UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, April 15, 2011

DURATION: 150 mins.

First Year – Engineering

MAT188T – Applied Linear Algebra

Examiner: G. Simpson

Exam Type: A   Calculator Type: 3

Total Marks: 100   This exam has 4 pages

INSTRUCTIONS:

There are ten (10) problems.
Write your answers in the exam booklet. Show all work.
Approved calculators are permitted. No other aids are permitted.

1. **10 Marks:** True or False (No Justification Required):

   (a) If A is a 3×3 matrix, then
   $$\det(17A) = 17^3 \det(A)$$

   (b)
   $$\text{If } A = \begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}, \quad \text{then } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \in \text{null}(A)$$

   (c) The reduced row echelon form of a matrix is unique

   (d) The set
   $$\{[r,\ -2,\ s,\ t]^T \mid r, s, t \in \mathbb{R}\}$$
   is not a subspace of $\mathbb{R}^4$

   (e) The matrix
   $$\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$
   is diagonalizable

   (f) If P is an $n \times n$ orthogonal matrix, then $\lambda = 2$ cannot be an eigenvalue of P

   (g) It is possible for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to map the unit square to a pentagon

   (h) If the rank of the $m \times n$ matrix A is m, then $AX = B$ always has a solution

   (i) If the rank of the $m \times n$ matrix A is n, then $AX = B$ always has a solution

   (j) If A is an $n \times n$ matrix such that $A^2 = 0$, then $\det(A) = 0$

2. **14 Marks:**

   (a) Find an equation for the plane passing through the points $A(0, -2, 1)$, $B(1, -1, -2)$ and $C(-1, 1, 0)$.

   (b) Find the distance from this plane to the point $P(1, 1, 1)$.

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3. **12 Marks:** Compute an orthogonal basis for the span of the three vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$$

4. **14 Marks:** Consider the matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Given that A row reduces to

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find bases for null(A) and col(A)

(b) What are the dimensions of null$(A)^{\perp}$ and im$(A)^{\perp}$?

5. **12 Marks:** Find a vector in the subspace U closest to X, where

$$U = \text{span}\left\{[-2, 1, 3]^T, [3, 0, 1]^T\right\}, \quad X = [1 - 1, 1]^T$$

6. **12 Marks:** Let

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 13 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

(a) Without making any computations, why is A diagonalizable?

(b) Find an orthogonal matrix P such that $P^T AP$ is diagonal.

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7. **10 Marks:** Assume the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}4\\-2\end{bmatrix}, \quad T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}-1\\-19\end{bmatrix}$$

Find a matrix A for T.

8. **6 Marks:**

(a) Given an $n \times n$ square matrix A, assume that $A^3 = 0$. Prove

$$(I - A)^{-1} = I + A + A^2$$

(b) For an $n \times n$ square matrix $A \neq 0$, assume that $A^n = 0$ for some $n = 1, 2, \ldots$ Find an expression for $(I - A)^{-1}$. Justify your answer.

9. **5 Marks:** Find a solution to the system of differential equations

$$\frac{df_1}{dx} = f_1 + f_2$$
$$\frac{df_2}{dx} = 4f_1 - 2f_2$$

satisfying the conditions $f_1(0) = 1$ and $f_2(0) = 6$.

10. **5 Marks:** Find the least squares approximating function of the form

$$y = A\cos(x) + B\sin(x)$$

for the data pairs, (x, y),

$$(0, 1), \quad (\pi/2, 2), \quad (\pi, 1/2), \quad (2\pi, -1/4)$$

END OF EXAM