UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Quiz 4

Date: Jun 4, 2025
Duration: 50 minutes

Course: MAT 187 (Calculus II)
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Instructions

- This is a Type A assessment and **does not** allow any external aids.
- Read all instructions carefully and **justify all your answers.** No points will be awarded for a correct answer without justification.
- Read each question carefully. **No clarification or content related questions will be answered.**

You may use the following space for scratch work or to continue your solutions if you run out of room. If you do so, please clearly indicate in the original question that part of your solution appears here.

1. (3 points) **True of False:** If a sequence $\{a_n\}_n$ is bounded, then it converges. Justify your answer or provide a counter-example.

Solution: False. A bounded sequence does not necessarily converge. A bounded **monotonic** sequence does.

Counter-example: Consider the sequence $a_n = (-1)^n$. This sequence is bounded, since all terms lie between -1 and 1.

However, it does not converge because it oscillates and does not approach a single limit:

$$a_1 = -1, \quad a_2 = 1, \quad a_3 = -1, \quad a_4 = 1, \quad \dots$$

2. (3 points) Suppose a power series **centered at 5** converges at $x = 2$. What can you say about its convergence at $x = 7$? Explain.

Solution: The power series is centered at $x = 5$, so its converges up to some distance from the center (the radius of convergence).

We are told that the series converges at $x = 2$, which is a distance of

$$|2 - 5| = 3$$

from the center. Therefore, the **radius of convergence is at least 3**.

Now consider the point $x = 7$, which is at a distance

$$|7 - 5| = 2$$

from the center.

Since $2 < 3$, and the series converges at a point farther from the center than 7, we can conclude that the series **must also converge at $x = 7$**.

3. (6 points) Starting from the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$, find a power series representation for the function:

$$f(x) = \frac{x}{(1-x)^2}.$$

Hint: Differentiate something!

Solution: We start with the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}, \quad \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1}.$$

So,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Now multiply both sides by x to get:

$$f(x) = \frac{x}{(1-x)^2} = x \cdot \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n.$$

Thus, the power series representation is:

$$\boxed{f(x) = \sum_{n=1}^{\infty} nx^n}, \quad |x| < 1.$$

4. (9 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}.$$

Find the radius of convergence and the interval of convergence of the series. Be sure to fully justify your answer, including testing any relevant endpoints.

Solution: We use the Ratio Test to find the radius of convergence. Let

$$a_n = \frac{(x-2)^n}{n \cdot 3^n}.$$

Then,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-2)^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n} \right| = \left| \frac{(x-2) \cdot n}{3(n+1)} \right|.$$

Taking the limit:

$$\lim_{n \to \infty} \left| \frac{(x-2) \cdot n}{3(n+1)} \right| = \left| \frac{x-2}{3} \right|.$$

The Ratio Test tells us the series converges when this limit is less than 1:

$$\left| \frac{x-2}{3} \right| < 1 \quad \Rightarrow \quad |x-2| < 3.$$

So, the **radius of convergence** is $R = 3$, and the series converges for $x \in (-1, 5)$.

Next, we test the endpoints.

At $x = -1$:

$$\sum_{n=1}^{\infty} \frac{(-1-2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is the alternating harmonic series. It converges by the alternating series test since $\frac{1}{n}$ is decreasing sequence that goes to zero.

At $x = 5$:

$$\sum_{n=1}^{\infty} \frac{(5-2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series. It diverges by the p-series test or the integral test.

Final Answer: Radius of convergence is $R = 3$. Interval of convergence is

$$\boxed{[-1, 5)}.$$

5. Consider a mechanical system in which a damped spring is compressed, released, and oscillates indefinitely. The *displacement amplitude* after each bounce is reduced by a constant factor r, where $0 < r < 1$, due to energy loss from damping. The system starts with amplitude A_0, and each cycle takes the same amount of time.

   Each oscillation releases energy proportional to the **square of the amplitude**. That is, the energy released in the first oscillation is $E_1 = kA_0^2$, the second is $E_2 = k(rA_0)^2$, and so on.

   (a) (6 points) Show that the *total* energy dissipated by the system over all time is finite, and find a formula for it in terms of k, A_0, and r.

   Solution: The total energy dissipated is the sum of the energy released during each oscillation:

   $$E_{\text{total}} = kA_0^2 + k(rA_0)^2 + k(r^2A_0)^2 + \cdots = kA_0^2\left(1 + r^2 + r^4 + r^6 + \cdots\right)$$

   This is a geometric series with first term 1 and ratio r^2, where $0 < r^2 < 1$. The sum of an infinite geometric series $\sum_{n=0}^{\infty}(r^2)^n$ is:

   $$\sum_{n=0}^{\infty} r^{2n} = \frac{1}{1-r^2}$$

   Thus,

   $$E_{\text{total}} = kA_0^2 \cdot \frac{1}{1-r^2}$$

   Since $0 < r < 1$, the denominator is positive and finite, so the total energy is finite.

   (b) (3 points) Explain what this means about how real-world damped systems behave. Can such systems dissipate an infinite amount of energy over time?

   Solution: This result shows that although the damped system continues to oscillate infinitely (in theory), the amount of energy lost in each cycle decreases rapidly. The total energy dissipated over all time converges to a finite value.

   In engineering terms, this means that a damped system cannot release infinite energy—it is bounded by the initial energy input. This aligns with physical intuition: once the initial energy is dissipated through damping, the system comes to rest or settles, and no further energy is lost.