Infectious Disease Modeling Modules: D2, D3

In this problem set, we examine a simple model for the spread of a contagious disease. Our model makes the following assumptions:

- The disease can only be transmitted by human-to-human contact in a single specific population;
- The size of the population remains constant in time (that is, no births or deaths occur and the population is cut out from the outside world);
- The disease confers no immunity after someone has recovered from infection.

Of course, these assumptions limit the applicability of our model to real-life epidemics, but they provide a good foundation upon which more complex and accurate models can be based. Let time be denoted by $t \in [0, \infty)$, measured in days. From the above assumptions, the population can be divided into two groups:

- $S(t)$ = fraction of the population that is susceptible to infection but not actually infected at time t, and
- $I(t)$ = fraction of the population that is infected at time t.

Our model requires two parameters with values in (0, 1): the infection rate a and the recovery rate b, both with units of $(\text{days})^{-1}$ (in practice, these numbers would need to be estimated using patient data). In words, we can think of $b\Delta t$ as the proportion of infected individuals recovering and returning to the susceptible group over a small time interval of length Δt. Similarly, we can think of $a\Delta t$ in the following way: out of all those susceptibles who came into contact with an infected individual during the small time interval, $a\Delta t$ is the proportion who actually end up getting infected as a result of that contact.

1. Explain why the following identity holds for all t:

$$S(t) + I(t) = 1$$

Your answer must be contained in one or two sentences.

A: People in the population can either be infected or not infected, in which case they are susceptible. Therefore, the sum of the proportion of the people who are infected and susceptible must equal one because the sum represents the entire population.

2. In this problem, we derive an ordinary differential equation that can be solved (or simulated) to find $I(t)$. For now, suppose that at every instant in time everyone in the population is in contact with one and only one other person. Every Δt days, the contacts change up. So the proportion of people whose contact is an infected person during a time interval is approximately $I(t)$ and the proportion of people whose contact is a susceptible person during a time interval is approximately $S(t)$.

(a) Let $\Delta I_{\text{recovered}}$ denote the change in $I(t)$ due only to recoveries occurring in the time interval $[t, t + \Delta t]$. Argue why we have

$$\Delta I_{\text{recovered}} \approx -b\Delta t I(t)$$

A: As stated earlier, $b\Delta t$ is the proportion of infected individuals recovering and becoming susceptible over Δt. To get the proportion of the total population that is recovering, multiply the proportion of infected individuals that are recovering by the proportion of individuals that are infected, meaning the expression we get is $b\Delta t I(t)$. Since this is the proportion of individuals coming out of the infected group, the proportion of individuals infected goes down by $b\Delta t I(t)$, hence the change $\Delta I_{\text{recovered}}$ is equal to $-b\Delta t I(t)$

(b) Let $\Delta I_{\text{infected}}$ denote the change in $I(t)$ due only to infections occurring in the time interval $[t, t + \Delta t]$. Argue why we have

$$\Delta I_{\text{infected}} \approx a\Delta t I(t)S(t)$$

A: For a susceptible person to be infected, their contact needs to be an infected person. If we were to choose a random person from the population, the probability of choosing a susceptible person is *S(t)*. Now we need to randomly choose their contact. The probability of randomly choosing an infected person is *I(t)*. The probability of choosing a susceptible person *and* an infected person to be a pair is $P(I \cap S) = I(t)S(t)$. The proportion of susceptible people in contact with an infected person that get infected is $a\Delta t$, hence the

Adding (2) to (3) and dividing both sides by Δt,

$$\frac{\Delta I}{\Delta t} = \frac{\Delta I_{\text{infected}}}{\Delta t} + \frac{\Delta I_{\text{recovered}}}{\Delta t} = aIS - bI$$

And by taking the limit as delta t approaches 0 of both sides:

$$\frac{dI}{dt} = aIS - bI$$

Using (1) to eliminate *S*, we can convert (5) to the following differential equation for $I(t)$ alone.

$$\frac{dI}{dt} = (a-b)I - aI^2$$

3. Recall that an equilibrium solution of (6) is a solution $I_0(t)$ of (6) that is constant in time. Give a condition on the ratio $R_0 \doteq a/b$ that is necessary for a nonzero equilibrium solution to exist. Write down the equilibrium solution in terms of R_0.

A: The ratio R_0 needs to be greater than 1 because $a > b$. This is because if $a < b$, then both the coefficients of I in $\frac{dI}{dt} = (a-b)I - aI^2$ are negative, meaning the only possibility for an equilibrium (the only way

Now, setting $\frac{dI}{dt}$ equal to 0:

$$0 = (a - b)I - aI^2$$

$$0 = I(a - b - aI)$$

Ignoring the solution of $I = 0$,

$$0 = a - b - aI$$

$$I = \frac{a - b}{a}$$

We need this expression in terms of $\frac{a}{b}$, so the obvious thing to do would be to split up the fraction and put what remains in terms of $\frac{a}{b}$:

$$I = 1 - \frac{b}{a}$$

$$I = 1 - \frac{1}{\frac{a}{b}}$$

$$I = 1 - \frac{1}{R_0}$$

4. How do you think solutions of (6) will behave as $t \to \infty$ when a and b are chosen so that $R_0 \in (0, 1)$? Answer in four sentences or less.

A: If R_0 is between 0 and 1, that means that $0 < \frac{a}{b} < 1$ meaning $a < b$ which would mean that, as explained earlier, $\frac{dI}{dt}$ is negative. As t approaches infinity, the function $I(t)$ is just going to keep on decreasing. However, I cannot become negative because if this were the case, $\frac{dI}{dt}$ would instantly become positive and I would instantly rise above the x-axis again. This means that $I(t)$ is going to asymptomatically approach 0 as that is the only way it can be decreasing forever but not negative.