Midterm II

November 12, 2024 9:23 AM

University of Toronto

Faculty of Applied Sciences and Engineering

MAT188 – Midterm II – Fall 2024

LAST (Family) NAME: ______________________________

FIRST (Given) NAME: ______________________________

Email address: ______________________________@mail.utoronto.ca

STUDENT NUMBER: ______________________________

Time: 90 mins.

1. **Keep this booklet closed** until an invigilator announces that the test has started. You may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **in the boxes**. There is ample space within each one. If you must use additional space, please use the blank pages at the end of this booklet and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 14 pages, excluding the cover page, and 6 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work.
7. No textbook, notes, or other outside assistance is allowed.
8. **No calculator** or equivalent devices allowed.

Question:	1	2	3	4	5	6	Total
Points:	13	7	11	9	6	14	60
Score:							

Part A

1. (13 points) Fill in the bubble for all statements that **must** be true. You don't need to include your work or reasoning. Some questions may have more than one correct answer. For those questions, you may get a negative mark for incorrectly filled bubbles.

   (a) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Then $\det A$ is

   ○ 0 ○ 10 ○ 12 ● 24

   (b) Which of the following sets of vectors are linearly independent?

   ○ $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2024 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ ○ $\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2023 \\ 2024 \end{bmatrix}, \begin{bmatrix} 2025 \\ 42 \end{bmatrix} \right\}$

   ● $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 12 \\ -2 \end{bmatrix} \right\}$ ● $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 188 \\ 2024 \end{bmatrix}, \begin{bmatrix} 0 \\ 186 \\ 188 \\ 2024 \end{bmatrix} \right\}$

   (c) Given a linear transformation $T : \mathbb{R}^9 \to \mathbb{R}^9$, which of the following are subspaces?

   ● $\{\vec{v} \in \mathbb{R}^9 \mid T^2(\vec{v}) = \vec{0}\}$
   ● $\ker(T) \cap \operatorname{im}(T)$
   ○ $\{\vec{v} \in \mathbb{R}^9 \mid T^2(\vec{v}) = \vec{e}_2\}$
   ● The image of T^2

(d) Let $T : \mathbb{R}^5 \to \mathbb{R}^6$ be an injective linear transformation and let A be the standard matrix of T. Choose all that apply.

$A_{6\times 5}$

- ● The row-echelon form of A has no column corresponding to a free variable.
- ● Every column in the row-echelon form of A is a pivot column.
- ● The row-echelon form of A has at least one row of zero.
- ○ Every row in the row-echelon form of A contains a leading one.
- ● The solution set to $A\vec{x} = \vec{0}$ only contains the zero vector.

(e) Assume that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is linear such that $\ker(T) = \text{span}(\vec{e_1})$. Choose all that apply.

rank - Nullity : $1 + \dim(\text{im}\, T) = 2$

- ○ T is injective but not surjective
- ○ T is surjective but not injective
- ● the kernel and image of T are both lines
- ○ the image of T is a plane in $\mathbb{R}^3$

(f) Let T be a linear transformation with the associated standard 6×6 matrix A. Which of the following are equivalent to T being invertible?

- ● $A\vec{x} = \vec{e}_1$ has a unique solution
- ○ $A\vec{e}_2 = \vec{e}_5$
- ○ $T(\vec{x}) = \vec{e}_3$ has no solution
- ○ $A\vec{e}_2 = \vec{0}$
- ● There is some basis of the image of T that has six vectors

(g) Suppose A is a 2×5 matrix and $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent vectors in $\mathbb{R}^5$. Choose all that apply to $\dim(\text{span}(A\vec{u}, A\vec{v}, A\vec{w}))$.

$A_{2\times 5}$

$T_A : \mathbb{R}^5 \to \mathbb{R}^2$

- ○ $\dim(\text{span}(A\vec{u}, A\vec{v}, A\vec{w}))$ **must** be 3.
- ○ $\dim(\text{span}(A\vec{u}, A\vec{v}, A\vec{w}))$ **could** be 3.
- ○ $\dim(\text{span}(A\vec{u}, A\vec{v}, A\vec{w}))$ **must** be 2.
- ● $\dim(\text{span}(A\vec{u}, A\vec{v}, A\vec{w}))$ **could** be $0, 1, 2$, but can't be 3.

2. (7 points) Fill in the blank. You do not need to include your computations or reasoning.

Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 5 \end{bmatrix}, \text{ and } \vec{u}_4 = \begin{bmatrix} -4 \\ 4 \\ 1 \\ -2 \end{bmatrix}.$$

Let $A = \begin{bmatrix} | & | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \\ | & | & | & | \end{bmatrix}$. You can use the fact that A row-reduces to $\begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Is the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ linearly independent? ○ Yes ● No

(b) If possible, find a nontrivial linear relation among the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and $\vec{u}_4$. If not possible, give a trivial linear relation

$$\vec{0} = \boxed{1}\,\vec{u}_1 + \boxed{1}\,\vec{u}_2 + \boxed{-1}\,\vec{u}_3 + \boxed{0}\,\vec{u}_4$$

(c) Find a basis $\mathcal{B}$ for $\text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ using only vectors from the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$.

$$\mathcal{B} = \boxed{\{\vec{u}_1, \vec{u}_2, \vec{u}_4\}}$$

(d) The dimension of $\text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$ is $\boxed{3}$

The dimension of $\text{Null}(A)$ is $\boxed{1}$

(e) Consider the linear transformation $T(\vec{x}) = A\vec{x}$. Choose all that apply.

○ T is injective ○ $\ker T$ is a plane ● $\ker T$ is a line ○ T is invertible

Part B

3. Assume that T is a linear transformation. Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ is a basis for $\mathbb{R}^4$ such that the matrix of T with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 12 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

(a) (2 points) Suppose $\vec{v} = \vec{b}_1 - \vec{b}_2 + \vec{b}_3$.

Then the $\mathcal{B}$-coordinates of $\vec{v}$ form the vector $\boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}$

(b) (3 points) Find $[T(\vec{v})]_{\mathcal{B}}$. Justify your answer.

$$[T(\vec{v})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 12 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 16 \\ 0 \end{bmatrix}$$

(c) (3 points) If $\vec{b}_1 = \vec{e}_2$ and $\vec{b}_3 = \vec{e}_1$, find the standard coordinates of $T(\vec{v})$. Justify your answer.

$$T(\vec{v}) = \vec{b}_1 + 16\vec{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 16 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(d) (3 points) Is $\vec{b}_2$ in the image of T? Justify your answer.

$$[\vec{b}_2]_{\mathcal{B}} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \notin \mathrm{Col}([T]_{\mathcal{B}})$$ because

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 12 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$$ is inconsistent.

4. State whether each statement is true or false by writing “True” or “False” in the small box, and provide a short and complete justification for your claim in the larger box. If you think a statement is true, explain why it must be true. If you think a statement is false, give a counterexample.

   (a) (3 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation that is not injective. Then T is also not surjective.

   True

   T inj $\Rightarrow \ker T = \{\vec{0}\} \Rightarrow \dim(\ker T) = 0$

   By rank-nullity $\dim(\operatorname{im} T) = 4 - 0 = 4$

   $\Rightarrow \operatorname{im}(T)$ is the entire $\mathbb{R}^4 \Rightarrow T$ is surj.

   Or we can say if $T(\vec{x}) = A\vec{x}$, A 4×4

   T inj $\Rightarrow$ rref has pivot in every col'n

   # rows = # col's $\Rightarrow$ rref A has pivot in every row

   $\Rightarrow A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^4$

   $\Rightarrow T$ is surj by def of surj.

   (b) (3 points) If S is the change of basis matrix from a basis $\mathcal{B}$ of $\mathbb{R}^3$ to the standard basis $\mathcal{E}$ of $\mathbb{R}^3$ then $\det(S)$ is non-zero.

   True

   $S = \begin{bmatrix} | & | & | \\ \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ | & | & | \end{bmatrix}_{3\times3}$ is a square matrix with L.I col's $\Rightarrow S$ is invertible.

   (c) (3 points) Let W be a 4-dimensional subspace of $\mathbb{R}^6$. Then every spanning set of W has exactly 4 vectors.

   False

   take $W = sp(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$

   and take $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_1 + \vec{e}_2\}$

   $sp(S) = W$. S is a spanning set for W with $\underline{\underline{5}}$ vectors

Usual rubric

5. In each part, give an **explicit** example of the mathematical object described or explain why such an object does not exist.

   (a) (2 points) A linear transformation T that is its own inverse.

   $T: \mathbb{R}^2 \to \mathbb{R}^2$ that reflects vectors w.r.t to the x axis (any reflection works)

   (b) (2 points) A set of 5 vectors whose span is a plane (a 2-dimensional subspace) in $\mathbb{R}^4$.

   $\{ \vec{e}_1, \vec{e}_2, \vec{e}_1 + \vec{e}_2, 2\vec{e}_1 + \vec{e}_2, 3\vec{e}_1 + \vec{e}_2 \}$

   (c) (2 points) A system of linear equations with 4 equations and 3 variables that has a unique solution and the determinant of its augmented matrix is non-zero.

   DNE

   Suppose A is the coefficient matrix of such a system and $[A|\vec{b}]$ be the augmented matrix. having unique sol'n implies that rref$[A|\vec{b}]$ has no pivot in the aug. col. Hence $[A|\vec{b}]$ does NOT row reduce to I_4 So $[A|\vec{b}]$ is not invertible $\Rightarrow \det([A|\vec{b}]) = 0$.

Part C

6. The motion of a robotic arm is controlled via a linear map T. The map takes in a set of instructions in form of a vector in $\mathbb{R}^5$ and generates the position of the tip of the robotic arm in $\mathbb{R}^3$. For the set of instructions $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ the arm travels to

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To simplify this we can write

$$T(\vec{v}) = \overbrace{\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}}^{A} \vec{v}. \sim \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) (3 points) Can the tip of the robotic arm explore the entirety of $\mathbb{R}^3$? Justify your answer. Yes.

$\dim(\operatorname{im}(T)) = \operatorname{rank}(A) = 3 \Longrightarrow \operatorname{im}(T) = \mathbb{R}^3$

(b) (3 points) Find a basis for $\ker T$. What is the dimension of $\ker T$?

$$\ker T = \{\underbrace{\vec{x} \in \mathbb{R}^5 / T(\vec{x}) = \vec{0}}_{A\vec{x} = \vec{0}}\} = \left\{ \begin{pmatrix} -x_3 + x_5 \\ -x_3 - x_5 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix}, \ x_3, x_5 \in \mathbb{R} \right\}$$

$$= \operatorname{Sp}\left(\underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{w}_1}, \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\vec{w}_2} \right)$$

$\vec{w}_1, \vec{w}_2$ are L.I.

$\{\vec{w}_1, \vec{w}_2\}$ is a basis for $\ker T$

$\dim(\ker T) = 2.$

Recall that the map is defined by $T(\vec{v}) = A\vec{v}$ where $A = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

(c) (1 point) Verify that the set of instructions $\begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ sends the tip of the arm to the origin.

$$A \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \vec{0}$$

(d) (3 points) Find a basis for the kernel of T that contains the vector $\begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

Take $\left\{ \overbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}}^{\vec{b}_1}, \overbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}^{\vec{b}_2} \right\}$.

$\dim \ker(T) = 2$ so any set with 2 L.I vectors in the kernel gives a basis.

$\vec{b}_1, \vec{b}_2 \in \ker T$, $\vec{b}_1, \vec{b}_2$ are L.I.

Recall that the map is defined by $T(\vec{v}) = A\vec{v}$ where $A = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

(e) (4 points) Suppose that the input instructions are only chosen from the subspace

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0 \right\}.$$

Can the tip of the robotic arm still explore all of $\mathbb{R}^3$? Justify your answer. Yes.

$$W = sp\left(\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{a}_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{a}_2}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{a}_3}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{a}_4} \right)$$

$$T(\vec{a}_1) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad T(\vec{a}_2) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad T(\vec{a}_3) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad T(\vec{a}_4) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$T(\vec{a}_i) \in im(T)$ $\qquad sp(T(\vec{a}_1), T(\vec{a}_2), \ldots, T(\vec{a}_4)) \underset{s.s}{\subseteq} im(T)$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix}$$

$\dim(sp(T(\vec{a}_1), \ldots, T(\vec{a}_3))) = 3 \Rightarrow$ it is the entire $\mathbb{R}^3$

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