Faculty of Applied Science & Engineering, University of Toronto
MAT188H1F - Linear Algebra
Fall 2016

Tutorial Problems 2

1 (a) Does the line through the point $P(1,2,-3)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ lie in the plane $2x_1 - x_2 - x_3 = 3$? Explain.

Solution: The vector equation of the line passing through $P(1,2,-3)$ with direction vector $\mathbf{d}$ is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \tag{1}$$

where $t \in \mathbb{R}$. Notice that $2(1+t) - (2+2t) - (-3-3t) = 3 + 3t \neq 3$, so that the line does not pass through the given plane.

1 (b) Is it true that every plane that contains the points $P(1,2,-1)$ and $Q(2,0,1)$ also contains the point $R(-1,6,-5)$? Explain.

Solution: Yes, since R is on the line through P and Q. To see this, note that the line through the points P and Q is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} -1 \\ 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \tag{3}$$

1 (c) Show that if a plane contains two distinct points P and Q, then it contains every point on the line through P and Q.

Solution: Notice that the equation of the line passing through $P(x_p, y_p, z_p)$ and $Q(x_q, y_q, z_q)$ is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} + t \begin{bmatrix} x_q - x_p \\ y_q - y_p \\ z_q - z_p \end{bmatrix} = \begin{bmatrix} (1-t)x_p + tx_q \\ (1-t)y_p + ty_q \\ (1-t)z_p + tz_q \end{bmatrix} \tag{4}$$

where $t \in \mathbb{R}$. Consider a plane $ax + by + cz + d = 0$ that contains P and Q. Therefore, the coordinates of P and Q satisfy the plane equation, so that:

$$ax_p + by_p + cz_p + d = 0 \tag{5}$$
$$ax_q + by_q + cz_q + d = 0 \tag{6}$$

Notice the following:

$$(1-t)(ax_p + by_p + cz_p + d) + t(ax_q + by_q + cz_q + d) = (1-t)(0) + t(0) = 0 \tag{7}$$

or equivalently,

$$a((1-t)x_p + tx_q) + b((1-t)y_p + ty_q) + c((1-t)z_p + tz_q) = 0 \tag{8}$$

which means that any point on the line passing through P and Q, i.e. has coordinates as in (4), satisfy the equation of the plane.

2. We define the distance between two points P and Q in $\mathbb{R}^n$ to be the length of the line segment from P to Q (or, equivalently, Q to P).

(a) Let P and Q be two points in $\mathbb{R}^3$. Show that the set of points that are equidistant (equal distance) from P and Q is a plane, determine a normal vector and a point on the plane.

Solution 1: Let $\mathbf{v} \in \mathbb{R}^3$. Then

$$\begin{aligned}
\mathbf{v} \text{ is equidistant from } P \text{ and } Q &\iff ||\mathbf{v} - P|| = ||\mathbf{v} - Q|| \\
&\iff ||\mathbf{v} - P||^2 = ||\mathbf{v} - Q||^2 \quad \text{(since lengths are non-negative)} \\
&\iff (\mathbf{v} - P) \cdot (\mathbf{v} - P) = (\mathbf{v} - Q) \cdot (\mathbf{v} - Q) \\
&\iff (\mathbf{v} - P) \cdot (\mathbf{v} - P) - (\mathbf{v} - Q) \cdot (\mathbf{v} - Q) = 0 \\
&\iff \big((\mathbf{v} - P) - (\mathbf{v} - Q)\big) \cdot \big((\mathbf{v} - P) + (\mathbf{v} - Q)\big) = 0 \\
&\iff (Q - P) \cdot \big(2\mathbf{v} - P - Q\big) = 0 \\
&\iff (Q - P) \cdot \big(\mathbf{v} - \tfrac{1}{2}(P + Q)\big) = 0
\end{aligned}$$

Here, to get from the 4th line to the 5th, we use a "difference of squares" formula: for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we have $\mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$.

Notice that the final line is the equation of a plane with normal vector $\overrightarrow{PQ} = Q - P$, containing the midpoint $\frac{1}{2}(P + Q)$ of P and Q.

(b) Find the scalar equation of the plane, each point of which is equidistant from the points $P(0, 1, -1)$ and $Q(2, -1, -3)$. Determine a normal vector and a point on the plane.

Solution: Using the result from part (a), we have the normal vector to be $\mathbf{n} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$, and a point on the plane is the midpoint of between P and Q, i.e. has coordinates $(1, 0, -2)$. An equation of the plane is thus: $2x - 2y - 2z = 6$.

(c) Consider the plane you found in part (b). Find the (shortest) distance from the point $R(1, 1, 1)$ to the plane and determine the point Q on the plane closest to the point $R(1, 1, 1)$.

Solution: The shortest distance is $||\text{proj}_{\mathbf{n}} \vec{PR}|| = || - \frac{1}{6} \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} || = \frac{\sqrt{3}}{3}$. The point on the plane closest to R is:

$$\vec{P} + \vec{PR} - \text{proj}_{\mathbf{n}} \vec{PR} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - (-\frac{1}{6} \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}) = \frac{2}{3} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \tag{9}$$

3. Let $\mathbf{d}$ be a non-zero vector in $\mathbb{R}^3$. Let $\text{proj}_{\mathbf{d}}(\mathbf{x})$ be the orthogonal projection of $\mathbf{x}$ onto $\mathbf{d}$.

(a) Show that for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

(i) $\text{proj}_{\mathbf{d}}(\mathbf{x} + \mathbf{y}) = \text{proj}_{\mathbf{d}}(\mathbf{x}) + \text{proj}_{\mathbf{d}}(\mathbf{y})$

(ii) $\text{proj}_{\mathbf{d}}(c\mathbf{x}) = c\,\text{proj}_{\mathbf{d}}(\mathbf{x})$

These are called the linearity properties of the orthogonal projection.

Solution: First we recall that $\text{proj}_{\mathbf{d}}(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$. Also, recall that the dot product has the following properties: for every $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$ and $c\in\mathbb{R}$ we have

$$(\mathbf{x}+\mathbf{y})\cdot\mathbf{d} = \mathbf{x}\cdot\mathbf{d}+\mathbf{y}\cdot\mathbf{d} \quad \text{and} \quad (c\mathbf{x})\cdot\mathbf{d} = c(\mathbf{x}\cdot\mathbf{d})$$

Therefore

$$\text{proj}_{\mathbf{d}}(\mathbf{x}+\mathbf{y}) = \frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{\mathbf{x}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} + \frac{\mathbf{y}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \text{proj}_{\mathbf{d}}(\mathbf{x}) + \text{proj}_{\mathbf{d}}(\mathbf{y}),$$
$$\text{proj}_{\mathbf{d}}(c\mathbf{x}) = \frac{(c\mathbf{x})\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = c\frac{\mathbf{x}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = c\ \text{proj}_{\mathbf{d}}(\mathbf{x})$$

(b) Notice by property (ii), $\text{proj}_{\mathbf{d}}(-\mathbf{x}) = -\text{proj}_{\mathbf{d}}(\mathbf{x})$ for every $\mathbf{x}\in\mathbb{R}^3$. Show algebraically that $\text{proj}_{-\mathbf{d}}(\mathbf{x}) = \text{proj}_{\mathbf{d}}(\mathbf{x})$ for every $\mathbf{x}\in\mathbb{R}^3$, and explain this result geometrically.

Solution: By dealing with the negative signs, we get

$$\text{proj}_{-\mathbf{d}}(\mathbf{x}) = \frac{\mathbf{x}\cdot(-\mathbf{d})}{\|-\mathbf{d}\|^2}(-\mathbf{d}) = (-1)^2\frac{\mathbf{x}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \text{proj}_{\mathbf{d}}(\mathbf{x})$$

since $\mathbf{x}\cdot(-\mathbf{d}) = -(\mathbf{x}\cdot\mathbf{d})$ and $\|-\mathbf{d}\|^2 = |-1|^2\|\mathbf{d}\|^2 = \|\mathbf{d}\|^2$.

Geometrically, this corresponds to the fact that $\mathbf{d}$ and $-\mathbf{d}$ generate the same line through the origin. The projections $\text{proj}_{\mathbf{d}}(\mathbf{x})$ and $\text{proj}_{-\mathbf{d}}(\mathbf{x})$ can both be defined as the point on this line which is nearest to $\mathbf{x}$, hence they are equal.

(c) Show algebraically that $\text{proj}_{\mathbf{d}}(\text{proj}_{\mathbf{d}}(\mathbf{x})) = \text{proj}_{\mathbf{d}}(\mathbf{x})$ for every $\mathbf{x}\in\mathbb{R}^3$, and explain this result geometrically.

Solution:

$$\begin{aligned}
\text{proj}_{\mathbf{d}}(\text{proj}_{\mathbf{d}}(\mathbf{x})) &= \text{proj}_{\mathbf{d}}\Big(\frac{\mathbf{x}\cdot\mathbf{d}}{||\mathbf{d}||^2}\mathbf{d}\Big)\\
&= \frac{\mathbf{x}\cdot\mathbf{d}}{||\mathbf{d}||^2}\frac{\mathbf{d}\cdot\mathbf{d}}{||\mathbf{d}||^2}\mathbf{d}\\
&= \frac{\mathbf{x}\cdot\mathbf{d}}{||\mathbf{d}||^2}\mathbf{d}\\
&= \text{proj}_{\mathbf{d}}(\mathbf{x})
\end{aligned}$$

Geometrically, this corresponds to the fact that projecting a vector proportional to $\mathbf{d}$ onto $\mathbf{d}$ is the vector itself.

(d) What can you say about the vectors $\mathbf{d}$ and $\mathbf{e}$ if $\text{proj}_{\mathbf{d}}(\text{proj}_{\mathbf{e}}(\mathbf{x})) = \mathbf{0}$ for every $\mathbf{x}\in\mathbb{R}^3$? Explain.

Solution:

(d) Assume that $\text{proj}_{\mathbf{d}}(\text{proj}_{\mathbf{e}}(\mathbf{x})) = \mathbf{0}$ for every $\mathbf{x}\in\mathbb{R}^3$. Then certainly $\text{proj}_{\mathbf{d}}(\text{proj}_{\mathbf{e}}(\mathbf{e})) = \mathbf{0}$. Since $\text{proj}_{\mathbf{e}}(\mathbf{e}) = \mathbf{e}$, we have

$$\mathbf{0} = \text{proj}_{\mathbf{d}}(\mathbf{e}) = \frac{\mathbf{e}\cdot\mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$$

Since we've assumed that $\mathbf{d}\neq\mathbf{0}$, this implies that $\mathbf{e}\cdot\mathbf{d}=0$. i.e. $\mathbf{e}$ and $\mathbf{d}$ are orthogonal.

Conversely, if $\mathbf{e}\cdot\mathbf{d}=0$ then $\text{proj}_{\mathbf{d}}(\text{proj}_{\mathbf{e}}(\mathbf{x}))=0$ for all $\mathbf{x}\in\mathbb{R}^3$. Indeed $\text{proj}_{\mathbf{d}}(\mathbf{x})$ is always proportional to $\mathbf{d}$, and the assumption $\mathbf{d}\cdot\mathbf{e}=0$ implies that $\text{proj}_{\mathbf{e}}(c\mathbf{d})=0$ for any $c\in\mathbb{R}$.

Geometrically this situation corresponds to projecting onto a line, then projecting the result onto a second line which is *perpendicular* to the first; the result will always be 0.

4. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$. Decide if the following statements are true or false. If the statement is true, prove it; if it's false, find a counterexample. You may find that drawing a picture helps.

(a) If $\mathbf{x}$ and $\mathbf{y}$ are non-zero, and a and b are positive, the angle between $a\mathbf{x}$ and $b\mathbf{y}$ is the same as the angle between $\mathbf{x}$ and $\mathbf{y}$.

Solution: True. The angle between $a\mathbf{x}$ and $b\mathbf{y}$ is:

$$\cos^{-1}\left(\frac{a\mathbf{x} \cdot b\mathbf{y}}{\|a\mathbf{x}\|\|b\mathbf{y}\|}\right) = \cos^{-1}\left(\frac{a\mathbf{x} \cdot b\mathbf{y}}{|a|\|\mathbf{x}\||b|\|\mathbf{y}\|}\right) = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \tag{10}$$

which is the same as the angle between $\mathbf{x}$ and $\mathbf{y}$.

(b) If $\mathbf{x}$ is orthogonal to both $\mathbf{y}$ and $\mathbf{z}$, then $\mathbf{x}$ is orthogonal to every vector in $\text{span}\{\mathbf{y}, \mathbf{z}\}$.

Solution: True. Let $\mathbf{w} = \mathbf{t_1}\mathbf{y} + \mathbf{t_2}\mathbf{z}$ be in the span of the two vectors. Notice that $\mathbf{x} \cdot \mathbf{w} = \mathbf{t_1}\mathbf{x} \cdot \mathbf{y} + \mathbf{t_2}\mathbf{x} \cdot \mathbf{z} = \mathbf{t_1}(\mathbf{0}) + \mathbf{t_2}(\mathbf{0}) = \mathbf{0}$, because $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $\mathbf{z}$. Thus, $\mathbf{x}$ is orthogonal to any vector in the span of $\mathbf{y}$ and $\mathbf{z}$.

(c) $||\mathbf{x} + \mathbf{y}|| = ||\mathbf{x} - \mathbf{y}||$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Solution: True. Notice

$$\begin{aligned}
& ||\mathbf{x} + \mathbf{y}|| = ||\mathbf{x} - \mathbf{y}|| && (11)\\
\iff & (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) && (12)\\
\iff & ((\mathbf{x} + \mathbf{y}) - (\mathbf{x} - \mathbf{y})) \cdot \mathbf{x} + ((\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y})) \cdot \mathbf{y} = \mathbf{0} && (13)\\
\iff & (2\mathbf{y}) \cdot \mathbf{x} + (\mathbf{2x}) \cdot \mathbf{y} = \mathbf{0} && (14)\\
\iff & (\mathbf{y}) \cdot \mathbf{x} + (\mathbf{x}) \cdot \mathbf{y} = \mathbf{0} && (15)\\
\iff & \mathbf{x} \cdot \mathbf{y} = \mathbf{0} && (16)\\
& && (17)
\end{aligned}$$