Duration: 100 minutes **Tuesday November 19, 2024**

MAT186 Calculus I
Universal Make-up Test

Full Name: Sol Utions ______________________________

Student number: ______________________________

Email : ______________________________@mail.utoronto.ca

Signature: ______________________________

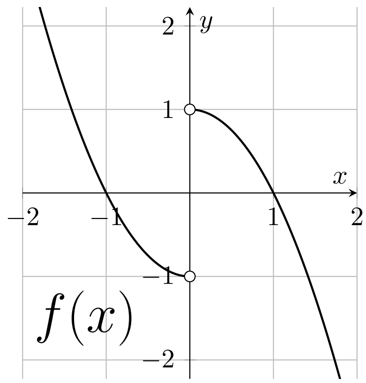
Instructions:

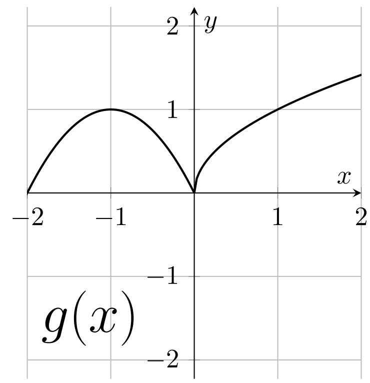
1. This test contains a total of 12 pages, and a total of 38 marks available.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this test, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

Use the following graphs to answer Question 1, Question 2, and Question 3 below.





1. $\lim_{x\to 0} g(x) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ -1.

● 0.

○ 1.

○ DNE.

Note that $g(x)$ is continuous at $x = 0$ and $g(0) = 0$.

2. $\lim_{x\to 0} f(g(x)) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ -1.

○ 0.

● 1.

○ DNE.

As x approaches 0, $g(x)$ approaches 0 from above. Therefore, $\lim_{x\to 0} f(g(x)) = \lim_{y\to 0^+} f(y) = 1$.

3. $\lim_{x\to 0} g(f(x)) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ -1.

○ 0.

● 1.

○ DNE.

As x approaches 0 from the left, $f(x)$ approaches -1 from above. Therefore, $\lim_{x\to 0^+} g(f(x)) = \lim_{y\to -1^+} g(y) = 1$.
As x approaches 0 from the right, $f(x)$ approaches 1 from below. Therefore, $\lim_{x\to 0^-} g(f(x)) = \lim_{y\to 1^-} g(y) = 1$. The limit exists since the left and right limits are equal.

Multiple Choice: No justification is required. Only your final answer will be graded.

4. Let $f(x) = a\ln(bx)$ where a is a positive constant and b is a negative constant. Which of the following statements are guaranteed to be true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ The domain of $f(x)$ is $(0, \infty)$.
- ● The graph of $f(x)$ has a vertical asymptote.
- ● $f^{-1}(0) = b^{-1}$.
- ○ $f'(x) = \dfrac{a}{bx}$.
- ● $f(x)$ is strictly decreasing on its domain.

- Since b is negative, the domain of f is $(-\infty, 0)$ for $\ln(bx)$ to be defined. The first choice is false.
- The second choice is true since f has a vertical asymptote at $x = 0$.
- Since $f(b^{-1}) = a\ln 1 = 0$ It follows that $f^{-1}(0) = b^{-1}$. The third choice is true.
- Since $f'(x) = \frac{a}{x}$, the fourth choice is false.
- Since $f'(x) < 0$ on its domain, we can conclude that f is strictly decreasing on its domain. Alternatively, we can reach the same conclusion from sketching the graph of f. Therefore, the fifth choice is true.

5. Let $f(x)$ be a strictly decreasing, differentiable function on $(-\infty, \infty)$. Suppose that $f(-3) = 2$, and $f(2) = -3$. Which of the following statements are guaranteed to be true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ $f(x) = -x - 1$.
- ● $\lim\limits_{x\to 2} f(x) = -3$.
- ○ $\lim\limits_{x\to\infty} f(x) \to -\infty$.
- ● There exists a value of x such that $f(x) = 0$.
- ● $f'(2) \leq 0$.

- The first choice is false. There are many functions with $f(-3) = 2$ and $f(2) = -3$.
- The second choice is true. All differentiable functions are also continuous.
- The third choice is false. Although f is decreasing, it may approach a finite limit as $x \to \infty$, if the graph of f has a horizontal asymptote.
- The fourth choice is true. Since f is continuous, this is guaranteed from the intermediate value theorem.
- The fifth choice is true. If f is strictly decreasing on its domain, then $f'(x) \leq 0$ on its domain.

Multiple Choice: No justification is required. Only your final answer will be graded.

6. The cross-sectional dimensions of a type of wooden board called a "2x4" are nominally 2 inches by 4 inches (hence the name). However, the actual dimensions of a 2x4 may be as small as 1.5 inches by 3.5 inches in practice. Assume that the dimensions are never larger than 2 inches by 4 inches.

Let E be the error of the the cross sectional area A of a 2x4. Which of the following statements are true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ● $5.25\ (\text{in})^2 \leq A \leq 8\ (\text{in})^2$
- ● $8 - A \leq 2.75$
- ○ $|A - 8| = 2.75$
- ○ $E = |A - 2.75|$.
- ● $E = |8 - A|$.
- ● $-2.75 \leq E \leq 2.75$

From the question, we are given that the area A of a board satisfies $5.25 \leq A \leq 8\ in^2$. Therefore, $8 - A \leq 2.75$. We are not guaranteed that $|A - 8| = 2.75$.

If E is the error in the area, the error is $E = |8 - A|$ since the true area of a board should be $8\ in^2$. Since $0 \leq 8 - A \leq 2.75$ from the problem, we also have $0 \leq E \leq 2.75$. This guarantees that the last choice is also satisfied as the error never exceeds -2.75 or 2.75.

7. Let $r(x)$ and $s(x)$ be functions defined on the interval $[1, 5]$ such that $1 \leq r(x) \leq 3$ and $x \leq s(x) \leq 2x$. Which of the values below are *guaranteed* to be upper bounds for $\dfrac{1}{r(x) + s(x)}$ on $[1, 5]$? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ● 1.
- ● $\frac{2}{3}$.
- ● $\frac{1}{2}$.
- ○ $\frac{1}{3}$.
- ○ $\frac{1}{5}$.
- ○ $\frac{1}{13}$.

From the inequalities given in the problem, and the domain of $[1, 5]$, we have

$$0 < x + 1 \leq r(x) + s(x) \leq 2x + 3.$$

Therefore, using properties of inequalities,

$$\frac{1}{r(x) + s(x)} \leq \frac{1}{x + 1}.$$

Since the domain is $[1, 5]$, $\frac{1}{x+1}$ is largest when $x = 1$, so we are guaranteed that

$$\frac{1}{r(x) + s(x)} \leq \frac{1}{2}$$

for all $x \in [1, 5]$. Therefore, any number greater than $\frac{1}{2}$ is an upper bound.

Short Answer: Organize your work in a reasonably neat and coherent way.

8. A population of deer was released on an uninhabited island in Northern Ontario in an effort to establish a permanent population of deer on the island. Over time, the population of deer increased. The table below summarizes the population by year.

Year	2020	2021	2022	2023	2024
Population	17	20	21	25	29

Let $P(t)$ give the population of deer on the island at time t measured in years after 2020.

(a) In one sentence, give a practical interpretation of $P(50)$ in terms of year and population of deer. [1 mark]

$P(50)$ is the number of deer on the island in 2070.

(b) In one sentence, give a practical interpretation of $P^{-1}(50)$ in terms of year and population of deer. [1 mark]

$P^{-1}(50)$ is the number of years needed for the population to grow to 50 deer.

(c) Use the table above to estimate $(P^{-1})'(25)$. Show your reasoning. [1 mark]

Recall that the derivative of the inverse $(P^{-1})'$ and P' are related by:

$$(P^{-1})'(25) = \frac{1}{P'(P^{-1}(25))}.$$

Since $P^{-1}(25) = 3$, we need to approximate $P'(3)$. From the table, we see that

$$P'(3) \approx P(4) - P(3) = 4.$$

Therefore, we can approximate

$$(P^{-1})'(25) \approx \frac{1}{4}.$$

(d) Using your answer from part (c), give a practical interpretation of $(P^{-1})'(25)$ in terms of year and population of deer. [1 mark]

When the deer population has 25 deer, it takes about 1/4 years (or 3 months) for the population to grow by one deer.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

9. Water leaking onto your bedroom floor creates a circular pool with an area that increases at the rate of 3 square centimetres per minute. How fast is the radius of the pool increasing when the radius is 10 centimetres? Be sure that you clearly define any variables (including units) that you introduce, and that you clearly identify what you are trying to solve for using mathematical notation. [3 marks]

Let r be the radius of the pool in cm. Its area A is $A = \pi r^2$ since it is circular.

By the chain rule, we conclude that

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Therefore, if $\frac{dA}{dt} = 3cm^2/min$ and $r = 10cm$, we conclude that

$$\frac{dr}{dt} = \frac{1}{2\pi r}\frac{dA}{dt} = \frac{3}{20\pi} cm/min$$

Final Answer (include units) $\boxed{\frac{3}{20\pi} cm/min}$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

10. Provide a careful argument using the Squeeze Theorem to evaluate $\lim_{x\to\infty} \dfrac{e^{-\sin x}}{x}$. [4 marks]

To use the Squeeze Theorem to evaluate the limit $\lim_{x\to a} f(x)$, we must identify two functions, $g(x)$ and $h(x)$ such that $g(x) \le f(x) \le h(x)$ in some interval around the point a, though not necessarily at a itself. Then, we must show that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x)$, such that $\lim_{x\to a} f(x)$ is "squeezed" between $f(x)$ and $g(x)$, and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = \lim_{x\to a} f(x)$.

In this case, we have $f(x) = \dfrac{e^{-\sin(x)}}{x}$, and $a \to \infty$. First, we identify appropriate bounding functions for $f(x)$. Consider:

$$-1 \le -\sin(x) \le 1 \implies \frac{1}{e} \le e^{-\sin(x)} \le e$$

Then, for $x \to +\infty$ $x > 0$, so

$$\frac{-1}{ex} \le \frac{e^{-\sin(x)}}{x} \le \frac{e}{x}$$

Taking $g(x) = \dfrac{-1}{ex}$ and $h(x) = \dfrac{e}{x}$, the next step is to evaluate the limits of $f(x)$, $g(x)$ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{-1}{ex} = 0$$
$$\lim_{x\to\infty} \frac{e}{x} = 0$$

Finally, since $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = 0$, and $g(x) \le f(x) \le h(x)$ for $x \to \infty$, we can conclude by the Squeeze Theorem that $\lim_{x\to a} f(x) = 0$.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

11. Show that there exists at least one solution $x \in (0, \infty)$ to the equation

$$x^{186} = x\sqrt{x^2 + \cos^2 x} + 32$$

[4 marks]

This question tests the use of the Intermediate Value Theorem. First, we re-arrange the equation and define a function such that

$$f(x) = x^{186} - x\sqrt{x^2 + cos^2(x)} - 32$$

and note that $f(x)$ is a continuous function over $x \in \Re$. Then we can reduce the problem to showing that $f(x) = 0$ at least once in its domain.

To do this, we search for locations $x = a$ such that $f(a) < 0$ and $x = b$ such that $f(b) > 0$. Once we have determined these points, we can use the IVT to conclude that there must exist some $x = c \in [a, b]$ such that $f(c) = 0$, as desired.

First, we notice that $f(0) = -32 < 0$, so we choose $a = 0$. Then, we notice that x^{186} very quickly dominates the behaviour of the function for $x > 1$. In fact, we can show that

$$\lim_{x\to\infty} x^{186} - x\sqrt{x^2 + cos^2(x)} - 32 \to \infty$$

Therefore, we can select some $b \gg 1$ such that $f(b) > 0$, and conclude by IVT that there must be at least one location $x = c$ where $f(c) = 0$, since $f(x)$ is continuous over the whole real line.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided

12. The force $F(h)$ due to gravity on an object at height h above sea level can be modelled by the initial value problem

$$\begin{cases} \dfrac{dF}{dh} = \dfrac{-2F}{R+h} \\ F(0) = -9.8m \end{cases}$$

where m is the mass of the object, and R is the radius of the earth.

(a) Use a linear approximation to estimate $F(h)$ near sea level. [3 marks]

Using the standard form for linear approximation, we seek an expression of the form

$$L(h) = \frac{dF}{dh}|_{x=0}(h-0) + F(0)$$

Substituting the expressions above, we obtain:

$$L(h) = \frac{-2F(0)}{R+0}h - 9.8m$$
$$L(h) = \frac{19.6}{R}h - 9.8m$$

Therefore, near sea level, $F(h) \approx \frac{19.6}{R}h - 9.8m$.

(b) Does your answer from part (a) give an underestimation or overestimation of the actual value of $F(h)$ near sea level? [3 marks]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ Underestimate.

● Overestimate.

To determine whether the linear approximation at $h = 0$ will be an overestimate or an underestimate, we consider the concavity of $F(h)$ at this point. Taking the derivative of $\frac{dF}{dh}$ we have

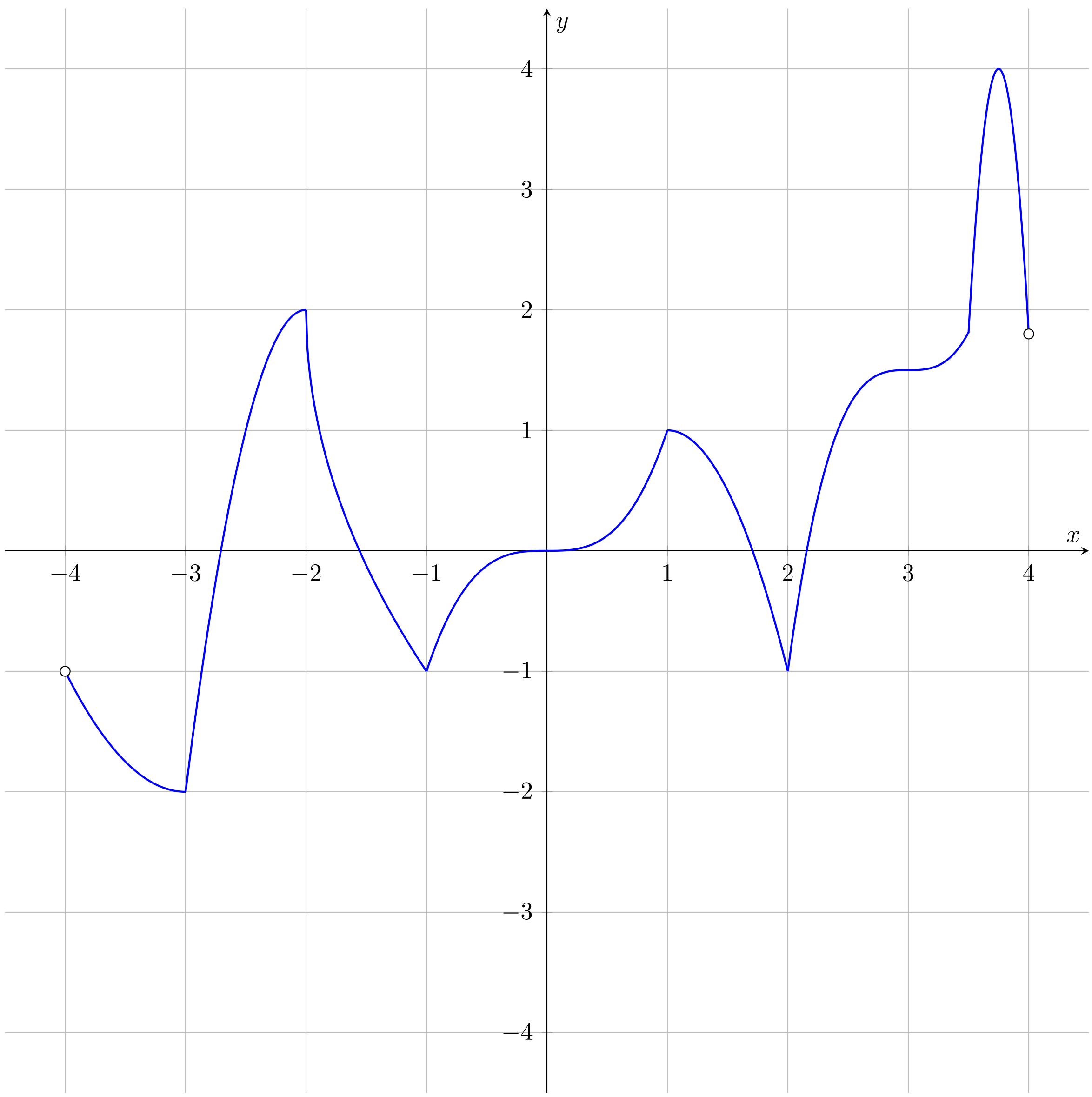
$$\frac{d^2F}{dh^2} = \frac{-2F'}{R+h} + \frac{2F}{(R+h)^2}$$

and evaluating at $h = 0$, we see that the second derivative is negative, therefore the function $F(h)$ is concave down, and the linear approximation is an overestimate of the value of the function in the region near $h = 0$.

Graphing: Only your final answer will be graded.

13. On the axes below, carefully sketch the graph of a function $f(x)$ whose domain is $(-4, 4)$, and that satisfies the properties below. [6 marks]

- $f(x)$ is continuous in its domain.
- $f(-2) = 2$ is a local maximum.
- $f'(-1)$ does not exist.
- $f(x)$ has a critical point at $x = 0$.
- $f(x)$ does not have a local extremum at $x = 0$.
- $f'(2)$ does not exist.
- $f(2) = -1$ is a local minimum.
- $f(x)$ has inflection points at $x = -1$ and $x = 3$.
- The range of $f(x)$ is $[-2, 4]$.



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If you wish to have this page marked, make sure to refer to it in your original solution.

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