University of Toronto
Faculty of Applied Science and Engineering

FINAL EXAMINATION, April 2007
First Year - CIV, CHE, IND, LME, MEC, MMS

MAT 187H1S, Calculus 2
Exam Type: A

Examiners: Y. Kim, G. Lynch, P. Milgram, R. Saghin

Last Name: _____

First Name: _____

Student Number: _____

Instructions:

- **The use of non-programmable calculators is permitted.**
- Answer all questions. Total marks: 100.
- Please have your student card ready for inspection and turn off all cellular phones.
- This paper has a total of 14 pages, including this cover page. Present your solutions (in other words, show your work!) in the space provided. Use the back of the preceding page if you need more space. The value of each question is indicated in square brackets beside each question number.
- Do not tear any pages out from this test.

FOR MARKER USE ONLY	
Question	Marks
1	
2	
3	
4	
5	
6	
7	
8	
Total	

1. Evaluate the following expressions and justify your answers.

(i) [5 marks] $\int \sin^3 x \cos^5 x \, dx$.

(ii) [5 marks] The exact value (no decimal approximations) of $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$.

(Question 1 continued)

(iii) [6 marks] The interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$.

(Question 1 continued)

(iv) [8 marks] $\displaystyle\int_1^{\infty} \frac{\arctan x}{x^2}\,dx.$

2. Let C be the parametric curve $x = 10 - t^2$, $y = t^3 - 12t$.

   (a) [4 marks] Show that C has two tangents at the point $(-2,0)$, and find the equations of both tangent lines.

   (b) [2 marks] Find the points where the tangent line is horizontal or vertical.

(Question 2 continued) Recall that $x = 10 - t^2$, $y = t^3 - 12t$.

(c) [3 marks] For what values of t is the curve concave upward or downward?

(d) [3 marks] Sketch the curve, using the information you have obtained from parts (a)-(c).

3. Find the general solution for the following differential equations; give your solution explicitly as y in terms of x.

(i) [6 marks] $4y'' + 8y' + 5y = 0$.

(ii) [6 marks] $\frac{dy}{dx} = \frac{1}{2}(e^{x/3} - y)$.

4. Determine whether the following series converges or diverges; justify your answer and clearly indicate the convergence/divergence test you are using. (If you do not state whether the series converges or diverges, you will not earn any part marks.)

(i) [4 marks] $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$.

(ii) [4 marks] $\sum_{n=0}^{\infty} \frac{7+2n^5}{(1+n^3)^2}$.

(Question 4 continued)

(iii) [4 marks] $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n^2+3n+5}{3n^2-6n+1}\right)^n$.

5. [10 marks] Find the critical points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. For each critical point, determine whether f has a local minimum, local maximum, or a saddle point.

6. [8 marks] Consider the polar curves $r = \sin 2\theta$ and $r = \sin \theta$. Sketch the graph of both curves and find the area of the region that lies inside both curves.

7. Let $f(x) = \sum_{n=0}^{\infty} x^n$, which converges for $|x| < 1$. Suppose we perform termwise differentiation on the series for $f(x)$.

   (a) [2 marks] What is the radius of convergence of the power series for $f^{(2007)}(x)$, the 2007th derivative of f? Briefly explain your answer.

   (a) [3 marks] Find a power series representation for $f(x) = \dfrac{x}{(1-x)^2}$.

   (c) [5 marks] Give the exact value of the sum $\sum_{n=0}^{\infty} (-1)^n \dfrac{n^2}{2^n}$.

8. [12 marks] Torricelli's Law states that

$$A(h)\frac{dh}{dt} = -a\sqrt{2gh}$$

where $h(t)$ is the depth of a fluid in the tank at time t, $A(h)$ is the cross-sectional area of the tank at height h above the exit hole, a is the cross-sectional area of the exit hole, and $g = 10$ m/sec^2 is the acceleration due to gravity.

Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm, how long will it take for the water to drain completely?

End of examination
(Available for scrap work. Do NOT tear out this page!)