University of Toronto
Faculty of Applied Sciences and Engineering

MAT188 – Midterm I – Fall 2023

LAST (Family) NAME: ____________________

FIRST (Given) NAME: ____________________

Email address: ____________________@mail.utoronto.ca

STUDENT NUMBER: ____________________

Time: 90 mins.

1. **Keep this booklet closed** until an invigilator announces that the test has started. You may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **in the boxes**. There is ample space within each one. If you must use additional space, please use the blank pages at the end of this booklet and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 16 pages, excluding the cover page, and 6 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work.
7. No textbook, notes, or other outside assistance is allowed.

Question:	1	2	3	4	5	6	Total
Points:	11	9	12	9	8	11	60
Score:							

1 Part A

1. (11 points) Fill in the bubble for all statements that **must** be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles.

   (a) Let $A = (-5, -1, 0)$, $B = (-7, -2, 1)$, $P = (3, 2, 1)$ and $Q = (-1, 0, 3)$.

   Then $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are

   ◯ Parallel and pointing the same direction

   ◯ Parallel and pointing opposite direction

   ◯ Perpendicular

   ◯ None of the above

   (b) Consider a line ℓ which passes through the point $(-3, -5, 4)$ and which is parallel to the line $\vec{x} = t\begin{bmatrix}3\\6\\7\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix}$. At which point does the line ℓ intersect the yz-plane?

   ◯ $(0, 1, 11)$ ◯ $(0, 11, 3)$

   ◯ $(3, 6, 7)$ ◯ $(0, 8, 10)$

   (c) Which one(s) of the following matrices is in reduced row echelon form (RREF)?

   ◯ $\begin{bmatrix}-7 & 0 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$ ◯ $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$

   ◯ $\begin{bmatrix}0 & 1 & 0 & 10\\ 0 & 0 & 1 & 3\end{bmatrix}$ ◯ $\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}$

(d) Suppose $B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ is the RREF of an augmented matrix of a system of linear equations. Then this system has

- ◯ Unique solution: $x = -1$, $y = 0$, $z = 3$.
- ◯ Infinity many solutions. One particular solution is $x = -1, y = -1, z = 3$.
- ◯ Unique solution: $x = -1$, $z = 3$.
- ◯ No solution

(e) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(\vec{x}) = A\vec{x}$, the first column of A is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and the third column of A is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Then $T(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix})$ is

- ◯ $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$
- ◯ $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$
- ◯ $\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$
- ◯ Not enough information

(f) Consider

$$A = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x - 4y + 8z = -2\} \qquad B = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 2y - z = 1\}$$

What is a geometrical description of the intersection of A and B? Choose at most one option.

○ A plane with normal vector $\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$ passing through the point $(-2, 0, 0)$

○ Two planes that do not pass through the origin.

○ A line parallel to $\{t \begin{bmatrix} -12 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}\}$.

○ None of the above

(g) Suppose A is a 2×3 matrix and rank $A = 2$. Suppose $\vec{b}$ is a vector in $\mathbb{R}^2$. Choose all that applies.

○ The linear system $A\vec{x} = \vec{b}$ may be inconsistent.

○ The general solution to $A\vec{x} = \vec{0}$ is parallel to the general solution to $A\vec{x} = \vec{b}$.

○ The linear system $A\vec{x} = \vec{0}$ has infinity many solutions.

○ The RREF of A has 3 leading entires.

(h) Let A be the product of matrices below. What is $a_{11} + a_{23}$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix} = A$$

○ 4 ○ 10 ○ 8 ○ None of the above

2. Fill in the blank. You don't need to include your computation or reasoning.

   (a) (3 points) Let A be the standard matrix of a linear transformation that rotates vectors in $\mathbb{R}^3$ counterclockwise, as seen from the positive z-axis, through $\pi/2$ around the z-axis. Let $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ be columns of A, so that $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$. Then

$$\vec{a}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad \vec{a}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

   (b) (3 points) Solve the equation

$$-4x - 6y + 9z = 3,$$

   and describe your solution in the vector-parametric form.

$$\vec{x} = t\begin{bmatrix} \\ \\ \\ \end{bmatrix} + s\begin{bmatrix} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad t, s \in \mathbb{R}$$

   (c) (3 points) Find a, b, and c such that $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ Write DNE for a, b and c if such values do not exist.

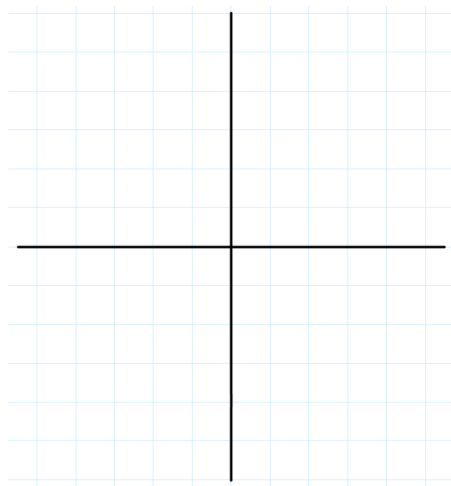
$a =$	$b =$	$c =$

Part B

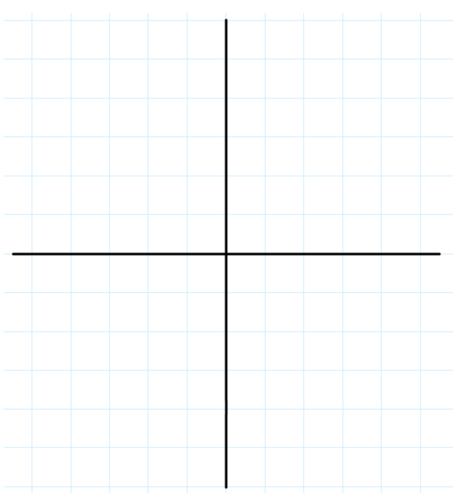
3. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with respect to the line $y = x$.

   (a) (2 points) The set V describes a letter in the alphabet. Accurately, draw this set. In your drawing, identify vectors with their tip when in standard position. Note that $\cup$ denotes the union of the two sets.

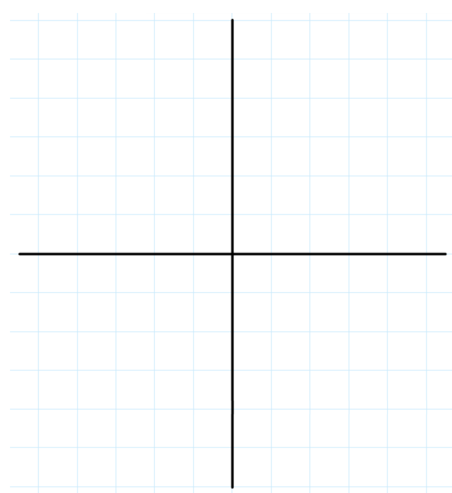
   $$V = \{t \begin{bmatrix} -2 \\ 3 \end{bmatrix} \mid 0 \le t \le 1\} \cup \{t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \mid 0 \le t \le 1\}$$



   (b) (4 points) Draw $S(V)$ and $T \circ S(V)$ on separate coordinate systems below. Clearly label the output of the following vectors: $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under S and $T \circ S$.



$S(V)$



$T \circ S(V)$

Recall that $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection with respect to the line $y = x$.

(c) (2 points) Let B be the standard matrix of T. Find B by computing the output of the standard vectors in $\mathbb{R}^2$. Justify your work. Write your final answer in the small box.

B=

(d) (4 points) Compute the standard matrix of $T \circ S$. Justify Your answer. Write your final answer in the small box.

4. State whether each statement is true or false by writing "True" or "False" in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true explain why it must be true. If you think a statement is false, give a counterexample.

   (a) (3 points) Consider a solution $\vec{x}_1$ of a linear system $A\vec{x} = \vec{b}$. If $\vec{x}_h$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution to $A\vec{x} = \vec{b}$.

   (b) (3 points) If the bottom row of a matrix A in reduced row-echelon form contains all 0's, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.

   (c) (3 points) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ be the columns of the matrix A. Suppose that $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then the equation $\vec{a}_4 = \vec{a}_1 + 2\vec{a}_2 + \vec{a}_3$ must hold.

5. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

   (a) (2 points) An equation of a plane that does not pass through the origin and is perpendicular to the vector $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

   (b) (2 points) A vector $\vec{u}$ and two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $\text{proj}_{\vec{u}}(\vec{v}_1) = \text{proj}_{\vec{u}}(\vec{v}_2)$.

   (c) (2 points) Two different linear transformations T and S such that $T \circ S = S \circ T$.

   (d) (2 points) A system of linear equations with three equations and three variables whose geometric interpretation is three planes in $\mathbb{R}^3$ intersecting at the y-axis.

Part C

6. A colony of ants travels between three different locations: their anthill, the strawberry patch, and by the puddle. Let's denote these by A, S, and P, respectively. The ants only walk on their constructed one-way paths, and they walk in a rhythmic pattern: Every minute, at the start of the minute, each ant will randomly follow one of the paths. If several paths are available, then an equal proportion of the ants will follow each of them. By the end of the minute, they are at their destination. The locations and the pre-constructed paths are shown in the diagram.

   Let x_1, x_2, and x_3 be the proportions of the ants who find themselves at A, S, and P, respectively, at the start of a minute; we collect this information in a distribution vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[d] & S \arrow[l, shift left] \\ P \arrow[ur] & \end{tikzcd}$$

   Let T be a transformation that transforms the distribution vector of ants at the start of the minute (right before ants start moving) to the distribution of vectors at the end of the minute (right after the ants have arrived at their destination).

   (a) (1 point) The domain of T is $\boxed{}$ and the codomain of T is $\boxed{}$.

   (b) (4 points) Find a matrix C such that $T(\vec{x}) = C\vec{x}$. Justify your work.

(c) (3 points) If all the ants are initially at the strawberry patch, what is the ants' distribution vector after 3 minutes? Justify your answer.

(d) (3 points) Is it possible that ants start at some initial distribution $\vec{x}$, and after one minute their distribution remains the same? If yes, find all such initial distributions. If no, why not?

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**