UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, APRIL 2016

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

EXAMINER: D. BURBULLA

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

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<u>**Instructions:**</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 12 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of eight questions, some with many parts. Attempt all of them. Each question is worth 10 marks. Marks for parts of a question are indicated in the question. **Total Marks: 80**
- PRESENT YOUR SOLUTIONS IN THE SPACE PROVIDED. You can use pages 11 and 12 for rough work. If you want anything on pages 11 or 12 to be marked you must indicate in the relevant previous question that the solution continues on page 11 or 12.

1. Let $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}$. Show $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal set, and write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

2. Let $A = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbf{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue.

3. Let $\mathbf{v}$ be a unit column vector in $\mathbf{R}^n$; let $A = I - 2\,\mathbf{v}\,\mathbf{v}^T$, where I is the $n \times n$ identity matrix. Each of the following five statements is true. Give a *brief, clear* explanation why, for 2 marks each.

(a) $\mathbf{v}^T\,\mathbf{v} = 1$.

(b) $(\mathbf{v}\,\mathbf{v}^T)\,\mathbf{v} = \mathbf{v}$.

(c) $\mathbf{v}$ is an eigenvector of A, corresponding to eigenvalue $\lambda = -1$.

(d) A is symmetric.

(e) A is orthogonal.

4.(a) [4 marks] Find

$$\det \begin{bmatrix} 1 & 0 & -2 & 4 \\ 2 & 1 & 5 & -2 \\ -1 & 3 & 2 & -1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

4.(b) [6 marks] Find all matrices A such that $A^2 = 0$ and

$$\operatorname{col}(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

5. Find the solution to the system of linear differential equations $\begin{cases} y_1' = 6y_1 + 2y_2 \\ y_2' = y_1 + 5y_2 \end{cases}$, where y_1, y_2 are functions of t, and $y_1(0) = 4,\ y_2(0) = -1$.

6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$.

7. Let $S = \text{span}\left\{[\ 2\ \ 1\ \ 0\ \ 0\ ]^T, [\ 0\ \ 1\ \ 0\ \ 2\ ]^T, [\ 1\ \ 1\ \ 1\ \ 1\ ]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 2 & -3 & 3 & 4 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

8. Let $A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$.

(a) [5 marks] Show that $\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is an eigenvector of A. What is the corresponding eigenvalue?

(b) [5 marks] Show that the eigenvalue of A that you found in part (a) is the only real eigenvalue of A, if $b \neq c$. BONUS: what happens if $b = c$?

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