UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 2017

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

EXAMINER: D. BURBULLA

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

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Instructions:

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 12 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of nine questions, some with many parts. Attempt all of them. Each question is worth 10 marks. Marks for parts of a question are indicated in the question. **Total Marks: 90**
- PRESENT YOUR SOLUTIONS IN THE SPACE PROVIDED. You can use pages 11 and 12 for rough work. If you want anything on pages 11 or 12 to be marked you must indicate in the relevant previous question that the solution continues on page 11 or 12.

1. Given that the reduced row echelon form of

$$A = \begin{bmatrix} 2 & 10 & 1 & 7 & 5 \\ -1 & -5 & 1 & 1 & -7 \\ 2 & 10 & 1 & 7 & 5 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 5 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. (No justification is required.)

(a) [1 mark] the rank of A Answer: ________

(b) [1 mark] $\dim(\mathrm{Row}(A))$ Answer: ________

(c) [1 mark] $\dim(\mathrm{Col}(A))$ Answer: ________

(d) [1 mark] $\dim(\mathrm{Null}(A))$ Answer: ________

(e) [1 mark] $\dim(\mathrm{Null}(A^T))$ Answer: ________

(f) [1 mark] A basis for the row space of A.

Answer: ________________

(g) [2 marks] A basis for the column space of A.

Answer: ________________

(h) [2 marks] A basis for the null space of A.

Answer: ________________

2. Let $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ 10 \\ 3 \\ -11 \end{bmatrix}$. Show $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal set, and write $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$.

3. Let $A = \frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbb{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue. Interpret your result geometrically.

4. Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 \\ 9x_1 + 2x_2 \end{bmatrix}.$$

(a) [5 marks] Draw the image of the unit square[1] under L and find its area.

(b) [5 marks] Find $L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

[1]The unit square is the square with the four vertices $(0, 0), (1, 0), (0, 1), (1, 1)$.

5. Let $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 6 \\ 3 & 2 & 7 & 1 \\ 3 & 6 & 7 & 4 \end{bmatrix}$. (a) [5 marks] Show that $S = \{\vec{x} \in \mathbb{R}^4 \mid A\,\vec{x} = A^T\,\vec{x}\}$ is a subspace of $\mathbb{R}^4$.

5.(b) [5 marks] Find a basis for S.

6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 7 & 1 & -1 \\ 1 & 7 & -1 \\ -1 & -1 & 7 \end{bmatrix}$.

7. Let $S = \text{span}\left\{[\begin{array}{cccc}1 & 0 & -1 & 1\end{array}]^T, [\begin{array}{cccc}3 & 1 & 1 & 1\end{array}]^T, [\begin{array}{cccc}2 & 0 & 1 & 1\end{array}]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

(b) [5 marks] Let $\vec{x} = \left[\begin{array}{cccc}2 & -3 & 3 & 4\end{array}\right]^T$. Find $\text{proj}_S(\vec{x})$.

8. Consider the plane Π in $\mathbb{R}^3$ with scalar equation $2x_1 - 3x_2 - 5x_3 = 0$ and the point with coordinates $Q(1, 3, 1)$. Find the point on the plane Π closest to the point Q.

9. Use the method of least squares to find the best fitting quadratic function for the five data points

$$(-2, 4.9),\ (-1, 2.1),\ (0, 0.9),\ (1, 1.9),\ (2, 5.1).$$

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The end.