Tuesday November 15	START: 13:10	DURATION: 110 mins

University of Toronto

Faculty of Applied Science & Engineering

MIDTERM EXAMINATION II
MAT188H1F
Linear Algebra

EXAMINERS: D. Burbulla, S. Cohen, D. Fusca, F. Lopez, M. Palasciano, M. Pugh, B. Schachter, S. Uppal

Last Name (PRINT): ______________________________

Given Name(s) (PRINT): ______________________________

Student NUMBER: ______________________________

Student SIGNATURE: ______________________________

EMAIL @mail.utoronto.ca: ______________________________

Instructions.

1. There are **58** possible marks to be earned in this exam. The examination booklet contains a total of 11 pages. It is your responsibility to ensure that *no pages are missing from your examination.* DO NOT DETACH ANY PAGES FROM YOUR EXAMINATION.
2. DO NOT WRITE ON THE QR CODE AT THE TOP RIGHT-HAND CORNER OF EVERY PAGE OF YOUR EXAMINATION.
3. For the full answer questions, WRITE YOUR SOLUTIONS ON THE FRONT OF THE QUESTION PAGES THEMSELVES. THE BACK OF EVERY PAGE WILL **NOT** BE SCANNED AND SEEN BY THE GRADERS.
4. Ensure that your solutions are LEGIBLE.
5. No aids of any kind are permitted. CALCULATORS AND OTHER ELECTRONIC DEVICES (INCLUDING PHONES) ARE NOT PERMITTED.
6. Have your student card ready for inspection.
7. There are no part marks for Multiple Choice (MC) questions.
8. You may use the two blank pages at the end for rough work. The last two pages of the examination WILL NOT BE MARKED unless you *clearly* indicate otherwise on the question pages.
9. For the full answer questions, show all of your work and justify your answers *but do not include extraneous information.*

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\\0\end{bmatrix}$, $T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}$, and $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

Determine $T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right)$.

(A) $\begin{bmatrix}1\\2\\1\end{bmatrix}$

(B) $\begin{bmatrix}1\\1\\-1\end{bmatrix}$

(C) $\begin{bmatrix}1\\4\\-1\end{bmatrix}$

(D) $\begin{bmatrix}1\\1\\0\end{bmatrix}$

(E) $\begin{bmatrix}0\\0\\0\end{bmatrix}$

Answer: C
Solution:

We start by noting that the three argument vectors, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\end{bmatrix}$, and $\begin{bmatrix}1\\0\\0\end{bmatrix}$, are linearly independent as we can use a linear combination of them to write the three unit vectors. This means that we can express $\begin{bmatrix}0\\1\\2\end{bmatrix}$ as $a\begin{bmatrix}0\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\end{bmatrix}$. We will find what a, b, and c are later, but for now, we know that $T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = aT\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) + bT\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) + cT\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ since T is a linear transformation. Thus,

$$T\left(\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = a\begin{bmatrix}0\\2\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\2 & 0 & 0\\0 & 1 & 0\end{bmatrix} \cdot \begin{bmatrix}a\\b\\c\end{bmatrix}$$

. Also, from before we know that

$$\begin{bmatrix}0\\1\\2\end{bmatrix} = a\begin{bmatrix}0\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix} \cdot \begin{bmatrix}a\\b\\c\end{bmatrix}$$

. Thus, we can find $\begin{bmatrix}a\\b\\c\end{bmatrix}$ by evaluating $\begin{bmatrix}0 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}^{-1} \cdot \begin{bmatrix}0\\1\\2\end{bmatrix}$.

To find $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$, we adjoin the identity matrix and do row operations.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Swap rows 1 and 3. $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

Subtract row 1 from row 2. $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

Subtract row 2 from row 3. $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Thus, we get $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, which means that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Therefore, $T\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ and the answer is **C**.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ a+b \end{bmatrix}$. If $T^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$, find $c+d$.

(A) -1
(B) 0
(C) 3
(D) 1
(E) 2

Answer: D
Solution:
We know that if $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ a+b \end{bmatrix}$ then $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$.
Thus if $T^{-1}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix}$.
This means that $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
To find $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$, we adjoin the identity matrix and do row operations.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

Subtract row 1 from row 2. $$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

Negate row 2. $$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Subtract twice row 2 from row 1. $$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Thus, we get $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$, which means that $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
If $c = -1$ and $d = 2$ then $c + d = 1$, and so the answer is **D**.

Part I - Multiple Choice. Clearly indicate your answer to each question by circling your choice. Each question is worth 2 marks.

For each question, choose the BEST option from the given options.

3. Which of the following functions $T : \mathbb{R}^2 \to \mathbb{R}$ are linear transformations?

(i) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x + 5y - 3.$

(ii) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x + 5y.$

(iii) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = |x|$

(A) (i) only
(B) (ii) only
(C) (i) and (ii) only
(D) (iii) only
(E) (ii) and (iii) only

Answer: B
Solution:
i) Does not satisfy $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = cT\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$. For example, take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $c = 3$.
ii) Works because we can represent T by the matrix $\begin{bmatrix} 2 & 5 \end{bmatrix}$ since $\begin{bmatrix} 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 2x + 5y$ and matrices are linear.
iii) Does not work as it does not satisfy $T\left(-\begin{bmatrix} x \\ y \end{bmatrix}\right) = -T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.
Thus the answer is **B**.

4. Let A, B, and C be $n \times n$ matrices. Which of the following statements are TRUE?

(i) If $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

(ii) If $A \neq \mathbf{0}$ and $AB = AC$, then $B = C$.

(iii) Either the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$ or there exist non-trivial solutions to the homogenous system $A\mathbf{x} = \mathbf{0}$.

(A) (i) and (ii) only
(B) (ii) only
(C) (i) and (iii) only
(D) (i), (ii), and (iii)
(E) none of (i), (ii), or (iii)

Answer: C
Solution:
i) If A and B differ in the j^{th} column then we can consider $\mathbf{x}$=$\mathbf{e}_j$. The output of $A \cdot \mathbf{x}$ will be the j^{th} column of A. The output of $B \cdot \mathbf{x}$ will be the j^{th} column of B. If these outputs are the same, the columns cannot differ.

ii) This is false. One counterexample is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Here, $A \neq 0$, the left and right sides are both $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, yet $B \neq C$.

iii) This is true. The forwards direction is true because if there were two different solutions to the equation we would have $A\mathbf{x_1} = \mathbf{b}$ and $A\mathbf{x_2} = \mathbf{b}$. Thus, $A\mathbf{x_1} - A\mathbf{x_2} = \mathbf{0}$ so $A(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0}$. The backwards direction is true because we can just choose $\mathbf{b} = \mathbf{0}$ and we have found our solution.

5. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be non-zero, non-parallel vectors in $\mathbb{R}^4$ and let $W = \text{span}\{\mathbf{u}, \mathbf{v}-\mathbf{u}, \mathbf{u}-\mathbf{w}, \mathbf{v}+\mathbf{w}\}$. Determine all possible values of $\dim(W)$.
(A) 4
(B) 1 or 2
(C) 2 or 3
(D) 1, 2, or 3
(E) 1, 2, 3, or 4

Answer: C
Solution:
We observe that since $2\mathbf{u}+(\mathbf{v}-\mathbf{u})-(\mathbf{u}-\mathbf{w})-(\mathbf{v}+\mathbf{w}) = \mathbf{0}$, then the set whose span makes up W is not linearly independent and so we can remove one of the vectors from the set and W will be unchanged. This also makes sense as all 4 vectors in the definition of W are linear combinations of only three vectors, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Thus, $\dim(W)$ cannot be 4. Also, if $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then $\mathbf{u}$ and $\mathbf{v}-\mathbf{u}$ are not parallel so $\dim(W)$ is at least 2. Thus, the only option that this leaves us **C**.

Part II - Short Answer Questions. Write your solutions in the space provided below each question.

1. Suppose you are given that the matrix $A = \begin{bmatrix} 3 & 6 & -1 & 5 & 1 \\ 1 & 2 & -1 & 2 & 0 \\ 2 & 4 & 0 & 3 & 1 \end{bmatrix}$ has row-echelon form $R = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Determine rank(A) and nullity(A). [2 marks]

The rank of a matrix is equal to the dimension of the row space, since the row space is invariant under row operations, and the dimension of the row space of R is clearly 2, rank(A)=2.

By the rank-nullity theorem, we know that since A has 5 columns, rank(A)+nullity(A)=5 and since rank(A)=2, nullity(A)=3.

(b) Find a basis for row(A). [2 marks]

Whenever we perform a row operation on A, the basis for the row space of the resulting matrix will still be a basis for the row space of the initial matrix. Thus, any basis for row(R) will be a basis for row(A).

It is clear that an adequate basis for row(R) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$, which would also then be an adequate basis for row(A).

(c) Find a basis for col(A). [2 marks]

Since the row rank of A is equal to the column rank of A, we need to choose two linearly independent columns for our basis. This means choosing two columns that are not multiples of each other, which means $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \right\}$ suffice.

(d) Find a basis for null(A). [2 marks]

We need to solve $A\mathbf{x} = 0$. We can perform row operations on this system until it is of the form $R\mathbf{x} = 0$. Thus, we get two constrains, that $x_1 = -x_4 - x_5$ and $x_3 = x_4 - x_5$. If we set $x_2 = a$, $x_4 = b$, and $x_5 = c$, we get that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -b - c \\ a \\ b - c \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, our basis for the null space of A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ as it is the basis to the solution space of the homogeneous equation $A\mathbf{x} = 0$.

2. (a) Let W be a subspace of $\mathbb{R}^n$. Define what it means for a set S of vectors in $\mathbb{R}^n$ to be a basis for W. [2 marks]

The definition of a basis of a subspace W is a set of vectors, $S \subset W$ such that

- S is linearly independent, and
- span(S)=W.

2. (b) Consider the subspace $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 + x_2 + x_3 = 0 \right\}$. Is the set $S = \left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$ a basis for W? Support your answer. [6 marks]

Answer: Yes.

Solution:

Notice that W is the set of solutions to the homogeneous equation $x_1 + x_2 + x_3 = 0$. Since the set of solutions requires two parameters, the dimension of W is 2. Notice that, for example, the set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W. Since the dimension of W is 2, any two linearly independent vectors from W will be a basis for W (cf. Tutorial Problems 5, 3(b)). The vectors in the set S are in W since -1+(-1)+2=0 and -3+2+1=0. They are also linearly independent since the matrix $\begin{bmatrix} -1 & -3 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$ has rank 2 so S is indeed a basis for W.

3. (a) Let W be a subspace of $\mathbb{R}^n$. Define $\dim(W)$. [2 marks]

The dimension of W is the number of vectors in every basis of W or the minimum number of vector required to span W.

3. (b) Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix}, \begin{bmatrix} c \\ 1 \\ 1 \end{bmatrix} \right\}$. For what value(s) of c is $\dim(W) = 2$. Support your answer. [6 marks]

Answer: c=-2

Solution:

We look for the value(s) of c such that $A = \begin{bmatrix} 1 & 1 & c \\ 1 & c & 1 \\ c & 1 & 1 \end{bmatrix}$ has rank 2.

Rearranging the matrix into its RREF form gives us the following.

$$\begin{bmatrix} 1 & 1 & c \\ 1 & c & 1 \\ c & 1 & 1 \end{bmatrix}$$

Subtract row 1 from row 2. $$\begin{bmatrix} 1 & 1 & c \\ 0 & c-1 & 1-c \\ c & 1 & 1 \end{bmatrix}$$

Subtract c times row 1 from row 3. $$\begin{bmatrix} 1 & 1 & c \\ 0 & c-1 & 1-c \\ 0 & 1-c & 1-c^2 \end{bmatrix}$$

Add row 2 to row 3. $$\begin{bmatrix} 1 & 1 & c \\ 0 & c-1 & 1-c \\ 0 & 0 & 2-c-c^2 \end{bmatrix}$$

At this point we can stop. We see that the rank of A may be 2 if the expression $2 - c - c^2 = (1-c)(c+2)$ is equal to 0. i.e. $c = 1$ or $c = -2$. However, if $c = 1$ the rank of A is 1 and the dimension of W is 1. If $c = -2$, the rank of A is 2 and so the only value of c for which the dimension of W is 2 is -2.

4. (a) Find the inverse of $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix}$. [4 marks]

We do this by adjoining the identity matrix and putting the matrix A into its RREF form. We get the following series of operations.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

Subtract twice row 1 from row 2. $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$

Add row 1 to row 3. $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right]$

Negate rows 2 and 3. $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$

Subtract twice row 3 from row 2. $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$

Subtract row 3 from row 1. $\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$

Add row 2 to row 1. $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & 3 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right]$

Thus the inverse of A is $\begin{bmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.

4. (b) Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Find a matrix C such that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$.
Suggestion: Use part (a). [4 marks]

We see that $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ (where A is the same as in part a). Thus, we get that $A^{-1} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. This immediately tells us that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} A^{-1} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$.

Hence, $C = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 & 3 \\ 4 & -1 & 2 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 12 & -2 & 5 \\ 2 & -1 & -1 \\ 16 & -3 & 8 \end{bmatrix}$.

5. Define a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by the following rule: $T(\mathbf{x})$ is the result of first rotating $\mathbf{x}$ counter-clockwise by $\frac{\pi}{4}$ and then multiplying by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Let S be the unit square in $\mathbb{R}^2$ with vertices $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(a) Find a matrix B such that $T(\mathbf{x}) = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^2$. [2 marks]

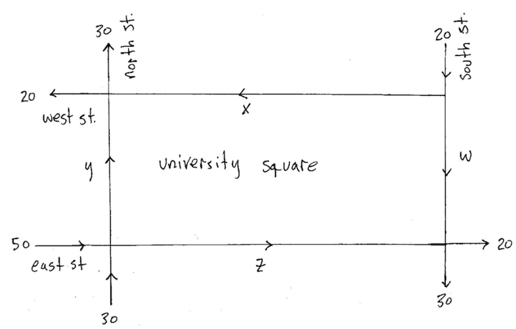
The rotation matrix for $\frac{\pi}{4}$ counterclockwise is given by $\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

From this, we get that to find T, we first apply the rotation, followed by A, so $B = A \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

(b) Find and sketch, as accurately as possible, the image of S under T. [6 marks]

We apply the matrix B to each vertex given in the problem and find that the vertices get mapped to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$, and $\begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, respectively. The square, then, gets mapped to the parallelogram with vertices at these points.

6. The diagram below is a map of a downtown area of a city. Each street is one-way in the direction of the respective arrows. The numbers represent the average number of cars per minute that enter or leave a given street at 6:00pm. The variables also represent the average numbers of cars per minute. Of course, unless there is an accident, the total number of cars entering any intersection must equal the total number leaving. Thus, at the intersection of West and North street we have $x + y = 50$.



(a) Starting at the intersection of West and North Street, and continuing clockwise around the square, write a system of linear equations that describes the traffic flow (assuming there are no accidents) and find all solutions to this system. You should find there are infinitely many solutions with one parameter. Use w as your free variable. [5 marks]

In the order indicated in the problem, we write a system of equation by added all the incoming cars and subtracting all of the leaving cars and setting that equal to 0. Thus we get

$$\begin{aligned} x + y - 20 - 30 &= 0 \\ 20 - x - w &= 0 \\ z + w - 20 - 30 &= 0 \\ 50 + 30 - y - z &= 0 \end{aligned}$$

Therefore, we get that $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 50 \\ 20 \\ 50 \\ 80 \end{bmatrix}$.

Row reducing this system of equations turns it into the system $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \\ 50 \\ 0 \end{bmatrix}$.

Our solution is therefore $x = 20 - w$, $y = 30 + w$, $z = 50 - w$, and w is a free variable.

(b) Nefarious Construction Co. wants to close down West Street for six months but the City Council refused to grant the permit. Explain why on the basis of the solutions to the system in part (a). [3 marks]

This has the same effect as setting $x = 0$, and by part a, this means that $w = 20$, so $y = 50$ and $z = 30$. This has the effect of causing massive traffic jams along North Street. Before, when we had a smaller value for w, such as 10, there was 20% less traffic among North Street and it could operate fine.