MAT186 Calculus I
Term Test 2
Solutions

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Instructions:

1. This test contains a total of 12 pages.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this test, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. $\lim_{x \to 1} \dfrac{\ln\left(\frac{1}{x}\right)}{\sin(\pi x)} =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ 0
- ○ DNE ($\to \infty$)
- ○ π
- ● $\dfrac{1}{\pi}$
- ○ $-\dfrac{1}{\pi}$

2. Consider following the ordinary differential equation (ODE) for an unknown function $y(t)$:

$$\frac{(y'(t))^2}{4} + (y(t))^2 = 1.$$

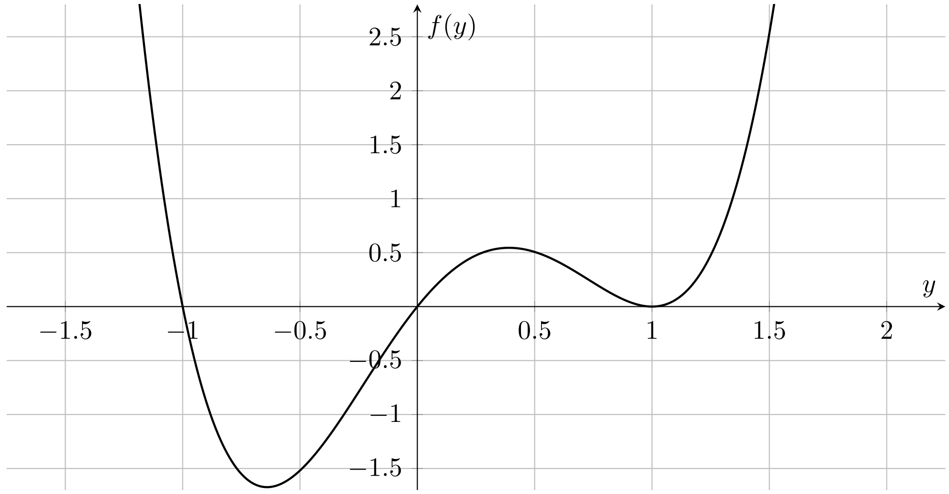
Which of the expressions below are solutions to this ODE? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ $y(t) = \sin(t)$
- ● $y(t) = \sin(2t)$
- ● $y(t) = \sin(2t + 2)$
- ○ $y(t) = \cos(t)$
- ● $y(t) = \cos(2t - 1)$

Multiple Choice: No justification is required. Only your final answer will be graded.

3. Consider the autonomous ordinary differential equation $\dfrac{dy}{dt} = f(y)$ where the graph of $f(y)$ vs. y is pictured below.



Which of the following statements are true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

○ If $y(0) = 0$ then the solution $y(t)$ is always strictly increasing.

● The equilibrium solution $y(t) = 1$ is semi-stable.

○ If $y(0) = 0.1$ then the solution $y(t)$ changes from concave down to concave up at some point $t > 0$.

● If $y(0) = 0.5$ then $\lim\limits_{t\to-\infty} y(t) = 0$.

● If $y(0) = 1.5$ then $\lim\limits_{t\to\infty} y(t) \to \infty$.

4. For the initial value problem

$$\begin{cases} y'(t) = (y-5)^2, \\ y(0) = 3, \end{cases}$$

which of the following step sizes Δt ensure that the approximation of $y(\Delta t)$ after **one step** of Euler's method does not cross the equilibrium solution? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

● $\Delta t = 0.2$

● $\Delta t = 0.4$

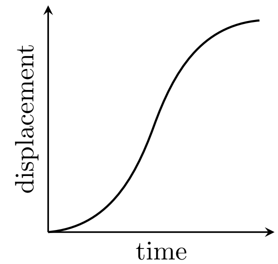
○ $\Delta t = 0.8$

○ $\Delta t = 1$

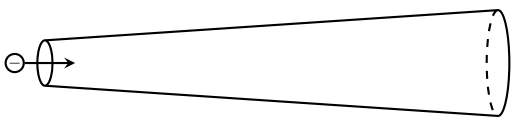
○ $\Delta t = 2$

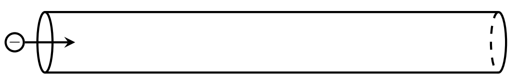
Multiple Choice: No justification is required. Only your final answer will be graded.

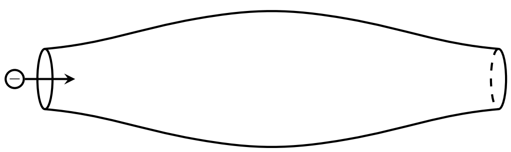
5. The (drift) velocity at which an electron travels in a copper wire is known to increase as the cross-sectional area of the wire decreases. The graph on the right shows the displacement of an electron (in millimetres) after it enters the leftmost part of a copper wire over time (in seconds). Which of the following wires pictured below is used? [1 mark]

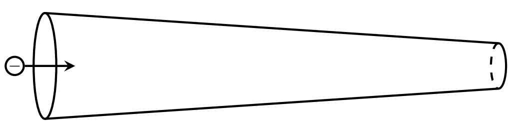


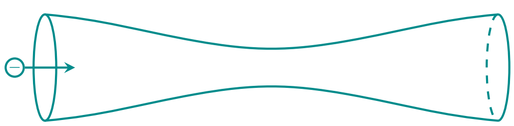
Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ 

○ 

○ 

○ 

● 

6. Kiki goes out for a morning walk. The table below summarizes Kiki's walking speed over a six second interval.

time (seconds)	0	1	2	3	4	5	6
speed (metres per second)	0.5	0.9	1.3	1.9	1.8	1.7	1.4

Which of the following represents a left-endpoint Riemann sum approximation of Kiki's distance travelled over the first six seconds. [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

○ 0.5+0.9+1.3+1.9+1.8+1.7+1.4

○ 0.9+1.3+1.9+1.8+1.7+1.4

● 2(0.5+1.3+1.8)

○ 2(1.3+1.8+1.4)

● 3(0.5+1.9)

Multiple Choice: No justification is required. Only your final answer will be graded.

7. Which of the following definite integrals is always underestimated with a left-endpoint Riemann sum? That is, a left-endpoint Riemann sum will always give a strictly lesser value than the definite integral. [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ● $\int_0^1 x\,dx$
- ○ $\int_0^1 1 - x\,dx$
- ○ $\int_{-2}^{-1} \frac{1}{x}\,dx$
- ○ $\int_0^1 1\,dx$
- ● $\int_0^1 e^{x^2}\,dx$

8. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(4 + \frac{k}{n}\right)^3 \frac{3}{n} = \int_{12}^{15} f(x)\,dx$ where $f(x) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $\frac{3}{16}x^3$
- ○ x^3
- ○ $3x^3$
- ● $\frac{1}{27}x^3$
- ○ $4x^3$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the boxes provided.

9. Let $T(t)$ denote the temperature (measured in °C) of a cup of tea at time t (measured in minutes). Suppose you are given that the temperature of a cup of tea, cooling off in a room with surrounding temperature of 20°C, is modelled by

$$\frac{dT}{dt} = -0.05(T - 20)$$

(a) Use the linear approximation to estimate the temperature of a cup of tea 48 seconds after the temperature is 70°C. [2 marks]

We seek to use linear approximation to estimate the temperature of the cup of tea (after 48 seconds), which is modeled as the solution to the above ODE. First of all, we define $t = 0$ to be the instant of time where we know the temperature is 70°C. Now notice that there is a disagreement between the time units used in the differential equation (minutes) and the time at which we want to estimate the temperature (seconds). However, note that 48 seconds $= \frac{4}{5}$ minutes, and so the problem translates into estimating said temperature at $t = \frac{4}{5}$ minutes. That being said, let us recall the general formula for the linear approximation around $t = 0$, evaluated at some time t^*,

$$T(t^*) \approx T(0) + T'(0)\big(t^* - 0\big).$$

The reason for using a linear approximation of $T(t)$ around $t = 0$ (instead of around $t =$ something else) is because $t = 0$ is the only point for which we have additional information about the solution, namely, $T(0)$. Then, it is enough to find the values of $T'(0)$, keeping in mind that $t^* = 48$ seconds $= \frac{4}{5}$ minutes, and that $T(0) = 70$°C. However, in order to find said value it is enough to evaluate the differential equation satisfied by T at $t = 0$, from where we get that

$$T'(0) = -\tfrac{1}{20}\big(T(0) - 20\big) = -\tfrac{1}{20}\big(70 - 20\big) = -\tfrac{5}{2}.$$

Gathering all the above information, we can now use the linear approximation formula to obtain that, after 48 seconds, the temperature of the cup of tea is approximately,

$$\begin{aligned} T(\tfrac{4}{5}) &\approx T(0) + T'(0)\big(\tfrac{4}{5} - 0\big) \\ &= 70 - \tfrac{5}{2} \cdot \tfrac{4}{5} \\ &= 68. \end{aligned}$$

Temperature ≈ $\boxed{68°\text{C}}$

(b) Is your answer in part (a) an underestimation or overestimation of the actual temperature 12 seconds after the temperature is 70°C. [3 marks: 1 mark for correct final answer; 2 marks for justification]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

● Underestimation.

○ Overestimation.

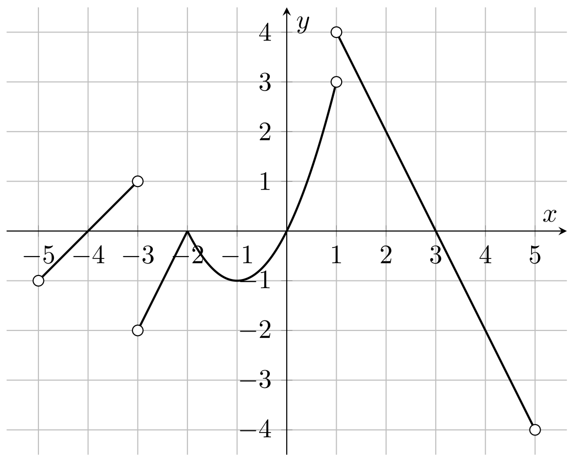
Generally speaking we know that a linear approximation of a function $f(x)$ will overestimate its exact value if $f(x)$ is concave down, and underestimate it if $f(x)$ is concave up around the study point, and so the question comes down to knowing whether $f(x)$ is concave up or down. In the present case we can calculate the second derivative of $T(t)$ by differentiating the given ODE above,

$$T''(t) = \frac{d}{dt}T'(t) = \frac{d}{dt}\Big(-0.05(T - 20)\Big) = -0.05T'(t) = (0.05)^2(T - 20).$$

Thus, noticing that $T(t) > 20$ for all $t \in [0, \frac{4}{5}]$, we deduce that $T''(t) = (0.05)^2(T - 20) > 0$ for all $t \in [0, \frac{4}{5}]$. Therefore, $T(t)$ is concave up on said interval, and hence the above linear approximation corresponds to an underestimate.

Short Answer: No justification is required. Only your final answer will be graded. Put your final answer in the boxes provided.

10. Below is the graph of $y = g'(x)$



Suppose that $g(x)$ is a continuous function everywhere.

(a) Determine all the interval(s) where $g(x)$ is strictly increasing and strictly decreasing on $[-5, 5]$. [2 marks]

NOTE: Do not get too wrapped up thinking about the sign of g' here: this may make you miss the interval endpoints.

$g(x)$ is strictly increasing on: $\boxed{[-4, -3] \text{ and } [0, 3]}$

$g(x)$ is strictly decreasing on: $\boxed{[-5, -4] \text{ and } [-3, 0] \text{ and } [3, 5]}$

(b) Determine all the point(s) at which $g(x)$ has a local maximum on $[-5, 5]$. [1 mark]

$g(x)$ has local maximum at $x = \boxed{-3 \text{ and } +3}$

(c) Does $g(x)$ achieve a global maximum on $[-5, 5]$?

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●). [1 mark]

● Yes. This is a consequence of the Extreme Value Theorem (see the formula sheet), which is applicable since $g(x)$ is continuous everywhere.

○ No.

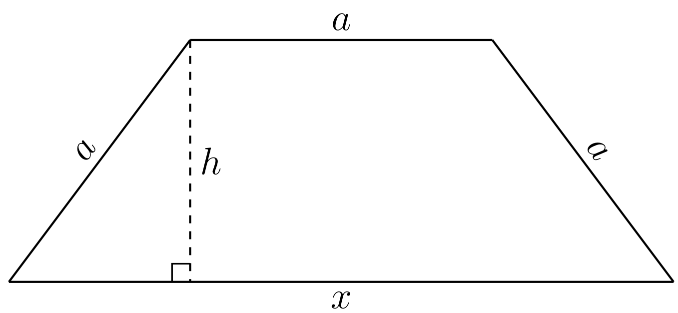
○ Not enough information to determine.

(d) Determine all the inflection point(s) of $g(x)$ on $[-5, 5]$. [1 mark]

$g(x)$ has a point of inflection at $x = \boxed{-2 \text{ and } -1 \text{ and } +1}$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the boxes provided.

11. You have a wire of length $3a$ with which you want to form a trapezoid with height h, three equal sides of length a, and a fourth side of length x. You would like to determine the value of $x \in [a, 3a]$ that maximizes the area of the trapezoid.



(a) Write a formula for the quantity to be maximized in terms of x, a, and h. No justification is required. [1 mark]

Note: you *do not need to know* the formula for the area of a trapezoid to do this problem. You can use the picture to express the area of the trapezoid in question as the sum of the area of a rectangle (of length a and width h) and the areas of two right triangles (of height h and base width $\frac{x-a}{2}$).

$$\boxed{ah + \frac{h}{2}(x-a)}$$

(b) Write any equations relating the independent variables in the formula from part (a). Use these equations to write the quantity to be maximized as a function f of a single variable. [2 marks]

First, using the Pythagorean Theorem, we know that

$$a^2 = h^2 + \left(\frac{x-a}{2}\right)^2. \tag{1}$$

This can be re-arranged to obtain

$$h = \sqrt{\frac{3a^2}{4} + \frac{x}{4}(2a-x)}.$$

This would give

$$f = f(x) = \frac{x+a}{4}\sqrt{3a^2 + x(2a-x)}.$$

Alternatively, we could use equation (1) to isolate for x in terms of h:

$$x = a + 2\sqrt{a^2 - h^2},$$

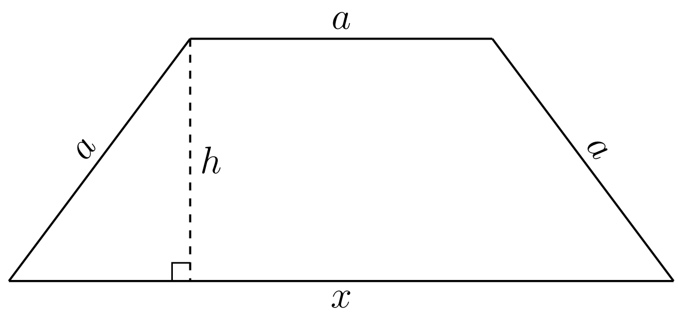
which gives

$$f = f(h) = h\left(a + \sqrt{a^2 - h^2}\right).$$

$$\boxed{f(x) = \frac{x+a}{4}\sqrt{3a^2 + x(2a-x)} \quad \textbf{OR} \quad f(h) = h\left(a + \sqrt{a^2-h^2}\right).}$$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the boxes provided.

11. You have a wire of length $3a$ with which you want to form a trapezoid with height h, three equal sides of length a, and a fourth side of length x. You would like to determine the value of $x \in [a, 3a]$ that maximizes the area of the trapezoid.



(c) Find, with justification, the global maximum of the function in part (b), and then determine the value of x that maximizes the area of the trapezoid. [3 marks]

Suppose we chose x to be our design variable, so we optimize $f(x)$. We use the Closed Interval Method, so we start by testing the endpoints:

$$f(a) = a^2,$$
$$f(3a) = 0.$$

In particular, $x = 3a$ is out of the race immediately. We then have to compute the critical points of $f(x)$:

$$\begin{aligned} 0 = f'(x) &= \frac{1}{4}\sqrt{3a^2 + x(2a - x)} + \frac{a + x}{4}\frac{a - x}{\sqrt{3a^2 + x(2a - x)}} \\ &= \frac{1}{4\sqrt{3a^2 + x(2a - x)}}\left(3a^2 + x(2a - x) + (a + x)(a - x)\right) \\ &= \frac{-1}{4\sqrt{3a^2 + x(2a - x)}}\left(x^2 - ax - 2a^2\right). \end{aligned}$$

The above is trivially simplified to obtain

$$0 = x^2 - ax - 2a^2 = (x - 2a)(x + a).$$

Since we do not allow $x = -a$, the only critical point here is

$$x_* = 2a.$$

We find that

$$f(x_*) = \frac{3a}{4}\sqrt{3a^2 + 2a(0)} = \frac{3\sqrt{3}}{4}a^2 > a^2,$$

(we have used that $3\sqrt{3} > 4$, which follows from taking the square root of $9 * 3 > 16!$), so x_* is the maximizer of $f(x)$ on $[a, 3a]$.

Alternatively, we could have chosen h to be our design variable. We can still use the Closed Interval Method to find the maximizer. This means we have to test the endpoints first. If $x \in [a, 3a]$, this means $h \in [0, a]$. We find that

$$f(0) = 0$$
$$f(a) = a\,(a + 0) = a^2.$$

In particular, we can definitely throw out $h = 0$ as a candidate for the maximizer!Now, we look for the critical points of $f(h)$. This requires solving

$$0 = f'(h) = a + \sqrt{a^2 - h^2} - \frac{h^2}{\sqrt{a^2 - h^2}}.$$

We start by multiplying all sides of the equation by $\sqrt{a^2 - h^2}$:

$$0 = a\sqrt{a^2 - h^2} + a^2 - 2h^2.$$

We then isolate the square root:

$$\frac{2}{a}h^2 - a = \sqrt{a^2 - h^2}.$$

Squaring both sides gives

$$\left(\frac{2}{a}h^2 - a\right)^2 = a^2 - h^2.$$

Expanding out the left-hand side and grouping like powers gives

$$4h^4 - 3a^2h^2 = 0.$$

Factoring this expression gives

$$0 = h^2\left(4h^2 - 3a^2\right),$$

which is easily solved to get $h = 0, \frac{a\sqrt{3}}{2}$. This means the only candidate maximizer we haven't counted already is

$$h_* = \frac{a\sqrt{3}}{2}.$$

We find that

$$f\left(h_*\right) = \frac{3\sqrt{3}}{4}a^2 > a^2 = f(a).$$

Therefore, h_* is the maximizer. This corresponds to

$$x = 2a.$$

Note how both choices of design variable ultimately give the same answer! Of course, on the exam, we'd only expect you to produce one of the answers above.

A final answer of $x = 2a$ with no justification or seeming to appear out of nowhere will not receive parts marks.

$x = \boxed{2a}$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

12. Let $f(x)$ be a twice differentiable function with $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 2$. Define

$$g(x) = \begin{cases} \dfrac{f(x)}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Give a *careful, well written* argument that shows $g'(0) = 1$. [4 marks]

First of all let's recall that, by definition, the derivative of a function $g(x)$ at a point x^* is given by the formula

$$g'(x^*) = \lim_{x\to x^*} \frac{g(x) - g(x^*)}{x - x^*}.$$

Now, by the definition of $g(x)$ given at the beginning of this question, we already know that $g(0) = 0$ and that for every x different than 0,

$$g(x) = \frac{f(x)}{x}.$$

Plugging this information into the definition of the derivative, we find that the derivative of $g(x)$ at $x^* = 0$ is given by

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\frac{f(x)}{x} - 0}{x - 0} \\ &= \lim_{x\to 0} \frac{f(x)}{x^2} \left[\to \frac{0}{0}\right], \end{aligned}$$

where for the last $\frac{0}{0}$ limit we have used the fact that $f(0) = 0$ along with the continuity of $f(x)$ at $x = 0$ ($f(x)$ is differentiable, so it's in particular continuous), so that $f(x) \to f(0) = 0$ as $x \to 0$. Then, since $f(x)$ is differentiable, in order to calculate the above limit we can use L'Hôpital's rule, provided that the resulting limit exists. Observe that using L'Hôpital's rule without checking the above conditions (differentiability of $f(x)$ and the $\frac{0}{0}$ limit) represents an unjustified use of said rule. Thus, we infer that,

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{f(x)}{x^2} \\ &\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{f'(x)}{2x} \end{aligned} \tag{2}$$

provided the limit of $f'(x)/2x$ exists. However, using that $f'(0) = 0$, we can now rewrite the above limit as

$$\begin{aligned} g'(0) &= \frac{1}{2} \lim_{x\to 0} \frac{f'(x)}{x} \\ &= \frac{1}{2} \lim_{x\to 0} \frac{f'(x) - 0}{x - 0} \\ &= \frac{1}{2} \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} \\ &= \frac{1}{2} f''(0) = 1, \end{aligned} \tag{3}$$

where we have used the fact that $f'(x)$ is differentiable at $x = 0$, so that, by definition of the derivative function

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = f''(0) = 2,$$

which is exactly what we used before. Since the resulting limit in equation (3) exists and equals one, we conclude that the application of L'Hôpital's rule was valid, and hence this is in fact the value of $g'(0)$.

Remark: Some students might have tried to apply L'Hôpital's rule a second time after arriving at (2), instead of identifying the last limit in (3) as the definition of the derivative of $f'(x)$ at $x = 0$, but the problem with doing this is that we end up with

$$g'(0) \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{f''(x)}{2},$$

assuming they have justified that $f'(x)/2x \to \frac{0}{0}$ as $x \to 0$ (which follows from the continuity of $f'(x)$ due to the twice differentiability of $f(x)$, so that they can ensure that $f'(x) \to 0$ as $x \to 0$), and that $f'(x)$ is differentiable, so they can apply L'Hôpital's rule again. However, since we do not know if $f''(x)$ is continuous at zero (the fact that $f'(x)$ is differentiable does not imply that $f''(x)$ is continuous), we cannot evaluate the above limit for $f''(x)$. More specifically, even if $f(x)$ is twice differentiable and $f''(0)$ exists and is equal to 2, the limit

$$\lim_{x\to 0} f''(x),$$

might not exist (due to the lack of continuity of $f''(x)$). Thus, we cannot directly evaluate that limit, plugging $f''(0) = 2$, to deduce that

$$g'(0) = \lim_{x\to 0} \frac{f''(x)}{2} \underset{\substack{\text{direct}\\\text{evaluation}}}{=} \frac{f''(0)}{2} = 1. \qquad \text{(not necessarily true)}$$

Alternative, but incorrect, approach: We noticed that a large portion of the students followed a different approach and we would like to acknowledge them only if they fully and properly justified their approach. First of all note that one could directly calculate the derivative of $g(x)$ by using the quotient rule

$$g'(x) = \frac{f'(x)x - f(x)}{x^2} \qquad \text{for } \; x \neq 0.$$

It is important to remark that this formula holds **only if** $x \neq 0$. Then, **assuming the continuity of** $g'(x)$ **at** 0 (which is not given), we could find $g'(0)$ by taking the limit of the above formula as $x \to 0$, from where we would get that

$$g'(0) = \lim_{x\to 0} \frac{f'(x)x - f(x)}{x^2} \left[\to \frac{0}{0}\right]$$

where for the last $\frac{0}{0}$ limit we have used the fact that $f(0) = 0$ and $f'(0) = 0$, along with the continuity of $f(x)$ and $f'(x)$ at $x = 0$ ($f(x)$ is twice differentiable, so in particular $f(x)$ and $f'(x)$ are continuous), so that both

$$f(x) \to f(0) = 0 \qquad \text{and} \qquad f'(x) \to f'(0) = 0$$

as $x \to 0$. Thus, since $f(x)$ is twice differentiable, in order to calculate the above limit we can use L'Hôpital's rule, provided that the resulting limit exists. Once again, the problem of using L'Hôpital's rule here is that we end up with

$$\begin{aligned} g'(0) &\overset{\text{l'H}}{=} \lim_{x\to 0} \frac{f''(x)x + f'(x) - f'(x)}{2x} \\ &= \lim_{x\to 0} \frac{f''(x)}{2} = \frac{2}{2} = 1, \end{aligned}$$

that requires **assuming the continuity of** $f''(x)$ **at** $x = 0$ (which is not given and leads to the same problems pointed out in the Remark in pink above).

A large number of students not only assumed all the continuity properties mentioned above but also missed the calculations for the last limit, making some terms disappear (or appear), taking limits only of some factors but not of the others so that the limit suddenly became 1. While we would acknowledge incorrect solutions that assumed extra (continuity) hypotheses and correctly justified the use of L'Hôpital's making up the limit to be 1 by wrong and unjustified calculations, making appear numbers or taking limits only of some factors and conveniently making the others disappear would not receive part marks if the question was to be graded..

IF NEEDED, USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

If you wish to have this page marked, make sure to refer to it in your original solution.

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