University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 3

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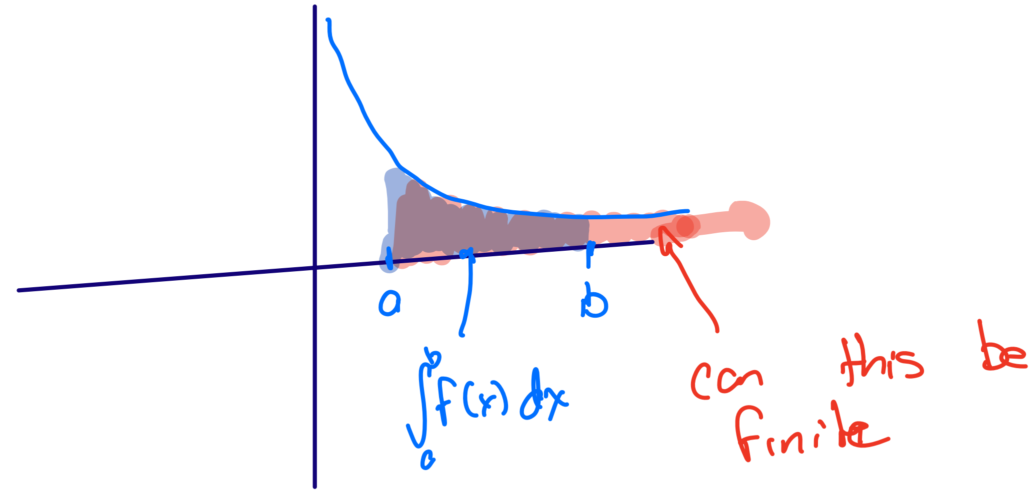
We will start 10 minutes past the hour. Use this time to make a new friend.

Improper Integrals - Infinite Bounds

$$\int_a^b f(x)\,dx$$



Can we take infinite bounds?

$$\int_a^\infty f(x)\,dx$$

why we want to do this?

ex// Radioactive decay: $N(t)$ be the number of radioactive particles

$$\frac{dN}{dt} = -\lambda e^{-\lambda t}$$

$\Leftarrow$ derivation later in the course

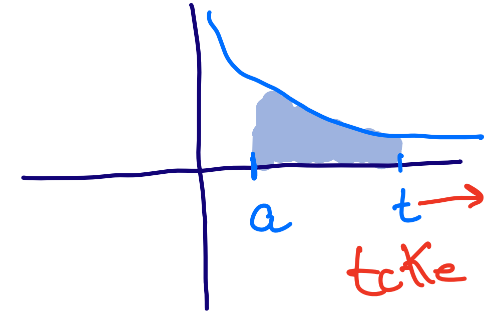
$\rightarrow \lambda$ is decay constant

$\rightarrow$ how many particles are lost after some time $t = a$

$$\int_a^\infty -\lambda e^{-\lambda t}\,dt$$

$\Leftarrow$ easier to compute

Def'n: ① $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \underbrace{\int_a^t f(x)\,dx}_{\text{well-defined definite integrals}}$



take $\lim_{t\to\infty}$

② $\displaystyle\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$

Def'n: We say $\int_a^\infty f(x)\,dx$ converges if $\lim_{t\to\infty}\int_a^t f(x)\,dx$ exists and equals finite value otherwise diverges.

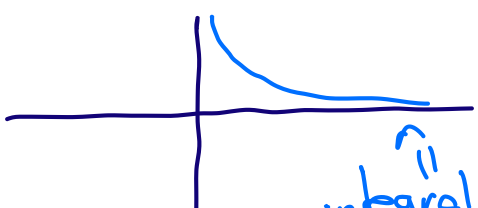
$\Rightarrow \lim \to \infty$

$\Rightarrow \lim \to -\infty$

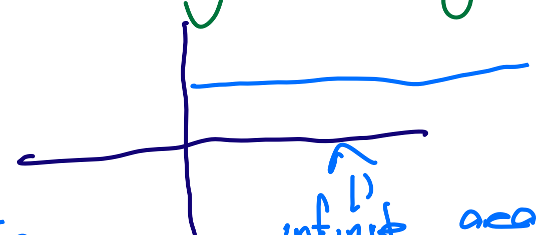
$\Rightarrow \lim$ DNE (osscilates)

} types of divergences

An integral $\int_a^\infty f(x)\,dx$ can only converge if $\lim_{x\to\infty} f(x) \to 0$



integral only finite if $\to 0$



infinite area $\int_a^\infty$

→ not just because $f(x)\to 0$ doesn't imply convergence

$\int_1^{\infty} \frac{1}{x^2}\,dx$

$\cancel{= -\frac{1}{x}\Big|_1^{\infty}}$

$\cancel{= -\frac{1}{\infty} - \left(-\frac{1}{1}\right)}$

$\cancel{= 1}$

$\Leftarrow$ infinity not
$\Leftarrow$ a number

Correct method

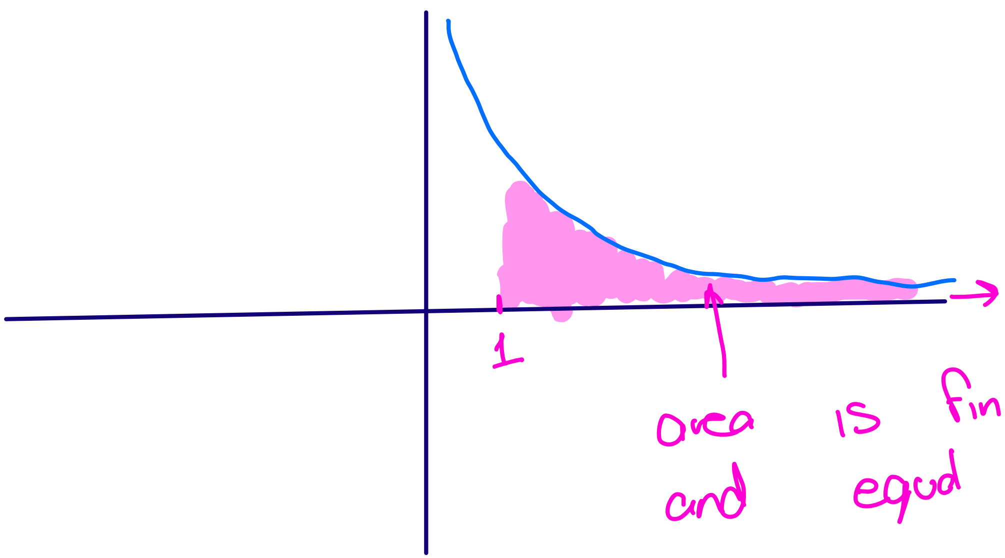
$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{x}\Big|_{x=1}^{x=t} \right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1 \right)$$

$$= 0 + 1$$

$$= 1$$



area is finite
and equal to 1

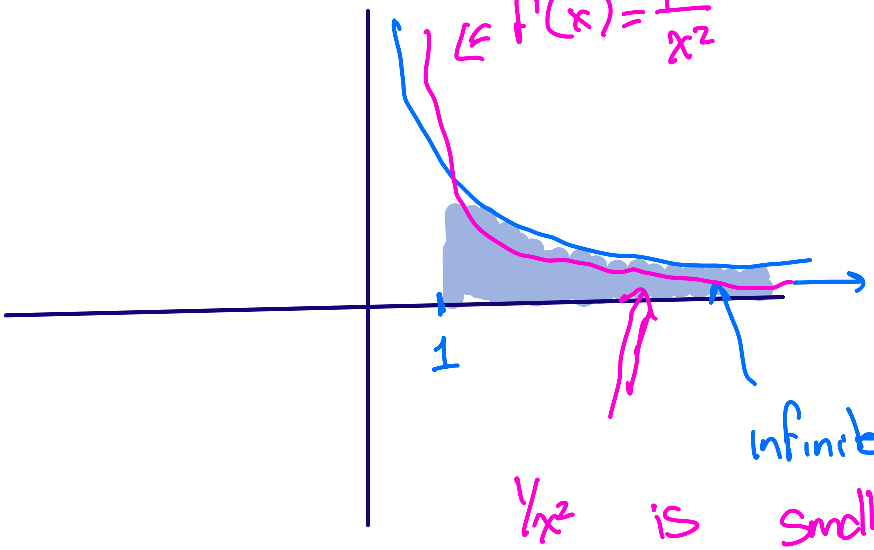
$\int_1^{\infty} \frac{1}{x} dx$

$= \lim\limits_{t \to \infty} \int_1^t \frac{1}{x} dx$

$= \lim\limits_{t \to \infty} \left(\ln(x) \Big|_{x=1}^{t} \right)$

$= \lim\limits_{t \to \infty} \left(\ln(t) - 0 \right)$

$= \infty$ $\quad \therefore$ integral diverges



$1/x^2$ is smaller and $\int_1^{\infty} \frac{1}{x^2}$ is finite as seen in prev. example

$$\int_{-\infty}^{0} xe^x \, dx$$ $\Leftarrow$ try this

$$\lim_{t \to -\infty} \int_t^0 x e^x \, dx$$

$$= \lim_{t \to -\infty} \left(xe^x - e^x \Big|_{x=t}^{x=0} \right)$$ $\Leftarrow$ integration by parts

$$= \lim_{t \to -\infty} \left(-1 - (te^t - e^t) \right)$$

$$\lim_{t \to -\infty} te^t = \lim_{t \to \infty} \frac{-t}{e^t} \overset{\text{l'hopital}}{=} \lim_{t \to \infty} \frac{-1}{e^t}$$

$$= -1 - 0 + 0$$

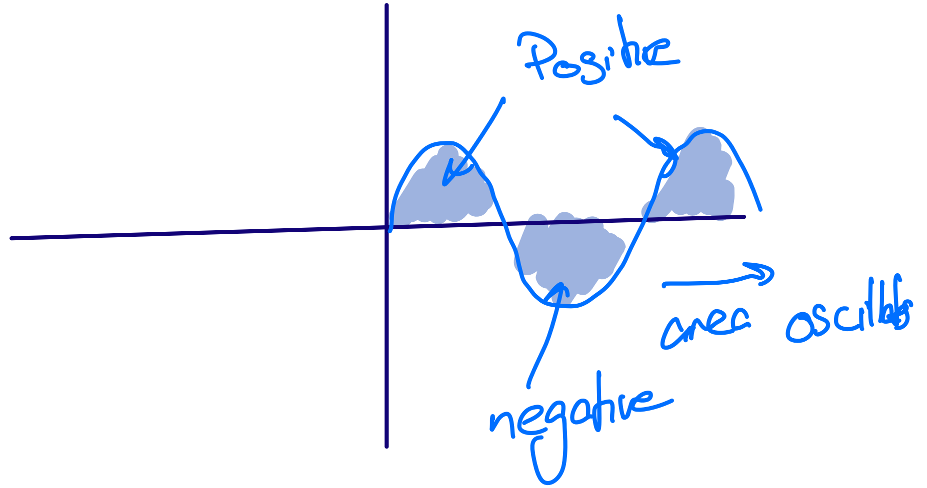
$$\int_0^\infty \sin(x)dx$$ $\Leftarrow$ try this

$$= \lim_{t\to\infty} \int_0^t \sin(x)dx$$

$$= \lim_{t\to\infty} \left(-\cos(t) + \cos(0)\right)$$

$$= 1 - \lim_{t\to\infty} \cos(t)$$ $\Leftarrow$ lim DNE

Diverges



$\int_{-\infty}^{\infty} \sin(x)dx$

$= \lim_{t\to\infty} \int_{-t}^{t} \sin(x)dx$

$= \lim_{t\to\infty} \left(-\cos(t) + \cos(-t)\right)$

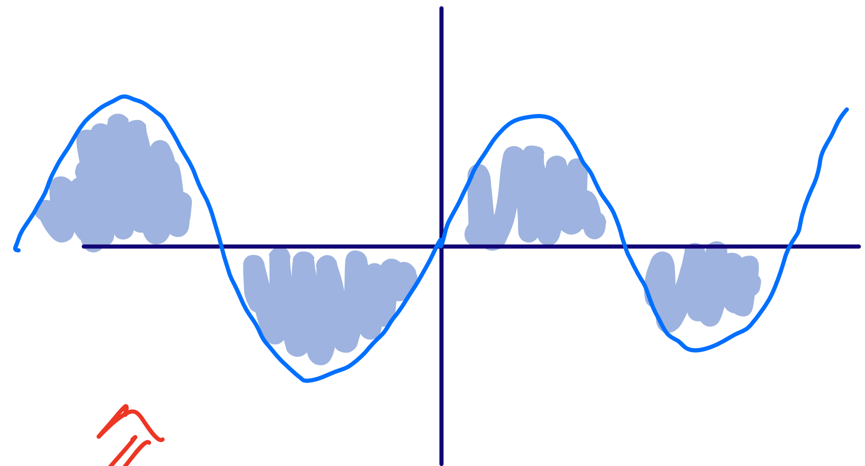
even function

$= \lim_{t\to\infty} \left(-\cos(t) + \cos(t)\right)$

$= \lim_{t\to\infty} (0)$

$= 0$

incorrect



doesn't make sense given last question

b/c if $\int_{-\infty}^{\infty} \sin(x) = 0$ then

$\int_{0}^{\infty} \sin(x) = \frac{0}{2} = 0$

Improper Integral - Infinite Bounds

Def'n: $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx := \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$

$$= \lim_{t\to\infty} \int_{t}^{a} f(x)\,dx + \lim_{t\to\infty}\int_{a}^{t} f(x)\,dx$$

$\Leftarrow$ deal with each ∞ using seperate limit

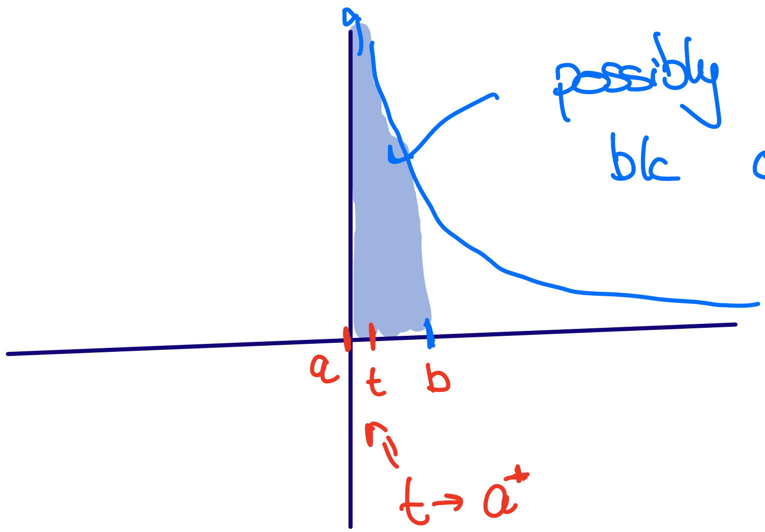
ex: $\displaystyle\int_{-\infty}^{\infty} \sin(x)\,dx = \underbrace{\int_{-\infty}^{0} \sin(x)\,dx}_{\text{DNE}} + \underbrace{\int_{0}^{\infty} \sin(x)\,dx}_{\text{DNE}}$

$\therefore$ diverges

$\Leftarrow$ split it at zero but can split anywhere

$\int_{-\infty}^{\infty} \frac{1}{x^2+4} dx$ $\Leftarrow$ try this at home

Improper Integral - Vertical Asmptote



Def'n: Suppose $f(x)$ has an asmptote at $x=a$

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

$$\int_b^a f(x)\,dx = \lim_{t \to a^-} \int_b^t f(x)\,dx$$

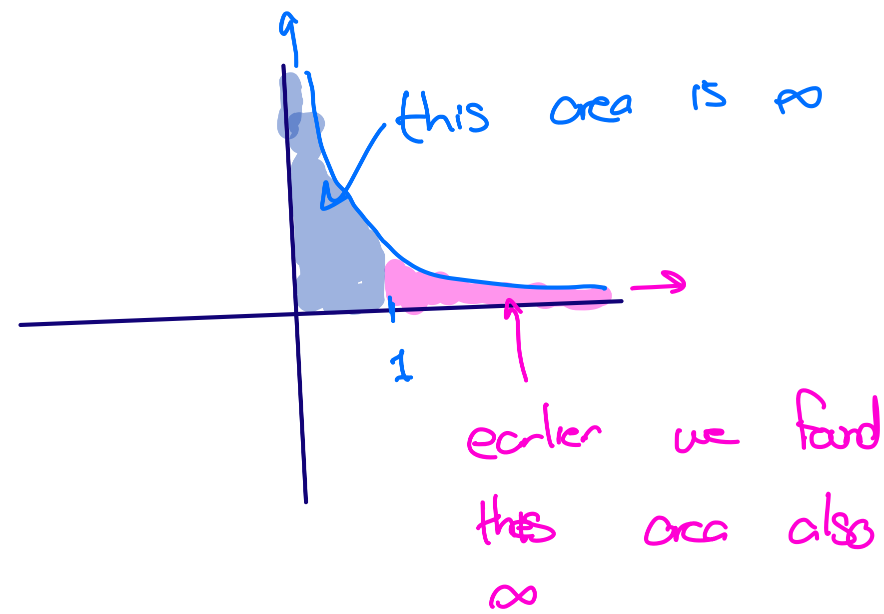
$\int_0^1 \frac{1}{x}\,dx$ $\Leftarrow$ vertical asymptote at 0

$= \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx$

$= \lim_{t\to 0^+} \ln(1) - \ln(t)$

$= -\lim_{t\to 0^+} \ln(t)$

$= \infty$ diverges



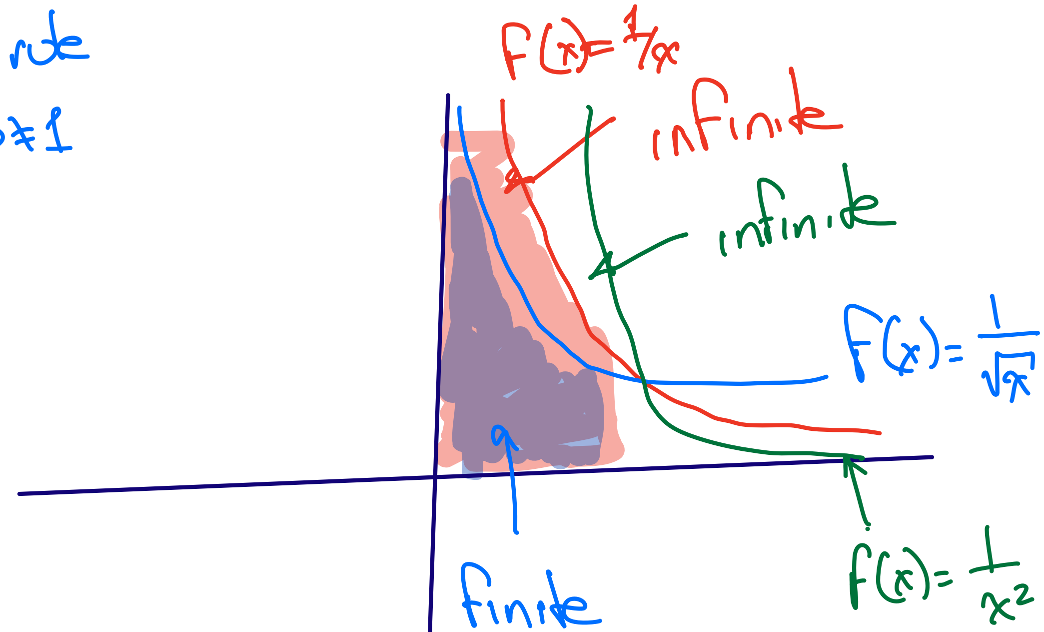
$\int_0^1 \frac{1}{x^p}\,dx$ ⇐ what values of $p \neq 1$ does integral converge

$= \lim\limits_{t \to 0^+} \int_t^1 \frac{1}{x^p}\,dx$ ⇙ power rule for $p \neq 1$

$= \lim\limits_{t \to 0^+} \left(\frac{1}{1-p} x^{1-p} \Big|_{x=t}^{x=1} \right)$

$= \frac{1}{1-p} \lim\limits_{t \to 0^+} \left(1 - t^{1-p} \right)$

$= \begin{cases} \frac{1}{1-p} & 1-p > 0 \\ \text{DNE} & 1-p < 0 \end{cases} \Rightarrow \begin{matrix} p < 1 \\ p > 1 \end{matrix}$



⇑

$f(x) = 1/x$ is the line between converge and divergence

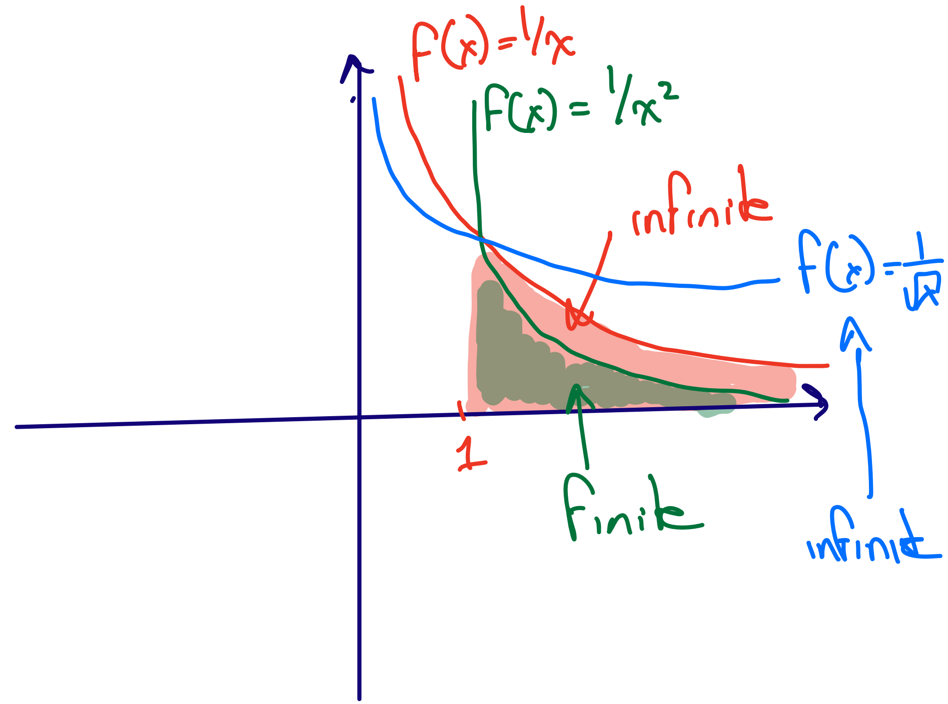
$$\int_1^\infty \frac{1}{x^p}\,dx \qquad p \neq 1$$

$$\lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p} x^{1-p} \Big|_{x=1}^{x=t} \right)$$

$$= \frac{1}{1-p} \lim_{t\to\infty} \left(t^{1-p} - 1 \right)$$

$$= \begin{cases} -\dfrac{1}{1-p} & 1-p<0 \qquad p>1 \\ \text{DNE} & 1-p>0 \qquad p<1 \end{cases}$$



p-Test

$\int_1^\infty \frac{1}{x^p}$ converges for $p > 1$

diverges for $p \leq 1$

$\int_0^1 \frac{1}{x^p}$ converge for $p < 1$

diverges for $p \geq 1$

→ $1/x$ diverges in both cases and is cut off point

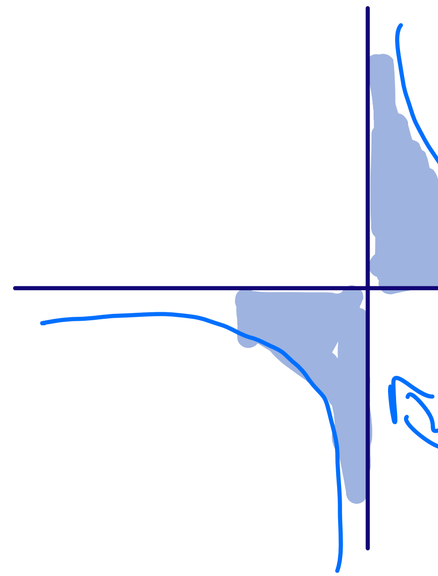
→ Careful! only for powers. $1/x$ not cut off point for arbitry functions

ex!! $\frac{1}{2x} < \frac{1}{x}$ but $\int_1^\infty \frac{1}{2x}\,dx$ DNE

$\int_{-1}^{1} \frac{1}{x}$ $\Leftarrow$ asymptote at $x=0$

$\rightarrow$ split into two limits

$$\underbrace{\int_{-1}^{0} \frac{1}{x}\,dx}_{\text{DNE}} + \underbrace{\int_{0}^{1} \frac{1}{x}\,dx}_{\text{DNE}}$$



$\cancel{-\infty - \infty = 0}$?

$\therefore \int_{-1}^{1} \frac{1}{x}\,dx$ diverges

Improper Integrals

Def'n: Suppose $f(x)$ has vertical asymptote at $x=a$

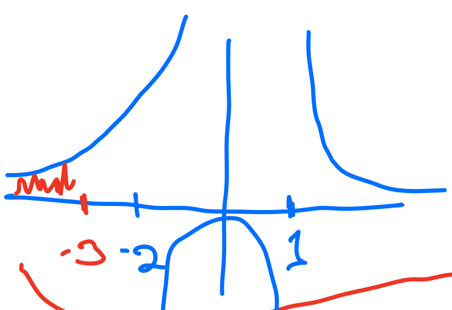
① $$\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx \qquad c \le a \le b$$

$$= \lim_{t\to a^-}\int_c^t f(x)\,dx + \lim_{t\to a^+}\int_t^b f(x)\,dx$$

② $$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{c} f(x)\,dx + \int_c^a f(x)\,dx + \int_a^b f(x)\,dx + \int_b^{\infty} f(x)\,dx}$$

→ every infinity uses seperate limit

→ if <u>any</u> diverge then integral diverges



ex: $$\int_{-\infty}^{\infty} \frac{1}{(x-1)(x+2)} = \int_{-\infty}^{-3} + \int_{-3}^{-2} + \int_{-2}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{\infty}$$

Comparison Test

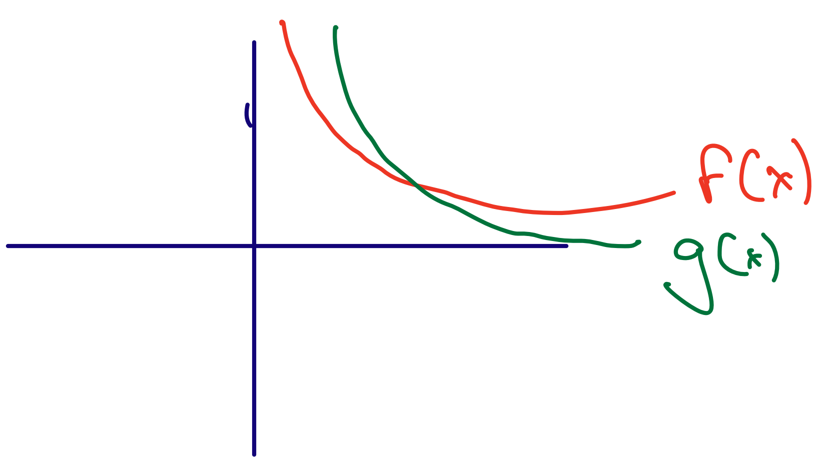
Thm: Let $0 \le f(x) \le g(x)$

① if improper integral $\to \int_a^{\infty} g(x)$ converges then $\int_0^{\infty} f(x)$ converges

$\to \int_a^{b} g(x)$ converges then $\int_a^{b} f(x)$ converges

② if improper integral $\to \int_a^{\infty} f(x)$ diverges then $\int_a^{\infty} g(x)$ diverges

$\to$



$\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ Converge and diverge?

→ educated guess: convergent b/c x^2 in denominator

→ $0 \le \frac{\sin^2(x)}{x^2}$, we can apply comparision test

→ need to find a convergent integral bigger than $\frac{\sin^2(x)}{x^2}$

$$\sin^2(x) \le 1$$

$$\frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$$

$\Rightarrow \int_1^{\infty} \frac{1}{x^2}\,dx$ converges (p-test)

$\therefore \int_1^{\infty} \frac{\sin^2(x)}{x^2}\,dx$ converges

$\int_1^\infty \frac{1}{x^2+4x}\,dx$ Converge or diverge?

→ educated guess: converge b/c $\frac{1}{x^2+4x} \sim \frac{1}{x^2}$ as $x \to \infty$ (x^2 term dominates denominator)

→ Find something bigger that converges

$4x > 0$

$\frac{x > 1}{\text{domain}}$

$x^2+4x > x^2$

$\frac{1}{x^2+4x} < \frac{1}{x^2}$

$\Rightarrow \int_1^\infty \frac{1}{x^2}\,dx$ converges $\Rightarrow \int_1^\infty \frac{1}{x^2+4x}\,dx$ converges

$$\int_1^\infty \frac{x}{x^2-\cos(x)}\,dx$$

→ note that $\frac{x}{x^2-\cos(x)} \geq 0$ for large x so we can use comparision test

→ educated guess: as $x \to \infty$ $\frac{x}{x^2-\cos(x)} \sim \frac{x}{x^2} = \frac{1}{x}$ which <u>diverges</u>

$$\frac{x}{x^2-\cos(x)} = \frac{1}{x-\frac{\cos(x)}{x}} > \frac{1}{x+1/x}$$

$$> \frac{1}{2x}$$

$\Rightarrow \int_1^\infty \frac{1}{2x}\,dx$ converges $\therefore \int_1^\infty \frac{x}{x^2-\cos(x)}\,dx$ diverges