models the function $L(t)$, the percentage of a particular task that has been learned after t months. Assume that 20% had been learned at $t = 0$.

(i) On the axes below, carefully sketch the solution to this initial value problem, clearly illustrating the long-term behaviour of the solution. Justify any intervals of concavity in your sketch in the blank space to the right of the given axes. [2.5 marks]



One way to determine the concavity of the solution is to differentiate both sides of the differential equation:

$$\frac{d}{dt}\frac{dL}{dt} = -\frac{1}{2}\frac{dL}{dt}$$
$$\frac{d^2L}{dt^2} = -\frac{1}{4}(100 - L)$$

When $L(t) < 100$, we see that $d^2L/dt^2 < 0$, which implies that the solution is concave down. Another approach would be to sketch the graph of L' vs. L and observe that L' is positive and decreasing.

(ii) Use Euler's method with step size $\Delta t = 0.5$ to estimate the percentage learned after one month. Show all your work. [2.5 marks]

We have, $\tilde{L}(0.5) = L'(0)(0.5) + L(0) = 40$, and $\tilde{L}(1) = \tilde{L}'(0.5)(0.5) + \tilde{L}(0.5) = 55$. Therefore, the percentage learned after one month is approximately 55%.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

13. The curve given by the equation

$$y^2 - 1 = 1 + x^2(y-1)$$

defines y implicitly as a function of x.

(a) Find $\dfrac{dy}{dx}$. [2 marks]

Implicitly differentiating both sides of the equation with respect to x using the product rule and chain rule gives:

$$2y\frac{dy}{dx} = 2x(y-1) + x^2\frac{dy}{dx}$$

After re-arranging, we find $\frac{dy}{dx} = \frac{2x(y-1)}{2y-x^2}$.

(b) Find the equation of the tangent line $T(x)$ to the curve at $(-\sqrt{2}, 0)$. [2 marks]

To find the equation of the tangent line, we consider a linear approximation to the relation at the point $x = a$, $T(x) = f'(a)(x-a) + f(a)$. Substituting $a = -\sqrt{2}$ and $y = f(a) = 0$, and using our result for dy/dx above, we have:

$$T(x) = \frac{2(-\sqrt{2})(0-1)}{2(0) - (-\sqrt{2})^2}(x+\sqrt{2}) = -\sqrt{2}(x+\sqrt{2})$$

$$T(x) = -\sqrt{2}(x+\sqrt{2})$$

(c) Determine all points (x, y) where the tangent line to the curve is horizontal. [2 marks]

We determine the points on the curve for which $\frac{dy}{dx} = 0$. Since

$$\frac{dy}{dx} = \frac{2x(y-1)}{2y-x^2},$$

$\frac{dy}{dx} = 0$ when $2x(y-1) = 0$ and $2y - x^2 \neq 0$. This occurs for $x = 0, y \neq 0$ and $y = 1, x \neq \pm\sqrt{2}$. To find the actual points (x, y), we use the original relation. For $x = 0$,

$$\begin{aligned} y^2 - 1 &= 1 + (0^2)(y-1) \\ y^2 &= 2 \\ y &= \pm\sqrt{2} \end{aligned}$$

For $y = 1$,

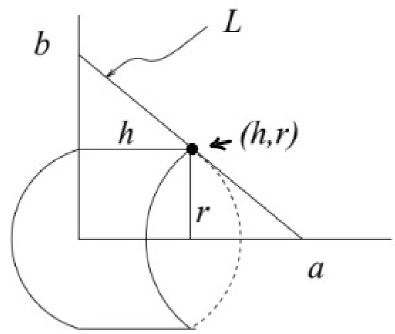
$$\begin{aligned} (1)^2 - 1 &= 1 + x^2((1)-1) \\ 0 &= 1 \end{aligned}$$

we conclude the points $(x, 1)$ are not on the curve.

Therefore, the two points where the tangent line equals zero are $(0, \sqrt{2})$ and $(0, -\sqrt{2})$.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

14. A cylinder of radius r and height h is stored under a slanted roof as pictured below. The point (h, r) lies on the straight line L given by the equation $ay + bx = ab$, where a and b are fixed positive numbers such that $a + b = 1$.



(a) Find an equation that represents h in terms of r and a. [2 marks]

Since (h, r) lies on the line $ay + bx = ab$, we have $h = x = a - \frac{a}{b}y$. Then, using that $a + b = 1$, we have

$$h = a - \frac{a}{1-a}r$$

You can also determine this relation using similar triangles. (Note that a is a constant parameter, and r is a variable...this matters in part (b)).

(b) Use the Closed Interval Method to determine the point (h, r) that maximizes the volume of the cylinder. Your answer may include the letter a. [4 marks]

Hint: The volume of a cylinder of radius r and height h is $V = \pi r^2 h$.

Using part (a) we have,

$$V(r) = \pi r^2 (a - \frac{a}{1-a}r)$$

where a is a constant parameter. .

From the Extreme Value Theorem, we know that $V(r)$ must take on its maximum value on some closed interval at an endpoint, or at some interior point of the interval. From our diagram, we see that the interval of interest for r is between $b = 1 - a$ (corresponding to a cylinder with 0 height), and 0, corresponding to a cylinder with 0 radius. Evaluating V at both of these locations returns $V = 0$, thus we expect our maximum volume to occur at a critical point inside this interval.

Differentiating $V(r)$ on $(0, 1 - a)$, we find

$$V'(r) = \pi \left(2ra - \frac{3r^2 a}{1-a} \right)$$

Solving $V'(r) = 0$, we obtain:

$$r = \frac{2}{3}(1-a)$$

To confirm this is a maximum, we can check the value of $V(r)$.

$$\begin{aligned} V(\frac{2}{3}(1-a)) &= \pi(\frac{2}{3}(1-a))^2(a - \frac{2a}{3(1-a)}(1-a)) \\ &= \frac{4}{27}\pi a(1-a)^2 \end{aligned}$$

For a between 0 and 1, this volume is positive; therefore, this is the maximum volume for the cylinder.

IF NEEDED, USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

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