University of Toronto
Faculty of Applied Sciences and Engineering

MAT188 – midterm II –Fall 2022

LAST (Family) NAME:	
FIRST (Given) NAME:	
Email address:	@mail.utoronto.ca
Utorid:	
STUDENT NUMBER:	

Time: 90 mins.

1. **Keep this booklet closed**, until an invigilator announces that the test has begun. However, you may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **into the boxes**. Ample space is provided within each box, however, if you must use additional space, please use the blank page at the end of this booklet, and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 14 pages, excluding the cover page, and 6 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work unless otherwise stated.
7. No textbook, notes, or other outside assistance is allowed.

Question:	1	2	3	4	5	6	Total
Points:	6	13	6	9	12	14	60
Score:							

Part A

1. (6 points) Fill in the circle for all statements that must be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. You may get a negative mark if you choose an incorrect option.

   (a) Is S a subspace of V? Choose ALL choices for which the answer is "yes".

   - ○ $S = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$, $V = \mathbb{R}^2$
   - ○ S is the unit cube, $V = \mathbb{R}^3$
   - ● S is a plane through origin in $\mathbb{R}^3$, $V = \mathbb{R}^3$
   - ○ $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, $V = \mathbb{R}^2$
   - ● S is the kernel of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^6$, $V = \mathbb{R}^5$
   - ○ S is the image of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^6$, $V = \mathbb{R}^5$

   (b) Assume the given statement and choose all the phrases that are true. "$A\vec{x} = \vec{0}$ has a unique solution, where A is an $n \times p$ matrix"

   - ● The linear transformation T defined by $T(\vec{x}) = A\vec{x}$ is injective.
   - ○ Columns of A span $\mathbb{R}^p$.
   - ○ $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^p$.
   - ○ There is at least one redundant vector among columns of A.
   - ○ The linear transformation T defined by $T(\vec{x}) = A\vec{x}$ is surjective.
   - ● If $A\vec{x} = \vec{b}$ is consistent then it has a unique solution.
   - ● Columns of A are linearly independent.
   - ● $n \geq p$.

2. Fill in the blank. You don't need to include your computation or reasoning.

(a) (4 points) Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & e & h \end{bmatrix} \qquad B = \begin{bmatrix} a & b & c \\ g & e & h \\ d+a & e+b & f+c \end{bmatrix} \qquad C = \begin{bmatrix} 2d+a & 2e+b & 2f+c \\ g & e & h \\ a & b & c \end{bmatrix}.$$

Let $T(\vec{x}) = A\vec{x}$, $S(\vec{x}) = B\vec{x}$ and $L(\vec{x}) = C\vec{x}$ be linear transformations given by left multiplication by matrices A, B and C respectively. Suppose $\det A = 4$. Let Ω be a cylinder of volume 5 in $\mathbb{R}^3$.

$\det C =$ $\boxed{8}$ the volume of $T \circ S(\Omega)$ is $\boxed{80}$ $\det(C^{-1}B) =$ $\boxed{-1/2}$

(b) (6 points) Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 8 \\ 11 \\ 16 \end{bmatrix}$, and $\vec{r} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$.

Choose all the correct options: The set $\{\vec{v}, \vec{w}, \vec{r}\}$ is

- ● linearly independent
- ○ linearly dependent
- ● a spanning set for $\mathbb{R}^3$
- ● a basis for $\mathbb{R}^3$

If possible find coefficients that make the relation below hold true. If not possible, write DNE in every box.

$$\vec{u} = \boxed{2}\,\vec{v} + \boxed{3}\,\vec{w}$$

(c) (3 points) Let $\mathcal{B} = (\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix})$ and $\mathcal{E} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be two bases for $\mathbb{R}^3$. Let S be a matrix such that $S[\vec{x}]_{\mathcal{E}} = [\vec{x}]_{\mathcal{B}}$ for any vector $\vec{x}$ in $\mathbb{R}^3$. find S^{-1}.

$$S^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 7 \end{bmatrix}$$

Part B

3. State whether each statement is true or false by writing "True" or "False" in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true explain why it must be true. If you think a statement is false, give a counterexample.

(a) (3 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^6$, given by $T(\vec{x}) = A\vec{x}$. Suppose rank $A = 2$. Then the kernel of T is a line in $\mathbb{R}^3$.

True

$\dim \ker T = 3 - \dim \operatorname{img} T$
$= 3 - 2 = 1$

a 1-dim subspace of $\mathbb{R}^3$ is a line

(b) (3 points) Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by $S(\vec{x}) = \begin{bmatrix} 5 & 1 & 15 \\ 0 & 2 & 22 \\ 0 & 0 & 10 \end{bmatrix} \vec{x}$.

Then S is invertible.

True

$\det \begin{bmatrix} 5 & 1 & 15 \\ 0 & 2 & 22 \\ 0 & 0 & 10 \end{bmatrix} = (5)(2)(10) = 100 \neq 0$

S has non-zero det hence it is invertible

4. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

(a) (3 points) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel and image are lines in $\mathbb{R}^3$.

DNE $\dim \operatorname{im} T + \dim \ker T = 3$

$1 + 1 \neq 3$

it is not possible to have 1 dim'l kernel & image.

(b) (3 points) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\ker(T)$ contains the vector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

many possible examples, including

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto \vec{0}$ or $T : \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{x} \mapsto \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \vec{x}$

(c) (3 points) Let T is the reflection about the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. A basis $\mathcal{B}$ for $\mathbb{R}^3$ such that $[T]_\mathcal{B}$ is diagonal.

$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$

$[T]_\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

5. Let $\mathcal{U} = \left(\vec{u}_1 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \vec{u}_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{7}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right)$ be a basis for the subspace V of $\mathbb{R}^5$.

(a) (4 points) Show that $\mathcal{U}$ is an orthonormal basis for $\mathbb{R}^3$.

we verify

$\vec{u}_1 \cdot \vec{u}_2 = 0, \quad \vec{u}_2 \cdot \vec{u}_3 = 0, \quad \vec{u}_1 \cdot \vec{u}_3 = 0$

and

$\|\vec{u}_1\| = \|\vec{u}_2\| = \|\vec{u}_3\| = \|\vec{u}_4\| = 1$

(b) (4 points) Let $W = \text{Span}(\vec{u}_1, \vec{u}_3)$ and $\pi_W : \mathbb{R}^5 \to \mathbb{R}^5$ be the orthogonal projection onto W. Let $\vec{x}$ be a vector in $\mathbb{R}^5$ such that $\vec{x} \cdot \vec{u}_1 = 2, \vec{x} \cdot \vec{u}_2 = 4$, and $\vec{x} \cdot \vec{u}_3 = -1$. Find $\pi_W(\vec{x})$. Put your final answer in the small box. Justify your answer.

$$\pi_W(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (x \cdot \vec{u}_3)\vec{u}_3$$

$$= 2\vec{u}_1 + (-1)\vec{u}_3$$

$\pi_W(\vec{x}) =$ $2\vec{u}_1 - \vec{u}_3$

(c) (4 points) Recall that $\mathcal{U} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ in part (a) is an orthonormal basis for V. Find $[\pi_W(\vec{x})]_\mathcal{U}$, the $\mathcal{U}$ coordinate of $\pi_W(\vec{x})$. Put your final answer in the small box. Justify your answer.

$$\pi_W(\vec{x}) = 2\vec{u}_1 + 0\vec{u}_2 - \vec{u}_3$$

$$[\pi_W(\vec{x})]_\mathcal{U} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$[\pi_W(\vec{x})]_\mathcal{U} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

Part C

6. Becoming more and more interested with lower-dimensional beings, Cordylia has discovered a way to build a machine which can transport part of herself down to lower dimensional space without causing irreparable damage to herself. She wishes to perform a test of her machine.

   Cordylia's sensors have discovered beings which exist in the subspace V of $\mathbb{R}^6$ defined by the following equations:

$$\begin{cases} x+y+z+u+v+w=0 \\ x+z+v=0 \\ y+u+w=0 \end{cases}$$

   (a) (3 points) Find a basis $\mathcal{B}_1$ for the subspace V.

$$\left[\begin{array}{cccccc|c} 1&1&1&1&1&1&0 \\ 1&0&1&0&1&0&0 \\ 0&1&0&1&0&1&0 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 1&0&1&0&1&0&0 \\ 0&1&0&1&0&1&0 \\ 0&0&0&0&0&0&0 \end{array}\right]$$

$$\begin{aligned} x &= -z-v \\ y &= -u-w \end{aligned}$$

z, u, v, w free

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} -1\\0\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} -1\\0\\0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\-1\\0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\-1\\0\\0\\0\\1 \end{pmatrix} \right\}$$

(b) (3 points) What is the dimension of V? What is the dimension of the orthogonal subspace $V^{\perp}$?

$\dim V = 4$, $\dim V + \dim V^{\perp} = 6 = \dim \mathbb{R}^6$

$\dim V =$ 4 $\qquad \dim V^{\perp} =$ 2

(c) (2 points) Find a basis $\mathcal{B}_2$ for $V^{\perp}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$V = \mathrm{Nul}\, A^T = (\mathrm{Col}\, A)^{\perp}$$

$$V^{\perp} = (\mathrm{Nul}\, A^T)^{\perp} = ((\mathrm{Col}\, A)^{\perp})^{\perp}$$

$$= \mathrm{Col}\,(A)$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(d) (3 points) Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ be the basis of $\mathbb{R}^6$ consisting of all vectors from the bases $\mathcal{B}_1$ of V and $\mathcal{B}_2$ of $V^\perp$. Let $P : \mathbb{R}^6 \to \mathbb{R}^6$ be the orthogonal projection onto the subspace V. Find the matrix representation of P with respect to the basis $\mathcal{B}$. Call this matrix B.

$$[P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) (3 points) To create her projection machine, Cordylia needs to know the standard matrix A for the projection map to the subspace V. Cordilya knows that there exists an invertible matrix S such that $A = S^{-1}BS$. Without computing it fully, give an expression for the matrix S.

S is change of basis from $\mathcal{E}$ to $\mathcal{B}$

$$S = \begin{bmatrix} | & & | \\ [\vec{e}_1]_{\mathcal{B}} & \cdots & [\vec{e}_6]_{\mathcal{B}} \\ | & & | \end{bmatrix}$$

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