University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 – Final –Winter 2023

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Exam Type: C **Time: 150 mins.**

1. **Keep this booklet closed**, until an invigilator announces that the test has begun.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **into the boxes**. Ample space is provided within each box, however, if you must use additional space, please use the blank page at the end of this booklet, and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 16 pages, excluding the cover page, and 8 questions.
5. **Do not remove any page from this booklet.**
6. **Remember to show all your work for part B and C** unless the question explicitly says not to. You don't need to justify your choices in part A.
7. You may use your one pre-written page of notes, written on faculty approved exam-aid sheet. Your notes should be handwritten. Any other note written on any other paper is considered academic dishonesty. You should write your name and student number on your pre-written note.
8. You may use a Casio FX-991 or Sharp EL-W516 calculator. Any other type of calculator on your desk is considered academic dishonesty. Please be ready to show your calculator to the invigilators if asked.

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	11	6	9	4	4	14	17	75
Score:									

Part A

1. Fill in the bubble for all statements that must be true. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles. You don't need to include your work or reasoning.

   (a) (1 point) Which option gives the correct setup for applying the integration by parts technique on $\int \ln x\, dx = uv - \int v\, du$.

   ○ $dv = \ln x\, dx,\ u = 1$

   ○ $dv = dx,\ u = \ln x$

   ○ $dv = \ln x\, dx,\ u = x$

   ○ None of the other options

   (b) (1 point) Consider the indefinite integral $\displaystyle\int \frac{1}{x^2\sqrt{x^2+2}}dx$. After first substituting $x = \sqrt{2}\tan(\theta)$, then $u = \sin(\theta)$, we arrive at which of the following integrals?

   ○ $\displaystyle\int \frac{1}{u}du$ ○ $\displaystyle\int \frac{1}{2u^2}du$ ○ $\displaystyle\int \frac{\sqrt{1-u^2}}{2u^2}du$ ○ None of the other options

   (c) (4 points) Suppose the power series $\sum_{n=5}^{\infty} c_n(x+3)^n$ converges at $x = 3$ and diverges at $x = -10$.

   (i) Select ALL values for which you can **guarantee** that the series is **convergent**.

   ○ $x = -9$ ○ $x = -11$ ○ $x = -2$ ○ $x = 2$ ○ $x = 9$ ○ $x = 16$

   (ii) Select ALL values for which you can **guarantee** that the series is **divergent**.

   ○ $x = -9$ ○ $x = -11$ ○ $x = -2$ ○ $x = 2$ ○ $x = 9$ ○ $x = 16$

(d) (1 point) Using the fact that $\frac{x}{2} \leq \sin(x)$ for all $x \in [0,1]$, $\displaystyle\int_0^1 \frac{\sqrt{x}}{\sin(x)}dx$ is:

○ Convergent ○ Divergent ○ Not enough information

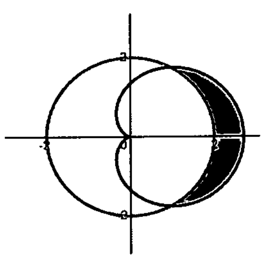
(e) (1 point) Suppose we want to approximate $\int_0^6 f(x)dx$ using the midpoint rule, M_n. Also, suppose $f''(x) = \dfrac{e^{\frac{x}{6}}\sin(2x)}{x+3}$. Among the given options, which one is the smallest value of n that ensures

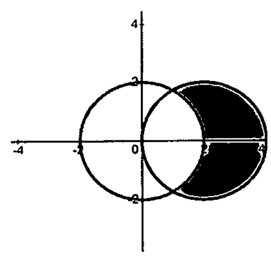
$$\left|\int_0^6 f(x)dx - M_n\right| < \frac{1}{100}?$$

○ 30 ○ 3 ○ 8 ○ 50

(f) (2 points) The following integral computes the area of which of the given regions?

$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (1+2\cos(\theta))^2 - 4\,d\theta$$


○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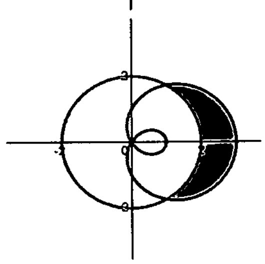
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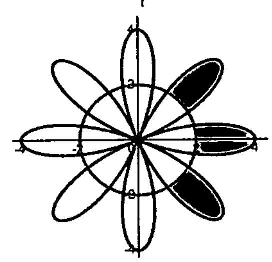

○

2. You don't need to include your computation or reasoning for this question. Parts of this question are independent of one another.

(a) (3 points) Consider the Riemann sum $\lim_{n\to\infty}\sum_{k=1}^{n}(n+2k)^3\frac{4}{n^4}$. The sum computes a definite integral $\int_a^b f(x)dx$.

$a =$ ☐ $b =$ ☐

Computing the sum $\lim_{n\to\infty}\sum_{k=1}^{n}(n+2k)^3\frac{4}{n^4}$ yields

○ 80 ○ 40 ○ 8 ○ 4

(b) Recall that $\ln x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n}$, with the radius of convergence $R = 1$.

(a) (4 points) Find the first three non-zero terms of the Taylor series of $f(x) = (x-2)\ln(\frac{x}{2})$ about $x = 2$. What is the radius of convergence R of this new series?

$f(x) =$ ☐ $+$ ☐ $+$ ☐ $+\cdots,$ $R =$ ☐

(b) (4 points) Find the first three non-zero terms of the Taylor series of $h(x) = \int_2^x f(t)dt$ centred at $x = 2$ **and** determine its radius of convergence R.

$h(x) =$ ☐ $+$ ☐ $+$ ☐ $+\cdots,$ $R =$ ☐

3. You don't need to include your computation or reasoning for this question.

A chemical C is formed during the reaction of two other chemicals, A and B. As C is formed, chemicals A and B are gradually eliminated such that the total mass of the chemicals remains constant.

(a) (2 points) Let $y(t)$ denote the mass of chemical C (in kg) at time t (in minutes), and suppose that it takes $\frac{1}{3}$ kg of A and $\frac{2}{3}$ kg of B to make 1 kg of C. If chemicals A and B have an initial amount of 20 kg each, determine the functions $m_A(t)$ and $m_B(t)$ that give the mass of the chemicals A and B respectively, in terms of $y(t)$, at time t.

$m_A(t) = \boxed{}$ $\qquad$ $m_B(t) = \boxed{}$

(b) (3 points) Suppose that C is formed at a rate that is proportional to the product of the remaining mass of A and B that are present at time t. After half an hour, C begins to transform into a new chemical D at a rate that is proportional to the square of the mass of C at time t.

Provide a piecewise initial value problem (IVP) that governs the mass of chemical C at time t, from $t = 0$ to $t = t_0$ (where $t_0 > 30$ and $y(t_0) = 0$).

final answer $\boxed{}$

(c) (1 point) Select ALL that apply.
Consider each of the ODEs in the piecewise IVP. They BOTH are:

○ first-order ○ autonomous ○ linear ○ separable

Part B

4. A population of insects in a region will grow at a rate that is proportional to their current population. Suppose $p(t)$ is the population at time t, measured in days.

   (a) (3 points) In the absence of any outside factors the population will triple in 14 days. Write down a differential equation that describes this population under these conditions. Your final answer should not include any parameter.

   the differential equation is $\boxed{}$

   (b) (2 points) On any given day there is a net migration into the area of 15 insects, and 23 are eaten by the local bird population. Write a differential equation that models the population.

   the differential equation is $\boxed{}$

   (c) (4 points) If there are initially 100 insects in the area, will the population survive or will it die out? Justify your answer. ◯ die out ◯ survive

5. (4 points) Suppose we want to approximate $\int_0^{2\pi} e^x \sin(x)dx$ with a combination of numerical methods on different intervals. If we use:

- Midpoint rule for $0 \leq x \leq \frac{\pi}{2}$
- Trapezoidal rule for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$
- Right Riemann Sum for $\frac{3\pi}{2} \leq x \leq \frac{7\pi}{4}$
- Left Riemann Sum for $\frac{7\pi}{4} \leq x \leq 2\pi$

Will our approximation be:

○ An overestimation ○ An underestimation

○ Either over or under the true value; it depends on how we partition the interval.

Provide a brief justification for your answer:

6. (4 points) Find the general solution of the differential equation

$$y' + (y+t)^2 = -1,$$

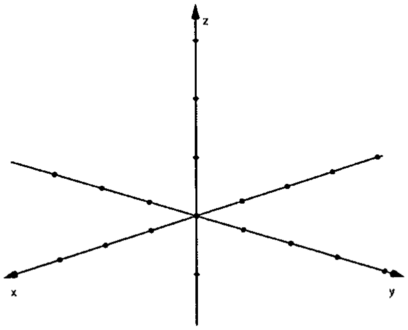
by making the substitution $v(t) = y(t) + t$. *Your final solution should be an explicit function for* $y(t)$.

the general solution is

Part C

7. Consider the curve $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t\rangle$ in $\mathbb{R}^3$.

   (a) (2 points) Sketch the curve in $\mathbb{R}^3$



   (b) (1 point) Draw the principal unit tangent vector at $\vec{r}(t)$ for a value for t of your choice.

   (c) (3 points) Find the unit tangent vector of this curve.

$\vec{T}(t) =$ $\boxed{}$

   (d) (3 points) Find a function $s(t)$ that gives the arclength between $t = 0$ and t at any given time t.

$s(t) =$ $\boxed{}$

Recall that $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle$, $t \in \mathbb{R}$.

(e) (3 points) Find the arclength parametrization of this curve.

arclength parametrization: $\boxed{}$

(f) (2 points) Compute the curvature of the curve at any given t.

the curvature is $\boxed{}$

8. Here we consider an initial value problem for a spring-mass model of the form

$$\begin{cases} ax'' + bx' + cx = f(t) \\ x(0) = A \\ x'(0) = B, \end{cases}$$

where $x(t)$ is measured in meters, with $x(t) > 0$ implying that the spring is stretched and $x(t) < 0$ meaning the spring is compressed.

(a) (5 points) Determine the initial value problem which corresponds to the following scenario (you don't need to justify your answer):

- The mass is 1 kg.
- The damping constant is an unknown non-negative parameter μ.
- The spring constant is 26 kg/s^2
- At $t = 0$ the spring is stretched 1 meter from its equilibrium and then released from rest.
- $f(t) = e^{-5t}\sin(wt)$, where w is an unknown parameter.

$\boxed{}\, x'' + \boxed{}\, x' + \boxed{}\, x = e^{-5t}\sin(wt)$, $\quad x(0) = \boxed{}$, $x'(0) = \boxed{}$

(b) (4 points) For this part of the problem assume $f(t) = 0$. Determine all possible values for $\mu \geq 0$ in which the mass will oscillate about its equilibrium for $t > 0$. *Oscillation does not necessarily imply constant amplitude.*

final answer $\boxed{}$

(c) (4 points) Suppose $f(t) = e^{-5t}\sin(wt)$, where w is an unknown parameter. Find all possible values for $\mu \geq 0$ and w such that the initial value problem solution $x(t)$ satisfies $\lim_{t\to\infty} x(t) = 0$.

final answer $\boxed{}$

(d) (4 points) Continue assuming that $f(t) = e^{-5t}\sin(wt)$. ChatGPT claims that regardless of the value of w, the particular solution of this ordinary differential equation is always of the form

$$x_p(t) = e^{-5t}(c_1\sin(wt) + c_2\cos(wt))$$

for some constants c_1 and c_2 which can be computed by the method of undetermined coefficients. Do you agree?

○ yes ○ no

If you agree, explain why. If you disagree, find a value of w for which the particular solution is not of the given form.

The purpose of this page is to provide additional space for your solutions. You may also use it as scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

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Trigonometric Functions

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$
$$\int \sec^3(x)\,dx = \frac{1}{2}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$$
$$\int \csc(x)\,dx = -\ln|\cot(x)+\csc(x)| + C$$
$$\int \tan(x)\,dx = -\ln|\cos x| + C$$
$$\int \cot(x)\,dx = \ln|\sin x| + C$$

Error formula for L_n and R_n

$|\text{Error}| \le \frac{M}{2n}(b-a)^2$

Where $M \ge |f'(x)|$ on $a \le x \le b$

Error formula for M_n

$|\text{Error}| \le \frac{M}{24n^2}(b-a)^3$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Error formula for T_n

$|\text{Error}| \le \frac{M}{12n^2}(b-a)^3$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Taylor Polynomial $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Remainder/Error $f(x) = p_n(x) + R_n(x)$ with $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

Common Taylor Series

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ $R = \infty$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ $R = \infty$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $R = 1$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ $R = \infty$

Polar coordinates $x = r\cos\theta$ $y = r\sin\theta$ $x^2+y^2=r^2$ $\tan\theta = \frac{y}{x}$

Unit Tangent $\vec{T}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$

Unit Normal $\vec{N}(t) = \frac{\vec{T}\,'(t)}{||\vec{T}\,'(t)||}$ **Curvature** $\kappa(t) = \frac{||\vec{T}\,'(t)||}{||\vec{r}\,'(t)||} = \frac{||\vec{r}\,'(t)\times\vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}$

Acceleration $$\vec{a}(t) = \vec{r}\,''(t) = \underbrace{\frac{d}{dt}||\vec{r}\,'(t)||}_{a_T}\,\vec{T}(t) + \underbrace{\kappa(t)||\vec{r}\,'(t)||^2}_{a_N}\,\vec{N}(t)$$