1 几何最优传输映射

1.1 Monge-Ampére 方程

问题 1.1 (Brenier). 给定 (Ω,μ) 和 $(\sum,\nu)$ 以及成本函数 $c(x,y)=\frac{1}{2}|x-y|^2$，最优传输映射 $T:\Omega\to\sum$ 是满足 *Monge-Ampére* 方程的 *Brenier* 势 $u:\Omega\to\mathcal{R}$ 的梯度映射。

$$\boxed{det\left(\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\right)=\frac{f(x)}{g\circ\bigtriangledown u(x)}} \tag{1}$$

问题 1.2 (Semi-discrete OT). 给定一个在 $\mathcal{R}^d$ 上的紧凸域 Ω，和 $p_1,p_2,\cdots,p_k$ 以及质量 $w_1,w_2,\cdots,w_k>0$，找到一个最优传输映射 $T:\Omega\to\{p_1,\cdots,p_k\}$，则 $vol(T^{-1}(p_i))=w_i$，使运输成本最小化

$$C(T):=\frac{1}{2}\int_{\Omega}|x-T(x)|^2\,\mathrm{d}x \tag{2}$$

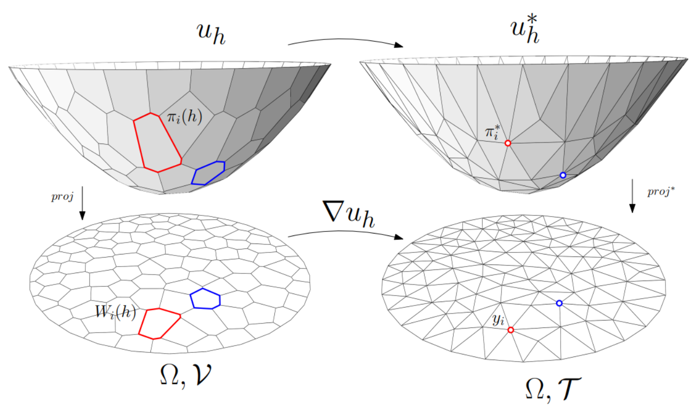

图 1: Brenier 图

根据 Brenier 定理，将有一个分段线性凸函数 $u:\Omega\to\mathbb{R}$、梯度映射给出了最佳运输映射。

定理 1.1 (Alexandrov 1950). 给定在 $\mathbb{R}^n$ 上的紧凸域 Ω，在 $\mathbb{R}^n$ 上互不相同的 $p_1,\cdots,p_k$，当 $A_1,\cdots,A_k>0$，使得 $\sum A_i=Vol(\Omega)$，则存在 *PL* 凸函数

$$f(x):=\max\{\langle x,p_i\rangle+h_i\mid i=1,\cdots,k\} \tag{3}$$

唯一的传输使得

$$Vol(W_i)=Vol(\{x\mid\bigtriangledown f(x)=p_i\})=A_i. \tag{4}$$

Alexandrov’s 证明是拓扑的，而不是变分。多年来人们一直开放寻找构造性的证明。

Algorithm: `ConvexHull`(P)

Input: A set P of points in the plane.

Output: A list $\mathcal{L}$ containing the vertices of $\mathcal{CH}(P)$ in clockwise order.

Sort the points by x-coordinate, resulting in a sequence $p_1, ..., p_n$.

Put the points p_1 and p_2 in a list $\mathcal{L}_{\text{upper}}$, with p_1 as the first point.

for $i \leftarrow 3$ **to** n **do**

  Append p_i to $\mathcal{L}_{\text{upper}}$.

  while $\mathcal{L}_{\text{upper}}$ *contains more than* 2 *points* **and** *the last three points in* $\mathcal{L}_{\text{upper}}$ *do not make a right turn* **do**

    Delete the middle of the last three points from $\mathcal{L}_{\text{upper}}$.

  end

end

Put the points p_n and p_{n-1} in a list $\mathcal{L}_{\text{lower}}$, with p_n as the first point.

for $i \leftarrow n-2$ **downto** 1 **do**

  Append p_i to $\mathcal{L}_{\text{lower}}$.

  while $\mathcal{L}_{\text{lower}}$ *contains more than* 2 *points* **and** *the last three points in* $\mathcal{L}_{\text{lower}}$ *do not make a right turn* **do**

    Delete the middle of the last three points from $\mathcal{L}_{\text{lower}}$.

  end

end

Remove the first and the last point from $\mathcal{L}_{\text{lower}}$ to avoid duplication of the points where the upper and lower hull meet.

Append $\mathcal{L}_{\text{lower}}$ to $\mathcal{L}_{\text{upper}}$, and call the resulting list $\mathcal{L}$.

return $\mathcal{L}$