UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

Quiz 5

Date: Jun 11, 2025
Duration: 50 minutes

Course: MAT 187 (Calculus II)
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Instructions

- This is a Type A assessment and **does not** allow any external aids.
- Read all instructions carefully and **justify all your answers.** No points will be awarded for a correct answer without justification.
- Read each question carefully. **No clarification or content related questions will be answered.**

You may use the following space for scratch work or to continue your solutions if you run out of room. If you do so, please clearly indicate in the original question that part of your solution appears here.

1. (2 points) Explain in words the qualitative difference between a stable and an unstable equilibrium.

Solution: A **stable equilibrium** is one where nearby solutions tend to move toward the equilibrium over time, meaning small disturbances away from die out and the solution settles back at the equilibrium. An **unstable equilibrium** is one where nearby solutions move away from the equilibrium, so small disturbances grow over time, never returning to the equilibrium.

2. (3 points) Can the ODE

$$y' - e^y y = e^x$$

be solved with the method of integrating factors? Justify your answer. Which types of ODE does the method of integrating factor apply to?

Solution: The method of integrating factors applies to **first-order, linear** differential equations of the form

$$y' + P(x)y = Q(x),$$

where both $P(x)$ and $Q(x)$ are functions of x only, and the equation is linear in y.

The given ODE is

$$y' - e^y y = e^x.$$

We observe that the term $e^y y$ depends nonlinearly on y, since it is the product of y and an exponential function of y. Therefore, the equation is *not linear* in y, and does not match the required form for applying an integrating factor.

3. (4 points) You're told that the salt concentration $C(t)$ in a well-mixed tank increases quickly at first and then levels off over time. Which of the following differential equations could model this behavior?

A. $\dfrac{dC}{dt} = -kC$ **B.** $\dfrac{dC}{dt} = k(C_{\text{in}} - C)$ **C.** $\dfrac{dC}{dt} = k$

Explain your reasoning.

Solution: Answer: B.

This equation models a situation where the concentration approaches a limiting value C_{in} over time. If the initial concentration is below C_{in}, then dC/dt is initially large and positive. As C approaches C_{in}, the rate of change decreases, and the concentration levels off.

Option A represents exponential decay, where the concentration decreases toward zero. **Option C** represents constant growth; the concentration increases linearly over time and never levels off.

4. (5 points) Consider the nonlinear differential equation

$$y'' = 6y^2$$

Determine whether the function

$$y(x) = \frac{1}{(Ax+B)^2}$$

satisfies the equation. Find any constraints on the constants A and B that are required for it to be a solution.

Solution: Let us compute the first and second derivatives of $y(x)$. First,

$$y' = \frac{d}{dx}\left((Ax+B)^{-2}\right) = -2A(Ax+B)^{-3}.$$

Next, the second derivative is

$$y'' = \frac{d}{dx}\left(-2A(Ax+B)^{-3}\right) = 6A^2(Ax+B)^{-4}.$$

Now we compute the right-hand side of the differential equation:

$$6y^2 = 6\left(\frac{1}{(Ax+B)^2}\right)^2 = 6(Ax+B)^{-4}.$$

We see that for the ODE to be satisfied, we need:

$$\begin{aligned} y'' = 6A^2(Ax+B)^{-4} &= 6(Ax+B)^{-4} = 6y^2 \\ A^2 &= 1 \\ \Rightarrow \qquad A &= \pm 1 \end{aligned}$$

Conclusion: The function $y(x) = \frac{1}{(Ax+B)^2}$ satisfies the differential equation $y'' = 6y^2$ provided that $A = \pm 1$. The constant B can be any real number.

5. (6 points) Solve the initial value problem

$$y' - 2y = e^{3x}, \qquad y(0) = 2$$

Solution: This is a linear first-order ODE and can be solved using the method of integrating factors. The standard form is

$$y' + P(x)y = Q(x),$$

with $P(x) = -2$ and $Q(x) = e^{3x}$

Compute the integrating factor.

$$\mu(x) = e^{\int -2\,dx} = e^{-2x}$$

Multiply both sides of the ODE by the integrating factor:

$$\begin{aligned} e^{-2x}y' - 2e^{-2x}y &= e^x \\ \frac{d}{dx}(e^{-2x}y) &= e^x \end{aligned}$$

Integrate both sides:

$$\int \frac{d}{dx}(e^{-2x}y)\,dx = \int e^x\,dx \quad \Rightarrow \quad e^{-2x}y = e^x + C \quad \Rightarrow \quad y(x) = e^{3x} + Ce^{2x}$$

Apply the initial condition $y(0) = 1$:

$$2 = e^0 + Ce^0 = 1 + C \Rightarrow C = 1$$

Final Answer:

$$\boxed{y(x) = e^{3x} + e^{2x}}$$

6. A cup of tea is initially at a temperature of 85°C and is left to cool in a room that is kept at 20°C. According to Newton's Law of Cooling, the rate at which the tea cools is proportional to the difference between its temperature and the ambient temperature.

   (a) (7 points) **Write** and **solve** an initial value problem for the temperature $T(t)$ of the tea at time t (in minutes), assuming the cooling constant is $k = 0.1$.

Solution: Newton's Law of Cooling gives the differential equation:

$$\frac{dT}{dt} = -k(T - T_{\text{env}}),$$

where $T_{\text{env}} = 20$ and $k = 0.1$. Substituting in:

$$\frac{dT}{dt} = -0.1(T - 20).$$

This is a separable equation. We separate variables and integrate:

$$\int \frac{1}{T-20}\,dT = -0.1 \int dt \quad \Rightarrow \quad \ln|T - 20| = -0.1t + C.$$

Exponentiating both sides:

$$|T - 20| = e^{-0.1t+C} = Ae^{-0.1t},$$

where $A = e^C > 0$, so:

$$T(t) = 20 + Ae^{-0.1t}.$$

Apply the initial condition $T(0) = 85$:

$$85 = 20 + A \Rightarrow A = 65.$$

Final answer:

$$\boxed{T(t) = 20 + 65e^{-0.1t}}.$$

   (b) (3 points) How long will it take for the tea to cool to 50°C?

Solution: Set $T(t) = 50$ and solve for t:

$$50 = 20 + 65e^{-0.1t} \Rightarrow 30 = 65e^{-0.1t} \Rightarrow e^{-0.1t} = \frac{6}{13}.$$

Take the natural logarithm:

$$-0.1t = \ln\left(\frac{6}{13}\right) \Rightarrow t = -\frac{1}{0.1}\ln\left(\frac{6}{13}\right).$$

Final answer:

$$\boxed{t = 10\ln\left(\frac{13}{6}\right)}$$