UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Quiz 1

Date: May 14, 2025 | **Course:** MAT 187 (Calculus II)
Duration: 50 minutes | **Examiner:** Armanpreet Pannu

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Instructions

- This is a Type A assessment and **does not** allow any external aids.
- Read all instructions carefully and **justify all your answers.** No points will be awarded for a correct answer without justification.
- Read each question carefully. **No clarification or content related questions will be answered.**

You may use the following space for scratch work or to continue your solutions if you run out of room. If you do so, please clearly indicate in the original question that part of your solution appears here.

1. (3 points) A student attempts to evaluate the indefinite integral $\int x \sin(x)\, dx$ using integration by parts. They choose $f(x) = \sin(x)$ and $g'(x) = x$. **Without solving the integral**, explain why this is likely not a good choice for integration by parts.

> **Solution:** With this choice of $f(x) = \sin(x)$ and $g'(x) = x$, we are differentiating $\sin(x)$, which gives $\cos(x)$, and integrating x, which gives $\frac{1}{2}x^2$. This results in more complicated expressions during the integration by parts process.
>
> A better choice is to let $f(x) = x$ and $g'(x) = \sin(x)$, because differentiating x simplifies to 1, and integrating $\sin(x)$ yields $-\cos(x)$, which simplifies the integrand. This leads to a simpler overall computation.

2. (3 points) In class, we evaluated the integral $\int \frac{1}{1+x^2} dx = \arctan(x)$ using trigonometric substitution. Explain why partial fraction decomposition is not a suitable method for evaluating this integral.

> **Solution:** The denominator $1 + x^2$ is an irreducible quadratic polynomial as it cannot be factored into linear terms using real numbers. Partial fraction decomposition only works when the denominator can be factored into simpler rational terms. While expressions like $\frac{1}{x^2+1}$ can appear as part of a partial fraction decomposition, they are considered terminal forms (can't be simplified further) and require a different approach—such as trigonometric substitution—for integration. Therefore, partial fraction decomposition is not suitable here.

3. (6 points) Use integration by parts to evaluate the following integral:

$$\int \arctan(x) dx$$

> **Solution:** Let $g'(x) = 1$ and $f(x) = \arctan(x)$, then:
>
> $$\begin{aligned} \int \arctan(x) dx &= x \arctan(x) - \int x \frac{1}{1+x^2} dx \qquad\qquad u = 1 + x^2 \\ &= x \arctan(x) - \frac{1}{2} \int \frac{1}{u} du \\ &= x \arctan(x) - \frac{1}{2} \ln|u| \\ &= x \arctan(x) - \frac{1}{2} \ln(1 + x^2) + C \end{aligned}$$

4. (8 points) Evaluate:

$$\int \frac{x^4 - x^3 - 1}{x^2 - x} dx$$

Solution: First we start with polynomial long division to get:

$$\frac{x^4 - x^3 - 1}{x^2 - x} = x^2 - \frac{1}{x(x-1)}$$

We decompose the remainder term as:

$$\begin{aligned} \frac{1}{x(x-1)} &= \frac{A}{x} + \frac{B}{x-1} \\ &= \frac{Ax - A + Bx}{x(x-1)} \end{aligned}$$

Which gives us the system of equations $A = -1$ and $A + B = 0$ so $B = 1$. Overall we get:

$$\begin{aligned} \frac{x^4 - x^3 - 1}{x^2 - x} &= x^2 + \frac{1}{x} - \frac{1}{x-1} \\ \int \frac{x^4 - x^3 - 1}{x^2 - x} dx &= \int x^2 dx + \int \frac{1}{x} dx - \int \frac{1}{x-1} dx \\ &= \frac{1}{3}x^3 + \ln|x| - ln|x-1| + C \end{aligned}$$

5. (10 points) Consider a cable suspended between two poles located at $x = -a$ and $x = a$. The cable forms a shape known as a *catenary*; however, for small sags and engineering approximations, this curve can be modeled by a simple parabola:

$$y(x) = \frac{h}{a^2}x^2$$

where a is half the horizontal distance between the poles, and h is the vertical sag.
Arc Length Formula: The arclength L of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Using the formula above and the given parabola, set up an integral that computes the total length of the cable from $x = -a$ to $x = a$. Then, evaluate the integral and simplify the result as much as possible.

Hint: $\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$

Solution: We are given the cable is modeled by the parabola: $y(x) = \frac{h}{a^2}x^2$ and therefore $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{h}{a^2}x^2\right) = 2\frac{h}{a^2}x$. Using the arc length formula:

$$\begin{aligned}
L &= \int_{-a}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_{-a}^{a} \sqrt{1 + 4\frac{h^2}{a^4}x^2}\,dx \\
&= 2\int_0^a \sqrt{1 + 4\frac{h^2}{a^4}x^2}\,dx \\
&= 2\int_0^a \sqrt{1 + k^2x^2}\,dx \qquad \text{Letting } k = 2\frac{h}{a^2}
\end{aligned}$$

Let $x = \frac{1}{k}\tan\theta$, so $dx = \frac{1}{k}\sec^2\theta\,d\theta$. Then:

$$\sqrt{1 + k^2x^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$$

Change of limits:

$$x = 0 \Rightarrow \theta = 0, \quad x = a \Rightarrow \theta = \tan^{-1}(ka)$$

Substituting:

$$L = 2\int_0^{\tan^{-1}(ka)} \sec\theta \cdot \frac{1}{k}\sec^2\theta\,d\theta = \frac{2}{k}\int_0^{\tan^{-1}(ka)} \sec^3\theta\,d\theta$$

Apply the provided formula:

$$\begin{aligned}
\int_0^{\tan^{-1}(ka)} \sec^3\theta\,d\theta &= \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}(ka)} \\
&= \frac{1}{2}\left(\sqrt{1 + k^2a^2}\cdot ka + \ln\left(\sqrt{1 + k^2a^2} + ka\right)\right)
\end{aligned}$$

Therefore:

$$L = \frac{2}{k}\cdot\frac{1}{2}\left(ka\sqrt{1 + k^2a^2} + \ln\left(\sqrt{1 + k^2a^2} + ka\right)\right)$$

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Recall $k = \frac{2h}{a^2} \Rightarrow \frac{1}{k} = \frac{a^2}{2h}$, so:

$$L = \frac{a^2}{2h}\left(\frac{2h}{a}\sqrt{1 + \frac{4h^2}{a^2}} + \ln\left(\sqrt{1 + \frac{4h^2}{a^2}} + \frac{2h}{a}\right)\right)$$

Simplify:

$$\boxed{L = a\sqrt{1 + \frac{4h^2}{a^2}} + \frac{a^2}{2h}\ln\left(\sqrt{1 + \frac{4h^2}{a^2}} + \frac{2h}{a}\right)}$$