Duration: 150 minutes **Wednesday December 14, 2022**

Faculty of Applied Science & Engineering
University of Toronto

MAT186 Calculus I
Final Exam

Full Name: ______________________________

Student number: ______________________________

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Signature: ______________________________

Instructions:

1. This test contains a total of 16 pages.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this exam, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. Suppose $f(x)$ is differentiable at $x = 0$. Which of the following statements must be true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ $f(0)$ is defined.
- ○ $\lim_{x \to 0} f(x)$ exists.
- ○ $f(x)$ is continuous at $x = 0$.
- ○ $f'(x)$ is differentiable at $x = 0$.

2. If $f(x) = \ln(2x + 3)$, then $(f^{-1})'(1) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $\dfrac{2\ln 5 + 3}{2}$
- ○ $\dfrac{5}{2}$
- ○ $\dfrac{2}{e}$
- ○ $\dfrac{e}{2}$

Multiple Choice: No justification is required. Only your final answer will be graded.

3. As an airplane takes off, the weight of the airplane decreases as its altitude increases. Let $f(x)$ be the weight of the plane in kilograms, at an altitude of x kilometres. Which of the statements below is a correct interpretation of $(f^{-1})'(2000)$? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ For a plane weighing 2000 kg, the exact amount the plane's altitude must change for its weight to increase by 1 kilogram. The quantity $(f^{-1})'(2000)$ is negative.
- ○ For a plane weighing 2000 kg, the approximate amount the plane's altitude must change for its weight to increase by 1 kilogram. The quantity $(f^{-1})'(2000)$ is negative.
- ○ For a plane weighing 2000 kg, the exact amount the plane's altitude must change for its weight to increase by 1 kilogram. The quantity $(f^{-1})'(2000)$ is positive.
- ○ For a plane weighing 2000 kg, the approximate amount the plane's altitude must change for its weight to increase by 1 kilogram. The quantity $(f^{-1})'(2000)$ is positive.

4. Consider the initial value problem

$$\begin{cases} P'(t) = P(P-1)(P-2)^2(P-3) \\ P(0) = 2.5 \end{cases}$$

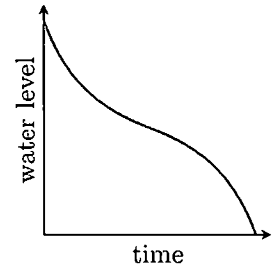
If $P(t)$ solves this initial value problem, then $\lim_{t\to\infty} P(t) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ 1
- ○ 2
- ○ 3
- ○ DNE ($\to +\infty$)

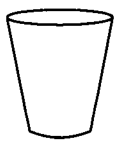
Multiple Choice: No justification is required. Only your final answer will be graded.

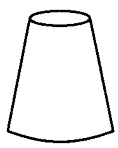
5. A water tank is initially full and the water is pumped out at a constant rate in litres per minute. The graph on the right shows the change of the water level inside the tank over time. Which of the following water tanks is used?

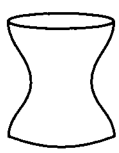


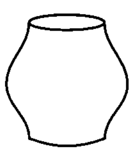
[1 mark]

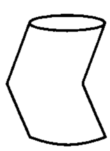
Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ 

○ 

○ 

○ 

○ 

Multiple Choice: No justification is required. Only your final answer will be graded.

6. A deer is walking in a forest. The table below summarizes the deer's walking speed over a six second interval.

time (seconds)	0	1	2	3	4	5	6
speed (metres per second)	0.1	0.7	0.9	1.1	0.8	0.6	0.5

Which of the following represents a right-endpoint Riemann sum approximation of the deer's distance travelled over the first six seconds. [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

○ 0.1+0.7+0.9+1.1+0.8+0.6

○ 0.1+0.7+0.9+1.1+0.8+0.6+0.5

○ 2(0.9+0.8+0.5)

○ 2(0.7+1.1+0.6)

○ 3(1.1+0.5)

7. Which of the following definite integrals is always overestimated with a right-endpoint Riemann sum? That is, a right-endpoint Riemann sum will always give a greater value than the definite integral. [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

○ $\int_0^1 2x\,dx$

○ $\int_0^1 1-2x\,dx$

○ $\int_{-2}^{-1} \frac{1}{2x}\,dx$

○ $\int_0^1 2\,dx$

○ $\int_0^1 e^{x^2}\,dx$

Multiple Choice: No justification is required. Only your final answer will be graded.

8. Which of the following sum of definite integrals represents the shaded area enclosed by the functions

$$f(y) = (y-1)^2, \qquad g(x) = x - 1 \quad \text{and} \quad h(x) = (x-2)^2 + 1.$$

pictured below. [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

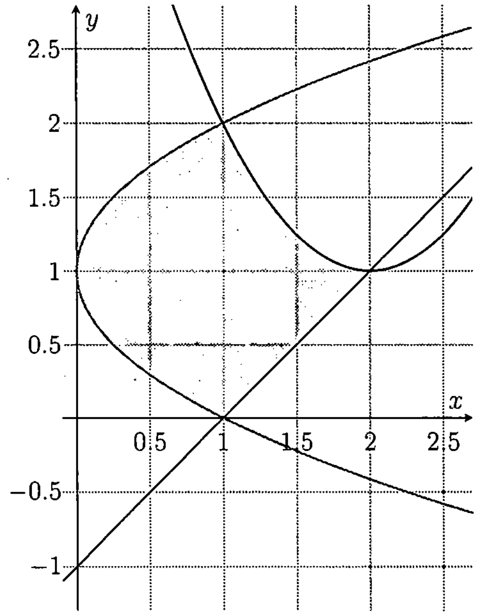
○ $\int_0^1 (\sqrt{x}+1) - (-\sqrt{x}+1)\,dx + \int_1^2 (x-1) - ((x-2)^2+1)\,dx$

○ $\int_0^1 (-\sqrt{x}+1) - (\sqrt{x}+1)\,dx + \int_1^2 (x-2)^2 + 1 - (x-1)\,dx$

○ $\int_0^1 (\sqrt{x}+1) - (-\sqrt{x}+1)\,dx + \int_1^2 (x-2)^2 + 1 - (x-1)\,dx$

○ $\int_0^1 (y+1) - (y-1)^2\,dy + \int_1^2 2 - \sqrt{y-1} - (y-1)^2\,dy$

○ $\int_0^1 (y+1) - (y-1)^2\,dy + \int_1^2 2 + \sqrt{y-1} - (y-1)^2\,dy$



Multiple Choice: No justification is required. Only your final answer will be graded.

9. Let $\rho(x)$ represent the traffic density on College street, measured in cars per kilometres, where x is the number of kilometres west of the intersection of College street and Spadina avenue. Which of the following is a correct interpretation of $\int_0^1 \rho(x)\,dx$? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ The total traffic density on College street from Spadina avenue to a point 1 kilometre west of Spadina avenue.
- ○ The total change in traffic density on College street from Spadina avenue to a point 1 kilometre west of Spadina avenue.
- ○ The total number of cars on College street from Spadina avenue to a point 1 kilometre west of Spadina avenue.
- ○ None of the above.

10. The area of a circle changes as a function of its radius, r, and its radius changes as a function of time $r = g(t)$. If $\frac{dA}{dr} = f(r)$, then the total change in area between $t = 0$ and $t = 1$ is = ______? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ $\int_{\pi(g(0))^2}^{\pi(g(1))^2} dA$
- ○ $\int_{g(0)}^{g(1)} f(r)\,dr$
- ○ $\int_0^1 f(g(t))g'(t)\,dt$
- ○ None of the above.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the boxes provided.

11. Let $f(x)$ represent the fuel efficiency, in kilometres per litres, of a car travelling at a speed of x kilometres per hour.

(a) What are the units of $f'(x)$? No justification is required for this part. [1 mark]

(b) Suppose $f'(99) = -0.35$. If you are driving at 99 kilometres per hour, should you speed up or slow down to be more fuel efficient? Briefly explain. Enter either "speed up" or "slow down" in the box provided as your final answer.
[3 marks: 1 mark for correct answer; 2 marks for explanation]

Answer:

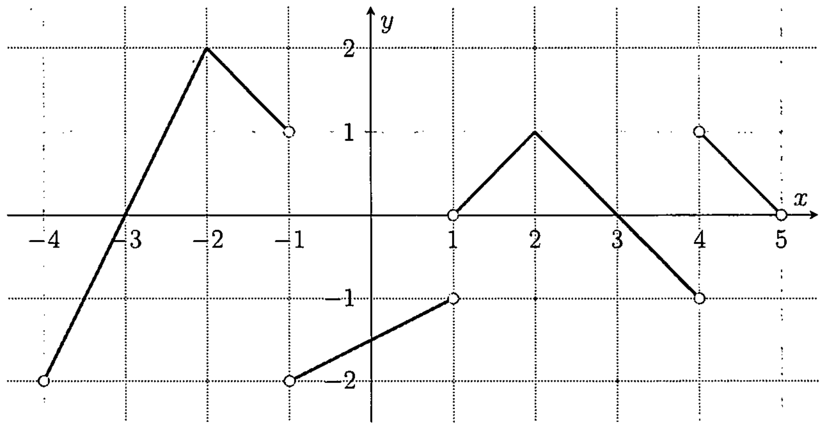
Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

12. Water leaking onto your bedroom floor creates a circular pool with an area that increases at the rate of 3 square centimetres per minute. How fast is the radius of the pool increasing when the radius is 10 centimetres? [5 marks]

Answer:

Short Answer: No justification is required. Only your final answer will be graded. Put your final answer in the boxes provided.

13. The graph of a function $f(x)$ is pictured below.



Let $F(x)$ be an antiderivative of $f(x)$, whose domain is the same as that of $f(x)$.

(a) Determine the interval(s) where $F(x)$ is strictly increasing and strictly decreasing. [1 mark]

$F(x)$ is strictly increasing on:

$F(x)$ is strictly decreasing on:

(b) Determine the point(s) at which $F(x)$ has a local maximum. [1 mark]

$F(x)$ has local maximum at $x =$

(c) Determine the interval(s) where $F(x)$ is concave up and concave down. [1 mark]

$F(x)$ is concave up on:

$F(x)$ is concave down on:

(d) Determine the inflection point(s) of $F(x)$. [1 mark]

$F(x)$ has a point of inflection at $x =$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

14. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$. The goal of this problem is to evaluate I without using antiderivatives of any trigonometric functions.

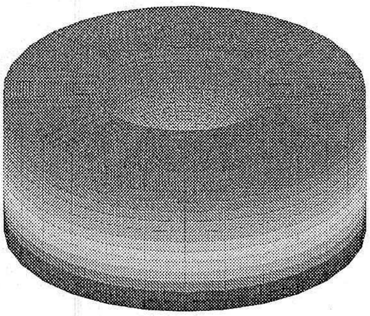
(a) Using the substitution $u = \dfrac{\pi}{2} - x$, show that I can be rewritten as $I = \displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$. Do not evaluate the integrals using antiderivatives. [2 marks]

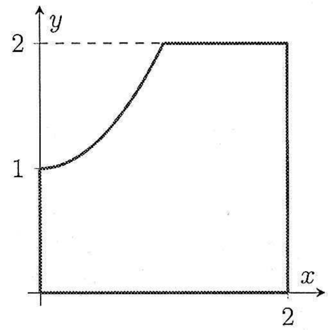
Hint: $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$.

(b) Using part (a), show that $I = \dfrac{\pi}{4}$, without using antiderivatives of any trigonometric functions. [2 marks]

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

15. A company hires you to solve the following problem. They would like to manufacture soap dishes with the shape shown in the figure below on the left. They require the base to be circular with a radius of 2 cm and that its total volume be 7.5π cm^3. You very cleverly notice that you can model a cross-section of the soap dish using the following functions, $f_m(x) = x^m + 1$ and $y = 2$.





(a) Write a sum of definite integrals that represents the volume of the solid obtained by rotating the area bounded by $f_m(x) = x^m + 1$, $y = 2$, $x = 2$ and both axes, pictured above on the right, about the y-axis. [3 marks]

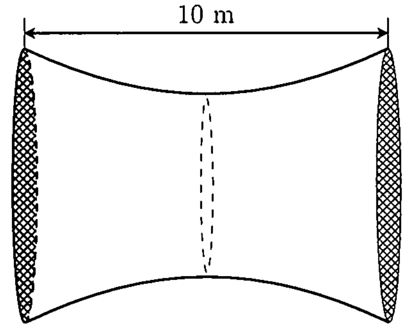
(b) Find the suitable value of $m \in \mathbb{R}$ to obtain the requested volume of 7.5π cm^3. [2 marks]

$m =$ $\boxed{}$

16. Consider a log of wood given in the figure below with length 10 metres. The cross-section x metres from the leftmost point of the log is a circle with radius $r(x) = 2 + 0.1(x-5)^2$ metres, where $x \in [0, 10]$. There is moss growing on the entire surface of the log *except* on the shaded portions at the endpoints. You would like to approximate the mass of moss growing on the log.

Suppose that the density of the moss growth, x metres from the leftmost point of the log, is given by $\rho(x) = 100(5x+3)$ grams per squared metre, for $x \in [0, 10]$. That is, the more you move to the right on the log, the more dense the moss growth is.

Write a Riemann sum approximation, with n subintervals, modelling the mass of moss growing on the log. Make sure to state your assumptions and explain all your steps. You are encouraged to annotate the figure to support your explanation. [5 marks]



16. EXTRA PAGE FOR Q16 ONLY, if needed.

IF NEEDED, USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

If you wish to have this page marked, make sure to refer to it in your original solution.

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Formula Sheet for MAT186 Final Exam

Trigonometric Formulas:

- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin x = \cos\left(\frac{\pi}{2} - x\right)$
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Logarithms and Exponentials:

- $\log_a a^x = x$
- $a^{\log_a x} = x$
- $a^{b+c} = a^b a^c$
- $\log_a bc = \log_a b + \log_a c$
- $(a^b)^c = a^{bc}$
- $\log_a b^c = c \log_a b$

Limits:

- $\lim\limits_{h \to 0} (1 + ah)^{\frac{1}{h}} = e^a$
- $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x = e^a$
- $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$