UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION, DECEMBER 2014
DURATION: 2 AND 1/2 HRS
FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

EXAMINERS: D. BURBULLA, P. ESKANDARI, M. LEIN, Y. LOIZIDES,
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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Full Name: ______________________________
Last First

Student Number: ______________________________

Signature: ______________________________

Instructions:

- ONLY THE **FRONT PAGES** WILL BE SCANNED. THE BACK PAGES WILL NOT BE SEEN BY THE EXAMINERS.
- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 10 pages (including this cover page). Make sure you have all of them. Do not tear any pages from this exam.
- You can use the back of the pages and page 10 for rough work.
- This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

PART I : No explanation is necessary.

1. Big Theorem, Final Exam Version: Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix}$$

be the matrix with the vectors in $\mathcal{A}$ as its columns, and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Decide if the following statements are equivalent to the statement, "A is invertible." Circle Yes if the statement is equivalent to "A is invertible," and No if it isn't.

Note: +1 for each correct choice; −1 for each incorrect choice; and 0 for each part left blank.

(a) $\mathcal{A}$ spans $\mathrm{col}(A)$. Yes No

(b) A^T is invertible. Yes No

(c) The reduced echelon form of A is I, the $n \times n$ identity matrix. Yes No

(d) T is onto. Yes No

(e) $\mathbf{0}$ is not in $\mathrm{col}(A)$. Yes No

(f) $\ker(T) = \{\mathbf{0}\}$. Yes No

(g) $\mathcal{A}$ is a basis for $\mathrm{row}(A)$. Yes No

(h) $\dim(\mathrm{col}(A)) = \dim(\mathrm{row}(A))$. Yes No

(i) $\mathrm{null}(\mathrm{adj}(A)) = \{\mathbf{0}\}$. Yes No

(j) $\lambda = 0$ is an eigenvalue of A. Yes No

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

2. Find the following:

   (a) [2 marks] $\dim(S^{\perp})$, if S is a subspace of $\mathbb{R}^6$ and $\dim(S) = 2$.

   (b) [2 marks] $\det(-2A^2B^T)$, if A and B are 3×3 matrices with $\det(A) = 1$ and $\det(B) = 3$.

   (c) [2 marks] all values of a such that $\mathbf{u} = \begin{bmatrix} 1 \\ a \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3a \\ -1 \\ -6 \end{bmatrix}$ are orthogonal.

   (d) [2 marks] $\det(A)$, if A is an orthogonal matrix.

   (e) [2 marks] the dimensions of the square matrix A, if the characteristic polynomial of A is

$$(x-3)^3(x-2)^2(x-1)^4(x+1).$$

3. Find all values of the parameter a for which the system of equations

$$\begin{array}{rcrcrcl} x_1 & + & ax_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & + & ax_3 & = & 1 \\ ax_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

has *(i)* no solution, *(ii)* a unique solution, *(iii)* infinitely many solutions.

4. Let $\mathbf{u} = [\ 1 \quad -2 \quad 3\ ]^T$; let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by $T(\mathbf{x}) = \text{proj}_{\mathbf{u}}\mathbf{x}$.

(a) [5 marks] Find the matrix of T.

(b) [5 marks] Find a basis for each of $\ker(T)$ and $\text{range}(T)$.

5. Find the solution to the system of linear differential equations

$$\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 6y_1 - 5y_2 \end{aligned}$$

where y_1, y_2 are functions of t, and $y_1(0) = 2,\ y_2(0) = 3$.

6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

7. Let $S = \text{span}\left\{[\ 1\ \ 1\ \ 0\ \ 1\ ]^T, [\ 0\ \ 1\ \ -1\ \ 1\ ]^T, [\ 2\ \ 0\ \ 1\ \ 1\ ]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

8. Let $A = \begin{bmatrix} 1/3 & 3/4 \\ 2/3 & 1/4 \end{bmatrix}$.

(a) [6 marks] Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

(b) [4 marks] Find and simplify A^n. (Bonus: what can you say about the entries of A^n as $n \to \infty$?)

This page is for rough work; it will not be marked.

The end.

University of Toronto
Faculty of Applied Science and Engineering

Final Examination, 11 December 2014

First Year, Program 5

MAT194F Calculus I

Exam Type A

No aids of any kind are permitted.
No calculators of any kind are permitted.

Time allowed: 2 ½ hours.

Each question is worth 10 marks out of a total of 100.

Examiners: P.C. Stangeby and F. Al-Faisal

1. (i) Find the derivatives of: (a) $5x$, (b) $3/x^2$, (c) $\cos^{-1}(2\sqrt{x})$, (d) $\ln(2x^{-2})$, (e) $3^{3/\sqrt{x}}$.

   (ii) Find all the anti-derivatives of each of:
   (a) $5x$, (b) $3/x^2$, (c) $\cos(2x)$, (d) $x\,e^{x^2}$, (e) 2^x.

2. Provide a $\delta-\varepsilon$ proof that $\lim_{x\to-2} x^2 = 4$.

3. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

4. Sketch the graph of $y = x^2e^{-x}$ indicating all significant features.

5. Two cars leave the same point at the same time. The first car travels at a steady 10 km/hr. The second car travels at a steady 20 km/hr. The first car travels north. The second car travels east for 1 hr and then turns south.
   (a) What is the rate of separation of the two cars after 2 hrs?
   (b) What is the rate of rotation of the line joining the two cars after 2 hrs?

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6. Evaluate: (a) $\int \frac{x^3 - 2\sqrt{x}}{x} dx$, (b) $\int \frac{\left(1+\sqrt{x}\right)^{2014}}{\sqrt{x}} dx$, (c) $\int \frac{e^x}{1-4e^{2x}} dx$,

(d) $\int \frac{\cos x}{5+\sin^2 x} dx$, (e) $\int \frac{1}{\sqrt{6x-x^2}} dx$

7. (a) Prove that $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$

(b) Find the volume of the solid obtained by revolving the region lying under the curve $y = x^{-1/2}\left(x^2-1\right)^{-1/4}$ and between the lines $x = \sqrt{2}$ and $x = 2$, about the x-axis. Express your answer in the form $k\pi^2$ and give the value of k.

8. (a) Find the equation of a curve that passes through the point (0, 1) and whose tangent at the point (x, y) has slope $e^x - y$.

(b) Find the solution of $y' - \left(\frac{1}{x} + 3x^2\right)y = xy^2$ given that $y(1) = 1$. Hint: Consider letting $z = 1/y$.

9. Evaluate: (a) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$, (b) $\lim_{x\to\infty} \sqrt[3]{x^3+x^2} - \sqrt[3]{x^3-x^2}$.

10. (a) Suppose that $f(x)$ is differentiable for all x and satisfies $xf(x) = x + \sin^{-1} x$. Find $f(0)$ and $f'(0)$.

(b) Find $y(x)$ to satisfy $y(x) = y'(x) + \int e^{2x} y(x)\,dx + \lim_{x\to-\infty} y(x)$ given $\lim_{x\to 0} y(x) = 0$ and $\lim_{x\to \ln(\pi/2)} y(x) = 1$.

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