Faculty of Applied Science & Engineering, University of Toronto

MAT188H1F - Linear Algebra

Fall 2016

Tutorial Problems 8

1. Suppose A is a 4×6 matrix that has reduced row-echelon form

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which of the following statements about A are guaranteed to be true, where A_i denotes the ith column of A.

You may want to consult the paragraph "Basis of the Column Space of a Matrix" on page 158 of the textbook before attempting this question.

(a) The columns of A are linearly independent.

(b) The system of equations $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^4$.

(c) The system of equations $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^4$.

(d) A_4 is a linear combination of A_1, A_2, and A_5.

(e) A_3 is a linear combination of A_1, A_2, and A_5.

(f) A_3 is a linear combination of A_1 and A_2.

(g) $A_4 = A_1 - 2A_2$.

Solution:

(a) *False, the columns of A are in $\mathbb{R}^4$ and there are 6 of them so they cannot be linearly independent.*

(b) *False, from R we see that A has rank 3 and so* $\dim \operatorname{col}(A) = 3$. *But $A\mathbf{x} = \mathbf{b}$ has a solution if and only if* $\mathbf{b} \in \operatorname{col}(A)$.

(c) *False, because* $\dim \operatorname{null}(A) = 3$ *so there are infinitely many solutions to* $A\mathbf{x} = \mathbf{0}$.

(d) *True, from R we see that A_1, A_2 and A_5 form a basis for* $\operatorname{col}(A)$.

(e) *True, from R we see that A_1, A_2 and A_5 form a basis for* $\operatorname{col}(A)$.

(f) *True, otherwise R would have a leading entry in its third column.*

(g) *False. We could have $R = A$ in which case this obviously fails.*

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2. Each row of the table below summarizes information about some matrix A. Fill in all the missing information from the table. Give reasons for your choices.

Size	$A\mathbf{x}=\mathbf{b}$ consistent $\forall\ \mathbf{b}$	$A\mathbf{x}=\mathbf{b}$ has at most one solution	$\dim(\text{col}(A))$	$\text{nullity}(A))$	$\text{rank}(A)$
3×4	yes				
5×5				1	
5×5			5		
3×2		yes			
4×4	yes				
5×4					3

Which of the rows, if any, describe an invertible matrix?

Solution:

Size	$A\mathbf{x}=\mathbf{b}$ consistent $\forall\ \mathbf{b}$	$A\mathbf{x}=\mathbf{b}$ has at most one solution	$\dim(\text{col}(A))$	$\text{nullity}(A)$	$\text{rank}(A)$
3×4	yes	no	3	1	3
5×5	no	no	4	1	4
5×5	yes	yes	5	0	5
3×2	no	yes	2	0	2
4×4	yes	yes	4	0	4
5×4	no	no	3	1	3

Let's analyze each row separately. But before we do that, let's make a few preliminary observations.

First, $\text{rank}(A) = \dim(\text{col}(A))$. **So the 4th and 5th columns will always have the same values.** Next, we shall be making constant use of the rank-nullity theorem (Theorem 8, page 161 of the textbook), which states that if A is an $m\times n$ matrix, then

$$n = \text{rank}(A) + \text{nullity}(A).$$

This allows us to determine the rank of A if we have its nullity and vice versa.
Consider $A\mathbf{x}=\mathbf{b}$ (here we retain that A is $m\times n$ so $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{b}\in\mathbb{R}^m$). Theorem 3, page 155 of he textbook tells us that this system will be consistent if and only if $\mathbf{b}$ is in the column space of A. Hence the system will be consistent for *every* $\mathbf{b}\in\mathbb{R}^m$ if and only if $\text{col}(A)=\mathbb{R}^m$, or equivalently, if and only if $\text{rank}(A)=m$. The system has at most one solution if and only if the columns of A are linearly independent.

The columns of A are linearly independent if and only if they form a basis for the column space of A. Because A has n columns, this last statement is equivalent to having $\text{rank}(A) = n$ (or, by the rank-nullity theorem, to having $\text{nullity}(A) = n - n = 0$). To summarize: **We can put "yes" under "$A\mathbf{x} = \mathbf{b}$ consistent $\forall$ $\mathbf{b}$" if and only if** $\text{rank}(A) = m$**. We can put "yes" under "$A\mathbf{x} = \mathbf{b}$ has at most one solution" if and only if** $\text{rank}(A) = n$**.**

- **Row 1:** If $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^{\mathbf{3}}$, then $\text{rank}(A) = 3$ and so $\dim(\text{col}(A)) = 3$ too. The rank-nullity theorem tells us that $\text{nullity}(A) = 4 - 3 = 1$. In particular, $\text{nullity}(A) \neq 0$, so A doesn't give unique solutions.
- **Row 2:** The rank-nullity theorem tells us that $\text{rank}(A) = 5 - 1 = 4$, so $\dim(\text{col}(A)) = 4$ too. As $\text{rank}(A) \neq 5$, we conclude that the system $A\mathbf{x} = \mathbf{b}$ cannot be consistent for every $\mathbf{b}$. For the same reason, the system will not have a unique solution when it is consistent.
- **Row 3:** We immediately get that $\text{rank}(A) = 5$ and so $\text{nullity}(A) = 5 - 5 = 0$ by the rank-nullity theorem. Next, because $\text{rank}A = 5$ and A is 5×5 the system $A\mathbf{x} = \mathbf{b}$ must be consistent and admit a unique solution for all $\mathbf{b} \in \mathbb{R}^5$.
- **Row 4:** We must have that $\text{nullity}(A) = 0$ and so $\text{rank}(A) = 2 - 0 = 2$. Then $\dim(\text{col}(A)) = 2$, too. Since $\text{rank}(A) \neq 3$, the system $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b} \in \mathbb{R}^{\mathbf{3}}$. On the other hand, because $\text{rank}(A) = 2$, the system will have a unique solution whenever it is consistent.
- **Row 5:** We conclude immediately that $\text{rank}(A) = 4$ and so $\dim(\text{col}(A)) = 4$ and so we get unique solutions in this case. And of course $\text{nullity}(A) = 4 - 4 = 0$.
- **Row 6:** The rank of A is not equal to either the number of rows or columns, so we don't get consistency or uniqueness for all systems here. And $\text{nullity}(A) = 4 - 3 = 1$ by the rank-nullity theorem.

Finally, let's determine which of these rows describe invertible matrices. First off, for a matrix to be invertible, it has to be square. So we need only consider the matrices described by rows 2, 3 and 5. Next, recall that an $n \times n$ is invertible if and only if its rank is n (or equivalently, if and only if its nullity is 0). From our table above, we see that rows 3 and 5 satisfy this condition, but row 2 does not. Thus only rows 3 and 5 describe invertible matrices.

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3. Let

$$A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 2 & 6 & 4 \\ 10 & 2 & 14 & 20 \\ 2\sqrt{2} & -\sqrt{2} & 0 & 4\sqrt{2} \\ \pi & e & \pi + 2e & 2\pi \\ \sqrt{2} & \sqrt{3} & \sqrt{2} + 2\sqrt{3} & 2\sqrt{2} \\ \ln 5 & 6 & \ln 5 + 12 & 2\ln 5 \\ -7 & 4 & 1 & -14 \\ 17 & -24 & -31 & 34 \\ 2 & 2 & 6 & 4 \end{bmatrix}$$

Without doing any row-reductions, answer the following questions.

(a) What is $\text{rank}(A)$? Explain.

(b) What is $\text{nullity}(A)$)?

(c) Is $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$? Explain.

(d) Find two different bases for row(A). Justify your answers.

(e) True or False: null(A) is equal to the set of solutions to the system

$$\begin{aligned} x_1 + x_2 + 3x_3 + 2x_4 &= 0 \\ 17x_1 - 24x_2 - 31x_3 + 34x_4 &= 0 \end{aligned}$$

Solution:

(a) Let A_1, A_2, A_3 and A_4 denote the columns of 1st, 2nd, 3rd and 4th columns of A, respectively. Then by inspection we see that $A_4 = 2A_1$ and $A_3 = A_1 + 2A_2$. Moreover, A_1 and A_2 are not multiples of each other. So we see that, among the columns of A, the first two are independent and span the other two. Thus $\{A_1, A_2\}$ is a basis for the column space of A, and therefore $\operatorname{rank}(A) = \dim \operatorname{col}(A) = 2$.

(b) Since A is 10×4, the rank-nullity theorem tells us that so $\operatorname{nullity}(A) = 4 - \operatorname{rank}(A) = 2$.

(c) No, because $\operatorname{rank}(A) \neq 10$.

(d) We know that $\dim \operatorname{row}(A) = \operatorname{rank}(A) = 2$, so any two linearly independent rows of A will form a basis. For instance, we may take the first and third rows, or the first and fourth, or ...

(e) **True**. The first and ninth rows of A are linearly independent (because they are not multiples of each other), and hence form a basis for row(A). This means, in particular, that all the other rows of A are linear combinations of these two. Thus we can perform row operations to A to transform it into

$$R = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 17 & -24 & -31 & 34 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then null(A) is equal to null(R) (because row operations do not alter the null-space). But null(R) is, by definition, equal to

$$\text{the set of all } \mathbf{x} \in \mathbb{R}^4 \text{ such that } R\mathbf{x} = \mathbf{0},$$

or equivalently,

$$\text{the set of all } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ such that } \begin{bmatrix} x_1 + x_2 + 3x_3 + 2x_4 \\ 17x_1 - 24x_2 - 31x_3 + 34x_4 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is precisely the set of solutions to the system

$$\begin{aligned} x_1 + x_2 + 3x_3 + 2x_4 &= 0 \\ -7x_1 + 4x_2 + x_3 - 14x_4 &= 0. \end{aligned}$$

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4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\2\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}$, and $T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}c+1\\2\\4\end{bmatrix}$.

(a) Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3$.

(b) Determine all values of c such that T invertible.

Solution:

(a) The columns of A are, in order, $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$, and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$. So, we need only find $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$.

We have

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}c+1\\2\\4\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\0\\1\end{bmatrix}.$$

Thus

$$A = \begin{bmatrix}1 & 0 & c\\1 & 1 & 0\\2 & 1 & 1\end{bmatrix}.$$

(b) The transformation T is invertible if and only if A is invertible and A is invertible iff $\operatorname{rank}(A) = 3$. You may verify

$$\begin{bmatrix}1 & 0 & c\\1 & 1 & 0\\2 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & c\\0 & 1 & -c\\0 & 0 & 1-c\end{bmatrix}.$$

so that A has rank 3 iff $c \neq 1$. So T is invertible as long as $c \neq 1$.
