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University of Toronto

Faculty of Applied Science & Engineering

MAT188 – Midterm II, Fall 2021

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Solutions

Question	Points	Score
1	12	
2	15	
3	18	
4	21	
5	14	
Total:	80	

1. (a) (8 points) Clearly circle all correct answers. There might be more than one correct answer. Choosing an incorrect answer may negatively affect your mark. You don't need to show your work.

   1. The determinant of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is

      (i) -1 (ii) 0 (iii) 1 (iv) -4 (v) none of above

      Solution: (i)

   2. Assume $A\vec{x} = \vec{0}$ has a unique solution, where A is an $n \times p$ matrix. Choose all the phrases that are true.

      (i) The linear transformation T defined by $T(\vec{x}) = A\vec{x}$ is one to one (injective).
      (ii) The columns of A span $\mathbb{R}^p$.
      (iii) $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^p$.
      (iv) There is at least one redundant vector among the columns of A.
      (v) The linear transformation T defined by $T(\vec{x}) = A\vec{x}$ is onto.
      (vi) If $A\vec{x} = \vec{b}$ is consistent then it has a unique solution.
      (vii) The columns of A are linearly independent.
      (viii) $n \geq p$.

      Solution: (i), (vi),(vii),(viii)

3. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T(\vec{v}) = \vec{e}_1$ and $T(\vec{u}) = 3\vec{e}_2$. $T(5\vec{u} - 2\vec{v})$ is

(i) $\begin{bmatrix} -2 \\ 151 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 \\ 15 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ (iv) not enough information

Solution: $\begin{bmatrix} -2 \\ 15 \end{bmatrix}$

(b) (4 points) Match the following linear transformations in $\mathbb{R}^2$ with their associated matrix.

I: Reflection about the line $y = x$.

II: Reflection about the x-axis.

III: Orthogonal projection onto the line $y = x$.

IV: Counterclockwise rotation by $\frac{\pi}{4}$ radians.

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

$C = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution: I-D, II-A, III-B, IV-C

2. For each part write your **final** answer in the provided box. You may use the blank area under each question to show your work. No justification is required.

   (a) (3 points) Give an example of a vector in $\mathbb{R}^3$ that is in $\text{Span}\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\end{bmatrix}\}$.

   Solution: take
   $$\begin{bmatrix}1\\2\\-1\end{bmatrix}$$
   any other linear combination of the give vectors work.

   (b) (3 points) Give an explicit example of the standard matrix of a linear transformation that is onto but not one to one.

   Solution: Take
   $$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$
   any matrix whose RREF has a free column work.

(c) (3 points) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the standard matrix of an orthogonal linear transformation. Find A^{-1}.

Solution: $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

(d) (3 points) Let $C = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}$. It is given that $\text{REF}(C) = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. Give a non-trivial linear relation among $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ or explain why that is not possible.

Solution:
For example

$$\vec{v}_3 = -8\vec{v}_1 + 2\vec{v}_2$$

Any nontrivial solution to $C\vec{x} = \vec{0}$ gives a valid nontrivial linear relation among $\vec{v}_i$'s.

(e) (3 points) Let $\mathcal{B} = \{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \}$ be a basis for $\mathbb{R}^3$. Apply Gram-Schmidt to $\mathcal{B}$ to find an orthonormal basis for $\mathbb{R}^3$.

Solution: Let $\vec{b}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Note that $\vec{b}_2$ is already orthogonal to $\vec{b}_1$. Take $\vec{u}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|}$ and $\vec{u}_2 = \frac{\vec{b}_2}{\|\vec{b}_2\|}$, $\vec{b}^{\perp} = \vec{b}_3 - proj_{(\vec{u}_1, \vec{u}_2)}\vec{b}_3 = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \\ \frac{5}{6} \end{bmatrix}$. Hence

$$\vec{u}_1 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{-\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} \frac{-\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

3. True or false? Clearly state "True" or "False" in the provided box. **Justify** you answer in the blank space under each question. You justification may be a numerical computation, a mathematical reasoning (proof) or a counter example.

   (a) (3 points) The matrix $\begin{bmatrix} 3 & 5 & 7 & 2 \\ 6 & 10 & 14 & 4 \\ 4 & 8 & 5 & 3 \\ 2 & 9 & 7 & 3 \end{bmatrix}$ is invertible.

   Solution: False. Let's call this matrix A. Note that the first two rows of A are multiple of each other. Hence columns of A^T are not linearly independent, which implies $\det(A^T) = \det(A) = 0$. Hence A is not invertible.

   (b) (3 points) Let A be an orthogonal matrix. Consider its QR factorization, that is $QR = A$. The matrix R is an identity.

   Solution: True. If A is orthogonal, columns of A are orthonormal. Hence, applying Gram-Schmidt to it result on the same vectors. That is $Q = A$, which implies R is identity.

(c) (3 points) Suppose A is an invertible matrix, then $(ABA^{-1})^4 = B^4$.

Solution: False, $(ABA^{-1})^4 = (ABA^{-1})(ABA^{-1})(ABA^{-1})(ABA^{-1}) = (AB(A^{-1}A)B(A^{-1}A)B(A^{-1}A)BA^{-1}) = (ABIBIBIBA^{-1}) = AB^4A^{-1}$ Take A any non-identity invertible matrix and any B for counter example.

(d) (3 points) The image of the linear transformation

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_2 + 5x_3 \\ -6x_1 + 4x_2 - 10x_3 \end{bmatrix}$ is 2-dimensional.

Solution: False. Consider the standard matrix A of this linear transformation

$$\begin{bmatrix} 3 & -2 & 5 \\ -6 & 4 & -10 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

There is only one pivot among the columns of $RREF(A)$. Hence dimension of the kernel of T is 1.

(e) (3 points) Let A be an $n \times n$ matrix. If $\ker A = \operatorname{im} A$ then $A^2\vec{x} = \vec{0}$ for all $\vec{x}$ in $\mathbb{R}^n$.

Solution: True. $A^2\vec{x} = A(A\vec{x})$. Now $A\vec{x}$ is in $\operatorname{im} A = \ker A$. Hence $A(A\vec{x}) = \vec{0}$

(f) (3 points) Let S be the set of all solutions to $2x + y - z = 0$ except for $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. S is a subspace of $\mathbb{R}^3$.

Solution: False. S is not closed under scalar multiplication. Consider $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \in S$ and $r = \frac{1}{2} \in \mathbb{R}$. $r\vec{v}$ is not in S.

4. Consider the linear transformation $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 3 & 0 & 6 & -9 \\ -2 & 2 & 4 & 0 \\ 0 & 1 & 4 & -3 \end{bmatrix}$.

   (a) (6 points) Find a basis for $ker(T)$. Show your work.

Solution:

$$A = \begin{bmatrix} 3 & 0 & 6 & -9 \\ -2 & 2 & 4 & 0 \\ 0 & 1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$\ker(T) = \{ \begin{bmatrix} -2x_3 + 3x_4 \\ -4x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix}, x_3, x_4 \text{ are free}\} = span(\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix})$$

The vectors $\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}$) are linearly independent and hence make a basis for $\ker T$.

   (b) (5 points) What is the dimension of $im(T)$? Justify.

Solution: We found a basis for kernel of T made of two vectors. Hence dimension of the kernel of T is 2. By rank nullity $\dim imT = 4 - \dim \ker T = 4 - 2 = 2$. Alternatively we can see that exactly two columns of $RREF(A)$ have pivots. Hence exactly two columns of A are linearly independent. Those columns make a basis for imT.

(c) (5 points) Using $\mathcal{B}$ to denote the basis you found in (a), can we find the $\mathcal{B}$-coordinates of $x = \begin{bmatrix} 1 \\ -7 \\ 4 \\ 3 \end{bmatrix}$? If yes, find it and show your work. If not, explain why.

Solution:
First note that $A\vec{x} = \vec{0}$. That is $\vec{x} \in \ker T$ and we can write is as a linear combination of any basis for the kernel of A. Let $\mathcal{B} = (\begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix})$. We solve for $r_1 \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 4 \\ 3 \end{bmatrix}$.

$$\begin{bmatrix} -2 & 3 & 1 \\ -4 & 3 & -7 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

(d) (5 points) Suppose a vector $\vec{v}$ in $\mathbb{R}^4$ satisfies $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find $[\vec{x} + \vec{v}]_{\mathcal{B}}$.

Solution: Note that change of coordinate is a linear transformation:
$[\vec{x} + \vec{v}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

5. Let $a \in \mathbb{R}$ and let $A = \begin{bmatrix} 0 & 1 & -a & 0 \\ 2 & a & -1 & 0 \\ 1 & 1 & 1 & 0 \\ a & 4 & \pi & 1 \end{bmatrix}$.

(a) (5 points) For what values of a is the matrix A invertible?

Solution: Computing the determinant using a Laplace expansion along the last column, we have $\det A = \det \begin{bmatrix} 0 & 1 & -a \\ 2 & a & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Computing this determinant using a Laplace expansion along the first row, we get:

$$(-1)(3) - a(2-a) = a^2 - 2a - 3 = (a-3)(a+1).$$

This is zero if and only if $a = -1$ or 3. So the matrix A is invertible if $a \neq -1, 3$.

(b) (4 points) Suppose the number a is chosen so that A is invertible, and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4\}$ be bases of a vector space V so that the matrix A is the change-of-basis matrix $S_{\mathcal{B}\to\mathcal{C}}$ from $\mathcal{B}$ to $\mathcal{C}$. Find the $\mathcal{C}$-coordinate of $\vec{b}_3$.

Solution:

$A = S_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & [\vec{b}_3]_{\mathcal{C}} & [\vec{b}_4]_{\mathcal{C}} \end{bmatrix}$, that is $[\vec{b}_3]_{\mathcal{C}} = \begin{bmatrix} -a \\ -1 \\ 1 \\ \pi \end{bmatrix}$

(c) (5 points) Explain why $\mathcal{B}$ and $\mathcal{C}$ must have a vector in common.

Solution: $S_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} & [\vec{b}_3]_{\mathcal{C}} & [\vec{b}_4]_{\mathcal{C}} \end{bmatrix}$. So if $S_{\mathcal{B}\to\mathcal{C}} = A$, then $[\vec{b}_4]_{\mathcal{C}} = \vec{e}_4$. This means that

$$\vec{b}_4 = 0\vec{c}_1 + 0\vec{c}_2 + 0\vec{c}_3 + 1\vec{c}_4,$$

so $\vec{b}_4 = \vec{c}_4$ and thus $\mathcal{B} \cap \mathcal{C} \neq \emptyset$.

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