Solutions to MAT188H1F - Linear Algebra - Fall 2014

Term Test 1 - September 30, 2014

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

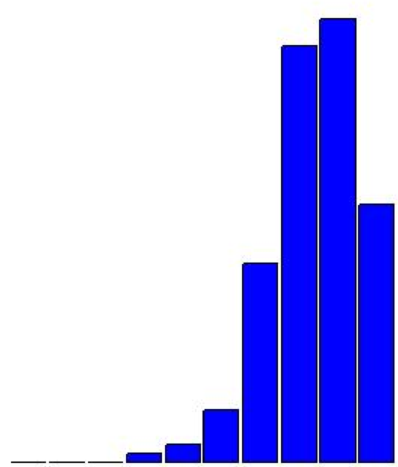
This test consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. The range on every question was 0 to 10. Q4 and Q5 had the highest averages; Q2 and Q8 the lowest.
2. All the questions on this test, except for 2(b), were very similar to homework problems. In 2(b) the usual process was reversed: given the solution what is the corresponding reduced augmented matrix?
3. In Q3 to Q8 you must explain your work fully to receive full marks. Many students simply wrote down some matrices or equations with no indication of what they were doing, or why they were doing it! The markers will not fill in the details for you; you must make it clear what you are doing.

Breakdown of Results: 962 students wrote this test. The marks ranged from 30% to 100%, and the average was 78.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	18.5%
A	50.3%	80-89%	31.8%
B	29.8%	70-79%	29.8%
C	14.2%	60-69%	14.2%
D	3.7%	50-59%	3.7%
F	1.9%	40-49%	1.3%
		30-39%	0.6%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



PART I : No explanation is necessary.

1. (avg: 7.6) Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 2 & 16 & 3 \\ 0 & 1 & -7 & -8 & 5 \end{bmatrix}, \ C = \begin{bmatrix} 1 & -2 & 8 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & -2 & 8 & -4 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

be the augmented matrices of four systems of linear equations. Answer the following questions. Note: there can be more than one answer per part. For this question all correct choices count as +1 and all incorrect choices count as −1.

(a) Which of the four above matrices are in echelon form? <u>B, C, D</u>

(b) Which of the four above matrices are in reduced echelon form? <u>B</u>

(c) Which of the four above matrices are augmented matrices of an inconsistent linear system of equations? <u>D</u>

(d) Which of the four above matrices are augmented matrices of a system of linear equations with a unique solution? <u>A, C</u>

(e) Which of the four above matrices are augmented matrices of an echelon system of linear equations? <u>B, C</u>

(f) Which of the four above matrices are augmented matrices of a triangular system of linear equations? <u>C</u>

2. (avg: 6.6) Each part is worth 5 marks.

(a) The general solution to a linear system of equations is

$$\begin{aligned} x_1 &= 4 + 6s_1 - 5s_2 \\ x_2 &= s_1 \\ x_3 &= -9 + 3s_2 \\ x_4 &= s_2 \end{aligned},$$

for parameters s_1, s_2. Express the solution as a linear combination of three vectors in $\mathbb{R}^4$.

Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -9 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

(b) Find a vector $\mathbf{b}$ in $\mathbb{R}^2$ and a matrix A with four columns each of which is a vector in $\mathbb{R}^2$, such that the equation $A\mathbf{x} = \mathbf{b}$ has solution

$$\mathbf{x} = \begin{bmatrix} 4 + 6s_1 - 5s_2 \\ s_1 \\ -9 + 3s_2 \\ s_2 \end{bmatrix}.$$

Solution: infinitely many possible answers. Easiest is to pick the reduced echelon matrix that produces the given solution:

$$A = \begin{bmatrix} 1 & -6 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$$

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

3. (avg: 8.4) Solve the system of linear equations

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ x_1 + x_2 - x_3 - 2x_4 = 2 \\ -2x_1 + 3x_3 + 5x_4 = -5 \end{cases}$$

by reducing its augmented matrix to reduced echelon form.

Solution: row reduce the augmented matrix.

$$\left[\begin{array}{rrrr|r} 1 & 2 & 1 & 3 & 0 \\ 1 & 1 & -1 & -2 & 2 \\ -2 & 0 & 3 & 5 & -5 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 5 & -2 \\ 0 & 4 & 5 & 11 & -5 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 2 & 5 & -2 \\ 0 & 0 & 3 & 9 & -3 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right]$$

This last matrix is the reduced echelon form of the augmented matrix of the given system of equations. Let $x_4 = s$ be a parameter. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-2s \\ s \\ -1-3s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

4. (avg: 9.2) A total of 275 people attend a concert. Ticket prices are \$12 for adults, \$10 for seniors and \$8 for students. The total revenue was \$3100. Determine how many adults, seniors and students attended the concert, given that the number of seniors who attended was twice the number of students.

   Solution: let x, y, z be the number of adults, seniors and students, respectively, that attended. We have

$$x + y + z = 275,\ 12x + 10y + 8z = 3100,\ y - 2z = 0.$$

   The augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 275 \\ 12 & 10 & 8 & 3100 \\ 0 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 275 \\ 0 & 2 & 4 & 200 \\ 0 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 175 \\ 0 & 1 & 2 & 100 \\ 0 & 0 & 4 & 100 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 25 \end{array}\right].$$

   So the unique solution is $x = 200, y = 50$ and $z = 25$. That is, 200 adults, 50 seniors, and 25 students attended the concert.

5. (avg: 9.2) Balance the chemical reaction

$$\mathrm{C_3H_8} + \mathrm{O_2} \longrightarrow \mathrm{CO_2} + \mathrm{H_2O}$$

which describes the oxidation of propane to produce carbon dioxide and water, by first setting up and then solving an appropriate homogeneous system of equations.

Solution: suppose the balanced reaction looks like

$$x_1\mathrm{C_3H_8} + x_2\mathrm{O_2} \longrightarrow x_3\mathrm{CO_2} + x_4\mathrm{H_2O}$$

Count the number of like atoms on each side of the reaction:

$$\begin{array}{rcccc} \text{Carbon:} & 3x_1 = x_3 & & 3x_1 - x_3 = 0 \\ \text{Hydrogen:} & 8x_1 = 2x_4 & \Leftrightarrow & 4x_1 - x_4 = 0 \\ \text{Oxygen:} & 2x_2 = 2x_3 + x_4 & & 2x_2 - 2x_3 - x_4 = 0 \end{array}$$

Reduce the augmented matrix for this homogeneous system:

$$\left[\begin{array}{cccr|c} 3 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccr|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccr|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cccr|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & -5 & 0 \\ 0 & 0 & 4 & -3 & 0 \end{array}\right].$$

Let $x_4 = s$, then

$$x_1 = \frac{s}{4},\ x_2 = \frac{5s}{4}, x_3 = \frac{3s}{4}.$$

To find the smallest whole integer solution, let $s = 4$, so that

$$x_1 = 1,\ x_2 = 5, x_3 = 3.$$

That is, the balanced chemical reaction is

$$\mathrm{C_3H_8} + 5\mathrm{O_2} \longrightarrow 3\mathrm{CO_2} + 4\mathrm{H_2O}$$

6. (avg: 7.4) Find a set of vectors $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^4$ that spans the solution set of the system of equations

$$\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ 3x_1 + 9x_2 - 11x_3 + 14x_4 = 0 \\ -2x_1 - 6x_2 - 6x_3 + 4x_4 = 0 \end{cases}$$

Solution: first solve the system, for example by row reduction on the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 3 & 9 & -11 & 14 & 0 \\ -2 & -6 & -6 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & 0 & -8 & 8 & 0 \\ 0 & 0 & -8 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let $x_2 = s_1, x_3 = s_2$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s_1 - s_2 \\ s_1 \\ s_2 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus every solution is a linear combination of $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$; take

$$\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

7. (avg: 7.6) Find all values of h such that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans $\mathbb{R}^3$ if

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ h \\ 4 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Solution: need to find all values of h such that *any* vector in $\mathbb{R}^3$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. That is, the vector equation

$$x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + x_3\,\mathbf{a}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

must be consistent for every choice of a, b, c. Reduce the corresponding augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 1 & h & 0 & b \\ -1 & 4 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & h-3 & -1 & b-a \\ 0 & 7 & 2 & c+a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 7 & 2 & c+a \\ 0 & h-3 & -1 & b-a \end{array}\right]$$

If $h = 3$ this matrix corresponds to a triangular system, which is always consistent. Now proceed with $h \neq 3$:

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 7 & 2 & c+a \\ 0 & h-3 & -1 & b-a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 7 & 2 & c+a \\ 0 & 0 & 2h+1 & (h-3)(c+a)-7(b-a) \end{array}\right].$$

As long as $2h+1 \neq 0$ this matrix corresponds to a triangular system which has a solution regardless of a, b, c. Thus: the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans $\mathbb{R}^3$ if

$$h \neq -\frac{1}{2}.$$

If $h = -\frac{1}{2}$, then the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 7 & 2 & c+a \\ 0 & 0 & 0 & (-7/2)(c+a)-7(b-a) \end{array}\right],$$

which represents an inconsistent system if (say) $a = -1, b = 0, c = -1$; that is, the bottom right hand corner is $7 \neq 0$. Answer: $h \neq -\frac{1}{2}$.

8. (avg: 6.5) Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

   (a) (2 marks) If a matrix has more rows than columns and is in echelon form then it must have at least one row of zeros at the bottom. ⊗ **True** ○ **False**

   Solution: let the number of rows of the matrix be n, let the number of columns be m. Then $n > m$, and the number of leading entries in the echelon form of the matrix can be at most m, ie one per column. So there are at least $n - m$ rows with no leading entries; they are zero rows.

   (b) (2 marks) For every matrix A with columns that span $\mathbb{R}^n$, $A\,\mathbf{x} = \mathbf{0}$ has non-trivial solutions. ○ **True** ⊗ **False**

   Solution: not true for $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (c) (2 marks) $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \right\}$. ○ **True** ⊗ **False**

   Solution: look at the middle components. $s \cdot 0 + t \cdot 0 = 0 \neq 3$

   (d) (2 marks) Every system of linear equations with more variables than equations must have infinitely many solutions. ○ **True** ⊗ **False**

   Solution: the system of two equations $x_1 + x_2 + x_3 = 1$ and $x_1 + x_2 + x_3 = 2$ in three variables has no solutions, because it is obviously inconsistent.

   (e) (2 marks) If a system of linear equations has the trivial solution then it must be a homogeneous system of equations. ⊗ **True** ○ **False**

   Solution: $A\,\mathbf{x} = \mathbf{b}$ and $\mathbf{x} = \mathbf{0}$ together imply $\mathbf{b} = \mathbf{0}$, since $A\,\mathbf{0} = \mathbf{0}$.

 The end.