Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 12

1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

(a) Is A orthogonal?

(b) Is A orthogonally diagonalizable?

(c) Does A satisfy $A^2 - A = I$?

Solution:

Notice that $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

(a) Yes. You may verify $A^T A = I$, where I is the 3×3 identity matrix.

(b) Yes. A is symmetric.

(c) A straightforward check shows that it does not.

2. Find numbers a, b, c, d such that the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & a \\ 1 & -1 & 0 & b \\ 1 & 1 & -\sqrt{2} & c \\ 1 & -1 & 0 & d \end{bmatrix}$$

is orthogonal.

Solution

Recall that A is orthogonal iff the columns of A are orthonormal. So, for the given A to be orthogonal we require $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ to be orthogonal to each of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \\ 0 \end{bmatrix}$, i.e.,

$$\begin{aligned} a + b + c + d &= 0 \\ a - b + c - d &= 0 \\ \sqrt{2}a - \sqrt{2}c &= 0 \end{aligned}$$

and that the length of $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is 2. Solving, we find that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 6 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.
Hint: There are only two eigenvalues of A, $\lambda_1 = 9$, and $\lambda_2 = 2$.

Solution:

Using the hint, we find bases for the two eigenspaces E_2 and E_9. Since A is symmetric, we should have $(E_2)^\perp = E_9$.

You may verify that $E_2 = \text{null} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, and $E_9 = \text{null} \begin{bmatrix} -6 & 1 & 1 & 2 \\ 1 & -6 & 1 & 2 \\ 1 & 1 & -6 & 2 \\ 2 & 2 & 2 & -3 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Notice that $(E_2)^\perp = E_9$ as predicted but the basis vector for E_2 are not orthogonal. Applying the Gram-Schmidt procedure to these vectors yields an orthogonal basis $\left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 1 \\ 2 \end{bmatrix} \right\}$ for E_2. Hence,

$$\left\{ \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 0 \\ -5 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} 1 \\ -6 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^4$ consisting of eigenvectors of A. So, the matrix P with these vectors as its columns is orthogonal and such that

$$\begin{aligned} P^T AP &= \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \\ &= D \end{aligned}$$

4. Let $b \neq 0$ and $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

Solution:
We begin by finding the characteristic polynomial of A to determine the eigenvalues of A:

$C_A(\lambda) = \det(A - \lambda I) = (\lambda - (a - b))(\lambda - (a + b))$

so that the eigenvalues of A are $\lambda = a \pm b$. Next, we find bases for each of the eigenspaces E_{a+b} and E_{a-b}:

$E_{a+b} = \text{null} \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$E_{a-b} = \text{null} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Again, since A is symmetric, we note that $(E_{a+b})^{\perp} = E_{a-b}$ as expected. Hence, $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A. So, the matrix P with these vectors as columns is orthogonal and such that

$$\begin{aligned} P^T A P &= \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} \\ &= D \end{aligned}$$

5. Find a and b to find the best-fitting equation of the form $y = ax + b2^x$ for the set of data points $(0, 1)$, $(1, 3)$, and $(2, 10)$.

Solution:

If the given points were on the curve $y = ax + b2^x$, then the system

$$\begin{aligned} b &= 1 \\ a + 2b &= 3 \\ 2a + 4b &= 10 \end{aligned}$$

would be consistent. It isn't, of course, so we find the least squares solution to the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}$. The least squares solution $\mathbf{x}$ solves the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ which you may verify has solution $\mathbf{x} = \begin{bmatrix} -\frac{13}{5} \\ 1 \end{bmatrix}$. Hence $y = -\frac{13}{5}x + 2^x$.