Practise Problem Set 5

MAT 187 - Summer 2025

These questions are meant for your own practice for quiz 5 and are not to be handed in. Some of these questions, or problems similar to these, may appear on the quizzes or exams. Therefore, solutions to these problems will not be posted but you may, of course, ask about these questions during office hours, or on Piazza.

Suggestions on how to complete these problems:

- Solution writing is a skill like any other, which must be practiced as you study. After you write down your rough solutions, take the time to write a clear readable solution that blends sentences and mathematical symbols. This will help you to retain, reinforce, and better understand the concepts.
- After you complete a practice problem, reflect on it. What course material did you use to solve the problem? What was challenging about it? What were the main ideas, techniques, and strategies that you used to solve the problems? What mistakes did you make at the first attempt and how can you prevent these mistakes on a Term Test? What advice would you give to another student who is struggling with this problem?
- Discussing course content with your classmates is encouraged and a mathematically healthy practice. Work together, share ideas, explain concepts to each other, compare your solutions, and ask each other questions. Teaching someone else will help you develop a deeper level of understanding. However, it's also important that reserve some time for self-study and self-assessment to help ensure you can solve problems on your own without relying on others.

1. Consider the differential equation:

$$\frac{dx}{dt} = x^2(1-x)$$

   (a) Compute the equilibrium points. Are they stable or unstable?

   (b) Sketch the direction field and describe the behaviour of the possible solutions (without solving).

   (c) Find the general solution.

2. For each of the following differential equations, a function $y(x)$ is proposed as a solution. Verify if the function satisfies the equation, or determine any unknown parameters, if present, that make $y(x)$ a solution.

   (a) $(y')^2 - y = 0$ Guess: $y(x) = Ax^2 + Bx + C$

(b) $y'' = (y')^2$ Guess: $y(x) = C\ln|x|$.

(c) $y'' - y'y = 0$ Guess: $y(x) = C_1\tan(C_2 x)$

(d) $yy'' - (y')^2 = 0$ Guess: $y(x) = Ae^{kx}$

3. Find the general solution to each of the following:

(a) $\dfrac{dy}{dx} = \dfrac{x}{1+y^2}$

(b) $\dfrac{dy}{dx} = \dfrac{y^2}{1+x^2}$

(c) $\dfrac{dy}{dx} = x^2y - 3x^2 + y - 3$

(d) $\dfrac{dy}{dx} = \dfrac{\cos(x)}{y}$

(e) $\dfrac{dy}{dx} = e^x y^2$

4. Find the general solution to each of the following:

(a) $\dfrac{dy}{dx} + y\tan(x) = \sec(x)$

(b) $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2,\ x > 0$

(c) $\dfrac{dy}{dx} + 4y = xe^{-4x}$

(d) $\dfrac{dy}{dx} - \dfrac{2}{x}y = x^3,\ x > 0$

5. Consider the following differential equations. In each case, solve the initial value problem and find the particular solution that satisfies the given condition.

(a) Solve $\dfrac{dy}{dx} = 2xy$, with $y(0) = 3$.

(b) Solve $\dfrac{dy}{dx} + y = x$, with $y(0) = 1$.

(c) Solve $\dfrac{dy}{dx} = \cos(x)$, with $y(0) = 2$.

(d) Solve $\dfrac{dy}{dx} = y(1-y)$, with $y(0) = \frac{1}{2}$.

6. An object with initial temperature T_0 is placed in an environment with a constant temperature T_{env}. According to Newton's Law of Cooling, the rate of change of the object's temperature is proportional to the difference between the object's temperature and the ambient temperature:

(a) Write down an IVP that describes the temperature of the object as a function of time $T(t)$.

(b) Identify some physical factors that may effect the constant of proportionality (also known as the cooling constant) in your ODE.

(c) **Without solving your ODE**, describe what happens to the temperature as time $t \to \infty$. Does the object ever reach an equilibrium temperature?

(d) Solve the IVP.

(e) Suppose a cup of coffee initially at 90°C is placed in a room where $T_{\text{env}} = 20°$ C. If the cooling constant (the constant of proportionality in your ODE) is $k = 0.05$, how long will it take for the coffee to cool down to 50°C.

(f) A scientist observes that a metal rod initially at 150°C cools to 100°C in 10 minutes when placed in an environment at 30°C. Estimate the cooling constant k.

7. A tank initially contains **100 liters** of fresh water. A brine solution with a salt concentration of **2 g/L** flows into the tank at a rate of **5 L/min**, and the well-mixed solution drains out at the same rate. However, in addition to this process, a secondary source adds extra salt at a constant rate of **3 g/min**.

Let $C(t)$ be the concentration of salt (g/L) in the tank at time t (in minutes).

(a) Write a differential equation for $C(t)$.

(b) Find the equilibrium concentration of salt in the tank.

(c) Determine what happens to $C(t)$ as $t \to \infty$. Does the concentration stabilize? Why or why not?

(d) Solve the differential equation for $C(t)$ given that the initial concentration in the tank is **0 g/L** (pure water).

(e) How long will it take for $C(t)$ to reach **90%** of its equilibrium value?

8. Let g be the acceleration due to gravity on the Earth's surface, and let R be the radius of the Earth. Assume that the Earth is a perfect sphere with uniform density.

Consider a tunnel drilled straight through the Earth from one point on the surface to the **diametrically opposite** point. Assume:

- The tunnel is **vacuum-sealed** (no air resistance).
- The gravitational force acting on an object inside the tunnel varies **directly** with its distance from the Earth's center.

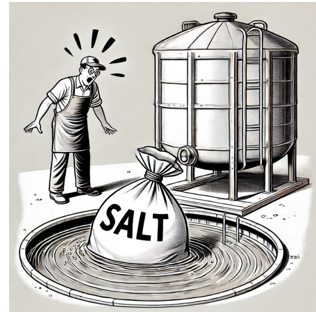
(a) Explain why the gravitational force inside the Earth behaves like a **restoring force** similar to a simple harmonic oscillator.

(b) Write the equation of motion for an object moving through the tunnel.

(c) Derive the time it takes for an object to travel from one end of the tunnel to the other. Express your answer in terms of known constants: g, R, and other relevant parameters.

(d) Find the maximum velocity of the object during its motion. Determine where along the path this velocity occurs.

9. A system of two 3000 L holding tanks is set up with a pipe bringing water to the first tank, another pipe taking water from the first tank to the second, and a final pipe draining away from the second tank, with each pipe moving 100 L/hr. Usually, the system holds nothing but pure water, with the tanks kept half-full.



One day (or so the story is told), an apprentice spilled a 60 kg bag of salt at the mouth of the pipe feeding into the first tank. To add further injury to the original injury, the bag itself flew from his grasp and partially clogged the pipe draining from the second tank (don't ask how – it just did) reducing the flow to 50L/hr.

Assume the pile of salt is being dissolved by the water at the rate of half of the remaining amount every $\ln\left(\frac{2}{3}\right)$ hours. As usual, assume everything mixes instantly.

(a) Write an IVP for the amount of salt in tank 1.

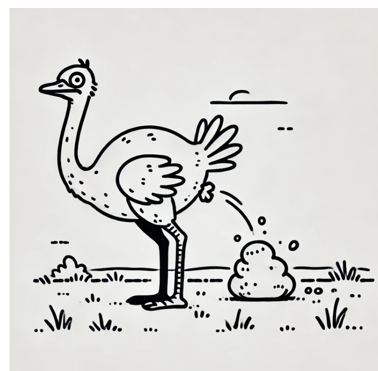
(b) Write an IVP for the amount of salt in tank 2.

(c) Determine the time at which tank 2 reaches full capacity.

(d) Solve for the amount of salt in tank 2 as a function of time.

(e) Find the amount of salt in tank 2 at the moment it fills up.

10. In a remote grassland of total area 100 hectares, a population of ostriches produces environmental waste that gradually accumulates in the soil. The waste breaks down naturally due to microbial activity and weathering. However, if too much waste accumulates, it begins to harm vegetation and disrupts the delicate ecosystem. Scientists model the poop as a function of time $P(t)$ (measured in kg per hectare) using the following dynamics:



- Each ostrich contributes 2 kg of waste per day to the environment.
- The number of ostriches is initially 50 and grows at a rate proportional to their current population, with a growth rate of 5
- Natural processes decay the waste at a rate proportional to the current level, with a decay constant of 0.1 per day.

(a) Write an IVP for $P(t)$, incorporating ostrich population growth and waste decay.

(b) Determine whether pollution will stabilize over time or grow indefinitely.

(c) Solve for P(t) explicitly in terms of time.

(d) If the ecosystem can only sustain 500 kg/ha of pollution before causing severe damage, how long until this threshold is crossed?

(e) Suppose conservationists introduce a natural cleaning mechanism that removes an additional 20 kg/day regardless of waste level. How does this change the long-term behavior of $P(t)$?