Faculty of Applied Science & Engineering, University of Toronto

MAT188H1F - Linear Algebra

Fall 2016

Tutorial Problems 7

Recall that one reason for the word "linear" in linear mappings (transformations) is that they preserve "linear" combinations (see Textbook, page 134). Here's a geometric interpretation on why the "linear" in "linear" mappings.

Theorem: Let A be an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$ (recall that the only linear mappings from $T : \mathbb{R}^n \to \mathbb{R}^m$ are matrix mappings - see Textbook, Theorem 3, page 136). Then T maps a line segment in $\mathbb{R}^n$ to a line segment in $\mathbb{R}^m$ or to a single point in $\mathbb{R}^m$.

Proof: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $\mathbf{z} = \mathbf{x} + t(\mathbf{y} - \mathbf{x}), t \in \mathbb{R}$ is the vector equation of the line through $\mathbf{x}$ and $\mathbf{y}$. The line segment joining $\mathbf{x}$ and $\mathbf{y}$ is the portion of the line corresponding to $0 \leq t \leq 1$. Then,

$$\begin{aligned} T(\mathbf{z}) &= A\mathbf{z} \\ &= T(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \\ &= A(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \\ &= A\mathbf{x} + t(A\mathbf{y} - A\mathbf{x}) \end{aligned}$$

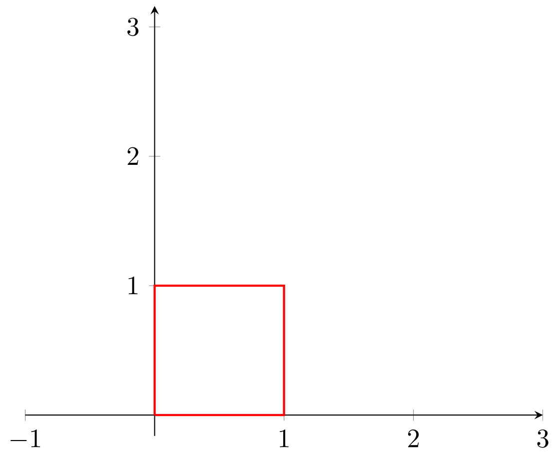
Notice that $A\mathbf{z} = A\mathbf{x} + t(A\mathbf{y} - A\mathbf{x})$ is the vector equation of the line through $A\mathbf{x}$ and $A\mathbf{y}$ in $\mathbb{R}^m$ and the line segment joining $A\mathbf{x}$ and $A\mathbf{y}$ is the portion of the line corresponding to $0 \leq t \leq 1$ If $A\mathbf{x} = A\mathbf{y}$, the equation describes the single point $A\mathbf{x}$ in $\mathbb{R}^m$.

The above Theorem is useful in computing images of subsets of $\mathbb{R}^n$.

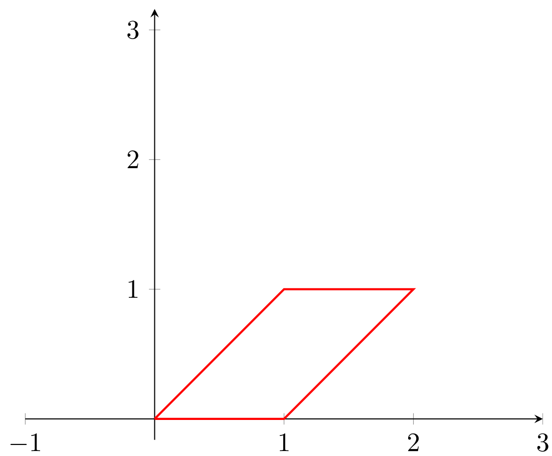
Defintion: If S is a subset of $\mathbb{R}^n$, the *image* of S under T is the set $T(S)$ consisting of all points $T(\mathbf{x})$ where $\mathbf{x} \in S$. In symbols, $T(S) = \{T(\mathbf{x}) \mid \mathbf{x} \in S\}$.

Example: Let S be the unit square in $\mathbb{R}^2$ with vertices $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find and sketch the image of S under T.

Solution: Using the above Theorem, we know that the line segment joining $\mathbf{x}$ and $\mathbf{y}$ is mapped to the line segment joining $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{y}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ under T; the line segment joining $\mathbf{y}$ and $\mathbf{z}$ is mapped to the line segment joining $T(\mathbf{y}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{z}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under T; the line segment joining $\mathbf{z}$ and $\mathbf{w}$ is mapped to the line segment joining $T(\mathbf{z}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T(\mathbf{w}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T; and the line segment joining $\mathbf{w}$ and $\mathbf{x}$ is mapped to the line segment joining $T(\mathbf{w}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T. Hence, T maps the unit square S



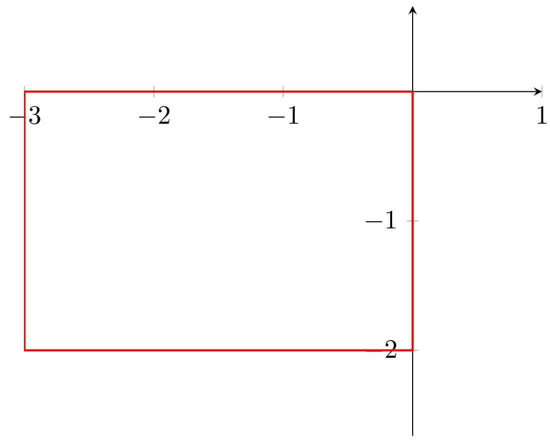
to the parallelogram



1. Let S be the unit square in $\mathbb{R}^2$ with vertices $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Find and sketch the image of S under T where

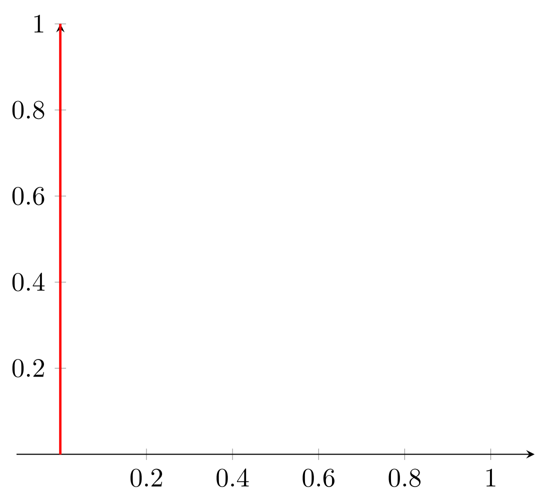
(i) $A = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$

Solution: The line segment joining $\mathbf{x}$ and $\mathbf{y}$ is mapped to the line segment joining $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{y}) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ under T; the line segment joining $\mathbf{y}$ and $\mathbf{z}$ is mapped to the line segment joining $T(\mathbf{y}) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $T(\mathbf{z}) = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ under T; the line segment joining $\mathbf{z}$ and $\mathbf{w}$ is mapped to the line segment joining $T(\mathbf{z}) = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ and $T(\mathbf{w}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ under T; and the line segment joining $\mathbf{w}$ and $\mathbf{x}$ is mapped to the line segment joining $T(\mathbf{w}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T. Hence, T maps the unit square S to



(ii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

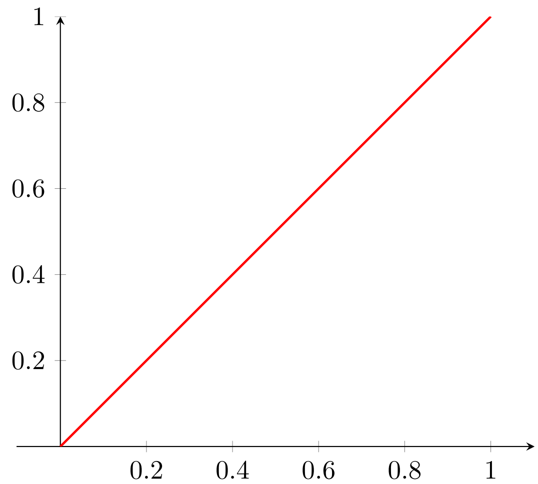
Solution: The line segment joining $\mathbf{x}$ and $\mathbf{y}$ is mapped to the line segment joining $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{y}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T; the line segment joining $\mathbf{y}$ and $\mathbf{z}$ is mapped to the line segment joining $T(\mathbf{y}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{z}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under T; the line segment joining $\mathbf{z}$ and $\mathbf{w}$ is mapped to the line segment joining $T(\mathbf{z}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(\mathbf{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under T; and the line segment joining $\mathbf{w}$ and $\mathbf{x}$ is mapped to the line segment joining $T(\mathbf{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T. Hence, T maps the unit square S to



(iii) $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

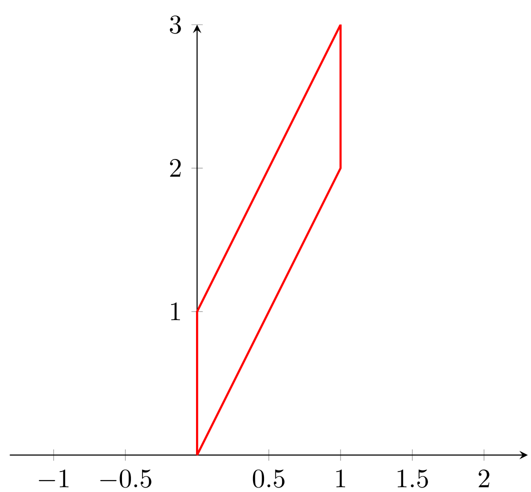
Solution: The line segment joining $\mathbf{x}$ and $\mathbf{y}$ is mapped to the line segment joining $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{y}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T; the line segment joining $\mathbf{y}$ and $\mathbf{z}$ is mapped to the line segment joining

$T(\mathbf{y}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{z}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T; the line segment joining $\mathbf{z}$ and $\mathbf{w}$ is mapped to the line segment joining $T(\mathbf{z}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T(\mathbf{w}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T; and the line segment joining $\mathbf{w}$ and $\mathbf{x}$ is mapped to the line segment joining $T(\mathbf{w}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T. Hence, T maps the unit square S to



(iv) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Solution: The line segment joining $\mathbf{x}$ and $\mathbf{y}$ is mapped to the line segment joining $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $T(\mathbf{y}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ under T; the line segment joining $\mathbf{y}$ and $\mathbf{z}$ is mapped to the line segment joining $T(\mathbf{y}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T(\mathbf{z}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ under T; the line segment joining $\mathbf{z}$ and $\mathbf{w}$ is mapped to the line segment joining $T(\mathbf{z}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T(\mathbf{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ under T; and the line segment joining $\mathbf{w}$ and $\mathbf{x}$ is mapped to the line segment joining $T(\mathbf{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ under T. Hence, T maps the unit square S to



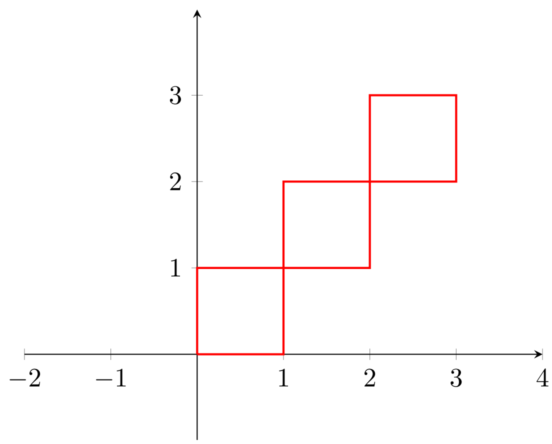
2. Find the matrix of the following reflections:

(i) $R_1 : \mathbb{R}^2 \to \mathbb{R}^2$, is a reflection in the line $x_1 - 3x_2 = 0$.

(ii) $R_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection in the line $2x_1 = x_2$.

(a) Find the matrix of the composition $R_1 \circ R_2 : \mathbb{R}^2 \to \mathbb{R}^2$ and show that it can be identified as a rotation and determine the angle of rotation.

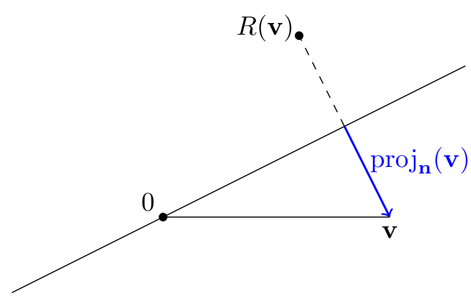
(b) Sketch, as accurately as possible, the image of the three unit squares pictured below under each of the transformations R_1, R_2, and $R_1 \circ R_2$.



Solution:

Let's consider a slightly more general picture, of reflection in the line ℓ defined by $ax_1 + bx_2 = 0$. Consider the normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$. Notice that the line ℓ is exactly the set of all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ such that $\mathbf{x} \cdot \mathbf{n} = 0$. The reflection of a point $\mathbf{v} \in \mathbb{R}^2$ in the line ℓ can be computed using $\mathbf{n}$:

$$R(\mathbf{v}) = \mathbf{v} - 2\ \mathrm{proj}_{\mathbf{n}}(\mathbf{v}) = \mathbf{v} - 2\frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n}$$



(i) For the line $x_1 - 3x_2 = 0$, where $\mathbf{n} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, we compute using the above formula that

$$R_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\frac{1}{10}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix},$$
$$R_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\frac{-3}{10}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

So, the corresponding matrix is $A = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$.

(ii) Similarly, for the line $2x_1 - x_2 = 0$ we have $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and we find

$$R_2\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-3/5\\4/5\end{bmatrix}, \qquad R_2\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4/5\\3/5\end{bmatrix}$$

The corresponding matrix is then $B = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$.

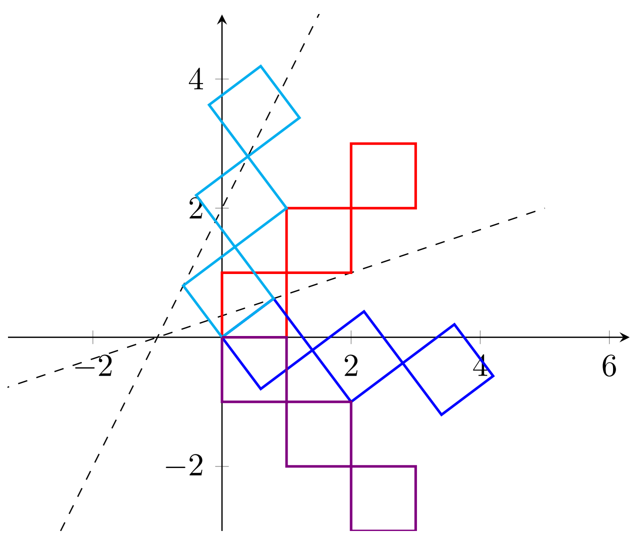
(a) The matrix of the composition $R_1 \circ R_2$ is the product

$$AB = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Recall that the matrix corresponding to a counter-clockwise rotation in angle θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Substituting $\theta = \frac{3\pi}{2}$, we get precisely $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Hence $R_1 \circ R_2$ is a counter-clockwise rotation in angle $\frac{3\pi}{2}$.

(b) We plot the image of the three unit squares under each of the transformations. Notice that, in each case, the images are also unit squares (i.e. angles and lengths have been preserved!).

Below are the: original image, image under R_1, image under R_2, and image under $R_1 \circ R_2$. Also plotted are the lines $x_1 - 3x_2 = 0$ and $2x_1 = x_2$.



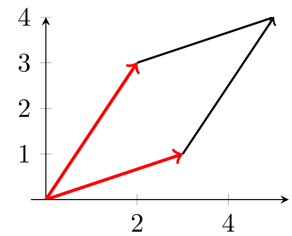
3. Let S be the unit square in $\mathbb{R}^2$ with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$, and $\begin{bmatrix}0\\1\end{bmatrix}$ and let P be the parallelogram with vertices $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$, and $\begin{bmatrix}5\\4\end{bmatrix}$.

(a) Find a matrix A such that the linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_1(\mathbf{x}) = A\mathbf{x}$ maps S onto P.

(b) Find a matrix B such that the linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_2(\mathbf{x}) = B\mathbf{x}$ maps P onto S.

(c) Find a matrix of the linear transformation $T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$.

Solution: The parallelogram P



(a) Any linear map T taking S to P must take the vertices of S to the vertices of P. Furthermore, T has to take vertices of S adjacent to the origin, i.e. (1,0) and (0,1), and send them to vertices of P which are adjacent to the origin, namely (3,1) and (2,3) which are drawn in red above. There are two ways to do this, which we will call $T_{1,1}$ and $T_{1,2}$. Either:

$$T_{1,1}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}, T_{1,1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$$

$$T_{1,2}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, T_{1,2}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$$

$T_{1,1}$ is given by the matrix $A_1 = \begin{bmatrix}3 & 2\\1 & 3\end{bmatrix}$, and $T_{1,2}$ is given by the matrix $A_2 = \begin{bmatrix}2 & 3\\3 & 1\end{bmatrix}$.

(b) By the same logic as above, we have two possible linear maps, given by:

$$T_{2,1}\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}, T_{2,1}\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T_{2.2}\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}, T_{2,2}\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

We can calculate these matrices directly, or we can notice that these linear maps are the inverses of the linear maps from part (i). Notice that $T_{2,1} \circ T_{1,1} = I$, the identity map, and $T_{2,2} \circ T_{1,2} = I$. This tells us that $T_{2,1} = (T_{1,1})^{-1}$ and $T_{2,2} = (T_{1,2})^{-1}$.

Theorem 2.4.6 tells us that if T is given by the matrix A, then T^{-1} is given by the matrix A^{-1}. This means that $T_{2,1}$ is given by the matrix $B_1 = A_1^{-1} = \frac{1}{7}\begin{bmatrix}3 & -2\\-1 & 3\end{bmatrix}$ and $T_{2,2}$ is given by the matrix $B_2 = A_2^{-1} = \frac{-1}{7}\begin{bmatrix}1 & -3\\-3 & 2\end{bmatrix}$.

(c) Recall that if T is the linear transformation given by the matrix A and S is the linear transformation given by the matrix B, then $T \circ S$ is given by the matrix AB. In part (ii), we found our matrices by taking the inverses of our matrices from part (i). The typical answer would then be that $T_2 \circ T_1$ is given by the matrix $A_1^{-1}A_1 = I$ or $A_2^{-1}A_2 = I$. Interestingly, we have two more possible matrices for $T_2 \circ T_1$: $A_2^{-1}A_1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ or $A_1^{-1}A_2 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. This second linear map is reflection along $y = x$.

4. Let $A = \begin{bmatrix} 1 & a+b & c \\ 1 & a+c & b \\ 1 & b+c & a \end{bmatrix}$, where a, b, c are not all equal.

(a) Determine rank(A).

(b) Find a basis for null(A).

(c) Find a basis for col(A).

Solution:

(a) Since the rank is the number of leading entries in a row echelon form for A, we must row reduce A to row echelon form.

$$\begin{bmatrix} 1 & a+b & c \\ 1 & a+c & b \\ 1 & b+c & a \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 1 & a+b & c \\ 0 & c-b & b-c \\ 0 & c-a & a-c \end{bmatrix}$$
$$= \begin{bmatrix} 1 & a+b & c \\ 0 & c-b & -(c-b) \\ 0 & c-a & -(c-a) \end{bmatrix}$$

Since a, b, c are not all equal, one of $c - a \neq 0$ and $c - b \neq 0$ must be true, otherwise $c = a$ and $c = b$ means they are all equal.

Case 1: $\mathbf{c - b \neq 0}$

Then A row reduces further:

$$A \longrightarrow \begin{bmatrix} 1 & a+b & c \\ 0 & c-b & -(c-b) \\ 0 & c-a & -(c-a) \end{bmatrix}$$
$$\xrightarrow{R_2/(c-b)} \begin{bmatrix} 1 & a+b & c \\ 0 & 1 & -1 \\ 0 & c-a & -(c-a) \end{bmatrix}$$
$$\xrightarrow{R_3-(c-a)R_2} \begin{bmatrix} 1 & a+b & c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Case 2: $\mathbf{c - a \neq 0}$

Then A row reduces further:

$$A \longrightarrow \begin{bmatrix} 1 & a+b & c \\ 0 & c-b & -(c-b) \\ 0 & c-a & -(c-a) \end{bmatrix}$$

$$\xrightarrow{R_3/(c-a)} \begin{bmatrix} 1 & a+b & c \\ 0 & c-b & -(c-b) \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & a+b & c \\ 0 & 1 & -1 \\ 0 & c-b & -(c-b) \end{bmatrix}$$

$$\xrightarrow{R_3-(c-b)R_2} \begin{bmatrix} 1 & a+b & c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) We continue row reducing:

$$A \longrightarrow \begin{bmatrix} 1 & a+b & c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-(a+b)R_2} \begin{bmatrix} 1 & 0 & a+b+c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Reading the solution off, we notice that x_3 is a parameter. Let $x_3 = t$. Then:

$$\begin{aligned} x_1 &= -(a+b+c)t \\ x_2 &= t \\ x_3 &= t \end{aligned}$$

So the general solution is $\mathbf{x} = t \begin{bmatrix} -a-b-c \\ 1 \\ 1 \end{bmatrix}$.

(c) Because the first and second column of the REF of A has leading ones, then the corresponding columns from A form a basis for the columnspace of A. That is, a basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} a+b \\ a+c \\ b+c \end{bmatrix} \right\}$.

5. As a teaching assistant for MAT188, you have two groups of students working on solving the same system of linear equations in tutorial. Group A tells you the solution is

$$x_a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

where $s, t \in \mathbb{R}$. Group B tells you the solution is

$$x_b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -7 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}$$

where $s, t \in \mathbb{R}$. Are the answers consistent? i.e. Are both groups correct?

For the answers to be consistent, we have to show that for any $s_a, t_a \in \mathbb{R}$, we can find $s_b, t_b \in \mathbb{R}$ such that:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + s_a \begin{bmatrix}-6\\2\\2\end{bmatrix} + t_a \begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix} + s_b \begin{bmatrix}-7\\2\\3\end{bmatrix} + t_b \begin{bmatrix}-6\\2\\2\end{bmatrix}$$

Let $s_a = t_a = 0$, so that $x_a = \begin{bmatrix}1\\0\\0\end{bmatrix}$. For the system to be consistent, we should find $s_b, t_b \in \mathbb{R}$ such that:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix} + s_b \begin{bmatrix}-7\\2\\3\end{bmatrix} + t_b \begin{bmatrix}-6\\2\\2\end{bmatrix}$$

or equivalently,

$$\begin{bmatrix}0\\1\\-1\end{bmatrix} = s_b \begin{bmatrix}-7\\2\\3\end{bmatrix} + t_b \begin{bmatrix}-6\\2\\2\end{bmatrix}$$

From the first row of the system above, we get $s_b = -(6/7)t_b$, and replacing back in the system above, we get:

$$\begin{bmatrix}0\\1\\-1\end{bmatrix} = (-6/7)t_b \begin{bmatrix}-7\\2\\3\end{bmatrix} + t_b \begin{bmatrix}-6\\2\\2\end{bmatrix} = t_b \begin{bmatrix}0\\2/7\\-4/7\end{bmatrix}$$

which means the system is inconsistent. Therefore, at least one of the groups is wrong.