UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING
MAT186H1-S Calculus I - Final

Examiner - Janelle Resch

Exam Type: A

Calculator Type: 4

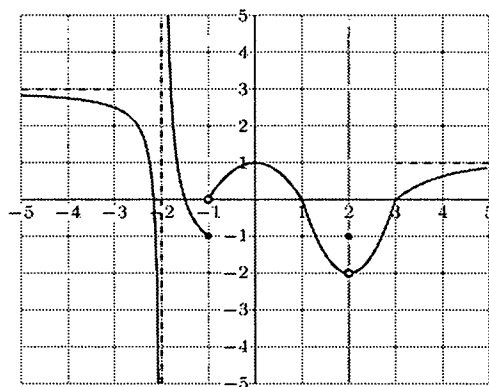Full Legal Name: _____

Student Number: _____

Email: _____

Signature: _____

Instructions:

1. **NO CALCULATORS PERMITTED.**

2. The final exam contains 17 pages. DO NOT DETACH ANY PAGES FROM THIS TEST.

3. The backside of the pages WILL NOT be scanned nor graded. Use the back side of pages for rough work only.

4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices are NOT to be in your pockets and CANNOT BE TAKEN TO THE WASHROOM.

5. Show all your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise notes; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.

6. You must use the methods learned in this course to solve all the problems.

7. Good luck and do your best!

Short Answer Problems:

1. Consider the graph of the function below and answer the following questions.

(a) Evaluate the following. [6 marks]

$$\lim_{x \to -1^-} f(x) = \qquad \lim_{x \to -1^+} f(x) = \qquad f(-1) =$$

$$\lim_{x \to 2^-} f(x) = \qquad \lim_{x \to 2^+} f(x) = \qquad f(2) =$$

(b) Does $\lim_{x \to -1} f(x)$ and $\lim_{x \to 2} f(x)$ exist? Justify your answer. [4 marks]

(c) For what values of x is $f(x)$ discontinuous? State the type of the discontinuity. [6 marks]

(d) Write the intervals where $f(x)$ is differentiable. [2 marks]

(e) Write the intervals where $f(x)$ is increasing and decreasing. [3 marks]
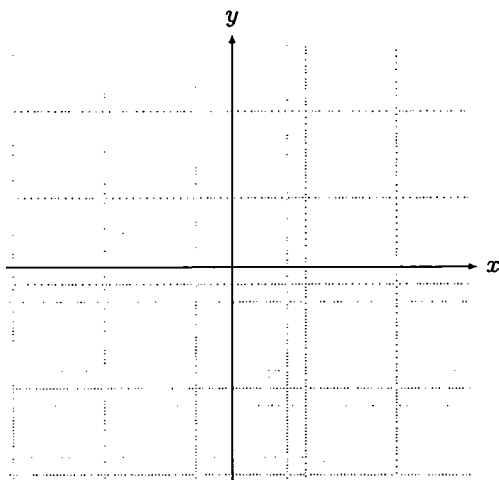
(f) State any critical numbers of $f(x)$ and the associated extrema value. [2 marks]

(g) State any points of inflection of $f(x)$. [1 mark]

(h) Write the intervals where $f(x)$ is concave up and concave down. [2 marks]

2. Consider the function $f(x) = |x|$. Does $f(x)$ have any critical points with corresponding extrema? Does $f(x)$ have any points of inflection. Justify your reasoning. [3 marks]

3. Write the formal definitions for the Riemann sum and definite integral. What is the difference between the Riemann sum and the definite integral. [4 marks]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $[a, b]$. State both parts of the Fundamental Theorem of Calculus. [2 marks]

5. Set up but do not evaluate the integral that would give the area between the curves $f(x) = x^2 - 9$ and $g(x) = 9 - x^2$. Sketch the region given in the question, make sure to label the curves. [5 marks]

6. Show that if $y(t) = y_0 e^{kt}$ with y_0 and k constants, then $\frac{dy}{dt} = ky$. [3 marks]
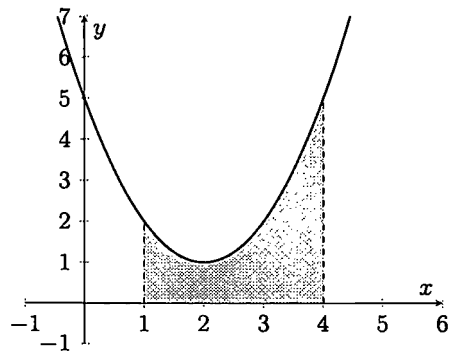
7. Consider the differential equation
$$\frac{dP}{dt} = P(P - K)$$
where K is a positive constant and $P(t)$ represents a population at time t.

(a) What are the equilibrium solutions? [2 marks]

(b) What is the physical interpretation if $\frac{dp}{dt} > 0$? [1 mark]

8. Set up but do not evaluate the integral to find the volume of the solid of revolution bounded by $x = 1$, $x = 4$ and $f(x) = x^2 - 4x + 5$ rotated about the x-axis. [5 marks]

Computational Problems:

1. Use any method to evaluate the following. Make sure to justify your work.

   (a) Evaluate $\lim\limits_{x \to \infty} \left(\dfrac{3x+1}{x - \sin(x)} \right)$. [5 marks]

   (b) Find the derivative of $f(x) = x^{\sin(x)}$. [5 marks]

2. Use any method to evaluate the following integrals and show all steps.

(a) $\int \left(x^e + e^{2x} + x^{\pi} + \pi^x + 1 \right) dx$ [2 marks]

(b) $\int_0^e \left(\dfrac{x^2 - 9}{x + 3} \right) dx$ [4 marks]

(c) $\int 3x^2 e^{x^3 - 1} \, dx$ [5 marks]

3. Solve one of the two integrals. **Yes, you only need to solve one of these to get full marks!** [6 marks].

(a) $\int_0^{\frac{\pi}{2}} \sin^4(x) \cos(x) \, dx$

(b) $\int_1^5 \frac{1}{\sqrt{2x-1}} \, dx$

Long Answer Problems:

1. Find all values of $a, b \in \mathbb{R}$ such that [8 marks]

$$\lim_{x \to 2} \left(\frac{ax^2 + bx + 2}{\ln(3 - x)} \right) = -1.$$

2. Consider the function
$$f(x) = \frac{x^2 - 4}{x^2 - 1}.$$

Perform all the steps we outlined to sketch the curve with calculus, and then sketch the curve. This includes listing the domain, any asymptotes, intervals of increase/decrease with justification, intervals of concavity with justification, and along with the sketch plotting and labeling any critical numbers, points of inflection, and intercepts. Make sure to justify relative extrema using the first or second derivative test. [12 marks]

Domain:

Therefore, the domain is _____ .

Intercepts:

Therefore, we have the y-intercept(s) _____ and x-intercept(s) _____ .

Asymptotes:

This gives vertical asymptote(s) _____ and horizontal asymptote(s) _____.

Critical Points and Increasing/Decreasing Intervals:

We have the critical point(s) _____

$f(x)$ is increasing on the interval _____

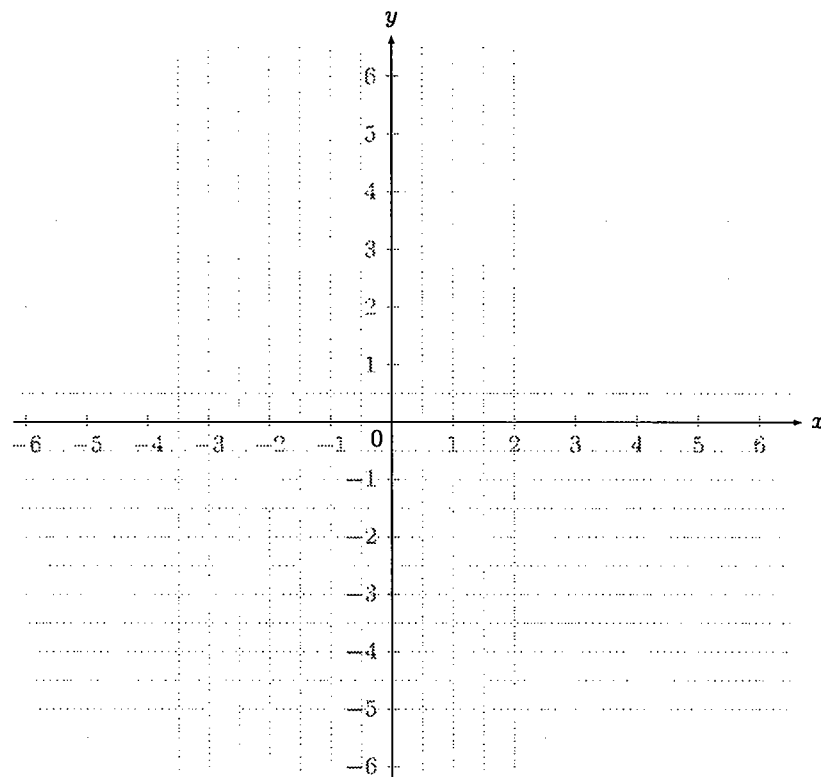$f(x)$ is decreasing on the interval _____

Points of Inflection and Concavity Intervals:
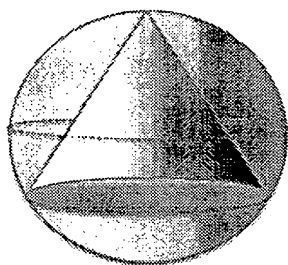
The inflection point(s) _____

$f(x)$ is concave up on the interval _____

$f(x)$ is concave down on the interval _____

Sketch of Graph:

3. Find the volume of the largest possible cone with radius r and height h inscribed in a sphere of radius R. What fraction of the volume of the sphere is occupied by the cone? Note that the volume of a sphere is $\frac{4}{3}\pi R^3$ and the volume of a cone is $\frac{1}{3}\pi r^2 h$. [10 marks]

4. Let $T(t)$ represent the temperature of an object at time t in a room with ambient temperature T_A.

(a) Recall Newton's law of cooling, which says the rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings. If $k > 0$ is the constant of proportionality, write the differential equation that describes the cooling law. [1 mark]

(b) State any equilibrium solutions. [1 mark]

(c) Solve for the family of solutions to this differential equation. [5 marks]

(d) Consider a cup of coffee in a room with a temperature of 20° C. The temperature of the coffee is initially 100° C but 10 minutes later, it has cooled to 60° C, determine the temperature of the coffee as a function of time. [5 marks]