

University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 15

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We will start 10 minutes past the hour. Use this time to make
a new friend.

Harmonic Oscillators

An undamped harmonic oscillator is any system that satisfies

$$\text{Force}(x) = -Kx$$

$$m\ddot{x} = -Kx$$

$$\boxed{m\ddot{x} + Kx = 0}$$

$$\Leftarrow F = ma$$

$$a = \frac{d^2x}{dt^2} = \ddot{x}$$

$$K > 0$$

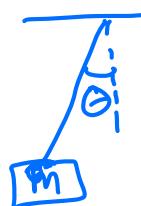
ex/

→ on ideal spring



x ← distance from equilibrium

→ a pendulum



θ ← for small θ , harmonic oscillation

→ LCR circuits with zero resistance



Solution to undamped harmonic oscillator:

$$m\ddot{x} + Kx = 0$$

$$\text{char. eq. : } mr^2 + K = 0$$

$$r = \pm \sqrt{-\frac{K}{m}} = \omega$$

Since $K, m > 0$, always pure imaginary

$$\omega = \sqrt{\frac{k}{m}}$$

we often write

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega^2 x = 0$$

A solution of form $C_1 \cos(\omega t) + C_2 \sin(\omega t)$ can be written as $A \cos(\omega t + \phi)$ (or $A \sin(\omega t + \phi)$)

PF:

$$A \cos(\omega t + \phi) \stackrel{?}{=} C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

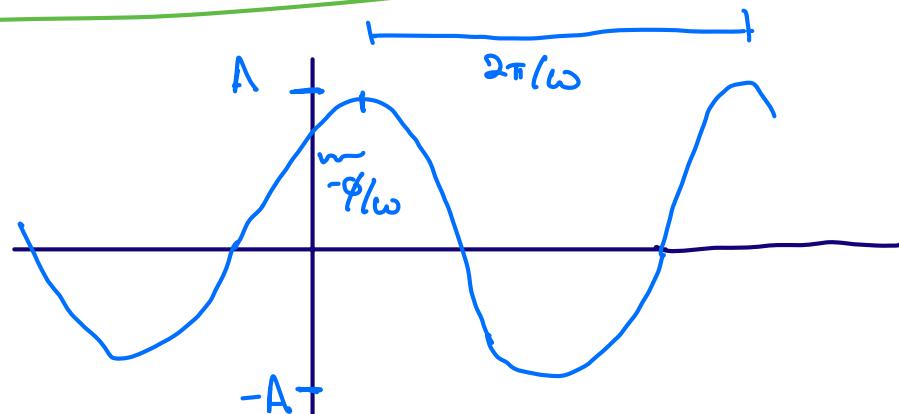
$$A \cos(\phi) \sin(\omega t) + A \sin(\phi) \cos(\omega t) \stackrel{?}{=} C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

$$\Rightarrow \begin{cases} C_1 = A \cos(\phi) \\ C_2 = A \sin(\phi) \end{cases} \Rightarrow A = \sqrt{C_1^2 + C_2^2}$$

$$\phi = \arctan\left(\frac{C_2}{C_1}\right)$$

$$x(t) = A \cos(\omega t + \phi)$$

↑
Amplitude ↑
Phase shift



$\underbrace{\omega^2}_{\text{constant}} \Rightarrow \omega = 2$ $v_0 = 0 \Rightarrow \text{velocity at } t=0$
 Solve the IVP $\ddot{x} + 4x = 0$, $x_0 = 10$, ~~$\dot{x}_0 = 0$~~ $\dot{x}(0) = 0$
 general solution:

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

$$10 = x_0 = x(0) = C_1(1) + C_2(0) \Rightarrow C_1 = 10$$

$$0 = \dot{x}(0) = -2C_1 \sin(2(0)) + 2C_2 \cos(2(0)) \\ = 2C_2 \Rightarrow C_2 = 0$$

$$x(t) = 10 \cos(2t)$$

↑
amplitude

Damped Harmonic Oscillator

→ include a force that opposes velocity

$$\text{Force} = -Kx - 2\gamma \dot{x}$$

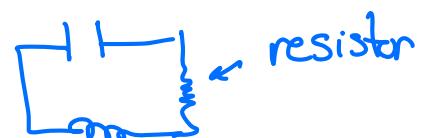
$\underbrace{}_{\text{restoring force}}$ $\underbrace{\phantom{-2\gamma \dot{x}}}_{\text{damping force}}$

$K > 0$
 $\gamma > 0$

$$m\ddot{x} + 2\gamma \dot{x} + Kx = 0$$

→ applies to

- laminar drag
- internal resistance of spring
- LCR circuits



→ Solution

$$mr^2 + 2\gamma r + k = 0$$

$$r = \frac{-(2\gamma) \pm \sqrt{(2\gamma)^2 - 4mk}}{2m}$$

$$r = -\frac{\gamma}{m} \pm \frac{1}{m} \sqrt{\gamma^2 - mk}$$

→ if $\gamma^2 - mk > 0$, 2 real roots
 $\gamma^2 > mk$ (γ is large) Overdamped

\rightarrow if $\gamma^2 - mk < 0$, complex roots

$\gamma^2 < mk$ (γ small) Underdamped

\rightarrow if $\gamma^2 - mk = 0$, 1 repeated real root

(γ "just-right") Critically damped

Overdamped Case

$$\Rightarrow \text{Chr eq. } mr^2 + 2\gamma r + k = 0 \quad (\gamma^2 - mk > 0)$$

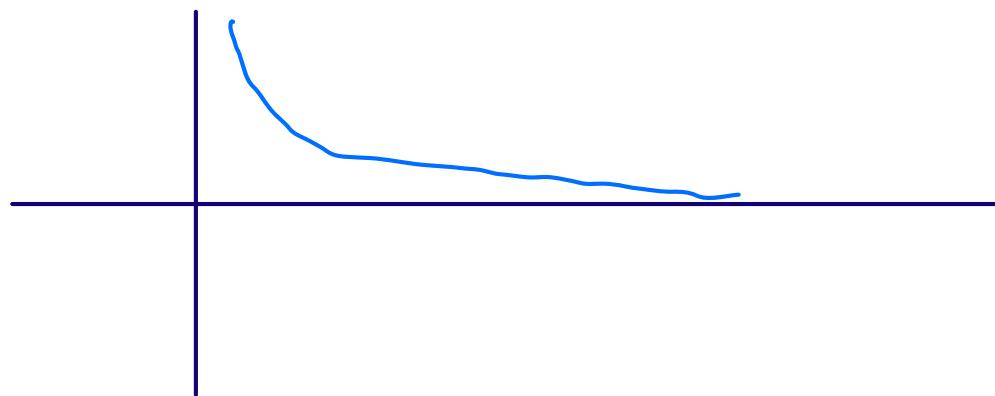
$$r_1 = -\frac{\gamma}{m} - \frac{1}{m} \sqrt{\gamma^2 - mk} < 0$$

$$r_2 = -\frac{\gamma}{m} + \frac{1}{m} \sqrt{\gamma^2 - mk} = -\frac{\gamma}{m} + \sqrt{\left(\frac{\gamma}{m}\right)^2 - \frac{k}{m}} < 0$$

} both roots
are negative

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

\rightarrow Overdamped, there
is no oscillations



Underdamped Case

$$\Rightarrow \text{char eq. } mr^2 + 2\zeta r + k = 0 \quad (\zeta^2 - mk < 0)$$

$$r_1 = -\frac{\zeta}{m} + i\sqrt{mk - \zeta^2} = -\frac{\zeta}{m} + i\omega_d$$

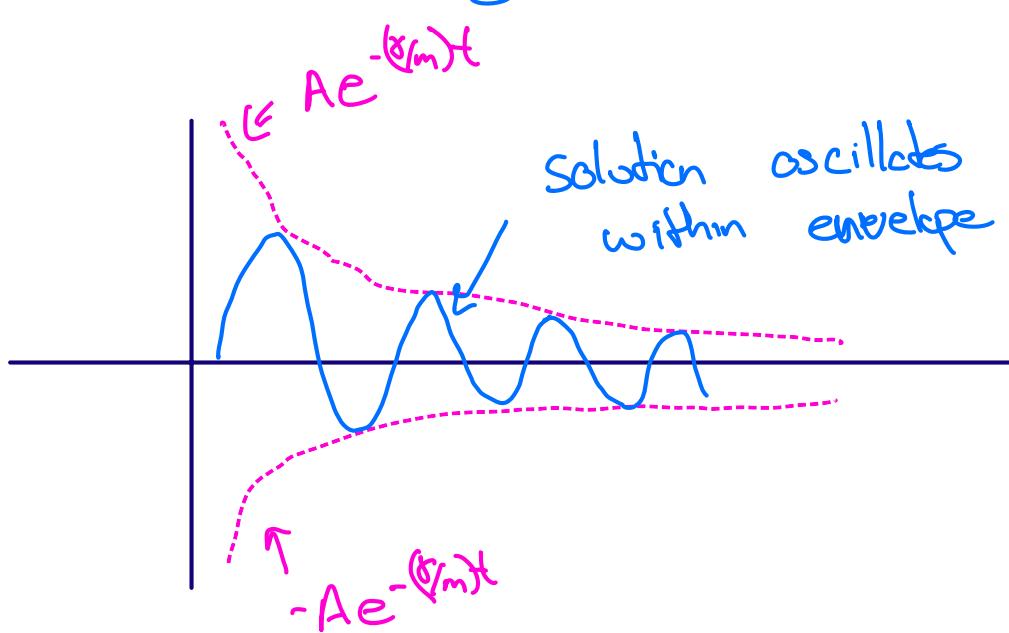
$$\text{led } \omega_d^2 = mk - \zeta^2$$

$$r_2 = -\frac{\zeta}{m} - i\sqrt{mk - \zeta^2} = -\frac{\zeta}{m} - i\omega_d$$

$$x(t) = e^{-(\zeta/m)t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

$$= e^{-(\zeta/m)t} A \cos(\omega_d t + \phi)$$

max value of $\cos = 1$
 $\max x(t) = A e^{(-\zeta/m)t}$
 $\min x(t) = -A e^{(-\zeta/m)t}$



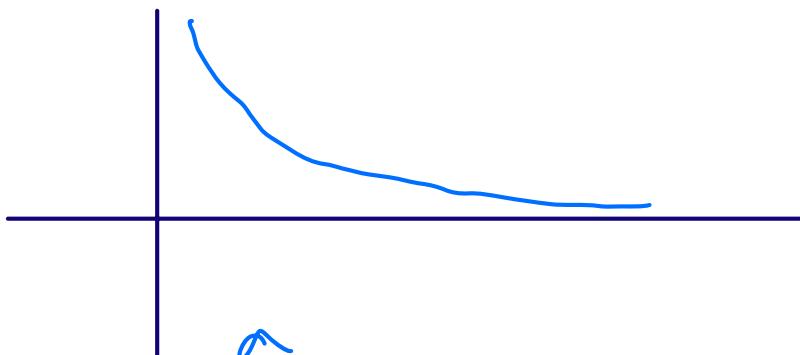
→ underdamped, there are oscillations that die down to zero

Critically Damped

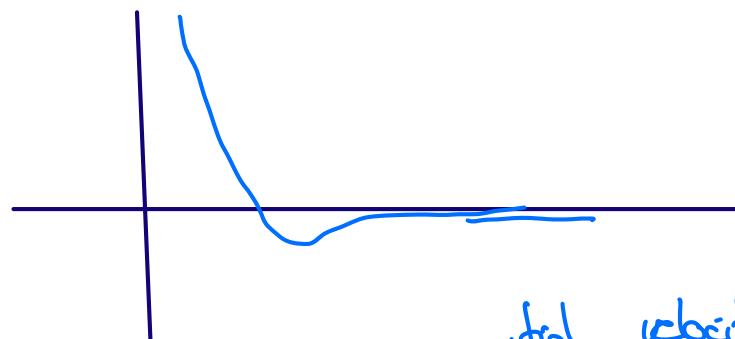
$$\Rightarrow \text{char eq. } mr^2 + 2\gamma r + k = 0 \quad (\gamma^2 - mk = 0)$$

$$\begin{aligned} r_1 &= -\frac{\gamma}{m} + \sqrt{0} \\ r_2 &= -\frac{\gamma}{m} - \sqrt{0} \end{aligned} \quad \left. \begin{array}{l} \text{repeated} \\ \text{root} \end{array} \right\}$$

$$\begin{aligned} x(t) &= C_1 e^{-\frac{\gamma}{m}t} + C_2 t e^{-\frac{\gamma}{m}t} \\ &= (C_1 + C_2 t) e^{-\frac{\gamma}{m}t} \end{aligned}$$



↑
Solution goes to
zero without oscillation



↑
if v_0 is large
initial velocity
then solution crosses
zero exactly once

Solve the initial value problem:

$$\ddot{x} + 6\dot{x} + 9x = 0, \quad x(0) = 2, \quad \dot{x}(0) = -1$$

⇒ char eq. $r^2 + 6r + 9 = 0$
 $r = -3$ (repeated root)

$$x(t) = (C_1 + C_2 t) e^{-3t}$$

$$\dot{x}(t) = (C_1 + C_2 t)(-3)e^{-3t} + C_2 e^{-3t}$$

$$2 = x_0 = x(0) = (C_1 + C_2(0))e^{-3 \cdot 0} = C_1$$

$$\Rightarrow C_1 = x_0 = 2$$

$$-1 = v_0 = \dot{x}(0) = -3(C_1 + C_2(0))e^{-3 \cdot 0} + C_2 e^{-3 \cdot 0} = C_1 + C_2$$

$$-3C_1 + C_2 = v_0 \quad \Leftrightarrow \quad C_2 = v_0 + 3x_0$$

$$\boxed{\begin{aligned} C_2 &= v_0 + 3x_0 \\ C_2 &= -1 + 6 \end{aligned}}$$

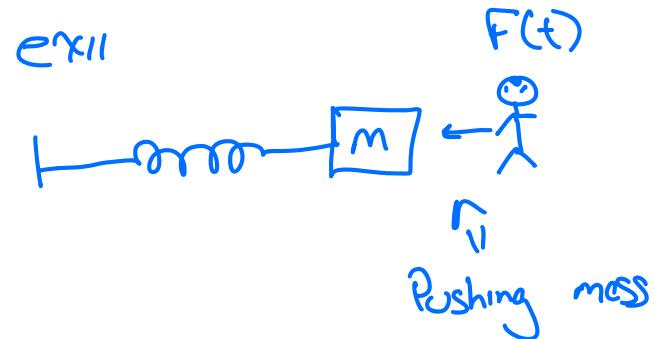
$$\boxed{x(t) = (2 + 5t)e^{-3t}}$$

Forced Harmonic Oscillators

A damped harmonic oscillator but with external driving force

$$m\ddot{x} + 2\gamma\dot{x} + Kx = F(t)$$

force damping spring force driving force



→ types of driving force

$$F(t) = t^2 + t + 3 \quad (\text{polynomial})$$

$$F(t) = Ae^{ct} \quad (\text{exponential})$$

$$F(t) = A\sin(\omega t) \quad (\text{sineoidal})$$

← most interesting
for harmonic
oscillators

Undamped Oscillator with Forcing

$$mx' + kx = F_0 \cos(\omega_f t)$$

Homogeneous solution: $x(t) = A \cos(\omega t)$

$$\omega = \sqrt{\frac{k}{m}}$$

↑ assume phase $\phi = 0$

Case 1: $\omega_f \neq \omega$

$$x_p(t) = B \cos(\omega_f t + \phi)$$

$$x(t) = A \cos(\omega t) + B \cos(\omega_f t + \phi)$$

Case 2: $\omega_f = \omega$

$$x_p(t) = Bt \cos(\omega_f t + \phi)$$

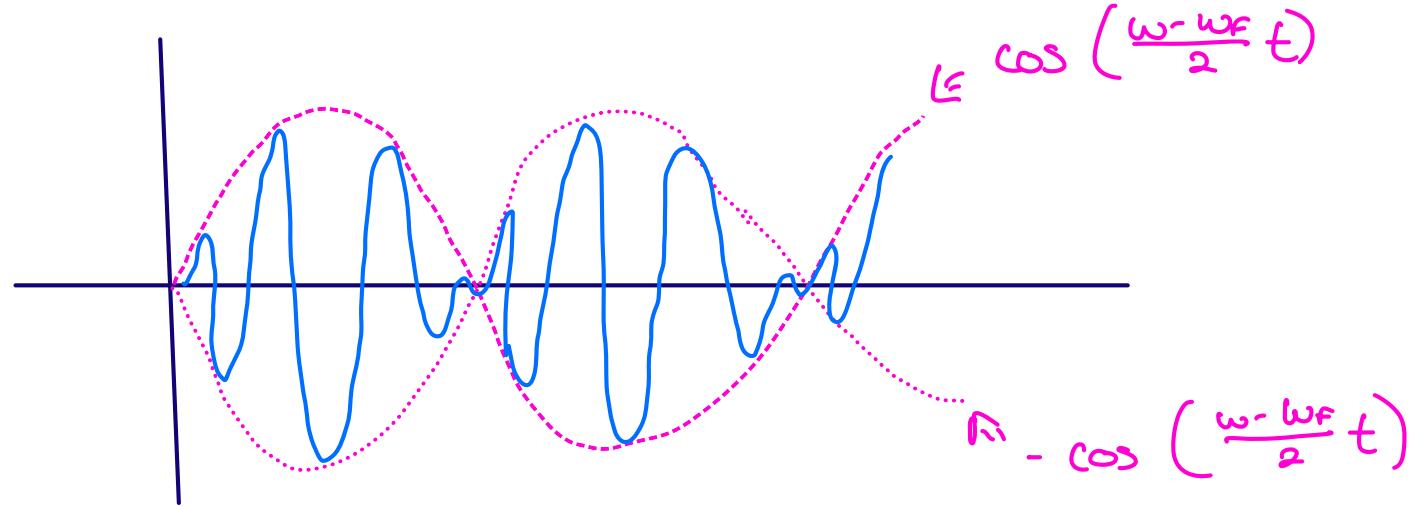
$$x(t) = A \cos(\omega t) + Bt \cos(\omega t + \phi)$$

Case 1: $\omega_f \neq \omega$ (non-resonant forcing)

$$x(t) = A \cos(\omega t) + B \cos(\omega_f t)$$

(for simplicity assuming $\phi = 0$ but solution looks same)

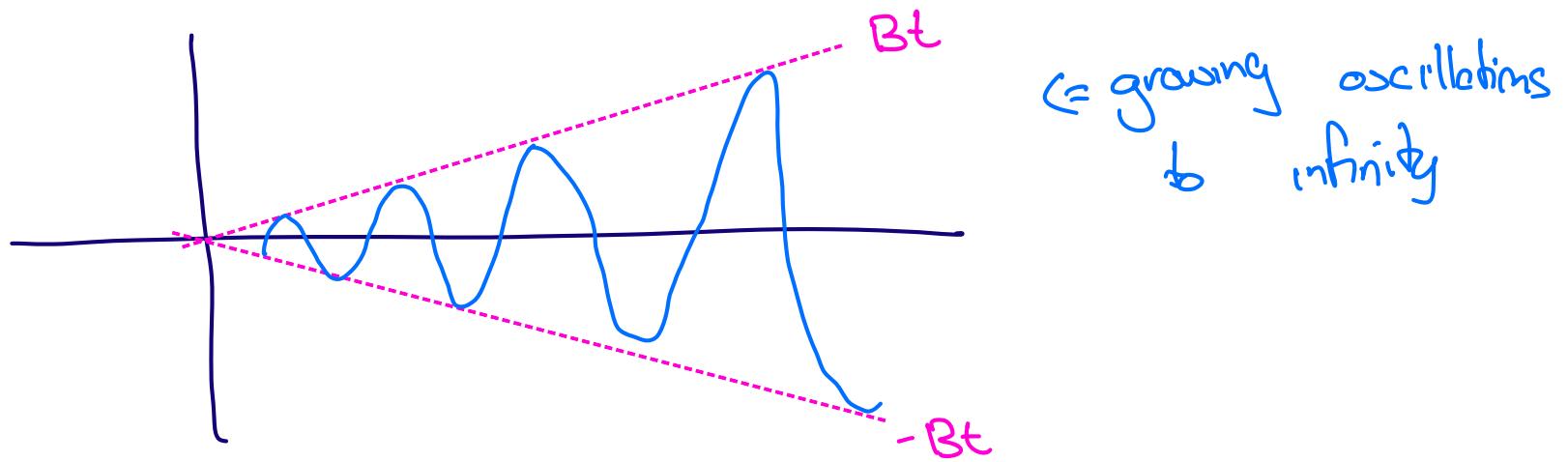
$$= (\text{some coeff}) \underbrace{\cos\left(\frac{(\omega + \omega_f)t}{2}\right)}_{\text{large freq}} \underbrace{\cos\left(\frac{(\omega - \omega_f)t}{2}\right)}_{\text{small freq.}} \leftarrow \text{trig identity}$$



- this is called "beats" phenomena
- most pronounced when ω is close to ω_f
 i.e. $|\omega - \omega_f|$ small so the envelope is lower freq

Case 2 : Resonance $\omega = \omega_f$

$$x(t) = A \cos(\omega t) + \underbrace{B t \cos(\omega t + \phi)}_{\text{dominating term for } t \text{ large}} \leftarrow B \text{ is constant, not dependent on initial condition}$$



Driven & Damped Harmonic Oscillators

→ the undamped oscillator is an approximation for very underdamped oscillator in small time

In general :

$$m\ddot{x} + 2\gamma\dot{x} + K = F_0 \cos(\omega_F t)$$

$$x(t) = \underbrace{x_c(t)}_{\begin{array}{l} \textcircled{1} \text{ under damped} \\ \textcircled{2} \text{ over damped} \\ \textcircled{3} \text{ critically} \end{array}} + \underbrace{x_p(t)}_{\begin{array}{l} \rightarrow \text{all three cases, } \cos(\omega_F t) \\ \text{not a complementary soln} \end{array}}$$

$$\Rightarrow x_p(t) = A_p \cos(\omega_F t)$$

(over) $x(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t} + A_p \cos(\omega_f t)$

(under) $x(t) = e^{-\frac{\gamma}{2}t} A \cos(\omega_f t + \phi) + A_p \cos(\omega_f t)$

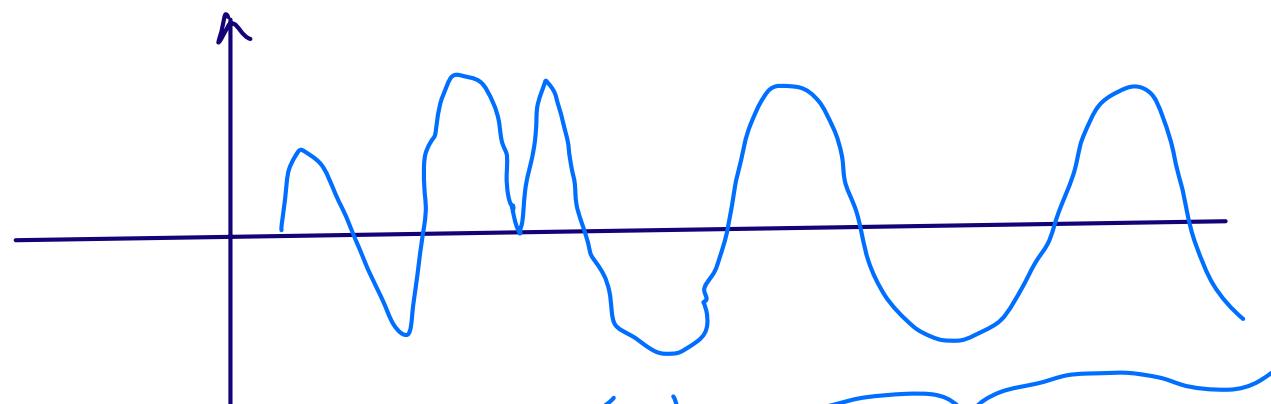
(critical) $x(t) = (C_1 + C_2 t) e^{-\frac{\gamma}{2}t} + A_p \cos(\omega_f t)$

in all cases

complementary sol'n

$\rightarrow 0$ for large t

$$x(t) \rightarrow A_p \cos(\omega_f t)$$



Initial behaviour
(random)

eventually with

systems driving

moves
freq.

Conclusion: real-world systems don't experience "beats"

Phenomena or true resonance

→ lightly-damped oscillators demonstrate behaviors
similar to resonance & beats for small time