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final-7056d

#791

Question	Points	Score
1	15	
2	24	
3	21	
4	18	
5	22	
Total:	100	

Q1a 3

1. (15 points) Clearly circle **all** correct answers. There might be more than one correct answer. Choosing an incorrect answer may negatively affect your mark. **You don't need to show your work.**

(a) Classify the following **linear** transformations as COULD be invertible, MUST be invertible, or NEVER invertible. Clearly circle your choice.

1. $T : \mathbb{R}^n \to \mathbb{R}^n$ where $\ker(T) = \{\vec{0}\}$

   COULD be invertible MUST be invertible NEVER invertible

2. $T : \mathbb{R}^n \to \mathbb{R}^n$ where $T(\vec{v}) = \vec{0}$ for some non-zero $\vec{v} \in \mathbb{R}^n$.

   COULD be invertible MUST be invertible NEVER invertible

3. $T : \mathbb{R}^n \to \mathbb{R}^n$ where $\dim(\operatorname{im} T) = n$.

   COULD be invertible MUST be invertible NEVER invertible

1 4 1

(b) Which of the following is a subspace of $\mathbb{R}^3$.

1. The solution set to $\cancel{x + y} + z = 1$.
2. The set of $\{A\cancel{\vec{x} \mid \vec{x}} \in \mathbb{R}^3\}$, where A is an 4×3 matrix.
3. The set $\{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = 5\vec{v}\}$ where A is an 3×3 matrix. ke $(A - 5I)$
4. The kernel of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$.

Q1b 4

3 & 4

Q1c 4

(c) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ be vectors in $\mathbb{R}^4$. Choose all correct statements.

1. $\text{sp}(\vec{v}_1, \vec{v}_2)$ can be 1-dimensional.
2. $\text{sp}(\vec{v}_1, \vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_2 - \vec{v}_1)$ can be 3-dimensional.
3. $\text{sp}(\vec{v}_1, \vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_2 - \vec{v}_1, \vec{v}_3)$ can be 4-dimensional
4. $\text{sp}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ can be 4 dimensional.

1, 2, 4

Q1d 2

(d) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Suppose $\det(A) = 5$, then the determinant of the matrix $\begin{bmatrix} a_{11} & -3a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

1. -15
2. 15
3. 0
4. insufficient information to decide

4

Q1e 2

(e) Assume $A^T = A$. Consider two distinct vectors $\vec{u} \neq \vec{v}$. Which of the following options is impossible:

1. $A\vec{u} = \vec{u}$ and $A\vec{v} = \vec{v}$ and $\vec{u} \cdot \vec{v} = 0$
2. $A\vec{u} = \vec{u}$ and $A\vec{v} = \vec{v}$ and $\vec{u} \cdot \vec{v} \neq 0$
3. $A\vec{u} = \vec{u}$ and $A\vec{v} = 2\vec{v}$ and $\vec{u} \cdot \vec{v} = 0$
4. $A\vec{u} = \vec{u}$ and $A\vec{v} = 2\vec{v}$ and $\vec{u} \cdot \vec{v} \neq 0$

4 only

Q2 24

2. For each part write your **final** answer in the provided box. You may use the blank area under each question to show your work. **Justify your answer**

(a) (4 points) Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $A = P^T D P$. Write your answer for D and P in the box.

$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

2a) Fully Correct Solution (4/4) **4**

$\lambda = 1, 5$

eigen vector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$

$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

(b) (4 points) Let B be a 3×3 matrix with eigenvalues 0, 1, and -2. What is $rank(B)$?

2

2b) Fully correct solution (4/4) **4**

~~B is similar~~

$B = S^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} S$ invertible matrix

so $BS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} S$

So B is similar to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ so they have equal ~~determinant~~ rank. Since similar matrices have same amount of pivots.

$\text{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 2 = \text{rank}(B)$

(c) (4 points) Let V be the subspace of $\mathbb{R}^3$ defined by the equation: $x_1 + x_2 + x_3 = 0$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto V. That is $T(\vec{x}) = proj_V \vec{x}$.

Find $T\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right)$.

$\boxed{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}}$

basis for V: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$ $v_2^\perp = \begin{bmatrix} -0.5 \\ -0.5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$u_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$

2c) Fully correct solution (4/4) 4

$proj_V \vec{x} = -\frac{1}{\sqrt{2}} u_1 + \frac{-9}{\sqrt{6}} u_2 = \begin{bmatrix} -1/2 & +9/6 \\ 1/2 & +9/6 \\ 0 & -18/6 \end{bmatrix} = \begin{bmatrix} \\ 1/2 + 3/2 \\ -3 \end{bmatrix}$

$\vec{x}$ is already in V.

(d) (4 points) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ and $\det(A) = 2$.

Let $B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ -2a_{21} & -2a_{22} & -2a_{23} & -2a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$. What is the determinant of $A^{-1}B^2$?

$\boxed{8}$

~~det(B) = -det(A) = 2 det(A)~~

B can be obtained by switching row 2 and 3 of A, and multiple row 3 by (-2)

2d) Fully correct solution (4/4) 4

So $\det(B) = -\det(A) \cdot (-2) = 2\det(A) = 4$

$\det(A^{-1}B^2) = \det(A^{-1}) \cdot \det(B)^2 = \det(A)^{-1} \cdot 4^2 = \frac{1}{2} 4^2 = 8$

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

(e) (4 points) Let $A_{2\times 2}$ be a matrix such that $A\vec{e}_1 = \vec{e}_1$ and $A(\vec{e}_1 + \vec{e}_2) = 2\vec{e}_1 + 2\vec{e}_2$. Find A^{-1}.

$$\boxed{\begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}}$$

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2e) Fully correct solution (4/4) **4**

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{2} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

(f) (4 points) Find an orthogonal basis for the kernel of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = A$

$$\boxed{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}}$$

$$\ker(A) = m \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + k \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad m, k \in \mathbb{R}$$

normalize

$$\text{basis}: \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

~~normalize~~

2f) Fully correct solution (4/4) **4**

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$

3. True or false. Clearly state "True" or "False" in the provided box. **Justify** you answer in the blank space under each question. You justification may be a numerical computation, a mathematical reasoning (proof), or a counterexample.

(a) (3 points) The product of two invertible matrices is invertible.

True

3 correct! (3/3) 3

invertible matrix $A \Leftrightarrow \det A \neq 0$

if A, B are invertible matrix,

$\det A \neq 0$, $\det B \neq 0$

$\det AB = \det A \cdot \det B \neq 0$ so $\det(AB) \neq 0$ so AB is invertible

$(AB)^{-1} = B^{-1}A^{-1}$ so inverse of AB exists.

(b) (3 points) Let A be an $n \times m$ matrix and $\vec{b} \in \mathbb{R}^n$. The system $A\vec{x} = \vec{b}$ has a unique solution exactly when (if and only if) $\vec{b} \in \operatorname{im}(A)$.

False

3 correct! (3/3) 3

let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\operatorname{im}(A) = \{\vec{0}\}$

Assume statement is correct, so if $\vec{b} \in \operatorname{im}(A)$, $A\vec{x} = \vec{b}$ has unique solution

$\vec{b} \in \{0\}$ so $\vec{b} \in \operatorname{im}(A)$

$A\vec{x} = \vec{b}$ has solution $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so solution isn't unique, so assumption is false

(c) (3 points) Every diagonalizable matrix is invertible.

False

3 correct! (3/3) 3

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diag... diagonal already so

$A = IDI$ where $D = A$

A^{-1} doesn't exist because $\det A = 0 - 0 = 0$ so A is not invertible

(d) (3 points) Let A be a square matrix such that $\det(A) = 0$, then there is a row of A that is a scalar multiple of another row of A.

$\boxed{\text{Fake}}$

~~$\det(A) = \det = \det(A^T) = 0$~~

3 correct! (3/3) **3**

~~rows of A is columns of A^T.~~

~~since $\det(A^T)=0$, $\det A^T$ is non invertible, so~~

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \end{bmatrix}$

$\det A = 0+0+0$ (Laplace expansion on 3rd column)

$= 0.$

However no row is scalar multiple of other rows:

$[1\ 2\ 0] \neq k_1[1\ 3\ 0]$, $[1\ 3\ 0] \neq [1\ 4\ 0]k_2$, $[1\ 2\ 0] \neq k_3[1\ 4\ 0]$, $k_1, k_2, k_3 \in \mathbb{R}$

(e) (3 points) If 0 is an eigenvalue of B then B is not surjective.

$\boxed{\text{True}}$

Let $v_1 \in \mathbb{R}^n$, $B =$ an $n \times m$ matrix

if 0 is an eigenvalue:

$Bv_1 = \lambda v_1 = 0v_1 = \vec{0}_n$ where $v_1 \neq \vec{0}_n$

since $v_1 \in \mathbb{R}^n$, $\vec{0}_n \in \mathbb{R}^m$,

$\vec{0}_n \in \mathbb{R}^n$ so $n = m$ so B is square matrix.

so B is $n \times n$ matrix

3 correct! (3/3) **3**

... $v_1 \neq \vec{0}_n$, $v_1 \in \ker(B)$ so $\ker(B) \neq \{0\}$

so $\dim(\ker(B)) \neq 0$ so $\mathrm{rank}(B) = n - \dim(\ker(B)) \neq n$

~~so $\mathrm{im}(A) \neq \mathbb{R}^n$~~

~~so there isn't a pivot for every column in B so B is not injective.~~

so there isn't a pivot for every row of B so B is not surjective because $\dim(\mathrm{im}(B)) = \mathrm{rank}(B) \neq n$ so $\mathrm{im}(B) \neq \mathbb{R}^n$ so B is not surjective

(f) (3 points) Suppose A is an $n \times n$ matrix such that A^2 is invertible, then A^5 is also invertible. $\boxed{\text{True}}$

If $\det A = 0$, $\det A^2 = 0$

If $\det A < 0$, $\det(A^2) = \det(A)^2 > 0$

If $\det A > 0$, $\det(A^2) = \det(A)^2 > 0$

~~so $\det(A \cdot A) = \det(A^2) = \det(A)\cdot\det(A)$~~

3 correct! (3/3) 3

so $\det(A^2) = \det(A \cdot A) = \det(A)\det(A) = \det(A)^2$

A^2 is invertible so $\det(A^2) \neq 0$

so $\det(A)^2 \neq 0$ so $\det(A) \neq 0$ so $\det(A)^5 \neq 0$

so $\det(A^5) \neq 0$ so A^5 is invertible

(g) (3 points) Let P be the standard matrix of the orthogonal projection onto an $n-1$ dimensional subspace V of $\mathbb{R}^n$. Then P is symmetric.

$\boxed{\text{True}}$

let V be subspace of $\mathbb{R}^n$, and $\dim(V) = n-1$

let T be an orthogonal projection to V in $\mathbb{R}^n$

$T(\vec{x}) = P\vec{x} = \text{proj}_V(\vec{x})$ can be written as $QQ^T\vec{x}$ where Q is a matrix with columns made up by an orthonormal basis of V

3 correct! (3/3) 3

~~because $\text{proj}_V(\vec{x}) = A(A^TA)^{-1}A^T\vec{x}$, and where A is a matrix with columns being the basis of V.~~

~~if A is orthon~~

so $P\vec{x} = QQ^T\vec{x}$ so $P = QQ^T$

so $P^T = (QQ^T)^T = (Q^T)^T \cdot Q^T = QQ^T = P$ so $P^T = P$ so P is symmetric

Q4 18

$21+6=27$

73

$(4 \cdot 2\ 8 \times 9$ $\quad 7 \cdot 3/9$

$-3 \cdot -2/9$

4. Let vectors $\vec{v}_1$ and $\vec{v}_2$ be linearly independent and

$$A = [\vec{v}_1 \quad \vec{v}_2] = QR = \begin{bmatrix} 1/3 & 8/9 \\ 2/3 & -2/9 \\ 0 & 1/3 \\ 2/3 & -2/9 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 6 \end{bmatrix}$$

(a) (7 points) If $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 7 \\ -3 \end{bmatrix}$, find $\text{proj}_{\text{span}(\vec{v}_1, \vec{v}_2)}(\vec{x})$.

Let $Q = [\vec{u}_1, \vec{u}_2]$, and u_1, u_2 are orthonormal since $u_1 \cdot u_2 = \vec{0}$, $\|u_1\| = \|u_2\| = 1$

$\text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{u}_1, \vec{u}_2)$ since v_1, v_2 are linear combination of u_1, u_2

$$\text{proj}_{\text{span}(\vec{v}_1,\vec{v}_2)}(\vec{x}) = \text{proj}_{\text{span}(\vec{u}_1,\vec{u}_2)}(\vec{x}) = (\vec{x}\cdot\vec{u}_1)\,\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\,\vec{u}_2$$

$$= -2\vec{u}_1 + \left(\frac{7}{3} + \frac{2}{3}\right)\vec{u}_2 = -2\vec{u}_1 + 3\vec{u}_2$$

$$= \begin{bmatrix} -2/3 + \cancel{24/9}\ 8/3 \\ -4/3 \quad -2/3 \\ 0 \quad + 1 \\ -4/3 \quad -2/3 \end{bmatrix} = \begin{bmatrix} 6/3 \\ -6/3 \\ 1 \\ -6/3 \end{bmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \\ -2 \end{pmatrix}$$

perfect explanation:)

4(a) correct 7/7 **7**

Check

$-\frac{2}{3}$ $-\frac{64}{3}$ $-\frac{66}{3}$

$$\begin{bmatrix}1\\2\\0\\2\end{bmatrix} \quad \frac{2+16}{6^3} \quad = \begin{bmatrix}1\\2\\0\\2\end{bmatrix} \quad \begin{bmatrix}8\\0\\2\\0\end{bmatrix}$$

14/3

Let vectors $\vec{v}_1$ and $\vec{v}_2$ be linearly independent and

$$\begin{matrix}16/3 & 8/3\\ -4/3 & -2/3\\ 2 & 1\\ -4/3 & -2/3\end{matrix} \qquad 2-24\begin{pmatrix}-22\\4\\-8\\4\end{pmatrix}$$

$$A = [\vec{v}_1 \ \ \vec{v}_2\,] = QR = \begin{bmatrix}1/3 & 8/9\\ 2/3 & -2/9\\ 0 & 1/3\\ 2/3 & -2/9\end{bmatrix}\begin{bmatrix}3 & 2\\ 0 & 6\end{bmatrix}$$

(b) (6 points) Let $\vec{q}_1$ be the first column of Q and $\vec{q}_2$ be the second column of Q. Let $\mathcal{B} = (\vec{q}_1, \vec{q}_2)$ be a basis for a subspace of $\mathbb{R}^4$. If $\vec{y} = 2\vec{v}_1 - 4\vec{v}_2$, what is $[\vec{y}]_{\mathcal{B}}$?

$T_{\mathcal{B}}^{\mathcal{E}} = [\vec{q}_1 \ \vec{q}_2]$, $v_1 = 3\cdot\vec{q}_1$, $v_2 = 2\cdot\vec{q}_1 + 6\cdot\vec{q}_2$

$\vec{y} = 2\cdot3\vec{q}_1 + -4(2\vec{q}_1 + 6\vec{q}_2) = 6\vec{q}_1 - 8\vec{q}_1 - 24\vec{q}_2 = -2\vec{q}_1 - 24\vec{q}_2$

$[\vec{y}]_{\mathcal{B}} = \cancel{\mathcal{B}} \begin{bmatrix}-2\\-24\end{bmatrix}$

4(b) correct (6/6) 6

(c) (5 points) Solve $A\vec{x} = \begin{bmatrix}8\\4\\2\\4\end{bmatrix}$.

let $\vec{p} = \begin{bmatrix}8\\4\\2\\4\end{bmatrix}$. $\vec{p} = 6\cdot\vec{q}_2 + 8\vec{q}_1$

check: $\begin{bmatrix}\cancel{6/3} & 8/3 + 16/3\\ & 16/3 + -4/3\\ & 0 + 2\\ & 16/3 - 4/3\end{bmatrix} = \begin{bmatrix}24/3\\12/3\\2\\12/3\end{bmatrix} = \begin{bmatrix}8\\4\\2\\4\end{bmatrix}$

$\vec{p} = 8\vec{q}_1 + 6\vec{q}_2 = A\vec{x} = QR\vec{x}$

$= [q_1 \ q_2]\begin{bmatrix}8\\6\end{bmatrix} = [q_1 \ q_2]R\vec{x}$

$\Rightarrow \begin{bmatrix}8\\6\end{bmatrix} = R\vec{x} \Rightarrow \left[\begin{array}{cc|c}3 & 2 & 8\\ 0 & 6 & 6\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}3 & 2 & 8\\ 0 & 1 & 1\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}3 & 0 & 6\\ 0 & 1 & 1\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1 & 0 & 2\\ 0 & 1 & 1\end{array}\right]$

$\vec{x} = \begin{bmatrix}2\\1\end{bmatrix}$

4(c) correct (5/5) 5

Q5 22

v_1 4 -8+5 $(2-\lambda)^2+1$ $(\lambda^2-4\lambda+5)$

2 1 / -1 4

-1 1 / -1 1

$\lambda=-2$ $2\lambda=-4$ 2 1 / -1 2 $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\lambda^2-(2+c)$ λ_{max} = 1.5

5. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ -1 & c \end{bmatrix}$, where $c \in \mathbb{R}$. $(2-\lambda)(1-\lambda)-1$ $-3\lambda+\lambda^2$ $2\lambda-3=0$

(a) (4 points) Verify that A has an eigenvalue of algebraic multiplicity two if and only if $c=0$ or $c=4$.

Σ almu = n = 2

5(a) correct (4/4) 4

$\det(A-\lambda I) = (2-\lambda)(c-\lambda)+1 = \lambda^2-(2+c)\lambda+(2c+1)$, using quadratic formula:

$\Rightarrow \lambda = \frac{2+c}{2} \pm \sqrt{(\frac{2+c}{2})^2-(2c+1)}$ has 1 solution iff discriminant $=0$

$(\frac{2+c}{2})^2-(2c+1)=0 \Rightarrow c=4,0$. If $c=4$ or 0, there is only one λ, so $n=\Sigma$ almu = almu $(\lambda)=2$. If $c=4$, $\lambda=3$, if $c=0$, $\lambda=1$

(b) (3 points) Pick one of the two values $c=0$ or $c=4$. Find the corresponding eigenvalue of A and a basis of the corresponding eigenspace for your choice of c.

let $c=0$.

~~$\ker(A-0I) = \ker(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix})$~~

5(b) correct (3/3) 3

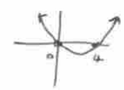
char$(A) = (2-\lambda)(-\lambda)+1 = \lambda^2-2\lambda+1 = (\lambda-1)(\lambda-1) \Rightarrow \lambda=1$

$\ker(A-\lambda I) = \ker(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}) =$ span$(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$

basis of eigenspace: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) (4 points) For which values of c is A diagonalizable (over $\mathbb{R}$)?

if $\lambda_1 \neq \lambda_2$, A is diagonalizable:



from (a): $\lambda = \frac{2+c}{2} \pm \sqrt{(\frac{2+c}{2})^2-(2c+1)} \Rightarrow$ $\lambda_1 \neq \lambda_2 \hookrightarrow (\frac{2+c}{2})^2-(2c+1) > 0$

$1+c+c^2/4-2c-1 = c^2/4-c = c(c/4-1) > 0 \Rightarrow c<0$ or $c>4$

if $0<c<4$, there is no real λ so A is not diagonalizable

if $c=0$, Σ gemu $=1$ so A is not diagonalizable

if $c=4$, Σ gemu $=1$ so A is not diagonalizable

so A is diagonalizable if and only if $c<0$ or $c>4$

5(c) correct (4/4) 4

5 4
2.5 4.5
AS SD

Check

(d) (4 points) When $c = 4.5$, eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ -1 & 4.5 \end{bmatrix}$ are 2.5 and 4. Diagonalize A by finding an invertible S and a diagonal D such that $A = SDS^{-1}$.

$$\ker(A-2.5I) = \ker\left(\begin{bmatrix} -0.5 & 1 \\ -1 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$\ker(A-4I) = \ker\left(\begin{bmatrix} -2 & 1 \\ -1 & 0.5 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$D = \begin{bmatrix} 2.5 & 0 \\ 0 & 4 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

5(d) correct (4/4) **4**

$$SDS^{-1} = \begin{bmatrix} 5 & 4 \\ 2.5 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 4 \\ 2.5 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{3} = \begin{pmatrix} 6/3 & 3/3 \\ -3/3 & 13.5/3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 4.5 \end{pmatrix}$$

(e) (4 points) Continue using $c = 4.5$. Consider a discrete dynamical system given by $\vec{x}_t = A\vec{x}_{t-1}$. Find a closed form for $\vec{x}_t$.

~~Let $\vec{x}_{t-1} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$~~ Let $\vec{x}_0 = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$$\vec{x}_t = A^t \vec{x}_0 = (SDS^{-1})^t \vec{x}_0 = SD^tS^{-1}\vec{x}_0 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2.5^t & 0 \\ 0 & 4^t \end{bmatrix}\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \cdot \frac{1}{3}\right) \cdot \vec{x}_0$$

$$= \begin{bmatrix} 2\cdot 2.5^t & 4^t \\ 2.5^t & 2\cdot 4^t \end{bmatrix}\left(\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 2\cdot 2.5^t & 4^t \\ 2.5^t & 2\cdot 4^t \end{bmatrix}\begin{bmatrix} 2k_1 - k_2 \\ -k_1 + 2k_2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2\cdot 2.5^t(2k_1 - k_2) + 4^t(-k_1 + 2k_2) \\ 2.5^t(2k_1 - k_2) + 2\cdot 4^t(-k_1 + 2k_2) \end{bmatrix}$$

$$\vec{x}_t = \begin{bmatrix} 2/3 \cdot 2.5^t(2k_1 - k_2) + 4^t/3\,(-k_1 + 2k_2) \\ 2.5^t/3\,(2k_1 - k_2) + \frac{2}{3}4^t(-k_1 + 2k_2) \end{bmatrix}$$

5(e) Correct (4/4) **4**

10 8

5 5

25

12.5

20

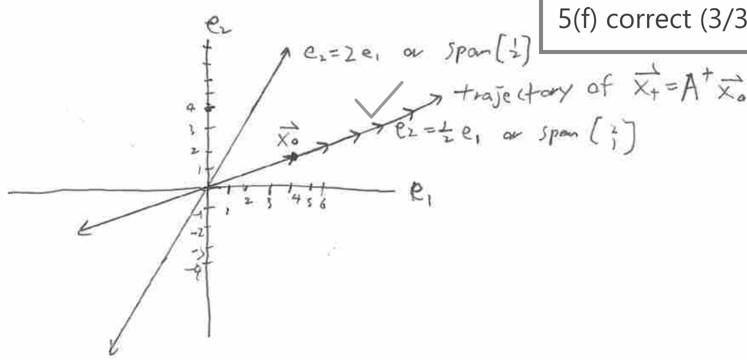
(f) (3 points) Continue using $c = 4.5$.. Suppose the initial state of the dynamical system $\vec{x}_t = A\vec{x}_{t-1}$ is $\vec{x}_0 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Draw a trajectory of this system. What will happen in the long run?

$2k_1 - k_2 = 8 - 2 = 6$

$-k_1 + 2k_2 = 0$

$$\vec{X}_t = \begin{bmatrix} 6 \cdot \frac{2}{3}\, 2.5^t \\ 6/3 \; 2.5^t \end{bmatrix} = \begin{bmatrix} 4 \cdot 2.5^t \\ 2 \cdot 2.5^t \end{bmatrix}$$

5(f) correct (3/3) 3



In long run $\vec{X}_t$ approaches $\begin{bmatrix} \infty \\ \infty \end{bmatrix}$ and will always be on the line $e_2 = \frac{1}{2} e_1$

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