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University of Toronto
Faculty of Applied Science & Engineering

Winter 2024/2025
110 Minutes

MAT187H1S Midterm 2 Solutions

Exam Reminders:

- Fill out your name, UTORid, and email address at the top of this page.
- Do not begin writing the exam until instructed to do so.
- As a student, you help create a fair and inclusive writing environment; unauthorized aids are prohibited and using one may result in you being charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. These devices may *not* be left in your pockets.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator.
- In the event of a fire alarm, do not check cell phones or other electronic devices unless authorized to do so.

Special Instructions:

- Write legibly and darkly.
- For questions with a boxed area, ensure your answer is completely within the box.
- Fill in your bubbles completely.

Good: ● A ○ B   Bad: ◐ A ⊙ B ⊠ C

Scoring

Question:	1	2	3	4	5	6	Total
Points:	8	12	6	4	7	5	42
Score:							

Formula Sheet

The following formulae are provided for your use during the exam.

Simpson's Rule

If p is a quadratic polynomial,

$$\int_a^b p(x)\,\mathrm{d}x = \frac{b-a}{6}\left(p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right).$$

Taylor's Remainder Theorem

Let T_n be the nth Taylor approximation of function f centered at a. Further, assume f has at least $n+1$ continuous derivatives. Then

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x and a.

Geometric Sum

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

1. For this question there is no need to show your work.

   (a) (2 points) Find an antiderivative of x^2e^x whose graph passes through $(0,0)$.

   $x^2e^x - 2xe^x + 2e^x - 2$

   (b) Consider the function $f(x) = x^3 \sin(x^2)$

   i. (1 point) The 11^{th} derivative of f at 0 is

   ● 0 ○ $1/3!$ ○ $-11!/3!$ ○ $11!/5!$ ○ $13!/5!$

   ii. (1 point) The 13^{th} derivative of f at 0 is

   ○ 0 ○ $1/3!$ ○ $-11!/3!$ ○ $11!/5!$ ● $13!/5!$

   (c) (2 points) Let f and g be differentiable functions and suppose both of their 6^{th} Taylor polynomials, centered at 3, are equal. Which statements are true? Mark all that apply.

   ○ It **must** be that $f(0) = g(0)$
   ● It **must** be that $f(3) = g(3)$
   ○ It **must** be that $f(x) = g(x)$ for all $x \in \mathbb{R}$
   ● It **could** be that $f(x) = g(x)$ for all $x \in \mathbb{R}$

   (d) (2 points) Which of the following improper integrals converge? Mark all that apply.

   ○ $\int_0^\infty e^x \, dx$ ● $\int_1^\infty \frac{1}{x^5} \, dx$ ○ $\int_0^1 \frac{1}{x^5} \, dx$
   ● $\int_1^\infty \frac{e^{-x}}{x} \, dx$ ○ $\int_{-\infty}^\infty \sin(x) \, dx$

   Scratch work:

2. Let $f(x) = e^{-x^2}$.

Let T be a power series representation of f centered at 0.

(a) (2 points) Write down $T(5)$ using Σ-notation.

$$T(5) \quad = \quad \boxed{\sum_{i=0}^{\infty} \frac{(-1)^i (5)^{2i}}{i!}}$$

(b) (2 points) Provide a well-written definition of the statement "the series $T(5)$ converges". *Do not determine whether $T(5)$ converges. Only write the definition of convergence.*

The series $T(5)$ converges if the sequence of partial sums $S_n = \displaystyle\sum_{i}^{n} \frac{(-1)^i (5)^{2i}}{i!}$ converges.

(c) (3 points) Does $T(5)$ converge? Provide a well-written justification. Your justification must cite any convergence tests that you use.

● $T(5)$ **converges** ○ $T(5)$ **diverges**

Justification:

Applying the ratio test to $T(5)$ we compute

$$\lim_{i\to\infty} \frac{\left|\frac{(-1)^{i+1}(5)^{2(i+1)}}{(i+1)!}\right|}{\left|\frac{(-1)^i(5)^{2i}}{i!}\right|} = \lim_{i\to\infty} \frac{i!}{(i+1)!} \cdot \frac{5^{2i+2}}{5^{2i}} = \lim_{i\to\infty} \frac{25}{i+1} = 0$$

Since $0 < 1$, the ratio test determines that $T(5)$ converges.

Let $F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x e^{-t^2}\,\mathrm{d}t$, let P be a power series representation of F centered at 0, and define

$$g(x) = \begin{cases} 1 & \text{if } x < 1 \\ e^{-x} & \text{if } x \geq 1 \end{cases}.$$

For the remaining parts, you may use the fact that $e^{-x^2} \leq g(x)$ for all $x \geq 0$.

(d) (2 points) Write down $P(5)$ using Σ-notation.

$$P(5) \quad = \quad \boxed{\sum_{i=0}^{\infty} \frac{(-1)^i (5)^{2i+1}}{(2i+1)\cdot i!}}$$

(e) (1 point) Compute $I = \int_0^5 g(x)\,\mathrm{d}x$. *No need to simplify.*

$$I \quad = \quad \boxed{1 + e^{-1} - e^{-5}}$$

(f) (2 points) Use g to find an upper bound for $F(5)$. Explain how you obtained your bound. *No need to simplify.*

$$F(5) \quad \leq \quad \boxed{1 + e^{-1} - e^{-5}}$$

Explanation:

Since $f(x) \leq g(x)$ for $x \geq 0$, we have that

$$\int_0^k f(x)\,\mathrm{d}x \leq \int_0^k g(x)\,\mathrm{d}x.$$

Letting $k = 5$ we arrive at our upper bound.

3. Let

$$f(x) = \frac{1}{x^2 + x - 6} \qquad \text{and} \qquad g(x) = \frac{A}{x-2} + \frac{B}{x+3}.$$

(a) (2 points) You would like to find $A, B \in \mathbb{R}$ so that $f(x) = g(x)$ for all x. Let $\vec{v}$ be the column vector with entries A, B. Find a matrix M and a vector $\vec{b}$ so that solving $M\vec{v} = \vec{b}$ will give you the correct values for A, B.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}}$$

(b) (2 points) You would like to compute $I = \displaystyle\int_0^\infty f(x)\,\mathrm{d}x$. Express I using only definite integrals and limits. *Do not evaluate any of the integrals/limits.*

$$I = \boxed{\lim_{k\to 2^-} \int_0^k f(x), \mathrm{d}x + \lim_{k\to 2^+} \int_k^3 f(x), \mathrm{d}x + \lim_{k\to\infty} \int_3^k f(x), \mathrm{d}x}$$

(c) (2 points) Let F be an antiderivative of f satisfying $F(6) = 10$.
Does $\displaystyle\lim_{x\to\infty} F(x)$ exist? Provide a well-written justification.

● $\displaystyle\lim_{x\to\infty} F(x)$ exists ○ $\displaystyle\lim_{x\to\infty} F(x)$ does not exist ○ Cannot be determined

Justification: Notice that $x^2 + x - 6 \geq x^2$ when $x \geq 6$. Therefore

$$0 < f(x) \leq \frac{1}{x^2}$$

when $x \geq 6$. Since F is an antiderivative of f (and because f is continuous on $[6, \infty)$), we know

$$\int_6^x f(t)\,\mathrm{d}t = F(x) + C$$

for a fixed constant C. By our previous inequality,

$$\lim_{x\to\infty} \int_6^x f(t)\,\mathrm{d}t \leq \lim_{x\to\infty} \int_6^x \frac{1}{t^2}\,\mathrm{d}t < \infty$$

with the last inequality following from the p-test. Because $f(x)$ is positive on $[0, \infty)$, by monotone convergence, $\displaystyle\lim_{x\to\infty} \int_6^x f(t)\,\mathrm{d}t$ converges and therefore $\displaystyle\lim_{x\to\infty} F(x)$ converges.

4. In this question, you are asked to draw, if possible, the graph of a function $f : [0, 2] \to \mathbb{R}$ subject to various restrictions. If no such function exists, you are to mark the option "No such f exists".

For both questions, you are to assume:

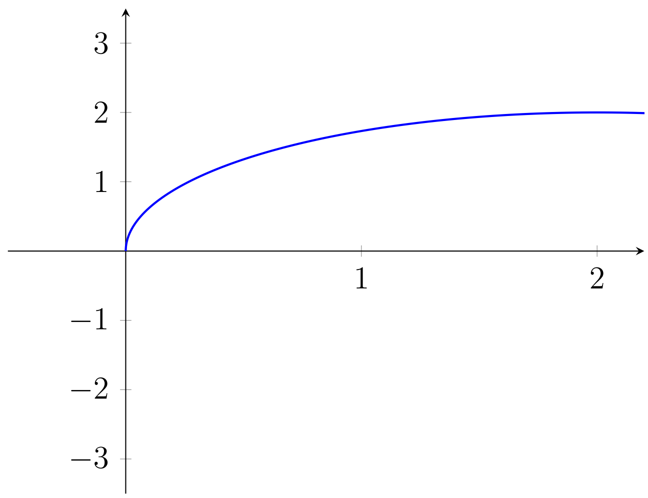
- f is continuous
- f is defined on its entire domain $[0, 2]$
- $f(0) = 0$

Note 1: in the following questions you are ***drawing*** f *but* ***integrating*** $1/f$.

Note 2: if you feel like your drawing is unclear, you may add a description in words below your graph.

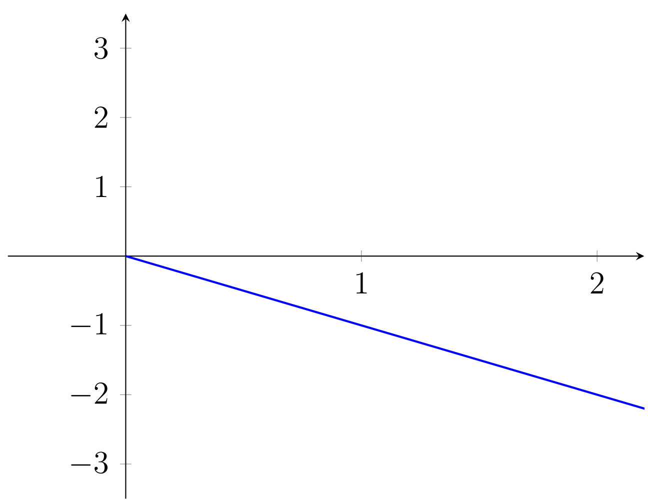
(a) (2 points) f satisfies that the improper integral $\displaystyle\int_0^2 \frac{1}{f(x)}\mathrm{d}x$ **converges** and is **positive**.

● Such an f exists ○ No such f exists



(b) (2 points) f satisfies that the improper integral $\displaystyle\int_0^2 \frac{1}{f(x)}\mathrm{d}x$ **diverges** to $-\infty$.

● Such an f exists ○ No such f exists



5. Shiwei is analysing the power series $P(x) = \sum_{n=0}^{\infty} n2^n x^n$. He is searching for the radius of convergence of P. His writeup for his instructor is as follows:

Dear Instructor, I applied the ratio test to find radius of convergence of $P(x)$.

(Step 1) First, I compute the limit

$$L = \lim_{n\to\infty} \frac{(n+1)2^{n+1}x^{n+1}}{n2^n x^n}.$$

(Step 2) Simplifying, I get

$$L = \lim_{n\to\infty} \frac{(n+1)2x}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot 2x.$$

(Step 3) Since $2x$ is a constant, I may pull it out of the limit and so we see

$$L = \lim_{n\to\infty} \frac{n+1}{n} \cdot 2x = 2x \cdot \lim_{n\to\infty} \frac{n+1}{n} = 2x \cdot 1 = 2x.$$

(Step 4) The ratio test requires $L < 1$. I see

$$L = 2x < 1 \qquad \Longrightarrow \qquad x < \frac{1}{2}.$$

(Step 5) Therefore $P(x)$ converges when $x \in (-\infty, \frac{1}{2})$.

(Step 6) Therefore the radius of convergence of $P(x)$ is ∞.

(a) (2 points) Select the answer that best describes Shiwei's work.

- ○ Shiwei reached the correct conclusion **and** his work is correct.
- ○ Shiwei reached the correct conclusion **but** his work is incorrect.
- ● Shiwei reached an incorrect conclusion.
- ○ It cannot be determined from the given information whether Shiwei's work/conclusion is correct or not.

(b) (2 points) What is the radius of convergence of P?

Radius of convergence $=$ 1/2

Scratch work:

(c) (3 points) Correct any mistakes Shiwei made by crossing out/modifying Shiwei's steps. If Shiwei made no mistakes, mark "Shiwei's answer has no mistakes".

○ Shiwei's answer has no mistakes

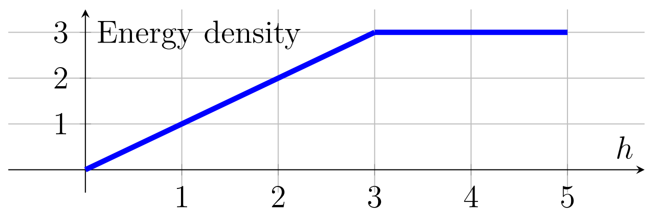
(Step 1)	First, I compute the limit $$L = \lim_{n\to\infty} \left| \frac{(n+1)2^{n+1}x^{n+1}}{n2^n x^n} \right|.$$
(Step 2)	Simplifying, I get $$L = \lim_{n\to\infty} \frac{(n+1)\,\lvert 2x \rvert}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \lvert 2x \rvert.$$
(Step 3)	Since $\lvert 2x \rvert$ is a constant, I may pull it out of the limit and so we see $$L = \lim_{n\to\infty} \frac{n+1}{n} \cdot \lvert 2x \rvert = \lvert 2x \rvert \cdot \lim_{n\to\infty} \frac{n+1}{n} = \lvert 2x \rvert \cdot 1 = \lvert 2x \rvert.$$
(Step 4)	The ratio test requires $L < 1$. I see $$L = \lvert 2x \rvert < 1 \qquad \Longrightarrow \qquad \lvert x \rvert < \frac{1}{2}.$$
(Step 5)	Therefore $P(x)$ converges when $\cancel{x \in (-\infty, \frac{1}{2})}$. $x \in (-1/2, 1/2)$
(Step 6)	Therefore the radius of convergence of $P(x)$ is $\cancel{\infty}$. $1/2$

6. A new solar plant is installing advanced batteries to provide power during the night. Each battery is $5m$ tall and looks roughly like a cone.

The cross-sectional area (in square metres) of a battery at h metres above the ground is given by

$$A(h) = \frac{1}{2\sqrt{h+4}}.$$

Because of the way the chemicals in the battery settle, the energy density of a fully charged battery varies as a function of height. Let $D(h)$ be the energy density at height h metres above the base of a fully charged battery (in units of giga-Joules per cubic metre). D is given by the following graph.



(a) (2 points) Set up a definite integral to compute the total energy, E, contained in a fully charged battery. *Do not evaluate your integral. Your final answer may contain the expressions* $A(h)$ *and* $D(h)$.

$$E = \boxed{\int_0^5 A(h) \cdot D(h)\, \mathrm{d}h}$$

(b) (3 points) Compute the total energy, E, contained in a fully charged battery. *No need to simplify your answer.*

$$E = \boxed{\frac{43}{3} - \frac{14\sqrt{7}}{3} = 9 - \tfrac{2}{3}(7)^{3/2} + \tfrac{2}{3}(4)^{3/2} = 9 + \frac{16}{3} - \tfrac{2}{3}(7)^{3/2} \approx 1.986}$$

Scratch work:

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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