University of Toronto
Faculty of Applied Science and Engineering
Final Examination, April 2019
Duration: 2 and 1/2 hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

Examiner: D. Burbulla

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

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Signature: ______________________________

Instructions:

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 12 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of eight questions, some with many parts. Attempt all of them. Marks for parts of a question are indicated in parentheses beside the question. **Total Marks: 80**
- Do not approximate your answers unless specifically instructed to do so.
- Present your solutions in the space provided; give full and complete explanations! You can use pages 10, 11 and 12 for rough work. If you want anything on pages 10, 11 or 12 to be marked you must indicate in the relevant previous question that the solution continues on page 10, 11 or 12.

1. [10 marks] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & -2 \\ 3 & 5 & 8 & 2 & 4 \\ 6 & 11 & 17 & 5 & -2 \\ -2 & 4 & 2 & 6 & -5 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the rank of A and a basis for each of $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$.

2. [10 marks] For this question, let A be a 4×7 matrix.

   (a) [5 marks] Find all possible values of $\dim(\text{im}(A))$ and the corresponding values of $\dim(\text{null}(A))$.

   (b) [5 marks; 1 mark for each part] Now suppose the rank of A is 4. Find the value of the following:

   (*i*) $\dim(\text{col}(A))$

   (*ii*) $\dim(\text{row}(A))$

   (*iii*) $\dim\left((\text{null}(A))^{\perp}\right)$

   (*iv*) $\dim\left(\text{null}(A^T)\right)$

   (*v*) $\dim\left(\text{null}(AA^T)\right)$

3. [14 marks] If A is an $m \times n$ matrix it can be proved that there is a unique $n \times m$ matrix A^* (called the *generalized inverse* of A) satisfying the following four conditions:

$$A\,A^*\,A = A,\ A^*\,A\,A^* = A^*,\ (AA^*)^T = AA^* \text{ and } (A^*A)^T = A^*A.$$

(a) [6 marks] Verify that the generalized inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is $A^* = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}$.

(b) [2 marks] Show that if A is an $n \times n$ invertible matrix, then $A^* = A^{-1}$.

(c) [2 marks] Show that if A is an $n \times n$ symmetric matrix such that $A^2 = A$, then $A^* = A$.

(d) [4 marks] Show that if $\vec{b} \in \mathbb{R}^m$ and $A\,A^*\vec{b} = \vec{b}$, then for any vector $\vec{y} \in \mathbb{R}^n$, $\vec{x} = A^*\vec{b} + (I - A^*A)\,\vec{y}$ is a solution to the matrix equation $A\,\vec{x} = \vec{b}$.

4. [6 marks] Find the least squares approximating line $y = z_0 + z_1\,x$ for the data points $(2,4), (4,3), (7,2), (8,1)$.

5. [10 marks] Prove the following:

   (a) [3 marks] If A and B are $m \times n$ matrices and $U = \{\vec{x} \in \mathbb{R}^n \mid A\,\vec{x} = B\,\vec{x}\}$, then U is a subspace of $\mathbb{R}^n$.

   (b) [4 marks] Eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

   (c) [3 marks] $\lambda = 0$ is an eigenvalue of the $n \times n$ matrix A if and only if A is not invertible.

6. [10 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}.$$

7. [10 marks] Let $U = \text{span}\left\{\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T\right\}$.

   (a) [5 marks] Find an orthogonal basis of U.

   (b) [5 marks] Let $\vec{x} = \begin{bmatrix} -2 & 1 & 0 & 3 \end{bmatrix}^T$. Find $\text{proj}_U(\vec{x})$.

 Continued...

8. [10 marks] Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation with matrix $A = \begin{bmatrix} \vec{e}_n & \vec{e}_{n-1} & \cdots & \vec{e}_2 & \vec{e}_1 \end{bmatrix}$, where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{n-1}, \vec{e}_n$ are the standard basis vectors of $\mathbb{R}^n$.

(a) [2 marks] If $\vec{x}$ is in $\mathbb{R}^n$, what is $T(\vec{x})$?

(b) [2 marks] What is A^2?

(c) [3 marks] Show that if λ is an eigenvalue of A then $\lambda = \pm 1$.

(d) [3 marks] Find the characteristic polynomial of A for $n \geq 2$.

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