Cooling Fins — Modules: E1-E4

For a body with total surface area A_s and uniform surface temperature T_s, the instantaneous rate of convective heat transfer out of the body is modeled by Newton's law of cooling:

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

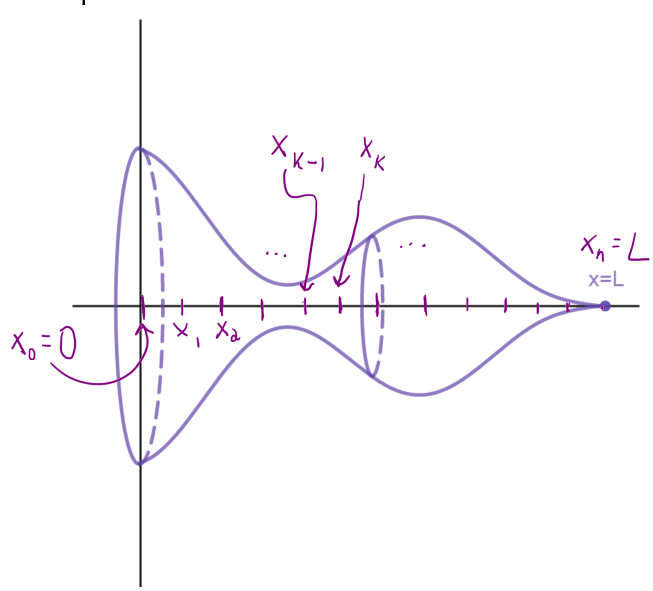
where h is a constant and T_∞ is the ambient temperature (temperature of the room). The units are as follows:

$$\begin{aligned} \dot{Q}_{\text{conv}} &: \text{Watts} \\ h &: \text{Watts/(meters squared} \cdot \text{degrees Kelvin)} \\ A_s &: \text{meters squared} \\ (T_s - T_\infty) &: \text{degrees Kelvin} \end{aligned}$$

What if we extended this model a bit? Say we constructed a cooling fin shaped as a solid of revolution. Let our fin be the solid created when the area between a function $r(x)$ and the x axis between $x = 0$ to $x = L$ is revolved around the x-axis. All distances are in meters. As a simplifying assumption, suppose the surface temperature of the fin $T_s = T_s(x)$ depends only on the horizontal distance from the origin (note this is no longer a uniform surface temperature). We want to find an expression for the instantaneous rate of convective heat transfer out of our entire cooling fin, $\dot{Q}_{\text{conv}}$.

1. Explain a plan to break this problem up into parts, providing the notation you will use.
2. Explain how you are approximating $\dot{Q}_{\text{conv}}$ on one of your individual parts. Be specific in explaining what is being approximated by what, and any choices you are making. Units are an important part of this explanation.
3. Add up your approximations to come up with an expression for the approximate value of $\dot{Q}_{\text{conv}}$ for the whole fin.
4. Use your answer to the previous part to explain how to obtain a definite integral formula for the $\dot{Q}_{\text{conv}}$ for the whole fin; use appropriate mathematical notation to relate the two.

Your answer to this problem will involve the functions $r(x)$ and $T_s(x)$ as well as the constants h, T_∞, and L. You should not assume $r(x)$ looks as pictured below.



1.) Divide the interval $0<x<L$ into n subintervals of width $\Delta x = L/n$.

On each subinterval, we will approximate the rate of convective heat transfer out of the surface area corresponding to that subinterval.

Given a subinterval $[x_{k-1}, x_k]$ we call the rate of convective heat transfer of the fin's surface area between $x = x_{k-1}$ and $x = x_k$ by the name $\dot{Q}_k$.

2.) To approximate $\dot{Q}_k$ we approximate the surface area ΔSA_k between x_{k-1} and x_k by employing the surface area differential $\pi\sqrt{1+(r'(x_k))^2}\,\Delta x \approx \Delta SA_k$
This approximation is known to be good for small values of Δx.

We also approximate the varying surface temperature $T_s(x)$ over this small bit of surface by taking the representative surface temperature $T_s(x_k)$ at the point x_k and supposing that the entire small bit of surface actually has this temperature. This approximation is good because the surface temperature can't vary by too much over a very small bit of surface.

Thus, the surface area between x_{k-1} and x_k is, by our simplifying assumptions, approximated by a body with surface area $\pi\sqrt{1+(r'(x_k))^2}\Delta x$ and uniform surface temperature $T_s(x_k)$.
Since this is exactly what we need to apply Newton's law of cooling, we can conclude that

$$\dot{Q}_k \approx h\pi\sqrt{1+(r'(x_k))^2}\cdot\left(T_s(x_k)-T_\infty\right)\cdot\Delta x$$

In this instance we know the units check out because the units check out in Newton's Law of Cooling.

3.) The heat energy (in Joules) leaving the fin due to convection each second is equal to the sum of the Joules of heat energy leaving each bit of surface area each second. Because of this we can add up each $\dot{Q}_k$ to obtain an approximation of $\dot{Q}_{conv}$, the total heat energy leaving the fin each second. Therefore

$$\dot{Q}_{conv} \approx \sum_{k=1}^{n} h\pi\sqrt{1+(r'(x_k))^2}\left(T_s(x_k)-T_\infty\right)\Delta x$$

4.) Because this approximation gets better and better as n, the number of subintervals, goes to infinity, we will find our definite integral formula for the $\dot{Q}_{conv}$ out of the whole fin by taking a limit as n goes to infinity.

$$\dot{Q}_{conv} = \lim_{n\to\infty}\sum_{k=1}^{n} h\pi\sqrt{1+(r'(x_k))^2}\left(T_s(x_k)-T_\infty\right)\Delta x$$

$$= \int_0^L h\pi\sqrt{1+(r'(x))^2}\left(T_s(x)-T_\infty\right)dx$$

This last step is justified by the limit definition of the definite integral.

We could also rewrite this by pulling constants out of the integral, if desired.

$$\boxed{\dot{Q}_{conv} = h\pi\int_0^L \sqrt{1+(r'(x))^2}\left(T_s(x)-T_\infty\right)dx}$$

On the assumption that the cooling fin is attached by its base at x=0 to the body it is trying to cool, and tapers to a point at x=L, we did not here compute the heat leaving either end. This assumption wasn't warranted by what was written in the prompt, so your response might be different. Since it doesn't require an integral to find those terms, we are just interested in this argument above.

	Clarity and conciseness of written exposition; consideration of audience	Good movement between English & Mathematics	Mathematical Thinking
Below expectations (0)	Writing is confusing, or shows signs of carelessness through high level of error; does not demonstrate application of engineering communication; only understandable to the writer.	Math is presented without context; it is difficult to reconstruct the meaning of equations.	No demonstration of any significant level of mathematical analysis at an appropriate level. No serious attempt to apply mathematics.
Shows potential (1)	Writing requires some effort to understand, or has some errors rarely affecting understanding; some evidence of communication principles but may be awkward or simplistic; offers sufficient information and demonstrates mostly logical development of ideas; generally directed to the instructor or TA.	While the writing is clear, the math is kept completely separate from it; requires some effort to understand how the math and writing are connected.	A serious attempt is made to apply appropriate math and explain mathematical reasoning, regardless of completeness or correctness.
Meets expectations (2)	Writing shows appropriate selection and balance of text and image (if necessary); demonstrates clarity at the paragraph, sentence, and word choice levels, with minimal error; writing applies principles in ways that enhance the reading experience; generally directed towards a fellow student who understands the math but not this particular instance.	The interaction of the text with the math is adequate; any transitions between math and text are appropriately placed and make sense; any equations and variables are introduced and explained in an organized manner; any solutions are interpreted in a real-world context where appropriate; no guesswork is needed by the reader.	The response shows a good ability to analyze a situation using math; relevant topics are applied with proper notation; any solutions are critically analyzed for sensibility; any obviously incorrect solutions are remarked upon and followed up with a sketch of an alternative idea or plan.
Above expectations; bonus (3)	The writing demonstrates professional polish through word choice, sentence structure choices, clarity, concision, proofreading, etc. This solution is presentable as a textbook example or exposition.	The math is merged very well with the writing; any equations and variables are embedded, clear, and concise; any transitions between text and mathematics aid understanding; any solutions are contextualized and explained in a professional manner.	The analysis is excellent, creating a large amount of insight into the prompt and demonstrates mastery of the topic at or above the course level.