NAME (Given then Family):

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University of Toronto
Faculty of Applied Science & Engineering

Winter 2024/2025
110 Minutes

MAT187H1S Midterm 1 Solutions

Exam Reminders:

- Fill out your name, UTORid, and email address at the top of this page.
- Do not begin writing the exam until instructed to do so.
- As a student, you help create a fair and inclusive writing environment; unauthorized aids are prohibited and using one may result in you being charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. These devices may *not* be left in your pockets.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator.
- In the event of a fire alarm, do not check cell phones or other electronic devices unless authorized to do so.

Special Instructions:

- Write legibly and darkly.
- For questions with a boxed area, ensure your answer is completely within the box.
- Fill in your bubbles completely.

Good: ● A ○ B Bad: ◐ A ⊙ B ⊠ C

Scoring

Question:	1	2	3	4	5	6	7	Total
Points:	6	8	4	8	5	14	7	52
Score:								

Formula Sheet

The following formulae are provided for your use during the exam.

Simpson's Rule

If p is a quadratic polynomial,

$$\int_a^b p(x)\,\mathrm{d}x = \frac{b-a}{6}\left(p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right).$$

Taylor's Remainder Theorem

Let T_n be the nth Taylor approximation of function f centered at a. Further, assume f has at least $n+1$ continuous derivatives. Then

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.

Geometric Sum

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

1. (a) (2 points) The point p, given in rectangular coordinates, is $(x, y) = (3, 3)$. Express the point p in polar coordinates in *two different ways*.

$(r, \theta) = \boxed{(\sqrt{18}, \pi/4)}$ or $(r, \theta) = \boxed{(-\sqrt{18}, -3\pi/4)}$

(b) (2 points) Which complex numbers have a *modulus* greater than 2? Mark all that apply.

- ○ $1 + i$
- ● $3i$
- ● -3
- ● $2 - 2i$
- ○ 0
- ○ $-1.99e^{7\pi i}$

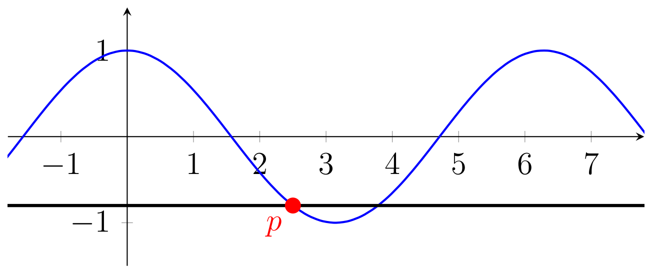
(c) (2 points) Given points $(1, a)$, $(2, b)$, $(3, c)$, and $(4, d)$, you construct an interpolating polynomial, p, using Lagrange's method. Given that $b < c$, which statements are true? Mark all that apply.

- ○ $p(2.5)$ **must** be larger than $f(2)$
- ● $p(2.5)$ **could** be larger than $f(2)$
- ○ $p(x)$ **must** be *strictly increasing* when $2 \leq x \leq 3$
- ● $p(x)$ **could** be *strictly increasing* when $2 \leq x \leq 3$
- ● $p(x)$ **must** be a continuous when $2 \leq x \leq 3$
- ● $p(2) = b$

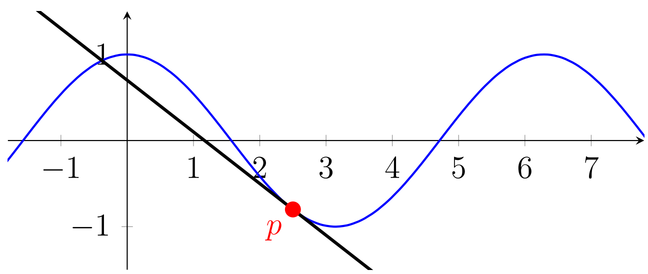
2. For each part of this question you are given the graph of $y = \cos(x)$. On this graph is marked the point $p \approx (2.5, -.8011)$. For each part, sketch the requested polynomial.

   Note: your sketch only needs to be approximate, but it should demonstrate the important behaviour of the requested polynomial.

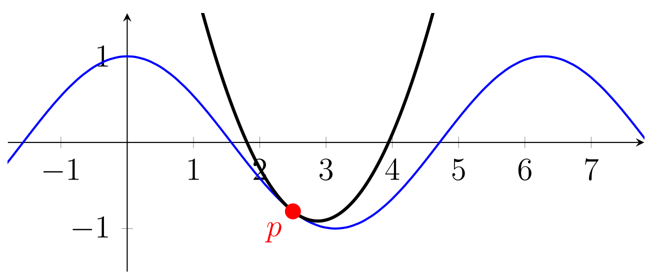
   (a) (2 points) Sketch a 0^{th} degree Taylor approximation of cosine centered at 2.5.



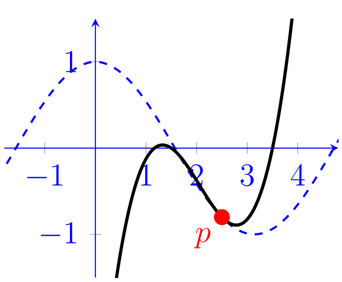
   (b) (2 points) Sketch a 1^{st} degree Taylor approximation of cosine centered at 2.5.

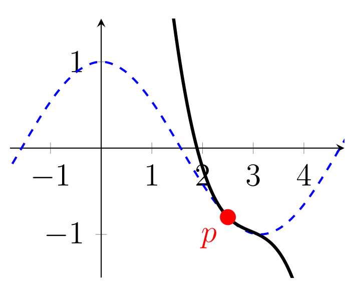


   (c) (2 points) Sketch a 2^{nd} degree Taylor approximation of cosine centered at 2.5.



   (d) (2 points) Which graph shows the 3^{rd} degree Taylor approximation of cosine centered at 2.5?

● 

○ 

○ None of the above

3. Tiffany is an artist. She loves painting cannon balls. Her idea for a new painting involves drawing the path of a cannon ball in flight and filling in the area between the bottom of the canvas and the arc of the cannon ball with gold leaf (an expensive alternative to paint).

   Since gold leaf is expensive, she wants to purchase exactly what she needs. To get an estimate, Tiffany has found a polynomial interpolation for the cannon ball's path (as traced out on the canvas).

   Let $f(x)$ represent the height of the cannon ball's path on the canvas in metres at a horizontal distance of x metres from the lower left corner of the canvas. Let

$$P(x) = -x^2 + x + 2$$

   be a quadratic approximation of f.

   For this question, you have the following information:

   - The canvas is $2m$ wide and $3m$ tall.
   - $|f(x) - P(x)| \leq \frac{1}{6}$ for $x \in [0, 2]$.

   (a) (2 points) Using the available information, give the best possible upper bound for the number of square metres of gold leaf that Tiffany needs to buy.

   Amount of gold leaf required $\leq$ $\boxed{11/3m^2 \approx 3.66m^2}$

   (b) (2 points) Using the available information, give the best possible lower bound for the number of square metres of gold leaf that Tiffany needs to buy.

   Amount of gold leaf required $\geq$ $\boxed{3m^2 \quad \text{or} \quad (1 + \frac{125}{36\sqrt{3}})m^2 \approx 3.0047m^2}$

   Scratch work:

4. A snail is climbing up a wall that is 10m tall. The snail starts crawling up the wall at a speed of 5m/h. Every hour, the snail gets distracted and its speed instantaneously decreases by half.

Let H_n be the height of the front of the snail's body after n hours of climbing. For $n \geq 3$, let W_n be the part of that distance travelled by the snail, starting from exactly three hours after it began its climb. For example, W_5 is the distance traveled by the snail between the end of hour 3 and the end of hour 5.

(a) (2 points) Express H_n using sigma notation (Σ notation).

$$H_n \quad = \quad \boxed{\sum_{i=0}^{n-1} 5(\tfrac{1}{2})^i}$$

(b) (2 points) Assume $n > 3$. Express W_n using sigma notation (Σ notation).

$$W_n \quad = \quad \boxed{\sum_{i=3}^{n-1} 5(\tfrac{1}{2})^i}$$

(c) (2 points) Assume $n \geq 3$. Express W_n in terms of H_n (i.e., your formula must involve H_n and *cannot* use sigma notation).

$$W_n \quad = \quad \boxed{H_n - H_3}$$

(d) (2 points) The snail's antennae stick out 1cm in front of its body. Will the snail ever touch the ceiling? Provide a well-written justification.

● The snail **will** touch the ceiling ○ The snail **won't** touch the ceiling

Justification:

If the snail's body reaches within 1cm of the ceiling, it will touch the ceiling. This will happen if $H_n \geq 9.99$ for some n. Using the Geometric sum formula, we see

$$H_n = 5\frac{1-(1/2)^n}{1-1/2} = 10(1-(1/2)^n).$$

From this formula we see $\lim_{n\to\infty} H_n = 10$ and so eventually $H_n \geq 9.99$. (In fact $H_7 > 9.99$.)

5. (5 points) Let f be a **concave up** and **decreasing** function on the interval $[0, 2]$. WolframAlpha reports

$$I = \int_0^2 f(x)\,\mathrm{d}x \approx 3.4587.$$

Each estimate below is produced using numerical integration with the same partition.

For each estimate below, identify which numerical integration technique was likely used by filling in the appropriate letter in the table below.

$$\begin{aligned} A &= 3.4427 \\ B &= 3.4906 \\ C &= 2.9142 \\ D &= 3.4589 \\ E &= 4.0670 \end{aligned}$$

E	Left-endpoint approximation
C	Right-endpoint approximation
B	Trapezoid rule
A	Midpoint rule
D	Simpson's rule

6. The function f has a Taylor series representation centered at $a = 2$ of

$$T(x) = 1 + 2(x-2) + 3(x-2)^2 + 4(x-2)^3 + 5(x-2)^4 + \cdots.$$

(a) (2 points) Express $T(x)$ using sigma notation (Σ notation).

$T(x) \quad = \quad \boxed{\sum_{i=0}^{\infty}(i+1)\cdot(x-2)^i}$

(b) (3 points) Does $T(x)$ converge when $x = 3$? Provide a well-written justification.

○ Converges ● Does not converge

Justification:

The series at $x = 3$ is $T(3) = 1 + 2 + 3 + 4 + \cdots$, with partial sums $\sum_{i=0}^{n} i$. These partial sums are bounded below by n, and so they diverge, which means $T(3)$ diverges.

(c) (3 points) Does $T(x)$ converge when $x = 2$? Provide a well-written justification.

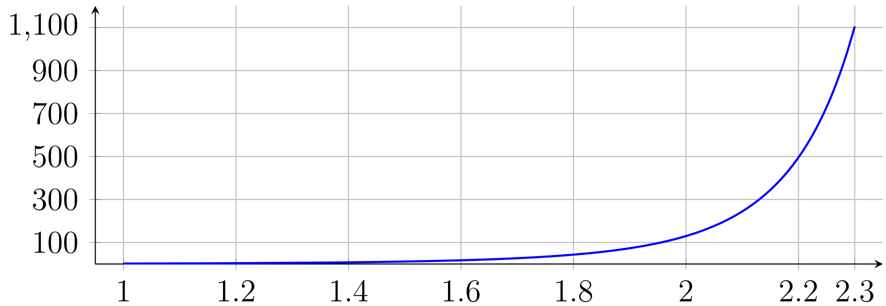
● Converges ○ Does not converge

Justification:

The series at $x = 2$ is $T(2) = 1 + 0 + 0 + \cdots = 1$, and so it is convergent.

This question uses the same f and T as the previous page.

(d) Let T_3 be the 3rd Taylor approximation of f centered at $a = 2$. Below is a graph of $y = f^{(4)}(x)$, the 4th derivative of f, from $x = 1$ to $x = 2.3$.



i. (2 points) Write down an expression for $T_3(x)$. (*No need to simplify/expand your expression.*)

$T_3(x) \quad = \quad \boxed{1 + 2(x-2) + 3(x-2)^2 + 4(x-2)^3}$

ii. (4 points) Give an upper bound for $|f(2.2) - T_3(2.2)|$. Provide a well-written justification.
Note 1: There is *no need to simplify* your upper bound.
Note 2: If you need to do scratch work, do so on the blank pages of the test. Only write your final, polished answer in the box.

$|f(2.2) - T_3(2.2)| \quad \leq \quad \boxed{\frac{500}{4!}(0.2)^4 \approx 0.033}$

Justification:

By Taylor's remainder theorem, $T_3(x) - f(x) = \frac{f^{(4)}(c)}{4!}(x-2)^4$ for some $c \in [2, 2.2]$. From the graph we can see $|f^{(4)}(c)| \leq 500$ when $c \in [2, 2.2]$. So, we get a bound of

$$|T_3(x) - f(x)| \leq \frac{500}{4!}(0.2)^4.$$

7. A *very fast* snail is heading straight towards you. Its distance from you at time t in seconds is given by $B(t)$ in metres. At time 0, the snail is 4 metres away from you travelling *towards you* at a speed of $3m/s$ and slowing down at a rate of $1m/s^2$. Further suppose that $B'''(t) < 0$ and that all 4th derivatives and up are negligible (i.e., close to zero).

   (a) (2 points) Give a formula for T_2, the 2nd Taylor approximation to $B(t)$ centered at $t = 0$.

   $T_2(t) \quad = \quad \boxed{4 - 3t + \frac{t^2}{2}}$

   (b) (2 points) Given the information above, what is your best estimate for when the snail will hit you?

   $t \quad = \quad \boxed{2}$

   (c) (3 points) Is your estimate from the previous part an underestimate or an overestimate? Provide a well written justification.

   ○ Underestimate ● Overestimate

   Justification:

   Since $B''' < 0$, we know that $T_3(t) = 4 - 3t + \frac{t^2}{2} - \alpha t^3$ for some $\alpha > 0$. With this higher order term, T_3 is decreasing *faster* than T_2, so T_2 provides an *overestimate*.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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