UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER 2017

DURATION: 2 AND 1/2 HRS

FIRST YEAR - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

EXAMINERS: D. BURBULLA, W. CLUETT, S. COHEN, S. LIU, M. PUGH, S. UPPAL, R. ZHU

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

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Instructions:

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 12 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of eight questions, some with many parts. Attempt all of them, but note that Question 4(c) offers a choice: do only ONE of the two options in 4(c)! Each question is worth 10 marks. Marks for parts of a question are indicated in the question. **Total Marks: 80**
- PRESENT YOUR SOLUTIONS IN THE SPACE PROVIDED. You can use pages 10, 11 and 12 for rough work. If you want anything on pages 10, 11 or 12 to be marked you must indicate in the relevant previous question that the solution continues on page 10, 11 or 12.

1. Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$; let $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

   Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal set. Show your work! Then write $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ and $\vec{u}_4$.

2.(a) [6 marks] Find all values of a for which the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & a & a \\ a & 2 & 1 \end{bmatrix}$ is **not** invertible.

2.(b) [4 marks] Let $A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 2 & 2 & 2 & 2 \\ 5 & 1 & 0 & 2 \\ 4 & -2 & 3 & 3 \end{bmatrix}$; $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\vec{v}$ is an eigenvector of A. What is the corresponding eigenvalue?

3. Let $\vec{x}$ and $\vec{y}$ be two non-zero vectors in $\mathbb{R}^n$.

   (a) [5 marks] Prove that if $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ then $\vec{x}$ and $\vec{y}$ are orthogonal. Illustrate this geometrically.

   (b) [5 marks] Prove that if $\vec{x}$ and $\vec{y}$ are orthogonal, then $\|\vec{x}+\vec{y}\| = \|\vec{x}-\vec{y}\|$. Illustrate this geometrically.

Continued...

4. Let A be an $m \times k$ matrix; let B be a $k \times n$ matrix.

(a) [3 marks] Show that $\text{nullity}(B) \leq \text{nullity}(A\,B)$. Recall: if X is a matrix, $\text{nullity}(X) = \dim(\text{null}(X))$.

(b) [3 marks] Use the Rank Theorem to show that $\text{rank}(AB) \leq \text{rank}(B)$.

(c) [4 marks] Show that if $\text{rank}(A) = k$ then $A^T A$ is invertible. (Hint: suppose $A^T A\,\vec{x} = \vec{0}$.)
OR: Show that if A is an $m \times m$ invertible matrix, then $\vec{x}^T A^T A\,\vec{x} > 0$ for all $\vec{x} \neq \vec{0}$ in $\mathbb{R}^m$.

5. Solve the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= 6x - y \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} &= 2x + 3y \end{cases}$$

for x and y as functions of t, given that $(x, y) = (5, 7)$ when $t = 0$.

6. If $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

7. Let $S = \operatorname{span}\left\{ [\ 1\ \ 1\ \ 0\ \ 1\ ]^T, [\ 0\ \ 1\ \ 1\ \ 1\ ]^T, [\ -1\ \ 1\ \ 1\ \ 1\ ]^T \right\}$.

(a) [3 marks] Find a basis for $S^\perp$, the orthogonal complement of S.

(b) [4 marks] Find the standard matrix of the linear transformation $\operatorname{perp}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

(c) [3 marks] What is the standard matrix of $\operatorname{proj}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$? (You can make use of part (b).)

8. Experimental data for the response, y, of an electronic device to an input, x in millivolts, is listed in the following table:

Trial i	1	2	3	4	5
Input x_i	2	3	4	5	6
Response y_i	5	7	8	11	12

Find the best fitting line with equation $y = a + b\,x$ for the given data.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

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