

University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 12

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We will start 10 minutes past the hour. Use this time to make a new friend.

Second order linear ODEs :

$$y'' + f(x)y' + g(x)y = h(x)$$

A second order linear ODE is called homogeneous if $h(x)=0$

→ a second order ODE has 2 unknown parameters (solution space is 2D)

$$\text{IVP: } \left. \begin{array}{l} y(0) = y_0 \\ y'(0) = v_0 \end{array} \right\} \begin{array}{l} \text{2 degrees of} \\ \text{freedom} \end{array}$$

For homogeneous linear ODEs, if $y_1(x)$ and $y_2(x)$ are solutions then any linear combination $y(x) = C_1 y_1(x) + C_2 y_2(x)$ is also a solution.

Pf: → plug into ODE: $y = C_1 y_1 + C_2 y_2$

$$y'' + f(x)y' + g(x)y \stackrel{?}{=} 0$$

$$(C_1 y_1 + C_2 y_2)'' + f(x)(C_1 y_1 + C_2 y_2)' + g(x)(C_1 y_1 + C_2 y_2) \stackrel{?}{=} 0$$

$$C_1 y_1'' + C_2 y_2'' + F(x) (C_1 y_1' + C_2 y_2') + g(x) (C_1 y_1 + C_2 y_2) \stackrel{?}{=} 0$$

$$C_1 \underbrace{(y_1'' + F(x)y_1' + g(x)y_1)}_{=0 \text{ b/c } y_1 \text{ is a solution}} + C_2 \underbrace{(y_2'' + F(x)y_2' + g(x)y_2)}_{=0} \stackrel{?}{=} 0$$

=0 b/c y_1 is
a solution

$$C_1 \cdot 0 + C_2 \cdot 0 \stackrel{\checkmark}{=} 0$$

$$\therefore y = C_1 y_1 + C_2 y_2$$

is a solution to
ODE

If y_1 & y_2 are two linearly independent solutions
to homogeneous second-order ODE, then

$y(x) = C_1 y_1(x) + C_2 y_2(x)$ is the general
solution

y_1 & y_2 are linearly independent if $y_1(x) \neq C y_2(x)$ for
any C in the whole interval of interest

$$y'' - 5y' + 6y = 0 \quad \Leftarrow \text{constant coeff.}$$

→ guess $y(t) = e^{rt}$

$$(e^{rt})'' - 5(e^{rt})' + 6(e^{rt}) = 0$$

$$\cancel{r^2 e^{rt}} - 5\cancel{r e^{rt}} + 6\cancel{e^{rt}} = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$$\Rightarrow r=3 \quad r=2$$

$$y_1(t) = e^{3t}$$

$$y_2(t) = e^{2t}$$

linearly independent

$$y(x) = C_1 e^{3t} + C_2 e^{2t}$$

General
Solution

\Leftarrow polynomial in r (called characteristic polynomial for the ODE)

Given any homogeneous ODE with constant coeff.

$$0 = a_0 y + a_1 y' + a_2 y'' + \dots + a_n y^{(n)}$$

→ guess solution $y = e^{rt}$

$$0 = a_0 (e^{rt}) + a_1 (e^{rt})' + \dots + a_n (e^{rt})^{(n)}$$

$$0 = a_0 \cancel{e^{rt}} + a_1 r \cancel{e^{rt}} + \dots + a_n r^n \cancel{e^{rt}}$$

$$0 = a_0 + a_1 r + a_2 r^2 + \dots + a_n r^n$$

⇐ called the
characteristic
polynomial

→ solutions: $y(t) = e^{r_1 t}$
 $y(t) = e^{r_2 t}$
 \vdots

⇐ r_1, r_2 are roots
of
char. polynomial

$$y'' - 3y' + 2y = 0, y(0) = 1 \text{ and } y'(0) = 0$$

\Rightarrow char. polynomial

$$0 = r^2 - 3r + 2$$

$$= (r-1)(r-2)$$

$\Rightarrow r=1 \quad r=2$ are sol'n

\Rightarrow general sol'n of ODE

$$y(t) = C_1 e^t + C_2 e^{2t}$$

$$\Rightarrow y'(t) = C_1 e^t + C_2 2e^{2t}$$

\Rightarrow apply I.C.

$$1 = y(0) = C_1 e^0 + C_2 e^{2 \cdot 0} = C_1 + C_2$$

$$0 = y'(0) = C_1 + 2C_2$$

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 + 2C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = -1 \end{cases}$$

$$\boxed{y(t) = 2e^t - e^{2t}}$$

$$y'' - 4y' + 4y = 0$$

→ char. poly.

$$0 = r^2 - 4r + 4$$

$$= (r-2)^2$$

$$\Rightarrow \boxed{r=2}$$

$$y(t) = C_1 e^{2t} + C_2 e^{2t} = \underbrace{(C_1 + C_2) e^{2t}}_{\text{one degree of freedom}}$$

→ need to find another linearly independent sol'n

$$\rightarrow \text{guess } y(t) = t e^{2t}$$

$$(t e^{2t})'' - 4(t e^{2t})' + 4(t e^{2t}) \stackrel{?}{=} 0$$

$$(e^{2t} + 2t e^{2t})' - 4(e^{2t} + 2t e^{2t}) + 4(t e^{2t}) \stackrel{?}{=} 0$$

$$\cancel{2e^{2t}} + \cancel{2e^{2t}} + \cancel{4t e^{2t}} - \cancel{4e^{2t}} - \cancel{8t e^{2t}} + \cancel{4t e^{2t}} \stackrel{?}{=} 0$$

= 0 so sol'n to ODE

$$\underbrace{y_1(t) = e^{2t} \quad y_2(t) = t e^{2t}}_{\text{linearly independent}}$$

$$\Rightarrow \boxed{y(t) = C_1 e^{2t} + C_2 t e^{2t}} \leftarrow \text{general solution}$$

Given a constant coeff. homogeneous ODE

$$0 = a_0 y + a_1 y' + \dots + a_n y^{(n)}$$

\Rightarrow char poly

$$0 = a_0 + a_1 r^2 + \dots + a_n r^n$$

$$0 = (r - r_1)^m \underbrace{\dots}_{\text{other roots}}$$

\Leftarrow if char poly has r_1 as m times repeated root then the solutions are

$$\left. \begin{array}{l} \text{m solutions from root } r_1 \text{ which was repeated m times} \\ y_1(t) = e^{r_1 t} \\ y_2(t) = t e^{r_1 t} \\ y_3(t) = t^2 e^{r_1 t} \\ \vdots \\ y_m = t^{(m-1)} e^{r_1 t} \end{array} \right\} \begin{array}{l} \text{not solutions if } r_1 \text{ is} \\ \text{not a repeated root} \end{array}$$

remaining solution from other roots $\left\{ \begin{array}{l} \cdot \\ \cdot \\ \cdot \end{array} \right.$

$$y'' + 2y' + y = 0, y(0) = 3 \text{ and } y'(0) = -5$$

\Rightarrow char. poly

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$r = -1$ repeated root

$$y_1(t) = e^{-t}$$

$y_2 = te^{-t}$
because of repeated root

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

$$\Rightarrow y'(t) = -C_1 e^{-t} + C_2 (e^{-t} - t e^{-t})$$

\Rightarrow apply I.V.

$$3 = y(0) = C_1$$

$$-5 = y'(0) = -C_1 + C_2$$

\Rightarrow

$$C_1 = 3$$

$$C_2 = -2$$

$$y(t) = 3e^{-t} - 2te^{-t}$$

$$y'' + 2y' + 5y = 0$$

\Rightarrow char. poly

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$r = -1 \pm \frac{\sqrt{-16}}{2}$$

$$r = -1 \pm 2\sqrt{-1}$$

$$\boxed{r = -1 \pm 2i}$$

$$y_1 = e^{(-1+2i)t}$$

$$y_2 = e^{(-1-2i)t}$$

\Leftarrow not real root
let $i = \sqrt{-1}$

} what does this mean?
 \rightarrow next lecture