Practise Problem Set 4

MAT 187 - Summer 2025

These questions are meant for your own practice for quiz 4 and are not to be handed in. Some of these questions, or problems similar to these, may appear on the quizzes or exams. Therefore, solutions to these problems will not be posted but you may, of course, ask about these questions during office hours, or on Piazza.

Suggestions on how to complete these problems:

- Solution writing is a skill like any other, which must be practiced as you study. After you write down your rough solutions, take the time to write a clear readable solution that blends sentences and mathematical symbols. This will help you to retain, reinforce, and better understand the concepts.
- After you complete a practice problem, reflect on it. What course material did you use to solve the problem? What was challenging about it? What were the main ideas, techniques, and strategies that you used to solve the problems? What mistakes did you make at the first attempt and how can you prevent these mistakes on a Term Test? What advice would you give to another student who is struggling with this problem?
- Discussing course content with your classmates is encouraged and a mathematically healthy practice. Work together, share ideas, explain concepts to each other, compare your solutions, and ask each other questions. Teaching someone else will help you develop a deeper level of understanding. However, it's also important that reserve some time for self-study and self-assessment to help ensure you can solve problems on your own without relying on others.

1. For each of the following sequences, determine whether the sequence converges or diverges. If it converges, find the limit. Show appropriate reasoning.
   (a) $a_n = \frac{3n^2+1}{n^2+5}$
   (b) $a_n = \frac{(-1)^n n}{n+1}$
   (c) $a_n = \sqrt{n^2+2n} - n$
   (d) $a_n = \frac{5+(-1)^n}{n}$
   (e) $a_n = \frac{n}{2^n}$
2. Analyze the following recursively defined sequences.
   Hint: If the sequence converges, $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. For recursively defined sequences, we can often plug this into the recursive relation to solve for the limit.
   (a) Let $a_1 = 2$, and define $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$.

i. Show that the sequence converges.

ii. Find $\lim_{n\to\infty} a_n$.

(b) Let $a_1 = 1$, and define $a_{n+1} = \frac{a_n}{1+a_n}$.

i. Show that $0 < a_n < 1$ for all $n \geq 2$.

ii. Show that the sequence converges.

iii. Determine $\lim_{n\to\infty} a_n$.

3. Determine if the following infinite series are convergent or divergent. If the series is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+2n}{3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n+1}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

Hint: Show that $\displaystyle\frac{1}{n^2+3n+2} = \frac{A}{n+1} - \frac{B}{n+2}$ for some constants $A, B \in \mathbb{R}$.

4. For the following series, use an integral comparison to determine if they are convergent or divergent. If convergent, find an upper and lower bound for the sum of the series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{n+1}$

Hint: Consider functions $f(x) = e^{-x}$ and $g(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$.

5. For each of the following series, determine whether the series converges or diverges. If it converges, state whether it converges absolutely or conditionally. Clearly indicate which test(s) you are using.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$

6. Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{for } p > 0.$$

(a) Use the left-endpoint approximation of $\int_1^N \frac{1}{x^p}\,dx$ to obtain a lower bound for the partial sum S_N.

(b) Use the right-endpoint approximation of $\int_1^N \frac{1}{x^p}\,dx$ to obtain an upper bound for the partial sum S_N.

(c) Use your answer in part (a) to justify why the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0 < p \leq 1$.

(d) Use your answer in part (b) to justify why the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

7. In this problem, you will derive the Taylor series expansion for $\ln(1 + x)$ using the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

(a) Find the Taylor series for $\frac{1}{1+x}$ by manipulating the geometric series.

(b) Integrate your result from part (a) term-by-term to find the Taylor series for $\ln(1+x)$, valid for $-1 < x < 1$.

(c) Determine the radius of converge for the Taylor series of $\ln(1+x)$. Does the Taylor Series converge at the end-points?

(d) Using your series from part (b), find the exact value of the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

8. Using the Taylor series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

(a) Compute the derivatives of e^x, $\sin x$, and $\cos x$ by differentiating term by term. What do the resulting series equal?

(b) Let $i = \sqrt{-1}$ denote the imaginary unit. Use the Taylor series expansions of e^x, $\cos(x)$, and $\sin(x)$ to derive and justify the identity: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. *This result is known as* ***Euler's formula****, and it plays a fundamental role in the study of complex numbers, both in this course and in future mathematics courses.*

(c) Find the first four non-zero terms of the Taylor series for $f(x) = e^x \sin(x)$.

9. To show, for example, that $\sin(x)$ equals its Taylor series for all $x \in \mathbb{R}$, we must show that $\lim_{n\to\infty} R_n(x) = 0$, where R_n is the remainder. To accomplish this, prove the following:

(a) Without using a calculator, justify why $\frac{3^7}{7!} < 1$.

(b) Use your result from part (a) to prove that $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

(c) Using a similar argument, prove that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real number x.

10. In systems and controls, a pure time delay of T seconds has the Laplace-domain transfer function:

$$H(s) = e^{-sT}$$

However, this function is not rational and is often approximated using power series.

(a) Write the first four nonzero terms of the Taylor series of e^{-sT} centered at $s = 0$.

(b) What is the radius of convergence of this series in s? Does it affect practical implementation?

(c) What does this approximation tell you about how the system behaves for small s (i.e., low frequencies)?

(d) What are the implications of truncating this series after a few terms in a control system?

11. A capacitor leaks charge such that only a fraction r of the previous charge remains each second, with $0 < r < 1$. The initial charge is Q_0.

(a) Write an expression for the charge lost in the n-th second.

(b) Use a series to compute the total charge lost over all time.

(c) Show that the total charge lost is finite, even though leakage continues indefinitely.