University of Toronto
Faculty of Applied Sciences and Engineering

MAT187 - Summer 2025

Lecture 13

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We will start 10 minutes past the hour. Use this time to make a new friend.

Complex Numbers

→ note that $x^2+1=0$ has no solution in the real numbers $x=\sqrt{-1}$?

Introduce a new number $i=\sqrt{-1}$ (imaginary unit)

→ do we need more numbers to be consistent ?

→ need to multiply numbers to get another number $\Rightarrow$ $\underbrace{b \cdot i = bi}$

real number a number

→ need add/subtract to get another number $\Rightarrow$ $a+bi$ ⇐ number

real number

The complex numbers are numbers of the form $a+bi$ where $a, b \in \mathbb{R}$

→ you can add/multiply/divide/subtract complex numbers to get another number

$$z_1 = a_1 + b_1 i \qquad z_2 = a_1 + b_2 i$$

$$\Rightarrow z_1 + z = (a_1 + b_1 i) + (a_2 + b_2 i)$$

$$= \underbrace{(a_1 + a_2) + (b_1 + b_2) i}_{\text{addition / subtraction} \Rightarrow \text{add / subtract real \& imaginary part}}$$

$$\Rightarrow z_1 \cdot z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$$

$$= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 \underbrace{i^2}_{-1}$$

$$= \underbrace{(a_1 a_2 - b_1 b_2)}_{\text{real part}} + \underbrace{(a_1 b_2 + b_1 a_2)}_{\text{imaginary part}} i$$

Given a complex number $z = a + bi$, define following:

① conjugate of z: $\bar{z} = a - bi$ ⇐ change sign of imaginary part

② real part of z: $\operatorname{Re}(z) = a = \frac{1}{2}(z + \bar{z})$

③ imaginary part of z: $\operatorname{Im}(z) = b = \frac{1}{2i}(z - \bar{z})$ ⇐ $\operatorname{Im}(z)$ is the coeff of i (which is a real number)

→ divide $z_1 = a_1 + b_1 i$

$z_2 = a_2 + b_2 i$

$z_2 \neq 0 \Rightarrow$ either $a_2 \neq 0$ or $b_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

⇐ multiply by conjugate of denominator

$$\frac{\bar{z}_2 z_1}{\bar{z}_2 z_2} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + (a_2 b_1 - a_1 b_2) i}{(a_2)^2 + (b_2)^2}$$

⇐ $z\bar{z}$ is always real

$$= \underbrace{\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}}_{\text{real part}} + \underbrace{\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}}_{\text{imaginary part}} i$$

Compute the following:

1. $(4+3i)-(2-i)$
2. $(2+3i)(1-i)$
3. $\frac{3+i}{1-2i}$

1) $(4+3i)-(2-i)$

$= (4-2)+(3-(-1))i$

$\boxed{= 2+4i}$

2) $(2+3i)(1-i)$

$= 2\cdot 1 + 2(-i) + (3i)(1) + (3i)(-i)$

$= 2 - 2i + 3i - 3i^2 \qquad \Leftarrow i^2 = -1$

$= 2 - 2i + 3i + 3$

$\boxed{= 5+i}$

3) $\frac{3+i}{1-2i} = \frac{(3+i)(1+2i)}{(1-2i)(1+2i)}$

$= \frac{3+6i+i+2i^2}{1+2i-2i-4i^2}$

$= \frac{1+7i}{1+4}$

$\boxed{= \frac{1}{5} + \frac{7}{5}i}$

→ How to do exponential?

$$e^{a+bi} = \underbrace{e^{a}}_{\text{real}} e^{bi} \quad \Leftarrow \text{property of exponentials}$$

$i = i$

$i^2 = -1$

$i^3 = i^2 i = -i$

$i^4 = i^2 i^2 = (-1)(-1) = 1$

$i^5 = i^4 i = i$

$\vdots$

→ what is e^{bi}? Taylor series!

$$e^{x} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

$$e^{bi} = 1 + (bi) + \frac{1}{2}(bi)^2 + \frac{1}{3!}(bi)^3 + \frac{1}{4!}(bi)^4 + \frac{1}{5!}(bi)^5 + \frac{1}{6!}(bi)^6 + \ldots$$

$$= 1 + bi - \frac{1}{2}b^2 - \frac{1}{3!}b^3 i + \frac{1}{4!}b^4 + \frac{1}{5!}b^5 i - \frac{1}{6!}b^6 + \ldots$$

$$= \underbrace{\left(1 - \frac{1}{2}b^2 + \frac{1}{4!}b^4 - \frac{1}{6!}b^6 + \ldots\right)}_{\text{real part}} + i \underbrace{\left(b - \frac{1}{3!}b^3 + \frac{1}{5!}b^5 + \ldots\right)}_{\text{imag part}}$$

$$e^{bi} = \cos(b) + i\sin(b)$$

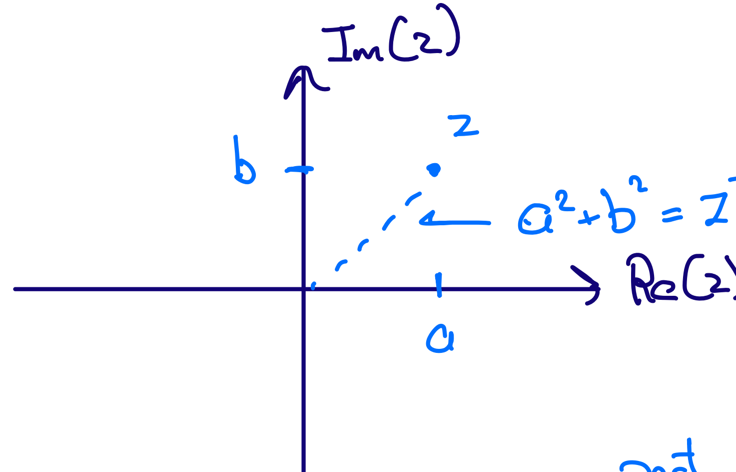
Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Complex exponentials: $e^{a+bi} = e^{a}\left(\cos(b) + i\sin(b)\right)$

Visualizing complex numbers

$\underbrace{z = a+bi}_{\text{2 real components}}$ $\Leftarrow$ rectangular (Cartesian) form of z

(same as $\mathbb{R}^2$)



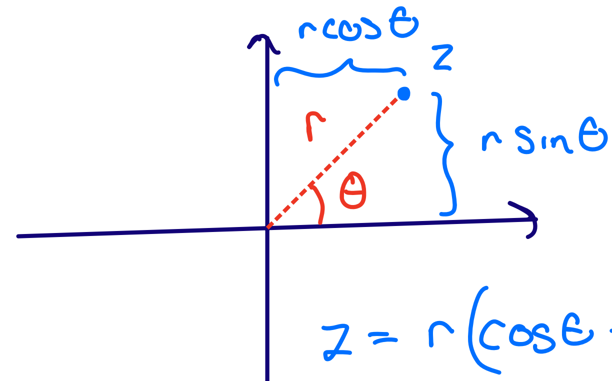
$a^2+b^2 = z\bar{z} = |z|^2$

Polar coordinates: define a point in $\mathbb{R}^2$ by the distance from origin and angle with x-axis

$\rightarrow \mathrm{mod}(z) = |z| = r = \sqrt{a^2+b^2}$

$\rightarrow \arg(z) = \theta = $ (use trig) $= \pm\arctan\left(\frac{b}{a}\right)$

$\Leftarrow$ re part



$\Leftarrow$ Im part

$z = r(\cos\theta + i\sin\theta)$

$\boxed{z = re^{i\theta}}$ $\Leftarrow$ polar representation of z

Multiplication in polar form

$z_1 = r_1 e^{i\theta_1}$

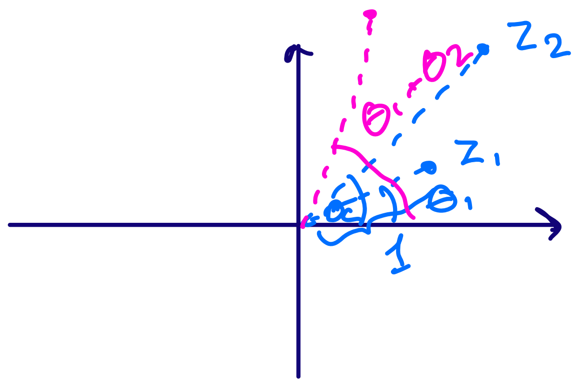
$z_2 = r_2 e^{i\theta_2}$

$\Rightarrow$

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$$

$$= \underbrace{(r_1 r_2)\, e^{i(\theta_1 + \theta_2)}}$$

Product of distance and add angles



Fundamental Thm. of algebra

→ every polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ has a complex root

→ every polynomial can be factored as product of linear terms $p(x) = (x - z_1)(x - z_2) \ldots (x - z_n) \Leftarrow z_i \in \mathbb{C}$

If $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a polynomial with real coeff. then if z is a root then $\bar{z}$ is also root

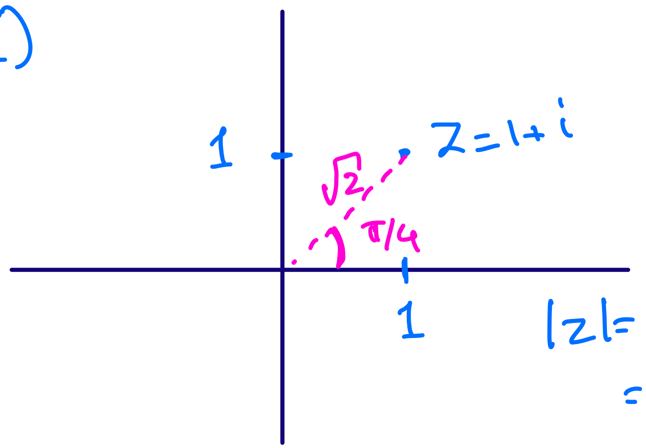
Convert to Polar Form

1. $1 + i$
2. $-\sqrt{3} - i$

Convert to Rectangular Form

1. $2e^{i\pi/4}$
2. $4e^{-i\pi/6}$

1)



$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$z = |z| e^{i\theta}$$

$$\boxed{z = \sqrt{2}\, e^{i\pi/4}}$$

$$y'' + 4y = 0$$

→ char. poly:

$$0 = r^2 + 4$$

$$r^2 = -4$$

$$r = \pm\sqrt{-4}$$

$$r = \pm 2i \quad \Leftarrow \text{pure imaginary root}$$

$\Leftarrow c_1, c_2 \in \mathbb{C}$

$$y(t) = c_1 e^{2it} + c_2 e^{-2it} \quad \Leftarrow \text{complex solution, can we get real solution?}$$

$$= c_1\left(\cos(2t) + i\sin(2t)\right) + c_2\left(\cos(-2t) + i\sin(-2t)\right)$$

$$= c_1\left(\cos(2t) + i\sin(2t)\right) + c_2\left(\cos(2t) - i\sin(2t)\right)$$

$$= \underbrace{(c_1 + c_2)\cos(2t) + (c_1 - c_2)\,i\sin(2t)}_{\text{to get real solution:}}$$

$(c_1 - c_2)i \in \mathbb{R} \Rightarrow c_1 - c_2$ is pure imag. $\Rightarrow$ real part equal

$c_1 + c_2 \in \mathbb{R} \Rightarrow$ imag part is negative of each other

$\Rightarrow$ $c_1 = \overline{c_2}$ will give you pure real solution

$$y(t) = (c_1 + \overline{c_1}) \cos(2t) + (c_1 - \overline{c_1}) i \sin(2t)$$

$$= \underbrace{2\operatorname{Re}(c_1)}_{A} \cos(2t) + \underbrace{(-2\operatorname{Im}(c_1))}_{B} \sin(2t)$$

$$\boxed{= A\cos(2t) + B\sin(2t)} \qquad A, B \in \mathbb{R}$$

2 linearly independant solutions

$\therefore$ general solution

$$y'' + 2y' + 5y = 0$$

→ char. poly:

$0 = r^2 + 2r + 5$

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$\boxed{r = -1 \pm 2i}$$ ⇐ complex roots

$$y(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t} \qquad C_1, C_2 \in \mathbb{C}$$

$$= e^{-t}\left(C_1 e^{2it} + C_2 e^{-2it}\right)$$

⇐ like in previous example, restrict $C_1 = \overline{C_2}$ and let $A = 2\,\mathrm{Re}(C_1)$, $B = -2\,\mathrm{Im}(C_1)$

⋮

$$\boxed{y(t) = e^{-t}\left(A\cos(2t) + B\sin(2t)\right)}$$

General solution

$A, B \in \mathbb{R}$

Summary: $y'' + by' + cy = 0$

$\rightarrow$ char poly $0 = r^2 + br + c$

① Distinct real roots r_1, r_2

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

② Repeate real root r

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

③ Complex roots $\underbrace{r_1 = a + bi \quad , \quad r_2 = a - bi}_{}$

$\rightarrow$ complex roots always come in conjugate pairs

$$y(t) = e^{at}\left(c_1 \cos(bt) + C_2 \sin(bt)\right)$$

$$y'' - 4y' + 13y = 0$$

$\rightarrow$ char. poly

$r^2 - 4r + 13 = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 4 \cdot 13}}{2}$

$= 2 \pm 3i$

$y(t) = e^{2t}\left(A\cos(3t) + B\sin(3t)\right) \quad A, B \in \mathbb{R}$