Midterm I Solution

February 13, 2024 1:38 PM

University of Toronto
Faculty of Applies Sciences and Engineering

MAT187 – Midterm I –Winter 2024

LAST (Family) NAME:	
FIRST (Given) NAME:	
Email address:	Solutions @mail.utoronto.ca
STUDENT NUMBER:	

Time: 90 mins.

1. **Keep this booklet closed**, until an invigilator announces that the test has begun. However, you may fill out your information in the box above before the test begins.
2. Please place your **student ID card** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
3. Please write your answers **inside the boxes** whenever provided. Ample space is provided within each box, however, if you must use additional space, please use the blank page at the end of this booklet, and clearly indicate in the given box that your answer is **continued on the blank page**. You can also use the blank pages as scrap paper. Do not remove them from the booklet.
4. This test booklet contains 14 pages, excluding the cover page, and 4 questions. If your booklet is missing a page, please raise your hand to notify an invigilator as soon as possible.
5. **Do not remove any page from this booklet.**
6. Remember to show all your work for part B and C. You don't need to justify your choices in part A.
7. No notes or outside help is allowed in your workspace.
8. No calculator is allowed.

Question:	1	2	3	4	Total
Points:	7	20	6	7	40
Score:					

Part A

1. (7 points) Fill in the bubble for all statements that must be true. Some questions may have more than one correct answer. You may get a negative mark for incorrectly filled bubbles. You don't need to include your work or reasoning.

$\int_1^5 x g'(x)\,dx = xg(x)\Big]_1^5 - \int_1^5 g(x)\,dx$

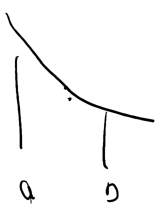
$= \Big[5g(5) - g(1)\Big] - (-10)$

$= (10+5) + 10 = 25$

(a) If $g(1) = -5$, $g(5) = 2$, and $\int_1^5 g(x)\,dx = -10$, evaluate the integral $\int_1^5 xg'(x)\,dx$.

○ 5 ● 25 ○ 15 ○ 10 ○ we don't need enough information

(b) You are working for a phone manufacturer named SESAMG and you are being asked to estimate the battery life of the newest product "eyePhone 15 megamax". You are given the following information:



The battery life is given by $\int_a^b f(x)\,dx$. $f(x)$ is a decreasing and concave up function. You need to be able to *guarantee* that the phone lasts at least as long as your estimate. At the same time you want to be as precise as possible. underestimate
Which numerical integration method should you use?

○ Leftpoint rule ○ Rightpoint rule ● Midpoint rule ○ Trapezoid rule

(c) Recall that the general solution to an ODE is a family that describes all possible solutions to that ODE.

Consider the ODE $y' = 6x(y-1)^{\frac{2}{3}}$. Choose ALL that apply.

○ The family of solutions $y(x) = 1 + (x^2 + C)^3$, with $C \in \mathbb{R}$, is the general solution of the ordinary differential equation $y' = 6x(y-1)^{\frac{2}{3}}$. $y=1$ is a sol'n not included in this family!

● $y' = 6x(y-1)^{\frac{2}{3}}$ has an equilibrium solution.

● The family $y(x) = 1 + (x^2 + C)^3$, with $C \in \mathbb{R}$, represent all the solutions to $y' = 6x(y-1)^{\frac{2}{3}}$ provided that $y \neq 1$ for any value of x.

○ None of the above.

(d) Let $f(x) = \frac{x-2}{x^3-5x^2+2x+8}$, and let D denote the domain of f. Consider the following function

$$g(x) = \begin{cases} \frac{x-2}{x^3-5x^2+2x+8} & \text{if } x \in D \\ -\frac{1}{6} & \text{otherwise} \end{cases}$$

$(x-2)(x^2-3x-4)$

$(x-2)(x-4)(x+1)$

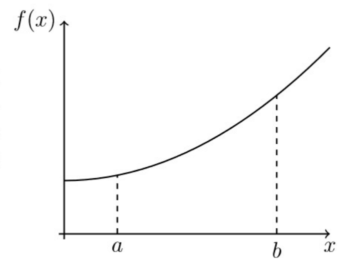
Find the **minimum** number of limits **required** to evaluate the following integral.

$x=-1$, $\underbrace{x=2}_{\text{removable}}$, $\underbrace{x=4}_{>3}$

$$\int_{-\infty}^{3} g(x)dx = \int_{-\infty}^{-2} + \int_{-2}^{-1} + \int_{-1}^{3}$$

◯ one ◯ two ● three ◯ four ◯ no limit is needed

(e) Consider the function plotted on the right. Recall that L_n, R_n denote the the Left and Right Riemann sum respectively and T_n and M_n denote the trapezoidal and midpoint rules respectively. Pick the one correct ordering:



- ◯ $L_{2024} \leq L_5 \leq M_{2024} \leq \int_a^b f(x)\,dx \leq T_5 \leq T_{2024}$
- ● $L_5 \leq L_{2024} \leq M_{2024} \leq \int_a^b f(x)\,dx \leq T_{2024} \leq T_{2000}$
- ◯ $L_5 \leq L_{2024} \leq \int_a^b f(x)\,dx \leq M_{2024} \leq T_{2024} \leq T_5$
- ◯ $T_{2024} \leq T_5 \leq M_{2024} \leq \int_a^b f(x)\,dx \leq L_5 \leq L_{2024}$
- ◯ $T_5 \leq T_{2024} \leq \int_a^b f(x)\,dx \leq M_{2024} \leq L_{2024} \leq L_5$

Part B

2. The parts of this question are NOT related.

   (a) (6 points) Find the following integrals. Justify your answer and write your final answer in the provided box.

$$\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx = \frac{3}{2}\int \frac{1}{x-1}\,dx + \left(-\frac{1}{2}\right)\int \frac{1}{x+1}\,dx + (-1)\int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{3}{2}\ln|x-1| + \left(-\frac{1}{2}\right)\ln|x+1| + (x+1)^{-1} + C$$

$$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2+2x+3 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

@ $x=1$ $\quad 6 = 4A \Rightarrow \boxed{A = \frac{3}{2}}$

@ $x=-1$ $\quad 2 = -2C \Rightarrow \boxed{C=-1}$

@ $x=0$ $\quad 3 = A - B - C$

$3 = \frac{3}{2} - B + 1 \Rightarrow B = -2 + \frac{3}{2}$

$B = \frac{-1}{2}$

$$\int \frac{dx}{x^2-2x+10} = \int \frac{dx}{(x-1)^2+3^2} = \frac{1}{9}\int \frac{3\sec^2\theta\,d\theta}{\tan^2\theta+1}$$

$\sin^2 + \cos^2 = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$x - 1 = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\tan\theta = \frac{x-1}{3}$

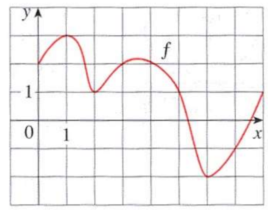
$\theta = \arctan\left(\frac{x-1}{3}\right)$

$-\pi/2 \leq \theta \leq \pi/2$

$$= \frac{1}{9}\int \frac{3\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C$$

$$= \frac{1}{3}\arctan\left(\frac{x-1}{3}\right) + C$$

(b) (3 points) The graph of a function is given on the right. Estimate $\int_0^8 f(x)\,dx$ by application of the trapezoidal rule using four subintervals ($n = 4$). Show your work.



$$\frac{2}{2}(f(0)+f(2)) + \frac{2}{2}(f(2)+f(4)) + \frac{2}{2}(f(4)+f(6)) + \frac{2}{2}(f(6)+f(8))$$

$$f(0)+2f(2)+2f(4)+2f(6)+f(8)$$

$$2 + 2\times 1 + 2\times 2 + 2(-2) + 1 = 5$$

(c) (6 points) For each integral, decide if it is convergent or divergent. Justify your answer. If the integral is convergent, find the value it converges to.

$\int_0^\infty \frac{1}{5+4x+4x^2}\,dx$ is $\leq \int \frac{1}{4x^2}dx$ ● convergent ○ divergent

Justify your choice.

$$\frac{1}{4x^2+4x+5} = \frac{1}{4}\left(\frac{1}{x^2+x+\frac{5}{4}}\right) = \frac{1}{4\left[(x+\frac{1}{2})^2+1\right]}$$

$$\int_0^\infty \frac{dx}{4\left[(x+\frac{1}{2})^2+1\right]} = \lim_{t\to\infty}\int_0^t \frac{dx}{4\left[(x+\frac{1}{2})^2+1\right]} = \lim_{t\to\infty}\frac{1}{4}\int_0^t \frac{\sec^2\theta\, d\theta}{\tan^2\theta+1}$$

$x+\frac{1}{2} = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$-\pi/2 \leq \theta \leq \pi/2$

$$= \lim_{t\to\infty}\frac{1}{4}\int_0^t d\theta$$

$$= \lim_{t\to\infty} \frac{1}{4}\left(\arctan(x+\tfrac{1}{2})\right]_0^t$$

$$= \lim_{t\to\infty}\frac{1}{4}\left[\arctan(t+\tfrac{1}{2}) - \arctan(\tfrac{1}{2})\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - \arctan(\tfrac{1}{2})\right)$$

$\int_{-\infty}^\infty 7e^{-|x|}\,dx$ is ● convergent ○ divergent

Justify your choice.

$$\int_{-\infty}^\infty 7e^{-|x|}dx = \int_{-\infty}^0 7e^{x}dx + \int_0^\infty 7e^{-x}dx$$

$$= \lim_{t\to-\infty} 7\int_t^0 e^x dx + \lim_{t\to\infty} 7\int_0^t e^{-x}dx$$

$$= \lim_{t\to-\infty} 7\left[e^0 - e^t\right] - \lim_{t\to\infty} 7\left[e^{-t} - e^0\right]$$

$$= 7\left[1-0\right] - 7\left[0-1\right]$$

$$= 7+7 = 14$$

$$\int_{-\infty}^\infty 7e^{-|x|}dx = 2\int_0^\infty 7e^{-x}dx$$

7 (convergent)

(d) (5 points) The amount of a type of bacteria in a a solution is modelled by

$$\frac{dy}{dt} = 30 - \frac{y(t)}{100},$$

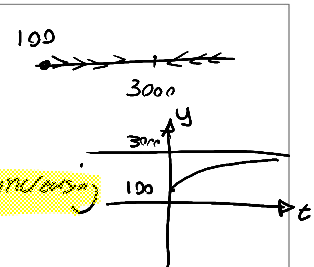
where $y(t)$ is measured in milligrams (mg) and t is measured in hours (h). Suppose we begin with 100 mg of bacteria in the solution. At this time, the amount of bacteria in the solution will ● increase ○ decrease.

Justify your choice. $y(0) = 100$

$y' = 30 - \frac{y}{100} = 0 \Rightarrow \frac{y}{100} = 30$

$\Rightarrow y = 3000$

@ $y = 100$, $y' > 0$ therefor y is increasing



Continue assuming that there is originally 100 mg of bacteria in the solution. Find $\lim_{t\to\infty} y(t)$.

$\lim_{t\to\infty} y(t) = 3000$

Explain, in plain English, what will happen to the amount of bacteria in the solution in the long run?

In the long run, the amount of bacteria will approach to 3000 mg but will never reach it.

$$e^{F(t)}y' + f(t)e^{F(t)}y = f(t)e^{F(t)}$$

$$\int \frac{d}{dt}\left(e^{F(t)}y\right)dt = \int f(t)e^{F(t)}dt = \int \frac{d}{dt}\left[e^{F(t)}\right]dt$$

$$e^{F(t)}y = e^{F(t)} + c$$

$$\boxed{y = 1 + ce^{-F(t)}}$$

3. The parts of this question **are related**.

(a) (3 points) Consider the differential equation $y' + f(t)y = f(t)$. Suppose $f(t)$ is continuous on the whole real line, with anti-derivative $F(t)$. Find the general solution of this ODE. Show your work and write your final answer in the provided box.

$y' = f(t) - f(t)y$

$y' = f(t)(1-y)$

$\frac{y'}{1-y} = f(t)$

We can use seprable method $(y \neq 1)$

$\int \frac{dy}{1-y} = \int f(t)dt$

$-\ln|1-y| = F(t) + c$

$\frac{1}{|1-y|} = e^{F(t)} c_1$ $\quad c_1 > 0$

$|1-y| = e^{-F(t)} c_2$

$1-y = \pm e^{-F(t)} c_2$

$y = 1 + e^{-F(t)} C$

$C \neq 0$

$$\boxed{y = 1 + Ce^{-F(t)} \quad C \in \mathbb{R}}$$

(b) (3 points) Now consider the initial value problem

$$\begin{cases} y' + f(t)y = f(t) \\ y(0) = 0. \end{cases}$$

Suppose $F(t)$ satisfies $F(0) = 1$ and $F(1) = 1$. If $y(t)$ solves the above initial value problem, then

- ○ $y(1) = -e$
- ○ $y(1) = \frac{1}{e}$
- ● $y(1) = 0$
- ○ $y(1) = 1$
- ○ $y(1) = e$

Justify your choice.

$0 = y(0) = 1 + Ce^{-F(0)} = 1 + \frac{C}{e} \Rightarrow \frac{C}{e} = -1 \Rightarrow \boxed{C = -e}$

$y(1) = 1 - e\,e^{-F(1)} = 1 - e\,e^{-1} = 0$

Part C

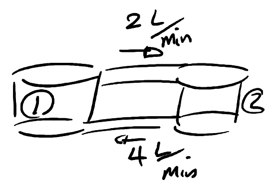
4. Two 500-litre tanks are connected to each other. Initially, both tanks contain 100 litres of pure water. For the duration of this experiment, tank 1 pumps its fluid into tank 2 at a rate of 2 $\frac{litres}{minute}$ and tank 2 pumps its fluid into tank 1 at a rate of 4 $\frac{litres}{minute}$. At $t = 0$ minutes, 10 grams of sugar is instantaneously poured into tank 1. We assume that the sugar in both tanks is perfectly dissolved and well-mixed at all times.

   (a) (2 points) Let $V_1(t)$ and $V_2(t)$ denote the volume of tank 1 and 2 at time t respectively. Then

   of the liquid in

   $V_1(t) = \boxed{100 + 2t}$ $\quad$ $V_2(t) = \boxed{100 - 2t}$

   (b) (3 points) Let $T_1(t)$ and $T_2(t)$ denote the amount of sugar measured in grams contained in tank 1 and tank 2 (respectively) t minutes after the sugar is added. Let C_{in} and C_{out} denote the concentration of sugar entering and leaving tank 1 at time t respectively. Describe C_{in} and C_{out} in terms of t, $T_1(t)$, and $T_2(t)$. Justify your work. Write your final answer in the provided box.



concentration of sugar entering (1) is the same as concentration of sugar in (2) @ time t, which is $\frac{T_2(t)}{V_2(t)}$. Vice versa, C_{out} is $\frac{T_1(t)}{V_1(t)}$.

$C_{in} = \boxed{\frac{T_2(t)}{V_2(t)}}$ $\quad$ $C_{out} = \boxed{\frac{T_1(t)}{V_1(t)}}$

x

dmount

(c) (2 points) Find an IVP that describes the rate of change of sugar in tank 1. Your ODE should only depend on t and $T_1(t)$. Justify your work. Write your final answer in the provided box.

Note that

$\frac{dT_1}{dt} = R_{in} - R_{out}$. By last part

$$\frac{dT_1}{dt} = 4\frac{T_2(t)}{V_2(t)} - 2\frac{T_1(t)}{V_1(t)} \quad *$$

$T_1(t) + T_2(t) = 10 \Rightarrow T_2(t) = 10 - T_1(t)$

Hence, plugging into *

$$\frac{dT_1}{dt} = 4\frac{10 - T_1}{V_2} + \frac{2T_1}{V_1}$$

By part 1, $V_1 = 100 + 2t$, $V_2 = 100 - 2t$. That is

$$\begin{cases} \frac{dT_1}{dt} = 4\frac{10 - T_1}{100 - 2t} + 2\frac{T_1}{100 + 2t} \\ T_1(0) = 10. \text{ gr} \end{cases}$$

This space is provided for additional space for your solutions if needed or to be used as a scrap paper. If you must use this space for your solutions, be sure to clearly indicate in the original question that your solution is continued on the overflow page. **Do not remove this page from the test booklet.**

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Trigonometric Functions

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$
$$\int \sec^3(x)\,dx = \frac{1}{2}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$$
$$\int \csc(x)\,dx = -\ln|\cot(x)+\csc(x)| + C$$
$$\int \tan(x)\,dx = -\ln|\cos x| + C$$
$$\int \cot(x)\,dx = \ln|\sin x| + C$$

Error formula for L_n and R_n

$$|\text{Error}| \le \frac{M}{2n}(b-a)^2$$

Where $M \ge |f'(x)|$ on $a \le x \le b$

Error formula for M_n

$$|\text{Error}| \le \frac{M}{24n^2}(b-a)^3$$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Error formula for T_n

$$|\text{Error}| \le \frac{M}{12n^2}(b-a)^3$$

Where $M \ge |f''(x)|$ on $a \le x \le b$

Average of $f(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$