University of Toronto
Faculty of Applies Sciences and Engineering

MAT187 – Midterm II –Winter 2023

LAST (Family) NAME: ______________________

FIRST (Given) NAME: Solutions

Email address: ______________________@mail.utoronto.ca

STUDENT NUMBER: ______________________

Time: 90 mins.

1. **Keep this booklet closed**, until an invigilator announces that the test has begun. However, you may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **into the boxes** whenever provided. Ample space is provided within each box, however, if you must use additional space, please use the blank page at the end of this booklet, and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 16 pages, excluding the cover page, and 7 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work for part B and C. You don't need to justify your choices in part A.
7. You may use your one pre-written page of notes, written on our exam-aid sheet that you printed from the Midterm page on Quercus with whatever you want **handwriting** (not typed) on both sides. Any other note written on any other paper is considered academic dishonesty. You should write your name and student number on your pre-written note.
8. You may use a Casio FX-991 (EX recommended; any suffix is acceptable, including ES, PLUS, ES PLUSC, MS, MSPLUS2) or Sharp EL-W516 (any suffix is acceptable, including TBSL, XG, XGB-SL) calculator. Any other type of calculator on your desk is considered academic dishonesty. Please be ready to show your calculator to the invigilators if asked.

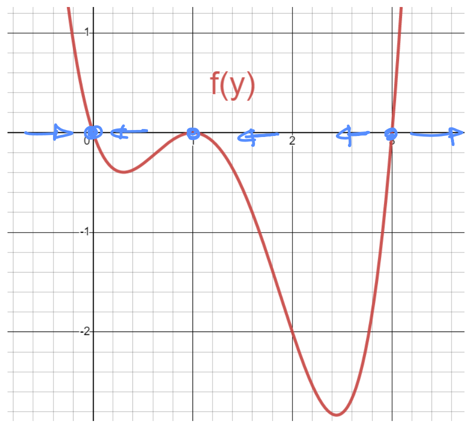
Question:	1	2	3	4	5	6	7	Total
Points:	10	9	12	6	4	10	9	60
Score:								

Part A

1. (10 points) Fill in the bubble for all statements that must be true. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles. You don't need to include your work or reasoning.

   (a) The graph of the function $f(y)$ is given below. How many constant solutions does the differential equation $y' = f(y)$ have?

Constant sol's are values of y @ which $y' = f(y) = 0$



○ 5 ○ 2 ○ 0 ○ 3 ○ Not enough information

   (b) Consider again the ODE $y' = f(y)$ with the same phase plot as given above in part (a). In each row of this table, make ONE choice.

$y = 0$	○ stable equilibrium	○ semi-stable equilibrium	○ unstable equilibrium	○ not an equilibrium
$y = 1$	○ stable equilibrium	○ semi-stable equilibrium	○ unstable equilibrium	○ not an equilibrium
$y = 3$	○ stable equilibrium	○ semi-stable equilibrium	○ unstable equilibrium	○ not an equilibrium

(c) Which one of the following gives the correct general solution to

$$y'' + y = \cos(3t) + 3\cos(t).$$

$r^2+1=0$

$r = \pm i \Rightarrow$ general sol'n has an oscillation factor with $\omega = 1$

↳ particular sol'n y_p contains $\cos(3t)$ or $\sin(3t)$

↳ since the frequency is shared with the complimentary sol'n, y_p contains $t\sin t$ or $t\cos t$.

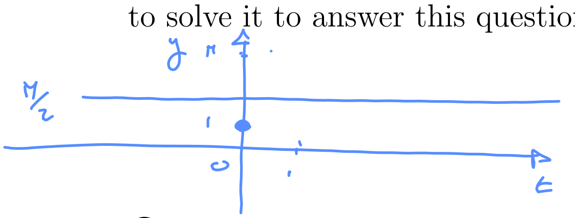
○ $y = C_1 e^t + C_2 e^{-t} - \frac{1}{8}\cos(3t) + \frac{3}{2} t \sin t$

○ $y = C_1 e^t + C_2 e^{-t} - \frac{1}{8}\cos(3t) + \frac{3}{2}\sin t$

● $y = C_1 \cos t + C_2 \sin t - \frac{1}{8}\cos(3t) + \frac{3}{2} t \sin t$

○ $y = C_1 \cos t + C_2 \sin t - \frac{1}{8}\cos(3t) + \frac{3}{2}\sin t$

(d) Consider the following ODE with the given constraints. Note that you don't need to solve it to answer this question. Choose ALL the correct statements.



$$\begin{cases} y' = \sin(y^2) \\ y(0) = 1 \\ y(1) = \pi \end{cases}$$

● The ODE is autonomous. $y' = f(y)$

○ There may be a solution.

● There is no solution. For such y to be a solution it needs to cross an equilibrium, which is not possible.

○ The ODE is linear.

(e) Let $Q(t)$ denote the amount of salt (measured in grams) in a large tank of salt water. The fluid pumped into the tank comes through two pipes. The first pipe is pure water, which is pumped in at 2 litres per minute. The second pipe is salt water with a concentration of 10 grams per litre, which is also pumped into the tank at a rate of 2 litres per minute. A third pipe removes well-mixed fluid from the tank at a rate of 1 litre per minute. Suppose the tank initially has 100 liters of pure water. Which of the following differential equations provides an appropriate model for the amount of salt in the tank at time t, while the tank is not full.

rate in − rate out

$Q'(t) = 10\,\frac{gr}{L} \times 2\,\frac{L}{m} - \frac{Q(t)}{100+3t}$

○ $\dfrac{dQ}{dt} = 5 - \dfrac{Q(t)}{100+t}$.

○ $\dfrac{dQ}{dt} = 20 - \dfrac{Q(t)}{100}$.

● $\dfrac{dQ}{dt} = 20 - \dfrac{Q(t)}{100+3t}$.

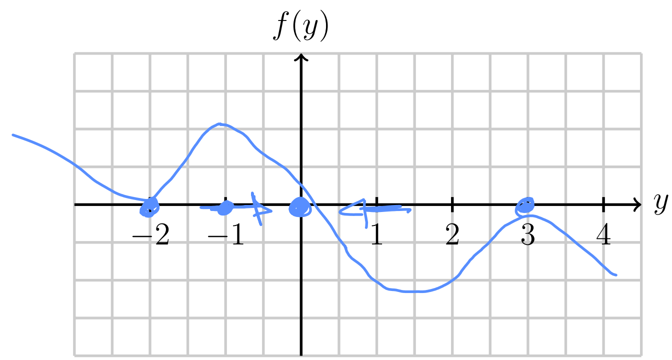
○ $\dfrac{dQ}{dt} = 40 - \dfrac{Q(t)}{100+3t}$.

Part B

2. Consider the autonomous ODE $y' = f(y)$.

(a) (4 points) Draw a phase plot for $y' = f(y)$ such that:

- The **only** equilibrium solutions to the ODE are at $y = -2$, $y = 0$, and $y = 3$.
- The equilibrium solutions at $y = 0$ is stable.
- The equilibrium solutions at $y = -2$ and $y = 3$ are **NOT unstable**.



(b) (3 points) Explain and justify how your phase plot fulfills these requirements.

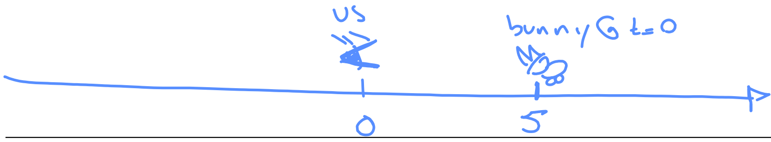
(1) Solutions to $y'=0$ are the equillibrium solutions to $y'=f(y)$
these are exactly where the f(y) value is zero in

(2) $y'>0$ for $y<0$ & $y'<0$ for $y>0$
So any solution near $y=0$ is atracted to it.

(3) For both $y=-2$ & $y=3$, y' has the same sign on both sides of the equilibrium, So they are neither stable nor unstable.

(c) (2 points) Consider the ODE corresponding to your phase plot in (a). Suppose $y(t)$ is a solution that satisfies $y(0) = -1$.

Then, $\lim_{t\to\infty} y(t)$ is equal to $\boxed{0}$. (You don't need to justify your answer.)



3. Our bunny Snuffles is moving along the x-axis, while you are standing still at $x = 0$ and facing the positive x-axis. Snuffles' position on the x-axis after t seconds is given by a function $f(t)$. Suppose $|f^{(3)}(t)| \le 3$, and suppose that $P_2(t) = -2t^2 - t + 5$ is the 2nd Taylor polynomial of $f(t)$, centred at $t = 0$.

(a) (1 point) What is the exact position of Snuffles at $t = 0$? $\boxed{5}$

(b) (1 point) Is Snuffles running towards you or away from you at $t = 0$? $f'(0) = p'(0) < 0$

● towards me ○ away from me

(c) (1 point) What is Snuffles' exact speed at $t = 0$? $\boxed{1 \ m/s}$

$|f'(0)| = |p'(0)|$

$f''(0) = p''(0) < 0$

p'' & p' have the same sign.

(d) (1 point) Is Snuffles speeding up or slowing down at $t = 0$?

● speeding up ○ slowing down

(e) (1 point) Is Snuffles' position at $t = 2$ equal to -5? ○ yes ● we don't know

(f) (1 point) Justify your choice in part (e).

$P_2(2) = -2(4) - 2 + 5 = -10 + 5 = -5$ approximation $f(2)$

(g) (3 points) Use Taylor's Theorem with Remainder to find the best upper bound you can, given the information in this question, for the value of $|R_2(t)| = |f(t) - P_2(t)|$. Your bound will be in terms of t.

$|R_2(t)| = \frac{|f^{(3)}(c)(t)^3|}{3!} \le \frac{|3t^3|}{3!} = \frac{3t^3}{6} = \frac{t^3}{2}$ $t > 0$ time

(h) (2 points) What are the **best** lower bound a and upper bound b that you can find for the position of Snuffles at $t = 2$. That is $a \le f(2) \le b$. Justify your answer.

$|f(2) - (-5)| < \frac{3 \times 8}{6} = \frac{8}{2} = 4$

$-4 \le f(2) + 5 \le 4$

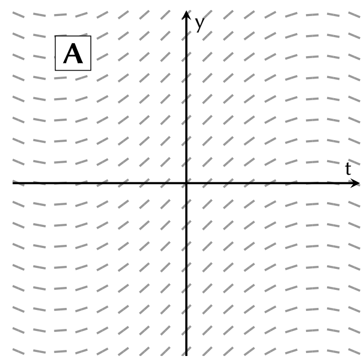
$-9 \le f(2) \le -1$

$\boxed{a = -9}$ $\boxed{b = -1}$

(i) (1 point) Is Snuffles ahead of you or behind you at time $t = 2$?

○ ahead of me ● behind me ○ not possible to tell

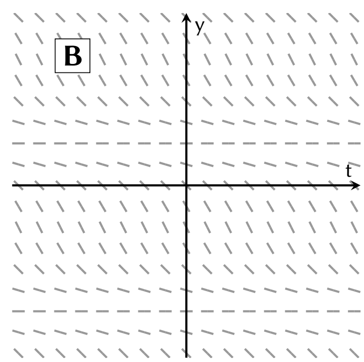
4. For each of the ODEs below, choose the matching direction field. Then explain your match in words.**Only justified choices will receive points. Just filling the box is not worth marks.**

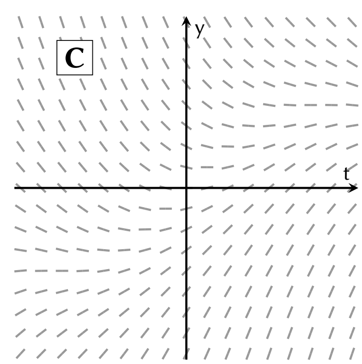


(a) (3 points) $y' = \sin(y) - 1$ matches direction field

$\boxed{B}$

Reasoning: This is an autonomous ODE so the slopes in the direction field should not depend on the value of t there is only one such direction field present.





(b) (3 points) $y' = y - \sin(t)$ matches direction field

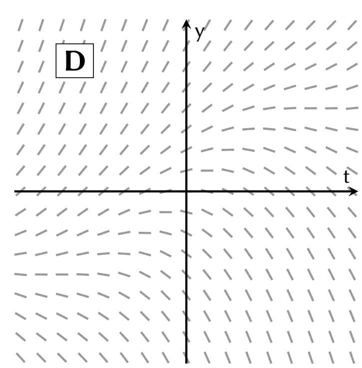
$\boxed{D}$

Reasoning: @ $(0,0)$ $y' = 0$ so C & D are possible.

$-1 \leq \sin t \leq 1$

$\lim_{y \to \infty} y - \sin t = \infty$ & $\lim_{y \to -\infty} y - \sin t = -\infty$

only D matches this behaviour.



5. Consider a spring attached at one end to the ceiling and to a block of mass 0.25 kg at the other end with gravity pulling down on it. Assume the spring's stiffness is given by $k = 9\ N/m$ and $g = 9.8 m/s^2$. In each part, put your final answer in the provided box.

(a) (2 points) If the system is undamped, and there are no other forces, write an ODE that governs the motion of the block. Clearly indicate what your variables measure. You don't need to solve it.

$y(t)$: distace from equilibrium of the spring with the mass attached

$my'' + cy' + ky = 0$

$0.25y'' + 9y = 0$

OR $y(t)$ = distance from equilibrium of the spring without mass attached

$my'' + ky = mg$

$0.25y'' + 9y = (0.25)(9.8)$

(b) (1 point) Now suppose there is a damping force with a damping (positive) constant c. Write an ODE that governs the motion of the block.

$0.25y'' + cy' + 9y = 0$ or $0.25y'' + cy' + 9y = (0.25)(9.8)$

(c) (1 point) For which values of c do we expect the oscillations to die out slowly over time? In other words, which values of c result in an underdamped oscillator?

$\Delta = c^2 - (4)(0.25)(9) < 0$

6. (a) (4 points) Find the solution to the following initial value problem.

$$\begin{cases} ty' - y = t^2 e^t \\ y(1) = 0 \end{cases}$$

$y' - \frac{y}{t} = \frac{t^2 e^t}{t}$

$y' + (-\frac{1}{t})y = te^t$ provided that $t \neq 0$

$\mu(t) = e^{\int -\frac{1}{t} dt} = e^{-\ln|t|} = \frac{1}{|t|} = \pm\frac{1}{t}$

since $t \neq 0$ there are two possi, ities for the interval over which this ODE has a sol'n. $(-\infty, 0)$ or $(0, +\infty)$ $y(1) = 0$ implies that we are interested in the $(0, +\infty)$ interval $\Rightarrow t > 0$

so $\mu(t) = \frac{1}{t}$

$\frac{1}{t}(y' + (\frac{-1}{t})y) = \frac{\cancel{t}}{\cancel{t}} e^t$

$\frac{d}{dt}\left(\frac{1}{t}y\right) = e^t$

$\int \frac{d}{dt}(\frac{1}{t}y)\, dt = \int \frac{e^t}{t} dt$

$\frac{1}{t}y = e^t + C$

$y = te^t + Ct$

$y(1) = 0 \Rightarrow 0 = e + C \Rightarrow C = -e$

$\boxed{y = te^t - et}$

(b) (2 points) For what **interval** of time is this solution valid? Justify your answer.

since $t \neq 0$ there are two possibilities for the interval over which this ODE has a sol'n. $(-\infty, 0)$ or $(0, +\infty)$ $y(1) = 0$ implies that we are interested in the $(0, +\infty)$ interval —

$(0, +\infty)$

Note for part (c): There is a typo in this question. The correct initial condition is $y(0)=\frac{\pi}{2}$. The given initial condition implies $\sin(y(0))=\sin\pi=0$. Hence the ODE is not defined for any function that satisfies the initial condition. We accept solutions that realize this and those who don't
1 2

(c) (4 points) Find the solution to the following initial value problem. State any assumption that you make.

$$\begin{cases} y' + \frac{t^2}{\sin(y)} = 0 \\ y(0) = \pi \end{cases}$$

1

$y' = -\frac{t^2}{\sin y}$

$(\sin y)\,y' = -t^2$

$\int \sin y\,dy = -\int t^2\,dt$

$-\cos y = -\frac{t^3}{3} + C$

$y(0)=\pi \quad -\cos\pi = C \Rightarrow C=1$

$\cos y = \frac{t^3}{3} + 1$

provided $0 \le y \le \pi$, we can take inverse "arccos" from both side to get

$y = \arccos\left(\frac{t^3}{3}+1\right)$

$-1 \le \frac{t^3}{3}+1 \le 1$

$-2 \le \frac{t^3}{3} \le 0$

$-6 \le t^3 \le 0$

$t \in [-\sqrt[3]{6}, 0]$

expression above makes sense provided that $t \in [-\sqrt[3]{6}, 0]$

2 For the ODE to make sense $\sin(y) \neq 0 \Rightarrow y \neq k\pi \quad k \in \mathbb{Z}$
The initial condition requires that $y(0)=\pi$. The ODE is not valid at $t=0$ for this function. No solution

Part C

7. In this problem you will work with a first-order ordinary differential equation that is initially neither separable or linear. The parts of this question are connected, however, each part can be done independently from the others. Consider the ODE:

$$yy' + x = \sqrt{x^2 + y^2}$$

(a) (2 points) Given the initial value $y(0) = a > 0$, without solving the ODE, show that $y'(0) = 1$.

at $t = 0$ $\quad y(0)y'(0) = \sqrt{y(0)^2} = y(0)$

$$y'(0) = \frac{y(0)}{y(0)} = 1$$

(b) (2 points) Let $(y(x))^2 = v(x) - x^2$. Rewrite the ODE in terms of $v(x)$. After this substitution, your new ODE should not contain y.

$y^2 = v - x^2$

$2yy' = v' - 2x$

$yy' = \dfrac{v' - 2x}{2}$

$$\frac{v' - 2x}{2} + x = \sqrt{x^2 + v - x^2}$$

$$\frac{v'}{2} \cancel{-x} \cancel{+x} = \sqrt{v}$$

$$v' = 2\sqrt{v}$$

(c) (2 points) Show that the implicit equation $(y(x))^2 = 2cx + c^2$, where c is the constant of integration, solves the original ODE $yy' + x = \sqrt{x^2 + y^2}$. State any condition that is required for your solution to work.

$y^2 = 2cx + c^2$

$\cancel{2}yy' = \cancel{2}c$

$c + x = \sqrt{x^2 + 2cx + c^2}$

$c + x = \sqrt{(x+c)^2}$

$c + x \overset{\checkmark}{=} |c + x|$

provided that

$c + x \geq 0$

(d) (3 points) Consider

$$\begin{cases} yy' + x = \sqrt{x^2 + y^2} \\ y(0) = 2. \end{cases}.$$

Show that the choice $c = -2$ in the implicit solution $(y(x))^2 = 2cx + c^2$ does NOT solve the IVP. Justify your answer. Hint: use part (a) or part (c).

$y(0) = 2$

$4 = 2c(0) + c^2$

$c = \pm 2$

if $c = -2$

$y^2 = -4x + 4$

$2yy' = -4$

$y' = \frac{-4}{2y}$

@ $x = 0$

$y'(0) = \frac{-4}{2y(0)}$

$y'(0) = \frac{-4}{4} = -1$

this contradicts the observation in (a)

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Trigonometric Functions

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$
$$\int \sec^3(x)\,dx = \frac{1}{2}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$$
$$\int \csc(x)\,dx = -\ln|\cot(x)+\csc(x)| + C$$
$$\int \tan(x)\,dx = -\ln|\cos x| + C$$
$$\int \cot(x)\,dx = \ln|\sin x| + C$$

Error formula for L_n and R_n

$|\text{Error}| \le \dfrac{M}{2n}(b-a)^2$

Where $M \ge |f'(x)|$ on $a \le x \le b$

Error formula for M_n

$|\text{Error}| \le \dfrac{M}{24n^2}(b-a)^3$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Error formula for T_n

$|\text{Error}| \le \dfrac{M}{12n^2}(b-a)^3$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Taylor Polynomial $\quad p_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Remainder/Error $\quad f(x) = P_n(x) + R_n(x) \quad$ with $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
for some c between a and x.

If $|f^{(n+1)}(x)| \le M$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$