Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 6

1 (a) Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 3 & -2 \\ 4 & 2 & -2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $B = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$. Find the augmented matrix of the linear system $BA\mathbf{y} + A\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

Solution:

$$A\mathbf{y} = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 3 & -2 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 0 \end{bmatrix} \tag{1}$$

Notice that $BA\mathbf{y} = \begin{bmatrix} 8x_1 \\ -7x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}$. Therefore, we have:

$$BA\mathbf{y} + A\mathbf{x} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} + A\mathbf{x} = \begin{bmatrix} 10 & -1 & 1 \\ 2 & -4 & -2 \\ 4 & 2 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \tag{2}$$

The augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 10 & -1 & 1 & 3 \\ 2 & -4 & -2 & 1 \\ 4 & 2 & -2 & 2 \end{array}\right] \tag{3}$$

1 (b) Let $A = (a_{ij})$ be a 2×3 matrix where $a_{ij} = i + j$, and $B = (b_{jk})$ be a 3×4 matrix where $b_{jk} = 2j + k$. Let $AB = (c_{ik})$ and find a formula for c_{ik} in terms of i and k.

Solution:
Recall that the $(i, k)^{\text{th}}$ entry of $C = AB$ is the dot product of the i^{th} row of A with the k^{th} column of B. The i^{th} row of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix} = \begin{bmatrix} i+1 & i+2 & i+3 \end{bmatrix}$$

since $a_{ij} = i + j$. And the k^{th} column of B is

$$\begin{bmatrix} b_{1k} \\ b_{2k} \\ b_{3k} \end{bmatrix} = \begin{bmatrix} 2+k \\ 4+k \\ 6+k \end{bmatrix}$$

since $b_{jk} = 2j + k$. Therefore $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k}$, which we find to be:

$$c_{ik} = (i+1)(2+k) + (i+2)(4+k) + (i+3)(6+k) = 3ik + 12i + 6k + 28$$

Note: since the matrices are of such small size, one can write out what A, B, and AB are. In fact:

$$AB = \begin{bmatrix} 49 & 58 & 67 & 76 \\ 64 & 76 & 88 & 100 \end{bmatrix}$$

However, from here it is not clear how one could recognize any patterns.

2 (a) If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, show that $A^2 = I$.

Solution: Simply compute $A^2 = AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

2 (b) What is wrong with the following argument? If $A^2 = I$, then $A^2 - I = 0$, so $(A - I)(A + I) = 0$, so that $A = I$ or $A = -I$.

Solution: Let's examine each claim.

The first is: "If $A^2 = I$, then $A^2 - I = 0$." This is fine: just subtract I from both sides of the equation.

The second claim is: "If $A^2 - I = 0$, then $(A - I)(A + I) = 0$." This is also fine: we have

$$(A - I)(A + I) = A^2 + AI - IA - I^2 = A^2 - I^2 = A^2 - I = 0.$$

[**Warning:** It is **not** true in general that $A^2 - B^2 = (A - B)(A + B)$! Indeed, what we have is $(A-B)(A+B) = A^2 + AB - BA - B^2$ — note that the order of multiplication matters! — and since $AB - BA \neq 0$ in general, we cannot say that the right hand side is equal to $A^2 - B^2$. In our case, we have $B = I$, and indeed $AB - BA = 0$ here. This is why the formula $A^2 - B^2 = (A - B)(A + B)$ ended up working in this argument.]

The final claim is: "If $(A - I)(A + I) = 0$, then $A = I$ or $A = -I$." **This is no good!** In general, matrices do not satisfy the rule "$AB = 0 \implies A = 0$ or $B = 0$". Indeed, take the matrix A from part **(a)**, and note that $A - I \neq 0$ and $A + I \neq 0$ while $(A - I)(A + I) = 0$. So this is where the error lies.

[**Aside:** Not only does the rule "$AB = 0 \implies A = 0$ or $B = 0$" fail for matrices, but it does so in a spectacular fashion: it is possible to give an example of an $A \neq 0$ such that $A^2 = 0$. Can you find one?]

3. For each of the following functions, decide if it is a linear mapping (linear transformation). If it is, express it as a matrix mapping (transformation). If it's not, find a specific example to show that one of the two conditions in the definition of a linear mapping on page 134 of the textbook does not hold.

(i) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + 2x_2 + x_3$

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} |x_1| \\ 2x_2 \\ 3x_1 + x_2 \end{bmatrix}$

(iii) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + x_2 + x_3 + 1 \end{bmatrix}$

(iv) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_3 \\ x_1 \\ 4x_2 \end{bmatrix}$

Solution: Recall that the only linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ are matrix mappings (see Textbook, Theorem 3, page 136). i.e. $T(\mathbf{x}) = A\mathbf{x}$ iff T is a linear transformation.

(i) $A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$.

(ii) T is not linear, consider the vectors $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$T(\mathbf{x}) + T(\mathbf{y}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = T(\mathbf{x} + \mathbf{y}).$$

(iii) T is not linear, consider the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and the scalar $4 \in \mathbb{R}$. Then

$$T(4\mathbf{x}) = \begin{bmatrix} 1 \\ 13 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 16 \end{bmatrix} = 4T(\mathbf{x}).$$

(iv) $A = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$.

4. Let T: $\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(a) Determine, if possible, $T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

(b) Suppose that it is also true that $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Determine $T\left(\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}\right)$.

Solution:

(a) If we can write $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then we can use the linearly of T to determine $T(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})$. Thus, we want to solve the following system of linear equations:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

It's easy to see that $a = -1$ and $b = 0$ is the unique solution here. (If this wasn't so clear, we could of course always setup the augmented matrix corresponding to the system and row reduce it.) Thus,

$$\begin{bmatrix}0\\1\\1\end{bmatrix} = -\begin{bmatrix}2\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix},$$

and so

$$T(\begin{bmatrix}0\\1\\1\end{bmatrix}) = T(-\begin{bmatrix}2\\1\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix}) = -T(\begin{bmatrix}2\\1\\1\end{bmatrix}) + 2T(\begin{bmatrix}1\\1\\1\end{bmatrix}) = -\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}1\\-5\end{bmatrix}.$$

(b) This time we want to consider the system

$$\begin{bmatrix}2\\2\\3\end{bmatrix} = a\begin{bmatrix}2\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

We can spot by inspection that $(a, b, c) = (0, 2, 1)$ solves this system, but let's nonetheless see how row reduction would work here:

$$\left[\begin{array}{ccc|c}2&1&0&2\\1&1&0&2\\1&1&1&3\end{array}\right] \xrightarrow[R_3-R_2]{R_1-2R_2} \left[\begin{array}{ccc|c}0&-1&0&-2\\1&1&0&2\\0&0&1&1\end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{ccc|c}0&-1&0&-2\\1&0&0&0\\0&0&1&1\end{array}\right] \xrightarrow[R_1\leftrightarrow R_2]{R_1\times(-1)} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&2\\0&0&1&1\end{array}\right].$$

So the system has the unique solution $(a, b, c) = (0, 2, 1)$. Thus

$$\begin{bmatrix}2\\2\\3\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix},$$

and consequently

$$T(\begin{bmatrix}2\\2\\3\end{bmatrix}) = T(2\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}) = 2T(\begin{bmatrix}1\\1\\1\end{bmatrix}) + T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = 2\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}3\\-5\end{bmatrix} = \begin{bmatrix}5\\-9\end{bmatrix}.$$

(c) **Method #1:** Look back at the third equation above:

$$\begin{bmatrix}3\\-5\end{bmatrix} = T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c\\f\end{bmatrix}.$$

What we can glean from this is that $T(\begin{bmatrix}0\\0\\1\end{bmatrix})$ gives us the third column of A. Similarly, the first and second columns of A will be $T(\begin{bmatrix}1\\0\\0\end{bmatrix})$ and $T(\begin{bmatrix}0\\1\\0\end{bmatrix})$, respectively. Now, just as in parts **(a)** and **(b)**, we can compute

$$T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = T(\begin{bmatrix}2\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix}) = T(\begin{bmatrix}2\\1\\1\end{bmatrix}) - T(\begin{bmatrix}1\\1\\1\end{bmatrix}) = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

$$T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = T(\begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}) = T(\begin{bmatrix}0\\1\\1\end{bmatrix}) - T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}1\\-5\end{bmatrix} - \begin{bmatrix}3\\-5\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$$

So the desired matrix is

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 3 & 0 & -5 \end{bmatrix}.$$

Method #2: Let's suppose that the desired matrix is

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

Now let's use the equation $T(\mathbf{x}) = A\mathbf{x}$ with several choices of $\mathbf{x}$ to get some equations involving the entries of A. For example, we have

$$\begin{aligned}
\begin{bmatrix} 1 \\ -2 \end{bmatrix} &= T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+c \\ d+e+f \end{bmatrix} \\
\begin{bmatrix} 1 \\ -5 \end{bmatrix} &= T(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b+c \\ e+f \end{bmatrix} \\
\begin{bmatrix} 3 \\ -5 \end{bmatrix} &= T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}.
\end{aligned}$$

We thus have 6 equations in the 6 unknowns a, b, c, d, e, f, and we can solve these as usual. It's actually easier here to not bother with setting up the augmented matrix, etc., and instead simply work this out by hand. From the last equation, we get $c = 3$ and $f = -5$. Then from the equation above that, we get $b+c = 1$ and $e+f = -5$, so $b = -2$ and $f = 0$. And finally, the first equation gives us $a = 1-b-c = 0$ and $d = -2-e-f = 3$. So the desired matrix is

$$A = \begin{bmatrix} 0 & -2 & 3 \\ 3 & 0 & -5 \end{bmatrix}.$$