University of Toronto, Faculty of Applied Science and Engineering

MAT187H1S – Calculus II – Final Exam Individual Part - April 16, 2019

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Time allotted: 100 minutes | No aids permitted

Total marks: 80

Instructions:

- Do not write on the QR code at the top of the pages.
- This test contains 16 pages and a detached booklet for multiple-choice questions and formula sheet. DO NOT DETACH ANY PAGES.
- You can use pages 12–15 to complete questions (**mark clearly** which questions you are answering).
- Calculators, cellphones, or any other electronic devices are not allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.
- DO NOT START the test until instructed to do so.

GOOD LUCK!

LONG ANSWER PART

11. You work for a roller coaster construction company and Canada's Wonderland is asking you to design a new ride. Your job is to design a specific part of the ride. **(21 marks)**

The trajectory should follow the curve given by

$$\vec{r}(t) = \left\langle \quad b\int_0^t \cos(u^2)\,du \quad , \quad ct \quad , \quad b\int_0^t \sin(u^2)\,du \quad \right\rangle \qquad \text{in metres,}$$

for $0 \leqslant t \leqslant 10$ seconds.

Consider parts (b)–(d) as independent.

(a) **(1 mark)** Show that $\vec{r}\,'(t) = \langle b\cos(t^2), c, b\sin(t^2)\rangle$.

(b) **(6 marks)** Riders should never exceed a speed of 50 m/s. What are the restrictions on b and c? Your answer should not include t. Justify your answer.

(c) **(6 marks)** Taking safety and durability into account, the curvature should not exceed $\frac{1}{10}$ m^{-1}. What are the restrictions on b and c? Your answer should not include t. Justify your answer.

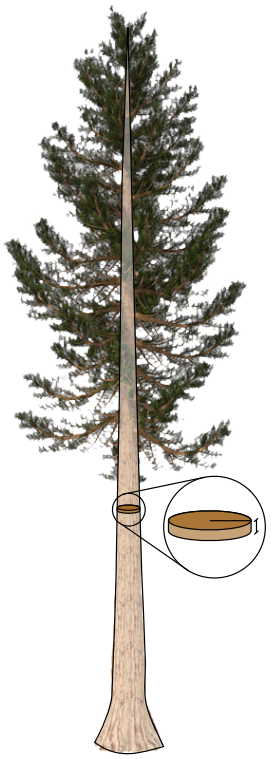
Note. For this question, do not use the restrictions that you found in (b).

(d) **(6 marks)** The roller coaster should exert a maximum acceleration of 40 m/s^2 in the normal direction. What are the restrictions on b and c? Your answer should not include t. Justify your answer.

Note. For this question, do not use the restrictions that you found in (b) and (c).

(e) **(2 marks)** Give one possible pair of values $b \neq 0$ and $c \neq 0$ that fulfil all the above design requirements. Your answer should not include t. Justify your answer.

12. A scientist wants to study the growth of the trunk of a Giant Sequoia. Assume the trunk has a circular cross section with radius $r(y)$ at the height of y and the total height of the tree is H (all units in metres). **(19 marks)**



(a) (7 marks) Assume that a Giant Sequoia's trunk is 50% water, uniformly distributed throughout the trunk.

Show that the work it took for the tree to bring water from the ground to fill the whole trunk was

$$W = \frac{\pi \rho g}{2} \int_0^H y\, r^2(y)\, dy,$$

where ρ is the mass density of water (in kg/m^3) and g is the acceleration due to gravity.

Your answer should not include just formulas, but also a step-by-step explanation about how the formulas were obtained.

Hint. Within a thin slice of the trunk as shown in the figure, work can be approximated as force times displacement.

(You can continue your answer to **(a)** on the next page)

You can continue your answer to **(a)** here.

(b) (6 marks) The scientist took some measurements of the tree:

- The tree is 18 m high.
- Measurements for the radius are given in the table on the right (all units are in metres).
- The function $yr^2(y)$ is increasing for $y \leqslant 2$.
- The function $yr^2(y)$ is decreasing for $y \geqslant 2$.

height y	radius r
0	2.0
2	1.3
4	0.8
6	0.56
8	0.43

height y	radius r
10	0.35
12	0.3
14	0.25
16	0.21
18	0.19

She wants to make sure to **overestimate** the work calculated in **(a)**.

Approximate as best as you can with the data available while making sure to overestimate it.

As before, don't give just formulas, but also **explain** all your decisions and procedures.

You don't have a calculator, so leave your answer unsimplified.

$W \approx$

(c) (6 marks) The scientist met with a statistician who analyzed the data and deduced that the shape of the tree is very well approximated by $r(y) = 3y^{-1.2}$ for $y \geqslant 2$.

Ignoring other physical constraints, can the tree grow infinitely tall and still need only a finite amount of work to bring water to the whole trunk? Justify your answer.

13. In this question we will study the continuity of a parameter of a second-order ordinary differential equation. **(20 marks)**

(a) (6 marks) Find the solution $x(t)$ of

$$\begin{cases} x''(t) - 2x'(t) + x(t) = 0 \\ x(0) = 0 \\ x'(0) = 1 \end{cases}$$

Justify your answer.

$x(t) =$

(b) (7 marks) Consider the initial-value problem

$$\begin{cases} x_h''(t) - 2x_h'(t) + (1-h^2)x_h(t) = 0 \\ x_h(0) = 0 \\ x_h'(0) = 1 \end{cases}$$

Assume $h > 0$. Find the solution $x_h(t)$. Justify your answer.

Hint. For this part only, you can assume $h > 0$ is a constant.

$x_h(t) =$

(c) (5 marks) Use your answer from **(b)** and find the limit $\lim_{h\to 0} x_h(t)$. Justify your answer.

$\lim_{h\to 0} x_h(t) =$

(d) (2 marks) Compare the results of questions **(a)** and **(c)**. What do you observe?

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