MAT 188 – Midterm I-Part B –Winter 2021

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Solutions

1. Let A be a 3×4 matrix with columns $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$. Let $\vec{b}$ be a vector in $\mathbb{R}^3$. Consider the equation $A\vec{x} = \vec{b}$.

   Suppose that after a few row reduction steps the augmented matrix $[A|\vec{b}]$ row reduces to $\begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   Find the general solution to $A\vec{x} = \vec{b}$. Justify your answer.

Solution: To write the general solution, first we row reduce $[A|\vec{b}]$ to RREF.

$$[A|\vec{b}] \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Suppose $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Then x_2 and x_4 are free variables, and x_1 and x_3 are basic.

Solving for basic variables in terms of free variables give us:

$x_1 = -2x_2 - 2x_4 - 1, x_3 = -x_4 + 2$, where x_2 and x_4 range over all real numbers. We can write the general solution in vector parametric form. Let $x_2 = t$ and $x_4 = s$ then

$$\vec{x} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

where t, s are in $\mathbb{R}$.

2. Let A and $\vec{b}$ be the same matrix and vector as in question 1. Recall that in Q11 part A you wrote $\vec{b}$ as a linear combination of columns of A.

   If you answered Q11 in part A:

   a) Write down all the answers you chose as correct for Q11 in part A.

   b) Justify each answer in (a) using your work in Q1.

   If you did NOT answer Q11 in part A:

   a) Write $\vec{b}$ as a linear combination of columns of A in three different ways, or explain why that is not possible. Justify your answer.

Solution: We computed the general solution to $A\vec{x} = \vec{b}$ to be

$$\vec{x} = t\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + s\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix} + \begin{bmatrix}-1\\0\\2\\0\end{bmatrix}$$

where t, s are in $\mathbb{R}$. Choosing $t = s = 0$ gives $\begin{bmatrix}-1\\0\\2\\0\end{bmatrix}$ as a solution, choosing $t = 1, s = 0$ gives $\begin{bmatrix}-3\\1\\2\\0\end{bmatrix}$, choosing $s = 1, t = 0$ gives $\begin{bmatrix}-3\\0\\1\\1\end{bmatrix}$ and finally choosing $s = t = 1$ gives $\begin{bmatrix}-5\\1\\1\\1\end{bmatrix}$ as a solution. Plugging in these solutions into $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]\vec{x} = \vec{b}$ gives: $-\vec{a}_1 + 2\vec{a}_3 = \vec{b}$, $-3\vec{a}_1 + \vec{a}_2 + 2\vec{a}_3 = \vec{b}$, $-3\vec{a}_1 + \vec{a}_3 + \vec{a}_4 = \vec{b}$ and $-5\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4 = \vec{b}$ respectively. Moreover, counting the number of pivots tells us $\operatorname{rank} A = \operatorname{rank}[A|\vec{b}] = 2$.