Function Properties

$\sin x$ max @ $\frac{\pi}{2}+2\pi k, k\in\mathbb{R}$; 0 @ $\pi+2\pi k$
$\cos x$ max @ $2\pi+2\pi k, k\in\mathbb{R}$; 0 @ $\frac{\pi}{2}+2\pi k$
$\tan x$ $x\in\mathbb{R}, x\neq\frac{\pi}{2}+n\pi$

Log $x>0$
Expo Range: $(0,\infty)$

Average Value of a Function

Mean value theorem
$f(x)$ continuous on $[a,b]$, there exist a point c:
$$f(c)=f_{avg}=\frac{1}{b-a}\int_a^b f(x)\,dx$$

V-sub Tips ☆change bound
☆ Derivative also appears
☆ Inner function
☆ Raised to the highest power
☆ Appears in denominator

Trig Identities
$\tan^2 x+1=\sec^2 x$ $\quad \csc^2 x-\cot^2 x=1$
$\sin 2x=2\sin x\cos x$ $\quad \tan 2x=\frac{2\tan x}{1-\tan^2 x}$
$=2\cos^2\theta-1$
$=1-\sin^2\theta$

Examination Aid Sheet
Faculty of Applied Science & Engineering

Both sides of the sheet may be used;
must be printed on 8.5” x 11” paper.

Subject: Calculus II MAT187

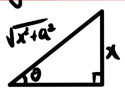
Candidate’s name: Kexin Qian

Candidate’s signature: JKQ

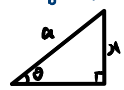
Common Integrals
$\int \ln x\,dx = x\ln x - x + C$
$\int a^x dx = \frac{a^x}{\ln a}+C$
$\int \frac{1}{x}dx=\ln|x|+C$
$\int \log_a x\,dx=\frac{x\log_a x - x}{\ln a}+C$

Common Derivatives
$\frac{d}{dx}e^x=e^x$
$\frac{d}{dx}\sin x=\cos x$
$\frac{d}{dx}\cos x=-\sin x$
$\frac{d}{dx}\tan x=\sec^2 x$
$\frac{d}{dx}\ln(g(x))=\frac{g'(x)}{g(x)}$
$\frac{d}{dx}e^{g(x)}=g'(x)e^{g(x)}$
$\frac{d}{dx}\csc x=-\csc x\cot x$
$\frac{d}{dx}\sec x=\sec x\tan x$
$\frac{d}{dx}\cot x=-\csc^2 x$

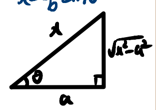
Trig Substitution



$x=\frac{a}{b}\tan\theta$



$x=\frac{a}{b}\sin\theta$



$x=\frac{a}{b}\sec\theta$

Integration By Parts

- $\int u\,dv = uv-\int v\,du$
- Log
  Inverse Trig
  Algebraic
  Trig
  Exponential

Top choice ⓤ

Partial Fraction Decomposition

1. Use long division → proper
2. Split into linear/quadratic factor
3. Linear: $\frac{P(x)}{Q(x)}=\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}\dots$
   Quadratic: $\frac{P(x)}{Q(x)}=\frac{Ax+B}{ax^2+bx+c}\dots$

☆ If repeated factor → increase degree of denominator: $\frac{\ }{p(x)}+\frac{\ }{p(x)^2}+\frac{\ }{p(x)^3}$
☆ Top → 1 degree < bottom

Improper Integral

① Infinite interval of integration
② Intergrand has vertical asymptote

$$\int_a^\infty f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$$
$$\int_{-\infty}^b f(x)dx=\lim_{a\to-\infty}\int_a^b f(x)dx$$

☆ converge → limit exist & finite
☆ diverge → limit does not exist

P-test

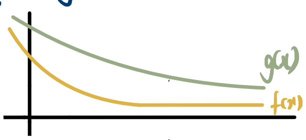
Case 1: $\int_1^\infty \frac{1}{x^p}dx$
- If $p>1$ → converge
- If $p\le 1$ → diverge

Case 2: $\int_0^1 \frac{1}{x^p}dx$
- If $p\ge 1$ → diverge
- If $p<1$ → converge

① $\int_a^b \frac{1}{(x-a)^p}dx$ ② $\int_a^b \frac{1}{(a-x)^p}dx$ same situation

Comparison Test

☆ $f(x)$ & $g(x)$ continuous ⊕ on $[a,\infty)$
☆ $f(x)\le g(x)$ over $[a,\infty)$



$g(x)$ converge → $f(x)$ converge
$f(x)$ diverge → $g(x)$ diverge

Exponential Test

$$\int_0^\infty e^{-ax}dx$$

- If $a>0$ → converge
- If $a\le 0$ → diverge

☆ Discontinuity at point
$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

Second Order ODE [Homogeneous]

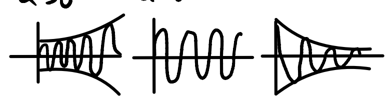
$ay''+by'+cy=0$
↓
2 solutions
↓
y_1 and y_2 (Linearly Independent)
↓
general solution
$y=c_1y_1+c_2y_2$

Discriminant:
$b^2-4ac\begin{cases}>0 & 2\text{ real}\\ =0 & 1\text{ real}\\ <0 & 2\text{ imaginary}\end{cases}$

Soln. Behavior

$\boxed{>0}$ $y(t)=c_1e^{r_1t}+c_2e^{r_2t}$
$\boxed{=0}$ $y(t)=c_1e^{r_1t}+c_2te^{r_1t}$
$\boxed{<0}$ $y(t)=e^{\alpha t}(c_1\cos(\beta t)+c_2\cos(\beta t))$
$r_1=\alpha+\beta i \quad r_2=\alpha-\beta i$

$\alpha>0$ $\quad\alpha=0$ $\quad\alpha<0$



Second Order ODE [Non-Homogeneous]

$ay''+by'+cy=f(t)$

Solve
1. Find general soln. y_c for homogeneous
2. Find particular soln. y_p
3. Add y_c and y_p for solution

Find y_p
- Guess form → adjust
- Works best if $f(t)$ = polynomials, sin/cos, expo

☆ Sum of 2 particular to 2 ODE is soln. to the sum of the ODEs
☆ Soln. is linear combination of terms contain $f(t)$
☆ After differentiation the linear combination of same term appear
☆ If not linear independent, multiply t until diff from homogeneous soln.

Method of Undetermined Coefficient

$f(t)$	Guess
k	A
$ax+b$	$Ax+B$
ax^2+bx+c	Ax^2+Bx+C
$ae^{\lambda x}$	$Ae^{\lambda x}$
$a\cos\beta x+b\sin\beta x$	$A\cos\beta x+B\sin\beta x$

same as this

☆ Include all terms even if B or C = 0
☆ If cos or sin above, need both terms

Steps
① Solve complementary equation and write general soln. ($f=0$)
② Guess y_p based on $f(t)$
③ Check if any term solves complementary, if so, $*t$
④ Sub y_p into ODE to find coefficient(s)
⑤ Add y_p and y_c

Nth Order Linear

$$y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\dots a_1(t)y'(t)+a_0(t)y(t)=f(t)$$

nth derivative
Order nth derivative
Ordinary only 1 independent variable
same func.

Estimation Methods

	overestimate	underestimate
Left	Strictly decreasing	" increasing
Right	strictly increasing	" decreasing
Midpoint	Always concave down	" up
Trapezoid	Always concave up	" down

$\sum_{i=1}^n f(x_i)\Delta x$, start @ 0 Ⓛ
$\sum_{i=1}^n f(x_i)\Delta x$, end @ last point Ⓡ
$\sum_{i=1}^n \left(\frac{x_i+x_{i+1}}{2}\right)\Delta x$ Ⓜ
Ⓣ $\sum_{i=1}^n \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)\dots+f(x_n)]$

First order separable

$y'(t)=g(y)f(t)$
Solving $\frac{dy}{dt}=g(y)f(t)$
$\frac{1}{g(y)}dy=f(t)dt$
$\int\frac{1}{g(y)}dy=\int f(t)dt$

Integrating factor

standard form $y'(t)+p(t)y=f(t)$
☆ Never lose solution eg. $\ln|y|\to y\neq 0$
I.F: $e^{\int p(x)dx}$ multiply all terms & integrate
$y\cdot(I.F)=\int(I.F)f(t)$

Modelling with Riemann

1. $x=\frac{b-a}{n}$
2. Representative points (x_k^*) $\begin{cases}R_n=x_0+k\Delta x\\ L_n=x_0+(k-1)\Delta x\\ M=x_k^*\end{cases}$
3. Use the function → if exact, then limit

Taylor Series & Radius of Convergence

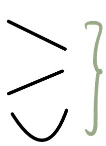
$$f(x)=\sum_{k=0}^\infty \frac{f^{(k)}(a)(x-a)^k}{k!} \text{ for } |x-a|<R$$

- Only if $\lim_{n\to\infty}R_n(x)=0$
- R → interval in which $f(x)$ = its Taylor series

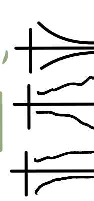
First-order Autonomous

$y'=\frac{dy}{dt}=f(y)$
☆ No independent variable
☆ Has phase plot
Qualitative Analysis
1. Find constant solution → $y(t)=c$ @ $\frac{dy}{dt}=0$
2. If $f(c)=0$ $y'=f(y)$, C is equilibrium

Stable
Unstable
Semistable



y & y' phase plot



☆ can connect to phase to determine type & plug values @ interval
☆ In direction field, same across t

Ratio Test

$\sum_{k=0}^\infty a_k$, Let $L=\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ $\quad a_k=\frac{f^{(k)}(a)(x-a)^k}{k!}$

$L<1\to\sum_{k=0}^\infty|a_k|$ absolutely converge (also $\sum_{k=0}^\infty a_k$ converge too)
$L>1\to\sum_{k=0}^\infty a_k$ diverge
$L=1\to$ Inconclusive

Modelling Area & Arc length

Arc length
$$\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

Area b/n 2 curves
$$A=\int_{left}^{right}(top-down)dx$$

Circle of radius r & centre (a, b)
$(x-a)^2+(y-b)^2=r^2$
$a>0$ Right $\quad b>0$ Up
↓ parametic
$x=r\cos\theta+a \quad y=r\sin\theta+b$

Rose curves
$r=a\sin(n\theta)$ odd → same petals
$r=a\cos(n\theta)$ even → double petals.
$a\neq0$ ☆ if $0<\theta<\pi$ → only 1 cycle
$n>1, \in\mathbb{Z}$ ∴ exact number of petals.

Power Series

Centred @ $x=a$

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

☆ Smooth func $f(x)$ → taylor → power

☆ Differentiable term by term on interval of convergence

Convergence of power series @ $x = x_0$

$\lim_{k\to\infty}\sum_{n=0}^{k} c_n(x_0-a)^n$ exist

Find where power series converge

Apply ratio test & find condition for x_0 under $p<1$

3 cases of power series convergence $\sum_{n=0}^{\infty}(x-a)^n$

Case 1 converge for all values of x

Case 2 diverge for all values of x except a

Case 3 Exists $R \in \mathbb{R}$ such that if $\begin{cases} |x-a|<R \to \text{converge} \\ |x-a|>R \to \text{diverge}\end{cases}$

Set of values x for series to converge = interval of convergence

Polar Coordinates/Graphs

(r, θ)

dependent, independent

☆ See 2nd page for information

Circle in polar coordinates

① Circle with centre $(a,0)$ & radius $|a|$
$r = 2a\cos\theta$

② circle with centre $(0,a)$ & radius $|a|$
$r = 2a\sin\theta$

r against θ → xy plane

Cartesian Graph to Polar Curve

1. Identify how r changes as θ grows
[Where r is ↑ or ↓]

2. Identify ordered pairs, (r,θ), where:
i) θ = radian measure that are easy to locate
$(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, etc)
ii) $r=0$ → points where curve goes through pole
iii) $r<0$ → point lie on opposite extension of terminal arm associated with θ

Motion Modelling

$\vec{r}(t)$ representation

$\vec{r}(t)$ position $\vec{r}''(t)$ acceleration

$\vec{r}'(t)$ velocity $\|\vec{r}'(t)\|$ speed

Acceleration

$$\vec{r}''(t) = \vec{a}(t) = a_T\vec{T}(t) + a_N\vec{N}(t)$$

a_T: tangential acceleration; a_N: normal acceleration

$a_T = \frac{d}{dt}\|\vec{r}'(t)\|$
$a_N = k\|\vec{r}'(t)\|^2$ } Not a vector

$a_T(t)>0$ $a_T(t)=0$ $a_T(t)<0$

Speeding up / No change in speed / slowing down

$a_N(t)>0$ $a_N(t)=0$ $a_N(t)<0$

Changing direction toward $\vec{N}(t)$ / Not turning / impossible

$\|\vec{r}(t)\|=1$
object moving in sphere
$\vec{r}'(t)$ constant
$\|\vec{r}'(t)\|$ constant
[constant velocity ⇒ constant speed]
$\|\vec{r}'(t)\|$ constant
$\vec{r}'(t)$ may not be constant
[constant speed, not constant V]

Spring Mass Systems

$my'' + cy' + ky = 0$ (sln)

m mass
c damping constant
k spring constant

Simple Harmonic Motion

$x(t) = c_1\cos\sqrt{\frac{k}{m}}t + c_2\sin\sqrt{\frac{k}{m}}t$

Characteristic Polynomial:

$mr^2 + cr + k = 0 \begin{cases} r_1 \\ r_2 \end{cases}$

Definitions:

Free → $f(t)=0$, homo $y''+w^2y=0$

Forced → $f(t)\neq 0$, non-homo

Period → $\frac{2\pi}{w}$ Frequency → $\frac{w}{2\pi}$ [$\frac{1}{\text{period}}$]

Angular Frequency (w) rad/s

$\boxed{c^2-4mk}$ $w = \frac{\sqrt{4mk-c^2}}{2m}$

Case 1 <0: Underdamping
$x(t) = e^{-\left(\frac{c}{2m}\right)t}(c_1\cos wt + c_2\sin wt)$

Case 2 >0 Overdamping
$x(t) = c_1e^{r_1t} + c_2e^{r_2t}$

Case 3 $=0$ Critical damping
$x(t) = (c_1 + c_2t)e^{-\frac{c}{2m}t}$

Restoring force $= -kx$

Damping force $= -cx'$

Alternating Series

• Alternating signs & decreasing terms in magnitude

Error for alternating series $\boxed{S}$

$$S = S_n + R_n = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

$|\text{Error}| = |S - S_N| \leq a_{N+1}$

$|R_n|$

☆ The rest of the term besides until S_n does not matter as they cancel/too small to be considered

Calc in Polar coordinates

Finding tangent line @ point

1. slope of tangent

$$\frac{dy/d\theta}{dx/d\theta} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

2. Point slope form

$$y - r(\theta)\sin\theta = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}(x - r(\theta)\cos\theta)$$

Finding area bounded by a curve

1. Circular sector $A_{\text{sector}} = \frac{r^2}{2}\theta$

2. $A = \int_{\theta_1}^{\theta_2}\frac{r^2}{2}d\theta$

Finding area between curves

For $\theta\in[\theta_1,\theta_2]$ & $r_2 \geq r_1$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2}(r_2)^2 - (r_1)^2\,d\theta$$

Vector-valued functions

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$
$= f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$
$= \begin{bmatrix} f(t) \\ g(t) \\ h(t)\end{bmatrix}$

Limit Derivative Integral } apply to each vector

Principal Unit-tangent vector

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ [Direction] [Speed @ t]

$\vec{r}(t)$ = curve that the object follows [motion]

☆ Points to where the object is moving
☆ $\vec{r}'(t)$ may not exist → if not smooth
☆ $\vec{r}'(t)$ tangent to curve at $\vec{r}(t)$
☆ If object speeds up, magnitude of $\vec{r}'(t)$ ↑, magnitude of $\vec{T}(t)$ does not ↑

Principal unit normal vector

$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$, $\vec{T}'(t)\neq\vec{0}$

• points to the curve is turning (inward)

Interpretation

$\vec{T}'(t_0) = 0 \begin{cases} \vec{r}(t) \text{ is a straight line} \\ \vec{r}(t) \text{ has no curvature} \\ \vec{r}(t) \text{ may change direction of curvature @ } t_0\end{cases}$

$\vec{T}(t)\perp\vec{N}(t)$ for a curve parametrized by $\vec{r}(t)$

Reparametrizing the curve

• Can change direction of curve, then change direction of T
• $r_1(t)$ & $r_2(t)$ are 2 reparametrizations of the same curve then they are not necessarily the same point on the curve

Curvature K

$K(t) = \frac{1}{R}$ for a curve $\vec{r}(t)$ & $\vec{r}'(t)\neq 0$,
R = largest circle that fits into the bend @ time t

Curvature @ time t

$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$

$\vec{r}(s)$ is in arclength parametrization

$K(t) = \left\|\frac{d\vec{T}}{ds}\right\|$

Another Formula

$K(t) = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$

Displacement / Distance / $\vec{r}'(t)$

Total displacement: $\vec{r}(b) - \vec{r}(a) = \int_a^b \vec{r}'(t)\,dt$

Distance traveled: $\int_a^b \|\vec{r}'(t)\|dt$

Parametric Equations

$t \to f(t) = x$, $t \to g(t) = y$ → $(f(t), g(t)) = (x,y)$

• Infinitely many pairs of functions f and g that results in same curve
↳ differ by rate & orientation

Orientation of the curve

• Direction of the plane curve as parameter (t) increases

Eliminating the Parameter

☆ Change from implicit to explicit relation
☆ May have range restrictions

• Isolate t from one equation
• sub into the other

Derivative of Parametric & x-y plane

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \leftrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Behaviour in tx and ty plane

x ↑ in tx plane → Right in xy-plane
x ↑ in ty plane → Up in xy-plane

Imaginary Numbers

$(a+bi)+(c+di) = (a+c)+(b+d)i$
$(a+bi)-(c+di) = (a-c)+(b-d)i$
$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$
$\frac{a+bi}{c+di} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$

$\bar{z} = a - bi$ conjugate

$|z| = \sqrt{a^2+b^2}$

$\bar{z}z = z\bar{z} = |z|^2$

Euler's identity: $e^{i\pi} = -1$, $e^{i2\pi} = 1$, $e^{ix} = \cos x + i\sin x$

Euler's formula: $z = r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$ → polar form

Example Parametrizations

① Semicircle $(2\cos(t), 2\sin(t))$, $t\in[0,\pi]$
② Circle (t^2, t); $t\in[0,\pi]$,
③ Ray $(\cos(2t), -\sin(2t))$ $t\in\mathbb{R}$
④ Line $(1+2t, 2-3t)$; $t\in\mathbb{R}$

$t<0$, $t>0$

Toolbox!

Newton's 2nd Law: $F = ma$ → $m\frac{dv}{dt} = -mg - bv(t)$ (constant)

Indeterminate form

$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $\infty-\infty$, $0\cdot\infty$
0^0, 1^∞, ∞^0

Complete the Square

1. Divide all terms by coefficient of x^2
2. Add $\left(\frac{b}{2}\right)^2$ and subtract $\left(\frac{b}{2}\right)^2$
$x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2 = \left(x+\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$

Derivative Rules

Product Rule $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ Chain Rule $f(g(x))' = f'(g(x))g'(x)$

Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$