MODULI SPACE OF INSTANTON SHEAVES ON FANO THREEFOLD V_4

XUQIANG QIN

ABSTRACT. We study semistable sheaves of rank 2 with Chern classes $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ on a Fano threefold V_4 of Picard number 1, degree 4 and index 2. We show that the moduli space of such sheaves is isomorphic to the moduli space of semistable rank 2, degree 0 vector bundles on a genus 2 curve. This also provides a natural smooth compactification of the moduli space of Ulrich bundles of rank 2 on V_4.

1. INTRODUCTION

Instanton bundles first appeared in [ADHM] as a way to describe Yang-Mills instantons on a 4-sphere S^4. They provide extremely useful links between mathematical physics and algebraic geometry. The notion of mathematical instanton bundle was first introduced on $\mathbb{P}^3$. Since then the irreducibility[T], rationality[MT] and smoothness[JV] of their moduli space were heavily investigated. Faenzi[Fa] and Kuznetsov[Ku12] generalized this notion to Fano threefolds, we recall

Definition 1.1. [Ku03] Let Y be a Fano threefold of index 2. An *instanton bundle of charge n* on Y is a stable vector bundle E of rank 2 with $c_1(E) = 0, c_2(E) = n$, enjoying the instantonic condition:

$$H^1(Y, E(-1)) = 0.$$

We mention that the charge $c_2(E) \geq 2$ [Ku12, Corollary 3.2]. The instanton bundles of charge 2 are called the *minimal instantons.*

In this paper, we will be concerned with minimal instantons and natural compactification of their moduli on a Fano threefold of degree 4 and index 2, which we denote by V_4. Such a threefold is an intersection of two quadrics in $\mathbb{P}^5$. The moduli spaces of minimal instanton bundles were discussed in [Ku12] and they were shown to be open subschemes of moduli space of rank 2 even degree bundles on a genus 2 curve C which is naturally associated to V_4 (see [Ku12, Theorem 5.10]).

On the other hand, Ulrich bundles are defined as vector bundles on a smooth projective variety X of dimension d so that

$$H^*(X, E(-t)) = 0$$

for all $t = 1, \ldots, d$. They first appeared in commutative algebra and entered the world of algebraic geometry via [ES]. The existence and moduli space of Ulrich bundles provide great amount of information about the original variety. For example, in the case when X is a smooth hypersurface, the existence of Ulrich bundles is equivalent to the fact that X can be defined set-theoretically by a linear determinant[B]. Inspired by [Ku12], Lahoz, Macrì and Stellari[LMS1][LMS2], studied moduli spaces of Ulrich bundles on cubic threefolds and

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fourfolds using derived categories. In a recent paper[CKL], the moduli space of stable Ulrich bundles of rank r were shown to be an open subscheme of moduli space of rank r degree $2r$ vector bundles on C. We will see on V_4, (minimal) instanton bundles and Ulrich bundles of rank 2 differ only by a twist of the very ample divisor $\mathcal{O}_{V_4}(1)$. Thus they share the same moduli space and compactifications.

Our first result shows that for minimal instantons on V_4, the instantonic condition is automatically satisfied.

Theorem 1.2. *Let E be a stable rank 2 vector bundle on V_4, with $c_1(E) = 0$ and $c_2(E) = 2$, then $H^1(V_4, E(n)) = 0$ for all $n \in \mathbb{Z}$. In particular, E is a (minimal) instanton bundle.*

In light of this result, we can generalized the notion of minimal instantons in the following way:

Definition 1.3. An *instanton sheaf* on V_4 is a semistable sheaf of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$.

We also mention that similar phenomenon was observed on cubic threefolds (See [D, Theorem 2.4]), but the proof used properties of the cubic.

Our next result is the classification of instanton sheaves of V_4. [D] classified instanton sheaves on cubic threefolds and proved that their moduli space is isomorphic to the blow-up of the intermediate Jacobian in the Fano surface of lines. We follow his method and prove that a parallel classification happens on V_4.

Theorem 1.4. *Let E be an instanton sheaf on V_4. If E is stable, then either E is locally free or E is associated to a smooth conic $Y \subset V_4$ such that we have the exact sequence:*

$$0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1) \to 0,$$

where θ is the theta-characteristic of Y.
If E is strictly semistable, then E is the extension of two ideal sheaves of lines.

Recall that a theta-characteristic of a non-singular curve Y is a line bundle L so that $L^{\otimes 2}$ is the canonical bundle. In the case when Y is a smooth conic, $Y \simeq \mathbb{P}^1$ and a theta-characteristic is just $\mathcal{O}_C(-1)$ where $\mathcal{O}_C(1)$ generates the Picard group of C.

Unfortunately, the method to study the moduli space in [D] does not transfer well to V_4. However, we note that $\mathcal{D}^b(V_4)$ has a semi-orthogonal decomposition:

$$\mathcal{D}^b(V_4) = \langle \mathcal{B}_{V_4}, \mathcal{O}_{V_4}, \mathcal{O}_{V_4}(1) \rangle$$

There is a genus 2 smooth curve C naturally associated to V_4 so that there is a natural choice of equivalence of triangulated categories $\Phi : \mathcal{D}^b(C) \simeq \mathcal{B}_{V_4}$ (see Section 2.2 for the precise definition). This functor is Fourier-Mukai by Orlov's result. Our second result connects instanton sheaves with rank 2 bundles on C.

Theorem 1.5. *Let E be an instanton sheaf on V_4, then $E \in \mathcal{B}_{V_4}$ and $\Phi^*(E)[-1]$ is a rank 2 semistable vector bundle of degree 0 on C. Moreover, if E is stable (strictly semistable), then the vector bundle $\Phi^*(E)[-1]$ is stable (strictly semistable).*

Using this relation, we construct a morphism between the moduli space M^{inst} of instanton sheaves on V_4 and the moduli space M of semistable vector bundles on C of rank 2 and degree 0 and prove it is an isomorphism.

Theorem 1.6. *There exists a morphism $\psi\colon M^{inst} \to M$ which is an isomorphism. As a result, the moduli space of instanton sheaves on V_4 is a $\mathbb{P}^3$-bundle over the Jacobian of C, thus a smooth projective variety of dimension 5.*

We believe our results can be generalized to find compactifications of moduli spaces of Ulrich bundles of higher ranks on V_4. Also similar ideas should work in finding the moduli space of instanton sheaves on Fano threefolds other than V_4 and cubics. For the case of Fano threefold V_5 of degree 5 and index 2, see [Q].

This paper is organized as follows. In the second section the reader can find some preliminary definitions and results that are used throughout the paper. In the third section we classify the instanton sheaves, showing the parallel result as on cubic threefolds holds. In the fourth section we connect instanton sheaves to vector bundles on C using derived category. In the last section we describe the moduli space of instantons on V_4.

Notations and conventions.

- We work over the complex numbers $\mathbb{C}$.
- Let E be a sheaf on V_4. We use $H^i(E)$ to denote $H^i(V_4, E)$ for simplicity. Also we use $h^i(E)$ to denote the dimension of $H^i(V_4, E)$ as a complex vector space.
- Let F be a sheaf or a representation with certain characterization, we will use $[F]$ to denote the point it corresponds to in the moduli space.

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2. Preliminaries

2.1. **Derived Categories.** Let X be an algebraic variety, we use $\mathcal{D}^b(X)$ to denote the derived categories of coherent sheaves on X. We denote $\mathrm{Ext}^p(F, G) = \mathrm{Hom}(F, G[p])$ and $\mathrm{Ext}^\bullet(F, G) = \oplus_{p\in\mathbb{Z}}\mathrm{Ext}^p(F, G)[-p]$. Recall that a sequence of full admissible triangulated subcategories of a triangulated category $\mathcal{T}$

$$D_1, \dots, D_n \subset \mathcal{T}$$

is semi-orthogonal if for all $j > i$

$$D_i \subset D_j^\perp$$

Such a sequence defines a semi-orthogonal decomposition of $\mathcal{T}$ is the smallest full subcategory of $\mathcal{T}$ containing $D_1, \dots, D_n$ is itself, in this case we use the notion $\mathcal{T} = \langle D_1, \dots, D_n\rangle$. An easy way to produce a semi-orthogonal decomposition is by using exceptional objects or collections.

Definitions 2.1. An object $F \in \mathcal{T}$ is called exceptional if $\mathrm{Ext}^\bullet(F, F) = \mathbb{C}$. A collection of exceptional objects $F_1, \dots, F_n$ is called an exceptional collection if $\mathrm{Ext}^p(F_j, F_i) = 0$ for all $j > i$ and all $p \in \mathbb{Z}$.

On a smooth Fano threefold V of index 2, Kodaira vanishing theorem implies:

$$H^i(V, \mathcal{O}_V) = 0$$

for all $i > 0$. Thus all line bundles on V are exceptional objects. Moreover, we can check $\{\mathcal{O}_V, \mathcal{O}_V(1)\}$ is an exceptional collection using the fact that V has index 2. We denote their left orthogonal complement by $\mathcal{B}_V$ and obtain the following semi-orthogonal decomposition:

$$\mathcal{D}^b(V) = \langle \mathcal{B}_V, \mathcal{O}_V, \mathcal{O}_V(1)\rangle.$$

Note an object $F \in \mathcal{D}^b(V)$ is in $\mathcal{B}_V$ if and only if

$$\operatorname{Hom}(\mathcal{O}_V, F[i]) = 0$$
$$\operatorname{Hom}(\mathcal{O}_V(1), F[i]) = 0$$

for all $i \in \mathbb{Z}$.

2.2. **Vector bundles.** In this section we recall several useful results about vector bundles on smooth projective varieties.

Proposition 2.2. [H1, Section 1] *Let X be a smooth projective variety of dimension at least 2 and E be a vector bundle of rank 2 on X. Suppose there exist a global section of E whose zero locus Y is of pure codimension 2, then we have an exact sequence:*

$$0 \to \mathcal{O}_X \to E \to I_Y \otimes \det(E) \to 0.$$

Theorem 2.3 (The Serre construction)**.** *Suppose X is a smooth projective variety of dimension at least 3. Let L be a invertible sheaf so that $h^1(L^{-1}) = 0$ and $h^2(L^{-2}) = 0$ and $Y \subset X$ a closed subscheme of pure codimension 2. We have an isomorphism* $\operatorname{Ext}^1(I_Y \otimes L, \mathcal{O}_X) = H^0(\mathcal{O}_Y)$*. The subscheme Y is the zero locus of a section of a vector bundle of rank 2 with determinant L if and only if Y is locally complete intersection and $\omega_Y = (\omega_X \otimes L)|_Y$.*

We recall the following useful result:

Proposition 2.4 (Mumford-Castelnuovo Criterion)**.** *Let F be a coherent sheaf on a projective variety X. Suppose $h^i(X, F(-i)) = 0$ for all $i \geq 1$, then $h^i(X, F(k)) = 0$ for all $i \geq 1$ and $k \geq -i$. Moreover F is generated by global sections.*

2.3. **Fano 3-fold V_4.** (See also [Ku12, Section 5.1]) A Fano threefold of degree 4 and index 2 is a smooth intersection of two quadrics in $\mathbb{P}^5$. We let V_4 be such a threefold. There is an associated genus 2 smooth curve C associated to V_4. We briefly recall its construction. Let V be a complex vector space of dimension 6 and A a vector space of dimension 2. A pair of quadrics gives a map $A \to S^2V^*$, so $\mathbb{P}(A)$ parametrizes a family of quadrics in $\mathbb{P}(V)$. There are 6 degenerate quadrics in this family, giving 6 points on $\mathbb{P}(A) \simeq \mathbb{P}^1$. C is defined to by the double cover of $\mathbb{P}(A)$ ramified at the 6 points. Clearly C is a curve of genus 2. We use $\tau : C \to C$ to denote its hyperelliptic involution.

By looking at the spinor bundles on quadrics in $\mathbb{P}(A)$, one can show there is a vector bundle $\mathcal{S}$ of rank 2 on $C \times V_4$. The associated Fourier-Mukai functor connects C with V_4 in the following way:

Theorem 2.5. [BO] *The Fourier-Mukai functor $\Phi_{\mathcal{S}} : \mathcal{D}^b(C) \to \mathcal{D}^b(V_4)$ provides an equivalence of $\mathcal{D}^b(C)$ onto $\mathcal{B}_{V_4}$.*

From now on, we use Φ to denote $\Phi_{\mathcal{S}}$ for simplicity.

Another way to understand the relation between C and V_4 and the vector bundle $\mathcal{S}$ is due to Mukai. It is shown that V_4 is the moduli space of rank 2 bundles on C with a fixed determinant ξ of odd degree. Then $\mathcal{S}$ is the universal family. We follow [Ku12]'s convention and assume $\deg\xi = 1$. Then

$$\det(\mathcal{S}) = \xi \boxtimes \mathcal{O}_{V_4}(-1).$$

We can also describe the Fano variety of lines $F(V_4)$ using this functor. First note it is straight forward to check that the ideal sheaf I_l of a line is an object in $\mathcal{B}_{V_4}$. Then

Theorem 2.6. [Ku12, Lemma 5.5] *There is an isomorphism $G : F(V_4) \xrightarrow{\sim} \mathrm{Pic}^0(C)$ given by*

$$G(l) = \Phi^{-1}(I_l[-1]).$$

In particular $\Phi^{-1}(I_l[-1])$ is a degree 0 line bundle on C.

This result will be crucial in our analysis of strictly semistable instanton sheaves.

Finally we provide some topological information that will be useful later on. Let $[h], [l], [p]$ be the class of a hyperplane section, a line and a point respectively. Then

$$H^2(V_4, \mathbb{Z}) \simeq \mathbb{Z}[h], \quad H^4(V_4, \mathbb{Z}) \simeq \mathbb{Z}[l], \quad H^6(V_4, \mathbb{Z}) \simeq \mathbb{Z}[p]$$

with $h \cdot l = p, h^2 = 4l, h^3 = 4p$.
The natural embedding $V_4 \subset \mathbb{P}^5$ provides a very ample divisor $\mathcal{O}_{V_4}(1)$. We will always use this polarization when talking about stability. A general section of $|\mathcal{O}_{V_4}(1)|$ is a del-Pezzo surface of degree 4, hence isomorphic to $\mathbb{P}^2$ blown up at 5 points in general position.
We compute the Todd class of V_4

$$\mathrm{td}(\mathcal{T}_{V_4}) = 1 + h + \frac{7}{3}l + p.$$

We also recall the Grothendieck-Riemann-Roch for the functor Φ.

Lemma 2.7. [Ku12, Lemma 5.2] *For any $F \in \mathcal{D}^b(C)$ we have*

$$\mathrm{ch}(\Phi(F)) = (2\deg(F) - \mathrm{rk}(F)) - \deg(F)h + \mathrm{rk}(F)l + \frac{\deg(F)}{3}p.$$

2.4. **Stability of sheaves.** Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(1)$ be a fixed ample line bundle. Let E be a coherent sheaf of rank r, then the slope of E is defined as:

$$\mu(E) = \frac{c_1(E) \cdot c_1(\mathcal{O}_X(1))^{n-1}}{r \cdot c_1(\mathcal{O}_X(1))^n}.$$

The sheaf E is called *(semi)stable* if it is torsion free and for any torsion free subsheaf $F \subset E$, we have

$$\frac{\chi(F(n))}{\mathrm{rk}(F)} (\leq) < \frac{\chi(E(n))}{\mathrm{rk}(E)}$$

for $n \gg 0$.
The sheaf E is called *μ-(semi)stable* if it is torsion free and for any torsion free subsheaf $F \subset E$, we have

$$\mu(F)(\leq) < \mu(E).$$

We have the following implications:

$$\mu - \text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu - \text{semistable}$$

When X is a smooth projective curve, any torsion free sheaf is locally free. We have the following criterion for semistability which is due to Faltings:

Lemma 2.8. [P, Exercise 2.8] *Let F, G be vector bundles on a curve X such that $H^i(F \otimes G) = 0$ for $i = 0, 1$, then both F and G are semistable.*

Let C be the associated curve of V_4. Then C has genus 2. We use M to denote the moduli space of semistable vector bundles of rank 2 and degree 0 on C. M was studied in detail in [NR]. Suppose J^1 is the moduli space of line bundles of degree 1. Then C is naturally embedded in J^1 as a divisor. We denote the corresponding divisor by Θ. To understand M, it suffices to understand the 3-dimensional subvariety $S \subset M$ consisting bundles with trivial determinant. [NR] showed S is naturally isomorphic to the projective space associated to $H^0(J^1, 2\Theta)$. From this it is not hard to conclude:

Theorem 2.9. [NR, Theorem 7.3] *M is canonically isomorphic to the space of positive divisors on J^1 algebraically equivalent to 2Θ. In particular, M is a projective bundle over the Jacobian of C.*

As a consequence, we have

Corollary 2.10. *M is a non-singular projective algebraic variety of dimension* 5.

2.5. **Instanton Sheaves.** Let Y be a Fano threefold of index 2. By definition an (minimal) *instanton bundle* is a stable vector bundle E of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$, enjoying the instantonic condition

$$H^1(Y, E(-1)) = 0.$$

We will see in Theorem 3.2 that the instantonic condition is automatically satisfied on V_4. We generalize this definition as follows:

Definitions 2.11. An *instanton sheaf* on V_4 is a semistable sheaf of rank 2 with Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$.

We use M^{inst} to denote the moduli space of such sheaves. It is clear that the moduli space of instanton bundles $MI_2(Y)$ is an open subscheme of M^{inst}.
We also recall two equivalent definitions of an Ulrich bundle.

Definitions 2.12. Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n of degree d. An *Ulrich bundle* E is a vector bundle on X satisfying

$$H^*(X, E(-t)) = 0$$

for all $t = 1, \ldots, n$. Equivalently, it is a vector bundle of rank r satisfying

$$H^i(X, E(t)) = 0$$

for all $t \in \mathbb{Z}$ and $0 < i < n$ and having Hilbert polynomial $P_E(t) = dr\binom{n+t}{t}$

We list a few well-known facts about Ulrich bundles that will be useful to us:

- There are no Ulrich line bundles on a variety X with $\mathrm{Pic}(X) = \mathbb{Z}\mathcal{O}_X(1)$.
- A Ulrich bundle is semistable. If it is not stable, it is an extension of Ulrich bundles of smaller ranks.

To see the relation between (minimal) instanton bundle and Ulrich bundle of rank 2 on V_4, we first recall the following result:

Proposition 2.13. [CKL, Proposition 3.4] *Let E be an Ulrich bundle of rank r on V_4, then $\mu(E) = 1$.*

As a result, if E is a rank 2 Ulrich bundle, $E(-1)$ is a semistable rank 2 bundle with Chern class $c_1 = 0$. Moreover, we have the Hilbert polynomial

$$P_{E(-1)}(t) = 8\binom{t+2}{3}.$$

Using Riemann-Roch, it is not hard to see $c_2(E(-1))=2$. Combine this with the following result:

Lemma 2.14. [Ku12] *Let E be an (minimal) instanton bundle on a Fano threefold of index 2. Then*

$$H^*(E(t))=0$$

for $t=0,-1,-2$.

We obtain

Proposition 2.15. *A vector bundle E on V_4 is an (minimal) instanton bundle if and only if $E(1)$ is a stable Ulrich bundle of rank 2.*

If we use M_2^{sU} to denote the moduli space of stable Ulrich bundles. It follows immediately from this result that:

$$M_2^{sU}\simeq MI_2(V_4)$$

To see what $MI_2(V_4)$ is, we recall that Kuznetsov gave the following description of $MI_n(V_4)$:

Theorem 2.16. [Ku12, Theorem 5.10] *Let $\mathcal{R}$ be a second Raynaud bundle. The moduli space $MI_n(V_4)$ of instantons of charge n is isomorphic to the moduli space of simple vector bundles $\mathcal{F}$ on C of rank n and degree 0 such that*

$$\begin{aligned}
&\mathcal{F}^*\simeq\tau^*\mathcal{F},\\
&H^0(C,\mathcal{F}\otimes\mathcal{S}_y)=0 \text{ for all } y\in V_4,\\
&\dim\operatorname{Hom}(\mathcal{F},\mathcal{R})=\dim\operatorname{Ext}^1(\mathcal{F},\mathcal{R})=n-2.
\end{aligned}$$

Remark 2.17. Apply this theorem to $n=2$ and note that in this case the last equation shows $\mathcal{F}$ is semistable by Lemma 2.8, we see

$$M_2^{sU}\simeq MI_2(V_4)\subset M.$$

See also [CKL, Theorem 4.14] for a similar result from the perspective of Ulrich bundles.

2.6. **Curves and surfaces of low degrees.** In this section we recall some results about varieties of (almost) minimal degrees in projective spaces. Recall a variety $V\subset\mathbb{P}^n$ is said to be *non-degenerate* if V is not contained in any hyperplane in $\mathbb{P}^n$.

Regarding the degree of a curve we have first the classical Castelnuovo Bound:

Theorem 2.18 (Castelnuovo Bound)**.** *Let $C\subset\mathbb{P}^n$ be an irreducible non-degenerate smooth curve of degree d and genus g. Let m,ϵ be the quotient and remainder when dividing $d-1$ by $n-1$, and $\pi(d,n)=(n-1)m(m-1)/2+m\epsilon$, then*

$$g\le\pi(d,n).$$

In fact this bound is sharp, and extremal curves (curves with $g=\pi(d,n)$) can be explicitly described (See Chapter 3 of [ACGH]). This result was improved by Eisenbud and Harris:

Theorem 2.19. [EH, Theorem 3.15] *With the same notation as above. For any d and $n\ge4$, set m_1,ϵ_1 to be the quotient and remainder when dividing $d-1$ by n. Let $\mu_1=1$ if $\epsilon_1=n-1$ and 0 otherwise. Let*

$$\pi_1(d,n)=\binom{m_1}{2}n+m_1(\epsilon_1+1)+\mu_1,$$

then

(1) if $g > \pi_1(d,n)$ and $d \geq 2n+1$, then C lies on a surface of degree $n-1$; and
(2) if $g = \pi_1(d,n)$ and $d \geq 2n+3$, then C lies on a surface of degree n or $n-1$.

We now look at surfaces of small degrees. It is well know that for an irreducible, non-degenerate variety $M \subset \mathbb{P}^n$ of dimension m, the minimal degree is $n-m+1$. (See [GH] Section 1.3). Minimal surfaces are well understood.

Theorem 2.20. *Every non-degenerate irreducible surface of degree $n-1$ in $\mathbb{P}^n$ is either a rational normal scroll or the Veronese surface in $\mathbb{P}^5$.*

The case of degree n surface in $\mathbb{P}^n$ is a little more involved.

Theorem 2.21. [N, Theorem 8] *Every non-degenerate irreducible surface S of degree $m \neq 8$ in $\mathbb{P}^m$ is one of the following:*
(1) Projection of a minimal surface in $\mathbb{P}^{m+1}$.
(2) A del-Pezzo surface with isolated double points.
(3) A cone with a smooth elliptic base curve.

Remark 2.22. When $m = 8$ we have two more kinds of surfaces, we will not need this case. The reader can find more details in [N].

3. Classification of Instanton Sheaves on V_4

We first show that for minimal instantons on V_4, the instantonic condition is redundant.

Lemma 3.1. *Let $S \subset \mathbb{P}^4$ be a del Pezzo surface of degree 4 and E a semistable vector bundle of rank 2 with Chern classes $c_1(E) = 0$ and $c_2(E) = 2$. If $h^0(E) = 0$, then $h^1(E(n)) = 0$ for $n \in \mathbb{Z}$ and $h^2(E(n)) = 0$ for $n \geq -1$. If $h^0(E) \neq 0$, then $h^0(E) = 1$, $h^1(E(n)) = 0$ for $n \leq -2$ and $n \geq 1$, $h^1(E(-1)) = h^1(E) = 1$ and $h^2(E(n)) = 0$ for $n \geq 0$.*

Proof. See [D, Lemma 2.2] □

Theorem 3.2. *Let E be a stable rank 2 vector bundle on V_4, with $c_1(E) = 0$ and $c_2(E) = 2$, then $H^1(V_4, E(n)) = 0$ for all $n \in \mathbb{Z}$. In particular, E is a (minimal) instanton bundle.*

Proof. Let $S \in |\mathcal{O}_{V_4}(1)|$ be a general hyperplane section of V_4. Then E_S is μ-semi-stable with respect to the polarization $\mathcal{O}_S(1)$ [M, Theorem 3.1].
Suppose $h^0(E_S) = 0$. Consider the short exact sequence:

$$0 \to E(n-1) \to E(n) \to E_S(n) \to 0.$$

Since $h^1(E_S(n)) = 0$ for $n \in \mathbb{Z}$, we have $h^1(E(n)) \leq h^1(E(n-1))$. Thus $h^1(E(n)) = 0$ for all $n \in \mathbb{Z}$ since $h^1(E(n)) = 0$ for $n \ll 0$.
Suppose $h^0(E_S) \neq 0$, we will try to get a contradiction. We claim $E(2)$ is generated by global sections. Using the same exact sequence above, we obtain $h^1(E(-n)) = 0$ for $n \geq 2$. Note $h^2(E) = h^1(E(-2)) = 0$ and $h^3(E) = h^0(E(-2)) = 0$. Since $\chi(E) = 0$, we have $h^1(E) = 0$ and the exact sequence:

$$0 \to E \to E(1) \to E_S(1) \to 0. \tag{3.3}$$

gives $h^1(E(1)) = h^1(E_S(1)) = 0$. We have then $h^3(E(-1)) = h^0(E(-1)) = 0$. Thus $E(2)$ is generated by global sections by Mumford-Castelnuovo criterion(Proposition 2.4).
If there exist a section of $E(2)$ which vanishes nowhere, then E is isomorphic to $\mathcal{O}_{V_4}(2) \oplus \mathcal{O}_{V_4}(-2)$ and $c_2(E) = -16$, which is absurd. We have then an exact sequence:

$$0 \to \mathcal{O}_{V_4}(-4) \to E(-2) \to I_Y \to 0 \tag{3.4}$$

where $Y \in V_4$ is a smooth curve of degree $c_2(E(2)) = 18$. We have $h^1(I_Y) = 0$, so the curve Y is connected. We have $\omega_Y = \mathcal{O}_Y(2)$ and $g(Y) = 19$. Finally, the curve Y is non-degenerate since E is stable. Using (3.3), (3.4), it is not hard to find $h^0(\mathcal{O}_Y(1)) = 7$. Thus the curve Y is the projection to $\mathbb{P}^5$ of a non-degenerate curve in $\mathbb{P}^6$ with degree 18 and genus 19. The next lemma shows this leads to a contradiction. □

Lemma 3.5. *Let $\tilde{Y} \subset \mathbb{P}^6$ be a non-degenerate curve of degree 18 and genus 19. Let $O \notin \tilde{Y}$ be a point so that the projection from O induce an embedding of $\tilde{Y}$ into $\mathbb{P}^5$. then the image Y of $\tilde{Y}$ cannot lie in the intersection V_4 of two quadrics in $\mathbb{P}^5$.*

Proof. We first apply Theorem 2.19 to $\tilde{Y}$. In this case, we have $d = 18$ and $n = 6$, then $m_1 = 2$ and $\epsilon_1 = 5 = 6 - 1$. So $\mu_1 = 1$. We compute $\pi_1(18, 6) = 6 + 2 \times 6 + 1 = 19 = g$. Thus by the second part of Theorem 2.19, $\tilde{Y}$ lies on a surface S of 5 or 6 in $\mathbb{P}^6$.

Degree 5 surfaces in $\mathbb{P}^6$ are the images of $\mathbb{F}_{1+2k}$, $k = 0, 1, 2$ of the morphisms τ_k induced by complete linear systems $|C_0 + (3+k)f|$, where C_0 is the unique section with $C_0^2 = -1-2k$ and f is a general fibre. We have $\tilde{Y} \in |3C_0 + (12 + 3k)f|$. Note when $k = 0, 1$, τ_k is an closed embedding while τ_2 contracts the section C_0 and the image is a cone.

Degree 6 surfaces in $\mathbb{P}^6$ are:

(1) A cone over smooth elliptic curve of degree 6 in $\mathbb{P}^5$.
(2) A del Pezzo surface of degree 6 with possibly isolated double points.
(3) A projection of $\mathbb{F}_{2k} \subset \mathbb{P}^7$ (k=0,1,2,3), embedded via the complete linear system $|C_0 + (3 + k)f|$, from a point outside of the surface.

Let π be the projection from $O \in \mathbb{P}^6$. Let $\pi(S)$ be the image of S under the rational map π. If $\pi(S)$ is one dimensional, then S is a cone with base $\tilde{Y}$, which is absurd, since S can only be a cone over a rational or elliptic curve by the classification above. Thus $\pi(S)$ is two dimensional. If S is a cone, then its vertex is different from the point O.

Now suppose $Y \subset \mathbb{P}^5$ is contained in V_4 and use $\overline{V_4} \subset \mathbb{P}^6$ to denote the cone with vertex O and base V_4.

Suppose $\overline{V_4}$ does not contain the surface S. Recall V_4 is the complete intersection of two smooth quadrics Q_0,Q_1. Use $\overline{Q_i}$ to denote the cone with vertex O and base Q_i. Then $\overline{V_4}$ is the intersection of $\overline{Q_i}$'s. Then S is not contained in at least one of the $\overline{Q_i}'s$, say $\overline{Q_0}$. Then $\overline{Q_0}$ cut the surface S at a curve of degree 10 or 12, which cannot contain $\tilde{Y}$, contradiction. So $S \subset \overline{V_4}$. We thus also have $\pi(S) \subset V_4$.

Suppose $O \notin S$. Denote the degree of π by d, then $\pi(S)$ is a surface of degree $5/d$ or $6/d$. On the other hand, $\pi(S)$ corresponds to a Cartier divisor $\mathcal{O}_{V_4}(l)$ where l is an integer, thus has degree $4l$. We have now a contradiction since $4dl = 5$ or $4dl = 6$ have no integral solutions.

Suppose $O \in S$. If S has degree 5, then S is one of the surfaces $\tau_k(\mathbb{F}_{1+2k})$ for $k = 0, 1, 2$. The fibre f passing through O is contracted by π. But we have $\tilde{Y}.f = 3$ and π cannot induced a closed embedding on $\tilde{Y}$. If S is a cone over a smooth elliptic curve E of degree 6 in $\mathbb{P}^5$. Then S is the image of a ruled surface over E by the morphism associated to a linear system numerically equivalent to $|C_0 + 6f|$, where C_0 is the section we obtained by blowing up the vertex ($C_0^2 = -6$). Then $\tilde{Y} \equiv mC_0 + 18f$, where $m = 3$ or 4. Then again if f is the general fibre passing through O, $\tilde{Y}.f = m > 1$ and this contradict the fact that π induced a closed embedding on $\tilde{Y}$. If S is a del Pezzo surface of degree 6 with possible isolated double points. S is then the image of $\mathbb{P}^2$ blowing up three points, i.e. S is the image of the linear system $|3H - E_1 - E_2 - E_3|$ where H is the pull-back of a hyperplane section in $\mathbb{P}^2$ and E_i are the exceptional divisors. Note the singularities occur when the points are not in general position. Then (compactification) of $\pi(S) \subset \mathbb{P}^5$ is the image of the blow up of S at O by the morphism associated to $|3H - E_1 - E_2 - E_3 - G|$, where G is the exceptional divisor of the blow-up at O. Then $\pi(S) \subset \mathbb{P}^5$ is one of the following:

(i) The Veronese surface $\mathbb{P}^2 \subset \mathbb{P}^5$.
(ii) $\mathbb{F}_0$ or $\mathbb{F}_2$ embedded into $\mathbb{P}^5$ as a minimal surface.
(iii) A del Pezzo surface of degree 5 with possibly isolated double points.

On the other hand, $\pi(S) \subset V_4$, and hence corresponds a Cartier divisor of the form $\mathcal{O}_{V_4}(l)$. Thus $\pi(S)$ has degree $4l$. This will immediately lead to a contradiction in the third case. In the first and second case, we will have $l = 1$. But remember a smooth hyperplane section of V_4 can only be del Pezzo surfaces of degree 4, which differ from the first two cases.

The case when S is (3) is more complicated. Note when $k \neq 3$, the secant variety of $\mathbb{F}_{2k} \subset \mathbb{P}^7$ is a proper subvariety of $\mathbb{P}^7$ and projection from a point off the secant variety induces an isomorphism. Since a rational normal scroll does not have a tri-secant line(not contained in the surface), any secant line can only meet the scroll at two points transversely. Similarly, a tangent line(not contained in the surface) can be tangent a one point and does not meet the scroll again. Moreover, two distinct secant lines, a tangent line and a secant line or two tangent lines cannot meet at any points other than a point of the scroll unless the scroll intersects the plane spanned by these two lines in a conic. In conclusion, S can either be smooth, or have only one double point or only one double line (which comes from a conic on $\mathbb{F}_{2k}$). Now suppose $O \in S$ is a smooth point. Then there exist a unique corresponding point, which we also call O on $\mathbb{F}_{2k}$. Then $\pi(S)$ is the image of a codimension one base-point-free linear system in $|C_0 + (3+k)f - E|$, where E is the exceptional divisor obtained by blowing up O. Thus $\pi(S) \subset V_4$ is a Cartier divisor of degree 5, which leads to a contradiction as before.

When O is the double point, then there are either two distinct points or a point and a tangent direction at it on $\mathbb{F}_{2k}$ corresponding to O. For simplicity of argument, we will only discuss the case of two distinct points, the other case can be handled similarly. Denote the two points by p_1, p_2, then $\pi(S)$ is the image of $\mathbb{F}_{2k}$ via the complete linear system $|C_0 + (3+k)f - E_1 - E_2|$ where E_1, E_2 are the exceptional divisors obtained by blowing up p_1, p_2. Note by our choice, p_1, p_2 cannot lie on a line contained in S nor a conic in S. When $k = 0$, there are no conics on S. When $k = 1$, C_0 is the only conic in S. When $k = 2$, $C_0 + f$ will be a conic for any general fibre f. When $k = 4$, any two points not on the same line will be on a conic (which is the union of the lines containing each point). It is not hard to check $\pi(S)$ is one of the following surfaces:

(A) Image of $\mathbb{F}_0$ associated to complete linear system $|C_0 + 2f|$.
(B) Image of $\mathbb{F}_2$ associated to complete linear system $|C_0 + 3f|$.
(C) Image of $\mathbb{F}_4$ associated to complete linear system $|C_0 + 4f|$.

We present in Table 1 what $\pi(S)$ is depending on the position of p_1, p_2. Note all $\pi(S)$ are

Table 1

k	Position of p_1, p_2	$\pi(S)$
0	p_1, p_2 not on a C_0	A
0	p_1, p_2 on a C_0	B
1	$p_1 \in C_0$ and $p_2 \notin C_0$	B
1	$p_1, p_2 \notin C_0$	A
2	$p_1, p_2 \notin C_0$	B

smooth surfaces in $\mathbb{P}^5$ of degree 4. On the other hand, $\pi(S) \subset V_4$ corresponds to a Cartier divisor $\mathcal{O}_{V_4}(l)$. By looking at the degree, we see $l = 1$. But then $\pi(S)$ has to be del Pezzo surfaces of degree 4, which leads to a contradiction.

When O is a point on the double line, then $k = 1, 2$ or 3. In each case, the strict transformation of the conic under the blow up of E_1, E_2 will be contracted by the system $|C_0 + (3+k)f - E_1 - E_2|$, and it is not hard to check $\pi(S)$ is (C) by computing the degree and self-intersection of the strict transformation for each k. Since $\pi(S) = (C)$ has degree 4, it corresponds to the Cartier divisor $\mathcal{O}_{V_4}(1)$, which means $\pi(S) \subset H$, where H is a hypersurface in $\mathbb{P}^5$. This is absurd since (C) is non-degenerate. This finishes the proof. □

[D] classified instanton sheaves on cubic threefolds. Now we classify instanton sheaves on V_4, closely following the argument of [D]. When the proof transfers almost verbatim, we will only point out the changes in our situation and refer the readers to [D].

Proposition 3.6. *Let E be an instanton sheaf on V_4. Let F be the double dual of E. Then either E is locally free or F is locally free with second Chern classes $c_2(F) = 1$ and $h^0(F) = 1$ or $F = H^0(F) \otimes \mathcal{O}_{V_4}$.*

Proof. See [D, Proposition 3.1]. The proof of [D, Proposition 3.1] used mainly general arguments about semistable sheaves on projective varieties and can be directly applied here. We highlight a few similarities between V_4 and a cubic threefold V_3 which allow us to mimic the argument:

(1) Both V_4 and V_3 are Fano threefolds of index 2.
(2) General hyperplane sections of both V_4 and V_3 are del-Pezzo surfaces using the anticanonical embedding.
(3) The following inequalities of Hilbert polynomials on V_3

$$\chi(E(n)) < 2\chi(\mathcal{O}_{V_3}(n))$$
$$\chi(E(n)) < 2\chi(I_p(n))$$

remain true on V_4, in fact

$$\chi(E(n)) = \frac{4}{3}n^3 + 4n^2 + \frac{8}{3}n$$
$$\chi(\mathcal{O}_{V_4}(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{7}{3}n + 1$$
$$\chi(I_p(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{7}{3}n + 1$$

where p is a point. □

Lemma 3.7. *Suppose θ is the theta-characteristic of a smooth conic $C \subset V_4$. We consider the sheaf E which is the kernel of the surjection $H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1)$. Then E is stable with Chern classes $c_1(E) = 0$,$c_2(E) = 2$ and $c_3(E) = 0$.*

Proof. See [D, Lemma 3.4]. Again the arguments in [D] can be applied because of the similarites highlighted in the previous proof. The only new fact we need is $2\chi(I_C(n)) < \chi(E(n))$. This is true since

$$\chi(I_C(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{1}{3}n$$

□

Theorem 3.8. *Let E be an instanton sheaf on V_4. If E is stable, then either E is locally free or E is associated to a smooth conic $Y \subset V_4$ such that we have the exact sequence:*

$$0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1) \to 0$$

where θ is the theta-characteristic of Y.
If E is strictly semistable, then E is the extension of two ideal sheaves of lines.

Proof. See [D, Theorem 3.5]. Again the arguments in [D] can be applied due to the highlights in the previous two proofs. The only new fact we need in this proof is

$$\chi(I_Z(n)) = \frac{2}{3}n^3 + 2n^2 + \frac{7}{3}n + 1 - l(Z)$$

when Z is a zero-dimensional subscheme. □

4. Relation to semistable rank 2 bundles on C

We now use the classification of instanton sheaves to understand their relation with semistable rank 2 bundles on C.

Lemma 4.1. *Let E be an instanton sheaf on V_4. Then $E \in \mathcal{B}_{V_4}$.*

Proof. It suffices to show that $H^*(E(-1)) = H^*(E) = 0$. If E is a stable vector bundle, the result follows from Theorem 3.2 and [Ku03, Lemma B.3].
If E is associated to a smooth conic, we have the short exact sequence:

$$0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1) \to 0$$

Since $H^*(\mathcal{O}_{V_4}(-1)) = H^*(\theta) = 0$, we immediately obtain $H^*(E(-1)) = 0$. On the other hand, we have $H^0(H^0(\theta(1)) \otimes \mathcal{O}_{V_4}) = 2$ and $H^0(\theta(1)) = 2$. It is clear that the map $H^0(H^0(\theta(1)) \otimes \mathcal{O}_{V_4}) \to H^0(\theta(1))$ is surjective. Moreover, $H^i(\mathcal{O}_{V_4}) = H^i(\theta(1)) = 0$ for all $i > 0$. Thus $H^*(E) = 0$.
If E is the extension of the ideal sheaves of lines in V_4, the result follows from the fact that $I_l \in \mathcal{B}_{V_4}$. □

Remark 4.2. It is worth noting that when E is associated to a smooth conic, the short exact sequence:

$$0 \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1) \to 0$$

can be understood as saying E is the left mutation $\mathbf{L}_{\mathcal{O}_{V_4}}(\theta(1))[-1]$. This will imply $E \in \mathcal{O}_{V_4}^{\perp}$ (hence $H^*(E) = 0$) immediately.

[Ku12, Theorem 5.10] proved that for any (minimal) instanton bundle E of on V_4, $\Phi^*(E)[-1]$ is a simple rank 2 degree 0 vector bundles on C. We generalized this result in the following theorem:

Theorem 4.3. *Let E be an instanton sheaf on V_4, then $\mathcal{F} := \Phi^*(E)[-1]$ is a semistable vector bundle of rank 2 and degree 0 on C.*

Proof. If E is a vector bundle, by [Ku12, Theorem 5.10], $\mathcal{F} = \Phi^*(E)[-1]$ is a rank 2 degree 0 vector bundle on C such that

$$\mathrm{Hom}_C(\mathcal{F}, \mathcal{R}) = \mathrm{Ext}^1_C(\mathcal{F}, \mathcal{R}) = 0$$

where $\mathcal{R}$ is a second Raynaud bundle. By Lemma 2.8, $\mathcal{F}$ is semistable.
If E is associated to a smooth conic Y, we have the exact triangle in $\mathcal{D}^b(V_4)$:

$$\theta(1)[-1] \to E \to H^0(\theta(1)) \otimes \mathcal{O}_{V_4}.$$

Apply the functor $\Phi^*(\cdot)[-1]$ and noting $\Phi^*(\mathcal{O}_{V_4}) = 0$, we obtain

$$\mathcal{F} = \Phi^*(\theta(1))[-2]$$

Recall as pointed out in the proof of [Ku12, Theorem 5.10], Φ^* is a Fourier-Mukai transform with the kernel $\mathcal{S}^* \otimes p_{V_4}^* \mathcal{O}_{V_4}(-2)[3]$. Thus the fiber of the object $\mathcal{F}$ at a point $x \in C$ is given by

$$\begin{aligned}\mathcal{F}_x &= H^{\bullet+1}(V_4, \mathcal{S}_x^* \otimes \theta(-1)) \\ &= H^{\bullet+1}(Y, \mathcal{S}_x^*|_Y \otimes \theta(-1)) \\ &= H^{-\bullet}(\mathbb{P}^1, \mathcal{S}_x|_Y \otimes \mathcal{O}_{\mathbb{P}^1}(1))\end{aligned}$$

Now $\mathcal{S}_x|_Y$ is a rank 2 bundle on $Y \simeq \mathbb{P}^1$ with degree -2. Moreover, we note $H^0(V_4, \mathcal{S}_x^*) = \mathbb{C}^4$ and the induced map $\mathcal{O}_{V_4}^{\oplus 4} \to \mathcal{S}_x^*$ is surjective (see [Ku12, Proposition 5.7]). Hence $\mathcal{S}_x^*|_Y$ as a sheaf on Y is generated by global sections. Thus

$$\mathcal{F}_x = \mathbb{C}^2[0]$$

for all $x \in C$. Hence $\mathcal{F}$ is a vector bundle of rank 2. By Lemma 2.7, we see $\mathcal{F}$ has degree 0.

To see $\mathcal{F}$ is semistable. Note a vector bundle $\mathcal{F}$ of rank 2 and degree 0 is not stable if and only if there is a nontrivial morphism $\mathcal{F} \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle of degree 0. By adjunction

$$\begin{aligned}\mathrm{Hom}(\mathcal{F}, \mathcal{L}) = \mathrm{Hom}(\Phi^*(\theta(1))[-2], \mathcal{L}) &= \mathrm{Hom}(\theta(1), \Phi(\mathcal{L})[2]) \\ &= \mathrm{Hom}(\theta(1), I_l[1]) \\ &= \mathrm{Ext}^1(\theta(1), I_l)\end{aligned}$$

where l is a line on V_4 by Theorem 2.6. Apply $\mathrm{Ext}^\bullet(\theta(1), -)$ to the short exact sequence $0 \to I_l \to \mathcal{O}_{V_4} \to \mathcal{O}_l \to 0$, we have

$$\mathrm{Hom}(\theta(1), \mathcal{O}_l) \to \mathrm{Ext}^1(\theta(1), I_l) \to \mathrm{Ext}^1(\theta(1), \mathcal{O}_{V_4})$$

Now the first space is zero since θ is supported on a smooth conic, while the last space is Serre dual to $\mathrm{Ext}^2(\mathcal{O}_{V_4}(2), \theta(1)) = H^2(\theta(-1)) = 0$. Thus $\mathrm{Ext}^1(\theta(1), I_l) = 0$ and $\mathcal{F}$ is in fact stable in this case.

If E is the extension of ideal sheaves of lines in V_4, then we have short exact sequence:

$$0 \to I_{l_1} \to E \to I_{l_2} \to 0 \tag{4.4}$$

Apply the functor $\Phi^*(\cdot)[-1]$ we have exact triangle

$$\Phi^*(I_{l_1})[-1] \to \mathcal{F} \to \Phi^*(I_{l_2})[-1] \tag{4.5}$$

By [Ku12, Lemma 5.5], $\Phi^*(I_{l_i})[-1]$ are line bundles of degree 0 on C, thus $\mathcal{F}$ is a strictly semistable rank 2 bundle of degree 0. ☐

Proposition 4.6. *Let E be a stable(strictly semistable) instanton sheaf on V_4, then $\Phi^*(E)[-1]$ is a stable(strictly semistable) vector bundle on C.*

Proof. By the above theorem, $\Phi^*(E)[-1]$ is always semistable. By the proof of the above theorem, we see for a strictly semistable instanton E, $\Phi^*(E)[-1]$ is a strictly semistable vector bundle on C. Suppose for an instanton sheaf E, $\mathcal{F} = \Phi^*(E)[-1]$ is a strictly semistable vector bundle on C. Let

$$0 \to \mathcal{L}_1 \to \mathcal{F} \to \mathcal{L}_2 \to 0$$

be a Jordan-Hölder filtration. Then $\mathcal{L}_1, \mathcal{L}_2$ have to be degree 0 line bundles. Apply the functor $\Phi(\cdot)[1]$, by [Ku12, Lemma 5.5], there exist two lines l_1, l_2 in V_4 so that

$$0 \to I_{l_1} \to E \to I_{l_2} \to 0$$

is exact. Hence E is a strictly semistable instanton sheaf. □

By now we have established a well-behaved correspondence between (semi)stable instanton sheaves on V_4 and (semi)stable rank 2 degree 0 vector bundles on C. Next is to use this correspondence to analyze the two moduli spaces.

5. Moduli Space of Instantons

We start by showing the smoothness of M^{inst}. To do this we first compute some related invariants.

Lemma 5.1. *Let θ be the theta characteristic of a smooth conic Y in V_4. Let E be the kernel of the natural surjection $H^0(\theta(1)) \otimes \mathcal{O}_{V_4} \to \theta(1)$. Then $\operatorname{Ext}^2(E,E) = 0$ and $\operatorname{Ext}^1(E,E)$ has dimension 5.*

Proof. By Theorem 4.3, $E = \Phi(\mathcal{F})[1]$, where $\mathcal{F}$ is a rank 2 bundle on C. Thus

$$\operatorname{Ext}^2(E,E) = \operatorname{Ext}^2(\Phi(\mathcal{F})[1], \Phi(\mathcal{F})[1]) = \operatorname{Ext}^2_C(\mathcal{F}, \mathcal{F}).$$

The last space is 0 since C is a curve.

Now $\operatorname{Ext}^3(E,E) \simeq \operatorname{Hom}(E, E(-2))^* = 0$ and $\operatorname{Hom}(E,E) = \mathbb{C}$. By Riemann-Roch, $\chi(E,E) = -4$. Thus $\operatorname{Ext}^1(E,E)$ is five dimensional. □

Lemma 5.2. *Let $l_1, l_2 \subset V_4$ be two lines. Then $\operatorname{Ext}^2(I_{l_1}, I_{l_2}) = 0$ and $\dim \operatorname{Ext}^1(I_{l_1}, I_{l_2}) = 1$ if $l_1 \neq l_2$ and 2 if $l_1 = l_2$.*

Proof. Since $I_{l_1}, I_{l_2} \in \mathcal{B}_{V_4}$, we have

$$\begin{aligned}\operatorname{Ext}^2(I_{l_1}, I_{l_2}) &= \operatorname{Ext}^2_{\mathcal{D}^b(C)}(\Phi^{-1}(I_{l_1}[-1]), \Phi^{-1}(I_{l_2}[-1])) \\ &= \operatorname{Ext}^2_C(\mathcal{L}_1, \mathcal{L}_2)\end{aligned}$$

where $\mathcal{L}_i$ is the line bundle corresponding to l_i as in Theorem 2.6. Since C is a curve, we see the above extension group is 0.
Moreover, $\operatorname{Ext}^3(I_{l_1}, I_{l_2}) \simeq \operatorname{Hom}(I_{l_2}, I_{l_1}(-2))^* = 0$. By Riemann-Roch, $\chi(I_{l_1}, I_{l_2}) = -1$. Thus the lemma follows. □

Let $N \geq 1$ be an integer and V be a complex vector space. Let Q be the Hilbert scheme of the quotient $V \otimes \mathcal{O}_X(-N) \to E$ of X with rank 2 and Chern classes $c_1(E) = 0$, $c_2(E) = 2$ and $c_3(E) = 0$ and L the natural polarization [Si]. The group $G = PGL(V)$ acts on Q and a suitable power of L is G-linearized. Let Q_c^{ss} be the $PGL(V)$-semistable points corresponding to quotients without torsion and Q_c the closure of Q_c^{ss} in Q. When the integer N and the vector space V are suitably chosen the following properties are satisfied. The map $V \otimes \mathcal{O}_X \to E(N)$ induces an isomorphism $V \simeq H^0(E(N))$ and $h^i(E(k)) = 0$ for $k \geq N$ and $i \geq 1$ and for all E in Q_c. The point $[E] \in Q_c$ is semistable iff the sheaf E

is semistable iff $E \in Q_c^{ss}$. The stabilizer of $[E]$ in $GL(V)$ is identified with the group of automorphisms of the sheaf E and moduli space is then the GIT quotient $Q_c^{ss}//G$.

Lemma 5.3. *With the above hypothesis, the scheme Q_c^{ss} is smooth.*

Proof. The tangent space of Q_c^{ss} at a point $[E]$ is isomorphic to $\mathrm{Hom}(F, E)$ where F is the kernel of the map $V \otimes \mathcal{O}_X(-N) \to E$. The scheme Q_c^{ss} is smooth at the point if $\mathrm{Ext}^1(F, E) = 0$. Consider the exact sequence:

$$\mathrm{Ext}^1(V \otimes \mathcal{O}_X(-N), E) \to \mathrm{Ext}^1(F, E) \to \mathrm{Ext}^2(E, E)$$

We then obtain an inclusion $\mathrm{Ext}^1(F, E) \to \mathrm{Ext}^2(E, E)$ since $h^1(E(N)) = 0$. It suffices then to prove $\mathrm{Ext}^2(E, E) = 0$. But this follows from the fact $E = \Phi(\mathcal{F})[1]$ where $\mathcal{F}$ is a sheaf on C, as we have seen in the proof of Lemma 5.1 and 5.2. □

Theorem 5.4. *The moduli space M^{inst} of semistable sheaves of rank 2 with Chern classes $c_1(E) = 0, c_2(E) = 2, c_3(E) = 0$ on V_4 is smooth of dimension 5.*

Proof. See [D, Theorem 4.6] □

We now construct a morphism from M^{inst} to M. M^{inst} is the GIT quotient $Q_c^{ss}//G$. Let $\mathcal{E}$ be a universal family on $Q_c^{ss} \times V_4$. For any $t \in Q_c^{ss}$, by Lemma 4.3 and Lemma 4.6, the map

$$\Psi : Q_c^{ss} \to M$$
$$t \mapsto [\Phi^*(\mathcal{E}_t)[-1]]$$

is well defined. Since $\Phi^*(-)[-1]$ is Fourier Mukai, Ψ is algebraic. To see this morphism is invariant under G, it suffices to check for any t that corresponds to strictly semistable instanton sheaf, i.e. $\mathcal{E}_t$ is the extension of I_{l_1} and I_{l_2}, we have $\Psi(t) = \Psi(t_0)$ where t_0 corresponds to $I_{l_1} \oplus I_{l_2}$. Applying the functor $\Phi^*(-)[-1]$ to the short exact sequence

$$0 \to I_{l_1} \to E \to I_{l_2} \to 0$$

we obtain

$$0 \to \Phi^*(I_{l_1})[-1] \to \Phi^*(E)[-1] \to \Phi^*(I_{l_2})[-1] \to 0$$

Recall $\Phi^*(I_{l_i})[-1]$ are line bundles of degree 0 thus $\Phi^*(E)[-1]$ lies in the S-equivalence class of $\Phi^*(I_{l_1})[-1] \oplus \Phi^*(I_{l_2})[-1]$. As a result Ψ descends to a morphism $\psi : M^{inst} \to M$.

Theorem 5.5. *$\psi \colon M^{inst} \to M$ is an isomorphism. As a result, the moduli space of instanton sheaves on V_4 is a projective bundle over the Jacobian of C.*

Proof. Since both M^{inst} and M are projective, ψ is proper. We claim ψ is injective. Let $\psi([E_1]) = \psi([E_2])$. By Proposition 4.6, either both E_i are stable or both E_i are strictly semistable. If E_i are stable, then $\psi([E_1]) = \psi([E_2])$ implies $E_1 \simeq E_2$, i.e. $[E_1] = [E_2]$. If E_i are strictly semistable, the injectivity follows from Theorem 2.6.

Any proper quasi-finite morphism is finite, so ψ is a finite morphism. Thus the image $\psi(M^{inst})$ has dimension 5, which must be all of M. So ψ is surjective. Since ψ is injective and M is integral, we see M^{inst} must be connected. Along with Theorem 5.4, we know M^{inst} is a smooth variety.

Let $[E] \in M^{inst}$ be a stable point, then the tangent space at $[E]$ is given by $\mathrm{Ext}^1(E, E)$. The tangent space at $\psi(E)$ (which is also stable) is given by $\mathrm{Ext}^1(\Phi^*(E)[-1], \Phi^*(E)[-1])$. Since $\Phi^* : \mathcal{B}_{V_4} \to \mathcal{D}^b(C)$ is an equivalence, ψ induces isomorphism between tangent spaces, so ψ is étale when restricted to the open stable locus. Since ψ is also injective and we are working over $\mathbb{C}$, ψ is an open immersion over the stable locus (See for example Stack Project 40.14). Now ψ is a bijective birational proper morphism, it has to be an isomorphism. □

References

[ACGH] E. Arbarello, M. Cornalba, P. Griffits, J. Harris, *Geometry of Algebraic Curves I*, Grundleheren der math. Wissenschaften,vol. 267, Srpinger Verlag (1985).

[ADHM] M. Atiyah, V. Drinfeld, N. Hitchin,Y. Manin, *Construction of instantons*, Phys. Lett. A, 65(1978),185–187.

[B] A. Beauville, *An introduction to Ulrich bundles*, European Journal of Mathematics, 4(2018):26–36.

[BO] A. Bondal, D. Orlov, *Semi-orthogonal decomposition fo algebraic varieties*, arXiv:alg-geom/9506012.

[CKL] Y. Cho, Y. Kim, K. Lee *Ulrich bundles on intersections of two 4-dimensional quadrics*, arXiv:1704.03352 2017.

[D] S. Druel, *Espace des modules des faisceaux de rang 2 semistables de classes de Chern $c_1 = 0, c_2 = 2$ et $c_3 = 0$ sur la cubique de $\mathbb{P}^4$*, Internat. Math. Res. Notices, 19:985–1004,2000.

[EH] D. Eisenbud, J. Harris, *Curves in projective spacce*, Les Presses de L'Université de Montréal(1982).

[ES] D. Eisenbud, F. Schreyer, *Resultants and Chow forms via exterior syzygies* , J. Amer. Math. Soc., 16(2003):537–579.

[Fa] D. Faenzi, *Even and odd instanton bundles on Fano threefolds of Picard number 1*, arXiv:1109.3858,2013.

[GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley Interscience(1994).

[H1] R. Hartshorne, *Stable vector bundles of rank 2 on $\mathbb{P}^3$*, Math. Ann., 238(1978):229-280.

[JV] M. Jardim, M. Verbitsky, *Trihyperkahler reduction and instanton bundles on $\mathbb{CP}^3$*, Compositio Math., 150(2011):1836-1868.

[Ku03] A. Kuznetsov, *Derived Categories of cubic and V_{14} threefolds*, Proc. Steklov Inst. Math., 246(2004):183–207.

[Ku12] A. Kuznetsov, *Instanton bundles on Fano threefolds*, Cent. Eur. J. Math., 10(2012):1198–1231.

[LMS1] M. Lahoz, E. Macrì, P. Stellari, *ACM bundles on cubic threefolds*, Algebr. Geom., 2(2015), 231–269.

[LMS2] M. Lahoz, E. Macrì, P. Stellari, *ACM bundles on cubic fourfolds containing a plane*, Brauer groups and obstruction problems: Moduli spaces and arithmetic, Birkhauser/Springer, 2017.

[M] M. Maruyama, *Boundedness of semi-stable sheaves of small ranks*, Nagoya Math. J.78(1980): 65-94.

[Mu] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Math. Studies 59, Princeton University Press (1966)..

[MT] D. Markushevich, A.S. Tikhomirov , *Rationality of instanton moduli*, arXiv:1012.4132868,2011.

[N] M. Nagata, *On rational surfaces I. Irreducible curves of arithmetic genus 0 or 1.*, Mem. College Sci. Univ. Kyoto Ser. A Math., Volume 32, Number 3 (1960), 351-370.

[NR] M. Narasimhan, S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Annals of Mathematics, 89(1969):14-51.

[P] M. Popa, *Generalized Theta linear series on moduli spaces of vector bundles on curves*, Handbook of moduli: Volume III, (2013): 219-255.

[Q] X. Qin, *Moduli space of instanton sheaves on the Fano threefold V_5*, arXiv:1810.04737, 2018.

[QZ] X. Qin,S. Zhang, *Moduli of quiver representations for exceptional collections on surfaces*, arXiv:1803.06533, 2018.

[Si] C. Simpson, *Moduli of representations of the fundamental group of a smooth variety*, Publ. Math. IHES, 79(1994):47-129.

[T] A.S. Tikhomirov, *Moduli of mathematical instanton vector bundles with odd c_2 on projective space*, Izv. Math., 76(2012):991-1073.

Department of Mathematics, Indiana University, 831 E. Third St., Bloomington, IN 47405, USA

E-mail address: qinx@iu.edu