

Tutorial Problems 11

1. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis for \mathbb{R}^n .

(a) If $\mathbf{x} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$ and $\mathbf{y} = b_1\mathbf{v}_1 + \dots + b_n\mathbf{v}_n$, show that

$$\mathbf{x} \cdot \mathbf{y} = a_1b_1 + \dots + a_nb_n.$$

(b) If $\mathbf{x} = a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n$, show that

$$\|\mathbf{x}\|^2 = a_1^2 + \dots + a_n^2.$$

(c) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal basis for \mathbb{R}^3 . If $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$, $\|\mathbf{x}\| = 5$, $\mathbf{x} \cdot \mathbf{v}_1 = 4$, and \mathbf{x} is orthogonal to \mathbf{v}_2 , what are the possible values of a_1, a_2 , and a_3 ?

Solution:

(a) We will use three important properties of the dot product:

$$(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{w} = \mathbf{u}_1 \cdot \mathbf{w} + \mathbf{u}_2 \cdot \mathbf{w}$$

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u}, \quad (c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w})$$

valid for any vectors $\mathbf{u}, \mathbf{w}, \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Applying the first two properties, we can expand

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= (a_1\mathbf{v}_1 + \dots + a_n\mathbf{v}_n) \cdot (b_1\mathbf{v}_1 + \dots + b_n\mathbf{v}_n) \\ &= (a_1\mathbf{v}_1) \cdot (b_1\mathbf{v}_1) + (a_1\mathbf{v}_1) \cdot (b_2\mathbf{v}_2) + \dots + (a_1\mathbf{v}_1) \cdot (b_n\mathbf{v}_n) \\ &\quad + (a_2\mathbf{v}_2) \cdot (b_1\mathbf{v}_1) + (a_2\mathbf{v}_2) \cdot (b_2\mathbf{v}_2) + \dots + (a_2\mathbf{v}_2) \cdot (b_n\mathbf{v}_n) \\ &\quad + \dots \\ &\quad + (a_n\mathbf{v}_n) \cdot (b_1\mathbf{v}_1) + (a_n\mathbf{v}_n) \cdot (b_2\mathbf{v}_2) + \dots + (a_n\mathbf{v}_n) \cdot (b_n\mathbf{v}_n) \end{aligned}$$

Since the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ form an orthonormal set, by definition we have

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1, & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

Hence, using the third property above,

$$(a_i\mathbf{v}_i) \cdot (b_j\mathbf{v}_j) = a_i b_j (\mathbf{v}_i \cdot \mathbf{v}_j) = \begin{cases} a_i b_j, & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

Therefore most terms in our expression for $\mathbf{x} \cdot \mathbf{y}$ are zero, except those where both indices are equal, leaving

$$\mathbf{x} \cdot \mathbf{y} = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

(b) Recall that $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$. Applying part (a) with $\mathbf{y} = \mathbf{x}$, we get

$$\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = a_1^2 + a_2^2 + \dots + a_n^2$$

(c) By part (a),

$$\mathbf{x} \cdot \mathbf{v}_1 = a_1, \quad \mathbf{x} \cdot \mathbf{v}_2 = a_2$$

So $\mathbf{x} \cdot \mathbf{v}_1 = 4$ implies that $a_1 = 4$. Also, \mathbf{x} is orthogonal to \mathbf{v}_2 means that $\mathbf{x} \cdot \mathbf{v}_2 = 0$, which implies that $a_2 = 0$. By part (b),

$$\|\mathbf{x}\|^2 = a_1^2 + a_2^2 + a_3^2$$

Substituting that $\|\mathbf{x}\| = 5$, $a_1 = 4$, and $a_2 = 0$, we get

$$5^2 = 4^2 + 0^2 + a_3^2$$

so that $a_3^2 = 9$. Therefore $a_3 = \pm 3$.

2. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$. Note that S is an orthogonal subset of \mathbb{R}^4 .

(a) Let $W = \text{Span}(S)$. Find $\text{proj}_W \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(b) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $W = \text{Span}(S)$. Find $\text{proj}_W(\mathbf{x})$ and a matrix A such that $\text{proj}_W(\mathbf{x}) = A\mathbf{x}$.

(c) What are the eigenvalues and eigenspaces of the matrix A from part (b)? Use geometric reasoning instead of finding the characteristic polynomial of A (i.e. you should decide what the eigenvalues and eigenspaces are by thinking about what A does to vectors). If A is diagonalizable, then write down an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution:

(a)

$$\text{proj}_W \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right) = \frac{1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 + 1 \cdot (-1)}{1^2 + 2^2 + 1^2 + (-1)^2} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + \frac{1 \cdot 0 + 1 \cdot (-1) + 1 \cdot 1 + 1 \cdot (-1)}{0^2 + (-1)^2 + 1^2 + (-1)^2} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

(b) In general,

$$\text{proj}_W(\mathbf{x}) = \frac{x_1 \cdot 1 + x_2 \cdot 2 + x_3 \cdot 1 + x_4 \cdot (-1)}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + \frac{x_1 \cdot 0 + x_2 \cdot (-1) + x_3 \cdot 1 + x_4 \cdot (-1)}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Pulling out a common denominator, this is

$$\frac{1}{21} (x_1 \cdot 1 + x_2 \cdot 2 + x_3 \cdot 1 + x_4 \cdot (-1)) \begin{bmatrix} 3 \\ 6 \\ 3 \\ -3 \end{bmatrix} + \frac{1}{21} (x_1 \cdot 0 + x_2 \cdot (-1) + x_3 \cdot 1 + x_4 \cdot (-1)) \begin{bmatrix} 0 \\ -7 \\ 7 \\ -7 \end{bmatrix}$$

which simplifies to

$$\frac{1}{21} \begin{bmatrix} (3)x_1 + (6)x_2 + (3)x_3 + (-3)x_4 \\ (6)x_1 + (19)x_2 + (-1)x_3 + (1)x_4 \\ (3)x_1 + (-1)x_2 + (10)x_3 + (-10)x_4 \\ (-3)x_1 + (1)x_2 + (-10)x_3 + (10)x_4 \end{bmatrix}.$$

It follows that

$$A = \frac{1}{21} \begin{bmatrix} 3 & 6 & 3 & -3 \\ 6 & 19 & -1 & 1 \\ 3 & -1 & 10 & -10 \\ -3 & 1 & -10 & 10 \end{bmatrix}.$$

(c) Remember a key fact about projection is that if $\mathbf{x} \in W$ then $\text{proj}_W \mathbf{x} = \mathbf{x}$ which means \mathbf{x} is an eigenvector of A with eigenvalue 1, while if $\mathbf{x} \in W^\perp$ then $\text{proj}_W \mathbf{x} = \mathbf{0}$ so that \mathbf{x} is an eigenvector of A with eigenvalue 0.

The vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ are in W and so are eigenvectors with eigenvalue 1. To find the eigenvectors with eigenvalue 0 we need a basis for W^\perp . This amounts to finding a basis for the null space of

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & -3 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

The basis vectors are $\begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. So we get the matrices

$$P = \begin{bmatrix} 1 & 0 & -3 & 3 \\ 2 & -1 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\}$ and let $W = \text{Span}(S)$.

(a) Show that S is an orthogonal basis for W .

(b) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Find $\text{proj}_W(\mathbf{x})$. What does your answer tell you about \mathbf{x} ?

Solution:

(a) Call the vectors in S $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. A straightforward check shows $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$, and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$ so that S is orthogonal and so linearly independent (see Theorem 1 from Section 7.1 of the textbook) and so an orthogonal basis for W .

(b) You may verify that $\text{proj}_W(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2} + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2} + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\|\mathbf{u}_3\|^2} = \mathbf{x}$ which tells you that $\mathbf{x} \in W$.

$$4. \text{ Let } W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- (a) Find an orthogonal basis for W .
- (b) Find an orthogonal basis for W^\perp .
- (c) Show that the basis you found in part (a) together with the basis you found in part (b) constitutes an orthogonal basis for \mathbb{R}^5 .

Solution:

- (a) Apply the Gram–Schmidt process to the spanning set:

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ \mathbf{v}_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ 3 \\ -2 \\ 3 \\ -7 \end{bmatrix} \\ \mathbf{v}_3 &= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 3/5 \\ -2/5 \\ 3/5 \\ -7/5 \end{bmatrix}}{\begin{bmatrix} 3/5 \\ 3/5 \\ -2/5 \\ 3/5 \\ -7/5 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 3/5 \\ -2/5 \\ 3/5 \\ -7/5 \end{bmatrix}} \begin{bmatrix} 3/5 \\ 3/5 \\ -2/5 \\ 3/5 \\ -7/5 \end{bmatrix} = \frac{1}{80} \begin{bmatrix} 85 \\ -75 \\ -30 \\ 5 \\ 15 \end{bmatrix}. \end{aligned}$$

The desired orthogonal basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Let $\mathbf{x} \in \mathbb{R}^5$ and note that

$$\begin{aligned}
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \text{ is in } W^\perp &\iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \\
&\quad x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\
&\iff x_1 + x_2 + x_4 - x_5 = 0 \\
&\quad x_1 - x_2 + x_5 = 0 \\
&\iff \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0} \\
&\iff \mathbf{x} \in \text{null} \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \right).
\end{aligned}$$

This shows that W^\perp is equal to the null-space of $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}$. (Observe that the rows of this matrix are the vectors in the spanning set of W .) To find a basis for this null-space, we proceed as usual. First we row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

So a general vector in the null-space looks like

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s/2 \\ -s/2 + t \\ -2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for W^\perp is given by $\left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, or more simply, by $\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. To get an

orthogonal basis, we apply the Gram–Schmidt process to this basis:

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \left[\begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right] = \frac{1}{6} \begin{bmatrix} -1 \\ 5 \\ -12 \\ 2 \\ 6 \end{bmatrix}.$$

So, finally, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W^\perp .

- (c) The claim here is that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}_1, \mathbf{u}_2\}$ is a basis for \mathbb{R}^5 . As this is a set of 5 vectors and \mathbb{R}^5 is 5-dimensional, it suffices to show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent. But this follows immediately from Theorem 1 from Section 7.1 of the textbook (which states that a set of orthogonal vectors is linearly independent): Indeed, by construction, the \mathbf{v}_i 's are orthogonal to each other and the \mathbf{u}_i 's are orthogonal to each other, and because the former live in W and the latter in W^\perp , the \mathbf{v}_i 's are orthogonal to the \mathbf{u}_i 's.
- (d) By the same reasoning as in part (c), the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m\}$ is linearly independent. However, here we do not know that $\dim \mathbb{R}^n = k + m$, so we still need to prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m\}$ spans \mathbb{R}^n . To this end, let $\mathbf{x} \in \mathbb{R}^n$. Then $\text{proj}_W(\mathbf{x}) \in W$, and so, because $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for W , we can write

$$\text{proj}_W(\mathbf{x}) = a_1 \mathbf{v}_1 + \dots + a_k \mathbf{v}_k \quad (*)$$

for some scalars $a_1, \dots, a_k \in \mathbb{R}$ (in fact, $a_i = \frac{\mathbf{x} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}$). Next, note that $\mathbf{x} - \text{proj}_W(\mathbf{x}) \in W^\perp$ so similarly we can write

$$\mathbf{x} - \text{proj}_W(\mathbf{x}) = b_1 \mathbf{u}_1 + \dots + b_m \mathbf{u}_m \quad (**)$$

for some scalars $b_1, \dots, b_m \in \mathbb{R}$. But now, by combining (*) and (**), we obtain

$$\mathbf{x} = \text{proj}_W(\mathbf{x}) + b_1 \mathbf{u}_1 + \dots + b_m \mathbf{u}_m = a_1 \mathbf{v}_1 + \dots + a_k \mathbf{v}_k + b_1 \mathbf{u}_1 + \dots + b_m \mathbf{u}_m.$$

Hence $\mathbf{x} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m\}$. So $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m\}$ is linearly independent and spans \mathbb{R}^n , and therefore is a basis. Consequently $\dim \mathbb{R}^n = k + m = \dim W + \dim W^\perp$.

[**Remark:** If we take the dot product of both sides of (**) with \mathbf{u}_i , we get

$$\mathbf{x} \cdot \mathbf{u}_i - \underbrace{\text{proj}_W(\mathbf{x}) \cdot \mathbf{u}_i}_{=0} = b_1 \underbrace{\mathbf{u}_1 \cdot \mathbf{u}_i}_{=0} + b_2 \underbrace{\mathbf{u}_2 \cdot \mathbf{u}_i}_{=0} + \dots + b_i (\mathbf{u}_i \cdot \mathbf{u}_i) + \dots + b_m \underbrace{\mathbf{u}_m \cdot \mathbf{u}_i}_{=0}.$$

Thus

$$b_i = \frac{\mathbf{x} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}.$$

This shows that

$$\mathbf{x} - \text{proj}_W(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \dots + \frac{\mathbf{x} \cdot \mathbf{u}_m}{\mathbf{u}_m \cdot \mathbf{u}_m} \mathbf{u}_m = \text{proj}_{W^\perp}(\mathbf{x}),$$

or equivalently, that $\mathbf{x} = \text{proj}_W(\mathbf{x}) + \text{proj}_{W^\perp}(\mathbf{x})$.]

5 (a) Apply the Gram-Schmidt process to the linearly *dependent* set of vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

5 (b) Use what you noticed in part (a) to explain what happens if you apply the Gram-Schmidt process to a set of vectors that is not linearly independent.

Solution:

(a) Call the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. First we leave \mathbf{u}_1 alone. The vector \mathbf{u}_2 becomes

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \end{bmatrix}.$$

The vector \mathbf{u}_3 becomes

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{u}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{14/3}{42/9} \begin{bmatrix} -\frac{5}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Remember that the Gram-Schmidt process takes vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and replaces them with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ which are orthogonal and have the same span. Since the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent, their span is at most two-dimensional. This means that the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is also at most two-dimensional. So one of the vectors, say \mathbf{v}_i is a linear combination of the other two. But \mathbf{v}_i is also orthogonal to the other two vectors, and so has to be $\mathbf{0}$.
