MAT186 Term Test 1 Soltuions

October 17, 2022

Multiple Choice Solutions

1. The inequality $-2 \leq x \leq 8$ can be expressed in the form $|x + a| \leq b$, where $a =$ _____ and $b =$ _____. [1 mark]

   Solution: $|x-(-a)| \leq b$, is an interval centered at $-a$ of radius b. Since the interval $-2 \leq x \leq 8$ is centered at $x = 3$, with radius 5 we have $a = -3$ and $b = 5$.

2. Let $f(x) = x^2 - 1$. Select *all* the numbers below that are upper bounds for $|f(x)|$ on the interval $[-\dfrac{3}{4}, \dfrac{1}{3}]$. [2 marks]

   Solution: $f(x) = x^2 - 1$ is an upward opening parabola with vertex at $(x, f(x)) = (0, -1)$ and roots at ± 1, therefore, $f(x)$ is strictly negative on the interval $[-\dfrac{3}{4}, \dfrac{1}{3}]$, therefore the maximum of $|f(x)|$ on this interval is 1. This implies the provided numbers which are upper bounds are 1 and $\frac{3}{2}$.

3. One of your professors (we're not naming names) has noticed that student's raw scores on a linear algebra test is a decreasing function of time they spend on their phones. This same professor noticed that, given a raw score, they could determine how much time a student spent on their phone. Which of the functions below could represent this professor's observations? [1 mark]

   Solution: We look for a function that is decreasing (and invertible).

   - $f(t) = 100 \sin(t + 50) + 100$ oscillates, and therefore is not decreasing (nor invertible).
   - $g(t) = 100 - \dfrac{(t-50)^2}{25}$, is increasing for $0 \leq t \leq 50$, and therefore does not correspond to the professor's observations.
   - $h(t) = 100\, e^{\frac{-t^2}{50}}$, this is the ONLY option which could represent the professor's observations.
   - $k(t) = 100 - 2^{-t}$, is invertible but increasing.

4. Recall that $\sin^{-1}(x)$ is the inverse of $\sin(x)$ whose domain is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $\cos^{-1}(x)$ is the inverse of $\cos(x)$ whose domain is restricted to $[0, \pi]$. We define a new inverse function as follows:

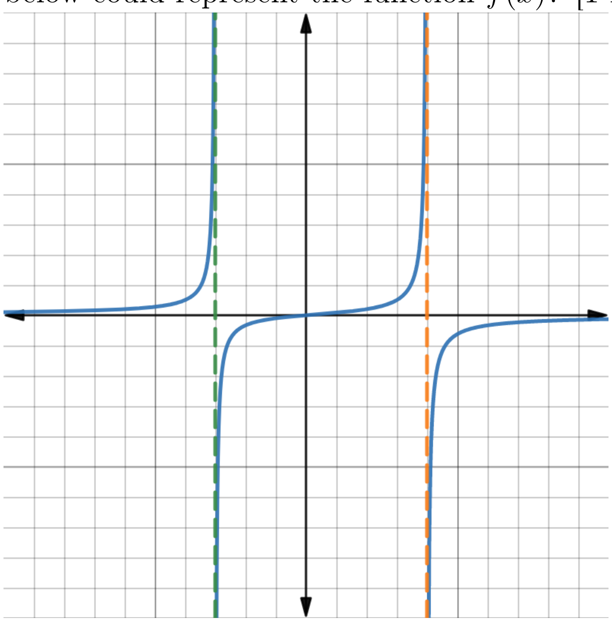
$\text{zin}^{-1}(x)$ is defined to be the inverse of $\sin(x)$ whose domain is restricted to $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

Fill in the blank: A solution to the equation $\cos(\text{zin}^{-1}(x)) = \sin(\cos^{-1}(x))$ *does not exist* when _____. [2 marks]

Solution: The range of $\text{zin}^{-1}(x)$ is $[\frac{\pi}{2}, \frac{3\pi}{2}]$, which implies that $\cos(\text{zin}^{-1}(x)) \leq 0$. In contrast, the range of $\cos^{-1}(x)$ is $[0, \pi]$, which implies that $\sin(\cos^{-1}(x)) \geq 0$. Therefore, equality of these two functions can only occur when $\cos(\text{zin}^{-1}(x)) = \sin(\cos^{-1}(x)) = 0$, which occurs when $x = \pm 1$.

Thus a solution does not exist when $x \in (-1, 0)$ or $x \in (0, 1)$.

5. Let a and b be positive constants, where $a \neq b$. The graph of a function $f(x)$ is pictured below, together with the vertical line $x = -a$, and the vertical line $x = b$. Which of the expressions below could represent the function $f(x)$? [1 mark]



Solution:

- $f(x) = \dfrac{x}{(x-a)(x-b)}$. This would have both vertical asymptotes at positive x values, so this is not the correct choice.
- $f(x) = \dfrac{x}{(x+a)(x-b)}$. For $x > b$ this function is positive, so it does not match the graph.
- $f(x) = \dfrac{-x}{(x+a)(x-b)}$. This is our function!
- $f(x) = \dfrac{-x}{(x-a)(x-b)}$. This would have both vertical asymptotes at positive x values, so this is not the correct choice.

6. $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right) = $ _____? [2 marks]

Solution: $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right) \longrightarrow \infty - \infty$, therefore we must try to manipulate this expression.

$$\begin{aligned}
\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right) &= \left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)\frac{\left(\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}\right)}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}\right)} \\
&= \frac{\left(\left(\sqrt{x+\sqrt{x}}\right)^2-\left(\sqrt{x-\sqrt{x}}\right)^2\right)}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}\right)} \\
&= \frac{(2\sqrt{x})}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}\right)}.
\end{aligned}$$

Factoring out a $\sqrt{x}$ from the denominator yields: $\dfrac{(2\sqrt{x})}{\left(\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}\right)} = \dfrac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}}$,

and thus

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right) = \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}} = \frac{2}{2} = 1.$$

7. Suppose that $f(x)$ and $g(x)$ are differentiable functions defined for all x, and satisfy the following properties:

- $f(-2) = g(-2)$
- $f'(x) < g'(x)$ for all x.

Which of the statements below are true? [2 marks]

- $f(x) < g(x)$ for all $x > -2$. This is true since $g(x)$ is growing faster than $f(x)$ everywhere on this interval and $g(-2) = f(-2)$, so $f(x)$ can never catch back up.
- $f(x) < g(x)$ for all $x < -2$. This is false, in fact, $f(x) > g(x)$ for all $x < -2$.
- The graphs of f and g do not intersect. They intersect at $x = -2$.
- The graphs of f and g intersect at exactly one point. This is true since $f(x) > g(x)$ for $x < -2$ and $g(x) > f(x)$ for $x > -2$.
- The graphs of f and g intersect at more than one point. No.

8. Let $f(x)$ be differentiable for all x, and let $g(x)$ be the tangent line to f at a point $a \neq 0$. Which of the expression(s) below are equal to $f'(a)$? [2 marks]

- $g'(a)$. $g(x)$ has a constant slope of $f'(a)$, therefore $g'(a) = f'(a)$, so this is true.
- $g(a+1) - g(a)$. $g(a+1) - g(a) = \frac{g(a+1)-g(a)}{1}$, so it is the slope of a secant line of $g(x)$, which is always equal to $g'(a) = f'(a)$, so this is true.
- $\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$. This yields $f'(0)$, so in general this is false.
- $\dfrac{f(x) - f(a)}{x - a}$ This is the slope of a secant line with an endpoint at $x = a$. In general this will not equal $f'(a)$, so this is false.
- $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a-h)}{2h}$.

$$\begin{aligned}\frac{f(a+h) - f(a-h)}{2h} &= \frac{1}{2}\frac{f(a+h) - f(a) + f(a) - f(a-h)}{h} \\ &= \frac{1}{2}\left(\frac{f(a+h) - f(a)}{h} + \frac{f(a) - f(a-h)}{h}\right).\end{aligned}$$

Therefore, $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a-h)}{2h} = \dfrac{1}{2}(2f'(a)) = f'(a)$.

True or False:

9. Let $f(x) = 2^x - \dfrac{10}{x}$. Then there exists a $c \in [-\dfrac{5}{2}, 3]$ such that $f(c) = 0$.

Solution: $f(1) = 2 - 10 = -8 < 0$ and $f(3) = 8 - (3 + \frac{1}{3}) > 0$. Since $f(x)$ is the difference of a power function and a rational function, it is continuous on its domain, which contains the interval $(0, \infty)$. Therefore, since $f(x)$ is negative at $x = 1$ and positive at $x = 3$, the Intermediate Value Theorem states that there must exist a point $c \in (1, 3)$ such that $f(c) = 0$. Therefore the statement is **true**.

Short Answer Solutions

10. Let $f(x)$ represent the fuel efficiency, in kilometres per liters, of a car travelling at a speed of x kilometres per hour.

(a) What are the units of $f'(x)$? [1 mark]

Solution: Since $f(x)$ has the units of $\dfrac{km}{L}$, and x is measured in $\dfrac{km}{h}$, implies that $f'(x)$ is measured in $\dfrac{km \cdot h}{L \cdot km} = \dfrac{h}{L}$.

(b) Suppose $f'(101) = 0.74$. If you are driving at 101 kilometres per hour, should you speed up or slow down to be more fuel efficient? Briefly explain. Enter either "speed up" or "slow down" in the box provided as your final answer.

Solution: You should **speed up**, since $f'(101) = 0.74 > 0$, means the graph is increasing at this point and therefore the car has an increased fuel efficiency when moving (at least slightly) faster than 101 $\frac{km}{h}$.

11. Let $f(x) = \begin{cases} 5^{2x-2} + 2, & x \leq 1 \\ \dfrac{3x^2 + x + 5}{Ax^2 + 2}, & 1 < x < 2, \\ \ln(Bx) + 1, & x \geq 2 \end{cases}$ where A and B are positive constants.

(a) Find all values of A ad B such that $f(x)$ is continuous at $x = 1$ and $x = 2$.

Solution: To have continuity at $x = 1$, we require $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$.
For $x \leq 1$, we have $f(x) = 5^{2x-2} + 2$, which is a constant plus the composition of an exponential function and a polynomial, therefore it is continuous on its domain.
Therefore, $\lim_{x \to 1^-} f(x) = f(1) = 3$. We now need to ensure $\lim_{x \to 1^+} f(x) = 3$ as well. In the interval $1 < x < 2$, $f(x)$ is a rational function, therefore, $\lim_{x \to 1^+} f(x) = \dfrac{3+1+5}{A+2}$. Setting this equal to 3 yields $\frac{1}{A+2} = \frac{1}{3}$, which implies $A = 1$.

Using $A = 1$ we now find B which makes $f(x)$ continuous at $x = 2$. Using again the continuity of rational functions, we have $\lim_{x \to 2^-} f(x) = \dfrac{12+2+5}{6} = \dfrac{19}{6}$. Using the continuity of logarithms, we have $\lim_{x \to 2^+} f(x) = \ln(2B) + 1$.

Setting $\ln(2B) + 1 = \frac{19}{6}$, implies $\ln(2B) = \frac{13}{6}$. Which implies $B = \dfrac{1}{2} e^{\frac{13}{6}}$

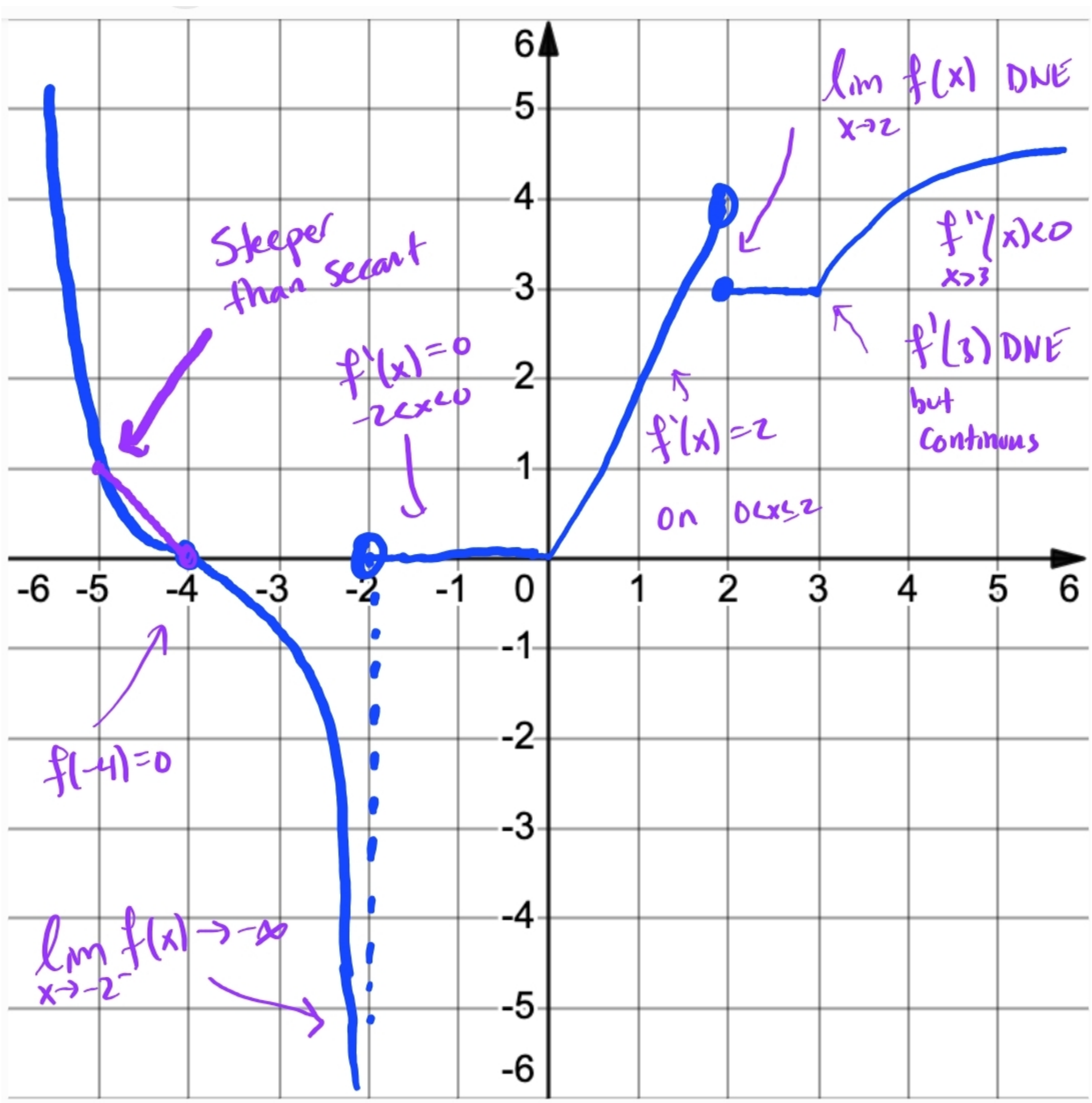
(b) Use the limit definition of the derivative to write an explicit expression for $f'(-1)$. *Your answer should not involve the letter f. Do not attempt to evaluate or simplify the limit.*

Solution: Using the definition of the derivative,

$$\begin{aligned} f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h \to 0} \frac{(5^{2(-1+h)-2} + 2) - (5^{2(-1)-2} + 2)}{h} \end{aligned}$$

12. On the axes below, sketch a well-labeled graph of a function f that satisfies the given properties. [5 marks]

- $f(-4) = 0$
- $f'(-5) < f(-4) - f(-5)$
- $\lim_{x \to -2^-} f(x) \to -\infty$
- $f'(x) = 0$ for $-2 < x < 0$
- $f'(x) = 2$ for $0 < x < 2$
- $\lim_{x \to 2} f(x)$ does not exist
- $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$ does not exist
- f is continuous at $x = 3$
- f is increasing for $x > 3$
- $f''(x) < 0$ for $x > 3$



13. We define the integer part Int(x) of a positive number $x > 0$ to be the largest integer number that is smaller than or equal to x. For example, Int$(1.3) = 1$ and Int$(\pi) = 3$. Determine

$$\lim_{x \to 0^+} \frac{x}{2} \mathrm{Int}\left(\frac{3}{x}\right).$$

[3 marks]

Hint: Give a certain theorem a squeeze.

Solution: By definition of the function Int, we have $\frac{3}{x} - 1 \leq \mathrm{Int}\left(\frac{3}{x}\right) \leq \frac{3}{x}$.

Therefore, for $x \geq 0$, we have $\frac{x}{2}\left(\frac{3}{x} - 1\right) \leq \frac{x}{2}\mathrm{Int}\left(\frac{3}{x}\right) \leq \frac{x}{2}\frac{3}{x}$, or equivalently

$$\frac{3}{2} - \frac{x}{2} \leq \frac{x}{2}\mathrm{Int}\left(\frac{3}{x}\right) \leq \frac{3}{2}.$$

Since this inequality holds for all $x \geq 0$, it also holds as we take a limit as $x \to 0^+$. Therefore,

$$\lim_{x \to 0^+}\left(\frac{3}{2} - \frac{x}{2}\right) \leq \lim_{x \to 0^+}\frac{x}{2}\mathrm{Int}\left(\frac{3}{x}\right) \leq \lim_{x \to 0^+}\frac{3}{2},$$

which yields

$$\frac{3}{2} \leq \lim_{x \to 0^+}\frac{x}{2}\mathrm{Int}\left(\frac{3}{x}\right) \leq \frac{3}{2},$$

and therefore, by the Squeeze Theorem, we have $\lim_{x \to 0^+}\frac{x}{2}\mathrm{Int}\left(\frac{3}{x}\right) = \frac{3}{2}$.