Duration: 100 minutes **Tuesday October 1, 2024**

MAT186 Calculus I
Term Test 1

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Instructions:

1. This test contains a total of 12 pages.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this test, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. Which of the intervals below represents the set of values of x for which $|x+1| < 1$? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ● $(-2, 0)$
- ○ $(0, 2)$
- ○ $(-2, 2)$
- ○ $(-\infty, -2) \cup (0, \infty)$
- ○ $(-\infty, 0) \cup (2, \infty)$

Explanation: The inequality $|x+1| < 1$ defines the set of points that are within 1 unit of -1, so $-2 < x < 0$.

2. Which of the intervals below represents the entire domain of $f(x) = \dfrac{\ln(\ln x)}{x-4}$? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $(-\infty, 4) \cup (4, \infty)$
- ○ $(0, 4) \cup (4, \infty)$
- ● $(1, 4) \cup (4, \infty)$
- ○ $(e, 4) \cup (4, \infty)$
- ○ $(4, \infty)$

Explanation: Recall that the domain of $\ln x$ is $x > 0$. Therefore, $\ln(\ln x)$ is defined when $\ln x > 0$, that is when $x > 1$. We must also exclude $x = 4$ in the domain of the function $\frac{\ln(\ln x)}{x-4}$. Therefore, the set of numbers in $(1, 4) \cup (4, \infty)$ is the domain.

Multiple Choice: No justification is required. Only your final answer will be graded.

Use the following information to answer Questions 3 & 4 below, and Question 5 on the next page.

Based on the principle of conservation of energy, an object's Escape Velocity represents the minimum speed it must be travelling to "escape" from the gravitational attraction of a nearby massive body, such as a planet. In particular, for some object escaping the gravitational attraction of a body with fixed mass $M \gg 0$ (measured in kilograms kg), the escape velocity v_e (measured in metres per second $\frac{m}{s}$) when the escaping object is a distance r (measured in meters m) from the centre of the massive body is given as:

$$v_e(r) = \sqrt{\frac{2GM}{r}}$$

In the equation, G is the positive-valued Universal Gravitational Constant.

3. What are the units of G? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $\sqrt{\frac{kg}{m}}$
- ○ $\frac{m}{kg}$
- ○ $\frac{m^2}{kg \cdot s}$
- ● $\frac{m^3}{kg \cdot s^2}$
- ○ G is unitless

Explanation: After rearranging the equation, the units of G must satisfy:

$$[G] = \frac{[v]^2[r]}{[M]} = \frac{[m^2/s^2][m]}{[kg]} = \frac{m^3}{kg \cdot s^2}$$

4. Which of the following are properties of the function $v_e(r)$? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ The domain is $r \in (0, \infty)$, and the range is $v_e \in \mathbb{R}$.
- ● The domain is $r \in (0, \infty)$, and the range is $v_e \in (0, \infty)$.
- ○ $v_e(r)$ is strictly increasing on its domain.
- ● $v_e(r)$ is strictly decreasing on its domain.
- ● $v_e(r)$ is an invertible function over its whole domain.

Explanation: Observe that $\sqrt{\frac{2GM}{r}}$ is non-negative if $r > 0$, and strictly decreasing. Furthermore, any function that is strictly decreasing over its domain is invertible over its domain.

Multiple Choice: No justification is required. Only your final answer will be graded.

Use the following information to answer Question 5 below. (This paragraph is repeated from the previous page.)

> Based on the principle of conservation of energy, an object's Escape Velocity represents the minimum speed it must be travelling to "escape" from the gravitational attraction of a nearby massive body, such as a planet. In particular, for some object escaping the gravitational attraction of a body with fixed mass $M \gg 0$ (measured in kilograms kg), the escape velocity v_e (measured in metres per second $\frac{m}{s}$) when the escaping object is a distance r (measured in meters m) from the centre of the massive body is given as:
>
> $$v_e(r) = \sqrt{\frac{2GM}{r}}$$
>
> In the equation, G is the positive-valued Universal Gravitational Constant.

5. Which of the statements below most accurately describes the meaning of the inverse of $v_e(r)$? [1 mark]

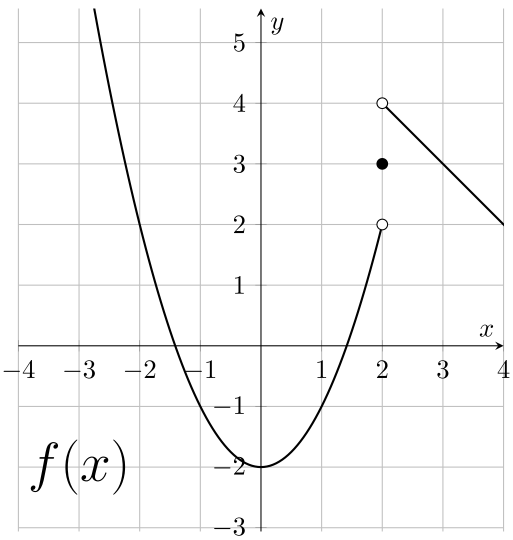
Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

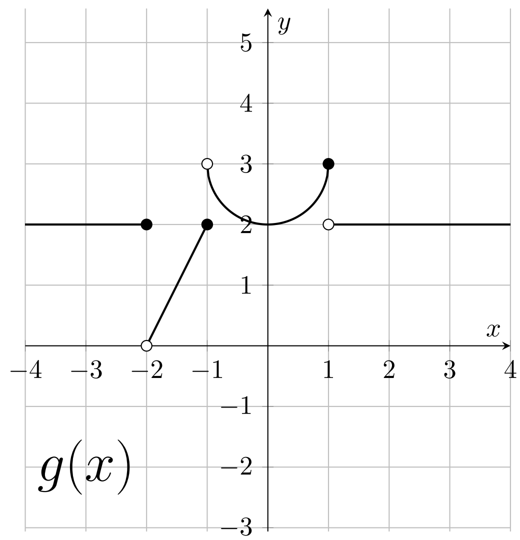
- ○ The inverse of $v_e(r)$ gives the accumulated distance r of an escaping object from a body with mass M when it is travelling at speed v_e.
- ● The inverse of $v_e(r)$ gives the distance r between a body of mass M and an escaping object such that the object will escape the gravitational attraction of the massive body when its speed is v_e.
- ○ The inverse of $v_e(r)$ gives the speed v_e required for an object to escape the gravitational attraction of a body with mass M when the distance between the massive body and the object is r.
- ○ The inverse of $v_e(r)$ has no meaning since $v_e(r)$ is not invertible on its domain.
- ○ The inverse of $v_e(r)$ has no meaning since $v_e(r)$ is not invertible on $r \in \mathbb{R}$.

Explanation: From the previous part, we know that $v_e(r)$ is an invertible function. As $v_e(r)$ gives velocity as a function of distance, the inverse of $v_e(r)$ will give distance as a function of velocity.

Multiple Choice: No justification is required. Only your final answer will be graded.

Use the following graphs to answer Question 6 and Question 7 below.





6. Which of the following statements are true? [2 marks]

- ● $f(x)$ is continuous at -2
- ● $\lim_{x\to 0} g(f(x))$ exists and $g(f(0))$ is defined.
- ○ $\lim_{x\to 0} g(f(x))$ does not exist but $g(f(0))$ is defined.
- ○ $\lim_{x\to 0} g(f(x))$ exists but $g(f(0))$ is undefined.
- ○ $g(f(x))$ is continuous at 0.

Solution: We see from the graph of f that it is continuous at $x = -2$. We also know that $g(f(0))$ is defined since $g(f(0)) = g(-2) = 2$. Furthermore, we know that $\lim_{x\to 0} g(f(x))$ exists since if x is in an interval around 0 (*but not exactly 0*), $f(x)$ is approaching -2 from above. Therefore, we conclude that,

$$\lim_{x\to 0} g(f(x)) = \lim_{y\to -2^+} g(y) = 0$$

using the graph of g. However, $g(f(x))$ is **not** continuous at $x = 0$ since $\lim_{x\to 0} g(f(x)) = 0 \neq g(f(0)) = 2$.

7. $\lim_{x\to -3} f(g(x)) =$ ______? [1 mark]

- ○ 1
- ○ 2
- ● 3
- ○ 4
- ○ does not exist

Solution: If x is in an interval around -3, $g(x)$ takes on the constant value 2. Therefore, near $x = -3$, we conclude that $f(g(x))$ takes on the constant value $f(2) = 3$. We conclude that $\lim_{x\to -3} f(g(x)) = 3$.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

8. Decide if the following statements are true or false. If the statement is true, explain why. If the statement is false, either give a formula for $f(x)$ or draw a possible graph of $f(x)$ that is a counterexample (that is, it satisfies the hypothesis of the statement but not the conclusion).

Indicate your final answer by **filling in exactly one circle** corresponding to your choice (unfilled ○ filled ●).

(a) If a is in the domain of $f(x)$, then $\lim_{x\to a} f(x) = f(a)$. [2 marks]

○ True.

● False.

If a is in the domain of $f(x)$, it is possible for $\lim_{x\to a} f(x) \neq f(a)$ (i.e. f is discontinuous at $x = a$) if either:

- $\lim_{x\to a} f(x)$ does not exist. For example, for $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$, the limit $\lim_{x\to 0} f(x)$ does not exist since the left and right limits at $x = 0$ differ.
- $\lim_{x\to a} f(x)$ exists but differs from $f(a)$. For example, if $g(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$, we have $\lim_{x\to 0} g(x) = 0$ but it is not equal to $g(0) = 1$.

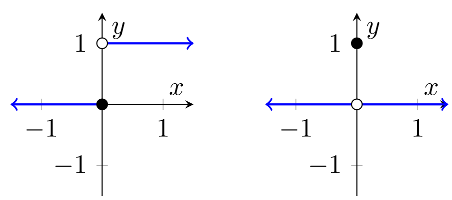


Figure 1: Graph of $f(x)$ (left) and $g(x)$ (right)

(b) If $\lim_{x\to 1} f(x) = 0$, then $f(0.99)$ is closer to 0 than $f(0.9)$. [2 marks]

○ True.

● False.

If

$$h(x) = \begin{cases} 0 & x \neq 0.99 \\ 1 & x = 0.99 \end{cases},$$

we have $\lim_{x\to 1} h(x) = 0$, but $h(0.99) = 1$ is further away from 0 than $h(0.9) = 0$ is.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

9. Suppose you have two pieces of wire that are exactly 10 cm long each. You hold the pieces together horizontally and cut at *approximately* their midpoint, leaving you with two pairs of wires; a pair on your left of equal length, and a pair on your right of equal length. The pair on your left may differ in length from the pair on your right. You then arrange the four pieces to form a rectangular frame.

How close to the midpoint does your cut need to be if you want the area inside the frame to be within 1cm^2 of 25cm^2? [4 marks]

Let x be distance from the midpoint where you have cut the wires. We now have two pieces of wire of length $5 - x$ and two pieces of wire of length $5 + x$. The area of the frame is therefore $(5 - x)(5 + x) = 25 - x^2$ (cm^2).

To ensure that the frame is within $\pm$1cm^2 of 25cm^2, we must ensure that

$$24 \leq 25 - x^2 \leq 26.$$

We have $25 - x^2 \leq 26$ is always true since $x^2 \geq 0$ for all x. To ensure that $24 \leq 25 - x^2$, we must have $x^2 \leq 1$, or $|x| \leq 1$.

Therefore, we must cut within 1cm of the midpoint to guarantee our area is within the acceptable error.

You must cut within $\boxed{\text{1cm of the midpoint}}$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the boxes provided.

10. Evaluate the limits below, if they exist, and give a careful argument that your answer is correct. Write "DNE" in the box for your final answer if the limit does not exist.

(a) $\lim_{x\to\frac{\pi}{2}^-} \cos x \cos(\tan x)$. [4 marks]

For all real x, we have

$$-1 \leq \cos x \leq 1.$$

Therefore, for $0 \leq x < \frac{\pi}{2}$, we have

$$-1 \leq \cos(\tan x) \leq 1.$$

Now, for $0 \leq x < \frac{\pi}{2}$, $\cos x$ is non-negative, and therefore

$$-\cos x \leq \cos x \cos(\tan x) \leq \cos x$$

Finally, since $\cos x$ is continuous at $\frac{\pi}{2}$, we have $\lim_{x\to\frac{\pi}{2}^-} -\cos x = 0 = \lim_{x\to\frac{\pi}{2}^-} \cos x$ from which we conclude

$$\lim_{x\to\frac{\pi}{2}^-} \cos x \cos(\tan x) = 0$$

by the Squeeze Theorem.

$\lim_{x\to\frac{\pi}{2}^-} \cos x \cos(\tan x) = \boxed{0}$

(b) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$. [4 marks]

Observe that since x is negative,

$$\sqrt{4x^2+x} = \sqrt{4x^2(1+\frac{1}{4x})} = \sqrt{4x^2}\sqrt{1+\frac{1}{4x}} = |2x|\sqrt{1+\frac{1}{4x}} = -2x\sqrt{1+\frac{1}{4x}}$$

where we have used $\sqrt{x^2} = |x|$ for all x, and $|x| = -x$ since x is negative.

Therefore, using the previous observation:

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = \lim_{x\to-\infty} \frac{3x}{-2x\sqrt{1+\frac{1}{4x}}-2x} = \lim_{x\to-\infty} \frac{3}{-2\sqrt{1+\frac{1}{4x}}-2} = \frac{3}{(-2)(1)-2} = -\frac{3}{4}$$

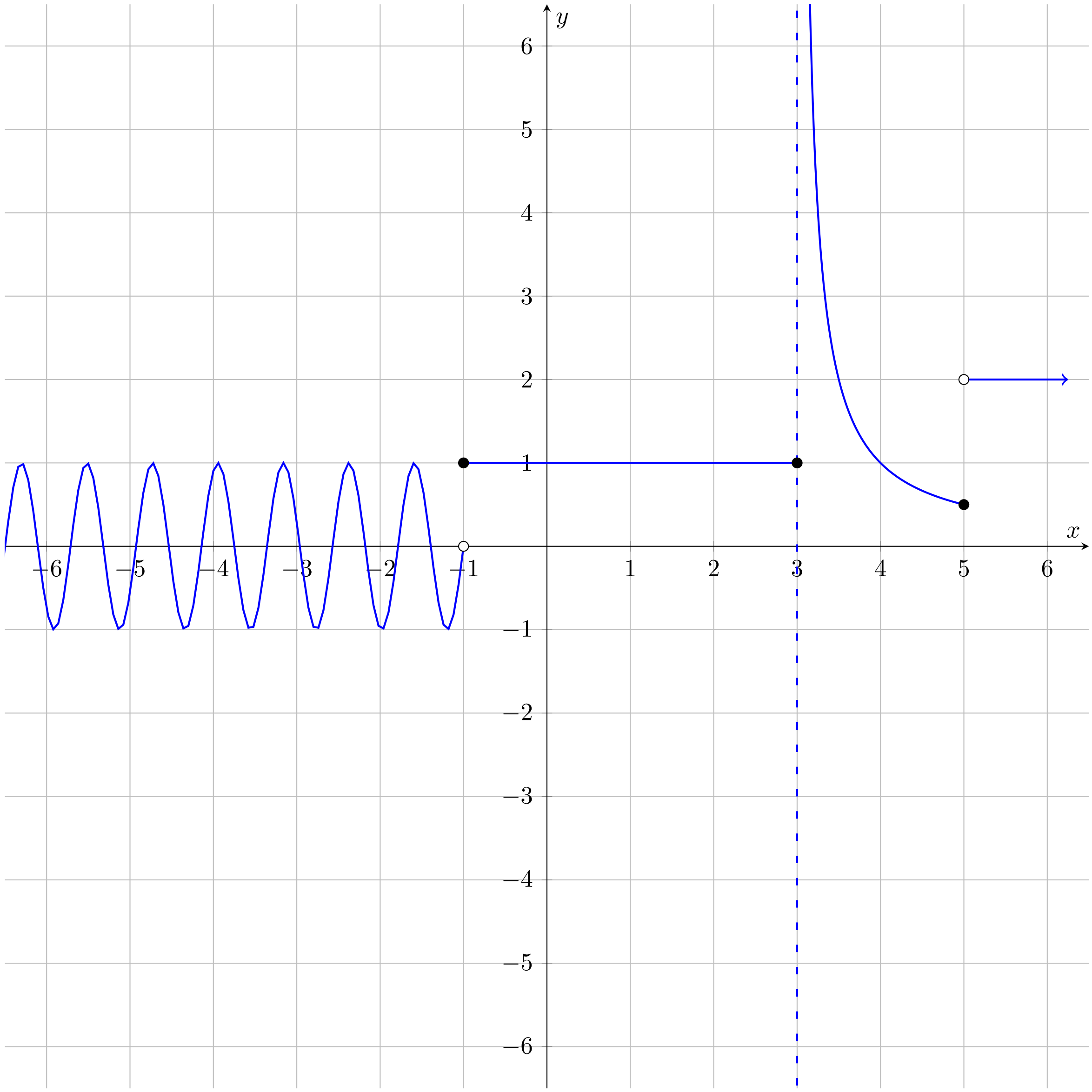
where we have used $\lim_{x\to-\infty} \frac{1}{4x} = 0$ to complete the calculation.

$\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = \boxed{-\frac{3}{4}}$

11. On the axes below, sketch a well-labeled graph of a function $f(x)$ defined everywhere that satisfies the given properties below. [6 marks]

- $f(-1) = 1$
- $\lim_{x \to 3+} f(x) \to \infty$
- $f(x)$ is invertible on $(3, 5)$
- $\lim_{x \to \infty} f(x) = 2$
- $f(x)$ is not continuous at $x = -1$
- $\lim_{x \to 3^-} f(x) = 1$
- $\lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x)$
- $\lim_{x \to -\infty} f(x)$ does not exist but $\lim_{x \to -\infty} f(x) \not\to \pm\infty$

There are many possible solutions. One possibility is shown below:



Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

12. Show that the equation

$$x^{186} + \frac{188}{x^2 + 1 + \cos^2 x} = 100$$

has at least two solutions. [6 marks]

First, observe that $x^2 + \cos^2 x + 1 \geq 1$ and therefore, $x^2 + 1 + \cos^2 x$ is strictly positive for all real numbers. We conclude that $x^2 + 1 + \cos^2 x$ is **never** zero, and our function

$$f(x) = x^{186} + \frac{188}{x^2 + 1 + \cos^2 x}$$

is a continuous function for all real x. Furthermore, our argument also shows that so $f(x) \geq x^{186}$ for all real x since $\dfrac{188}{x^2 + 1 + \cos^2 x}$ is a positive number.

At this point, there are several methods that can be used to reach the solution.

Method 1 - "Strict" IVT Argument: We know that $f(0) = \frac{188}{2} = 96$ and $f(2) \geq 2^{186}$. Since

$$f(0) = 96 < 100 < 2^{186} < f(2)$$

and $f(x)$ is continuous, the Intermediate Value Theorem guarantees a solution $f(c) = 100$ with $0 < c < 2$. To show that there is a **second** solution, observe that $f(-2)$ also satisfies $f(-2) \geq 2^{186}$. Therefore, the Intermediate Value Theorem guarantees a solution $f(d) = 100$ with $-2 < d < 0$. We are guaranteed $c \neq d$ as c is positive and d is negative, so $f(x) = 100$ has at least two distinct solutions.

Method 2 - IVT-Like Argument: We observe that $\lim_{x\to\infty} f(x) \to \infty$ and similarly, $\lim_{x\to-\infty} f(x) \to \infty$ since $f(x) \geq x^{186}$ for all real x. Therefore, since $f(0) = 96$, $100 > 96$ and f is continuous, $f(x) = 100$ has must have at least one positive and one negative solution.

Method 3 - A Symmetry Argument: Finally, we can observe that $f(x)$ is an even function (that is $f(-x) = f(x)$). This is because $(-x)^n = x^n$ for any even n and $\cos(-x) = \cos x$. Therefore, if we can show that $f(c) = 100$ for some value of $c \neq 0$ (using Solution 1 or 2), we will also have $f(-c) = f(c) = 100$. Solutions $-c$ and c must be distinct since $c \neq 0$.