University of Toronto
Faculty of Applied Sciences and Engineering

MAT188 – Midterm I – Fall 2024

LAST (Family) NAME: ____________________

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Email address: ____________________@mail.utoronto.ca

STUDENT NUMBER: ____________________

Solutions

Question:	1	2	3	4	5	6	7	Total
Points:	12	9	10	6	9	6	8	60
Score:								

1 Part A

1. (12 points) Fill in the bubble for all statements that **must** be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. For those questions, you may get a negative mark for incorrectly filled bubbles.

(a) The magnitude of $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is

○ 4 ○ 2 ● $\sqrt{6}$ ○ 6

Solution: ch0-LS2: **I can perform algebraic arithmetic on vectors in a Euclidean vector space Rn (e.g., addition, scalar multiplication, norm, and dot product).** (Computation) ch0-COM-vecarith.

$$\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\| = \sqrt{(1)^2+(0)^2+(1)^2+(2)^2} = \sqrt{6}$$

(b) Consider the sets ℓ_1 and ℓ_2:

$$\ell_1 = \left\{ t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \mid t_1 \in \mathbb{R} \right\}$$

$$\ell_2 = \left\{ t_2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mid t_2 \in \mathbb{R} \right\}$$

Choose all that apply.

○ ℓ_1 and ℓ_2 represent lines in $\mathbb{R}^2$

○ ℓ_1 and ℓ_2 are parallel

● ℓ_1 and ℓ_2 intersect at a point

○ ℓ_1 and ℓ_2 do not intersect

Solution: Ch1-LS8: **I can represent a line in R2 and R3 in vector form.** (Conceptual) ch0-CON-vecline

Ch1-LS9: **Given an algebraic description of a line in R2 or R3, I can visualize the line. (Visual/Geometry)** ch0-VG-line

First note that the direction vectors are not parallel. So $\ell_1 \cap \ell_2$ is either a point or empty.

$$\ell_2 \ni \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \in \ell_1$$

(c) For a system with two distinct equations and four variables, which of these **could** be true of the solution set? Mark all that apply

Possibilities for pivot positions in RREF are

$$\begin{array}{c} \begin{matrix} x & y & z & w & \ \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \end{array}\right] \end{array} \text{ or } \begin{array}{c} \begin{matrix} x & y & z & w & \ \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & & & & \\ & & & & 1 \end{array}\right] \end{array} \text{ or } \begin{array}{c} \begin{matrix} x & y & z & w & \ \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & & & & \\ & & & & \end{array}\right] \end{array}$$

- ● The solution set could be empty
- ○ The solution set could contain a single vector
- ○ The solution set could have exactly one parameters
- ● The solution set could have exactly two parameters
- ● The solution set could have exactly three parameters

Solution: Ch1-LS3: **Given the REF of an augmented matrix, I can determine the type of solution set the system has (no solution, a unique solution, infinitely many solutions).** (Conceptual) ch1-CON-augsoltype//

(d) For what value(s) of a will the following linear system be inconsistent?

$$\begin{aligned} ax + y &= 1 \\ 2x + (3a+1)y &= -2 \end{aligned}$$

○ $a = 1$

○ $a = -1$

○ $a = \frac{2}{3}$

○ No such value exists

Solution: Ch1-LS3: **Given the REF of an augmented matrix, I can determine the type of solution set the system has (no solution, a unique solution, infinitely many solutions).** (Conceptual) ch1-CON-augsoltype//

aug. matrix $\left[\begin{array}{cc|c} a & 1 & 1 \\ 2 & 3a+1 & -2 \end{array}\right]$ $a = \frac{2}{3}$ yeilds $\left[\begin{array}{cc|c} \frac{2}{3} & 1 & 1 \\ 2 & 3 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} \frac{2}{3} & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$

which is inconsistent.

All other values yield a consistent system.

(e) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Let $\vec{v} \in \mathbb{R}^3$. Choose all that apply.

○ $A\vec{v}$ is not defined

● $A\vec{v}$ is a vector in $\mathbb{R}^2$

○ $A\vec{v}$ is a vector in $\mathbb{R}^3$

● $A\vec{v}$ is parallel to all columns of A.

○ $A\vec{v}$ is perpendicular to all columns of A.

$A\vec{v}$ is a linear combo of col's of A.
col's of A are in $\mathbb{R}^2$

Solution: Ch1-LS10: **I can interpret matrix-vector multiplication in terms of a linear combination of the columns of the matrix.** (Conceptual) ch1-CON-matlincom

Say $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$A\vec{v} = v_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + v_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + v_3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1+v_2+v_3 \\ v_1+v_2+v_3 \end{bmatrix} = (v_1+v_2+v_3)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This is a scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Fill in the blank. You do not need to include your computation or reasoning.

(a) (2 points) Suppose M is a linear transformation given by left multiplication by a 4×5 matrix. The domain of M is $\boxed{\mathbb{R}^5}$ and the codomain of M is $\boxed{\mathbb{R}^4}$

Solution: Suppose M is a linear transformation given by left multiplication by a 4×5 matrix. The domain of M is $\mathbb{R}^5$ and the codomain of M is $\mathbb{R}^4$.

Ch2-LS1: Given a linear transformation in any form, I can determine its domain and codomain. (Conceptual) ch2-CON-domcodom

(b) (1 point) Suppose $\begin{bmatrix}1\\3\\6\end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2 + 5\vec{v}_3$. Let $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$ be a matrix that has $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ as its columns.

Then $A\begin{bmatrix}2\\3\\5\end{bmatrix}$ is $\boxed{\begin{bmatrix}1\\3\\6\end{bmatrix}}$.

Solution: By MVP theorem $A\begin{bmatrix}2\\3\\5\end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2 + 5\vec{v}_3 = \begin{bmatrix}1\\3\\6\end{bmatrix}$

Ch1-LS10: I can interpret matrix-vector multiplication in terms of a linear combination of the columns of the matrix. (Conceptual) ch1-CON-matlincom

(c) (2 points) Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation satisfying $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-3\\7\end{bmatrix}$ and for all $a \in \mathbb{R}$,

$$T\left(\begin{bmatrix}-2+a\\-4+3a\end{bmatrix}\right) - T\left(\begin{bmatrix}a\\-a\end{bmatrix}\right) = \begin{bmatrix}6+8a\\-14+12a\end{bmatrix},$$

The second column of the standard matrix of T is $\boxed{}$.

Solution: There are many different ways of seeing this

Ch2-LS4: Given a linear transformation T in any form I can find T(x) for any vector x. (Computation) ch2-COM-lintransoutput

Since T is a linear transformation we have

$$T\left(\begin{bmatrix}-2+a\\-4+3a\end{bmatrix}\right) - T\left(\begin{bmatrix}a\\-a\end{bmatrix}\right) = T\left(\begin{bmatrix}-2+a-a\\-4+3a+a\end{bmatrix}\right) = T(\left(\begin{bmatrix}-2\\-4+4a\end{bmatrix}\right) = \begin{bmatrix}6+8a\\-14+12a\end{bmatrix}$$

Since this holds for all a, in particular it is true for $a = 1/2$ which gives us:

$$T(\left(\begin{bmatrix}-2\\-2\end{bmatrix}\right) = \begin{bmatrix}10\\-8\end{bmatrix}$$

Which since T is linear it implies

$$T(\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\-4\end{bmatrix}$$

We can write

$$T(\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T(\begin{bmatrix}1\\2\end{bmatrix}) + T(\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right) = \begin{bmatrix}-3\\7\end{bmatrix} + \begin{bmatrix}5\\-4\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}.$$

(d) (4 points) An angle between two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{R}^3$ is defined to be the angle between any two vectors perpendicular to each plane. Using this definition, calculate the angle between the planes given by equations $\mathcal{P}_1\colon\ 3x + 4y + 5z = 12$ and $\mathcal{P}_1\colon\ 7x + y = 8$. You can leave your answer in terms of $\cos^{-1}$.

The angle between the planes is

Solution: Notice that $\vec{n}_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ and $\vec{n}_2 = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$ are the normal vectors for $_1$ and $\mathcal{P}_2$ respectively. Let α denote the angle between the two normal vectors. Then

$$\cos(\alpha) = \frac{\vec{n}_1\vec{n}_2}{||\vec{n}_1||||\vec{n}_2||} = \frac{21+4}{\sqrt{50}\sqrt{50}} = \frac{1}{2}$$

Hence $\alpha = \cos^{-1}(1/2) = \pi/3$.

Ch0-LS6: I can compute the angle between any two vectors. (Computation) ch0-COM-angle

Find the vector parametric equation of the intersection line of the planes $\mathcal{P}_1$ and $\mathcal{P}_2$, described in set notation.

$\mathcal{P}_1 \cap \mathcal{P}_2 =$

Solution: Consider the system of linear equations given by the normal equations of $\mathcal{P}_1$ and $\mathcal{P}_2$. Solving the system and expressing the solution in vector form yields:

$$\{t \begin{bmatrix} 1 \\ -7 \\ 5 \end{bmatrix} + \begin{bmatrix} 4/5 \\ 12/5 \\ 6 \end{bmatrix} \mid t \in \mathbb{R}\}.$$

Ch2-LS1: I can perform row reduction on any matrix and reduce it to REF and RREF. (Computation) ch1-COM-rref

Ch2-LS3: Given an augmented matrix of a linear system, I can find the general solution to the system. (Computation) ch1-COM-augset

Ch2-LS7:Given a system of linear equations in R2 or R3, I can visualize the system and its general solution. (Visual/Geometry) ch1-VG-gslinsys

Consider the vectors perpendicular to $\mathcal{P}_1$ and $\mathcal{P}_2$ you chose to calculate the angle between the planes. Call them $\vec{v}_1$ and $\vec{v}_2$ respectively. Let $\vec{d}$ be a nonzero vector in $\mathcal{P}_1 \cap \mathcal{P}_2$. Then

- [x] $\vec{d}$ perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.
- [ ] $\vec{v}_1, \vec{v}_2$ are parallel to $\vec{d}$.
- [ ] $\vec{d}$ is parallel to at least one of $\vec{v}_1$ and $\vec{v}_2$.
- [ ] None of the given statements are true.
- [ ] $\vec{v}_1, \vec{v}_2$ and $\vec{d}$ are all perpendicular to each other.

Solution: $\vec{d}$ is on $\mathcal{P}_1$, hence it is perpendicular to $\vec{v}_1$, it is also on $\mathcal{P}_2$ and hence it is perpendicular to $\vec{v}_2$. Hence

$\vec{d}$ perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

Ch0-LS11: Given an algebraic description of a plane, I can visualize the plane. (Visual/Geometry) ch0-VG-plane

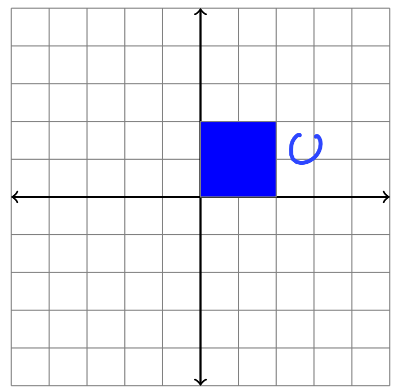
Part B

3. Consider the following linear transformations:

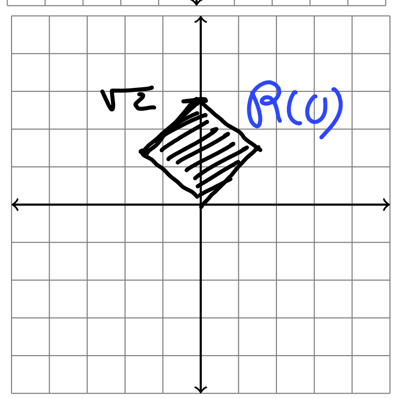
$R : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counter-clockwise about the origin by $\frac{\pi}{4}$ (or, if you prefer, 45°).

$T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors with respect to the line ℓ where

$$\ell = \{t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ t \in \mathbb{R}\}.$$



The following square has corners at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is called the unit square.



(a) (2 points) Draw the output of the unit square after applying R.

Solution:
Ch2-LS5: Given a linear transformation with domain and codomain in R2 or R3 I can visualize the effect of the transformation on the unit square or the unit cube. (Visual/Geometry) ch2-VG-unitlintrans
Ch2-LS7: I can visualize the effect of the geometric linear transformations on R2 and R3. (Visual/Geometry) ch2-VG-geotrans

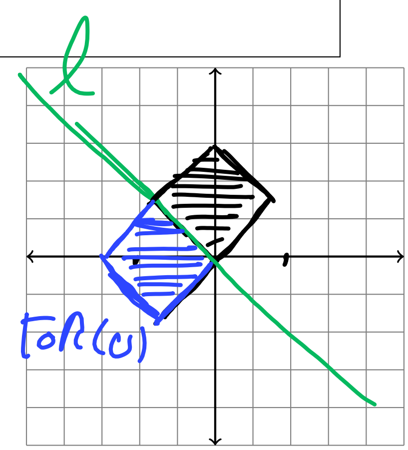
(b) (2 points) Let A be the standard matrix of R. That is $R(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^2$. Find A. Justify your answer.

A= $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Solution:
Ch2-LS6: I can find (not memorize) the standard matrix representation of the geometric transformations (rotation, reflection, projection, shear, scaling) on R2. (Conceptual) ch2-CON-geotrans

$$A = \sqrt{2}/2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

(c) (2 points) Draw the output of the unit square after applying $T \circ R$



Solution:
Ch2-LS9: I can visualize the effect of the composition of two linear transformations on a given vector in R2 and R3. (Visual/Geometry) ch2-VG-comp

(d) (2 points) Let C be the standard matrix of $T \circ R$. That is $T \circ R(\vec{x}) = C\vec{x}$. Find C. Justify your answer.

We apply T to col's of A.

$$T\left(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\right) = -\begin{bmatrix} +\frac{\sqrt{2}}{2} \\ +\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$T\left(\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\boxed{C= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}$$

Solution: Ch2-LS8: Given two linear transformations T and S, in any form, I can compute the standard matrix of T ∘ S or S ∘ T or show why they are not defined. (Conceptual) ch2-CON-lintranscomp

(e) (2 points) Find a matrix B such that $BA = C$. Justify your answer.

By the definition of composition of LTs

B is the standard matrix of T.

$B = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$, we can find $T(\vec{e}_i)$'s geometrically.

$$\boxed{B= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}$$

Solution: Ch2-LS6: I can find (not memorize) the standard matrix representation of the geometric transformations (rotation, reflection, projection, shear, scaling) on R2. (Conceptual) ch2-CON-geotrans

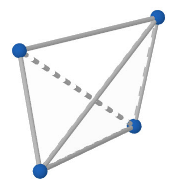
4. Given a set of n particles in $\mathbb{R}^3$ with position vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ and masses $m_1, m_2, ..., m_n$, the position vector of the center of mass of the system is

$$\vec{v}_{cm} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n),$$

where $M = m_1 + m_2 + \cdots + m_n$.

Consider four particles with position vectors

$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}2\\1\\-1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-3\\1\\0\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}-2\\1\\-2\end{bmatrix}$$



forming a pyramid in the space.

We want to distribute a total mass of 9 kg among these particles so that the position vector of the centre of the mass becomes $\vec{v}_{cm} = \frac{1}{9}\begin{bmatrix}4\\4\\1\end{bmatrix}$. You can assume that the mass of the rest of the pyramid itself is negligible.

(a) (2 points) Set up a system of linear equations whose general solution is the set of all the possible values of m_1, m_2, m_3, m_4 that give us the desired center of mass.

On the one hand we have: $\frac{1}{9}(m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_3\vec{v}_3) = \vec{v}_{cm}$

On the other we have $m_1 + m_2 + m_3 + m_4 = 9$

Solution: General LS: I can correctly use mathematical notation. (Writing) chg-WRIT-matno

putting all together:

$$\begin{cases} m_1 + 2m_2 - 3m_3 - 2m_4 = 4 \\ m_2 + m_3 + m_4 = 4 \\ m_1 - m_2 - 2m_4 = 1 \\ m_1 + m_2 + m_3 + m_4 = 9 \end{cases}$$

(b) (3 points) Find a matrix A and a vector $\vec{b}$ such that your system is equivalent to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 9 \end{bmatrix}$$

Solution: General LS: I can correctly use mathematical notation. (Writing) chg-WRIT-matno

(c) (1 point) You can assume that the matrix A in the previous part row-reduces to the 4×4 identity matrix. That is

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

this implies $A\vec{m} = \vec{b}$ has a unique sol'n

Is it possible to distribute the mass of 9 kg among these particles so that the position vector of the centre of the mass becomes $\vec{v}_{cm} = \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$? Justify your choice.

Solution: Ch1-LS5: Given the REF of an augmented matrix, I can determine the type of solution set the system has (no solution, a unique solution, infinitely many solutions). (Conceptual) ch1-CON-augsoltype

- [x] Yes
- [ ] No

5. State whether each statement is true or false by writing "True" or "False" in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true, explain why it must be true. If you think a statement is false, give a counterexample.

(a) (3 points) If the sum of two vectors in $\mathbb{R}^n$ is $\vec{0}$, they are parallel.

T

Solution: General LS:

Given a statement I can decide whether it is true or false, and correctly justify my decision. (Conceptual) chg-CON-tf

$\vec{v} + \vec{w} = \vec{0} \Rightarrow \vec{v} = -\vec{w}$

Two vectors are parallel if one is a scalar mult of the other.

(b) (3 points) A vector is called a unit vector if its magnitude (or norm) is one. The intersection between the set of unit vectors perpendicular to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$ and the set of unit vectors perpendicular to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$ is has exactly two vectors.

T

Solution: Given a statement I can decide whether it is true or false, and correctly justify my decision. (Conceptual) chg-CON-tf
AND (ch0-WRIT-sets, ch0-COM-vecarith)

$\vec{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \Rightarrow v_1 = 0$

$\vec{v} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow v_2 = 0$

$\|\vec{v}\| = 1 \Rightarrow \sqrt{v_3^2} = 1$

$\Rightarrow v_3 = \pm 1$

possible pts are $(0,0,1)$; $(0,0,-1)$

(c) (3 points) Let $\vec{c}$ be a solution to $A\vec{x} = \vec{0}$, and $\vec{y}$ be a solution to $A\vec{x} = \vec{b}$. Then $\vec{c} + \vec{y}$ is a solution to $A\vec{x} = \vec{b}$.

T

$A(\vec{c}+\vec{y}) = A\vec{c} + A\vec{y} = \vec{0} + \vec{b} = \vec{b}$

Solution: Given a statement I can decide whether it is true or false, and correctly justify my decision. (Conceptual) chg-CON-tf AND (ch0-WRIT-sets, ch0-COM-vecarith)

6. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

(a) (2 points) A linear transformation T which maps the set $L = \left\{ \begin{bmatrix}8\\4\end{bmatrix} + t\begin{bmatrix}2\\1\end{bmatrix},\ t \in \mathbb{R} \right\}$ to the set $T(L) = \left\{ t\begin{bmatrix}1\\3\\6\end{bmatrix}, t \in \mathbb{R} \right\}$.

$= \left\{ t\begin{bmatrix}2\\1\end{bmatrix},\ t \in \mathbb{R} \right\}$

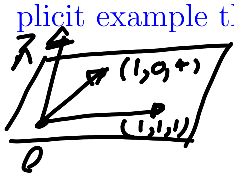
Solution:

There are so many possibilities. Any map T that sends $\begin{bmatrix}2\\1\end{bmatrix}$ to a nonzero multiple of $\begin{bmatrix}1\\3\\6\end{bmatrix}$ will do. All we need is

$T(2\vec{e}_1+\vec{e}_2) = \begin{bmatrix}1\\3\\6\end{bmatrix}$. Take $T(\vec{e}_1) = \begin{bmatrix}0\\0\\3\end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix}1\\3\\0\end{bmatrix}$. So $T(\vec{x}) = \begin{bmatrix}0 & 1\\0 & 3\\3 & 0\end{bmatrix}\vec{x}$.

(b) (2 points) The normal equation of a plane in $\mathbb{R}^3$ that passes through the origin and contains the points $(1, 0, 4)$ and $(1, 1, 1)$.

Solution: Given a description of a mathematical object I can find an explicit example that satisfies that definition. (Conceptual) chg-CON-exp



$\vec{n}\cdot\begin{bmatrix}1\\0\\4\end{bmatrix}=0$, $\vec{n}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}=0$ $\Rightarrow \begin{cases} n_1+4n_3=0 \\ n_1+n_2+n_3=0 \end{cases} \rightarrow \left[\begin{array}{ccc|c}1&0&4&0\\1&1&1&0\end{array}\right]$

$\sim \left[\begin{array}{ccc|c}1&0&4&0\\0&1&-3&0\end{array}\right]$. Take $\vec{n} = \begin{bmatrix}-4\\3\\1\end{bmatrix}$ $\boxed{-4x+3y+z=0}$

(c) (2 points) A linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that stretches the x-axis by a factor of 5 and does not change the y-axis.

$T(\vec{e}_1) = 5\vec{e}_1$

$T(\vec{e}_2) = \vec{e}_2$

$T(\vec{x}) = \begin{bmatrix}5 & 0\\0 & 1\end{bmatrix}\vec{x}$

Solution: Given a description of a mathematical object I can find an explicit example that satisfies that definition. (Conceptual) chg-CON-exp

Part C

7. Methane is CH_4 (that is, one carbon atom and four hydrogen atoms) and propane is C_3H_8 (three carbon, eight hydrogen). When a mixture of the two is burned in the right conditions, the chemical reaction produces only water (H_2O) and carbon dioxide (CO_2):

$$a \cdot CH_4 + b \cdot C_3H_8 + c \cdot O_2 \mapsto d \cdot CO_2 + e \cdot H_2O$$

   (a) (2 points) Create a system of equations to represent a balanced reaction (that is, the number of atoms of each type on one side must equal the number from the other side).

   (b) (1 point) Put your variables in alphabetical order and find the augmented matrix corresponding to this system.

\# of atoms on the left = # of atoms on the right

$$\begin{array}{l} C \\ H \\ O \end{array} \begin{cases} a + 3b = d \\ 4a + 8b = 2e \\ 2c = 2d + e \end{cases} \Rightarrow \begin{cases} a + 3b - d = 0 \\ 4a + 8b - 2e = 0 \\ 2c - 2d - e = 0 \end{cases}$$

$$\begin{array}{ccccc} a & b & c & d & e \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & -1 & 0 & 0 \\ 4 & 8 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & 0 \end{array}\right]$$

$$a = -2d + \frac{3}{2}e$$

$$b = d - \frac{1}{2}e$$

$$c = d + \frac{1}{2}e$$

d, e free

(c) (1 point) Find the RREF of the augmented matrix of your system.

(d) (2 points) Solve the system and describe the solution in set notation.

Solution: I can perform row reduction on any matrix and reduce it to REF and RREF. (Computation) ch1-COM-rref

Given an augmented matrix of a linear system, I can find the general solution to the system. (Computation) ch1-COM-augset

I can correctly use mathematical notation. (Writing) chg-WRIT-matno

$$\left\{ t\begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \;\middle|\; t, s \text{ in } \mathbb{R} \right\}$$

(e) (2 points) Why does the solution have the number of parameters that appear? Interpret your answer in terms of the given chemical reaction.

There are two independent reactions: The burning of CH_4 & the burning of C_3H_8; there are two parameters each control the balance required for the burning of each.

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