# MAT187 - Calculus II - Winter 2015

## Term Test 2 - March 10, 2015

Time allotted: 100 minutes.

Aids permitted: None.

Total marks: ~~50~~ 45

**Full Name:** SOLUTIONS, THE
Last First

**Student Number:**

**Email:** @mail.utoronto.ca

**Instructions**

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages (including this title page). Make sure you have all of them.
- You can use pages 12–14 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 12–14.

GOOD LUCK!

**PART I** No explanation is necessary. **(10 marks)**

**1.** Consider the differential equation

$$y' = (y^2 - 3y + 2)3t^2.$$

Write this equation in separable form:

$$\int \frac{1}{y^2-3y+2}\, dy = \int 3t^2\, dt.$$

**2.** Consider the separable differential equation

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dt} = 1.$$ $\Rightarrow \int \frac{1}{\sqrt{1-y^2}}\,dy = \int 1\,dt \Rightarrow$

$\arcsin(y) = t + c$

Then

$y(t) =$ $\sin(t+c)$ .

**3.** Consider the differential equation

$$y' + \tan(t)y = \cos(t).$$ $\mu(t) = e^{\int \tan t\, dt}$

$= e^{\ln|\sec t|}$

What is the integrating factor $\mu(t)$?

$\mu(t) =$ $|\sec t|$ .

**4.** Consider the differential equation

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}.$$ $(t^2 y)' = \cos t$

$t^2 y = \sin t + c$

The integrating factor is $\mu(t) = t^2$. Then the general solution is

$y(t) =$ $\frac{\sin(t)+c}{t^2}$ .

**5.** Consider the differential equation

$$t^2y'' + 7ty' + 9y = 0.$$ $p(p-1)\cancel{t^p} + 7p\cancel{t^p} + 9\cancel{t^p} = 0$

$p^2 + 6p + 9 = 0$

$(p+3)^2 = 0$

If we look for a solution of the form $y = t^p$, then

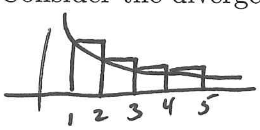
$p =$ $-3$

**6.** Circle the correct option. The series $\sum_{k=33}^{\infty} \frac{(-1)^k}{k^2}$

$\Sigma \frac{1}{k^2}$ converges (integral or p-series tests)

**(a)** converges absolutely. **(b)** converges conditionally. **(c)** diverges.

**7.** Consider the divergent series



$$\sum_{k=1}^{\infty} \frac{1}{k}$$

We want to add the first $N$ terms: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N}$ to make sure that we obtain a sum larger than 42.

$> \int_1^{N+1} \frac{1}{t} dt = \ln(N+1) > 42 \Rightarrow N+1 > e^{42}$

$N > e^{42} - 1$

Then we need:

$N \geqslant$ $e^{42}$

**8.** Consider the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

$\leftarrow$ Alternating. $\therefore$ error $< |a_{n+1}|$

So, let $\frac{1}{\sqrt{N+1}} < \frac{1}{1000}$ $\therefore \sqrt{N+1} > 1000$

We can approximate the series by adding the first $N$ terms: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \cdots + \frac{(-1)^{N+1}}{\sqrt{N}}$. $N > 10^6 - 1$

To make sure that the error is smaller than $\frac{1}{1000}$, we need

$N \geqslant$ 1,000,000

**9.** Recall that when we approximate a function $f(x)$ by $p_n(x)$, the Taylor polynomial of degree $n$ centered at $a = 0$, then the remainder is

$$f(x) - p_n(x) = R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1},$$

$f^{(5)}(x) = 3^5 e^{3x}$

on $[0,1]$, $f^{(5)}(x) \leq 3^5 \cdot e^3$

where $c$ is a point between $x$ and 0.

When we approximate $f(x) = e^{3x}$ at $x = 1$ by $p_4(1) = \frac{131}{8}$, the error we make is

error $\leqslant$ $\frac{243e^3}{120} \cdot 1$

(your answer should not depend on $n$, $x$ or $c$)

**10.** Consider the differential equation

$$y' = y^2 - (t-4)^3.$$

$y'(4) = 0^2 - 0^3 = 0$; $y'' = 2y \cdot y' - 3(t-4)^2$ $[y''(4) = 0]$;

$y''' = 2(y' \cdot y' + y \cdot y'') - 6(t-4)$ $[y'''(4) = 0]$;

$y^{IV}(t) = 2(3y' \cdot y'' + y \cdot y''') - 6$ $[y^{IV}(4) = -6]$.

If $y(4) = 0$, then the solution $y(t)$ has

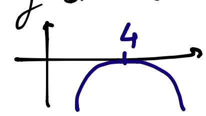
**(a)** a relative minimum at $t = 4$.

**(b)** a relative maximum at $t = 4$.

**(c)** an inflection point at $t = 4$.

**(d)** none of the other options.

$y^{IV}(4) = -6 \Rightarrow \begin{cases} y''' < 0 \text{ for } t > 4 \\ y''' > 0 \text{ for } t < 4 \end{cases} \Rightarrow \begin{cases} y'' < 0 \text{ for } t > 4 \\ y'' < 0 \text{ for } t < 4 \end{cases} \Rightarrow y$ concave downward



**PART II** **Justify** your answers.

**11.** Consider a population of jelly fish which satisfy the following growth model: **(10 marks)**

$$P'(t) = r\big(P(t) - T\big)\big(P(t) - K\big)^2 \qquad \text{where } 0 < T < K.$$

Initially, the population is $P_0 \geqslant 0$. We assume $r > 0$

**(a)** (2 marks) What are the equilibrium solutions?

These are the solutions where $P(t)$ is constant.

$\therefore P'(t) \equiv 0$.

so, $r(P(t) - T)(P(t) - K)^2 \equiv 0$

**Answer :** $P(t) \equiv T$ or $K$ $P(t) \equiv 0$ is equilibrium due to reality.

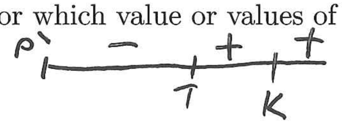
**(b)** (2 marks) For which value or values of $P_0$ will the population grow without bound?

If $P_0 > K$, then $P' > 0$ and is a polynomial, so $P$ will increase without bound.

Since solutions cannot cross equilibrium values, there are no other values of $P_0$ that will give us the result.

**Answer :** $P_0 \in (K, \infty)$

**(c)** (2 marks) For which value or values of $P_0$ will the population become extinct?




Again, $P'$ is a polynomial.

**Answer :** $P_0 \in [0, T)$

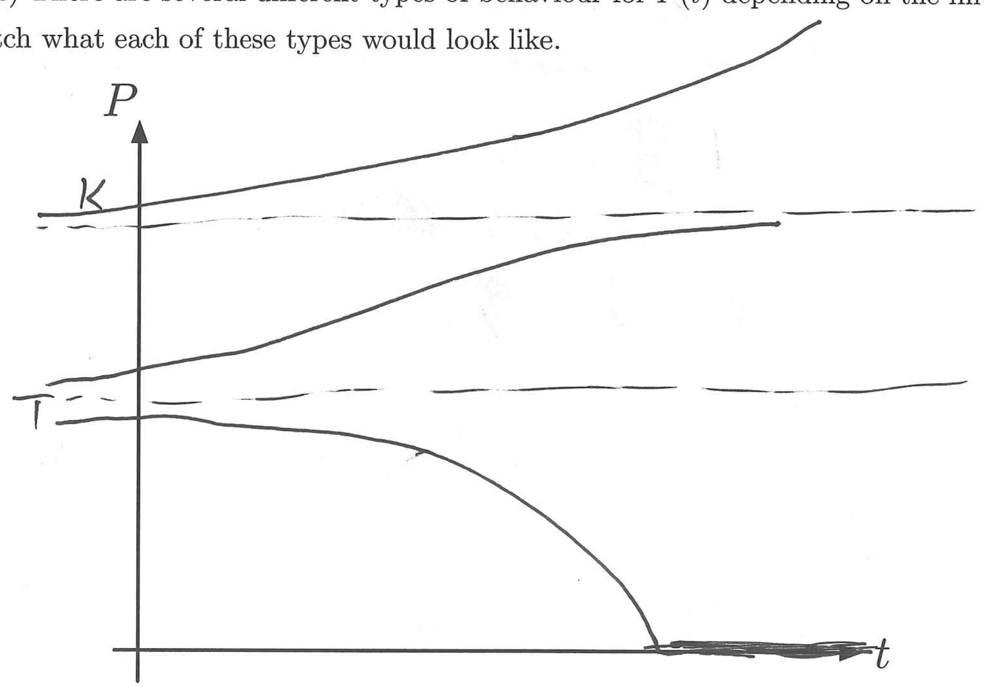
**(d)** (1 mark) For the values of $P_0$ you found in **(c)**, will the population become extinct in a finite amount of time?

$P'$ becomes larger as $P$ decreases, so the function decreases more rapidly. $\therefore$ It cannot be moving toward an asymptote.

so, it reaches $P(t)=0$, rather than approaching it as an asymptote

| **Answer** : (Circle the correct option) | Yes | No |
|---|---|---|

**(e)** (3 marks) There are several different types of behaviour for $P(t)$ depending on the initial value $P_0$. Sketch what each of these types would look like.



**12.** Let $r, K > 0$. Find the solution of **(5 marks)**

$$K\frac{dP}{dt} = rP(K-P)$$
$$P(0) = \frac{K}{2}$$

You can assume that the solution satisfies $0 \leqslant P(t) \leqslant K$.

The equation is separable:

$$\int \frac{K}{P(K-P)}\, dP = \int r\, dt$$

By Partial Fractions:

$$\int \frac{1}{P} + \frac{1}{K-P}\, dP = \int r\, dt$$

$$\ln P - \ln(K-P) = rt + C$$

[No abs. value since $0 \leq P(t) \leq K$]

$$\ln[P/(K-P)] = rt + C$$

$$P/(K-P) = e^{rt+C}$$

To solve for $C$, use $P(0) = \frac{K}{2}$:

$$1 = e^{C}$$

$$\therefore C = 0$$

so, $P/(K-P) = e^{rt}$

$$P = K \cdot e^{rt} - P \cdot e^{rt}$$

$$P(1 + e^{rt}) = K \cdot e^{rt}$$

**Answer** : $P(t) = K \cdot e^{rt} / (1 + e^{rt})$

**13.** Let $r > 0$. Examine the following series for convergence: **(4 marks)**

$$\sum_{k=1}^{\infty} \frac{k^k}{k!\, r^k}.$$

Fill in the space below and justify your answer. Don't worry about the boundary point.

Ratio Test!

$$\lim_{k\to\infty} \left[\frac{(k+1)^{k+1}}{(k+1)!\, r^{k+1}}\right] \Big/ \left[\frac{k^k}{k!\, r^k}\right]$$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1}}{k^k} \cdot \frac{k!}{(k+1)!} \cdot \frac{r^k}{r^{k+1}}$$

$$= \lim_{k\to\infty} \frac{(k+1)^{\cancel{k+1}}}{k^k} \cdot \frac{\cancel{k!}}{\cancel{(k+1)}(\cancel{k!})} \cdot \frac{\cancel{r^k}}{\cancel{r^k} \cdot r}$$

$$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k \cdot \frac{1}{r}$$

$$= \frac{e}{r}$$

converges for $\frac{e}{r} < 1$

diverges for $\frac{e}{r} > 1$

**Answer :** Series converges for $r >$ $e$

Series diverges for $r <$ $e$

**14.** Consider the function $f(x) = e^x \sin(x)$. **(6 marks)**

(a) (3 marks) Find the Taylor polynomial of degree 3 to approximate $f(x)$ near $x = 0$.

$$P_3(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x^2) + \frac{f'''(0)}{3!}(x^3)$$

$f(x) = e^x \sin x \qquad f(0) = 1 \cdot 0 = 0$

$f'(x) = e^x \sin x + e^x \cos x \qquad f'(0) = 1 \cdot 0 + 1 \cdot 1 = 1$

$f''(x) = \cancel{e^x \sin x} + e^x \cdot \cos x + e^x \cos x \cancel{- e^x \sin x}$

$= 2e^x \cos x \qquad f''(0) = 2 \cdot 1 \cdot 1 = 2$

$f'''(x) = 2e^x \cos x - 2e^x \sin x \qquad f'''(0) = 2 \cdot 1 \cdot 1 - 2 \cdot 1 \cdot 0 = 2$

**Answer** : $p_3(x) =$ $0 + x + x^2 + \frac{1}{3}x^3$

(b) (3 marks) Using part (a), we can approximate $e^{\frac{\pi}{2}}$ by $p_3\left(\frac{\pi}{2}\right)$. Give an upper bound for the error of this approximation. You can use the formula from question 9.

$$R_3(x) = \frac{f^{(4)}(c)}{4!} \cdot x^4$$

$$f^{(4)}(x) = \cancel{2 \cdot e^x \cos x} - 2e^x \sin x - 2e^x \sin x - \cancel{2e^x \cos x}$$

$$= -4e^x \sin x$$

on $[0, \frac{\pi}{2}]$, $e^x \leq e^{\frac{\pi}{2}}$ and $\sin x \leq 1$.

$$\therefore \left| R_3\left(\frac{\pi}{2}\right) \right| \leq \frac{4 \cdot e^{\frac{\pi}{2}} \cdot 1}{4!} \cdot \left(\frac{\pi}{2}\right)^4$$

**Answer** : error $\leqslant \dfrac{\pi^4}{96} \cdot e^{\frac{\pi}{2}}$

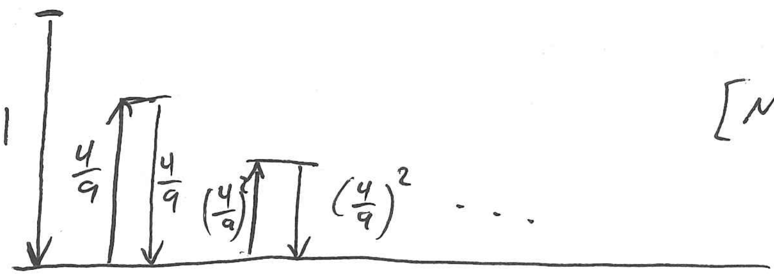
**15.** A ball is bouncing on the ground on a planet with gravitational constant $g$. **(10 marks)**

Assume that it takes the same time for the ball to go from the ground up to a height $h$ as it takes to drop from a height $h$ to the ground. The time for each of these is $\sqrt{\frac{2h}{g}}$ seconds.

Each time the ball bounces to $\frac{4}{9}$ of the height of the previous bounce.

Initially it is dropped from a height of 1 metre.

(a) (4 marks) Find the total distance travelled by the ball.




[Not to scale]

$$\therefore \text{Distance} = 1 + 2\cdot\frac{4}{9} + 2\cdot\left(\frac{4}{9}\right)^2 + 2\cdot\left(\frac{4}{9}\right)^3 + \cdots$$

$$= 1 + 2\cdot\sum_{n=1}^{\infty}\left(\frac{4}{9}\right)^n$$

$$= 1 + 2\cdot\frac{\left(4/9\right)}{\left(1-\frac{4}{9}\right)}$$

$$= 1 + 2\cdot\frac{\left(4/9\right)}{\left(5/9\right)}$$

$$= 1 + \frac{8}{5}$$

$\boxed{\textbf{Answer}: \text{Total distance} = \frac{13}{5} \text{ metres}}$

**(b)** (4 marks) Let $T_n$ be the total elapsed time it takes from the beginning when the ball is dropped until the ball hits the floor for the $n^{\text{th}}$ time. Find a formula for $T_n$.

$T(1) = \sqrt{\frac{2\cdot 1}{g}}$ (ball only falls)

$T(2) = \sqrt{\frac{2}{g}} + 2\sqrt{\frac{2}{g}\cdot\frac{4}{9}}$ (ball has to rise, then fall)

$T(3) = \sqrt{\frac{2}{g}} + 2\sqrt{\frac{2}{g}\cdot\frac{4}{9}} + 2\sqrt{\frac{2}{g}\cdot\left(\frac{4}{9}\right)^2}$ etc.

$\therefore T(n) = \sqrt{\frac{2}{g}} + 2\sum_{k=1}^{n-1}\sqrt{\frac{2}{g}\cdot\left(\frac{4}{9}\right)^k} = \sqrt{\frac{2}{g}} + 2\cdot\sum_{k=1}^{n-1}\sqrt{\frac{2}{g}}\cdot\left(\frac{2}{3}\right)^k = \sqrt{\frac{2}{g}}\left[1 + 2\sum_{k=1}^{n-1}\left(\frac{2}{3}\right)^k\right]$

$\sqrt{\frac{2}{g}}\left[-1 + 2\sum_{k=0}^{n-1}\left(\frac{2}{3}\right)^k\right]$

$\sqrt{\frac{2}{g}}\left[-1 + 2\cdot\frac{1-(2/3)^n}{1-2/3}\right]$

**Answer** : $T_n = \sqrt{\frac{2}{g}}\left[5 - 6\left(\frac{2}{3}\right)^n\right]$

**(c)** (2 marks) Does the ball ever stop bouncing? If so, how long does it take?

$\lim_{n\to\infty} T(n) = \lim_{n\to\infty} \sqrt{\frac{2}{g}}\left[5 - 6\left(\frac{2}{3}\right)^n\right]$

$= 5\cdot\sqrt{\frac{2}{g}}$

**Answer** : Yes, after $5\cdot\sqrt{\frac{2}{g}}$ seconds

**(Bonus)** What is the average speed of the ball? **(2 marks)**

Average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{(a)}{(c)} = \dfrac{\left(\frac{13}{5}\right)m}{\left(5\sqrt{\frac{2}{g}}\right)s}$

**Answer** : Average Speed = $\frac{13}{25}\sqrt{\frac{g}{2}}$ m/s