Midterm II

March 19, 2024 10:08 AM

University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 – Midterm II –Winter 2024

LAST (Family) NAME: _____

FIRST (Given) NAME: _____

Email address: _____@mail.utoronto.ca

STUDENT NUMBER: _____

Time: 90 mins.

1. **Keep this booklet closed** until an invigilator announces that the test has begun. You may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **inside the boxes** whenever provided. Ample space is provided within each box. However, if you must use additional space, please use the blank pages at the end of this booklet, and clearly indicate in the given box that your answer is **continued on the blank pages**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 12 pages, excluding the cover page, and 5 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work for parts B and C. You don't need to justify your choices in part A.
7. No notes or outside help is allowed in your workspace.
8. No calculator is allowed.

Question:	1	2	3	4	5	Total
Points:	8	9	7	6	10	40
Score:						

Part A

1. (8 points) Fill in the bubble next to each statements that must be true. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles. You don't need to include your work or reasoning.

   (a) Consider a function $f(x)$ that has derivatives of every order and that is defined for all real numbers. Let $p_{2024}(x)$ and $p_{2025}(x)$ denote the 2024th and 2025th Taylor polynomial of $f(x)$ centred at $a = 1$. Is it possible that $p_{2024}(x)$ and $p_{2025}(x)$ are the same polynomial?

   ◯ It can NEVER happen that $p_{2024}(x)$ and $p_{2025}(x)$ are the same polynomial.

   ● It SOMETIMES happens that $p_{2024}(x)$ and $p_{2025}(x)$ are the same polynomial.

   ◯ It is ALWAYS the case that $p_{2024}(x)$ and $p_{2025}(x)$ are the same polynomial.

   ◯ This CAN'T BE ANSWERED in general as possibly $p_{2024}(x)$ and/or $p_{2025}(x)$ might not be defined.

   (b) For which of the given value(s) of x does the following series diverge?

   $$\sum_{n=0}^{\infty} \frac{(-1)^n \sin^n(x)}{n+1}$$

   alternating test

   ◯ $x = 0$ ◯ $x = \pi$ ◯ $x = \frac{7\pi}{4}$ ◯ $x = \frac{\pi}{2}$ ● $x = \frac{3\pi}{2}$

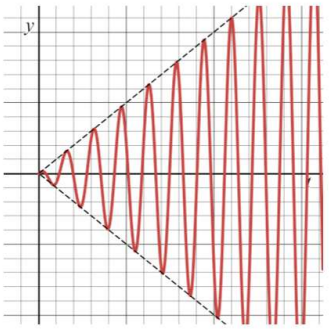
   Ls ratio test

   (c) Suppose $\sum_{n=0}^{\infty} c_n(x+1)^n$ converges at $x = 2$ and diverges at $x = 3$. Decide whether it converges or diverges at the following values of x.

   centre is −1 $R < 3$

$x = -1$	● Converges	◯ Diverges	◯ We can't tell
$x = -3$	● Converges	◯ Diverges	◯ We can't tell
$x = -8$	◯ Converges	● Diverges	◯ We can't tell

(d) Which of the following ordinary differential equations can have a solution with the following graph? The dashed lines are $y = \pm t$.



$r^2 + 4 = 0$

$r = \pm 2i$

$y_p = t\sin(2t)$

- ○ $y'' + 4y = t$
- ○ $y'' - 2y' + 2y = 0$
- ○ $y'' + 4y = \sin(4t)$
- ● $y'' + 4y = -4\sin(2t)$

(e) Let f be a function defined over all real numbers and with derivatives of all orders. Assume that the third Taylor polynomial. $p_3(x)$ of $f(x)$, centred at $x = 1$, is given by

$$p_3(x) = 1 - \frac{1}{2}(x-1)^2 + 20(x-1)^3.$$

Select the one true statement.

$f(1) = 1$

$f'(1) = 0$

$f''(1) = \left(\frac{-1}{2}\right)(2) = -1$

$f'''(1) = (20)(6) = 120$

- ○ $f(x)$ is increasing and concave up at $x = 1$.
- ○ $f(x)$ is decreasing and concave up at $x = 1$.
- ○ $f(x)$ is decreasing and concave down at $x = 1$.
- ○ $f(x)$ is increasing and concave up at $x = 1$.
- ● None of the above statements are true.

Part B

2. (a) (5 points) Suppose we are given a homogeneous, linear second-order initial value problem (IVP), with initial data provided at $t = 0$. We are told that the solution is

$$y(t) = \sin(9t) + \cos(9t).$$

Find the initial value problem. Justify your answer.

roots of char poly are $\pm 9i$

$r^2 + 81 = 0 \Longrightarrow y'' + 81y = 0$

general sol'n $y = C_1 \sin(9t) + C_2 \cos(9t)$

$y(0) = C_2 = 1$

$y'(0) = 9C_1 = 9 \Rightarrow C_1 = 1$

$y' = 9C_1\cos(9t) - 9C_2\sin(9t)$

wait $C_1 = C_2 = 1$

The ODE is $\boxed{1}\, y'' + \boxed{0}\, y' + \boxed{81}\, y = \boxed{0}$

The initial values are $\boxed{y(0) = 1,\ y'(0) = 9}$

(b) (4 points) Suppose the general solution of the ordinary differential equation

$$ay'' + by' + cy = f(t)$$

is

$$y_g(t) = k_1 e^{2t} + k_2 e^{-2t} - e^t.$$

Find constants a, b, c and a possible function $f(t)$, that would give y_g as the solution.

$ar^2 + br + c = (r-2)(r+2) = r^2 - 4 \Longrightarrow y'' - 4y = f(t)$

$y_p = -e^t$ plug in $y'' - 4y = f(t)$

$-e^t + 4e^t = 3e^t \Rightarrow \boxed{f(t) = 3e^t}$

$a = \boxed{1}$ $b = \boxed{0}$ $c = \boxed{-4}$ $f(t) = \boxed{3e^t}$

3. Consider the series $\sum_{n=0}^{\infty} a_n$ with $0 + \frac{1}{2} + 0 + \frac{3^2}{2^3} + 0 + \frac{5^2}{2^5} + 0 + \dots$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ n^2/2^n, & \text{if } n \text{ is odd} \end{cases}$$

(a) (2 points) The ratio test cannot be applied to $\sum_{n=0}^{\infty} a_n$. Explain why.

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ some a_n's are zero!

specifically $\frac{a_{2n+1}}{a_{2n}}$ is not defined

(b) (2 points) Find a series $\sum_{n=0}^{\infty} b_n$ such that

i) Each b_n is found somewhere in $\{a_n\}$.

ii) If $\sum_{n=0}^{\infty} a_n$ converges to S, $\sum_{n=0}^{\infty} b_n$ also converge to S.

iii) The ratio test can be applied to $\sum_{n=0}^{\infty} b_n$

remove zeros from the sequence! $\sum_{n=0}^{\infty} \frac{(2n+1)^2}{2^{2n+1}}$

$b_n =$ $\boxed{\frac{(2n+1)^2}{2^{2n+1}}}$

(c) (3 points) Does $\sum_{n=0}^{\infty} a_n$ converge? Justify your answer. yes

$\lim_{n\to\infty} \left| \frac{(2n+3)^2 \left(2^{2n+1}\right)}{2^{2n+3} (2n+1)^2} \right| = \lim_{n\to\infty} \left| \frac{4n^2+12n+9}{4[4n^2+4n+1]} \right| = \frac{1}{4} < 1$

4. Parts of this question are unrelated

(a) (3 points) Find the radius of convergence R of $\sum_{n=0}^{\infty} \frac{(x-5)^n}{4^n}$.

ratio test or notice

$= \sum_{n=1}^{\infty} \left(\frac{x-5}{4}\right)^n$ is a geometric series $\frac{1}{1}$ convergent, if

$\left|\frac{x-5}{4}\right| < 1 \Rightarrow |x-4| < 4$

$R =$ 4

(b) (3 points) Let $f(x)$ be a function whose 4-th derivative is given by

$$f^{(4)}(x) = \frac{3!\cos x}{x^2}. \quad < \frac{3!}{4^2}$$

If you want to use the 3rd Taylor polynomial $p_3(x)$, centred at $a = 4$, to approximate $f(6)$, which is the smallest error bound that you can guarantee? Justify your choice.

- ○ $|f(6) - p_3(6)| \leq 1/9$
- ● $|f(6) - p_3(6)| \leq 1/4$
- ○ $|f(6) - p_3(6)| \leq 1/3$
- ○ $|f(6) - p_3(6)| \leq 4/27$

$R_3(6) = \frac{f^{(4)}(c)(6-4)^4}{4!} \leq \frac{3! \cdot 2^4}{4^2 \, 4!}$

$4 \leq c \leq 6$ $= \frac{2^4}{4^3} = \frac{4^2}{4^3} = \frac{1}{4}$

Part B

5. A damped spring-mass system is modelled by

$$mx'' + bx' + kx = 0,$$

where m is the mass, $b > 0$ is the damping constant, $k > 0$ is the spring (stiffness) constant and $x(t)$ is displacement from the equilibrium position.

(a) (2 points) Determine a condition between the damping constant b and parameters m and k so that the solution to the ODE leads to oscillating motion.

$b^2 - 4mk < 0 \Rightarrow b^2 < 4mk$

(b) (3 points) The general solution of the ODE, assuming that the solution leads to oscillatory motion, is in the form

$$e^{At}(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)), \quad \text{where } C_1, C_2 \text{ in } \mathbb{R} \tag{1}$$

Find A and ω_d.

$mr^2 + br + k = 0 \Rightarrow r_1, r_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2m}$

$\dfrac{\sqrt{(-1)(4ac - b^2)}}{2m} = i^{\circ} \dfrac{\sqrt{4ac - b^2}}{2m}$

$A =$ $-b/2m$ and $\omega_d =$ $\dfrac{\sqrt{4k - b^2}}{2m}$

(c) (3 points) Suppose the damping constant b is zero, and that an external force $f(t) = \sin(3t)$ is applied to the system. The mass-spring system is now modelled by

$$mx'' + kx = \sin(3t)$$

If $m = 1$ kg/s and $k = 9$ N/m find the complementary solution, a particular solution and the general solution to the system.

$x'' + 9x = \sin(3t)$ $\quad y_p = At\cos(3t)$ find A.

$r^2 = -9$

$r = \pm 3i$

The complimentary solution is $C_1\cos(3t) + C_2\sin(3t)$

A particular solution is $y_p = -\frac{1}{6}t\cos(3t)$

The general solution is $C_1\cos(3t) + C_2(\sin(3t)) - \frac{1}{6}t\cos(3t)$

(d) (2 points) Explain what will happen to the mass-spring system in part (c) in the long run.

$\lim_{t\to\infty} y = \lim_{t\to\infty}\left(C_1\cos(3t) + C_2(\sin(3t)) - \frac{1}{6}t\cos(3t)\right) = \infty$

The frequency of the outside force matches the internal frequency of the system. In the long run, the magnitude of the oscillation will increase without a bound.

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

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Trigonometric Functions

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \sec^3(x)\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

$$\int \csc(x)\,dx = -\ln|\cot(x)+\csc(x)| + C$$

$$\int \tan(x)\,dx = -\ln|\cos x| + C$$

$$\int \cot(x)\,dx = \ln|\sin x| + C$$

Error formula for L_n and R_n

$|\text{Error}| \le \dfrac{M}{2n}(b-a)^2$

Where $M \ge |f'(x)|$ on $a \le x \le b$

Error formula for M_n

$|\text{Error}| \le \dfrac{M}{24n^2}(b-a)^3$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Error formula for T_n

$|\text{Error}| \le \dfrac{M}{12n^2}(b-a)^3$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Taylor Polynomial $\quad p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Remainder/Error $\quad f(x) = P_n(x) + R_n(x)$ $\quad$ with $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

If $|f^{(n+1)}(x)| \le M$, then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$