University of Toronto
Solutions to **MAT188H1F TERM TEST**
of **Tuesday, October 30, 2012**
Duration: 100 minutes

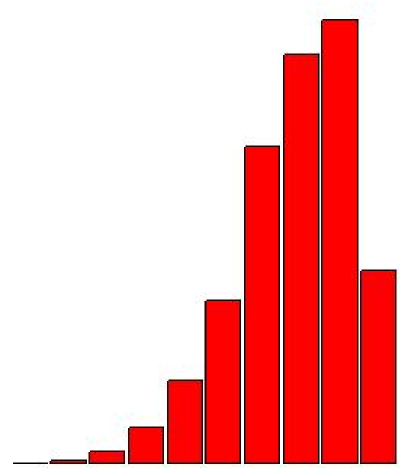
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.
Instructions: Make sure this test contains 6 sheets with questions on both sides. Do not tear any pages from this test. Present your solutions to all 11 questions in the space provided. The value for each question is indicated in parantheses beside the question number. **Total Marks: 60**

General Comments about the Test:

1. Since more than a third of the class got A, and almost two-thirds received at least B, the results on this test are generally quite good. But if your test mark is much less than 3 times your quiz mark, your performance is slipping, and you may be in trouble, since things always seem much harder in Chapter 4.
2. It was quite amazing how many students could *not* solve Question 2 correctly.
3. Questions 1, 2, 5, 6, 7, 8 and 9 can all be double-checked; these questions should all have been aced.
4. $\det(A + B) = \det(A) + \det(B)$ is false, but this was used by some students in some of their calculations, for example in Questions 5 and 10(c).
5. There are still students putting = between reduced matrices, or using implication ($\Rightarrow$) incorrectly. This is just throwing away marks.

Breakdown of Results: 987 students wrote this test. The marks ranged from 16.7% to 100%, and the average was 72.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	11.7%
A	38.4%	80-89%	26.7%
B	24.6%	70-79%	24.6%
C	19.1%	60-69%	19.1%
D	9.8%	50-59%	9.8%
F	8.0%	40-49%	5.0%
		30-39%	2.1%
		20-29%	0.7%
		10-19%	0.2%
		0-9%	0.0%



1. [4 marks] Find the inverse of the matrix $A = \begin{bmatrix} 2 & -2 & -1 \\ 2 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

Solution 1: use the adjoint formula. Since $\det(A) = -(-1)^3 = 1$,

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \text{adj}(A).$$

The 9 cofactors of A are:

$$\begin{aligned}
C_{11} &= \det\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} & C_{12} &= -\det\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} & C_{13} &= \det\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \\
&= 0 & &= 0 & &= -1 \\
C_{21} &= -\det\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} & C_{22} &= \det\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} & C_{23} &= -\det\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix} \\
&= 0 & &= -1 & &= 2 \\
C_{31} &= \det\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix} & C_{32} &= -\det\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} & C_{33} &= \det\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \\
&= -1 & &= -2 & &= 2
\end{aligned}$$

So

$$A^{-1} = \text{adj}(A) = [Cij]^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & -2 \\ -1 & 2 & 2 \end{bmatrix}.$$

Solution 2: use the Gaussian algorithm.

$$(A|I) = \left[\begin{array}{rrr|rrr} 2 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} -1 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -2 & -1 & 1 & 0 & 0 \end{array}\right] \to$$

$$\left[\begin{array}{rrr|rrr} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 2 \\ 0 & -2 & -1 & 1 & 0 & 2 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & 1 & -2 & -2 \end{array}\right] \to$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & 2 & 2 \end{array}\right] = (I|A^{-1})$$

2. [6 marks] Solve the following system of linear equations

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & x_4 & - & 2x_5 & = & 5 \\ 2x_1 & - & x_2 & + & x_3 & - & x_4 & + & x_5 & = & 2 \\ x_1 & + & 4x_2 & + & 5x_3 & + & 4x_4 & - & 7x_5 & = & 13 \end{array}$$

by first finding the reduced row-echelon form of its augmented matrix.

Solution: reduce the augmented matrix:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 2 & 1 & -2 & 5 \\ 2 & -1 & 1 & -1 & 1 & 2 \\ 1 & 4 & 5 & 4 & -7 & 13 \end{array}\right] \to \left[\begin{array}{rrrrr|r} 1 & 1 & 2 & 1 & -2 & 5 \\ 0 & -3 & -3 & -3 & 5 & -8 \\ 0 & 3 & 3 & 3 & -5 & 8 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 1 & -2 & 5 \\ 0 & 1 & 1 & 1 & -5/3 & 8/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1/3 & 7/3 \\ 0 & 1 & 1 & 1 & -5/3 & 8/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = s, x_4 = t, x_5 = u$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7/3 - s + u/3 \\ 8/3 - s - t + 5u/3 \\ s \\ t \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 7/3 \\ 8/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 1/3 \\ 5/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Not required; just neater.}}.$$

3. [5 marks] Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 2 \\ -1 & 2 & 0 & 3 \\ 2 & -1 & 2 & 1 \end{bmatrix}.$$

Calculate $\det(A)$.

Solution: use properties of determinants to simplify your work.

$$\det \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 2 \\ -1 & 2 & 0 & 3 \\ 2 & -1 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 4 & 1 & 6 \\ 2 & 7 & 2 & 8 \\ -1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 7 \end{bmatrix} = (-1)(-1)^4 \det \begin{bmatrix} 4 & 1 & 6 \\ 7 & 2 & 8 \\ 3 & 2 & 7 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 4 & 1 & 6 \\ -1 & 0 & -4 \\ -5 & 0 & -5 \end{bmatrix} = -(-1)^3 \det \begin{bmatrix} -1 & -4 \\ -5 & -5 \end{bmatrix} = 5 \det \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} = (5)(-3) = -15$$

4. [6 marks] Find all the values of c for which the system of equations

$$\begin{array}{rcrcrcr} x_1 & + & c\,x_2 & + & c\,x_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 4 \\ c\,x_1 & + & c\,x_2 & + & x_3 & = & 2 \end{array}$$

has no solution.

Solution: let A be the coefficient matrix. If $\det(A) \neq 0$, then A is invertible and the system will have a unique solution.

$$\det \begin{bmatrix} 1 & c & c \\ 1 & -1 & 1 \\ c & c & 1 \end{bmatrix} = -1 + 3c^2 - 2c = (3c+1)(c-1).$$

If $c = 1$, then the system is

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 2 \end{array}$$

which has infinitely many solutions, since the first and third equations are identical.

If $c = -1/3$, then the system is equivalent to the system

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & - & x_3 & = & 6 \\ x_1 & - & x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & - & 3x_3 & = & -6 \end{array}$$

which is inconsistent, since the second plus the third equation gives $2x_1 - 2x_3 = -2$, while the first plus the third equation gives $4x_1 - 4x_3 = 0$. **Answer:** $c = -1/3$

5. [5 marks] Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}.$$

Solution: need the characteristic polynomial of A.

Accept either $\det(\lambda I - A)$ or $\det(A - \lambda I)$. Working with the former:

$$\begin{aligned}
\det\begin{bmatrix} \lambda-3 & 4 & -2 \\ -1 & \lambda+2 & -2 \\ -1 & 5 & \lambda-5 \end{bmatrix} &= \det\begin{bmatrix} \lambda-2 & 2-\lambda & 0 \\ -1 & \lambda+2 & -2 \\ 0 & 3-\lambda & \lambda-3 \end{bmatrix} \\
&= (\lambda-2)(\lambda-3)\det\begin{bmatrix} 1 & -1 & 0 \\ -1 & \lambda+2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \\
&= (\lambda-2)(\lambda-3)\det\begin{bmatrix} 1 & 0 & 0 \\ -1 & \lambda+1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \\
&= (\lambda-2)(\lambda-3)(\lambda+1-2) \\
&= (\lambda-2)(\lambda-3)(\lambda-1);
\end{aligned}$$

so the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

Alternate Approach: simply expanding $\det(\lambda I - A)$ gives

$$\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

which has to be factored! If the expansion is incorrect then the factoring my be very difficult!

6. [7 marks] Given that the eigenvalues of

$$A = \begin{bmatrix} 0 & 4 & 6 \\ 9 & 0 & 9 \\ 6 & 4 & 0 \end{bmatrix}$$

are $\lambda_1 = -6$ and $\lambda_2 = 12$, find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Solution: find the eigenvectors.

$$(-6I - A|O) = \left[\begin{array}{ccc|c} -6 & -4 & -6 & 0 \\ -9 & -6 & -9 & 0 \\ -6 & -4 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 3 & 2 & 3 & 0 \\ 3 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];$$

so there are two (basic) eigenvectors corresponding to $\lambda_1 = -6$. For example, any two of

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix},$$

or their multiples, will do.

$$(12I - A|O) = \left[\begin{array}{ccc|c} 12 & -4 & -6 & 0 \\ -9 & 12 & -9 & 0 \\ -6 & -4 & 12 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 6 & -2 & -3 & 0 \\ 3 & -4 & 3 & 0 \\ 3 & 2 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 6 & -2 & -3 & 0 \\ 0 & -6 & 9 & 0 \\ 0 & 6 & 9 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 6 & -2 & -3 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 6 & 0 & -6 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];$$

so an eigenvector corresponding to $\lambda_2 = 12$ is

$$\begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

Then (for example) take

$$\underbrace{P = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -3 & 3 \\ 1 & 1 & 2 \end{bmatrix}, \; D = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 12 \end{bmatrix}.}_{\text{Columns of } P \text{ must correspond to correct diagonal entries of } D.}$$

7. [6 marks] Show that the triangle ΔPQR with vertices $P(1,1,1),\ Q(2,1,0),\ R(4,2,2)$ is a right angle triangle.

Solution 1:

$$\overrightarrow{PQ} = \begin{bmatrix} 2-1 \\ 1-1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix};\ \overrightarrow{PR} = \begin{bmatrix} 4-1 \\ 2-1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix};\ \overrightarrow{QR} = \begin{bmatrix} 4-2 \\ 2-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Observe that $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 2 + 0 - 2 = 0$, so the sides PQ and QR are orthogonal. That is, the interior angle at Q is $\pi/2$.

Solution 2: Observe that

$$\|\overrightarrow{PQ}\|^2 + \|\overrightarrow{QR}\|^2 = 1 + 1 + 4 + 1 + 4 = 11 = \|\overrightarrow{PR}\|^2,$$

which means the lengths of the three sides in ΔPQR satisfy the Pythagorean Theorem, whence the triangle is a right triangle.

8. [5 marks] If

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \vec{d} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix},$$

express $\vec{v}$ in the form $\vec{v} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_1$ is parallel to $\vec{d}$ and $\vec{v}_2$ is orthogonal to $\vec{d}$.

Solution: use the projection formula.

$$\vec{v}_1 = \text{proj}_{\vec{d}}\vec{v} = \frac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\vec{d} = \frac{3+2-1}{9+1+1} = \frac{4}{11}\vec{d} = \frac{1}{11}\begin{bmatrix} 12 \\ 4 \\ -4 \end{bmatrix}.$$

Then

$$\vec{v}_2 = \vec{v} - \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 12 \\ 4 \\ -4 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -1 \\ 18 \\ 15 \end{bmatrix}.$$

9. [6 marks] Given that $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are (basic) eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},$$

find all the solutions to the following system of differential equations:

$$\begin{array}{rcrcr} f_1'(x) & = & f_1(x) & + & 2f_2(x) \\ f_2'(x) & = & 4f_1(x) & + & 3f_2(x) \end{array}$$

where f_1 and f_2 are functions of x.

Solution: you can find the eigenvalues by using the definition of eigenvector:

$$AX_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -X_1 \Rightarrow \lambda_1 = -1;$$

$$AX_2 = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5X_2 \Rightarrow \lambda_2 = 5.$$

Or you can find them directly:

$$\det \begin{bmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{bmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Either way, the general solution is $F = c_1 X_1 e^{\lambda_1 x} + c_2 X_2 e^{\lambda_2 x}$. Thus

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-x} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5x};$$

that is,

$$f_1(x) = c_1\, e^{-x} + c_2\, e^{5x}$$

and

$$f_2(x) = -c_1\, e^{-x} + 2\, c_2\, e^{5x}.$$

10. [6 marks] Indicate if the following statements are True or False, and give a brief explanation why.

(a) [2 marks] If A is a square matrix such that $\operatorname{adj}(A)$ is the zero matrix, then $\det(A) = 0$.

True **False**

Solution: True.

$$\begin{aligned} A\operatorname{adj}(A) = \det(A)I \quad &\Rightarrow \quad O = \det(A)I \\ &\Rightarrow \quad \det(A) = 0 \end{aligned}$$

Alternate Solution: Every cofactor of A is zero, so the cofactor expansion along any row or column of A must be zero; that is $\det(A) = 0$.

(b) [2 marks] If A is a 3×3 matrix such that $A^T = -A$, then $\det(A) = 0$.

True **False**

Solution: True.

$$\begin{aligned} A^T = -A \quad &\Rightarrow \quad \det(A^T) = \det(-A) \\ &\Rightarrow \quad \det(A) = (-1)^3 \det(A) \\ &\Rightarrow \quad \det(A) = -\det(A) \\ &\Rightarrow \quad \det(A) = 0 \end{aligned}$$

(c) [2 marks] If A is a 3×3 matrix such that $A^3 + 4A = 7I$, then $\det(A) = 0$.

True **False**

Solution: False.

$$\begin{aligned} A^3 + 4A = 7I \quad &\Rightarrow \quad A(A^2 + 4I) = 7I \\ &\Rightarrow \quad \det(A(A^2 + 4I)) = \det(7I) \\ &\Rightarrow \quad \det(A)\det(A^2 + 4I) = 7^3 \neq 0 \\ &\Rightarrow \quad \det(A) \neq 0 \end{aligned}$$

11. [4 marks] Suppose A is an invertible matrix. Indicate if the following statements are True or False, and give a brief explanation why.

(a) [2 marks] A and A^{-1} have the same eigenvalues. **True** **False**

Solution: False. Consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

It has eigenvalue $\lambda = 2$, repeated. But its inverse is

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix},$$

which has eigenvalue $\mu = 1/2$, repeated, and $\mu \neq \lambda$.

(b) [2 marks] A and A^{-1} have the same eigenvectors. **True** **False**

Solution: True.
Recall that A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A. Then

$$AX = \lambda X \Leftrightarrow X = A^{-1}(\lambda X) \Leftrightarrow X = \lambda A^{-1}X \Leftrightarrow \frac{1}{\lambda}X = A^{-1}X.$$

Thus X is an eigenvector of A if and only if X is an eigenvector of A^{-1}.