ME 5405 Machine Vision

Mathematical Morphology

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1. Introduction to Mathematical Morphology

- Mathematical morphology is an approach to digital image processing based on "*shape".*
- It has an **algebraic system of operators**, which is useful because compositions of its operators can be formed that, when acting on complex shapes, are able to decompose them into their meaningful parts and separate them from their extraneous parts.
- Such a system of operators and their compositions permit the underlying shapes to be identified and optimally reconstructed from their distorted, noisy forms.
- Furthermore, they permit each shape to be understood in terms of decomposition, each entity of the decomposition being some suitably simple shape.
- In this chapter, we shall deal with sets in Z^2, where Z in integer.

2. Some Basic Definitions

Let *A* and *B* be sets in Z^2 with components $a = (a_1, a_2)$ & $b = (b_1, b_2)$, respectively

- The <u>translation</u> of *A* by *x* is defined as

$$(A)_x = \{c \mid c = a + x, \quad for \quad a \in A\} \qquad (1)$$

- The <u>reflection</u> of *B* is defined as

$$\breve{B} = \{x \mid x = -b, \quad for \quad b \in B\} \qquad (2)$$

- The <u>complement</u> of set *A* is

$$A^c = \{x \mid x \notin A\} \qquad (3)$$

- The <u>difference</u> of two sets *A* & *B*

$$A - B = \{x \mid x \in A, x \notin B\} = A \bigcap B^c \qquad (4)$$

2. Some Basic Definitions - Illustration

- The <u>translation</u> of *A* by *x* is defined as

$$(A)_x = \{c \mid c = a + x, \quad for \quad a \in A\}$$

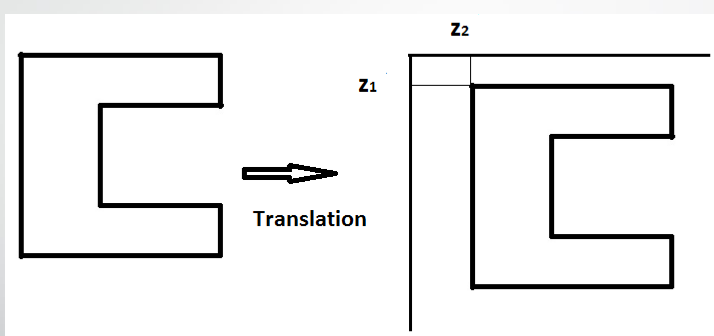


Fig. 1(a): The translation operation shifts the origin of the image to some (z_1, z_2) point.

2. Some Basic Definitions - Illustration

- The reflection of *B* is defined as

$$\breve{B} = \{x | x = -b, \quad for \quad b \in B\}$$

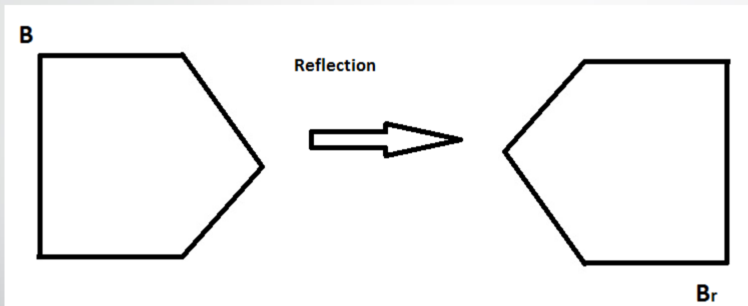


Fig. 1(b): Reflection operation

2. Some Basic Definitions - Illustration

- The complement of set *A* is

$$A^c = \{x | x \notin A\}$$

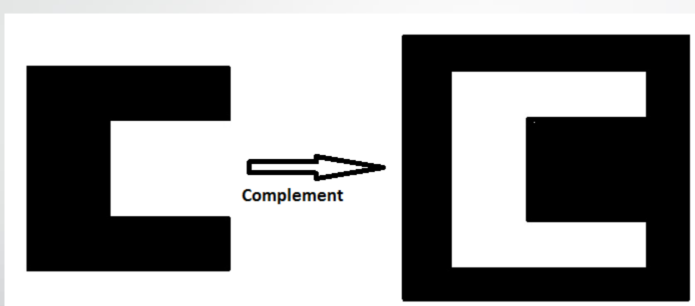


Fig. 1(c): The Complement operation

2. Some Basic Definitions - Illustration

- The difference of two sets *A* & *B*

$$A - B = \{x | x \in A, x \notin B\} = A \cap B^c$$

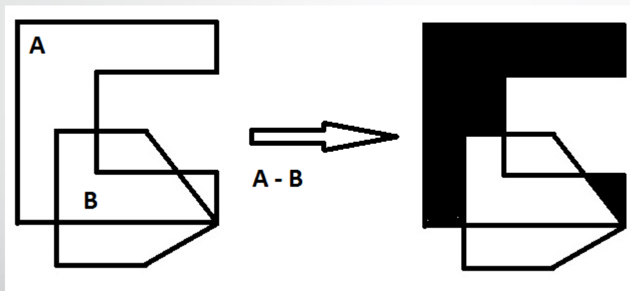


Fig. 1(d): The difference between A and B

3. Dilation

Let *A* and *B* be sets in Z^2, (or E^n in general)

The dilation of *A* by *B* is

$$A \oplus B = \{C \in E^n | c = a + b \;\; for\, some\;\; a \in A \;\; and \;\; b \in B\} \quad (5)$$

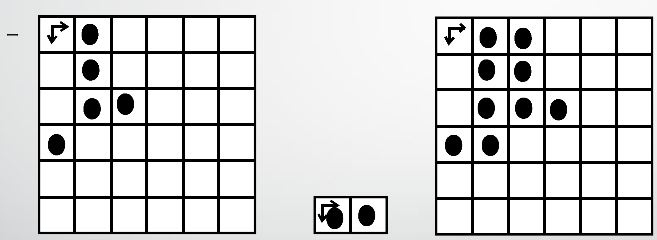


Fig. 2 Example of Dilation

A* =** {(0,1), (1,1), (2,1), (2,2), (3,0)} ; ***B = {(0,0), (0,1)};
***A⊕ B* =** {0,1), (1,1), (2,1), (3,0), (0,2), (1,2), (2,2), (2,3), (3,1)}

3.1. Dilation - Other Properties

$$A \oplus B = B \oplus A \tag{6}$$

<Proof>

$$\begin{aligned} A \oplus B &= \left\{ C \in E^N \mid C = a + b,\ for\ some\ a \in A\ and\ b \in B \right\} \\ &= \left\{ C \in E^N \mid C = b + a,\ for\ some\ a \in A\ and\ b \in B \right\} \\ &= B \oplus A \end{aligned}$$

3.2 Dilation - Other Properties

Dilation can also be defined as:

$$A \oplus B = \bigcup_{b \in B} (A)_b \tag{7}$$

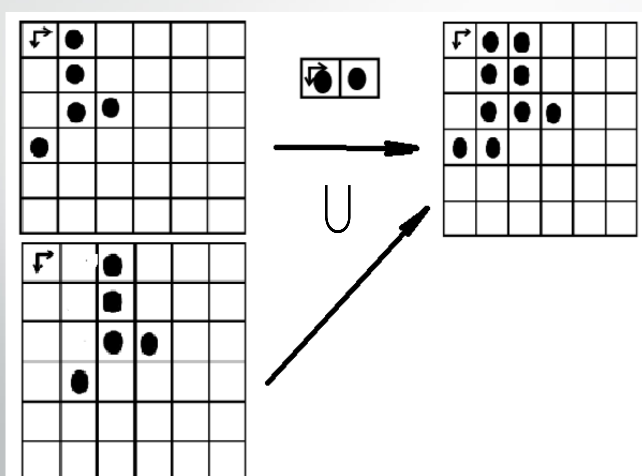


Fig. 2(a) Example of Dilation by Union of Sets

3.3. Dilation - Other Properties

Translation Invariant of Dilation, or

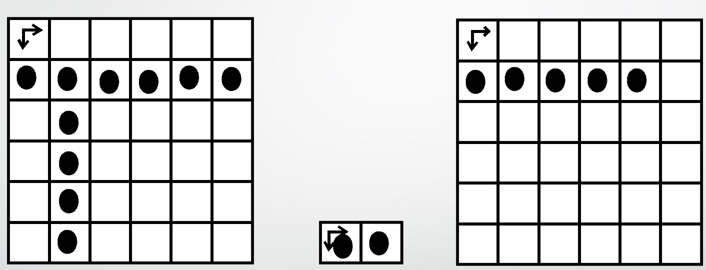
$$(A)_x \oplus B = (A \oplus B)_x \qquad (8)$$

The set A translated by x, then dilated by the structural element B, is the same as A is first dilated by B and the end result is translated by *x*

4.0. Erosion

The erosion of *A* by *B* is defined by:

$$A \ominus B = \{x \in E^N \mid x + b \in A \text{ for every } b \in B\} \qquad (9)$$



A ***B*** ***A ⊖ B***

Fig. 3 Example of Erosion

4.0. Erosion

Another definition of Erosion

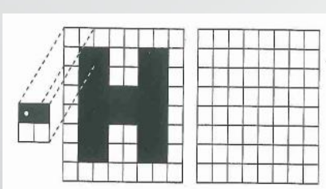
$$A \ominus B = \{x \in E^n |\ \text{for every } b \in B, \text{ there exists an } a \in A \text{ such that } x = a - b\} \qquad (10)$$

Erosion of an image *A* by a structuring element *B* is the set of all elements $x \in E^n$ for *B* translated to x is contained in *A*

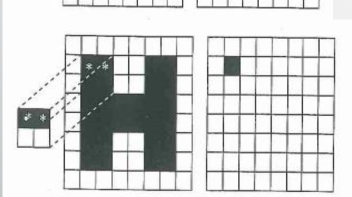
$$A \ominus B = \{x \in E^n | (B)_x \subseteq A\} \qquad (11)$$

4.0. Erosion – an Illustration (Fig. 4)

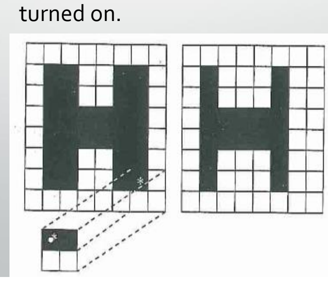
Fig. 5 shows the steps of erosion



(a) Not all the pixels in the structure element (SE) is overlapped with the image. The pixel under the origin of the SE is not turned on.



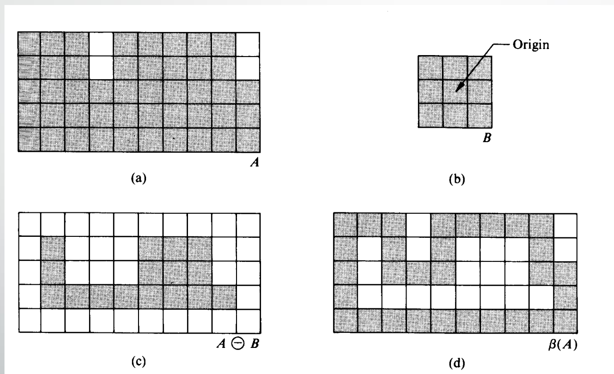
(b) All the pixels in the SE is contained in the image, hence the pixel under the origin of the SE is turned on.



(c) The process is repeated till the last pixel in the image is considered. The final eroded image is shown.

4.0. Erosion – Boundary Extraction

An example of erosion to extract the border of an image is shown in Fig. 5



$$\beta(A) = A - (A \ominus B)$$

5.0. More Characteristics

1. Noise Removal

$$J = I \bigcap (I \oplus N_4) \qquad \textbf{(12)}$$

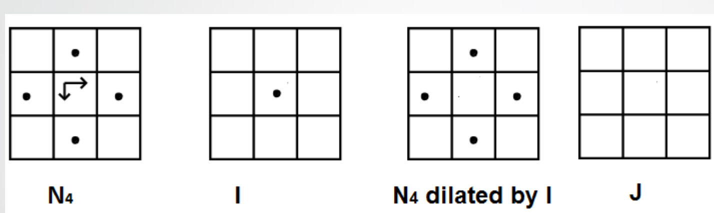


Fig. 6, the structuring element for noise removal.
where *I* is the input image and N_4 is the set of 4 neighbours belonging to the pixel (0,0).

J consists of only those points in *I* that have at least one of its 4 neighbours. All 4-neighbours-less pixels have been removed. Note that pixels on a diagonal line of one-pixel width will also be eliminated in *J* because they are not 4-connected.

5.0. More Characteristics

2. Dilation - Union

Dilating by a structuring element that can be represented as a union of two sets is the union of the dilation:

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C) \tag{13}$$

<Proof>

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

$$\begin{aligned} (B \cup C) \oplus A &= \bigcup_{x \in (B \cup C)} (A)_x \\ &= \left[\bigcup_{x \in B} (A)_x \right] \cup \left[\bigcup_{x \in C} (A)_x \right] \\ &= (A \oplus B) \cup (A \oplus C) \end{aligned}$$

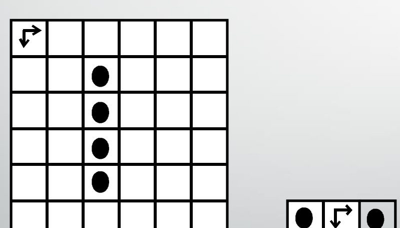
5.0. More Characteristics

3. Structuring Element Containing Origin

Dilating a set A by a structuring element containing the origin produces a result guaranteed to contain A.

Q: What happen if the structuring element does not contain the origin?

A: The dilation may have nothing in common with the set being dilated.



A B

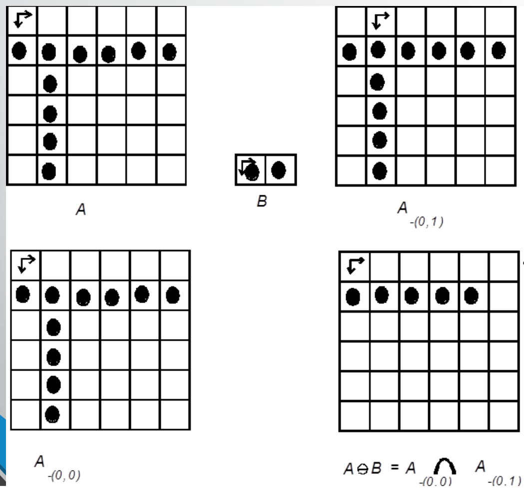
Fig. 7

Try the dilation of A by the structuring element B shown in Fig. 7.

5.0. More Characteristics

4. Erosion - Intersection

Whereas dilation can be represented as a union of translations, erosion can be represented as an intersection of the negative translations:



$$A \ominus B = \bigcap_{b \in B} A_{-b} \qquad (14)$$

Fig. 8

6.0. Dilation and Erosion – Some Relationships

1. **<u>Erosion – Dilation Duality</u>**

Dilation and Erosion transformations bear a marked similarity in that what one does to the image foreground, the other does to the image background.

Theorem:

$$(A \ominus B)^c = A^c \oplus \breve{B} \qquad (15)$$

$$(A \oplus B)^c = A^c \ominus B \qquad (16)$$

6.0. Dilation and Erosion – Some Relationships

1. Erosion – Dilation Duality

Dilation and Erosion transformations bear a marked similarity in that what one does to the image foreground, the other does to the image background.

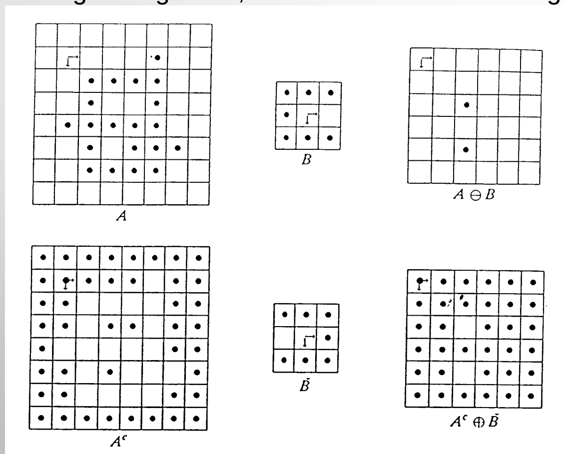


Fig. 9 : Duality relation between erosion and dilation

6.0. Dilation and Erosion – Some Relationships

2. Decomposition

a. Erosion of a set that itself has a decomposition as the intersection of two sets is the intersection of the erosions.

$$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K) \quad (17)$$

Fig. 10 shows an example

6.0. Dilation and Erosion – Some Relationships

2. Decomposition (a)

$$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K) \quad (17)$$

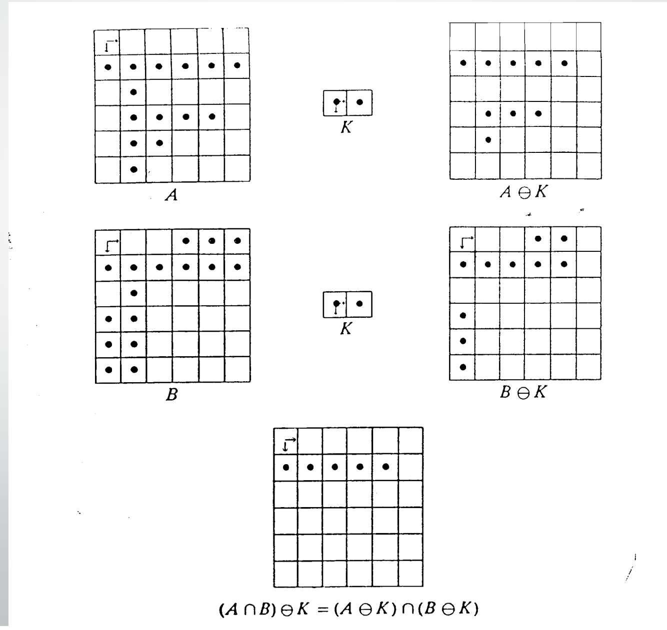


Fig. 10 An instance of the relationship shown in Eqn(17)

6.0. Dilation and Erosion – Some Relationships

2. Decomposition

(b) The erosion of a set by a structuring element that has a decomposition as the union of two sets is the intersection of the erosions:

$$A \ominus (K \cup L) = (A \ominus K) \cap (A \ominus L) \quad (18)$$

However,

$$A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C) \quad (19)$$

Please see Fig. 11 for an illustration.

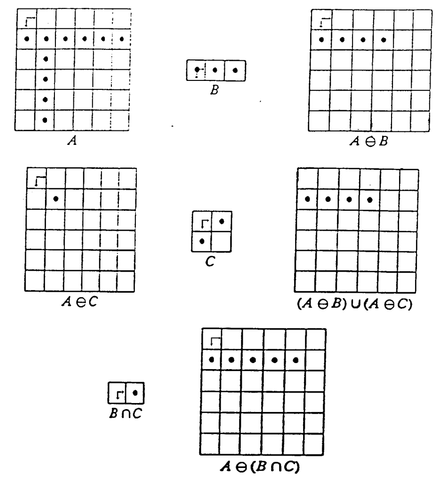
6.0. Dilation and Erosion – Some Relationships

2. Decomposition (b)

Fig. 11. An instance in which $A \ominus (B \cap C)$ is a proper superset of $(A \ominus B) \cup (A \ominus C)$ thereby showing that the general relation

$$A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C)$$

Cannot be made any stronger.



7.0. Dilation and Erosion Summary

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

$$(A \cup B) \oplus C = (A \oplus C) \cup (B \oplus C)$$

$$A \oplus B = \bigcup_{b \in B} A_b$$

$$A \subseteq B \Rightarrow A \oplus C \subseteq B \oplus C$$

$$(A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$$

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

$$(A \oplus B)^c = A^c \ominus \bar{B}$$

$$A \oplus B_1 = (A \oplus B)_1$$

$$A \oplus B = B \oplus A$$

$$(A \ominus B) \ominus C = A \ominus (B \oplus C)$$

$$(A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C)$$

$$A \ominus B = \bigcap_{b \in B} A_{-b}$$

$$A \subseteq B \Rightarrow A \ominus C \subseteq B \ominus C$$

$$(A \cup B) \ominus C \supseteq (A \ominus C) \cup (B \ominus C)$$

$$A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C)$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$$

$$A \ominus B_t = (A \ominus B)_{-t}$$

8.0. Opening and Closing

Dilation and Erosion are usually employed in pairs. The result of successively applied dilations and erosions is the elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features.

The particular significance of employing successively applied dilations and erosions to perform image transforms is **idempotent**, i.e., their re-application effects no further changes to the previously transformed results.

8.0. Opening and Closing

The opening of image B by structuring element K is denoted by $B \circ K$ and is given by equation (20), and Fig. 12 and 13 give some examples.

$$B \circ K = (B \ominus K) \oplus K \qquad (20)$$

The closing of image B by structuring element K is denoted by $B \bullet K$, and is given by equation (21), and Fig 15 shows some examples.

$$B \bullet K = (B \oplus K) \ominus K \qquad (21)$$

If B is unchanged by opening it with K, then we say that B is open with respect to K or B is open under K.

If B is unchanged by closing it with K, then we say that B is closed with respect to K or B is closed under K.

8.0. Opening and Closing - Example

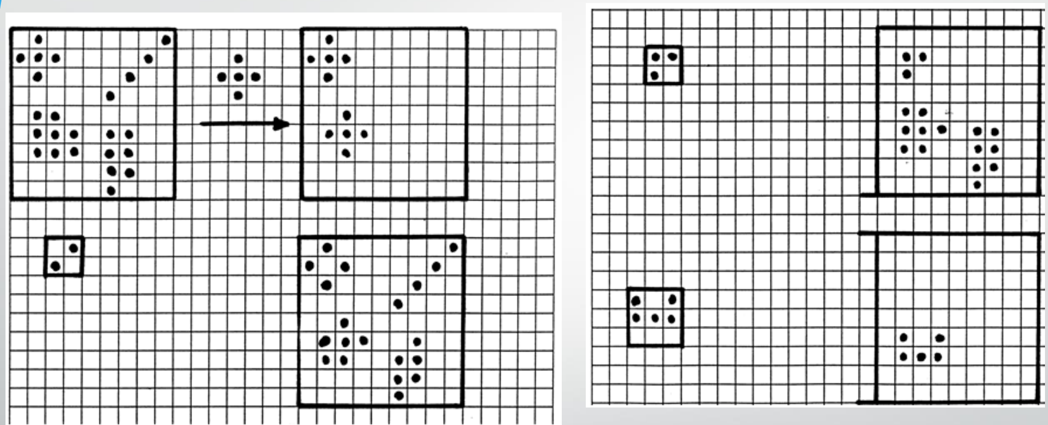


Fig. 12 Examples of opening

8.0. Opening and Closing - Example

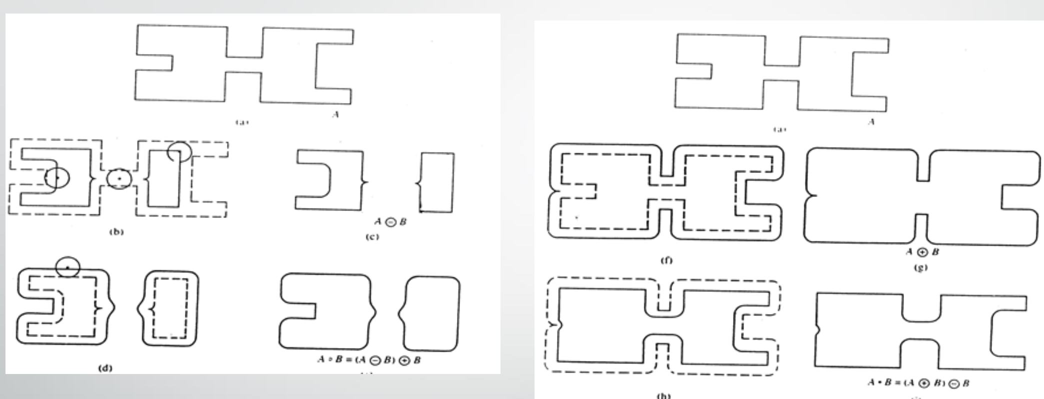


Fig 13 Illustrations of Opening and Closing operations

9.0. Applications

Example 1:

The shape F consists of a disk-like body and an elongated ellipse like handle.

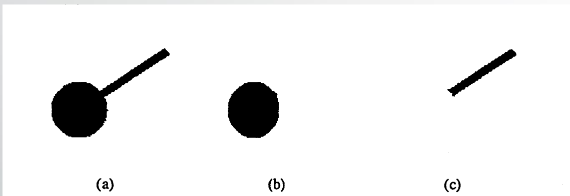


Fig. 14 Extraction of the body and handle of a shape F by opening with L for the body and taking the residue of the opening for the handle: (a) F, (b) $F \circ L$ and (c) $F - F \circ L$

Steps: (a) Shape F;

(b) $F \circ L$ where L is a small-disk structure element with radius just larger than the width of the ellipse;

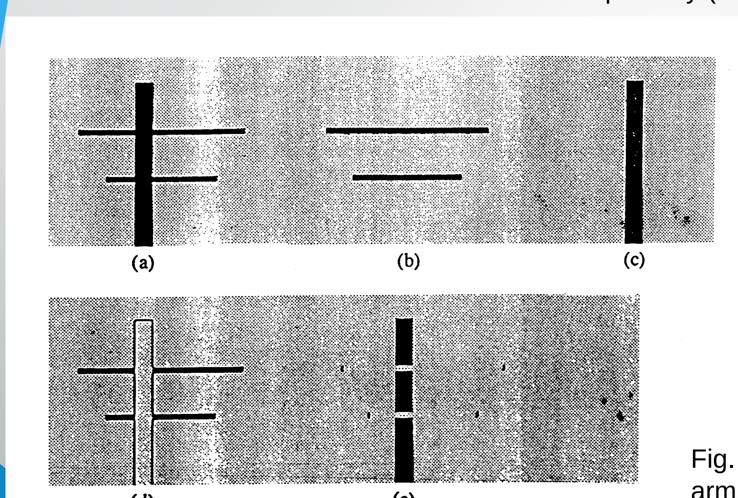
(c) $F - F \circ L$

Note that the order of the operations is important. The larger primitive must be obtained first and the smaller ones second.

9.0. Applications

Example 2:

The arm and trunk can be extracted separately (Fig. 15).

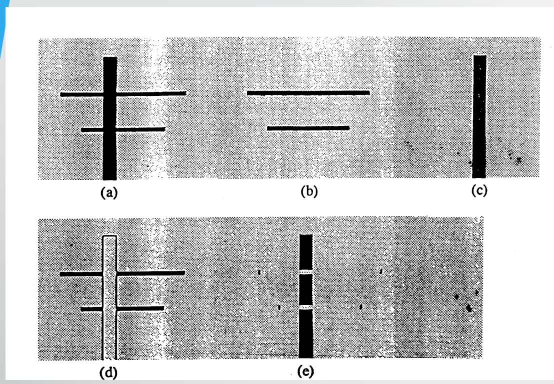


Trunk: open with a smaller trunk like structuring element whose length is larger than the width of each of the arms.

Arms: open with a smaller horizontally oriented structure element whose length is just larger than the width of the trunk.

Fig. 15 Extraction of the trunk and arms of a shape F by opening with vertically and horizontally oriented structuring elements

9.0. Applications



Trunk: open with a smaller trunk like structuring element whose length is larger than the width of each of the arms.

Arms: open with a smaller horizontally oriented structure element whose length is just larger than the width of the trunk.

9.0. Applications

Example 3:

Using opening operation to extract features of interest.

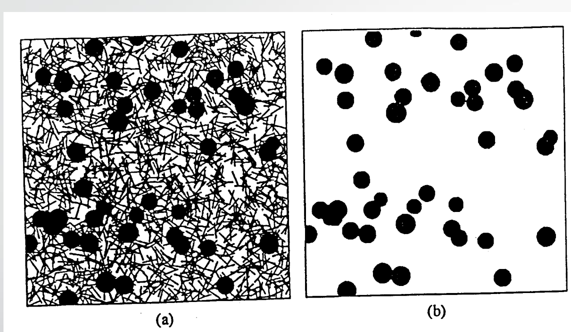


Fig. 16 (a) A binary image. (b) Opening of (a) with a disk structuring elements.

(a) Binary disks with average diameter of 35 pixels.

(b) Opening of image (a) by a disk of diameter of 13 pixels.

Examples

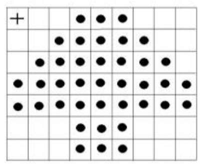


Fig. Ex 1(a)
Image A

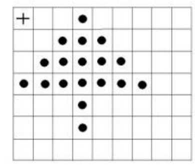


Fig. Ex 1(b)
Resulting
Image B

Fig. Ex 1(a) shows the image A, design a structure element and the associated morphological operation (s) that will give the resulting image shown in Fig. Ex 1(b).

Examples

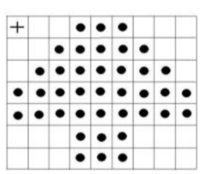


Fig. Ex 1(a)
Image A

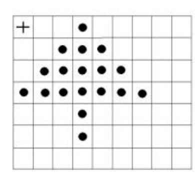
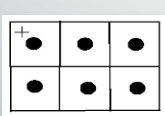


Fig. Ex 1(b)
Resulting
Image B

Fig. Ex 1(a) shows the image A, design a structure element and the associated morphological operation (s) that will give the resulting image shown in Fig. Ex 1(b).



Comparing Images A and B, we can see that B can be obtained from A by removing two pixels in the horizontal direction and one pixel in the vertical direction. Therefore, we can use the following structure element and perform an Erosion on Image A.

Examples

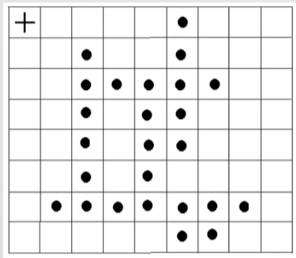


Fig. Ex 2(a)
Image C

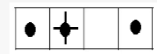


Fig. Ex2(b)
Structure
Element D

Show the resulting image after the morphological operation: ***C o D***