University of Toronto
Faculty of Applied Sciences and Engineering

MAT188 – Midterm I – Fall 2024

LAST (Family) NAME: ______________________

FIRST (Given) NAME: ______________________

Email address: ______________________@mail.utoronto.ca

STUDENT NUMBER: ______________________

Time: 90 mins.

1. **Keep this booklet closed** until an invigilator announces that the test has started. You may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **in the boxes**. There is ample space within each one. If you must use additional space, please use the blank pages at the end of this booklet and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 18 pages, excluding the cover page, and 7 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work.
7. No textbook, notes, or other outside assistance is allowed.
8. **No calculator** or equivalent devices allowed.

Question:	1	2	3	4	5	6	7	Total
Points:	12	9	10	6	9	6	8	60
Score:								

1 Part A

1. (12 points) Fill in the bubble for all statements that **must** be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. For those questions, you may get a negative mark for incorrectly filled bubbles.

   (a) The magnitude of $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is

   ○ 4 ○ 2 ○ $\sqrt{6}$ ○ 6

   (b) Consider the sets ℓ_1 and ℓ_2:

$$\ell_1 = \left\{ t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \mid t_1 \in \mathbb{R} \right\}$$

$$\ell_2 = \left\{ t_2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mid t_2 \in \mathbb{R} \right\}$$

   Choose all that apply.

   ○ ℓ_1 and ℓ_2 represent lines in $\mathbb{R}^2$ ○ ℓ_1 and ℓ_2 are parallel

   ○ ℓ_1 and ℓ_2 intersect at a point ○ ℓ_1 and ℓ_2 do not intersect

(c) For a system with two distinct equations and four variables, which of these **could** be true of the solution set? Mark all that apply

- ◯ The solution set could be empty
- ◯ The solution set could contain a single vector
- ◯ The solution set could have exactly one parameters
- ◯ The solution set could have exactly two parameters
- ◯ The solution set could have exactly three parameters

(d) For what value(s) of a will the following linear system be inconsistent?

$$\begin{aligned} ax + y &= 1 \\ 2x + (3a+1)y &= -2 \end{aligned}$$

◯ $a = 1$ ◯ $a = -1$

◯ $a = \frac{2}{3}$ ◯ No such value exists

(e) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Let $\vec{v} \in \mathbb{R}^3$. Choose all that apply.

- ◯ $A\vec{v}$ is not defined
- ◯ $A\vec{v}$ is a vector in $\mathbb{R}^2$
- ◯ $A\vec{v}$ is a vector in $\mathbb{R}^3$
- ◯ $A\vec{v}$ is parallel to all columns of A.
- ◯ $A\vec{v}$ is perpendicular to all columns of A.

2. Fill in the blank. You do not need to include your computation or reasoning.

(a) (2 points) Suppose M is a linear transformation given by left multiplication by a 4×5 matrix. The domain of M is $\boxed{}$ and the codomain of M is $\boxed{}$

(b) (1 point) Suppose $\begin{bmatrix}1\\3\\6\end{bmatrix} = 2\vec{v}_1 + 3\vec{v}_2 + 5\vec{v}_3$. Let $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$ be a matrix that has $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ as its columns.

Then $A\begin{bmatrix}2\\3\\5\end{bmatrix}$ is $\boxed{}$.

(c) (2 points) Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation satisfying $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}-3\\7\end{bmatrix}$ and for all $a \in \mathbb{R}$,

$$T\left(\begin{bmatrix}-2+a\\-4+3a\end{bmatrix}\right) - T\left(\begin{bmatrix}a\\-a\end{bmatrix}\right) = \begin{bmatrix}6+8a\\-14+12a\end{bmatrix},$$

The second column of the standard matrix of T is $\boxed{}$.

(d) (4 points) An angle between two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{R}^3$ is defined to be the angle between any two vectors perpendicular to each plane. Using this definition, calculate the angle between the planes given by equations $\mathcal{P}_1\colon\ 3x + 4y + 5z = 12$ and $\mathcal{P}_1\colon\ 7x + y = 8$. You can leave your answer in terms of $\cos^{-1}$.

The angle between the planes is $\boxed{}$

Find the vector parametric equation of the intersection line of the planes $\mathcal{P}_1$ and $\mathcal{P}_2$, described in set notation.

$\mathcal{P}_1 \cap \mathcal{P}_2 = \boxed{}$

Consider the vectors perpendicular to $\mathcal{P}_1$ and $\mathcal{P}_2$ you chose to calculate the angle between the planes. Call them $\vec{v}_1$ and $\vec{v}_2$ respectively. Let $\vec{d}$ be a nonzero vector in $\mathcal{P}_1 \cap \mathcal{P}_2$. Then

- ○ $\vec{d}$ perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.
- ○ $\vec{v}_1, \vec{v}_2$ are parallel to $\vec{d}$.
- ○ $\vec{d}$ is parallel to at least one of $\vec{v}_1$ and $\vec{v}_2$.
- ○ None of the given statements are true.
- ○ $\vec{v}_1, \vec{v}_2$ and $\vec{d}$ are all perpendicular to each other.

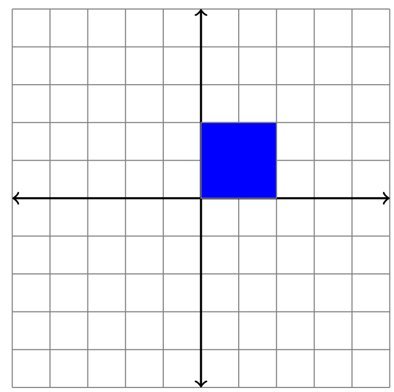
Part B

3. Consider the following linear transformations:

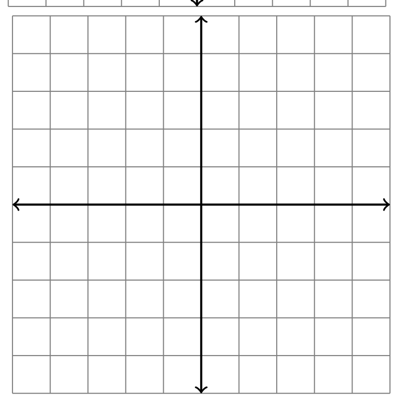
$R : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors counter-clockwise about the origin by $\frac{\pi}{4}$ (or, if you prefer, 45°).

$T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors with respect to the line ℓ where

$$\ell = \{t \begin{bmatrix} 1 \\ -1 \end{bmatrix} , t \in \mathbb{R}\}.$$



The following square has corners at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is called the unit square.

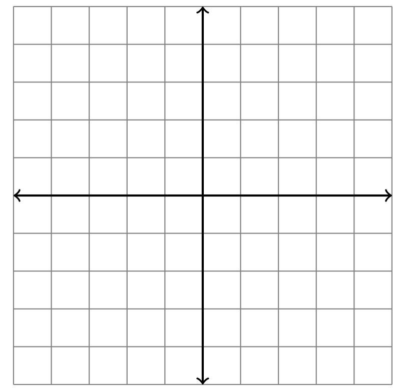


(a) (2 points) Draw the output of the unit square after applying R.

(b) (2 points) Let A be the standard matrix of R. That is $R(\vec{x}) = A\vec{x}$, for all $\vec{x} \in \mathbb{R}^2$. Find A. Justify your answer.

A=

(c) (2 points) Draw the output of the unit square after applying $T \circ R$



(d) (2 points) Let C be the standard matrix of $T \circ R$. That is $T \circ R(\vec{x}) = C\vec{x}$. Find C. Justify your answer.

C=

(e) (2 points) Find a matrix B such that $BA = C$. Justify your answer.

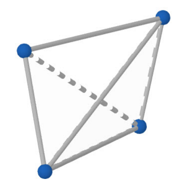
B=

4. Given a set of n particles in $\mathbb{R}^3$ with position vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ and masses $m_1, m_2, ..., m_n$, the position vector of the center of mass of the system is

$$\vec{v}_{cm} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n),$$

where $M = m_1 + m_2 + \cdots + m_n$.

Consider four particles with position vectors



$$\vec{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}2\\1\\-1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-3\\1\\0\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}-2\\1\\-2\end{bmatrix}$$

forming a pyramid in the space.

We want to distribute a total mass of 9 kg among these particles so that the position vector of the centre of the mass becomes $\vec{v}_{cm} = \frac{1}{9}\begin{bmatrix}4\\4\\1\end{bmatrix}$. You can assume that the mass of the rest of the pyramid itself is negligible.

(a) (2 points) Set up a system of linear equations whose general solution is the set of all the possible values of m_1, m_2, m_3, m_4 that give us the desired center of mass.

(b) (3 points) Find a matrix A and a vector $\vec{b}$ such that your system is equivalent to $A\vec{x} = \vec{b}$.

$A =$ and $\vec{b} =$

(c) (1 point) You can assume that the matrix A in the previous part row-reduces to the 4×4 identity matrix. That is

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Is it possible to distribute the mass of 9 kg among these particles so that the position vector of the centre of the mass becomes $\vec{v}_{cm} = \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$? Justify your choice.

○ Yes ○ No

5. State whether each statement is true or false by writing “True” or “False” in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true, explain why it must be true. If you think a statement is false, give a counterexample.

   (a) (3 points) If the sum of two vectors in $\mathbb{R}^n$ is $\vec{0}$, they are parallel.

   (b) (3 points) A vector is called a unit vector if its magnitude (or norm) is one. The intersection between the set of unit vectors perpendicular to $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in $\mathbb{R}^3$ and the set of unit vectors perpendicular to $\begin{bmatrix}0\\1\\0\end{bmatrix}$ in $\mathbb{R}^3$ is has exactly two vectors.

   (c) (3 points) Let $\vec{c}$ be a solution to $A\vec{x} = \vec{0}$, and $\vec{y}$ be a solution to $A\vec{x} = \vec{b}$. Then $\vec{c} + \vec{y}$ is a solution to $A\vec{x} = \vec{b}$.

6. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

   (a) (2 points) A linear transformation T which maps the set $L = \left\{ \begin{bmatrix} 8 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ t \in \mathbb{R} \right\}$ to the set $T(L) = \left\{ t \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}, t \in \mathbb{R} \right\}$.

   (b) (2 points) The normal equation of a plane in $\mathbb{R}^3$ that passes through the origin and contains the points $(1, 0, 4)$ and $(1, 1, 1)$.

   (c) (2 points) A linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that stretches the x-axis by a factor of 5 and does not change the y-axis.

Part C

7. Methane is CH_4 (that is, one carbon atom and four hydrogen atoms) and propane is C_3H_8 (three carbon, eight hydrogen). When a mixture of the two is burned in the right conditions, the chemical reaction produces only water (H_2O) and carbon dioxide (CO_2):

$$a \cdot CH_4 + b \cdot C_3H_8 + c \cdot O_2 \mapsto d \cdot CO_2 + e \cdot H_2O$$

   (a) (2 points) Create a system of equations to represent a balanced reaction (that is, the number of atoms of each type on one side must equal the number from the other side).

   (b) (1 point) Put your variables in alphabetical order and find the augmented matrix corresponding to this system.

(c) (1 point) Find the RREF of the augmented matrix of your system.

(d) (2 points) Solve the system and describe the solution in set notation.

(e) (2 points) Why does the solution have the number of parameters that appear? Interpret your answer in terms of the given chemical reaction.

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

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