University of Toronto

Faculty of Applied Science and Engineering

Final Examination, April 2018

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

Examiner: D. Burbulla

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

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Instructions:

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- This exam contains 14 pages, including this cover page, printed two-sided. Make sure you have all of them. Do not tear any pages from this exam.
- This exam consists of ten questions, some with many parts. Attempt all of them. Each question is worth 10 marks. Marks for parts of a question are indicated in the question. **Total Marks: 100**
- PRESENT YOUR SOLUTIONS IN THE SPACE PROVIDED. You can use pages 12, 13 and 14 for rough work. If you want anything on pages 12, 13 or 14 to be marked you must indicate in the relevant previous question that the solution continues on page 12, 13 or 14.

1. Fnd the following:

   (a) [2 marks] $\dim(S^{\perp})$, if S is a subspace of $\mathbb{R}^7$ and $\dim(S) = 4$.

   (b) [4 marks] $\det(-2A\,B^{-1}\,A^T)$, if A and B are 3×3 matrices, with $\det(A) = 3$ and $\det(B) = 5$.

   (c) [2 marks] the characteristic polynomial of $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$.

   (d) [2 marks] $\det \begin{bmatrix} 4 & 3 & 10 & -3 \\ 13 & 5 & e & -5 \\ -9 & 4 & \pi & -4 \\ 14 & -1 & \sqrt{2} & 1 \end{bmatrix}$

2. Let $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$. Show $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal set, and write $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$.

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3. Let $A = \frac{1}{17}\begin{bmatrix} 8 & 15 \\ 15 & -8 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbb{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue. Interpret your result geometrically.

4. Let S be the set of vectors $\vec{x}$ in $\mathbb{R}^4$ such that

$$\det \begin{bmatrix} 1 & 0 & 1 & x_1 \\ 2 & 1 & 0 & x_2 \\ 3 & 1 & 1 & x_3 \\ 5 & 2 & -1 & x_4 \end{bmatrix} = 0.$$

(a) [5 marks] Explain why S is a subspace of $\mathbb{R}^4$.

(b) [5 marks] Find a basis for S and its dimension.

5. Solve the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= -x + 4y \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} &= 8x - 5y \end{cases}$$

for x and y as functions of t, given that $(x, y) = (6, 12)$ when $t = 0$.

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6. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

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7. Let $S = \operatorname{span}\left\{[\ 1\ \ 0\ \ 1\ \ 0\ ]^T, [\ 0\ \ 1\ \ 2\ \ 0\ ]^T, [\ 1\ \ 1\ \ 1\ \ -2\ ]^T\right\}$.

(a) [3 marks] Find a basis for $S^{\perp}$, the orthogonal complement of S.

(b) [4 marks] Find the standard matrix of the linear transformation $\operatorname{perp}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

(c) [3 marks] What is the standard matrix of $\operatorname{proj}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$? (You can make use of part (b).)

8. Prove the following:

(a) [5 marks] if A is an invertible $n \times n$ matrix with eigenvalue λ, then λ^{-1} is an eigenvalue of A^{-1}.

(b) [5 marks] if A is a 7×5 matrix with rank 4, then the matrix $A^T A$ is not invertible.

9. Use the method of least squares to find the best fitting quadratic function $f(x) = a + bx + cx^2$ for the five data points $(x, y) = (-2, 3)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$ and $(2, 8)$.

10. Find all linear transformations $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the range of L is span $\left\{ \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \right\}$ and the null space (or kernel) of L is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

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