An Introduction to Magnetic Trap of Cold Atom

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1 Abstract

In this article, I first introduce the Zeeman effect, which magnetic trapping of neutral atoms is due to. Then I depict the 2-dimensional TOP trap, and solve the Schrödinger equation numerically and analytically. I point out that atoms have tendency to condense to the minimum point of the potential.

2 Introduction

Cold atoms system enjoys wide attentions in recent years because many interesting features have been explored in this system after the Bose-Einstein condensation was produced in laboratory in 1995 and this system is a good candidate to the quantum computer. Bosons are particles with integer spin. The wave function for a system of identical bosons is symmetric under interchange of any two particles. Unlike fermions, which have half-odd-integer spin and antisymmetric wave functions, bosons may occupy the same single-particle state. For a uniform gas of free particles, the relevant quantities are the particle mass m, the number of particles per unit volume n, and the Plank constant $h = 2\pi\hbar$. The condensation temperature T_c is:

$$T_c = C\frac{\hbar^2 n^{2/3}}{mk_B} \tag{1}$$

Here C is a numerical factor. Bose-Einstein condensation says that under the condensation temperature, the Bose gases will begin to condensate into the ground state. In order to observe the condensation we must put the gases into a potential such that the area the minimum potential energy locates in is the area the ground state locates in.

In this article, I first talk about the Zeeman effect. Then, I plot the 2-dimensional, rather than 3-dimensional TOP trap, for convenience.

3 Zeeman Effect

To take account the effect of an external magnetic field on the energy levels of an atom we must add to the hyperfine Hamiltonian the Zeeman energies arising from the interaction of the magnetic moments of the electron and the nucleus with the magnetic field. If we take the magnetic field **B** to lie in the z direction, the total Hamiltonian is thus

$$H_{spin} = A\mathbf{I} \cdot \mathbf{J} + CJ_z + DI_z \tag{2}$$

The constants C and D are given by

$$C = g\mu_B B \tag{3}$$

$$D = -\frac{\mu}{I}B \tag{4}$$

Since $|D/C| \sim m_e/m_p \sim 10^{-3}$, for many applications D may be neglected. At the same level of approximation the g factor of the electron may be put equal to 2. Because of its importance we consider a nuclear spin of 3/2. We diagonalize H_{spin} in a basis consisting of the eight states $|m_I, m_J\rangle$ where $m_I = \pm 3/2, \pm 1/2$ and $m_J = \pm 1/2$. The hyperfine interactionmay be expressed as

$$\mathbf{I} \cdot \mathbf{J} = J_z I_z + \frac{1}{2}(I_+ J_- + I_- J_+) \tag{5}$$

The Hamiltonian conserves the z component of the total angular momentum, and therefore it couples only states with the same value of the sum $m_I + m_J$. The energies of the states $|3/2, 1/2\rangle$ and $|-3/2, -1/2\rangle$ are easily calculated, since these states do not mix with any others. They are[1]

$$E(3/2, 1/2) = \frac{3}{4}A + \frac{1}{2}C + \frac{3}{2}D \tag{6}$$

$$E(-3/2, -1/2) = \frac{3}{4}A - \frac{1}{2}C - \frac{3}{2}D \tag{7}$$

which are linear in the magnetic field

The states $|m_I, -1/2\rangle$ and $|m_I = 1, 1/2\rangle$ are mixed. Therefore to calculate the energies of the states we need to diagonalize only 2×2 matrices. Let us first consider the matrix for $m_I + m_J = 1$, corresponding to states $|3/2, -1/2\rangle$ and $|1/2, 1/2\rangle$. The matrix elements of the Hamiltonian are

$$\begin{pmatrix} -\frac{3}{4}A - \frac{1}{2}C + \frac{3}{2}D & \frac{\sqrt{3}}{2}A \\ \frac{\sqrt{3}}{2}A & \frac{1}{4}A + \frac{1}{2}C + \frac{1}{2}D \end{pmatrix}$$

and the eigenvalues are

$$E = -\frac{A}{4} + D \pm \sqrt{\frac{3}{4}A^2 + \frac{1}{4}(A + C - D)^2} \tag{8}$$

For the states $|-3/2, 1/2\rangle$ and $|-1/2, -1/2\rangle$ the matrix is obtained from the one above by the substitution $C \to -C$ and $D \to -D$. The matrix for the states $|1/2, -1/2\rangle$ and $|-1/2, 1/2\rangle$ is

$$\begin{pmatrix} -\frac{1}{4}A\frac{1}{2}(C-D) & A \\ A & -\frac{1}{4}A - \frac{1}{2}(C-D) \end{pmatrix}$$

and the eigenvalues are

$$E = -\frac{A}{4} \pm \sqrt{A^2 + \frac{1}{4}(C - D)^2} \tag{9}$$

When D is neglected, the energy levels are given for $m_I + m_J = \pm 2$ by

$$E(3/2, 1/2) = A(\frac{3}{4} + \frac{b}{2}) and E(-3/2, -1/2) = A(\frac{3}{4} - \frac{b}{2}) \tag{10}$$

where $b = \frac{C}{A}$, for $m_I + m_J = \pm 1$ by

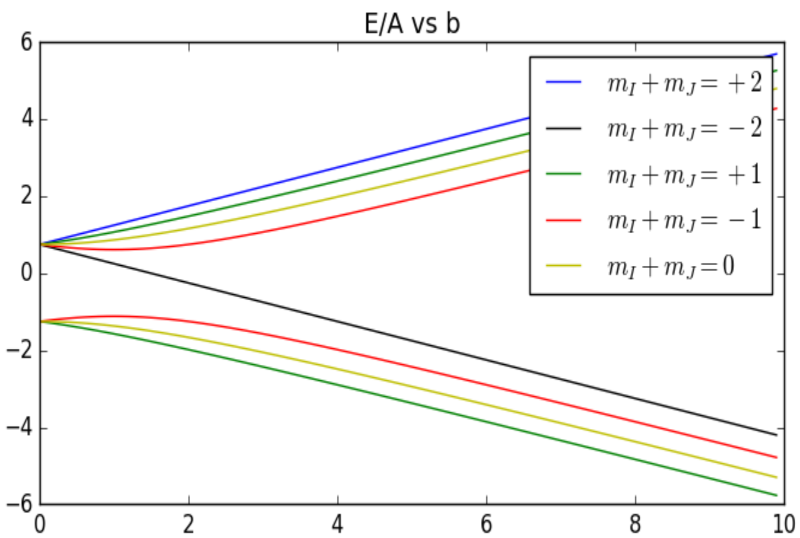
$$E = A(-\frac{1}{4} \pm \sqrt{\frac{3}{4} + \frac{1}{4}(1 + b)^2}) \tag{11}$$

and

$$E = A(-\frac{1}{4} \pm \sqrt{\frac{3}{4} + \frac{1}{4}(1 - b)^2}) \tag{12}$$

and for $m_I + m_J = \pm 0$ by

$$E = A(-\frac{1}{4} \pm \sqrt{1 + \frac{b^2}{4}}) \tag{13}$$



The figure above is the energies of hyperfine levels of an alkali atom with $I = 3/2$ and $A > 0$ in a magnetic field. At high magnetic fields, $b \gg 1$, the leading contributions to these expressions are $\pm Ab/2 = \pm C/2$ corresponding to the energy eigenvalues $\pm \mu_B B$ associated with the electronic spin. These calculations may easily be generalized to other values of the nuclear spin.

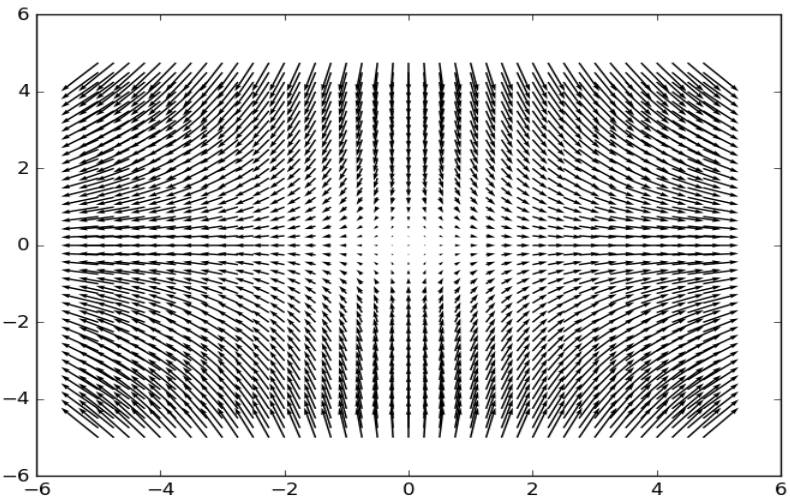
4 The TOP trap

Bose-Einstein condensation in dilute gases was first achieved in experiments using a modified quadrupole trap known as the time-averaged orbiting potential(TOP) trap. The instantaneous field is given by

$$\mathbf{B} = (B'x + B_0 cos\omega t, B'y + B_0 sin\omega t, -2B'z) \tag{14}$$

For convenience here we just consider the 2-dimensional case

$$\mathbf{B} = (B'x + B_0 cos\omega t, -B'y - B_0 sin\omega t) \tag{15}$$



To determine the effective potential we first evaluate the instantaneous strength of the magnetic field, which is given by

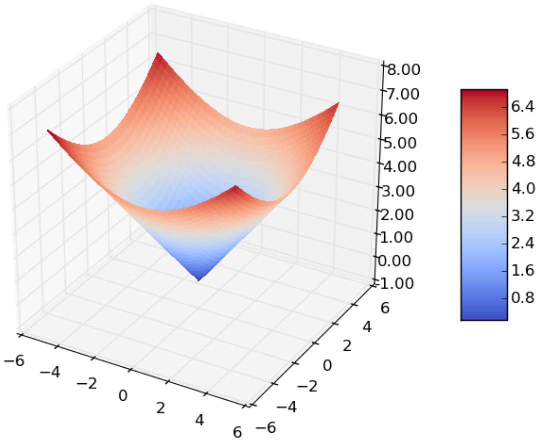
$$B(t) = [(B_0 cos\omega t + B'x)^2 + (B_0 sin\omega t + B'y)^2]^{1/2} \tag{16}$$

$$\approx B_0 + \frac{B'^2}{2B_0}(x^2 + y^2) + B'(xcos\omega t + ysin\omega t) \tag{17}$$

where the latter form applies for small distances from the node of the quadrupole field, $r \ll |B_0/B'|$. The time average, $< B >_t$, of the magnitude is

$$< B >_t = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} dt B(t) \tag{18}$$

$$\approx B_0 + \frac{B'^2}{2B_0}(x^2 + y^2) \tag{19}$$



Therefore the potential energy can be expressed as

$$V(x, y) = V_0 + \frac{1}{2} m \Omega^2 (x^2 + y^2) \tag{20}$$

where $\Omega^2 = \frac{\mu B'^2}{m B_0}$. It is obviously that this potential is a isotropic 2-dimensional harmonic oscillator potential well.

5 Solution to The Schrödinger Equation

5.1 Analytical Solution

The Schrödinger equation with the potential given by the last section is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{1}{2} m \Omega^2 (x^2 + y^2) \psi = E \psi \tag{21}$$

We can employ the unit in which $\hbar = m = B' = B_0 = \mu = 1$, then the equation reduce to

$$-\frac{1}{2} \nabla^2 \psi + \frac{1}{2}(x^2 + y^2) \psi = E \psi \tag{22}$$

We can easily get

$$\psi_{m,n} = (2^{m+n} m!n!\pi)^{-1/2} e^{-(x^2+y^2)} H_m(x) H_n(y) \tag{23}$$

$$E_{m,n} = m + n + 1 \tag{24}$$

where $H_n(x)$ is the Hermite polynomials and $m, n = 0, 1, 2 \cdots$.

The energy of the ground state is $E = 1$ and the wave function reduces to $\psi_{0,0} = \pi^{-1/2} e^{-(x^2+y^2)}$.

5.2 Numerical Solution

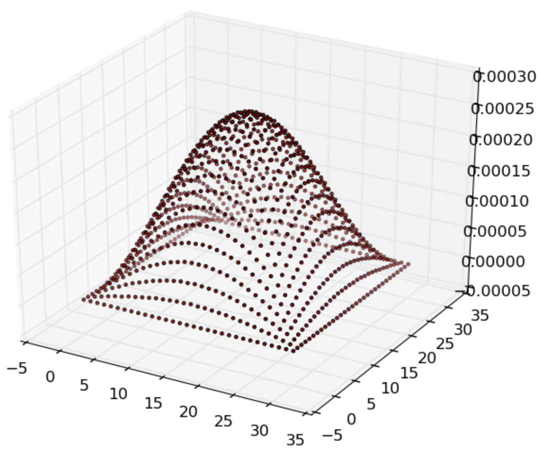
The Schrödinger equation we get above is

$$-\frac{1}{2}\nabla^2\psi + \frac{1}{2}(x^2 + y^2)\psi = E\psi \tag{25}$$

Discretizing the variable $x = i\Delta x, y = j\Delta y, \Delta x = \Delta y = \delta$, we get[2]

$$\psi(i,j) = \frac{1}{2}\frac{\psi(i+1,j) + \psi(i-1,j) + \psi(i,j+1) + \psi(i,j-1)}{2 - \frac{1}{2}\delta^4(i^2+j^2) - \delta^2 E} \tag{26}$$

The approach we take to get the solution is to begin with some initial guess for the solution, $\psi_0 = Constant$, say. Then we use the formula above to obtain the new ψ. This procedure will be repeated until the difference between the new ψ and the old ψ is sufficiently small. We consider the circumstance where the atoms cannot rush out off the region under observation. After calculating we can plot the wave function of the ground state ψ.



6 Conclusion

We illustrate that in the TOP trap, each atom behaves like a harmonic oscillator and the atoms cloud has a tendency to condense to the region where the minimum of the potential energy locates.

7 Appendix

1.The program *zeeman.py* is to plot the Zeeman effect.
2.The program *magnetic.py* is to plot the magnetic field.
3.The program *wavefunction.f90* is to compute the wave function.
4.The program *draw.py* is to plot the wave function.

References

[1] C.J.Pethick and H.Smith, *Bose-Einstein condensation*, Second edition, Cambridge University Press, Cambridge, (2008)

[2] Nicholas J.Giordano and Hisao Hakanishi, *Computational Physics*, Second edition, Tsinghua University Press, Beijing, (2007)