UNIVERSITY OF TORONTO, FACULTY OF APPLIED SCIENCE AND ENGINEERING

MAT187H1S - Calculus II

Final Exam - April 26, 2017

EXAMINERS: S. COHEN, A. FENYES, B. GALVÃO-SOUSA, P. MILGRAM, D. PANCHENKO, F. PARSCH, N. SARQUIS

Time allotted: 150 minutes

No Aids permitted

Total marks: 80

Full Name: _____
Last First

Student ID: _____

Email: _____ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 12 pages and a detached **formula sheet**. Make sure you have all of them.
- You can use pages 10–12 for rough work or to complete a question (**Mark clearly**).

DO NOT DETACH ANY PAGE.

GOOD LUCK!

PART I. No explanation is necessary. **(20 Marks)**

1. **(2 marks)** $\displaystyle\int_0^5 \sqrt{25-x^2}\,dx =$ $\boxed{}$.

2. **(2 marks)** Consider the function

$$f(x) = \frac{4x^3+5x^2-11x-22}{x^2-4}.$$

It can be decomposed into

$$f(x) = ax + b + \frac{c}{x-2} + \frac{d}{x+2}$$

with

$a =$ $\boxed{}$ $\qquad$ $c =$ $\boxed{}$

$b =$ $\boxed{}$ $\qquad$ $d =$ $\boxed{}$

3. **(2 marks)** Consider a continuous function $f(x)$ which satisfies

$$\int_{-b}^{b} f(x)\,dx = 1 + \frac{1}{b} \qquad \text{for all } b > 1.$$

Circle the **one** correct conclusion.

(a) There is enough information to decide that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ must converge.

(b) There is enough information to decide that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ must diverge.

(c) There is not enough information to decide about the convergence of $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.

4. **(2 marks)** Let $f(x)$ be twice differentiable satisfying $f(0) = 6$, $f(1) = 7$ and $f'(1) = 6$. Then

$$\int_0^1 xf''(x)\,dx = \boxed{}.$$

5. **(2 marks)** Give an example of a power series with radius of convergence $R = 3$.

$$\sum_{n=5}^{\infty} \boxed{}$$

Continued...

6. **(2 marks)** $\lim_{x\to 0} \dfrac{24\cos(x) + 12x\sin(x) - 24 + x^4}{x^6} =$ $\boxed{}$.

7. **(2 marks)** The vector-valued function $\vec{r}(t) = \left(3 - t^3, 0, 3 - t^3\right)$ traces (circle the correct conclusion)

(a) a straight line. **(c)** a circle. **(e)** a helix.

(b) a cubic. **(d)** an ellipse. **(f)** a spiral.

8. **(2 marks)** Consider a vector-valued function $\vec{r}(t) = \left(\cos(t) \quad , \quad \dfrac{t^2}{\sqrt{2}} + \dfrac{t^3}{3} \quad , \quad \sin(t)\right)$.

Then the curvature at $t = 0$ is

$$\kappa(0) = \boxed{}$$

9. **(2 marks)** The solution of

$$\begin{cases} y''(t) + 2y'(t) - 3y(t) = 0 \\ y(0) = 0 \\ y'(0) = 4 \end{cases}$$

is $y(t) =$ $\boxed{}$.

10. **(2 marks)** Consider the same differential equation

$$y''(t) + 2y'(t) - 3y(t) = 0.$$

Give an example of a solution that satisfies $y(t) > 0$ and $\lim_{t\to\infty} y(t) = 0$.

$y(t) =$ $\boxed{}$.

Continued...

PART II.

11. You are working for Texas Instruments and need to program the function $\arctan(x)$ into a calculator. The calculator can only add, subtract, multiply, and divide numbers. One way to do it is to use Taylor Series. **(20 Marks)**

(a) **(3 marks)** Use a substitution to show that $\arctan(x) = \displaystyle\int_0^x \frac{1}{1+u^2}\,du$.

(b) **(7 marks)** What is the Taylor series for $\arctan(x)$ centred at $a = 0$? Justify your answer.

Hint. Recall the geometric series, as given in the formula sheet.

Continued...

(c) (5 marks) What is the radius of convergence of this series? Justify your answer.

(d) (5 marks) We want to compute $\arctan(\frac{1}{2})$. How many terms do we have to add to make sure that the error is smaller than 2^{-8}? Justify your answer.

Hint. Once you have an inequality describing the number of terms needed, guess and check to find the smallest number of terms that satisfies the inequality.

Continued...

12. Consider a drone performing some acrobatics with position **(20 Marks)**

$$\vec{r}(t) = \Big(x(t), y(t), z(t)\Big) = \Big(3\cos(t) \quad , \quad 3\sin(t) \quad , \quad 2(t-1)^2 + 1\Big),$$

where t is measured in seconds and the position is measured in metres.

(a) (5 marks) What is the minimum altitude $z(t)$ flown by the drone? Justify your answer.

(b) (5 marks) What is the magnitude of the acceleration?

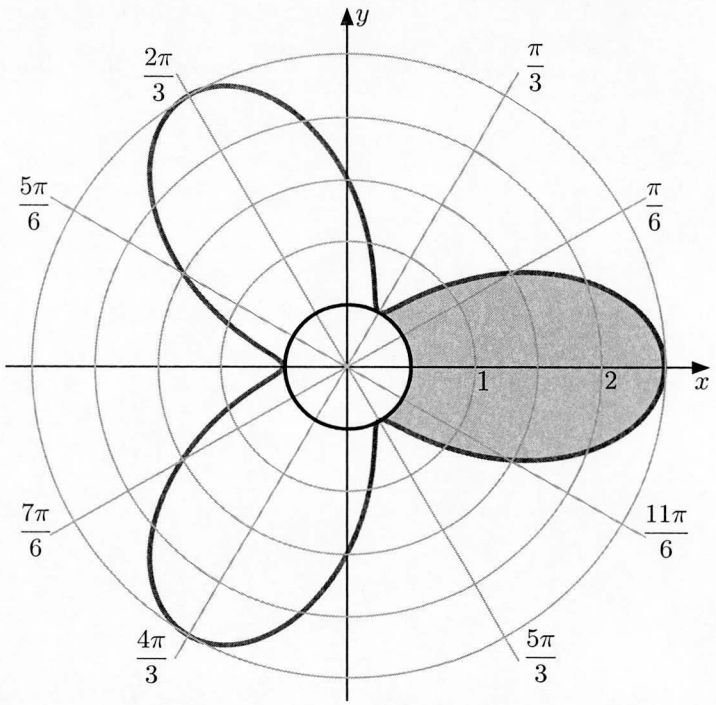
Continued...

(c) (10 marks) The drone works up to an altitude of only 163m. What is the total distance travelled by the drone until it stops working? Justify your answer. The answer can be in terms of $\int_a^b \sec^3(u)\,du$ without having to compute this integral.

Continued...

13. Consider a boat propeller with the shape **(20 Marks)**

$$r = 1.5 + \cos(3\theta)$$



(a) (5 marks) We can fit a circle centred at the origin inside this curve. What is the maximum radius of such a circle? Justify your answer.

(b) (7 marks) What is the tangent line for the first positive angle θ where the curve touches the centre circle found in **(a)**? Justify your answer.

Hint. $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Continued...

(c) (8 marks) What is the area of the shaded blade of the propeller excluding the centre circle found in **(a)**? Justify your answer.

Continued...

USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

If you wish to have this page marked, make sure to refer to it in your original solution.

USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

If you wish to have this page marked, make sure to refer to it in your original solution.

Continued...

USE THIS PAGE FOR ROUGH WORK.

The end.