University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture

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We will start 10 minutes past the hour. Use this time to make a new friend.

Non-Homogenous ODE

Consider ODEs of the form

$$y'' + by + c = \underbrace{f(t)}_{\text{forcing term}}$$

$\Leftarrow$ non-homogenous ODE with constant coeff.

$\rightarrow$ we will consider the following forcing terms

1. polynomials
2. exponential
3. sin / cosine
4. combinations i.e. $e^t \cos(2t)$

$y'' - y = e^{2x}$

$\rightarrow$ guess $y = Ae^{2x}$

$(Ae^{2x})'' - (Ae^{2x}) \stackrel{?}{=} e^{2x}$

$4A\cancel{e^{2x}} - A\cancel{e^{2x}} \stackrel{?}{=} \cancel{e^{2x}}$

$4A - A = 1 \implies \boxed{A = \frac{1}{3}}$

$\boxed{y_p(x) = \frac{1}{3}e^{2x}}$ is a particular solution

$\rightarrow$ not general solution. What are other solutions?

$\rightarrow$ guess $y(x) = y_p(x) + y_c(x)$

$(y_p(x) + y_c(x))'' - (y_p(x) + y_c(x)) \stackrel{?}{=} e^{2x}$

$$\underbrace{\left(y_p''(x) - y_p(x)\right)}_{\cancel{e^{2x}}} + \left(y_c''(x) - y_c(x)\right) \stackrel{?}{=} e^{2x}$$

$$+ \left(y_c''(x) - y_c(x)\right) \stackrel{?}{=} \cancel{e^{2x}}$$

$$y_c''(x) - y_c(x) = 0$$ $\Leftarrow$ y_c is a solution to homogenous ODE

$$y'' - y = 0$$

$\rightarrow$ char. poly

$$r^2 - 1 = 0$$

$$r = \pm 1$$

$$\boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$$ $\Leftarrow$ complementary solution (solution to homogenous version of ODE)

$$\boxed{y(x) = \frac{1}{3} e^{2x} + c_1 e^{x} + c_2 e^{-x}}$$ $\Leftarrow$ general solution

Method of Undermined Coefficients

Given $y'' + by' + cy = f(t)$

Guess a particular solution $y_p(t)$

$f(t)$	Guess
e^{at}	Ae^{at}
c	A
$at+b$	$At+B$
at^2+bt+c	At^2+Bt+C
$at^3+\dots$	$At^3+Bt^2+\dots$
$\vdots$	
$\sin(\omega t)$	$A\cos(\omega t)+B\sin(\omega t)$
$a\sin(\omega t)+b\cos(\omega t)$	$A\cos(\omega t)+B\sin(\omega t)$

Poly.

General solution to $y'' + by' + cy = f(t)$ is

$$y(t) = y_p(t) + y_c(t)$$

y_p = Particular solution

y_c = solution to $\underline{y'' + by' + cy = 0}$

$$y'' - 3y' + 2y = \text{[redacted]}\ t + 0$$

→ find complementary solution

$$y'' - 3y' + 2y = 0 \implies 0 = r^2 - 3r + 2r = (r-1)(r-2)$$

$$y_c(t) = c_1 e^{t} + c_2 e^{2t}$$

→ find particular solution

$$y_p(t) = At + B$$

$$(At+B)'' - 3(At+B)' + 2(At+B) = t$$

$$0 - 3A + 2At + 2B = t + 0$$

$$2At + (2B - 3A) = t + 0$$

$$2A = 1 \qquad 2B - 3A = 0$$

$$\boxed{A = \tfrac{1}{2}} \qquad \boxed{B = \tfrac{3}{4}} \qquad \boxed{y_p = \tfrac{1}{2}t + \tfrac{3}{4}}$$

General solution: $\boxed{y(t) = \frac{1}{2}t + \frac{3}{4} + c_1 e^{t} + c_2 e^{2t}}$

$$y'' + y = \cos(2\overset{t}{\cancel{x}})$$

→ complementary solution

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$$

$$y_c(t) = c_1 \cos(t) + c_2 \sin(t)$$

→ particular solution $y_p = A\sin(2t) + B\cos(2t)$

$$\big(A\sin(2t) + B\cos(2t)\big)'' + \big(A\sin(2t) + B\cos(2t)\big) = \cos(2t) + 0\sin(2t)$$

$$-4A\sin(2t) - 4B\cos(2t) + A\sin(2t) + B\cos(2t) = \cos(2t)$$

$$-3A\sin(2t) - 3B\cos(2t) = \cos(2t)$$

$$\Rightarrow A = 0 \;,\; B = -\frac{1}{3} \Rightarrow \boxed{y_p = -\frac{1}{3}\cos(2t)}$$

General solution: $\boxed{y(t) = -\frac{1}{3}\cos(2t) + c_1\cos(t) + c_2\sin(t)}$

$$y'' - 3y' + 2y = e^{2x}$$

Particular Solution

guess $y_p(x) = Ae^{2x}$

$$(Ae^{2x})'' - 3(Ae^{2x})' + 2(Ae^{2x}) \stackrel{?}{=} e^{2x}$$

$$4Ae^{2x} - 6Ae^{2x} + 2Ae^{2x} \stackrel{?}{=} e^{2x}$$

$$0 \stackrel{?}{=} e^{2x} \quad \Leftarrow \text{no solution?}$$

Problem: e^{2x} is solution to homogeneous equation

$\Rightarrow$ LHS will equal zero

IF forcing term is solution to homogeneous equation then guess won't work!

$\therefore$ always solve complementary solution first

Complementary Solution

$r^2 - 3r + 2 = 0 \Rightarrow r = 1, r = 2$

$$y_c(x) = C_1 e^{x} + C_2 e^{2x}$$

Particular Solution

$\rightarrow$ guess $\quad y_p(x) = A\boxed{x}e^{2x} \quad \Leftarrow$ add an x

$$(Axe^{2x})'' - 3(Axe^{2x})' + 2(Axe^{2x}) \stackrel{?}{=} e^{2x}$$

$$A(2xe^{2x} + e^{2x})' - 3A(2xe^{2x} + e^{2x}) + 2Axe^{2x} \stackrel{?}{=} e^{2x}$$

$$A(4xe^{2x} + 2e^{2x} + 2e^{2x}) - 3A(2xe^{2x} + e^{2x}) + 2Axe^{2x} \stackrel{?}{=} e^{2x}$$

$$\underbrace{(4A - 6A + 2A)}_{=0}xe^{2x} + \underbrace{(2A + 2A - 3A)}_{A}e^{2x} \stackrel{?}{=} e^{2x}$$

$$Ae^{2x} \stackrel{?}{=} e^{2x}$$

$$\boxed{A = 1} \qquad \boxed{y_p = xe^{2x}}$$

$$\boxed{y(x) = xe^{2x} + c_1e^{2x} + c_2e^{x}}$$

$$y'' - 3y' + 2y = e^x \cos(x)$$

Complementary Solution

$\rightarrow$ from before $\quad y_c = C_1 e^{2x} + C_2 e^x$

Particular Solution

$\rightarrow$ guess $\quad y_p(x) = e^x \left(A\cos(x) + B\sin(x) \right)$

$$\Rightarrow y_p'(x) = e^x \left((A+B)\cos(x) + (B-A)\sin(x) \right)$$

$$y_p''(x) = e^x \left(2B\cos(x) - 2A\sin(x) \right)$$

$$y_p'' - 3y_p' + 2y_p \overset{?}{=} e^x \cos(x)$$

$$e^x\left(2B\cos(x) - 2A\sin(x)\right) - 3e^x\left((A+B)\cos(x) + (B-A)\sin(x)\right)$$
$$+ 2e^x\left(A\cos(x) + B\sin(x)\right) \overset{?}{=} e^x\cos(x)$$

$$e^x \underbrace{\left(2B - 3A - 3B + 2A\right)}_{-B-A} \cos(x) + e^x \underbrace{\left(-2A - 3B + 3A + 2B\right)}_{A-B} \sin(x) \overset{?}{=} e^x \cos(x)$$

$$A-B=0$$
$$-B-A=1 \Rightarrow A=-\frac{1}{2} \qquad B=-\frac{1}{2}$$

$$y_p = e^x\left(-\frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)\right)$$

$$\boxed{y(x) = e^x\left(-\frac{1}{2}\cos(x) - \frac{1}{2}\sin(x)\right) + C_1e^{2x} + C_2e^x}$$

$$y'' + by' + cy = f(t)$$

$f(x)$	guess	$f(x)$ is sol'n to homogeneous	$xf(x)$ is sol'n to homogeneous
e^{ax}	Ae^{ax}	Axe^{ax}	Ax^2e^{ax}
$\sin(\omega x)$	$A\sin(\omega x) + B\cos(\omega x)$	$Ax\sin(\omega x) + Bx\cos(\omega x)$	
$ax^2 + bx + c$	$Ax^2 + Bx + C$		
$e^{ax}\sin(\omega x)$	$e^{ax}\left(A\sin(\omega x) + B\cos(\omega x)\right)$		
$e^{\gamma x}(ax^2 + bx + c)$	$e^{\gamma x}(Ax^2 + Bx + C)$		

$\vdots$

$y'' + 4y = \sin(x)$ Complementary: $y_c(x) = C_1\cos(2x) + C_2\sin(2x)$

guess: $y_p(x) = A\cos(x) + B\sin(x)$

$y'' + 4y = \sin(2x)$

$y_p(x) = Ax\cos(2x) + Bx\sin(2x)$

$y'' + 4y = x^2 + 1$

$y_p(x) = Ax^2 + Bx + C$

$y'' + 4y = x^2\sin(x)$

$y_p(x) = (Ax^2 + Bx + C)\left(D\cos(x) + E\sin(x)\right)$

$y'' + 4y = x^2e^x\sin(x)$

$y_p(x) = (Ax^2 + Bx + C)(e^x)(E\cos(x) + F\sin(x))$