University of Toronto
Faculty of Applied Sciences and Engineering

MAT188 – Midterm I – Fall 2023

LAST (Family) NAME: ______________________________

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STUDENT NUMBER: ______________________________

Solutions

Question:	1	2	3	4	5	6	Total
Points:	11	9	12	9	8	11	60
Score:							

1 Part A

1. (11 points) Fill in the bubble for all statements that **must** be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles.

(a) Let $A = (-5, -1, 0)$, $B = (-7, -2, 1)$, $P = (3, 2, 1)$ and $Q = (-1, 0, 3)$.

Then $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are

◯ Parallel and pointing the same direction

◯ Parallel and pointing opposite direction

◯ Perpendicular

◯ None of the above

Solution:
Parallel and pointing the same direction
LS: I can describe sets using the set builder notation. (Writing) ch0-WRIT-sets

(b) Consider a line ℓ which passes through the point $(-3, -5, 4)$ and which is parallel to the line $\vec{x} = t\begin{bmatrix}3\\6\\7\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix}$. At which point does the line ℓ intersect the yz-plane?

◯ $(0, 1, 11)$ ◯ $(0, 11, 3)$

◯ $(3, 6, 7)$ ◯ $(0, 8, 10)$

Solution: Answer: (0,1,11)
LS: Given an algebraic description of a line in R2 or R3, I can visualize the line. (Visual/Geometry)

(c) Which one(s) of the following matrices is in reduced row echelon form (RREF)?

○ $\begin{bmatrix} -7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$

Solution: All except for $\begin{bmatrix} -7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

LS: I can determine when a matrix is in REF or RRFF form. (Conceptual) ch1-CON-rref

(d) Suppose $B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ is the RREF of an augmented matrix of a system of linear equations. Then this system has

- ◯ Unique solution: $x = -1$, $y = 0$, $z = 3$.
- ◯ Infinity many solutions. One particular solution is $x = -1, y = -1, z = 3$.
- ◯ Unique solution: $x = -1$, $z = 3$.
- ◯ No solution

Solution: Infinity many solutions. One particular solution is $x = -1, y = -1, z = 3$.
LS: Given an augmented matrix of a linear system, I find the general solution to the system. (Computation)

(e) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(\vec{x}) = A\vec{x}$, the first column of A is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and the third column of A is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Then $T(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix})$ is

- ◯ $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$
- ◯ $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$
- ◯ $\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$
- ◯ Not enough information

Solution:
$\begin{bmatrix} 3 \\ -4 \end{bmatrix}$
LS: I can interpret matrix-vector multiplication in terms of a linear combination of the columns of the matrix. (Conceptual) ch1-CON-matlincom

(f) Consider

$$A = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid x - 4y + 8z = -2\} \qquad B = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid 2y - z = 1\}$$

What is a geometrical description of the intersection of A and B? Choose at most one option.

◯ A plane with normal vector $\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$ passing through the point $(-2, 0, 0)$

◯ Two planes that do not pass through the origin.

◯ A line parallel to $\{t \begin{bmatrix} -12 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}\}$.

◯ None of the above

Solution: A line parallel to $\{t \begin{bmatrix} -12 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}\}$.

LS: Given a system of linear equations in R2 or R3, I can visualize the system and its general solution. (Visual/Geometry)

(g) Suppose A is a 2×3 matrix and rank $A = 2$. Suppose $\vec{b}$ is a vector in $\mathbb{R}^2$. Choose all that applies.

◯ The linear system $A\vec{x} = \vec{b}$ may be inconsistent.

◯ The general solution to $A\vec{x} = \vec{0}$ is parallel to the general solution to $A\vec{x} = \vec{b}$.

◯ The linear system $A\vec{x} = \vec{0}$ has infinity many solutions.

◯ The RREF of A has 3 leading entires.

Solution: The general solution to $A\vec{x} = \vec{0}$ is parallel to the general solution to $A\vec{x} = \vec{b}$.
The linear system $A\vec{x} = \vec{0}$ has infinity many solutions.

LS: I can identify the solution type of a system based on information about the rank of its augmented and coefficient matrix. (Conceptual) ch1-CON-ranksoltype

LS: Given a system of linear equations in R2 or R3, I can visualize the system and its general solution. (Visual/Geometry) ch1-VG-gslinsys

(h) Let A be the product of matrices below. What is $a_{11} + a_{23}$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix} = A$$

○ 4 ○ 10 ○ 8 ○ None of the above

Solution:

10

LS: Given matrices A and B, I can compute AB, if defined, by finding the ij-th entry of AB directly by computing the dot product between the ith row of A and the jth column of B. (Computation) ch2-COM-matdotprod

2. Fill in the blank. You don't need to include your computation or reasoning.

(a) (3 points) Let A be the standard matrix of a linear transformation that rotates vectors in $\mathbb{R}^3$ counterclockwise, as seen from the positive z-axis, through $\pi/2$ around the z-axis. Let $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ be columns of A, so that $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$. Then

$$\vec{a}_1 = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad \vec{a}_3 = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Solution:

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) (3 points) Solve the equation

$$-4x - 6y + 9z = 3,$$

and describe your solution in the vector-parametric form.

$$\vec{x} = t \begin{bmatrix} \\ \\ \\ \end{bmatrix} + s \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad t, s \in \mathbb{R}$$

Solution: One possible vector parametric form is

$$\vec{x} = t \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 0 \\ 0 \end{bmatrix} \qquad t, s \in \mathbb{R}$$

(c) (3 points) Find a, b, and c such that $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$ Write DNE for a, b and c if such values do not exist.

$a =$	$b =$	$c =$

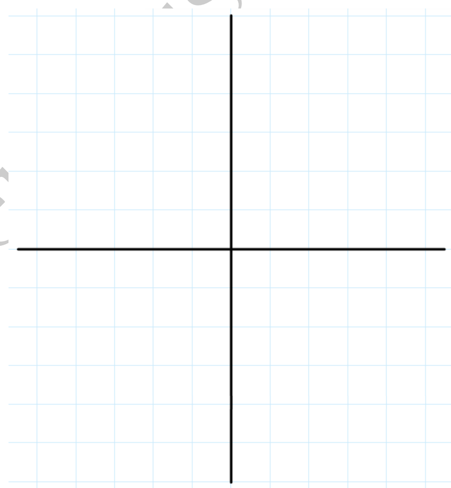
Solution: $a = 3, b = -1, c = 0$

Part B

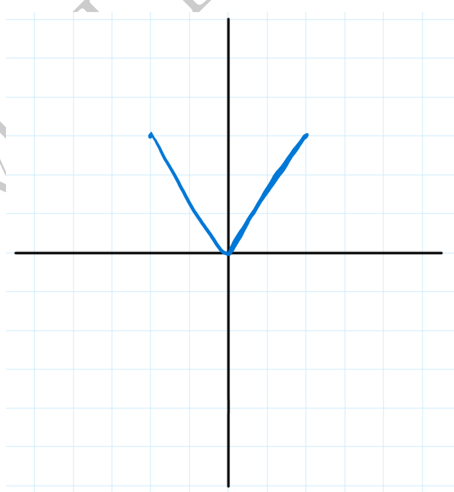
3. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with respect to the line $y = x$.

   (a) (2 points) The set V describes a letter in the alphabet. Accurately, draw this set. In your drawing, identify vectors with their tip when in standard position. Note that $\cup$ denotes the union of the two sets.

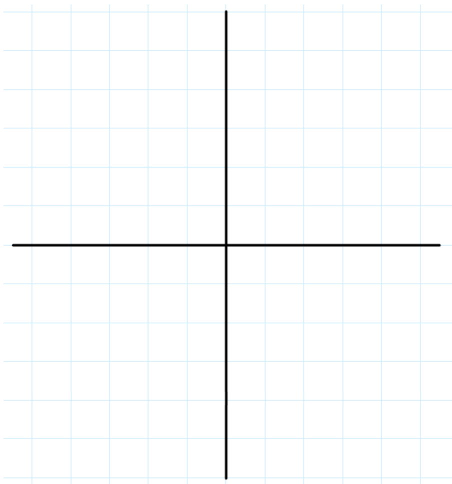
$$V = \{t \begin{bmatrix} -2 \\ 3 \end{bmatrix} \mid 0 \leq t \leq 1\} \cup \{t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \mid 0 \leq t \leq 1\}$$



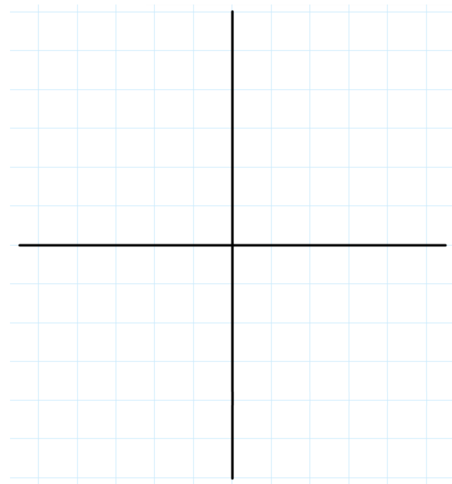
Solution:



   (b) (4 points) Draw $S(V)$ and $T \circ S(V)$ on separate coordinate systems below. Clearly label the output of the following vectors: $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under S and $T \circ S$.

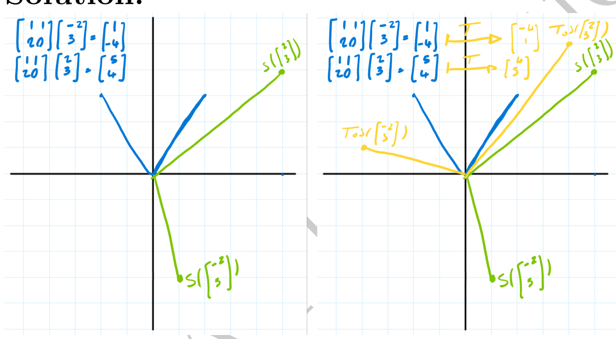


$S(V)$



$T \circ S(V)$

Solution:



Recall that $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection with respect to the line $y = x$.

(c) (2 points) Let B be the standard matrix of T. Find B by computing the output of the standard vectors in $\mathbb{R}^2$. Justify your work. Write your final answer in the small box.

B=

Solution: Note that $B = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$. We need to find $T(\vec{e}_1), T(\vec{e}_2)$ either geometrically or algebraically. To see the output algebraically, take $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ be a vector on the line $y = x$. Then

$$T(\vec{e}_1) = 2proj_{\vec{u}}(\vec{e}_1) - \vec{e}_1 = \vec{e}_2$$

$$T(\vec{e}_2) = 2proj_{\vec{u}}(\vec{e}_2) - \vec{e}_2 = \vec{e}_1$$

Hence $T(\vec{e}_1) = \vec{e}_2$ and $T(\vec{e}_2) = \vec{e}_1$ and

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(d) (4 points) Compute the standard matrix of $T \circ S$. Justify Your answer. Write your final answer in the small box.

Solution: We see that

$$T \circ S(\vec{x}) = T(\vec{S}(\vec{x})) = T(A\vec{x}) = B(A\vec{x}))$$

Hence, the standard matrix for $T \circ S$ is BA

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Alternatively, we can find the standard matrix of $T \circ S$ directly by computing $T \circ S(\vec{e}_1)$ and $T \circ S(\vec{e}_2)$.

$$T \circ S(\vec{e}_1) = T(B\vec{e}_1) = T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T \circ S(\vec{e}_2) = T(B\vec{e}_2) = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

4. State whether each statement is true or false by writing "True" or "False" in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true explain why it must be true. If you think a statement is false, give a counterexample.

   (a) (3 points) Consider a solution $\vec{x}_1$ of a linear system $A\vec{x} = \vec{b}$. If $\vec{x}_h$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{x}_1 + \vec{x}_h$ is a solution to $A\vec{x} = \vec{b}$.

   Solution:
   True. We can plugin $\vec{x}_1 + \vec{x}_h$ into $A\vec{x} = \vec{b}$, to see if the equation holds.

   $$A(\vec{x}_1 + \vec{x}_h) = A\vec{x}_1 + A\vec{x}_h = \vec{b} + \vec{0} = \vec{b}.$$

   The first equality is by properties of matrix-vector multiplication, and the second holds because $\vec{x}_1$ is a solution to $A\vec{x} = \vec{b}$ and $\vec{x}_h$ is a solution to $A\vec{x} = \vec{0}$.

   (b) (3 points) If the bottom row of a matrix A in reduced row-echelon form contains all 0's, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.

   Solution: False. For instance consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) (3 points) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ be the columns of the matrix A. Suppose that $\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ then the equation $\vec{a}_4 = \vec{a}_1 + 2\vec{a}_2 + \vec{a}_3$ must hold.

Solution: True. Note that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a particular solution to the system of linear equation with the augmented matrix A hence $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \vec{a}_4$.

5. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

   (a) (2 points) An equation of a plane that does not pass through the origin and is perpendicular to the vector $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

   Solution: $x - 2y - z = c, c \neq 0$

   (b) (2 points) A vector $\vec{u}$ and two vectors $\vec{v}_1$ and $\vec{v}_2$ such that $\text{proj}_{\vec{u}}(\vec{v}_1) = \text{proj}_{\vec{u}}(\vec{v}_2)$.

   Solution: Note that there are many possible examples.
   $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

   (c) (2 points) Two different linear transformations T and S such that $T \circ S = S \circ T$.

   Solution:
   Take T to be the rotation counterclockwise through $\pi/2$ in $\mathbb{R}^2$, and S to be rotation counterclockwise through $\pi/3$ in $\mathbb{R}^3$.

(d) (2 points) A system of linear equations with three equations and three variables whose geometric interpretation is three planes in $\mathbb{R}^3$ intersecting at the y-axis.

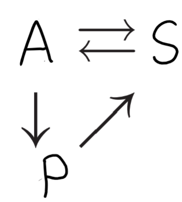
Solution: There are many possibilities including

$$\begin{aligned} y &= 0 \\ z &= 0 \\ z + y &= 0 \end{aligned}$$

Part C

6. A colony of ants travels between three different locations: their anthill, the strawberry patch, and by the puddle. Let's denote these by A, S, and P, respectively. The ants only walk on their constructed one-way paths, and they walk in a rhythmic pattern: Every minute, at the start of the minute, each ant will randomly follow one of the paths. If several paths are available, then an equal proportion of the ants will follow each of them. By the end of the minute, they are at their destination. The locations and the pre-constructed paths are shown in the diagram.

   Let x_1, x_2, and x_3 be the proportions of the ants who find themselves at A, S, and P, respectively, at the start of a minute; we collect this information in a distribution vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.



   Let T be a transformation that transforms the distribution vector of ants at the start of the minute (right before ants start moving) to the distribution of vectors at the end of the minute (right after the ants have arrived at their destination).

   (a) (1 point) The domain of T is ______ and the codomain of T is ______.

   Solution: $T : \mathbb{R}^3 \to \mathbb{R}^3$

   (b) (4 points) Find a matrix C such that $T(\vec{x}) = C\vec{x}$. Justify your work.

   Solution: Based on the diagram, we build the following system of linear equations

$$\begin{aligned} y_1 &= x_2 \\ y_2 &= 1/2x_1 + x_3 \\ y_3 &= 1/2x_1 \end{aligned}$$

   Reformulating in terms of matrix vector multiplication, we get

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Hence, $\vec{y} = T(\vec{x}) = C\vec{x}$, where $C = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix}$

(c) (3 points) If all the ants are initially at the strawberry patch, what is the ants' distribution vector after 3 minutes? Justify your answer.

Solution: We want to compute $T(T(T(\vec{e}_2)))$.

$$T(T(T(\vec{e}_2))) = T(T(C\vec{e}_2)) = T(T(\begin{bmatrix}1\\0\\0\end{bmatrix})) = T(\begin{bmatrix}0\\1/2\\1/2\end{bmatrix}) = \begin{bmatrix}1/2\\1/2\\0\end{bmatrix}$$

(d) (3 points) Is it possible that ants start at some initial distribution $\vec{x}$, and after one minute their distribution remains the same? If yes, find all such initial distributions. If no, why not?

Solution: We need to find a vector $\vec{x}$ that satisfies $C\vec{x} = \vec{x}$, or equivalently $C\vec{x} - \vec{x} = C\vec{x} - I\vec{x} = (C - I)\vec{x} = \vec{0}$. We compute

$$C - I = \begin{bmatrix}-1 & 1 & 0\\1/2 & -1 & 1\\1/2 & 0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

That is $C\vec{x} = \vec{0}$ is consistent with infinitely many solutions.

$$\{t\begin{bmatrix}2\\2\\1\end{bmatrix} \quad t \in \mathbb{R}\}$$

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