Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 5

1. Consider the subset

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+b+3c \\ a+b+2c \\ a+2b+3c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

of $\mathbb{R}^4$.

(a) Show that W is a subspace of $\mathbb{R}^4$. **Hint**: The easiest way to do this is to show W is the span of some vectors in $\mathbb{R}^4$ (c.f. Textbook, Theorem 2, page 18). This will also help you answer part (b) below.

Solution:

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+b+3c \\ a+b+2c \\ a+2b+3c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \tag{1}$$

$$W = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \tag{2}$$

$$W = \left\{ a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} \mid a, b, c \in \mathbb{R} \right\} \tag{3}$$

where

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

Therefore, for any vector $x \in W$, we have $x = a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3}$, for some $a, b, c \in \mathbb{R}$, so that $W = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Using Theorem 2, page 18, it follows that W is a subspace of $\mathbb{R}^4$.

(b) Find a basis for W. **Hint**: Use Textbook, Theorem 3, page 20.

Solution: In part (a), we have proved that $W = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Notice that $\mathbf{v_3} = \mathbf{v_2} + \mathbf{v_1}$. Therefore, using Theorem 3, page 20, it follows that $W = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$. Also, notice that $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent, because of the following:

$$a\mathbf{v_1} + b\mathbf{v_2} = 0 \tag{4}$$

$$\iff \begin{bmatrix} a+b \\ 2a+b \\ a+b \\ a+2b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

which gives, $b = -a$ and $b = -2a$, so that $a = 0$ and $b = 0$ is the only solution for this system of equations. Therefore, $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. With this, and using the definition of basis set, it follows that $\mathbf{v_1}$ and $\mathbf{v_2}$ form a basis for W.

(c) Determine the dimension of W.

Solution: The dimension of W is 2 because the basis set of W consists of 2 vectors.

2. Let $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}$ be vectors in $\mathbb{R}^n$. Suppose that $\mathbf{x_3} = \mathbf{x_1} - \mathbf{x_2}$ and $\mathbf{x_4} = 2\mathbf{x_1} + 3\mathbf{x_2} - \mathbf{x_3}$. Suppose $W = \text{span}\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}\}$.

(a) What are the possible dimensions of W?

Solution: To determine the possible dimensions of W, we should find the possible number of linearly independent vectors in W. Since any vector in W is a linear combination of $\mathbf{x_1}$, $\mathbf{x_2}$, $\mathbf{x_3}$, and $\mathbf{x_4}$, then we should find which vectors among these vectors are linearly independent. Notice that $\mathbf{x_3} = \mathbf{x_1} - \mathbf{x_2}$ and $\mathbf{x_4} = 2\mathbf{x_1} + 3\mathbf{x_2} - \mathbf{x_3} = \mathbf{x_1} + 4\mathbf{x_2}$. Therefore, $\mathbf{x_3}$ and $\mathbf{x_4}$ are linearly dependent on $\mathbf{x_1}$ and $\mathbf{x_2}$. Therefore, $W = \text{span}\{\mathbf{x_1}, \mathbf{x_2}\}$. With this, the possible dimensions of W are 0 (if $\mathbf{x_1} = \mathbf{x_2} = 0$), 1 (if $\mathbf{x_1}$ and $\mathbf{x_1}$ are non-zeros but linearly dependent), or 2 (if $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent).

(b) Suppose $\dim(W) = 2$. Must $\{\mathbf{x_3}, \mathbf{x_4}\}$ be linearly independent?

Solution: If $\dim(W) = 2$, then $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent. Thus, $a\mathbf{x_1} + b\mathbf{x_2} = 0$ if and only if $a = b = 0$. Suppose that there exist $a', b' \in \mathbb{R}$ such that $a'\mathbf{x_3} + b'\mathbf{x_4} = 0$. In the following, we will show that $a' = b' = 0$, so that $\mathbf{x_3}$ and $\mathbf{x_4}$ must be linearly independent.

$$a'\mathbf{x_3} + b'\mathbf{x_4} = 0 \tag{6}$$
$$a'(\mathbf{x_1} - \mathbf{x_2}) + b'(\mathbf{x_1} + 4\mathbf{x_2}) = 0 \tag{7}$$
$$(a' + b')\mathbf{x_1} + (-a' + 4b')\mathbf{x_2} = 0 \tag{8}$$
$$\tag{9}$$

But $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent, so that $(a' + b') = 0$ and $(-a' + 4b') = 0$, which is equivalent to $a' = b' = 0$. Therefore, $\mathbf{x_3}$ and $\mathbf{x_4}$ must be linearly independent.

3. Let W be a subspace of $\mathbb{R}^n$ of dimension k. Show that

(a) Any set of k vectors that span W must also be linearly independent and so form a basis for W.

Solution: Let $W = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subspace of $\mathbb{R}^n$ of dimension k. Assume towards a contradiction that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ are not linearly independent. This means that one of these vectors can be written as a linear combination of the rest. Without loss of generality, assume that this vector is $\mathbf{v_k}$, i.e. $\mathbf{v_k} = \mathbf{a_1}\mathbf{v_1} + \cdots + \mathbf{a_k}\mathbf{v_k}$. Then, using Theorem 3, page 20, $W = \text{span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{k-1}}\}$. With this, $\dim(W) \leq k - 1 < k$, a contradiction that completes the proof. Therefore, $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ are k linearly independent vectors of W. It follows that they form a basis for W.

(b) Any linearly independent set of k vectors from W must also span W and so form a basis for W.

Hint: You may find Textbook, Lemma 6, page 98 helpful.

Solution: Let $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ be a basis for W, because $\dim(W) = k$. Notice that $\text{span}\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\} = \mathbf{W}$, due to definition of basis. Let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a set of k vectors in W that are linearly independent. Let $\mathbf{b} \in W$ be an arbitrary vector. Due to Lemma 6, page 98, $\{\mathbf{b}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ are linearly dependent vectors. So, there exist $c_0, c_1, \ldots, c_k$, not all zero, such that:

$$c_0\mathbf{b} + c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = 0 \tag{10}$$

If $c_0 = 0$, then $c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = 0$ where $c_1, c_2, \ldots, c_k$ are not all zero. But this contradicts the fact that $\{v_1, \ldots, v_k\}$ is linearly independent! Therefore, $c_0 \neq 0$. Thus, for any arbitrary $\mathbf{b} \in W$, we can write:

$$-\frac{c_1}{c_0}\mathbf{v_1} - \cdots - \frac{c_k}{c_0}\mathbf{v_k} = \mathbf{b} \tag{11}$$

so that $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ span W. It follows that $\{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ also forms a basis for W.

This result is a "half-is-good-enough" type theorem. It says that IF you know that the dimension of a subspace W is k any set of k linearly independent vectors from W is automatically a basis for W (i.e. you needn't check they span W); and any set of k vectors from W that span W is automatically a basis for W (i.e. you needn't check they are linearly independent). For example, since the dimension of $\mathbb{R}^3$ is 3, any set of 3 linearly independent vectors from $\mathbb{R}^3$ is a basis for $\mathbb{R}^3$.

4. Forming bases that contain a given set of vectors.

(a) Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find a third vector $\mathbf{x}_3$ such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $\mathbb{R}^3$.

Solution: Using the result from Problem 3, we only need to find 3 linearly independent vectors. Notice that $\mathbf{x_1}$ and $\mathbf{x_2}$ are linearly independent. Let $\mathbf{x_3} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. We can check that the solution to the system $a\mathbf{x_1} + b\mathbf{x_2} + c\mathbf{x_3} = 0$ has only the trivial solution, so $\{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}\}$ are linearly independent, and thus, form a basis for $\mathbb{R}^3$

(b) Given linearly independent vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$, is it always possible to find a third vector $\mathbf{x}_3$ such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $\mathbb{R}^3$? Explain.

Solution: Notice that $\dim(\mathbb{R}^3) = 3$, so that we need 3 linearly independent vectors to form a basis $\mathbb{R}^3$ by question 3. Now, we know $\{\mathbf{x_1}, \mathbf{x_2}\}$ do not span $\mathbb{R}^3$, i.e. $\text{span}\{\mathbf{x_1}, \mathbf{x_2}\} \neq \mathbb{R}^{\mathbf{3}}$ which means we can always find a vector $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \notin \text{span}\{\mathbf{x_1}, \mathbf{x_2}\}$. This would mean that the set $\{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ is linearly independent (why?) set of 3 vectors from $\mathbb{R}^3$ and so a basis for $\mathbb{R}^3$.

(c) Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find a third vector $\mathbf{x}_3$ such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent in $\mathbb{R}^4$ and then find a fourth vector $\mathbf{x}_4$ such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a basis for $\mathbb{R}^4$.

Solution: Let $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{x}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Construct a system of 4 equations and 4 unknowns using: $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$. Notice that the only solution for this system is the trivial solution, i.e. $c_1 = c_2 = c_3 = c_4 = 0$. Therefore, $\{x_1, x_2, x_3, x_4\}$ are linearly independent, and so, $\{x_1, x_2, x_3\}$ are also linearly independent.

(d) Given linearly independent vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^4$, is it always possible to find vectors $\mathbf{x}_3$ and $\mathbf{x}_4$ such that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is a basis for $\mathbb{R}^4$? Explain.

Solution: We mimic part (b) above. Notice that $\dim(\mathbb{R}^4) = 4$, so that we need 4 linearly independent vectors to form a basis for $\mathbb{R}^4$ by question 3. Now, we know $\{\mathbf{x_1}, \mathbf{x_2}\}$ does not span $\mathbb{R}^4$, i.e. $\text{span}\{\mathbf{x_1}, \mathbf{x_2}\} \neq \mathbb{R}^{\mathbf{4}}$ which means that we can always find a vector $\mathbf{x} \in \mathbb{R}^4$, $\mathbf{x} \notin \text{span}\{\mathbf{x_1}, \mathbf{x_2}\}$. Then, the set $\{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ is linearly independent (why?). We know that $\{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ does not span $\mathbb{R}^4$ either, i.e. $\text{span}\{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\} \neq \mathbb{R}^{\mathbf{4}}$, which means that we can always find a vector $\mathbf{y} \in \mathbb{R}^4$, $\mathbf{y} \notin \text{span}\{\mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$. Then again the set $\{\mathbf{y}, \mathbf{x}, \mathbf{x_1}, \mathbf{x_2}\}$ is linearly independent set of 4 vectors from $\mathbb{R}^4$ and so a basis for $\mathbb{R}^4$.

Notice that this procedure holds in general. Given k linearly independent vectors $\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ in $\mathbb{R}^n$ you can always find vectors $\mathbf{x_{k+1}}, \ldots, \mathbf{x_n}$ in $\mathbb{R}^n$ so that $\{\mathbf{x_1}, \ldots, \mathbf{x_k}, \mathbf{x_{k+1}}, \ldots \mathbf{x_n}\}$ is a basis for $\mathbb{R}^n$.