University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 6

Instructor: Arman Pannu

We will start 10 minutes past the hour. Use this time to make a new friend.

Sequences

A sequence is an infinite list of numbers

$a_1, a_2, a_3, a_4 \ldots$

ex// $0, 0, 0, 0, \ldots$ $\quad a_n = 0$

$1, 2, 3, 4, 5,$ $\quad a_n = n$

$1, 3, 5, 7$ $\quad a_n = 2n-1$

$2^1, 4, 8, 16, 32$ $\quad a_n = 2^n$

$-1, 1, -1, 1, -1$ $\quad a_n = (-1)^n$

} explicitly defined sequences

$a_1 = 0, \quad a_{n+1} = a_n$

$a_1 = 1, \quad a_{n+1} = a_n + 1$

$a_1 = 1, \quad a_{n+1} = a_n + 2$

$a_1 = 2, \quad a_{n+1} = 2a_n$

$a_1 = -1, \quad a_{n+1} = -a_n$

Fibonnici $\quad a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + a_{n-1}$ } recursively defined

Sequence Convergence

We say a sequence converges to L if

$$\lim_{n\to\infty} a_n = L$$

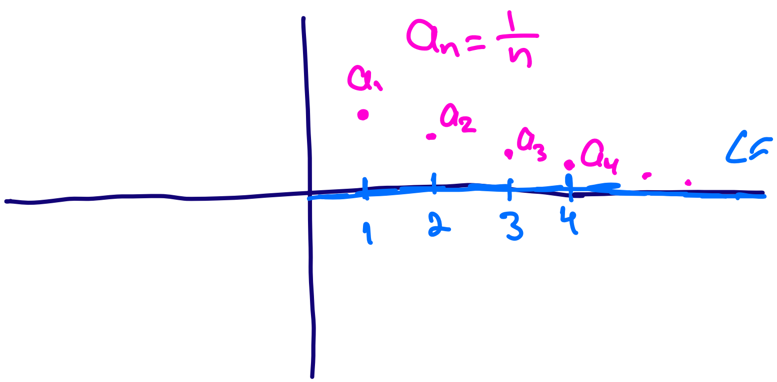
$\to$ sequence diverges if $\lim_{n\to\infty} a_n$ doesn't exist

ex// ① $a_n = 0$ $(0, 0, 0, \ldots)$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$ converges

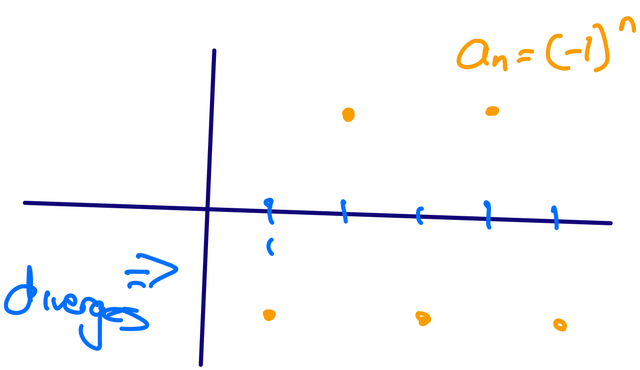
② $a_n = n$ $(1, 2, 3, \ldots)$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$ diverges

③ $a_n = \frac{1}{n}$ $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ converges

④ $a_n = (-1)^n$ $(-1, 1, -1, 1, \ldots)$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n = \text{DNE}$ diverges



$\Leftarrow$ sequence get closer to zero

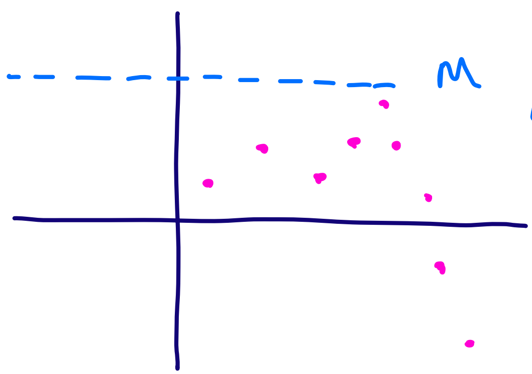


diverges $\Rightarrow$

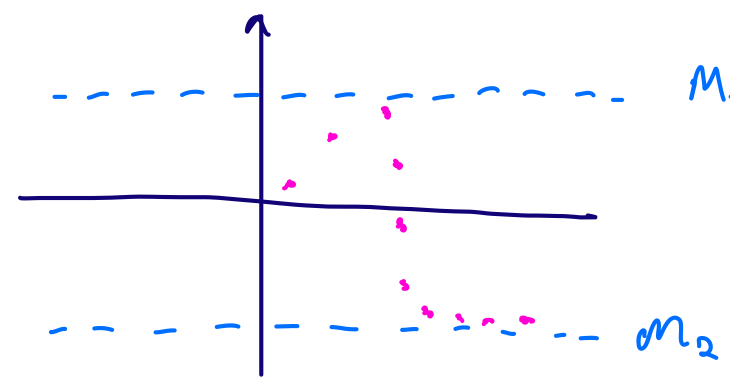
Monotone Convergence Theorem

Given a sequence $\{a_n\}_n$

- $\rightarrow \{a_n\}$ is bounded above if $a_n < M$ for all n
- $\rightarrow \{a_n\}$ " " below " $a_n > M$ for all n
- $\rightarrow \{a_n\}$ is bounded if bounded below and above



$\Leftarrow$ bounded above
sequence doesn't cross line at M

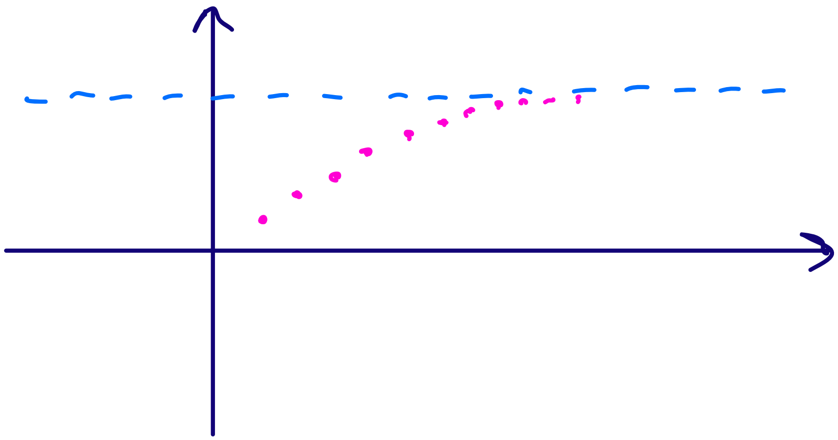


$\Rightarrow$ bounded

<u>Monotone convergence theorem</u>

If $\{a_n\}$ is increasing and bounded above then it converges

or decreasing " " below " " "



$\angle$= increasing and bounded

$\therefore$ must converge to some finite value

Series

A series is a sum of numbers

$$a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i \quad \Leftarrow \text{finite series}$$

$$a_1 + a_2 + a_3 + \ldots \qquad \sum_{i=1}^{\infty} a_i \quad \Leftarrow \text{infinite series}$$

ex// infinite series:

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{\infty} 0 = 0+0+0+0+\ldots \overset{?}{=} 0 \\ \displaystyle\sum_{i=1}^{\infty} i = 1+2+3+4+\ldots = ? \end{array}\right\} \text{can we make sense of this}$$

Def'n: given a series $\sum_{n=1}^{\infty} a_n$, define K_{th} <u>partial sum</u> as the sum of first K terms

$$S_K = \sum_{n=1}^{K} a_n = a_1 + a_2 + \ldots + a_{K-1} + a_K$$

Def'n: A series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $\{S_k\}_k$ converges

ex// $a_n = 0 \Rightarrow S_k = \underbrace{0 + \dots + 0}_{k\text{-times}} = 0$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} 0 = 0$$

$$\therefore \sum_{n=1}^{\infty} 0 = 0$$

$a_n = n \Rightarrow S_1 = 1$

$S_2 = 1+2 = 3$

$S_3 = 1+2+3 = 6$

$\vdots$

$S_k = \frac{k(k+1)}{2}$

$$\Rightarrow \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k(k+1)}{2} = \infty$$

$$\therefore \sum_{n=1}^{\infty} n \text{ diverges}$$

$$S_k = 1 + 2 + 3 + \dots + (k-2) + (k-1) + k$$

$k+1$

$k+1$

$k+1$

ex// Geometric Series: $a \neq 0$ some constant

$$\sum_{n=0}^{\infty} ar^n = a(r^0 + r^1 + r^2 + r^3 + \dots)$$

ex// $a = 1$, $r = 1/2$ $\Rightarrow$ $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Convergence?

$$S_k = ar^0 + ar^1 + \dots + ar^k$$

$$rS_k = ar + ar^2 + \dots + ar^{k+1}$$

$$= -ar^0 + \underbrace{ar^0 + ar + ar^2 + \dots + ar^k}_{S_k} + ar^{k+1}$$

$$rS_k = S_k + ar^{k+1} - a$$

$$S_k = \frac{ar^{k+1} - a}{r-1} \Leftarrow r \neq 1$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{ar^{k+1} - a}{r-1} = \boxed{\begin{cases} \frac{a}{1-r} & r<1 \\ \text{DNE} & r \geq 1 \end{cases} = \sum_{n=0}^{\infty} ar^n}$$

$\rightarrow$ for $r = 1$ $\Rightarrow$ $(a + a + a + \dots)$ diverges

Divergence Test

Given a series $\sum_{n=1}^{\infty} a_n$, if series converges then $\lim_{n\to\infty} a_n = 0$

$\to$ if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ doesn't converge

ex// $\sum_{n=1}^{\infty} n = 1+2+3+\ldots$ $\quad \lim_{n\to\infty} n \neq 0$ $\quad \therefore \sum_{n=1}^{\infty} n$ doesn't converge

$\sum_{n=1}^{\infty} n^2 = 1+4+9+16+\ldots$ $\quad \lim_{n\to\infty} n^2 \neq 0$ $\quad \therefore$ series doesn't converge

$\to$ converse is not true i.e. if $\lim_{n\to\infty} a_n = 0$ then series $\sum_{n=1}^{\infty} a_n$ may or <u>may not</u> converge

ex// $a_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} a_n = 0$ but $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$

$$\underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \ldots \quad \ldots$$

Integral Test

Let f be a positive continous decreasing function s.t. $\lim_{x\to\infty} f(x) = 0$

then $\int_1^\infty f(x)\,dx$ converges <u>if and only if</u> $\sum_{n=1}^{\infty} f(n)$ converges

ex//

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}} \Longleftrightarrow \underbrace{\int_1^\infty \frac{1}{x}\,dx}_{\text{divergent}}$$

$\Leftarrow$

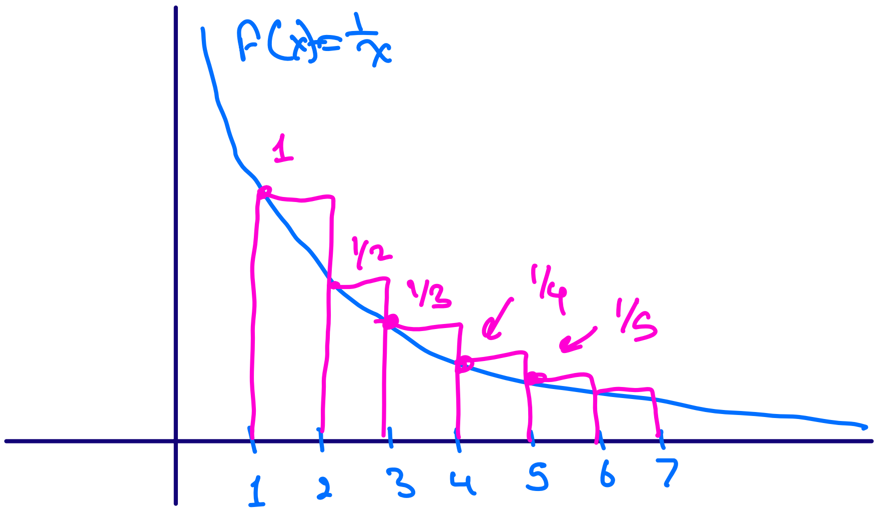
<u>P-Series</u>

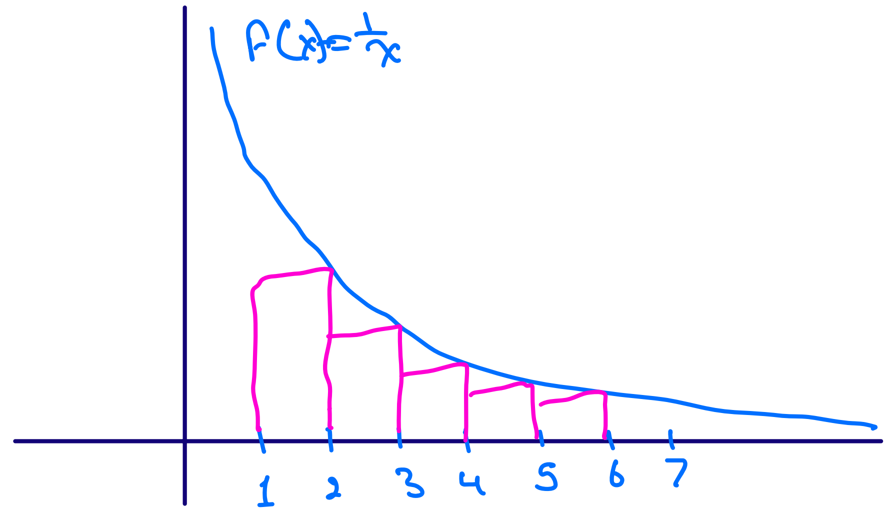
$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^P}}_{\substack{P>1 \text{ converges} \\ P\le \text{ diverges}}} \Longleftrightarrow \underbrace{\int_1^\infty \frac{1}{x^P}\,dx}_{\substack{P>1 \text{ converge} \\ P\le 1 \text{ diverges}}}$$

$\Leftarrow$

→ note that integral test doesn't tell us the value of the series

Intuition ?





→ left end-point approx with $\boxed{\Delta x = 1}$

$$\sum f(\text{left endpoint})\,\Delta x$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx$$

left endpoint is overestimate

→ right endpoint with $\Delta x = 1$

$$\sum_{n=2}^{\infty} \frac{1}{n} \leq \int_1^{\infty} \frac{1}{x}\,dx$$

underestimate

→ using this we can find upper/lower bounds for $\sum_{n=1}^{\infty} a_n$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$

$\rightarrow$ done above

converges for $p > 1$

diverges for $p \leq 1$

$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)} \quad \Longleftrightarrow \quad \int_2^{\infty} \frac{1}{x\ln(x)}\,dx$ $\rightarrow$ $x\ln(x)$ is positive, decreasing, $\rightarrow 0$, continuous

$$= \lim_{k\to\infty} \int_2^{k} \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{k\to\infty} \ln(\ln(x)) \Big|_{x=2}^{x=k}$$

$$= \lim_{k\to\infty} \Big(\ln(\ln(k)) - \ln(\ln(2)) \Big) \boxed{= \infty}$$

$\rightarrow$ by integral test, series diverges

Comparison Test

Given series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ whoe $0 \leq a_n \leq b_n$

① if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges

② if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges

$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ $\sim \sum \frac{1}{n^2}$ which converges, so try to upper bound

$\frac{1}{n^2+4} < \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\Rightarrow$ by comparision $\sum \frac{1}{n^2+4}$ converges

$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+2}$ $\sim \sum \frac{1}{n}$ diverges, so try to lower bound

$$\frac{n^2}{n^3+2} \leq \frac{n^2+1}{n^3+2} \Rightarrow \frac{1}{n+2/n^2} \leq \frac{n^2+1}{n^3+2} \Rightarrow \frac{1}{n+n} \leq \frac{1}{n+2/n^2} \leq \frac{n^2+1}{n^3+2}$$

$$\frac{1}{2n} \leq \frac{n^2+1}{n^3+2}$$

$\to$ by comparison, series diverges since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges

$\sum_{n=1}^{\infty} \frac{3}{5^n+1}$

$$\frac{3}{5^n+1} \leq \frac{3}{5^n}$$

$\to$ geometric series $\sum_{n=1}^{\infty} 3\left(\frac{1}{5}\right)^n$ converges, by comparision series converges

Ratio Test

Given series $\sum_{n=1}^{\infty} a_n$, let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ then

① if $L < 1$, then series is convergent

② if $L > 1$, then series is divergent

③ if $L = 1$, then series may or may not be convergent (inconclusive)

Intuition: if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

→ as $n \to \infty$ $|a_{n+1}| \sim L|a_n|$

→ eventually the series looks like a geometric series $a_n \sim L^n$

→ converges for $L<1$, diverges $L>1$

→ for $L=1$ ratio test inconclusive

$\sum_{n=1}^{\infty} \frac{3^n}{n!}$ $\Rightarrow a_n = \frac{3^n}{n!}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}/(n+1)!}{3^n/n!} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}\, n!}{(n+1)!\; 3^n} \right| = \lim_{n\to\infty} \left| \frac{3}{n+1} \right| = 0$$

$L = 0 < 1 \Rightarrow$ series converges by ratio test

$\sum_{n=1}^{\infty} \frac{n^2}{5^n}$ $a_n = \frac{n^2}{5^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2\; 5^n}{5^{n+1}\; n^2} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{5n^2} \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+1}{5n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1 + 2/n + 1/n^2}{5} \right| = \frac{1}{5} < 1$$

$\therefore$ Converges by ratio test

$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}\,(n+1)!\; n^n}{(n+1)^{(n+1)}\; 2^n\, n!} \right| = \lim_{n\to\infty} \left| \frac{2\; n^n}{(n+1)^n} \right|$$

$$= \frac{2}{e} < 1$$

$\rightarrow$ Series converges

Alternating Series

An alternating series is a series of the form

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \dots \qquad \text{For } a_n \geq 0$$

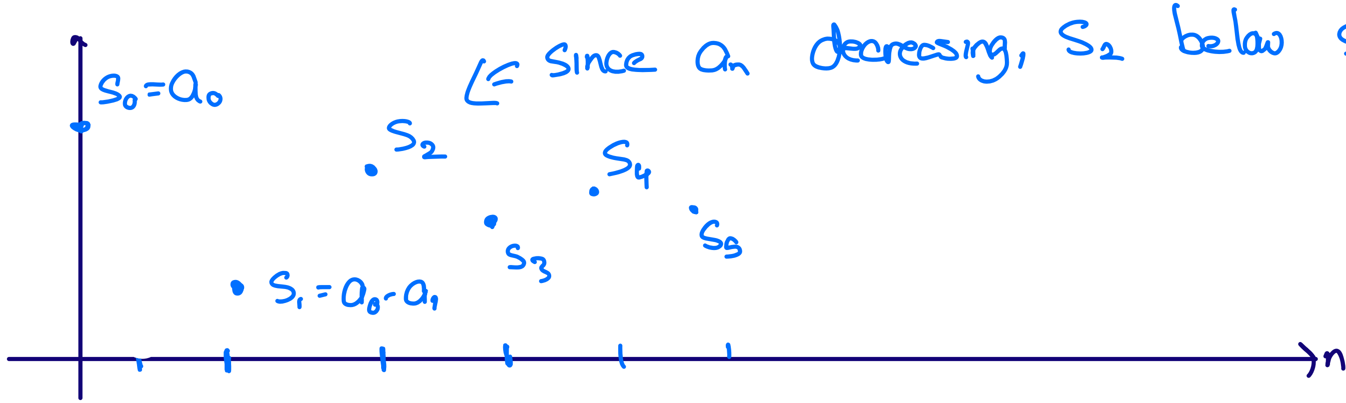
or

$$\sum_{n=0}^{\infty} (-1)^{n+1} a_n = -a_0 + a_1 - a_2 + \dots$$

Alternating Series Test

Given series $\sum_{n=0}^{\infty} (-1)^n a_n$, if a_n is decreasing and $\lim_{n\to\infty} a_n = 0$ then the alternating series converges

$\Leftarrow$ since a_n decreasing, S_2 below S_1



ex// alternating harmonic series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

→ by AST, $\frac{1}{n} \to 0$ and is decreasing

∴ alternating harmonic series converges

We say a series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent

ex// $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ → convergent

→ not absolutely convergent (b/c $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ doesn't converge)

If a series is absolutely convergent then its convergent