Tuesday December 13 **START: 14:00** **DURATION: 150 mins**

University of Toronto

Faculty of Applied Science & Engineering

FINAL EXAM
MAT188H1F
Linear Algebra

EXAMINERS: D. Burbulla, S. Cohen, D. Fusca, F. Lopez, M. Palasciano, M. Pugh, B. Schachter, S. Uppal

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Instructions.

1. There are **80** possible marks to be earned in this exam. The examination booklet contains a total of 14 pages. It is your responsibility to ensure that *no pages are missing from your examination.* DO NOT DETACH ANY PAGES FROM YOUR EXAMINATION.
2. DO NOT WRITE ON THE QR CODE AT THE TOP RIGHT-HAND CORNER OF EVERY PAGE OF YOUR EXAMINATION.
3. For the full answer questions, WRITE YOUR SOLUTIONS ON THE FRONT OF THE QUESTION PAGES THEMSELVES. THE BACK OF EVERY PAGE WILL **NOT** BE SCANNED AND SEEN BY THE GRADERS.
4. Ensure that your solutions are LEGIBLE.
5. No aids of any kind are permitted. CALCULATORS AND OTHER ELECTRONIC DEVICES (INCLUDING PHONES) ARE NOT PERMITTED.
6. Have your student card ready for inspection.
7. There are no part marks for Multiple Choice (MC) questions.
8. You may use the two blank pages at the end for rough work. The last two pages of the examination WILL NOT BE MARKED unless you *clearly* indicate otherwise on the question pages.
9. For the full answer questions, show all of your work and justify your answers *but do not include extraneous information.*

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

1. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6$, then $\det \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{bmatrix} =?$

(A) 6
(B) -6
(C) -12
(D) 12
(E) 0

2. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. The area of the parallelogram determined by $A\mathbf{x}$ and $A\mathbf{y}$ is

(A) 2
(B) 4
(C) 6
(D) 8
(E) 10

3. For which pair of linear transformations $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ below is $T_1 \circ T_2 = T_2 \circ T_1$?

(i) T_1 is the orthogonal projection on the x_1 axis, and T_2 is the orthogonal projection on the x_2 axis.

(ii) T_1 is the rotation through an angle θ_1, and T_2 is the rotation through an angle θ_2.

(iii) T_1 is the orthogonal projection on the x_1 axis, and T_2 is the rotation through an angle θ.

(A) (ii) and (iii)
(B) (i) and (iii)
(C) (i) and (ii)
(D) (i) only
(E) (ii) only

This portion of the page is left blank for your rough work, if necessary. Nothing written in this space will be graded or considered.

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

4. Suppose A is a 3×5 matrix such that each of the vectors $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ belong to null(A). Which of the following statements are TRUE?

(i) The rows of A are linearly dependent.

(ii) The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

(iii) If the system $A\mathbf{x} = \mathbf{b}$ has a solution, it is unique.

(A) (i) only
(B) (iii) only
(C) (i) and (ii) only
(D) (ii) only
(E) (i) and (iii) only

5. Let A be an $m \times n$ matrix with reduced row-echelon form R. Which of the following statements are TRUE?

(i) $\text{col}(A^T) = \text{row}(R)$.

(ii) $\text{row}(A^T) = \text{col}(R)$.

(iii) $\text{rank}(A) + \text{nullity}(A^T) = m$.

(A) (ii) and (iii)
(B) (i) and (iii)
(C) (i) and (ii)
(D) (i) only
(E) (iii) only

This portion of the page is left blank for your rough work, if necessary. Nothing written in this space will be graded or considered.

Part II - Short Answer Questions. Write your solutions in the space provided below each question.

1. (a) Suppose that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is an orthogonal set of vectors in $\mathbb{R}^3$, and let $A = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3]$. True or False: The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$. Give a reason for your answer. [2 marks]

1. (b) Suppose that $B = [\mathbf{b}_1\, \mathbf{b}_2\, \mathbf{b}_3]$ is an orthogonal matrix. True or False: $||\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3||^2 = 3$. Give a reason for your answer. [2 marks]

1. (c) Let A be an $n \times n$ matrix. Define what it means for a vector $\mathbf{x} \in \mathbb{R}^n$ to be an eigenvector of A. [2 marks]

1. (d) Let $A = \begin{bmatrix} 0 & 3 & -3 \\ 2 & 2 & -2 \\ -4 & -1 & 1 \end{bmatrix}$. Is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ an eigenvector of A? If so, what is its eigenvalue? [2 marks]

1. (e) Suppose that the characteristic polynomial of a diagonalizable matrix A is $C_A(\lambda) = \lambda^3(\lambda - 1)(\lambda + 2)^2$. What is rank($A$)? [2 marks]

1. (f) Recall that a matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Write both A^2 and A^{188} as product of P, P^{-1}, and D. [2 marks]

1. (g) Give an example of a 2×2 matrix A, that has only 0's and 1's as its entries, that is diagonalizable but not invertible. [2 marks]

1. (h) Give an example of a 2×2 matrix A, that has only 0's and 1's as its entries, that is invertible but not diagonalizable. [2 marks]

2. Suppose $A = \begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix}$, and the reduced row-echelon form of A is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Determine a, b, and c.

[6 marks]

3. Consider the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ c \end{bmatrix} \right\}$.

(a) For what value(s) of c is S a basis for $\mathbb{R}^3$? [4 marks]

(b) For what value(s) of c does the system

$$\begin{array}{ccccccc} x_1 & + & & & 2x_3 & = & 0 \\ x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & + & cx_3 & = & 0 \end{array}$$

have non-trivial solutions? [2 marks]

(c) For what value(s) of c is the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & c \end{bmatrix}$ invertible? [2 marks]

4. Consider the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(a) Find an orthogonal basis for $W^{\perp}$. [6 marks]

(b) Find a matrix A such that $\text{proj}_{W^{\perp}}(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^4$. [4 marks]

5. (a) Define what it means for an $n \times n$ matrix A to be orthogonally diagonalizable. [2 marks]

5. (b) Does there exist a 3×3 symmetric matrix A with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, and $\lambda_3 = 7$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$? If so, find A; if not, explain why not. [6 marks]

6. Consider the linear system

$$\begin{aligned} x_1 \quad & & = 2 \\ x_1 + x_2 & = 3 \\ x_2 & = c \end{aligned}$$

(a) Determine the value of c for which there exists a solution. What is the solution for this c? [1 mark]

(b) For every value of c there is a best approximate (least squares) solution. Find this solution. [5 marks]

(c) Let c_0 represent the value of c you found in part (a), and let $\mathbf{x}(c)$ represent the value of c you found in part (b) . What is $\lim_{c \to c_0} \mathbf{x}(c)$? [2 marks]

7. Suppose that camels reproduce every month and that the new offspring must wait two months before they begin to reproduce. If none of the camels die, and if the young camels are twice as fertile as the old ones, then the number of camels in the nth month is determined from the number of camels in the two previous months by the equation $x_n = x_{n-1} + 2x_{n-2}$, $n \geq 2$, where x_n represents the number of camels after n months. *If nothing else, you should be able to do parts (a) and (d) of this problem. So please don't give up just because this is a word problem.*

(a) Assume that $x_0 = 1$ and $x_1 = 1$. Find x_2. Now find x_3. [1 mark]

(b) Your classmate suggests studying this problem using matrix multiplication by writing:

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} \qquad \text{for } n \geq 2.$$

Using your classmate's suggestion, compute $\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ and $\begin{bmatrix} x_3 \\ x_2 \end{bmatrix}$. *You'll need to use that $x_0 = 1$ and $x_1 = 1$, of course.* [2 marks]

(c) Show that, in general, $\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$. [2 marks]

(d) Find an invertible matrix P and a diagonal matrix D so that $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$. [4 marks]

(e) Compute A^{n-1}. [2 marks]

(f) Use your answer in parts (c) and (e) to find a formula for x_n assuming that $x_0 = 1$ and $x_1 = 1$. [2 marks]

(g) What is $\lim_{n\to\infty} x_n$? [1 mark]

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