Basic differentials

$$v = \frac{ds}{dt} = \dot{s}$$

$$a = \frac{dv}{dt} = \dot{v},\ \ a = \frac{d^2s}{dt^2} = \ddot{s}$$

After some manipulation:

$$v\,dv = a\,ds,\ \ \dot{s}\,d\dot{s} = \ddot{s}\,ds$$

Non-constant acceleration

Get $s(t)$ and $v(t)$ in different situations:

A/T: $a = a(t) = \frac{dv}{dt}$

$$a(t)\,dt = dv \rightarrow \int_0^t a(t)\,dt = \int_{v_0}^{v} dv$$

$$\therefore\ v(t) = v_0 + \int_0^t a(t)\,dt$$

Similarly: $s(t) = s_0 + \int_0^t v\,dt$, and combining:

$$\therefore\ s(t) = s_0 + v_0 t + \int_0^t \left(\int_0^t a(t)\,dt\right) dt$$

(Do integral in 2 steps!)

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A/V: $a = a(v) = \frac{dv}{dt}$

$$t(v) = \int_0^t dt = \int_{v_0}^{v} \frac{1}{a(v)}\,dv$$

Then, solve for $v(t)$ and integrate to obtain $s(t)$. Alternatively, plug into $v\,dv = a\,ds$ instead:

$$v\,dv = a(v)\,ds \rightarrow \int_{v_0}^{v} \frac{v}{a(v)}\,dv = \int_{s_0}^{s} ds$$

$$\therefore\ s(v) = s_0 + \int_{v_0}^{v} \frac{v}{a(v)}\,dv$$

to obtain expression without t.

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A/S: $a = a(s)$

$$v\,dv = a\,ds \rightarrow \int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a(s)\,ds$$

$$\therefore\ v(s)^2 = {v_0}^2 + 2\int_{s_0}^{s} a(s)\,ds$$

then, with $v(s) = \frac{ds}{dt} \rightarrow \int_0^t dt = \int_{s_0}^{s} \frac{1}{v(s)}\,ds$

$$\therefore\ t(s) = \int_{s_0}^{s} \frac{1}{v(s)}\,ds$$

Rearrange to obtain $s(t)$ and then take derivative: $v(t) = s'(t)$.

Constant acceleration $a = a_c$

$$s = s_0 + v_0(t - t_0) + \frac{1}{2}a_c(t - t_0)^2$$

$$v = v_0 + a_c(t - t_0)$$

$$v^2 = {v_0}^2 + 2a_c(s - s_0)$$

$$s = s_0 + \frac{v_0 + v}{2}(t - t_0)$$

Sign of a

Positive:

1. $v > 0$: speeding up
2. $v < 0$: slowing down

Negative:

1. $v > 0$: slowing down
2. $v < 0$: speeding up

Relative Motion

Only works in x-y; B/A is B **with respect to** A.

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

Rectangular (x-y)

Unit vectors: $\hat{i}$ in $+x$ and $\hat{j}$ in $+y$

Projectile Motion

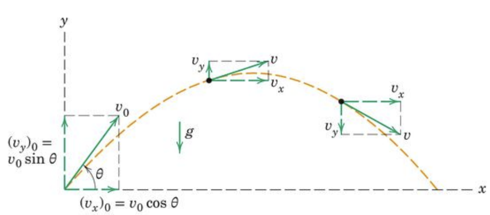
$$a_x = 0,\ \ a_y = -g$$

Therefore, using constant a equations above:

$$x = x_0 + v_{0_x}t \qquad y = y_0 + v_{y_0}t - \frac{1}{2}gt^2$$

$$v_x = v_{0_x} \qquad v_y = v_{0_y} - gt$$

where: $v_{0_x} = v_0\cos\theta$ and $v_{0_y} = v_0\sin\theta$



Normal-Tangential (n-t)

Unit vectors: $\hat{e_t}$ in $+\vec{v}$ direction and $\hat{e_n}$ towards center of curvature ($\hat{e_n} \perp \hat{e_t}$).

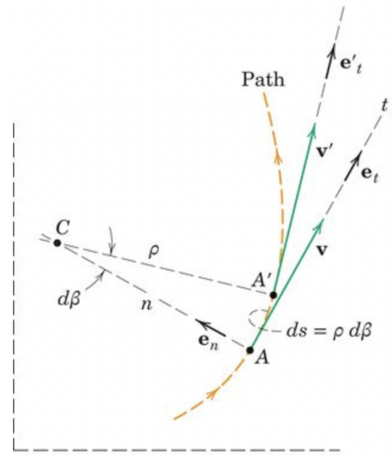
$$s = \rho\beta,\ \ v = \rho\dot{\beta} \text{ for constant } \rho$$

$$a_t = \dot{v} = \ddot{s}$$

$$a_n = v\dot{\beta} = \rho\dot{\beta}^2 = \frac{v^2}{\rho}$$

$$a_t\,ds = v\,dv \ \left(\text{from } a_t = \frac{dv}{dt},\ \ v = \frac{ds}{dt}\right)$$

Also, $a_t = \frac{d}{dt}(v = \rho\dot{\beta}) \ \rightarrow\ a_t = \rho\ddot{\beta} + \dot{\rho}\dot{\beta}$ to find $\dot{\rho}$



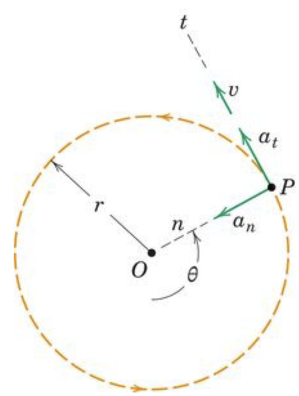
Circular Motion in n-t

ρ becomes constant r, β becomes θ:

$$v = r\dot{\theta}$$

$$a_n = v\dot{\theta} = r\dot{\theta}^2 = \frac{v^2}{r}$$

$$a_t = \dot{v} = r\ddot{\theta}$$



Polar (r-θ)

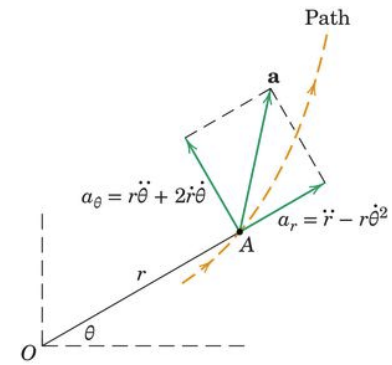
Unit vectors: $\hat{e_r}$ in $+\vec{r}$ direction and $\hat{e_\theta}$ in $+\theta$ direction ($\hat{e_\theta} \perp \hat{e_r}$).

$$\vec{v} = v_r\hat{e_r} + v_\theta\hat{e_\theta},\ \text{where } v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$\vec{a} = \dot{\vec{v}} = a_r\hat{e_r} + a_\theta\hat{e_\theta},\ \text{where } a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$



Aside: unit vector manipulation

$$d\hat{e_r} = \hat{e_\theta}d\theta \qquad d\hat{e_\theta} = -\hat{e_r}d\theta$$

$$\frac{d\hat{e_r}}{d\theta} = \hat{e_\theta} \qquad \frac{d\hat{e_\theta}}{d\theta} = -\hat{e_r}$$

$$\dot{\hat{e_r}} = \dot{\theta}\hat{e_\theta} \qquad \dot{\hat{e_\theta}} = -\dot{\theta}\hat{e_r}$$

Circular Motion in r-θ

r becomes constant:

$$v_r = 0 \qquad a_r = -r\dot{\theta}^2$$

$$v_\theta = r\dot{\theta} \qquad a_\theta = r\ddot{\theta}$$

Note: same as in n-t, but $a_r = -a_n$ as θ and t directions same but r and n directions opposite.

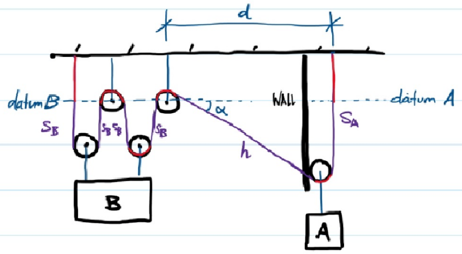
Units & Symbols

10^{12} Tera, 10^9 Giga, 10^6 Mega, 10^3 Kilo; 10^{-3} milli, 10^{-6} μicro, 10^{-9} nano, 10^{-12} pico

- **U**: Work [J]
- **T**: Kinetic energy [J]
- **V**: Potential energy [J]
- $\vec{\omega}$**,** $\vec{\alpha}$: Angular vel. [rad/s], accel. [rad/s^2]
- $\vec{G}$ or $\vec{L}$: Linear momentum [$kg \cdot \frac{m}{s}$, $N \cdot s$]
- $\vec{H}$: Angular momentum (point H_O, mass center H_G) [$kg \cdot \frac{m^2}{s}$]

Constrained Motion (pulleys/blocks)

Each block needs own datum, measures position *in direction of motion*. Then, differentiate the parts of the rope $l_{\text{rope}} = \ldots$.



$\ell_{rope} = 4s_B + s_A + h + \ldots$ (red portions)

$0 = 4v_B + v_A + \frac{dh}{dt}$

$h^2 = d^2 + s_A^2 \rightarrow h = \sqrt{d^2+s_A^2}$ and $h = \frac{d}{\sin\alpha}$

$\frac{dh}{dt} = \frac{1}{\cancel{2}} \frac{1}{\underbrace{\sqrt{d^2+s_A^2}}_{h}} \cancel{2} s_A \dot{s}_A = \frac{s_A}{h} \dot{s}_A = \sin\alpha\, v_A$

$\therefore\ 0 = 4v_B + v_A + \sin\alpha\, v_A$

Kinetics ($\sum \vec{F} = m\vec{a}$)

Forces:

- $W = \frac{Gm_1m_2}{r^2} = mg$
- N (always $\perp$ to contact surface)
- $F_{f,s} \leq \mu_s N,\ F_{f,k} = \mu_k N$
- $F_e = -kx$

Curvilinear:
n-t: $\sum F_n = ma_n,\ \sum F_t = ma_t,\ \sum F_b = 0$
r-θ: $\sum F_r = ma_r,\ \sum F_\theta = ma_\theta,\ \sum F_z = m\ddot{z}$
(refer to kinematics for a_n, a_t, a_r, a_θ)

Work *(all scalar!)*

- Gravity: $-mg(y_2 - y_1)$
- Constant applied force: $\int_{s_1}^{s_2} P\cos\theta\, ds$
- Spring: $-\frac{1}{2}k({s_2}^2 - {s_1}^2)$
- Constant friction: $-\mu_k N(s_2 - s_1)$

Generally:

$$U_{1\to 2} = \int_{s_1}^{s_2} F\cos\theta_{F,ds}\, ds$$

Interpretation: U_N done by surface on object, U_g done by gravity on object

Work-Energy *(for problems w/ F, Δs, v)*
$T_1 + \sum U_{1\to 2} = T_2$

Conservation of Energy
$T_1 + V_1 + (\sum U_{1\to 2}) = T_2 + V_2$

Linear Momentum and Impulse

Change in linear momentum is impulse.

$$\vec{G} = m\vec{v} \quad \text{and} \quad \dot{\vec{G}} = m\dot{\vec{v}} = \sum \vec{F}$$

$$\therefore\ \vec{G}_1 + \int_{t_1}^{t_2} \sum \vec{F}\, dt = \vec{G}_2$$

Conserved if no external forces (impulses):
$\sum G_1 = \sum G_2,\ \sum m_1\vec{v}_1 = \sum m_2\vec{v}_2$

Angular Momentum

Moment of linear momentum about a point:

$$\vec{H_O} = \vec{r} \times \vec{G} = \vec{r} \times m\vec{v}$$

$$|\vec{H_O}| = r_O mv\sin\theta \text{ (in } \hat{k} \text{ dir.)} = r_O mv_\theta$$

$$\dot{\vec{H_O}} = \sum \vec{M_O} = \sum \vec{r_O} \times \vec{F}$$

$$\therefore\ (\vec{H_O})_1 + \sum \int_{t_1}^{t_2} M_O\, dt = (\vec{H_O})_2$$

Conserved if no external moments:
$\sum (H_O)_1 = \sum (H_O)_2$

Moments: $\vec{M_O} = \vec{r_O} \times \vec{F},\ |\vec{M_O}| = Fd$

Systems of Particles

Kinetics

$$\vec{r_G} = \frac{\sum m_i\vec{r_i}}{m} \text{ (center of mass)}$$

$$\dot{\vec{r_G}} = \frac{\sum m_i\dot{\vec{r_i}}}{m} = \frac{\sum m_i\vec{v_i}}{m}$$

$$\ddot{\vec{r_G}} = \frac{\sum m_i\ddot{\vec{r_i}}}{m} = \frac{\sum m_i\vec{a_i}}{m} = \frac{\sum \vec{F_i}}{m}$$

$$\therefore\ \sum \vec{F} = m\ddot{\vec{r_G}} = m\vec{a_G}$$

Work and Energy

$\sum (T_1)_i + \sum \int \vec{F_i} \cdot d\vec{r} = \sum (T_2)_i$

Momentum and Impulse

$\sum m_i(\vec{v_i})_1 + \sum \int_{t_1}^{t_2} \vec{F_i}\, dt = \sum m_i(\vec{v_i})_2$

Math

Quadratic: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Radius of curvature:
$\rho_{xy} = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ and $\rho_{r\theta} = \frac{[r^2+(\frac{dr}{d\theta})^2]^{\frac{3}{2}}}{r^2+2(\frac{dr}{d\theta})^2 - r(\frac{d^2r}{d\theta^2})}$

Trig:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Rigid Body (only need 2 points)

Translation

$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$
$\therefore\ \vec{v}_B = \vec{v}_A + \cancelto{0}{\vec{v}_{B/A}}$ and $\vec{a}_B = \vec{a}_A + \cancelto{0}{\vec{a}_{B/A}}$

Rotation About Fixed Axis

$\vec{\omega} = \dot{\theta}$ and $\vec{\alpha} = \ddot{\theta}$ points in $\hat{k}$ direction (in/out of page), same for *all* points on rigid body.

$$\omega = \frac{d\theta}{dt},\ a = \frac{d\omega}{dt}$$

After some manipulation (analogous to linear):

$$\omega\, d\omega = \alpha\, d\theta,\ \dot{\theta}$$

For **constant angular acceleration** $\alpha = \alpha_c$, same as linear (but $s \to \theta$, $v \to \omega$, $a_c \to \alpha_c$).

Finding v and a of a point P

$\vec{v}_P = \vec{v}_{P/O} = \omega r_{P/O}\ \hat{e_\theta}$
$\vec{a}_P = -\omega^2 r\ \hat{e_r} + \alpha r\ \hat{e_\theta}$ (or in n-t: $\omega^2 r\ \hat{e_n} + \alpha r\ \hat{e_t}$)

Vectorially (note that $\vec{\omega} \times \vec{r}_{P/O}$ is $\vec{v}_{P/O}$):

$$\vec{v}_P = \vec{\omega} \times \vec{r}_{P/O}$$
$$\vec{\alpha}_P = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O}) + \vec{\alpha} \times \vec{r}_{P/O}$$
$$= \vec{\alpha} \times \vec{r}_{P/O} - \omega^2\vec{r}_{P/O} \text{ (simplified form)}$$

Relative Velocity

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$
$$\vec{v}_B = \vec{v}_A + \vec{w} \times \vec{r}_{B/A}$$

Find ω with $\omega = \frac{|\vec{v}_{B/A}|}{|\vec{r}_{B/A}|} = \frac{|\vec{v}_B - \vec{v}_A|}{|\vec{r}_{B/A}|}$.

Relative Acceleration

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$
$$\vec{a}_B = \vec{a}_A + [\ \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A})\ ]$$

Find α with $\alpha = \frac{|(\vec{a}_B)_t - (\vec{a}_A)_t|}{|\vec{r}_{B/A}|}$.
Rolling without slip: $\vec{a}_G = \alpha R\ \hat{\imath},\ \vec{a}_{cntct} = \omega^2 R\ \hat{\jmath}$

Kinetics

$$\sum \vec{F} = m\vec{a}_G \text{ and } \sum \vec{M}_G = I_G\vec{\alpha}$$

I_G: **rod**: $\frac{1}{12}mL^2$ and $I_{end} = \frac{1}{3}mL^2$, **disk**: $\frac{1}{2}mR^2$, **ring**: mR^2, **rect. plate**: $\frac{1}{2}m(a^2+b^2)$, **rad. of gyra.** ($K = \sqrt{\frac{I_G}{m}}$): mK^2

Parallel axis theorem: $I_A = I_G + md^2$

Made by Joe Dai in LaTeX for MIE100 in 2022

Work-Energy

$$T = \frac{1}{2}I_{IC}\omega^2 = \frac{1}{2}I_G\omega^2 + \frac{1}{2}m{v_G}^2$$

Work same, except $U_{f,s} = 0$ and $U_M = \int_{\theta_1}^{\theta_2} M\, d\theta$

Momentum and Impulse

Same principle holds, with below:

$$\vec{G} = m\vec{v}_G,\ \vec{H_G} = I_G\vec{\omega} \text{ and } \dot{\vec{H_G}} = \sum \vec{M}_G = I_G\vec{\alpha}$$

Note: still holds if $_G$ replaced with $_{IC}$ or $_O$! Use this for rolling wheel!

Vibrations

Parallel springs: $k_{\text{eq}} = \sum k_i$
Series springs: $\frac{1}{k_{\text{eq}}} = \sum \frac{1}{k_i}$ OR $\frac{k_1k_2}{k_1+k_2}$
Period $\tau = \frac{2\pi}{\omega_n}$, Frequency $f = \frac{\omega_n}{2\pi}$
$\sin\theta \approx \theta$ for small θ

General solutions $x(t)$ are as follows:

Undamped Free ($\ddot{x} + {\omega_n}^2 x = 0$)

$A\sin(\omega_n t) + B\cos(\omega_n t)$, where $\omega_n = \sqrt{\frac{k}{m}}$, *OR* $C\sin(\omega_n t + \phi)$, $\phi = \tan^{-1}(\frac{B}{A})$, $C = \frac{B}{\sin\phi} = \frac{A}{\cos\phi}$
Amplitude/max disp. is C

Hanging mass: $y_{\text{eq}} = \frac{mg}{k}$; pendulums: $\omega_n = \sqrt{g/l}$

Damped Free ($m\ddot{x} + c\dot{x} + kx = 0$)

Check sign of: $(\frac{c}{2m})^2 - \frac{k}{m}$
Crit ($= 0$): $(A + Bt)e^{\lambda t}$, where $\lambda = -\frac{c}{2m} = -\omega_n$
Overdamp (> 0): $Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
Underdamp (< 0): $De^{-\frac{c}{2m}t}\sin(\omega_d t + \phi)$, where $\omega_d = \sqrt{\frac{k}{m} - (\frac{c}{2m})^2} = \omega_n\sqrt{1-\zeta^2}$, $\zeta = \frac{c}{c_{\text{crit}}}$

Undamped Forced ($m\ddot{x} + kx = F_o\sin(\omega_o t)$)

$A\sin(\omega_n t) + B\cos(\omega_n t) + X\sin(\omega_o t)$, where $X = \frac{F_o/k}{1 - {\omega_o}^2/{\omega_n}^2}$
Magnification factor $MF = \frac{X}{F_o/k} = \frac{1}{1 - {\omega_o}^2/{\omega_n}^2}$
In-phase ($MF > 1$) if $\frac{\omega_o}{\omega_n} < 1$, out-of-phase ($MF < 0$) if $\frac{\omega_o}{\omega_n} > 1$, and resonance (VA) if $\omega_o = \omega_n$

Periodic Support Disp. $\delta(t) = \delta_o\sin(\omega_o t)$

Obtain x by subtract $x(t)$ and $\delta(t)$ (or vice versa) to get similar to $m\ddot{x} + kx = k\delta_o\sin(\omega_o t)$
$C\sin(\omega_n t + \phi) + X\sin(\omega_o t)$ where $X = \frac{\delta_o}{1 - {\omega_o}^2/{\omega_n}^2}$

Rigid Bodies

Use moments: $\sum \vec{M}_o = I_o\vec{\alpha} \to I_o\ddot{\theta} - \sum \vec{M}_o(\theta) = 0$