University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 2

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We will start at 6:10, use this time to make a new friend

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$$= \int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

$$= \int \frac{1}{\cancel{\cos\theta}} \cancel{\cos\theta} \, d\theta$$

$$= \int d\theta$$

$= \theta$ $x = \sin\theta \Rightarrow \theta = \arcsin(x)$

$$\boxed{= \arcsin(x)}$$

Trigonometric Substitution

→ use following identies

$\sin^2\theta + \cos^2\theta = 1$ ⇐ use $x = a\cos\theta$ or $x = a\sin\theta$ $a^2 - x^2$ in integrand

$\Rightarrow 1 - \sin^2\theta = \cos^2\theta$

$1 + \tan^2\theta = \sec^2\theta$ ⇐ use $x = a\tan\theta$ when $a^2 + x^2$ in integrand

$\sec^2\theta - 1 = \tan^2\theta$

Useful identies

$\frac{d}{dx}\tan(x) = \sec^2 x \Rightarrow \int \sec^2 x\,dx = \tan(x) + C$

$\frac{d}{dx} = \cot(x) = -\csc^2 x \Rightarrow \int \csc^2 x\,dx = -\cot(x) + C$

$\int \sec(x)\tan(x)\,dx = \sec(x) + C$

$\int \csc(x)\cos(x)\,dx = -\csc(x) + C$

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

$x = 2\cos\theta \implies \theta = \arccos\left(\frac{x}{2}\right)$

$dx = -2\sin\theta\, d\theta$

try using $x = 2\sin\theta$ (get some answer)

$$= \int \frac{\sqrt{4-4\cos^2\theta}}{4\cos^2\theta}(-2\sin\theta)\,d\theta$$

$$= -\int \frac{2\sin\theta}{4\cos^2\theta}\,2\sin\theta\,d\theta$$

$$= -\int \tan^2\theta\,d\theta \quad \Leftarrow \quad \tan^2\theta = \sec^2\theta - 1$$

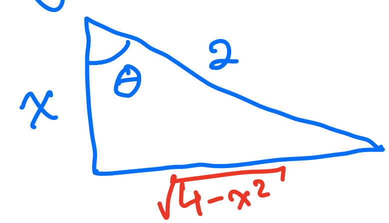
$$= -\int (\sec^2\theta - 1)\,d\theta$$

$$= -\tan\theta + \theta + C \quad \Leftarrow \text{not good}$$

$$= \cancel{-\tan\left(\arccos\left(\tfrac{x}{2}\right)\right)} + \arccos\left(\tfrac{x}{2}\right) + C$$

$$= -\frac{\sqrt{4-x^2}}{x} + \arccos\left(\tfrac{x}{2}\right) + C \quad ✓$$

$$\frac{\text{adj}}{\text{hyp}} = \cos\theta = \frac{x}{2}$$



$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{4-x^2}}{x}$$

$$\int \frac{1}{1+x^2}\,dx \qquad \Leftarrow \; x = 1\tan\theta$$

$$dx = \sec^2\theta \, d\theta$$

$$= \int \frac{1}{1+\tan^2\theta} \sec^2\theta \, d\theta$$

$$= \int \frac{1}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta} \, d\theta$$

$$= \int 1 \, d\theta$$

$$= \theta + C$$

$$x = \tan\theta$$

$$\Rightarrow \theta = \arctan(x)$$

$$\boxed{= \arctan(x) + C}$$

$$\int \frac{x^2}{x^2+9}\,dx$$

$x = 3\tan\theta$

$dx = 3\sec^2 d\theta$

$$= \int \frac{9\tan^2\theta}{9\tan^2\theta+9}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^2\theta}\, 3\sec^2\theta\, d\theta$$

$$= 3\int \tan^2\theta\, d\theta$$

$$= 3(\tan\theta - \theta) + C \qquad \Leftarrow \tan\theta = \frac{x}{3}$$

$$= x - 3\arctan\left(\frac{x}{3}\right) + C$$

$$\int \frac{x^2}{x^2+9}dx$$

$$\begin{array}{r|l} & 1 \\ \hline x^2+9 & x^2+0x+0 \\ & -(x^2+0x+9) \\ \hline & 0+0-9 \end{array}$$

$$= \frac{x^2}{x^2+9} = 1 + \text{remainder}$$

$$= 1 + \frac{-9}{x^2+9}$$

$$= \int 1 dx - \int \frac{9}{x^2+9} dx$$

$$= x - \underbrace{\int \frac{9}{x^2+9} dx}_{\int \frac{\cancel{9}}{\cancel{9}y^2+\cancel{9}} 3dy} \quad \Leftarrow \quad \begin{array}{l} x = 3y \\ dx = 3dy \end{array}$$

$$= \int \frac{1}{y^2+1} 3dy$$

$$= 3\arctan\left(\frac{x}{3}\right)$$

$$\Rightarrow \quad x - 3\arctan\left(\frac{x}{3}\right) + C$$

Partial Fraction Decomposition

→ Integral of a rational function $\frac{Q(x)}{P(x)}$ ⇐ $Q(x), P(x)$ are Polynomials

① polynomial long division : $H(x) + \frac{R(x)}{P(x)}$ ⇐ $R(x)$ is lower degree than $P(x)$

② Decompose $\frac{R(x)}{P(x)}$

→ $\frac{R(x)}{(x-a_1)\dots(x-a_n)} = \frac{A_1}{x-a_1} + \dots + \frac{A_n}{x-a_n}$ ⇐ independent linear factors

→ $\frac{R(x)}{(x-a_1)^n \dots} = \frac{B_1}{x-a_1} + \frac{B_2}{(x-a_1)^2} + \dots + \frac{B_n}{(x-a_1)^n} + \dots$

irreducible quadratic ⇒ → $\frac{R(x)}{\underline{(x^2+b_1x+c_1)} \dots} = \frac{C_1x+D_1}{(x^2+b_1x+c_1)} + \dots$

→ repeated quadratic roots also possible

$$\int \frac{x^3-3x^2+2x+2}{x^2-3x+2}\,dx$$

$$\begin{array}{r|l} & x \\ x^2-3x+2 & x^3-3x^2+2x+2 \\ & -(x^3-3x^2+2x\quad) \\ \hline & 0+0+0+2 \end{array}$$

$\leftarrow$ remainder

$$\Rightarrow \boxed{x+\frac{2}{x^2-3x+2}}$$

$$= \int x\,dx + \int \frac{2}{x^2-3x+2}\,dx$$

$$\frac{2}{x^2-3x+2} = \frac{2}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$$

$$= \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$\frac{2}{(x-1)(x-2)} = \frac{(A+B)x+(-B-2A)}{(x-1)(x-2)}$$

$$\Rightarrow \begin{array}{l} A+B=0 \\ -B-2A=2 \end{array} \Rightarrow \begin{array}{l} A=-2 \\ B=2 \end{array}$$

$$= \int x\,dx - 2\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx$$

$$= \frac{1}{2}x^2 - 2\ln|x-1| + 2\ln|x-2|$$

$$\int \frac{x-2}{(2x-1)^2(x-1)}\,dx$$

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}$$

$$= \frac{A(2x-1)(x-1) + B(x-1) + C(2x-1)^2}{(2x-1)^2(x-1)}$$

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{(2A+4C)x^2 + (-3A+B-4C)x + (A-B+C)}{(2x-1)^2(x-1)}$$

$$\begin{aligned} 2A+4C &= 0 \\ -3A+B-4C &= 1 \\ A-B+C &= -2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 2 \\ B &= 3 \\ C &= -1 \end{aligned}$$

$$2\int \frac{1}{2x-1}\,dx + 3\int \frac{1}{(2x-1)^2}\,dx - 1\int \frac{1}{x-1}\,dx$$

$$= \ln|2x-1| - \frac{3}{2}\,\frac{1}{2x-1} - \ln|x-1| + \text{constant}$$

$$\int \frac{2x+1}{(x-1)(x^2+1)}\,dx$$ $\Leftarrow x^2+1$ is irreducible quadratic factors

$$\frac{2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x-1)}{(x-1)(x^2+1)}$$

$$\frac{2x+1}{(x-1)(x^2+1)} = \frac{(A+B)x^2 + (C-B)x + (A-C)}{(x-1)(x^2+1)}$$

$$\begin{aligned} A+B &= 0 \\ C-B &= 2 \\ A-C &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 3/2 \\ B &= -3/2 \\ C &= 1/2 \end{aligned}$$

$$\int \frac{3/2}{x-1}\,dx + \frac{1}{2}\int \frac{-3x+1}{(x^2+1)}\,dx$$

$$= \frac{3}{2}\ln|x-1| - \frac{3}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$u = x^2$ sub

trig sub (arctan)

$$= \frac{3}{2}\ln|x-1| - \frac{3}{4}\ln|x^2+1| + \frac{1}{2}\arctan(x) + C$$