University of Toronto
Faculty of Applied Sciences and Engineering

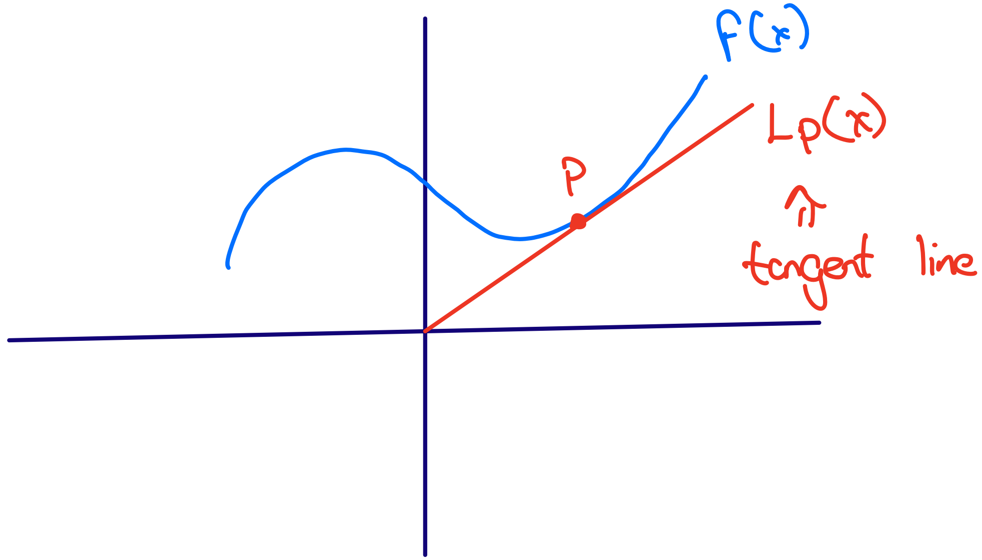
MAT187 - Summer 2025

Lecture 5

Instructor: Arman Pannu

We will start 10 minutes past the hour. Use this time to make a new friend.

Derivative and Linear Appoximations



→ $f'(x)$ is slope of tangent (or rate of change)

$$\underbrace{L_p(x) = f'(p)(x-p) + f(p)}_{\text{tangent line goes through point } (p, f(p))}$$

→ derivative is a tool to compute a linear approximation for $f(x)$

→ closer we are to point p, the better the approximation (tangent line is local approx)

Taylor Polynomials

→ can we approximate $f(x)$ with a higher degree polynomial?

$$f(x) \stackrel{?}{=} a_0 + a_1(x-p) + a_2(x-p)^2 + \dots + a_n(x-p)^n$$ ⇐ match derivatives to compute coeff

⇖ plug-in $x=p$

$$\boxed{f(p) = a_0} + \underbrace{a_1(p-p)}_{=0} + \dots$$

$$\Rightarrow f'(x) = a_1 + 2a_2(x-p) + \dots + n a_n (x-p)^{n-1}$$

⇖ plug-in $x=p$

$$\boxed{f'(p) = a_1}$$

$$\Rightarrow f''(x) = 2a_2 + \dots + n(n-1)(x-p)^{n-2}$$

$$f''(p) = 2a_2 \Rightarrow \boxed{a_2 = \frac{1}{2} f''(p)}$$

$$\vdots$$

$$\Rightarrow f^{(n)}(x) = n(n-1)\dots(1)\, a_n$$

$$f^{(n)}(p) = n!\, a_n \Rightarrow \boxed{a_n = \frac{1}{n!} f^{(n)}(p)}$$

Def'n: Given function f the n'th Taylor polynomial centered at $x=p$ is:

$$T_{n,p}(x) = f(p) + f'(p)(x-p) + \frac{f''(p)}{2}(x-p)^2 + \dots + \frac{f^{(n)}(p)}{n!}(x-p)^n$$

→ the taylor polynomial is a n^{th} degree approximation of $f(x)$

→ approximation is better the closer we are to $x=p$ (local approximation)

→ taylor poly. matches derivatives of f at $x=p$

ex// ① $f(x) = 3x^2 + 2x + 1$ centered at $x=0$

$T_{0,0}(x) = \underline{1}$ ⇐ best <u>constant</u> approx near $x=0$

$T_{1,0}(x) = 2x+1$ ⇐ best <u>linear</u> approx near $x=0$

$T_{2,0}(x) = 3x^2+2x+1 = f(x)$

$T_{3,0}(x) = 0x^3 + 3x^2 + 2x + 1 = f(x)$

② $f(x) = e^x$ centered at $x=0$

$$T_{n,0} = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$\vdots \qquad \vdots$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$

$$\boxed{T_{n,0}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n}$$

$e^x \approx 1 + x$ ⇐ linear approx

$e^x = 1 + x + \frac{1}{2}x^2$ ⇐ quadratic

} more accurate for small x $(x \to 0)$

$f(x) = 3x^2 + 2x + 1$

$$T_{2,0}(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$f(0) = 3(0)^2 + 2(0) + 1 = \boxed{1}$

$f'(x) = 6x + 2 \Rightarrow \boxed{f'(0) = 2}$

$f''(x) = 6 \qquad \boxed{f''(0) = 6}$

Find the Taylor polynomials of $\sin(x)$ centered at 0 up to the 4th degree.

$$
\begin{aligned}
f(x) &= \sin(x) & \qquad f(0) &= 0 \\
f'(x) &= \cos(x) & f'(0) &= 1 \\
f''(x) &= -\sin(x) \quad \Longrightarrow & f''(0) &= 0 \\
f'''(x) &= -\cos(x) & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin(x) & f^{(4)}(0) &= 0 \\
& & \vdots \quad &= 1 \\
& & &= 0 \\
& & &= -1 \\
& & &\ \vdots
\end{aligned}
$$

$$T_{1,0}(x) = f(0) + \frac{f'(0)}{1!}(x-0) \boxed{= x}$$

$\sin(x) \approx x$ for small x

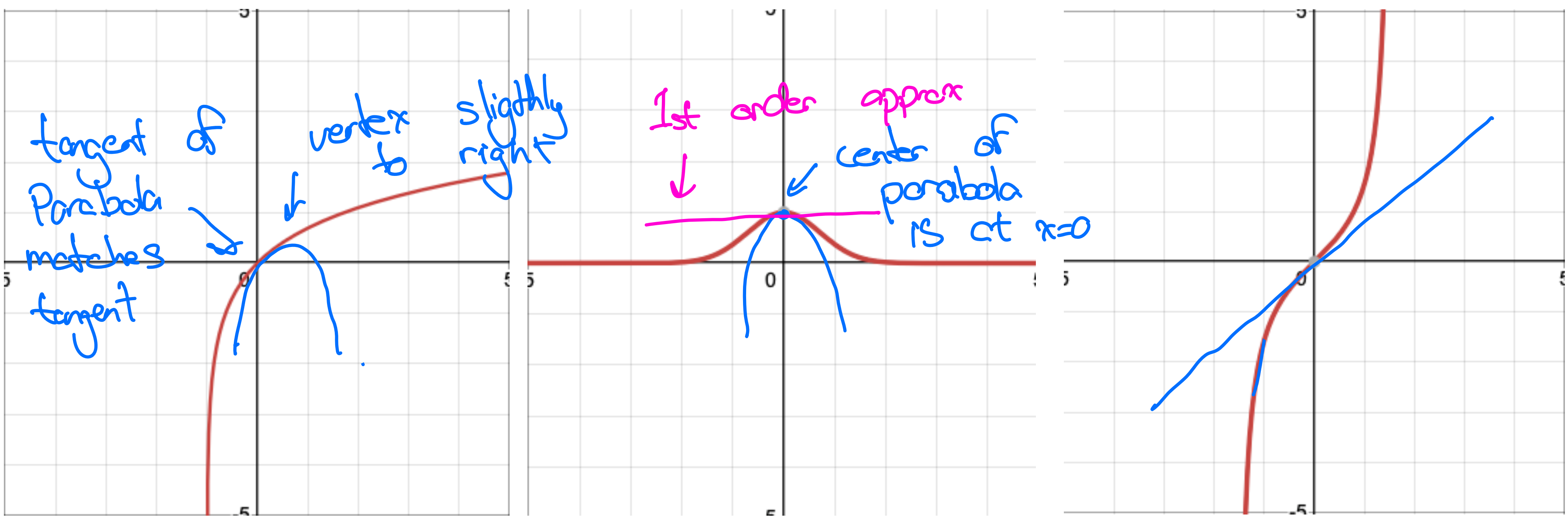
$$T_{2,0}(x) = \quad \ldots \quad = x$$

$$T_{3,0}(x) = \quad \ldots \quad = x - \frac{x^3}{3!}$$

$$T_{4,0}(x) = \quad \ldots \quad = x - \frac{x^3}{3!}$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \ldots$$

Without computing, sketch the 2nd order Taylor polynomial centered at 0 for the functions $\ln(1+x)$, e^{-x^2} and $tan(x)$



inflection point at zero so 2nd taylor polynomial is most likely linear

Taylor's Remainder Theorem

Let $R_k(x) = f(x) - T_{k,a}(x)$ $\Leftarrow$ error / remainder term

$$\boxed{R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}} \quad \text{where } c \in [a, x]$$

→ when $x \to a$ $(x-a)^{k+1} \to 0$ faster for larger k

⇒ error term smaller for larger k

⇒ larger order Taylor polynomials generally are better approx

→ for each x, there is different $c \in [a, x]$ which makes equation true

→ use an upper bound for $f^{(k+1)}(c)$ independent of c

When using the third degree Taylor polynomial to approximate $\sin(0.2)$, what is the maximum error?

$$\sin(x) \approx x - \frac{1}{3!}x^3$$

$$\sin(0.2) \approx 0.2 - \frac{1}{6}(0.2)^3 \approx 0.198666\ldots$$

$$R_3(x) = \frac{f^{(4)}(c)}{4!}(0.2-0)^4 \qquad c \in [0, 0.2]$$

$$= \frac{\sin(c)}{4!}(0.2)^4 \qquad \Leftarrow \quad -1 \leq \sin(c) \leq 1$$

$$|R_3(x)| \leq \frac{(0.2)^4}{4!} = 0.000066\ldots$$