MAT 186
Quiz 7

1. Graph the function

AB6	CF2	CF6	CF12	CF13	CS10	MW2

$$y = \frac{x^3 + 1}{|x^2 - 1|}$$

Be thorough and organized - analyze and label all of the important features of the function and make sure that it is easy to follow your reasoning.

(A) The function itself

We see absolute values, we take cases:

$$y = \begin{cases} \dfrac{x^3+1}{x^2-1}, & |x| > 1 \\ -\dfrac{x^3+1}{x^2-1}, & |x| < 1 \end{cases}$$
$$= \begin{cases} \dfrac{x^2-x+1}{x-1}, & |x| > 1 \\ -\dfrac{x^2-x+1}{x-1}, & |x| < 1 \end{cases}$$
$$= \begin{cases} x + \dfrac{1}{x-1}, & |x| > 1 \\ -x - \dfrac{1}{x-1}, & |x| < 1 \end{cases}$$

All of the above can be written as three branches $(x < -1, -1 < x < 1, x > 1)$, which is just as correct.

There is clearly one issue at $x = 1$ and a different one at $x = -1$. We check the limits on both:

$$\lim_{x\to -1^-} y = \lim_{x\to -1^-} x + \frac{1}{x-1} = -1 - \frac{1}{2} = -\frac{3}{2}$$
$$\lim_{x\to -1^+} y = \lim_{x\to -1^+} -x - \frac{1}{x-1} = 1 + \frac{1}{2} = \frac{3}{2}$$
$$\lim_{x\to 1^-} y = \lim_{x\to 1^-} -x - \frac{1}{x-1} = -1 - \left[\frac{1}{0^-}\right] = +\infty$$
$$\lim_{x\to 1^+} y = \lim_{x\to 1^+} x + \frac{1}{x-1} = 1 + \left[\frac{1}{0^+}\right] = +\infty$$

This gives a jump discontinuity at $x = -1$ and a vertical asymptote at $x = 1$. Note that we only needed one of the limits at $x = 1$, but it's helpful for the graphing to have both.

End behaviour is relatively simple to deduce from the cases, as the branch $|x| > 1$ includes the limits to both directions. So, $y = x$ is an oblique asymptote as $x \to \pm\infty$. We can also tell that the function will be a little above the asymptote for $x \to +\infty$ and a little below it for $x \to -\infty$.

(B) The derivative

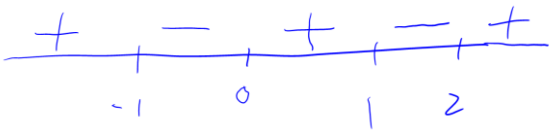
Differentiating by branch:

$$y' = \begin{cases} 1 - \dfrac{1}{(x-1)^2}, & |x| > 1 \\ -1 + \dfrac{1}{(x-1)^2}, & |x| < 1 \end{cases}$$

$$= \begin{cases} \dfrac{(x-1)^2 - 1}{(x-1)^2}, & |x| > 1 \\ \dfrac{1 - (x-1)^2}{(x-1)^2}, & |x| < 1 \end{cases}$$

$$= \begin{cases} \dfrac{x(x-2)}{(x-1)^2}, & |x| > 1 \\ \dfrac{x(2-x)}{(x-1)^2}, & |x| < 1 \end{cases}$$

It should take a little bit of time and a lotta bit of care to set up the number line here, but we get:



For future reference, $f(0) = 1, f(2) = 3$, with both points being local minima.

(C) The second derivative

$$y'' = \begin{cases} \dfrac{2}{(x-1)^3}, & |x| > 1 \\ -\dfrac{2}{(x-1)^3}, & |x| < 1 \end{cases}$$

Not much need for a number line here, but you should absolutely use one if it helps. In either case, the result is that the function is concave up for $x > -1$ and concave down for $x < 1$.

(D) The graph