University of Toronto
MAT186H1F TERM TEST
Tuesday, October 15, 2013
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Instructions: Answer all questions. Present your solutions in the space provided; use the backs of the pages if you need more space. Do not use L'Hopital's rule on this test. The value for each question is indicated in parantheses beside the question number. **Total Marks: 60**

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CHECK YOUR TUTORIAL:

● TUT0101	○ TUT0102	○ TUT0103	○ TUT0104
○ TUT0105	○ TUT0106	○ TUT0107	○ TUT0108

MARKER'S REPORT:

QUESTION	MARK
Q1	7
Q2	6
Q3	7
Q4	8
Q5	8
Q6	7
Q7	5
Q8	3
TOTAL	51

1. [8 marks; 4 marks for each part.] Find and simplify $\frac{dy}{dx}$ if

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 = \sec^2 x - \tan^2 x$

(a) $y = \dfrac{\tan x}{1 + x \tan x}$.

$$\frac{dy}{dx} = \frac{(\sec^2 x)(1+x\tan x) - (\tan x)\left((1)(\tan x) + (x)(\sec^2 x)\right)}{(1+x\tan x)^2}$$

$$= \frac{\sec^2 x + x\sec^2 x \tan x - \tan^2 x - x\sec^2 x \tan x}{(1+x\tan x)^2}$$

$$= \frac{\sec^2 x - \tan^2 x}{(1+x\tan x)^2}$$

$$\frac{dy}{dx} = \frac{1}{(1+x\tan x)^2}$$

(b) $y = \ln\sqrt{\dfrac{4x-3}{3x+9}}$, for $x > 3/4$.

$$y = \ln\left(\frac{4x-3}{3x+9}\right)^{\frac{1}{2}}$$

$$y = \frac{1}{2}\ln\left(\frac{4x-3}{3x+9}\right)$$

$$y = \frac{1}{2}\ln(4x-3) - \frac{1}{2}\ln(3x+9)$$

$$y' = \frac{1}{2}\cdot\frac{1}{4x-3}\cdot 4 - \frac{1}{2}\cdot\frac{1}{3x+9}\cdot 3$$

$$= \frac{\cancel{4}^{\,2}}{\cancel{2}(4x-3)} - \frac{\cancel{3}}{2(\cancel{3}x+\cancel{9})^{3}}$$

$$y' = \frac{2}{4x-3} - \frac{1}{2x+6}$$

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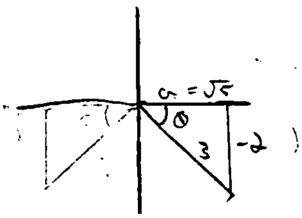
2. [7 marks] Given that $\theta = \sin^{-1}\left(-\frac{2}{3}\right)$ find the *exact* values of the following:

(a) [3 marks] $\cos\theta$

$\theta = \sin^{-1}\left(-\frac{2}{3}\right)$

$\Rightarrow \sin\theta = -\frac{2}{3}$



$\cos\theta = \frac{a}{h}$

2

$3^2 = (-2)^2 + a^2$

$9 - 4 = a^2$

$\pm\sqrt{5} = a$

$\cos\theta = \frac{\sqrt{5}}{3}$

why is $\cos\theta > 0$?

(b) [4 marks] $\cos(2\theta)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= (\cos\theta)^2 - (\sin\theta)^2$

$= \left(\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{-2}{3}\right)^2$

$= \frac{5}{9} - \frac{4}{9}$

4

$\cos 2\theta = \frac{1}{9}$

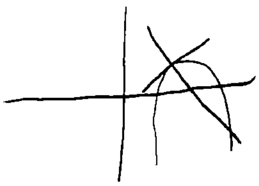
3. [7 marks] At what point does the normal line to the graph of $f(x) = -4 + 2x + 4x^2$ at the point $(1, 2)$ intersect the parabola a second time?

Normal Line to the graph at (1,2)

$f'(x) = 2 + 8x$

$f'(1) = 2 + 8(1)$

$= 10$



Normal $\perp$ Tangent

$m_N = -\frac{1}{10}$

$y = -\frac{1}{10}x + b$

=> plug in (1,2)

$2 = -\frac{1}{10}(1) + b$

$b = 2 + \frac{1}{10}$

$b = \frac{21}{10}$

confusing notation. write $-x$ instead of $-1x$

$Y = \frac{-1x + 21}{10}$ normal line

=> it hits the parabola at (1,2), we know this, we want the second point

when does

$y = \frac{-x+21}{10}$ equal $f(x) = -4 + 2x + 4x^2$

$-\frac{x}{10} + \frac{21}{10} = -4 + 2x + 4x^2$

$0 = 4x^2 + 2x + \frac{x}{10} - 4 - \frac{21}{10}$

$0 = 4x^2 + \frac{21x}{10} - \frac{61}{10}$

=> solve this quadratic using the quadratic formula

multiply each term by 10 => $0 = 4x^2 + \frac{21x}{10} - \frac{61}{10}$

$0 = 40x^2 + 21x - 61$

$x = \frac{-21 \pm \sqrt{21^2 - 4(40)(-61)}}{2(40)}$

$x = \frac{-21 \pm \sqrt{10201}}{80}$

$x = \frac{-21 \pm 101}{80}$

$x_1 = 1$ $x_2 = \frac{-122}{80}$

we have this already

$x_2 = -\frac{61}{40}$

plug into $f(x)$

$f\left(\frac{-61}{40}\right) = -4 + 2\left(\frac{-61}{40}\right) + 4\left(\frac{-61}{40}\right)^2$

$= -4 - \frac{61}{20} + 4 \cdot \frac{3721}{1600}$

$= \frac{-80 - 61}{20} + \frac{3721}{400}$

$= \frac{-141}{20} + \frac{3721}{400}$

$= \frac{-2820 + 3721}{400}$

$= \frac{901}{400}$

$\boxed{\Rightarrow P_2\left(\frac{-61}{40}, \frac{901}{400}\right)}$

4. [8 marks; 4 marks for each part.] Find the following limits:

4

(a) $\lim_{x\to 2} \frac{\sqrt{3x+10}-\sqrt{4x+8}}{x-2}$

$= \lim_{x\to 2} \frac{\sqrt{3x+10}-\sqrt{4x+8}}{x-2} \cdot \left[\frac{\sqrt{3x+10}+\sqrt{4x+8}}{\sqrt{3x+10}+\sqrt{4x+8}}\right]$ ✓

$= \lim_{x\to 2} \frac{(3x+10)-(4x+8)}{(x-2)(\sqrt{3x+10}+\sqrt{4x+8})}$

$= \lim_{x\to 2} \frac{-x+2}{(x-2)(\sqrt{3x+10}+\sqrt{4x+8})}$

$= \lim_{x\to 2} \frac{-\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{3x+10}+\sqrt{4x+8})}$

$= \lim_{x\to 2} -\frac{1}{\sqrt{3x+10}+\sqrt{4x+8}}$

$= \frac{-1}{\sqrt{3(2)+10}+\sqrt{4(2)+8}}$

$= -\frac{1}{\sqrt{6+10}+\sqrt{8+8}}$

$= \frac{-1}{\sqrt{16}+\sqrt{16}}$

$= \frac{-1}{4+4}$

$= -\frac{1}{8}$ ✓

(b) $\lim_{x\to 1} \left(\frac{3}{x-1} - \frac{6}{x^2-1}\right)$

4

$= \lim_{x\to 1} \left(\frac{3}{x-1} - \frac{6}{(x-1)(x+1)}\right)$

$= \lim_{x\to 1} \left(\frac{3(x+1)-6}{(x-1)(x+1)}\right)$

$= \lim_{x\to 1} \left(\frac{3x+3-6}{(x-1)(x+1)}\right)$

$= \lim_{x\to 1} \frac{3x-3}{(x-1)(x+1)}$

$= \lim_{x\to 1} \frac{3\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$

$= \lim_{x\to 1} \frac{3}{x+1}$

$= \frac{3}{1+1}$

$= \frac{3}{2}$

5. [8 marks; 4 marks for each part.] Find the following limits:

(a) $\lim_{x \to 0} \frac{\sin(x^2)}{x}$.

$$= \lim_{x \to 0} \frac{\sin(x^2)}{x} \cdot \left[\frac{x}{x}\right]$$

$$= \lim_{x \to 0} x \cdot \frac{\sin(x^2)}{x^2}$$

$\Rightarrow \lim_{t \to 0} \frac{\sin t}{t} = 1$ $\quad = (0) \cdot (1)$

$$= 0$$

$$\cancel{-1 \leq \sin(x^2) \leq 1}$$

$$\cancel{-\frac{1}{x^2} \leq \frac{\sin(x^2)}{x^2} \leq \frac{1}{x^2}}$$

$$\cancel{\lim_{x \to 0} -\frac{1}{x^2} \leq \lim_{x \to 0} \frac{\sin(x^2)}{x^2} \leq \lim_{x \to 0} \frac{1}{x^2}}$$

4

(b) $\lim_{x \to -\infty} \frac{\sqrt{x^2 + 11x}}{8 - 10x}$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2+11x}}{-10x+8}$$

divide by x

$\Rightarrow$
$$= \lim_{x \to -\infty} \frac{\frac{\sqrt{x^2+11x}}{x}}{\frac{-10x+8}{x}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{\frac{x^2}{x^2} + \frac{11x}{x^2}}}{-10 + \frac{8}{x}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{1 + \frac{11}{x}}}{-10 + \frac{8}{x}}$$

$$= \frac{-\sqrt{1+0}}{-10+0}$$

$$= \frac{-\sqrt{1}}{-10}$$

$$= \frac{1}{10}$$

4

6. [7 marks] Find the values of a and b so that

$$f(x) = \begin{cases} ax^2 + 2x - 1 & \text{if } x \le -2, \\ x^2 + bx & \text{if } x > -2 \end{cases}$$

is differentiable for all x.

$$f(x) = \begin{cases} ax^2 + 2x - 1 & \text{if } x \le -2 \\ x^2 + bx & \text{if } x > -2 \end{cases}$$

let $g(x) = ax^2 + 2x - 1$

let $h(x) = x^2 + bx$

- $f(x)$ is differentiable for all x if $f(x)$ is continuous and the $\lim_{x \to 2} f(x)$ exists, that is $g(-2) = h(-2)$ and $\lim_{x \to -2^-} g(x) = \lim_{x \to -2^+} h(x)$. In addition, $g'(-2) = h'(-2)$

$g'(x) = 2ax + 2$

$h'(x) = 2x + b$

$g'(-2) = 2a(-2) + 2$
$= -4a + 2$

$h'(-2) = -4 + b$

$g(x) = ax^2 + 2x - 1$

$g(-2) = a(-2)^2 + 2(-2) - 1$
$= 4a - 4 - 1$
$= 4a - 5$

$h(x) = x^2 + bx$

$h(-2) = (-2)^2 + b(-2)$
$= 4 - 2b$

$4a - 5 = 4 - 2b$

② $4a + 2b = 9$

$\Rightarrow g'(-2) = h'(-2)$

$-4a + 2 = -4 + b$

① $6 = b + 4a$

① $6 = b + 4a$; ① - ②

② $9 = 2b + 4a$

$-3 = -b + 0a$

$-3 = -b$

! $\boxed{b = 3}$ sub into ①

$6 = 3 + 4a$

$3 = 4a$

! $\boxed{a = \frac{3}{4}}$

$\lim_{x \to -2^-} g(x)$

$= \lim_{x \to -2^-} \frac{3}{4}x^2 + 2x - 1$

$= -2$

$\lim_{x \to -2^+} h(x)$

$= \lim_{x \to 2^+} x^2 + 3x$

$= -2$

5

7.(a) [3 marks] State the Intermediate Value Theorem.

- the intermediate value theorem states that if a function is continuous on a closed interval $[a,b]$, there is some value k in between $f(a)$ and $f(b)$ such that $f(c)=k$ where c is in the interval (a,b).

4

7.(b) [4 marks] Use the Intermediate Value Theorem to explain, clearly and concisely, why the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[1, 2]$.

$x^3+x^2-2x=1$

$x^3+x^2-2x-1=0$ must have at least one solution in the interval $[1,2]$

let $f(x)=x^3+x^2-2x-1$

$f(1) = 1^3+1^2-2(1)-1$
$= 1+1-2-1$
$= 2-2-1$
$f(1) = -1$

$f(2) = 2^3+2^2-2(2)-1$
$= 8+4-4-1$
$f(2) = 7$

4

$\Rightarrow$ since $f(x)$ is a polynomial, it is continuous on $x \in R$

$\Rightarrow$ because $f(1) < 0$ and $f(2) > 7$, the function x^3+x^2-2x-1 must cross the x-axis at least once in the interval $[1,2]$

(in terms of IVT) $\Rightarrow$ let $k=0$, let c be in $(1,2)$, by I.V.T. $f(c) = k = 0$

because $f(a) < f(c) < f(b)$

8. [8 marks] Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(x, y) = (1, -2/5)$ if $\frac{y}{x+2y} = x^9 - 3$.

$$\frac{y}{2x+2y} = x^9 - 3$$

$$\frac{dy}{dx} \Rightarrow \left[\frac{(y')(2x+2y) - (y)(2+2y')}{(2x+2y)^2}\right] = 9x^8$$

plug in $(1, -\frac{2}{5}) \Rightarrow \left[\frac{(y')(2(1)+2(-\frac{2}{5})) - (-\frac{2}{5})(2+2y')}{(2(1)+2(-\frac{2}{5}))^2}\right] = 9(1)^8$

$$\Rightarrow \left[\frac{2y' \cancel{-\frac{4}{5}y'} + \frac{4}{5} \cancel{+\frac{4}{5}y'}}{(2-\frac{4}{5})^2}\right] = 9$$

$$\Rightarrow \frac{2y' + \frac{4}{5}}{(\frac{6}{5})^2} = 9$$

$$\Rightarrow \frac{\frac{10y'+4}{\cancel{5}}}{\frac{36}{5^{\cancel{2}}}} = 9$$

$$\Rightarrow 10y' + 4 \cdot \frac{5}{36} = 9$$

$$\Rightarrow 50y' + 20 = 9(36)$$

$$50y' = 324 - 20$$

$$y' = \frac{304}{50} = \frac{152}{25}$$

$$\boxed{y' = 6.08}$$

$\frac{d^2y}{dx^2}$

$$\frac{(y')(2x+2y) - (y)(2+2y')}{(2x+2y)^2} = 9x^8$$

$$2xy' + 2y\cancel{y'} - 2y - \cancel{2}yy' = 9x^8(2x+2y)^2$$

$$2xy' - 2y = 9x^8(2x+2y)^2$$

$$2xy' = 9x^8(2x+2y)^2 + 2y$$

$$y' = \frac{9x^8(2x+2y)^2 + 2y}{2x}$$

$$\frac{d^2y}{dx^2} \Rightarrow y'' = \frac{\left(\left[(72x^7)(2x+2y)^2 + (9x^8)(2(2x+2y)\cdot(2+2y'))\right] + 2y'\right)(2x) - (2)(9x^8(2x+2y)^2 + 2y)}{4x^2}$$

plug in $6.08 = y'$ and $(1, -\frac{2}{5})$ $\Rightarrow$ $y'' = \frac{\left(\left[(72)(1.44) + (9)(2.4 \times 14.16)\right] + 12.16\right) - 2(9(1.44) - 0.8)}{4}$

$$y'' = 99.344 = 99.\frac{43}{125} = \boxed{\frac{12418}{125}}$$

3