

University of Toronto  
Faculty of Applied Sciences and Engineering

## **MAT187 - Summer 2025**

### Lecture 15

Instructor: Arman Pannu

We will start 10 minutes past the hour. Use this time to make a new friend.

# Harmonic Oscillators

An undamped harmonic oscillator is  
any system that satisfies

$$F(x) = -Kx$$

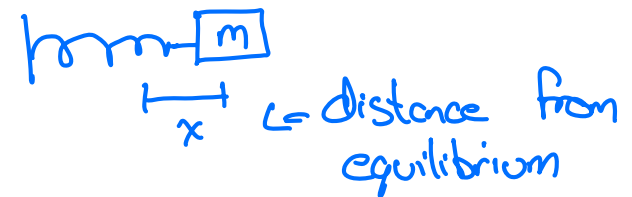
$$m\ddot{x} = -Kx$$

$$\boxed{m\ddot{x} + Kx = 0}$$

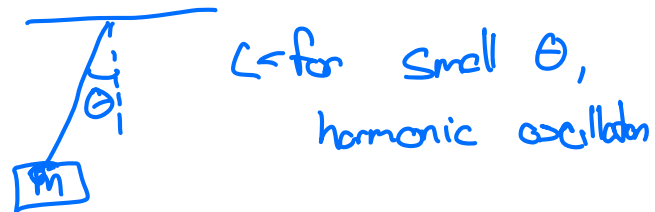
$$\begin{aligned} \hookrightarrow F &= ma \\ a &= \frac{d^2x}{dt^2} = \ddot{x} \end{aligned}$$

$$K > 0$$

ex://  
 $\rightarrow$  an ideal spring



$\rightarrow$  a pendulum



$\rightarrow$  LCR circuits with  
zero resistance



Solution to undamped harmonic oscillator:

$$m\ddot{x} + Kx = 0$$

$$\text{char. eq. : } mr^2 + K = 0$$

$$r = \pm \sqrt{-\frac{K}{m}} = \omega \text{ since}$$

$K, m > 0$ , always pure  
imaginary

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\omega = \sqrt{\frac{k}{m}}$$

we often write

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

A solution of form  $C_1 \cos(\omega t) + C_2 \sin(\omega t)$  can be written as  $A \cos(\omega t + \phi)$  (or  $A \sin(\omega t + \phi)$ )

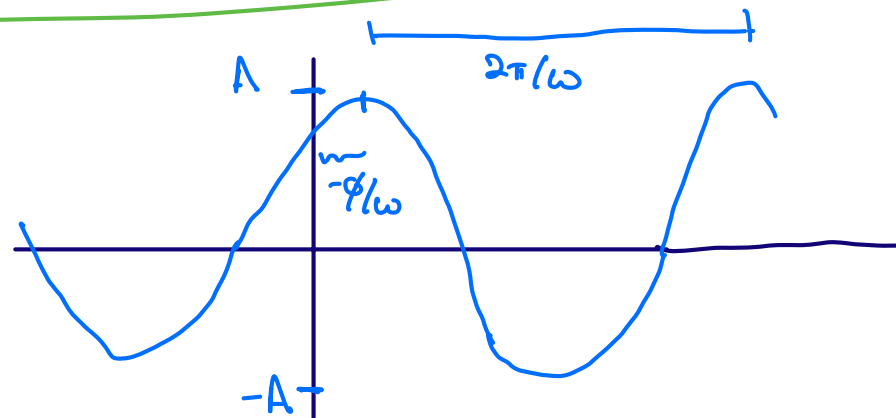
PF:  $A \cos(\omega t + \phi) \stackrel{?}{=} C_1 \cos(\omega t) + C_2 \sin(\omega t)$

$$A \cos(\phi) \sin(\omega t) + A \sin(\phi) \cos(\omega t) \stackrel{?}{=} C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\Rightarrow \begin{cases} C_1 = A \cos(\phi) \\ C_2 = A \sin(\phi) \end{cases} \Rightarrow \begin{cases} A = \sqrt{C_1^2 + C_2^2} \\ \phi = \arctan\left(\frac{C_2}{C_1}\right) \end{cases}$$

$$x(t) = A \cos(\omega t + \phi)$$

$\uparrow$  amplitude                       $\uparrow$  phase shift



Solve the IVP  $\ddot{x} + 4x = 0$ ,  $x_0 = 10$ ,  $\cancel{v_0 = 0}$   
 $\omega^2 \Rightarrow \omega = 2$   $v_0 = 0 \Rightarrow$  velocity at  $t=0$   
 $\dot{x}(0) = 0$   
 $x(0) = 10$

general solution:

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

$$10 = x_0 = x(0) = C_1(1) + C_2(0) \Rightarrow \boxed{C_1 = 10}$$

$$0 = \dot{x}(0) = -2C_1 \sin(2(0)) + 2C_2 \cos(2(0))$$
$$= 2C_2 \Rightarrow \boxed{C_2 = 0}$$

$$x(t) = 10 \cos(2t)$$

↑  
amplitude

# Damped Harmonic Oscillator

→ include a force that opposes velocity

$$\text{Force} = \underbrace{-Kx}_{\text{restoring force}} - \underbrace{2\gamma\dot{x}}_{\text{damping force}} \quad \begin{array}{l} K > 0 \\ \gamma > 0 \end{array}$$

$$m\ddot{x} + 2\gamma\dot{x} + Kx = 0$$

→ solution

$$mr^2 + 2\gamma r + K = 0$$

$$r = \frac{-(2\gamma) \pm \sqrt{(2\gamma)^2 - 4mK}}{2m}$$

$$r = -\frac{\gamma}{m} \pm \frac{1}{m} \sqrt{\gamma^2 - mK}$$

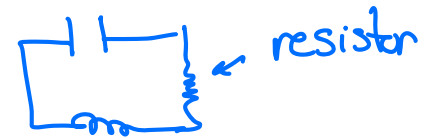
→ if  $\gamma^2 - mK > 0$ , 2 real roots  
 $\gamma^2 > mK$  ( $\gamma$  is large) **Overdamped**

→ applies to

→ laminar drag

→ internal resistance of spring

→ LCR circuits



$\rightarrow$  if  $\gamma^2 - mk < 0$ , complex roots  
 $\gamma^2 < mk$  ( $\gamma$  small) Underdamped  
 $\rightarrow$  if  $\gamma^2 - mk = 0$ , 1 repeated real root  
 $(\gamma \text{ "just-right"})$  Critically damped

### Overdamped Case

$\Rightarrow$  char eq.  $m\ddot{x} + 2\gamma\dot{x} + k = 0$  ( $\gamma^2 - mk > 0$ )

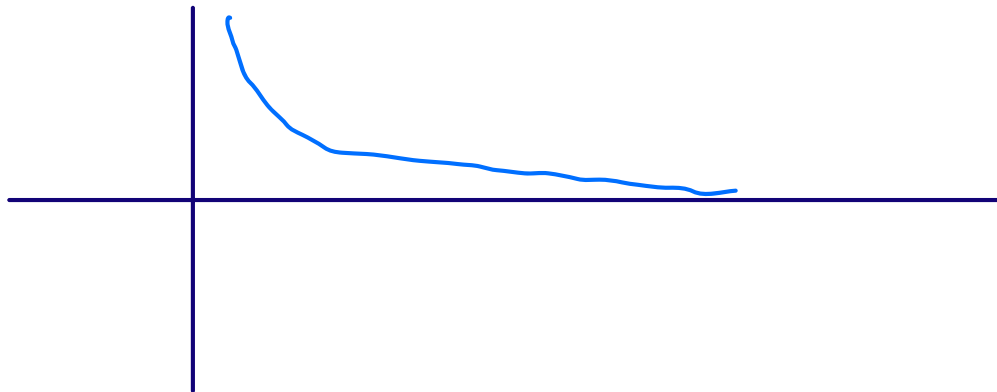
$$r_1 = -\frac{\gamma}{m} - \frac{1}{m} \sqrt{\gamma^2 - mk} < 0$$

$$r_2 = -\frac{\gamma}{m} + \frac{1}{m} \sqrt{\gamma^2 - mk} = -\frac{\gamma}{m} + \sqrt{\left(\frac{\gamma}{m}\right)^2 - \frac{k}{m}} < 0$$

} both roots are negative

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$\rightarrow$  overdamped, there is no oscillations



## Underdamped Case

$$\Rightarrow \text{char eq. } mr^2 + 2\gamma r + k = 0 \quad (\gamma^2 - mk < 0)$$

$$\text{let } \omega_d^2 = mk - \gamma^2$$

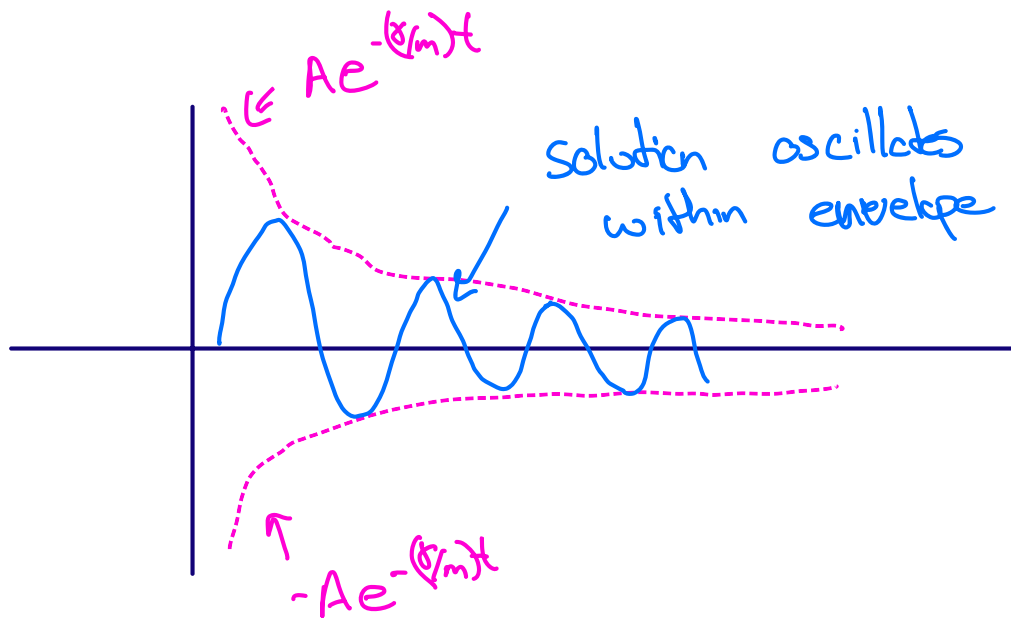
$$r_1 = -\frac{\gamma}{m} + i\sqrt{mk - \gamma^2} = -\frac{\gamma}{m} + i\omega_d$$

$$r_2 = -\frac{\gamma}{m} - i\sqrt{mk - \gamma^2} = -\frac{\gamma}{m} - i\omega_d$$

$$x(t) = e^{-\frac{\gamma}{m}t} \left( C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t) \right)$$

$$= e^{-\frac{\gamma}{m}t} A \cos(\omega_d t + \phi)$$

$$\left. \begin{array}{l} \text{max value of } \cos = 1 \\ \text{max } x(t) = A e^{(-\gamma/m)t} \\ \text{min } x(t) = -A e^{(-\gamma/m)t} \end{array} \right\}$$



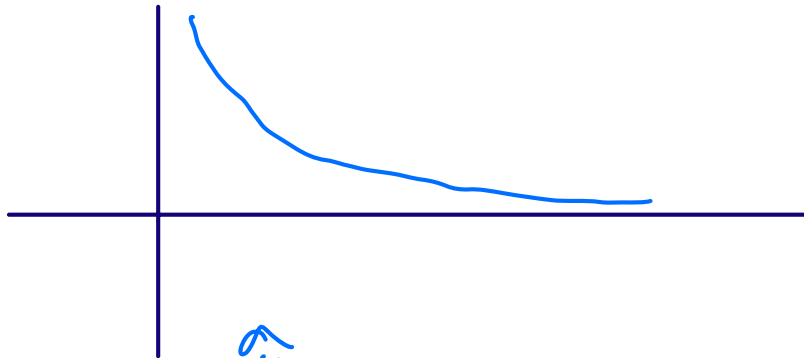
→ underdamped, there are oscillations that die down to zero

## Critically Damped

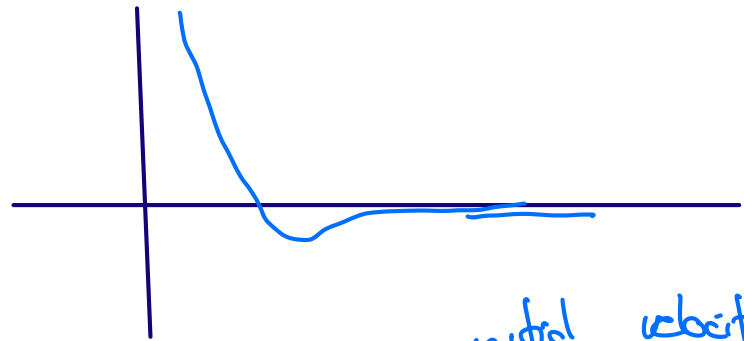
$\Rightarrow$  char eq.  $mr^2 + 2\gamma r + k = 0$  ( $\gamma^2 - mk = 0$ )

$$\left. \begin{aligned} r_1 &= -\frac{\gamma}{m} + \sqrt{0} \\ r_2 &= -\frac{\gamma}{m} - \sqrt{0} \end{aligned} \right\} \begin{array}{l} \text{repeated} \\ \text{root} \end{array}$$

$$\begin{aligned} x(t) &= c_1 e^{-(\gamma/m)t} + c_2 t e^{-(\gamma/m)t} \\ &= (c_1 + c_2 t) e^{-(\gamma/m)t} \end{aligned}$$



$\nearrow$   
Solution goes to  
zero without oscillation



$\nearrow$   
if  $v_0$  is large  
then solution crosses  
zero exactly once

Solve the initial value problem:

$$\ddot{x} + 6\dot{x} + 9x = 0, \quad x(0) = 2, \quad \dot{x}(0) = -1$$

$$\Rightarrow \text{char eq. } r^2 + 6r + 9 = 0$$

$$r = -3 \quad (\text{repeated root})$$

$$x(t) = (C_1 + C_2 t) e^{-3t}$$

$$x'(t) = (C_1 + C_2 t)(-3)e^{-3t} + C_2 e^{-3t}$$

$$2 = x_0 = x(0) = (C_1 + C_2(0))e^{-3 \cdot 0} = C_1$$

$$\Rightarrow C_1 = x_0 = 2$$

$$-1 = v_0 = x'(0) = -3(C_1 + C_2(0))e^{-3 \cdot 0} + C_2 e^{-3 \cdot 0} = C_1 + C_2$$

$$-3C_1 + C_2 = v_0$$

$$\Leftarrow C_1 = x_0$$

$$\begin{aligned} C_2 &= v_0 + 3x_0 \\ C_2 &= -1 + 6 \end{aligned}$$

$$x(t) = (2 + 5t) e^{-3t}$$

# Forced Harmonic Oscillators

A damped harmonic oscillator but  
with external driving force

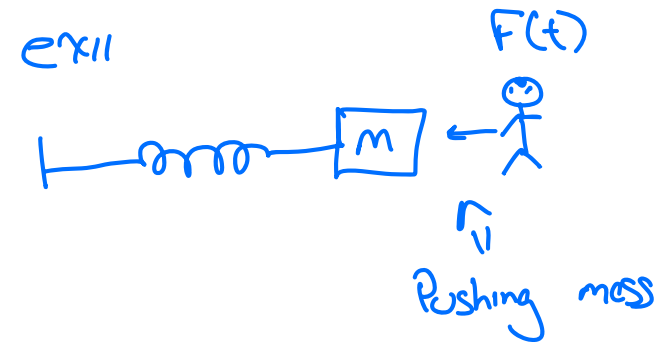
$$\underbrace{m}_{\text{force}} \ddot{x} + \underbrace{2\gamma}_{\text{damping}} \dot{x} + \underbrace{K}_{\text{spring force}} x = \underbrace{F(t)}_{\text{driving force}}$$

→ types of driving force

$$F(t) = t^2 + t + 3 \quad (\text{polynomial})$$

$$F(t) = Ae^{ct} \quad (\text{exponential})$$

$$F(t) = A \sin(\omega t) \quad (\text{sinusoidal})$$



← most interesting  
for harmonic  
oscillators

# Undamped Oscillator with Forcing

$$m\ddot{x} + Kx = F_0 \cos(\omega_f t)$$

Homogeneous solution:  $x(t) = A \cos(\omega t)$   $\omega = \sqrt{\frac{K}{m}}$   
 $\hat{=}$  assume phase  $\phi = 0$

Case 1:  $\omega_f \neq \omega$

$$x_p(t) = B \cos(\omega_f t + \phi)$$

$$x(t) = A \cos(\omega t) + B \cos(\omega_f t + \phi)$$

Case 2:  $\omega_f = \omega$

$$x_p(t) = B t \cos(\omega_f t + \phi)$$

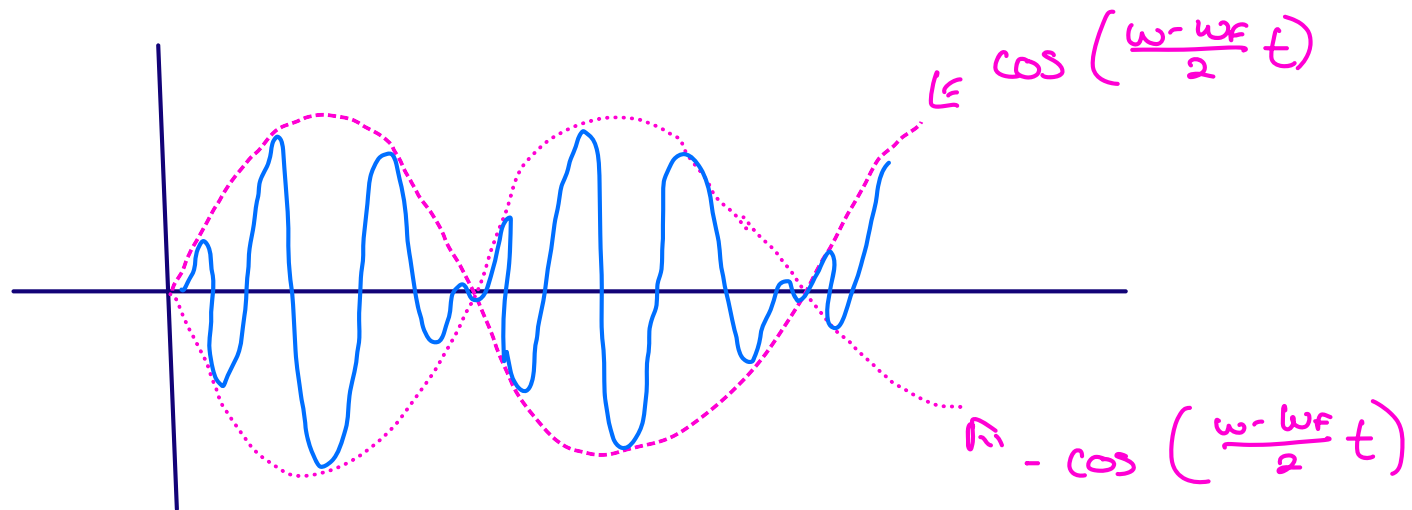
$$x(t) = A \cos(\omega t) + B t \cos(\omega_f t + \phi)$$

Case 1:  $\omega_f \neq \omega$  (non-resonant forcing)

$$x(t) = A \cos(\omega t) + B \cos(\omega_f t)$$

$$= (\text{some coeff}) \cos\left(\underbrace{\frac{(\omega + \omega_f)t}{2}}_{\text{large freq}}\right) \cos\left(\underbrace{\frac{(\omega - \omega_f)t}{2}}_{\text{small freq.}}\right)$$

(for simplicity assume  $\phi = 0$ )  
but solution looks same)  
 $\leftarrow$  trig identity

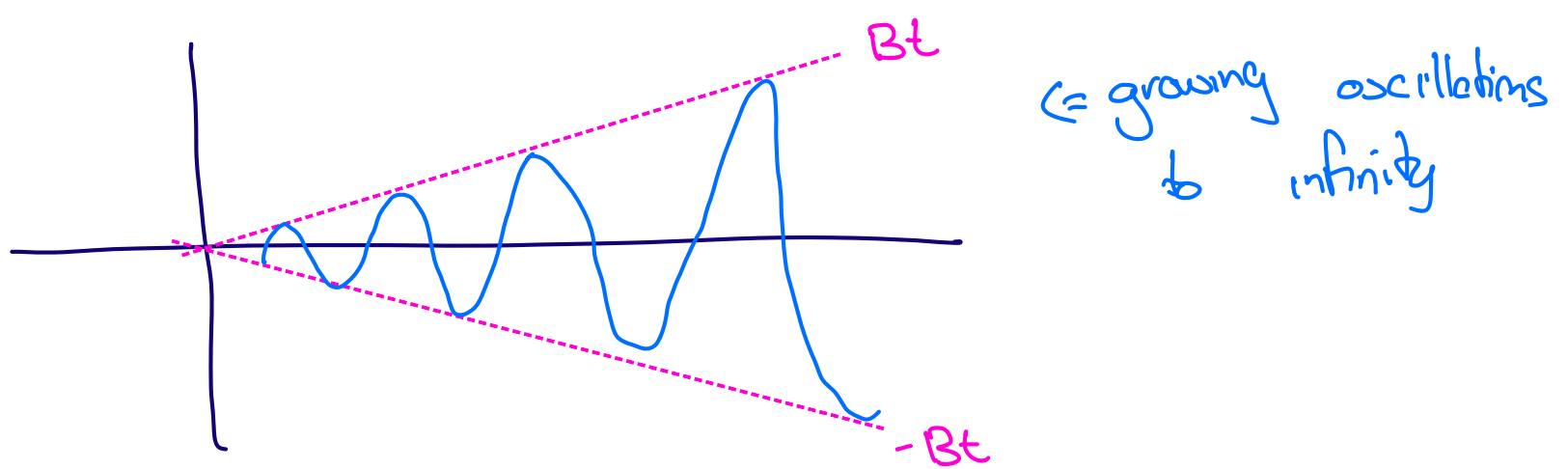


→ this is called "beats" phenomena

→ most pronounced when  $\omega$  is close to  $\omega_F$   
 i.e.  $|\omega - \omega_F|$  small so the envelope  
 is lower freq

Case 2: Resonance  $\omega = \omega_F$

$$x(t) = A \cos(\omega t) + \underbrace{Bt \cos(\omega t + \phi)}_{\substack{\text{dominating term} \\ \text{for } t \text{ large}}} \Leftrightarrow B \text{ is constant, not} \\ \text{dependent on initial condition}$$



## Driven & Damped Harmonic Oscillators

→ the undamped oscillator is an approximation for very underdamped oscillator in small time

In general:

$$m\ddot{x} + 2\gamma\dot{x} + K = F_0 \cos(\omega_F t)$$

$$x(t) = \underbrace{x_c(t)}_{\text{homogeneous solution}} + \underbrace{x_p(t)}_{\text{particular solution}}$$

① under damped

② over damped

③ critically

→ all three cases,  $\cos(\omega_F t)$  not a complementary sol'n

$$\Rightarrow x_p(t) = A_p \cos(\omega_F t)$$

(over)  $x(t) = c_1 e^{-\gamma_1 t} + c_2 e^{-\gamma_2 t} + A_p \cos(\omega_f t)$

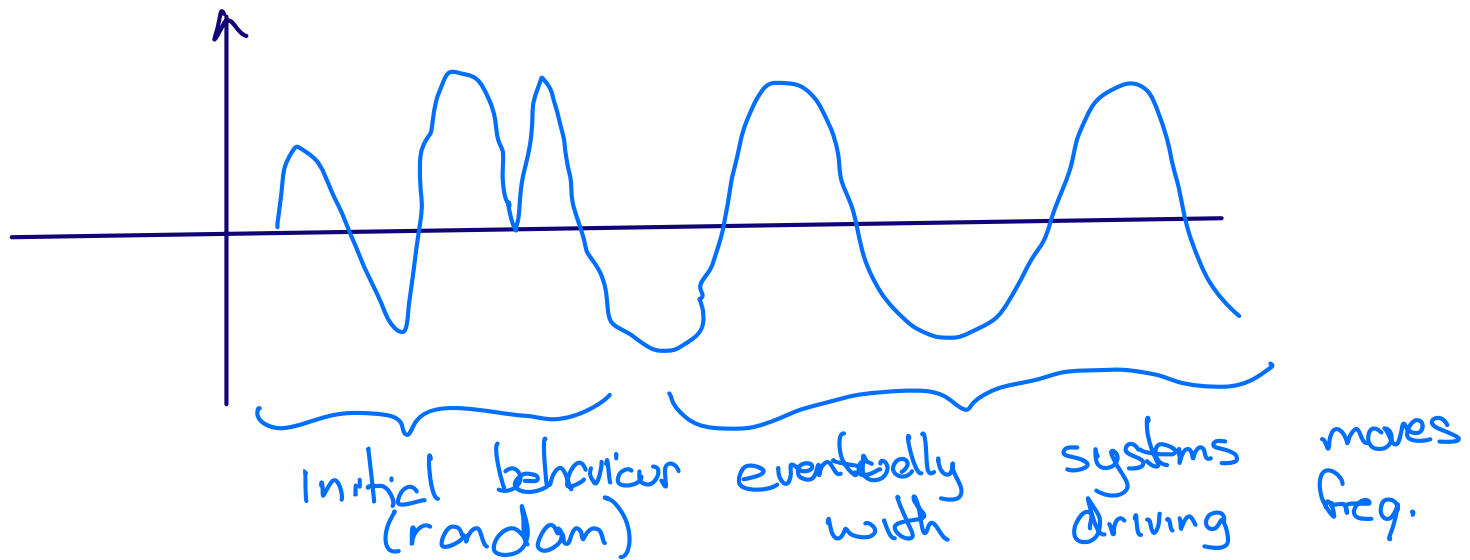
(under)  $x(t) = e^{-\frac{\gamma}{\omega_n} t} A \cos(\omega_f t + \phi) + A_p \cos(\omega_f t)$

(critical)  $x(t) = (c_1 + c_2 t) e^{-\frac{\gamma}{\omega_n} t} + A_p \cos(\omega_f t)$

→ in all cases  
complementary sol'n

→ 0 for large  $t$

$$x(t) \rightarrow A_p \cos(\omega_f t)$$



Conclusion: real-world systems don't experience "beats"

Phenomena or true resonance

→ lightly-damped oscillators demonstrates behaviours

Similar to resonance & beats for small time