PART I No explanation is necessary. **(10 marks)**

1. Consider the solid of revolution generated by revolving the region between two functions $f(x) \leqslant g(x)$ for $x \in [a,b]$ around the $x-$axis. Then its volume is given by (circle **one** choice)

(a) $\int_a^b (g(x) - f(x))\,dx$ **(c)** $\int_a^b \pi(g(x) - f(x))^2\,dx$

(b) $\int_a^b 2\pi x(g(x) - f(x))\,dx$ **(d)** $\int_a^b \pi(g(x)^2 - f(x)^2)\,dx$

2. Consider $\int_0^{\frac{\pi}{2}} \sin^{83} x \cos^{83} x\,dx$ and make a substitution to obtain

$$\int_0^{\frac{\pi}{2}} \sin^{83} x \cos^{83} x\,dx = \int_a^b f(u)\,du.$$

The substitution is

$u =$ sin x cos x

$a =$ 0 OR 1

$b =$ 1 0

3. On the integral of question **2**, the integrand becomes

$f(u) = u^{83}(1-u^2)^{41}$ OR $-u^{83}(1-u^2)^{41}$

4. A radioactive material decayed by 10% in 50 years.

Its half-life is $50 \cdot \frac{\log(.5)}{\log(.9)}$ for any log base.

5. Let $a > 0$ and consider the region bounded by the graph of $y = ae^{-ax}$ and the $x-$axis on the interval $[0, \infty)$.

Its area is 1 .

6. Consider the rational function $\dfrac{4x^2 - 2x^2 + x}{(x+1)(x-2)^3(x^2+9)^2}$. When using **partial fractions**, we write this function as a sum of the following terms (**circle all that apply**):

(a) $\dfrac{A}{x}$ (d) $\dfrac{D}{(x+1)}$ (g) $\dfrac{G}{(x-2)}$ (j) $\dfrac{J}{(x^2+9)}$ (m) $\dfrac{Mx+N}{(x^2+9)}$

(b) $\dfrac{B}{x^2}$ (e) $\dfrac{E}{(x+1)^2}$ (h) $\dfrac{H}{(x-2)^2}$ (k) $\dfrac{K}{(x^2+9)^2}$ (n) $\dfrac{Ox+P}{(x^2+9)^2}$

(c) $\dfrac{C}{x^3}$ (f) $\dfrac{F}{(x+1)^3}$ (i) $\dfrac{I}{(x-2)^3}$ (l) $\dfrac{L}{(x^2+9)^3}$ (o) $\dfrac{Qx+R}{(x^2+9)^3}$

7. Consider two functions $f(x)$ and $g(x)$ satisfying $0 \leqslant f(x) \leqslant g(x)$ for $x \in (0, \infty)$.

Assume that $\int_1^\infty g(x)\,dx$ **converges**. Then $\int_1^\infty f(x)\,dx$

(a) converges (b) diverges (c) we cannot tell

8. Consider two functions $f(x)$ and $g(x)$ satisfying $0 \leqslant f(x) \leqslant g(x)$ for $x \in (0, \infty)$.

Assume that $\int_1^\infty g(x)\,dx$ **diverges**. Then $\int_1^\infty f(x)\,dx$

(a) converges (b) diverges (c) we cannot tell

9. Recall that when approximating the integral $\int_a^b f(x)\,dx$ using the trapezoid rule, we make an error of at most $E_T \leqslant \frac{K(b-a)}{12}(\Delta x)^2$, where $K = \max_{x\in[a,b]} |f''(x)|$ and $\Delta x = \frac{b-a}{n}$.

To approximate the integral $\int_0^1 e^{(x^2)}\,dx$ with a maximum error of $\frac{e}{32}$, I should choose

$n \geqslant$ 4 $\left[f''(x) = (2+4x^2)e^{x^2} , \ |f''(x)| \leq 6 \right]$

10. A free-hanging rope forms a catenary: a curve which satisfies

$$y''(x) = \frac{1}{a}\sqrt{1 + (y'(x))^2} \qquad \text{for } x \in [-b, b].$$

Assume that for this rope, $y'(b) = -y'(-b) = \frac{10}{a}$. Then the length of the rope is

$$L = \int_{-b}^{b} \sqrt{1 + (y'(x))^2}\,dx = a\int_{-b}^{b} y''(x)\,dx = a\left[y'(x)\right]_{-b}^{b} = 20$$

(express the length as a number explicitly)

PART II **Justify** your answers.

11. You are working at a biology lab with a population of bacteria which grows **(10 marks)** proportionally to its population. Moreover, the population doubles its size every hour.

(a) Assuming that you start with P_0 million bacteria, find a formula for the population of bacteria after t hours.

Solution #1.

$$P(t) = P_0 \cdot 2^t \text{ million bacteria.}$$

Solution #2.

$$P(t) = P_0 e^{kt}$$

Since the population doubles every hour, we have $P(1) = 2P(0) = 2P_0$

$$\Leftrightarrow \cancel{P_0} e^k = 2\cancel{P_0}$$

$$\Leftrightarrow e^k = 2$$

$$\Leftrightarrow k = \ln 2$$

So $P(t) = P_0 e^{t \ln 2}$ million bacteria.

(b) You start with 100 million bacteria and you have two containers. Each can hold 300 million bacteria. Your job is to grow as many bacteria as you can in 2 hours.

What is the best way to divide the bacteria in the two containers? Justify your answer.

(**Hint.** This question is not hard)

Method I:

Splitting the population does not change its rate of growth. As long as neither container reaches its maximum, we expect $2^2 \cdot 100$ $= 400$ million bacteria after 2 hours.

$\therefore$ Any split that allows us to have no more than 300 million bacteria/container will work. So, each container should have at most $\frac{300}{2^2} = 75$ million bacteria.

Method II:

Let the containers initially hold x and y million bacteria (so, $x+y=100$).

After two hours, we have

$$x \cdot 2^2 + y \cdot 2^2 = 4(x+y) = 400 \text{ million bacteria}$$

as long as $4x \leq 300$ and $4y \leq 300$.

$$\therefore \; x \leq 75 \;, \quad y \leq 75$$

$$100 - x \leq 75$$

$$x \geq 25.$$

So, containers begin with x and $100-x$ million bacteria, with $25 \leq x \leq 75$.

12. Compute the following integrals. **(10 marks)**

(a) Let $b, \omega > 0$. Calculate $\int_0^b e^{-x}\sin(\omega x)\,dx. = I$

Integration by Parts twice ("bootstrapping")

Easier

$f' = e^{-x} \quad g = \sin(\omega x)$

$f = -e^{-x} \quad g' = \omega\cos(\omega x)$

$I = -e^{-x}\sin(\omega x)\Big|_0^b + \omega\int_0^b e^{-x}\cos(\omega x)dx$

$= -e^{-b}\sin(\omega b) + \omega\int_0^b e^{-x}\cos(\omega x)dx$

$f' = e^{-x} \quad g = \cos(\omega x)$

$f = -e^{-x} \quad g' = -\omega\sin(\omega x)$

$= -e^{-b}\sin(\omega b) + \omega\left[-e^{-x}\cos(\omega x)\Big|_0^b - \omega\int_0^b e^{-x}\sin(\omega x)dx\right]$

$= -e^{-b}\sin(\omega b) + \omega(-e^{-b}\cos(\omega b) + 1) - \omega^2 I$

$\therefore\ I = \frac{1}{\omega^2+1}\left[-e^{-b}\sin(\omega b) - \omega e^{-b}\cos(\omega b) + \omega\right]$

Tougher

use $f' = \sin(\omega x) \quad g = e^{-x}$

$f = -\frac{1}{\omega}\cos(\omega x) \quad g' = -e^{-x}$

$I = -\frac{1}{\omega}e^{-b}\cos(\omega b) + \frac{1}{\omega}\int_0^b e^{-x}\cos(\omega x)dx$

$f' = \cos(\omega x) \quad g = e^{-x}$

$f = \frac{1}{\omega}\sin(\omega x) \quad g' = -e^{-x}$

which leads to the same answer as above.

(b) Calculate $\displaystyle\int_0^1 \frac{\arcsin(x)\sqrt{1-x^2}}{\cos\left(\arcsin(x)\right)}\,dx. = I$

(**Hint.** Use a substitution)

Quick method: $\cos(\arcsin(x)) = \sqrt{1-x^2}$, either by triangles  or by unit circle  $(-, x)$

$\therefore I = \int_0^1 \arcsin(x)\,dx$, then IBP $f' = 1$, $g = \arcsin x$

$f = x$, $g' = \frac{1}{\sqrt{1-x^2}}$

By substitution:

Let $u = \arcsin x$ $\qquad x = \sin u$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad dx = \cos u\,du$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \frac{u\,(1-\sin^2 u)}{\cos(u)}\,du$$

$$= \int_0^{\frac{\pi}{2}} \frac{u \cdot \cos^{\cancel{2}} u}{\cancel{\cos u}}\,du$$

$f = u \qquad g' = \cos u$

$f' = 1 \qquad g = \sin u$

$$= u \sin u \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin u$$

$$= \frac{\pi}{2}\cdot 1 - 0\cdot 0 + \left[\cos u \Big|_0^{\frac{\pi}{2}}\right]$$

$$= \frac{\pi}{2} + 0 - 1$$

$$= \frac{\pi}{2} - 1$$

13. Let $u(t)$ be the temperature in ^oC at the Pearson airport t years after March 1, 2000. **(10 marks)**

Then the average temperature for the first decade (2000-2010) is

$$\text{Average temperature} = \frac{1}{10}\int_0^{10} u(t)\,dt.$$

(a) Let $a < b$. What is the average temperature from March 1 of the year $2000 + a$ to September 1 of the year $2000 + b$?

Sep. 1 is $\frac{1}{2}$ year after Mar. 1 of the same year.

So, Mar. 1, 2000+a to Sep. 1, 2000+b is $b - a + \frac{1}{2}$ years.

$\therefore$ Average temperature $= \dfrac{1}{b-a+\frac{1}{2}} \displaystyle\int_a^{b+\frac{1}{2}} u(t)\,dt$.

(b) Assume that $u(t) = 5 + 30e^{-t}\sin(2\pi t)$. If this temperature pattern holds forever, what is the limiting average temperature?

The limiting average temp. is found by taking (surprisingly) the limit of the average temperature from 0 to N, as N goes to ∞.

$$Avg = \lim_{N \to \infty} \frac{1}{N} \int_0^N u(t)\,dt$$

$$= \lim_{N \to \infty} \frac{1}{N} \int_0^N 5 + 30e^{-t} \sin(2\pi t)\,dt$$

$$= \lim_{N \to \infty} \frac{1}{N} \left[5N + \frac{30}{1+4\pi^2} \left(-e^{-N} \sin(2\pi N) - 2\pi e^{-N} \cos(2\pi N) + 2\pi \right) \right]$$

from 12(a), $b = N$, $\omega = 2\pi$

$= 5$, as all others go to 0 by Squeeze Thm.

14. Consider the function $f(x) = \frac{p}{x^p}$. Consider the solid created by rotating this function around the $x-$axis over the interval $[1, \infty)$.

(a) Calculate the volume of the solid. **(7 marks)**

By washers, the volume is

$$V = \int_{1}^{\infty} \pi \cdot [f(x)]^2 \, dx$$

$$= \lim_{N\to\infty} \pi \int_1^N \frac{p^2}{x^{2p}} \, dx$$

$$= \lim_{N\to\infty} \pi \cdot p^2 \int_1^N x^{-2p} \, dx$$

$$= \lim_{N\to\infty} \pi p^2 \left(\frac{x^{1-2p}}{1-2p} \right)\Big|_1^N$$

$$= \lim_{N\to\infty} \frac{\pi p^2}{1-2p} \left(N^{1-2p} - 1 \right)$$

This converges for $p > \frac{1}{2}$ to give $V = \frac{\pi p^2}{2p-1}$.

[The only other value for which V converges is $V = 0$ when $p = 0$.]

(b) Find the value of p that minimizes the volume of this solid. **(3 marks)**

Assuming $p > \frac{1}{2}$ (otherwise, $p = 0$ gives the minimum), we start with

$$V = \frac{\pi p^2}{2p-1}$$

$$\frac{dV}{dp} = \pi \cdot \frac{2p(2p-1) - p^2(2)}{(2p-1)^2} = 0$$

$$\frac{2\cancel{\pi}(p^2 - p)}{(2p-1)^2} = 0$$

$p = 0, \frac{1}{2}, 1$ are the critical values.

$\frac{dV}{dp}$	V undefined		$-$		$+$
	0		$\frac{1}{2}$		1

$\therefore$ The minimum value occurs at $p = 1$ [where $V = \pi$].