UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Final exam, Apr 21, 2023, 150 Minutes INSTRUCTOR: S. Zabanfahm

First Name: (write as legibly possible within the boxes)

Familly Name:

Student Id Number:

Reminder:

- No textbook notes, calculator or other aids are allowed.

Question	Maximum Mark
1	6
2	6
3	3
4	3
5	5
6	3
7	3
8	3
Total	32

Date: April 21st.

Question 1. *(7 points)Determine if the following statements are* ***True*** *or* ***False****. In case that they are false, provide a counter example.*

(1) *If matrix A is in reduced row-echelon form, then at least one of the entries in each column must be 1.*
(2) *Given that U and V are subspaces of $\mathbb{R}^n$, then $U \cap V$ is a subspace of $\mathbb{R}^n$.*
(3) *If A is an orthogonal matrix, then $det(A) = 1$.*
(4) *If A is an $n \times n$ matrix with n distinct eigenvalues, then A is symmetric.*
(5) *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map corresponding to projection to a subspace W, and $S : \mathbb{R}^n \to \mathbb{R}^n$ is reflection about $W^\perp$, then $T \circ S$ is not invertible.*
(6) *The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.*

Question 2. *(6 points) Give an example of*

(a) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $rank(A) = 2$ and $tr(A) = 3$.

(b) A 3×3 matrix A, such that $AA^T = 4I_3$.

(c) Linear map $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $rank(S \circ T) = 2$ and $rank(T \circ S) = 2$.

(d) A 4×4 upper triangular matrix A, such that A is orthogonally diagonalizable and $det(A) = tr(A)$.

(e) A matrix that is diagonalizable but not orthogonally diagonalizable.

(f) A matrix with no real eigenvalues.

Question 3. *(3 points) Consider the plane* $2x_1 - 2x_2 + 4x_3 = 0$*. Find a basis* $\mathcal{B}$ *for this plane such that* $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$*, where*

$$v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Question 4. *(3 points) Find the projection of* $e_3 \in \mathbb{R}^4$ *on the subspace spanned by*

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Question 5. *(5 points) Let*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

(1) Find an orthonormal eigenbasis for A.
(2) compute A^{10}.

Question 6. *(3 points) Find the QR factorization of the matrix*

$$A = \begin{bmatrix} 3 & 25 \\ 0 & 0 \\ 4 & -25 \end{bmatrix}.$$

Question 7. *(3 points) Find the inverse of the following matrix:*

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 2 & -3 & -4 & 5 \\ 2 & -3 & 4 & -5 \\ 2 & 3 & -4 & -5 \end{bmatrix}$$

(Hint: Note that the columns of A are orthogonal to each other.)

Question 8. *(3 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be an orthogonal transformation. With real distinct eigenvalues λ_1, λ_2 each with algebraic multiplicity 2.*

(1) What is the relation between λ_1 and λ_2

(2) Suppose that $T \circ T \neq I_4$, is it possible for T to be orthogonally diagonalizable?

Briefly Justify your answer for each part.