UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2018

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

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Instructions:

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 12 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of eight questions, some with many parts. Attempt all of them. Each question is worth 10 marks. Marks for parts of a question are indicated in the question. **Total Marks: 80**
- Do not approximate your answers unless specifically instructed to do so.
- Present your solutions in the space provided; give full and complete explanations! You can use pages 10, 11 and 12 for rough work. If you want anything on pages 10, 11 or 12 to be marked you must indicate in the relevant previous question that the solution continues on page 10, 11 or 12.

1. Given that the reduced row echelon form of

$$A = \begin{bmatrix} 1 & 3 & -1 & 7 & -6 \\ 5 & 15 & 4 & 5 & 27 \\ 3 & 9 & 6 & -9 & 39 \\ -6 & -18 & 2 & 5 & -23 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the rank of A and a basis for each of $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$.

2. For this question, let A be a 5×7 matrix.

(a) [6 marks] Find all possible values of $\dim(\mathrm{col}(A))$ and the corresponding values of $\dim(\mathrm{null}(A))$.

(b) [4 marks; 1 mark for each part] Now suppose $\dim(\mathrm{null}(A)) = 3$. Find the value of

(*i*) $\dim(\mathrm{im}(A))$

(*ii*) $\dim(\mathrm{row}(A))$

(*iii*) $\dim\left((\mathrm{null}(A))^{\perp}\right)$

(*iv*) $\dim\left(\mathrm{null}(A^T)\right)$

3. Show that the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & -7 & 2 \\ -12 & -24 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -3 & 1 \\ -4 & -7 & 2 \\ -16 & -30 & 9 \end{bmatrix}$$

(a) [4 marks] have the same characteristic polynomial

(b) [4 marks] and the same rank,

(c) [2 marks] but are *not* similar.

4. Consider the four data points $(x, y) = (0, -1), (1, 1), (2, 7), (3, 4)$.

(a) [6 marks] Find the least squares approximating line $y = z_0 + z_1\, x$ for the given data points.

(b) [4 marks] Find the cubic polynomial $f(x) = a + bx + cx^2 + dx^3$ such that each of the given data points satisfies the equation $y = f(x)$. (This is *not* a least squares approximating problem.)

5.(a) [4 marks] Is the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ such that $\det \begin{bmatrix} 2 & -1 & a \\ 3 & 1 & b \\ 1 & -1 & c \end{bmatrix} = 0$ a subspace of $\mathbb{R}^3$?

5.(b) [3 marks] Prove that if $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, then $\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 = 4\,\vec{x}\cdot\vec{y}$.

5.(c) [3 marks] Prove: if λ is an eigenvalue of the invertible matrix A, then λ^{-1} is an eigenvalue of A^{-1}.

6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}$.

7. Let $U = \text{span}\left\{\begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T\right\}$.

(a) [5 marks] Find an orthogonal basis of U.

(b) [5 marks] Let $\vec{x} = \begin{bmatrix} 2 & 0 & -1 & 3 \end{bmatrix}^T$. Find $\text{proj}_U(\vec{x})$.

8. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation with matrix $A = \begin{bmatrix} \vec{0} & \vec{e}_1 & \vec{e}_2 & \dots & \vec{e}_{n-1} \end{bmatrix}$, where $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}, \vec{e}_n$ are the standard basis vectors of $\mathbb{R}^n$.

(a) [2 marks] If $\vec{x}$ is in $\mathbb{R}^n$, what is $T(\vec{x})$?

(b) [2 marks] What is A^2?

(c) [4 marks] Let $m \geq 1$ be a whole number. Find a basis for each of $\text{col}(A^m)$ and $\text{null}(A^m)$.

(d) [1 mark] What is the least value of m such that A^m is equal to the zero matrix?

(e) [1 mark] Find the characteristic polynomial of A^m, for $m \geq 1$.

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