University of Toronto
Faculty of Applies Sciences and Engineering

MAT188 – Midterm I –Fall 2022

LAST (Family) NAME:____

FIRST (Given) NAME:____

Email address: ____________________________@mail.utoronto.ca

STUDENT NUMBER:____

Solutions

Question:	1	2	3	4	5	6	Total
Points:	6	9	13	12	8	15	63
Score:							

Part A

1. (6 points) Fill in the bubble for all statements that must be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles.

(a) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 3 & 0 \end{bmatrix}$ Compute $AB + C$.

○ $\begin{bmatrix} 2 & 3 \\ 14 & 1 \\ 7 & 1 \end{bmatrix}$ ○ Not defined ○ $\begin{bmatrix} 2 & 3 \\ 14 & 1 \\ 7 & 1 \end{bmatrix}$ ● None of the above

Solution:

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 11 & 2 \\ 4 & -1 \end{bmatrix}$$

$$AB + C = \begin{bmatrix} 1 & 0 \\ 11 & 2 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 14 & 1 \\ 7 & -1 \end{bmatrix}$$ **(Ch1:S11, ch2:S12)**

(b) Suppose $B = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the RREF of an augmented matrix of a system of linear equations. Then this system has **(Ch1: S4)**

○ Unique solution: $x = 0$, $y = -1$, $z = 3$.

● Infinity many solutions. One particular solution is $x = 5, y = -1, z = 3$.

○ Infinity many solutions. One particular solution is $x = -1, y = 3, z = 0$.

○ No solution

Solution: Solving the system we get $y = -1$,$z = 3$ and x is free. Hence the system has infinity many solutions. One particular solution is $x = 5$,$y = -1$,$z = 3$. **(Ch1: S4)**

(c) Let $\vec{e}_1$ and $\vec{e}_2$ be standard vectors in $\mathbb{R}^2$. Which one is an accurate geometric description of the set $\{c_1\vec{e}_1 + c_2\vec{e}_2 \,|\, 0 \leq c_1, c_2 \leq 1\} \cap \{k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,|\, k \in \mathbb{R}\}$. **(week 1, S2)**

- ○ A filled square in $\mathbb{R}^2$
- ○ A hollow square in $\mathbb{R}^2$
- ○ A plane
- ● A line segment in $\mathbb{R}^2$

Solution: This is the intersection of the unit square and the $x = y$ line in $\mathbb{R}^2$, which is a line segment.

(d) Consider

$$A = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 1x - 2y + 4z = -1 \} \qquad B = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 2y - z = 3 \}$$

Which one is the MOST accurate geometrical description of $A \cap B$? Choose at most one option. **(Week 1: S1, S6, S7, S9, Ch1 S6)**

- ○ A plane with normal vector $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ passing through the point $(-1, 0, 0)$
- ○ Two planes with normal vectors $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ that do not pass through the origin.
- ○ A line with a direction vector $\begin{bmatrix} -10 \\ 1 \\ 2 \end{bmatrix}$ that passes through the point $(-4, 3/2, 0)$.
- ● None of the above.

Solution:

(e) Suppose A is a 3×4 matrix and $rank A = 3$. Consider the linear system $A\vec{x} = \vec{0}$. **(Ch1: S7)**

- ○ The linear system $A\vec{x} = \vec{0}$ may be inconsistent.
- ● The linear system $A\vec{x} = \vec{0}$ is consistent.
- ● The linear system $A\vec{x} = \vec{0}$ has infinity many solutions.
- ○ We don't have enough information to be able to say anything about the solution type of the linear system $A\vec{x} = \vec{0}$.

2. Fill in the blank. You don't need to include your computation or reasoning.

(a) (3 points) Let A be the standard representation of a linear transformation that reflects vectors in $\mathbb{R}^3$ with respect to the xy-plane. Suppose $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, that is $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are columns of A. Then

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

(Ch2: S8, S4)

(b) (3 points) Find a and b such that $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ Write DNE for both a and b if you thing such values do not exist.

$\boxed{a = \text{DNE}}$ $\qquad$ $\boxed{b = \text{DNE}}$

(Ch1: S9)

(c) (3 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Describe all solutions of $A\vec{x} = \vec{0}$ in vector parametric form

$$\vec{x} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad t, s, k \in \mathbb{R}$$

(Ch1: S3)

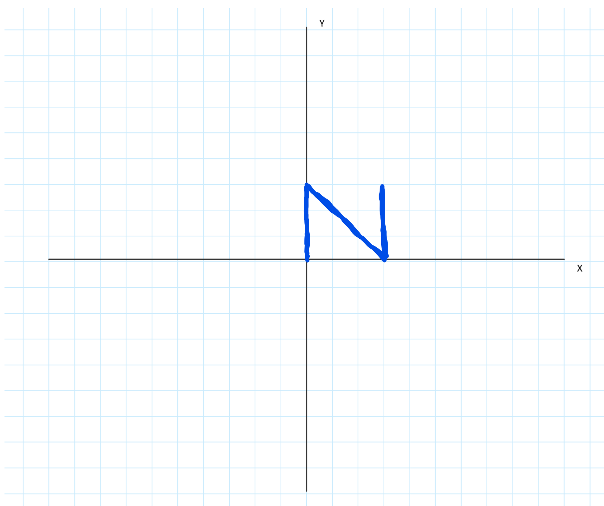
Part B

3. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with respect to $y = -x$.

   (a) (3 points) Accurately, draw the set

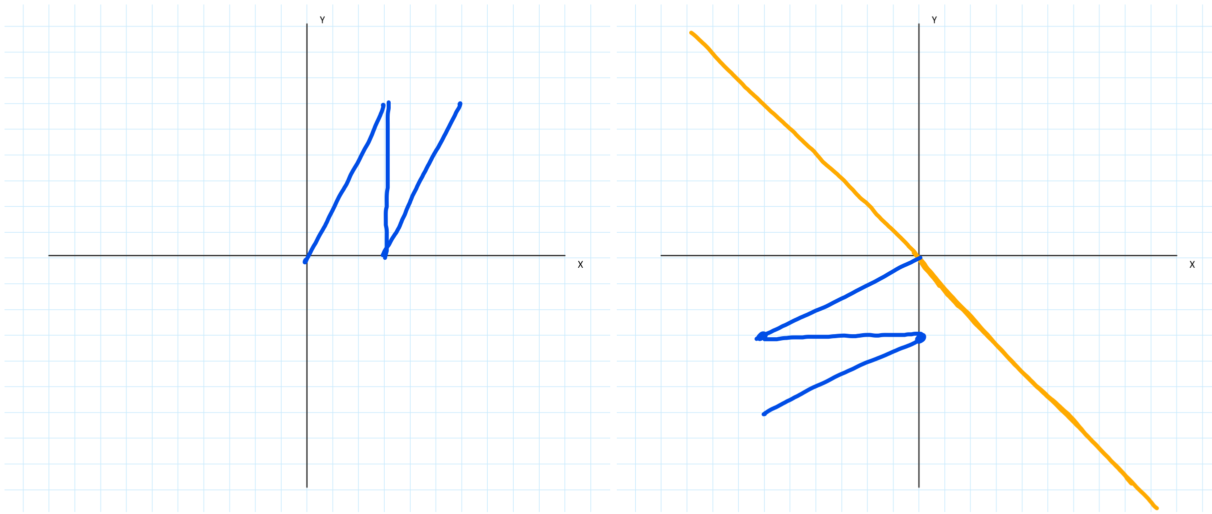
   $$N = \{t \begin{bmatrix} 0 \\ 3 \end{bmatrix} \mid 0 \le t \le 1\} \cup \{t \begin{bmatrix} 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \mid 0 \le t \le 1\} \cup \{t \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \mid 0 \le t \le 1\}$$

   Hint: the resulting image is a letter in the alphabet. **(Week1:S7,S2)**



Solution:

   (b) (4 points) Draw $S(N)$ and $T \circ S(N)$ on separate coordinate systems below. Label all vertices. **(Ch2: S10)**



Solution:

(c) (2 points) Recall that $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with respect to $y = -x$. Let B be the standard matrix of T. Calculate B. Justify your work. Write your final answer in the small box. **(Ch2: S7,S2)**

Solution: Note that $B = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$. We need to find $T(\vec{e}_1), T(\vec{e}_2)$ either geometrically or algebraically. $T(\vec{e}_1) = -\vec{e}_2$ and $T(\vec{e}_2) = -\vec{e}_1$

$$B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

B=

(d) (4 points) Compute the standard matrix of $T \circ S$. **(Ch2: S11, S12)**

Solution:

$$BA = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix}$$

4. State whether each statement is true or false by writing "True" or "False" in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true explain why it must be true. If you think a statement is false, give a counterexample.

   (a) (3 points) If T is a linear transformation, then $T(\vec{0}) = \vec{0}$, that is T preserves the origin. **(Ch1: S2)**

   Solution: True. $T(\vec{0}) = T(0\vec{v}) = 0T(\vec{v}) = \vec{0}$, where $\vec{v}$ is any vector in the domain of T.

   (b) (3 points) If the bottom row of a matrix A in reduced row-echelon form contains all 0's, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions. **(Ch1: S4)**

   Solution: False. For instance consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) (3 points) Suppose $\begin{bmatrix} 1 & 3 & 4 \\ 0 & a & 1 \\ 1 & 0 & c \end{bmatrix}$ is invertible. Then the system of linear equations with the augmented matrix $\begin{bmatrix} 1 & 3 & 4 & -1 \\ 0 & a & 1 & 1 \\ 1 & 0 & c & 3 \end{bmatrix}$ is consistent. **(Ch2: S16)**

Solution:

True. The unique solution is $\begin{bmatrix} 1 & 3 & 4 \\ 0 & a & 1 \\ 1 & 0 & c \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

(d) (3 points) If $A = [\vec{u}\ \vec{v}\ \vec{w}\ \vec{x}]$ and $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then the equation $\vec{x} = -\vec{u} - 2\vec{v} + \vec{w}$ must hold. **(Ch1: S9, S5, S12)**

Solution: True. Note that $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ is a particular solution to the system of linear equation with augmented matrix A hence $[\vec{u}\ \vec{v}\ \vec{w}] \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} = \vec{x}$, that is $\vec{x} = -\vec{u} - 2\vec{v} + \vec{w}$

5. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

   (a) (2 points) An equation of a plane that does not pass through the origin and is perpendicular to that vector $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. **(week 1: S8, S9)**

   Solution: $x - 2y = 4$

   (b) (2 points) The standard matrix of a linear transformation that is onto but not one to one. **(week 1: S21, S22)**

   Solution:
   $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

   (c) (2 points) A matrix A with the property that $A = A^{-1}$.

   Solution: Any reflection matrix work! $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (d) (2 points) A system of linear equations with three variables and three equations that has exactly two solutions.

   Solution: Such an example does not exist. By the Solution Type theorem, every system has either no solution or one solution or infinitely many solutions.

Part C

6. The color of light can be represented in a vector $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ in $\mathbb{R}^3$ where R is the amount of red, G is the amount of green, and B is the amount of blue. The eye of a healthy human receives the light and sends it to the brain. The human brain transform the incoming signal into the signal $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$ that can be interpreted by our brain.

$$\begin{aligned} N : \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{bmatrix} R \\ G \\ B \end{bmatrix} &\mapsto \begin{bmatrix} I \\ L \\ S \end{bmatrix} = \begin{bmatrix} \frac{R+G+B}{3} \\ R - G \\ B - \frac{R+G}{2} \end{bmatrix} \end{aligned}$$

(a) (2 points) Is N linear? Justify your answer by either using either the definition of a linear transformation directly, or by referring to a theorem covered in the course.

Solution: Yes. Note that $N(\begin{bmatrix} R \\ G \\ B \end{bmatrix}) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$. Every map that is given by left multiplication by a matrix is linear by Linear Transformations and Matrices theorme.

(b) (2 points) Is it possible for two distinct colour to be perceived as the same colour by the brain? Justify your answer. Your justification should use precise mathematical terminology.

Solution: No. Two distinct colours re perceived as the same colour by the brain if and only if N is NOT injective. Row reducing the matrix we got in (1) we get pivots in every columns. Hence N is injective.

(c) (3 points) Given $\vec{v} = \begin{bmatrix} I \\ L \\ S \end{bmatrix}$, is it possible to recover the $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ that is transformed to $\vec{v}$. If no, explain why. If yes, explain why and give a concise instruction on how to do so.

Solution: Given $\vec{v} = \begin{bmatrix} I \\ L \\ S \end{bmatrix}$, it is possible to recover the $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ that is transformed to $\vec{v}$ if and only if N is invertible if and only if its standard matrix representation row reduces to identity. Row reduction shows that

$$A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix} \sim I_3$$

The signal transferred into $\vec{v}$ is hence calculated by $A^{-1}\vec{v}$.

(d) (3 points) There are genetic, and not very uncommon, disorders that may cause the human brain to perceive distinct colours as the same colour. One such disorder can be replicated through the use af a pair of eyeglasses that transform the light via a linear transformation T. We know that

$$T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad T(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \qquad \text{and} \quad T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Find the standard matrix presentation of T.

Solution: To find the standard matrix of T, we need the output of the standard vectors. Observe that $\vec{e}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{e}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix}$. Hence,

$T(\vec{e}_2) = T(\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix}) = T(\begin{bmatrix}1\\1\\0\end{bmatrix}) - T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}0\\3\\1\end{bmatrix} - \begin{bmatrix}2\\4\\1\end{bmatrix} = \begin{bmatrix}-2\\-1\\0\end{bmatrix}$

$T(\vec{e}_3) = T(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix}) = T(\begin{bmatrix}1\\1\\1\end{bmatrix}) - T(\begin{bmatrix}1\\1\\0\end{bmatrix}) = \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}0\\3\\1\end{bmatrix} = \begin{bmatrix}2\\-2\\1\end{bmatrix}$

Therefore the standard matrix of T is

$$\begin{bmatrix}2 & -2 & 2\\4 & -1 & -2\\1 & 0 & 1\end{bmatrix}$$

(e) (2 points) Suppose a person with healthy eyes is wearing the glasses described in the previous part. Find the standard matrix represntation of a linear transformation that an $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ vector goes through as it passes through the glasses and then sent to the brain.

Solution: $\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 \\ -1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 4 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7/3 & -1 & 1/3 \\ -2 & -1 & 4 \\ -2 & 3/2 & 1 \end{bmatrix}$

(f) (3 points) Continue assuming that a person with healthy eyes is wearing the glasses described in the previous part. Find the set of all $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ vectors in $\mathbb{R}^3$ that are perceived as $\begin{bmatrix} I \\ L \\ S \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by the brain. Give a complete geometrical description of this set

Solution:

$$\left[\begin{array}{ccc|c} 7/3 & -1 & 1/3 & 1 \\ -2 & -1 & 4 & 1 \\ -2 & 3/2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & & & 55/72 \\ & 1 & & 13/12 \\ & & 1 & 65/72 \end{array}\right]$$

$\left\{ \begin{bmatrix} 55/72 \\ 13/12 \\ 65/72 \end{bmatrix} \right\}$ is a point.

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

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