University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, DECEMBER, 2012

Duration: 2 and 1/2 hours

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - LINEAR ALGEBRA

Exam Type: A

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SURNAME: (as on your T-card) ____________________

YOUR FULL NAME: ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________

QUESTION	MARK
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
Q9	
TOTAL	

Calculators Permitted: Casio 260, Sharp 520 or TI 30.

INSTRUCTIONS: Attempt all questions. Present your solutions in the space provided. Use the backs of the sheets if you need more space. Do not tear any pages from this exam. Make sure your exam contains 10 pages.

MARKS: Questions 1, 2, 3 and 4 are worth 10 marks.

Questions 5, 6, 7, 8 and 9 are each worth 12 marks.

If a question has multiple parts, the marks for each part are indicated in parentheses beside the question number; except for Question 2, for which each part is worth 2 marks.

TOTAL MARKS: 100

1. Find the following:

   (a) [5 marks] a vector equation for the line of intersection common to the two planes with equations $x + y - z = 5$ and $2x + y + 2z = 2$.

   (b) [5 marks] the shortest distance between the two parallel lines $\mathbb{L}_1$ and $\mathbb{L}_2$

$$\mathbb{L}_1 : \begin{cases} x &= 1 + t \\ y &= 0 - t \\ z &= 1 + t \end{cases} ; \quad \mathbb{L}_2 : \begin{cases} x &= 2 - s \\ y &= 3 + s \\ z &= 1 - s \end{cases}$$

   where s and t are parameters.

2.(a) [5 marks] Find the area of the triangle ΔPQR passing through the three points

$$P(1,3,1),\ Q(2,2,2),\ R(0,2,0).$$

2.(b) [5 marks] Let $U = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \mid xyz = 0 \right\}$.

1. Is U non-empty?

2. Is U closed under scalar multiplication?

3. Is U closed under vector addition?

4. Is U a subspace of $\mathbb{R}^3$?

3. Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) If $\lambda = 0$ is an eigenvalue of the $n \times n$ matrix A then A is not invertible.

True or **False**

(b) $\dim\left(\operatorname{im}\begin{bmatrix} 1 & 1 & 1 & -2 \\ 3 & 6 & 1 & 1 \\ 5 & 8 & 3 & -3 \end{bmatrix}\right) = 3$ **True** or **False**

(c) If U is a subspace of $\mathbb{R}^7$ and $\dim U = 4$ then $\dim U^{\perp} = 3$. **True** or **False**

(d) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$. **True** or **False**

(e) If $\{X_1, X_2, X_3\}$ is linearly independent, then so is $\{X_2, X_3\}$. **True** or **False**

4. Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & -1 & 2 \\ 2 & -1 & 3 & -2 & 2 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

5. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 5y \\ 2x + y \end{bmatrix}.$$

(a) [6 marks] Draw the image under T of the unit square, and calculate its area.

(b) [6 marks] Find the formula for $T \circ R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ if R is a rotation of $\pi/6$ clockwise around the origin.

6. Let $Q = \frac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix}$.

(a) [4 marks] Find the eigenvalues of Q.

(6) [8 marks] Find a basis for each eigenspace[1] of Q and plot the eigenspaces of Q in the plane, indicating which eigenspace corresponds to which eigenvalue of Q.

[1]Recall: if A is an $n \times n$ matrix, the eigenspace of A corresponding to λ is $E_\lambda(A) = \{X \in \mathbb{R}^n \mid AX = \lambda X\}$.

7. Let $U = \text{span}\left\{\left[\begin{array}{cccc} 0 & 1 & -1 & 0 \end{array}\right]^T, \left[\begin{array}{cccc} 1 & 0 & 0 & -1 \end{array}\right]^T, \left[\begin{array}{cccc} 1 & -1 & 0 & 0 \end{array}\right]^T\right\}$;
let $X = \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \end{array}\right]^T$. Find:

(a) [6 marks] an orthogonal basis of U.

(b) [6 marks] $\text{proj}_U(X)$.

8. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

9. Suppose $\vec{u}$ and $\vec{v}$ are two orthogonal unit vectors in $\mathbb{R}^3$. Let $A = \left[\ \vec{u} \mid \vec{v} \mid \vec{u} \times \vec{v}\ \right]$ and let $B = \vec{u}\,\vec{u}^T + \vec{v}\,\vec{v}^T + (\vec{u} \times \vec{v})\,(\vec{u} \times \vec{v})^T$.

(a) [6 marks] Explain why A must be an orthogonal matrix.

(b) [6 marks] Explain why B must be I, the 3×3 identity matrix.