UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, April 2007
First Year
MAT188H1 S – LINEAR ALGEBRA
Calculator Type: 3
Exam Type: A
Examiner – H. Witteman

Instructions: Write your solutions in this booklet.
If additional space is required, use exam booklets provided.

Given Name(s)/First Name(s) ______________________________

Family Name/Surname/Last Name ______________________________

Student Number ______________________________

Question	Marks	Out of
1		5
2		4
3		1
Part A: Chapter 3 Subtotal		10
4		4
5		4
6		4
7		4
8		3
9		3
10		3
Part B: Chapter 4 Subtotal		25
Total		**35**

Part A: Chapter 3 Material [10 marks]

1. **[5 marks]**

(a) **[1 mark]** Find the equation of the plane parallel to the plane $2x - 3y + 4z = 5$ and passing through the point $(1,1,1)$.

(b) **[2 marks]** Find the shortest distance between the two lines:
$$X_1 = (2,1,0) + s(1,1,-2)$$
$$X_2 = (3,3,0) + t(1,1,-2)$$

(c) **[2 marks]** Find two points C on the line through $A(1,0,1)$ and $B(1,1,0)$ such that $\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\|$.

2. **[4 marks]**

(a) **[2 marks]** Given a linear transformation **T** such that $\mathbf{T}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{T}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$,

find **A**, the matrix representation of **T**.

(b) **[2 marks]** Given a transformation $\mathbf{R}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ where $\mathbf{R}_\theta$ is the rotation through angle θ:

(i) **[1 mark]** Give **A**, the matrix representation of $\mathbf{R}_\theta$.

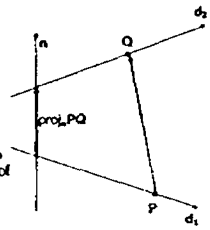
(ii) **[1 mark]** Is $\mathbf{R}_\theta$ linear? Why or why not?

3. **[1 mark]** State whether the following statement is <u>true</u> or <u>false</u>. If it is true, show or explain why. If it false, explain why or give a counterexample.

   If the plane $ax+by+cz = k$ passes through the origin, then $k=0$.

Problem Type 1: Steps

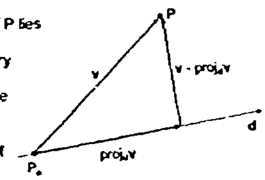
- What is the shortest distance between two skew lines $X_1=P+sd_1$ and $X_2=Q+td_2$?
  - Step 1: Choose one point on each line: P and Q.
  - Step 2: Use points to define directed line segment PQ.
  - Step 3: Define vector orthogonal to both lines by taking cross product of two direction vectors: $n=d_1 x d_2$.
  - Step 4: Project PQ onto vector n.
  - Step 5: Take magnitude of projection.



1

Problem Type 2a: Steps

- What is the distance between a point P in space and a line $X = P_0+td$?
  - Step 1 (optional): Check if P lies on line.
  - Step 2: Choose an arbitrary point on the line: P_0.
  - Step 3: Define directed line segment $v=P_0P$.
  - Step 4: Take $proj_d v$.
  - Step 5: Take magnitude of $v - proj_d v$.



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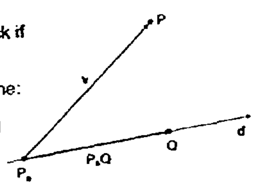
Problem Type 2b: Steps

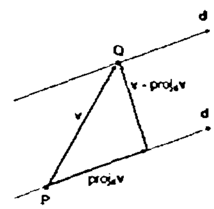
- Find the point Q on line $X = P_0+td$ closest to a point P in space.
  - Step 1 (optional): Check if P lies on line.
  - Step 2: Choose an arbitrary point on the line: P_0.
  - Step 3: Define directed line segment $v=P_0P$.
  - Step 4: $P_0Q = proj_d v$.
  - Step 5: Calculate coordinates of Q.



3

Problem Type 3: Steps

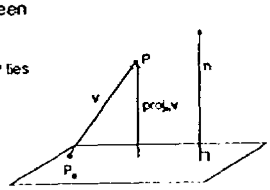
- What is the shortest distance between two parallel lines $X_1=P+sd$ and $X_2=Q+td$?
  - Step 1: Choose one point on each line: P and Q.
  - Step 2: Use points to define $v = PQ$.
  - Step 3: Project v onto d.
  - Step 4: Take magnitude of $v - proj_d v$.



4

Problem Type 4a: Steps

- What is the distance between a point P in space and a plane $ax+by+cz=k$?
  - Step 1 (optional): Check if P lies on plane.
  - Step 2: Choose an arbitrary point on the plane: P_0.
  - Step 3: Define directed line segment $v=P_0P$.
  - Step 4: Take $proj_n v$.
  - Step 5: Take magnitude of $proj_n v$.



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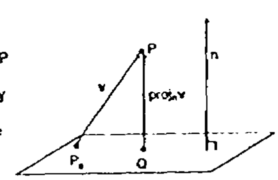
Problem Type 4b: Steps

- Find the point Q on plane $ax+by+cz=k$ closest to a point P in space.
  - Step 1 (optional): Check if P lies on plane.
  - Step 2: Choose an arbitrary point on the plane: P_0.
  - Step 3: Define directed line segment $v=P_0P$.
  - Step 4: $QP = proj_n v$.
  - Step 5: Calculate coordinates of Q.



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Part B: Chapter 4 Material [25 marks]

4. **[4 marks]** Given the subset $S = \left\{ \mathbf{X} \in \mathbf{R}^3 \mid \mathbf{X} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ where s and t are real numbers:

(a) **[2 marks]** Show that S is a subspace of R^3.

(b) **[1 mark]** What is $\dim(S)$?

(c) **[1 mark]** Which of the following subsets is equivalent to S?
$T = \operatorname{span}\{(1,0,0),(0,1,0)\}$
$U = \{(1,0,0),(0,1,0)\}$

5. **[4 marks]** Given the set of vectors $\{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}$:

(a) **[1 mark]** Is the set linearly independent? Why or why not? Support your answer.

(b) **[1 mark]** Does the set span $\mathbf{R}^4$? Why or why not? Support your answer.

(c) **[1 mark]** Is the set a basis for $\mathbf{R}^4$? Why or why not? Support your answer.

(d) **[1 mark]** Is the vector $(1,0,0,1)$ in the span of the set? Why or why not? Support your answer.

6. **[4 marks]**

Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 & 1 \end{bmatrix}$ with **RREF(A)** $= \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$:

(a) **[2 marks]** Give a basis for:

(i) the rowspace of **A**

(ii) the columnspace of **A**

(iii) the nullspace of **A**

(iv) the image of **A**

(c) **[1 mark]** Give the dimension of:

(i) the rowspace of **A**

(ii) the columnspace of **A**

(iii) the nullspace of **A**

(iv) the image of **A**

(d) **[1 mark]** Complete the statements:

(i) the rowspace of **A** is a subspace of ____

(ii) the columnspace of **A** is a subspace of ____

(iii) the nullspace of **A** is a subspace of ____

(iv) the image of **A** is a subspace of ____

7. **[4 marks]** Given $\mathbf{U} = \text{span}\{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}, \mathbf{X} = (3,-1,6,2)$:

(a) **[2 marks]** Find an orthogonal basis for **U**.

(b) **[1 mark]** Find a basis for $\mathbf{U}^{\perp}$.

(c) **[1 mark]** Find $proj_U X$ and $proj_{U^{\perp}} X$.

8. **[3 marks]** Find the best approximation to a solution for the inconsistent system:

$$x_1 - x_2 = 3$$

$$2x_1 + x_2 = -1$$

$$x_1 + 5x_2 = -4$$

(Hint: Use the matrix equation **$(A^T A)Z = A^T B$**.)

9. **[3 marks]** Given the data set

x	-1	0	1	2
y	-2	-1	2	2

find the least squares approximating polynomial of degree 1 ("linear regression") to fit the data.
(Hint: Use the matrix equation $(\mathbf{M}^T\mathbf{M})\mathbf{Z}=\mathbf{M}^T\mathbf{Y}$.)

10. **[3 marks]** For each of the following statements, state whether it is true or false. If it is true, show or explain why. If it false, explain why or give a counterexample.

(a) Any line in $\mathbf{R}^3$ that passes through the origin is a subspace of $\mathbf{R}^3$.

(b) It is possible for a 5x7 matrix to have a nullspace of dimension 6.

(c) It is possible to have a subspace U of $\mathbf{R}^{11}$ such that $\dim(U^{\perp}) = 11$.