BLOW UPS OF $\mathbb{P}^n$ AS QUIVER MODULI FOR EXCEPTIONAL COLLECTIONS

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ABSTRACT. Suppose P_m^n is the blow up of $\mathbb{P}^n$ at a linear subspace of dimension m, we provide a new method to show it is a quiver moduli space by constructing the blow-up as morphism between moduli spaces.

1. INTRODUCTION

Let X be a smooth projective variety over an algebraically closed field $\mathbf{k}$ of characteristic 0. Recall that objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ in the bounded derived category of coherent sheaves $\mathcal{D}^b(\mathrm{coh}(X))$ forms a exceptional collection if

(1) $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i[m]) = \mathbf{k}$ if $m = 0$ and is 0 otherwise;
(2) $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[m]) = 0$ for all $m \in \mathbb{Z}$ if $j < i$

An exceptional collection is strong if in addition: $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[m]) = 0$ for all i, j if $m \neq 0$. It is full if the smallest triangulated subcategory of $\mathcal{D}^b(\mathrm{coh}(X))$ containing $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is itself.

In this paper we are only concerned with the case when the objects $\mathcal{E}_i$ are line bundles and the exceptional collection is strong. In this situation, we can consider the finite dimensional associative algebra

$$\mathcal{A} = \mathrm{End}(\oplus_{i=1}^n \mathcal{E}_i)$$

It is well known that if the collection is full, there is an exact equivalence of derived categories

$$\mathbf{R}\mathrm{Hom}(\oplus_{i=1}^n \mathcal{E}_i, -) : \mathcal{D}^b(\mathrm{coh}(X)) \to D^b(\mathrm{mod}\text{-}\mathcal{A})$$

whose inverse is given by

$$(-) \otimes^L (\oplus_{i=1}^n \mathcal{E}_i) : D^b(\mathrm{mod}\text{-}\mathcal{A}) \to D^b(\mathrm{coh}(X))$$

This gives a non-commutative interpretation of the derived category of X. We note that when we input the structure sheaf $\mathcal{O}_x$ of a close point $x \in X$ into the first functor, we obtain:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}(\oplus_{i=1}^n \mathcal{E}_i, \mathcal{O}_x) &= \mathrm{Hom}(\oplus_{i=1}^n \mathcal{E}_i, \mathcal{O}_x) \\ &= \oplus_{i=1}^n (\mathcal{E}_i^\vee)_x\end{aligned}$$

Note since $\mathcal{A}^{op} = \mathrm{End}\big(\oplus_{i=1}^n (\mathcal{E}_i^\vee)\big)$ is a finite dimensional algebra, there exist a bound quiver (Q, I) such that giving an $\mathcal{A}^{op}$-module is equivalent to giving a representation of (Q, I).

King[King] proved that when restricted by a stability condition θ, the moduli space $M_\theta(\alpha)$ of semistable representations of a bound quiver (Q, I) with any dimension vector α is a projective scheme. Moreover, if $I = 0$, i.e. the quiver has no relations, the moduli space $M_\theta^S(\alpha)$ of stable representation is a smooth projective variety. In our situation, for each

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point $x \in X$, one can associate the representation $\oplus_{i=1}^{n}(\mathcal{E}_i^{\vee})_x$ of $\mathcal{A}^{op}$, which has dimension vector $(1, \dots, 1)$. This provides a tautological map

$$T_0 : X \to \mathcal{R}ep_{(1,\dots,1)}(Q)$$

to the moduli stack of representation of (Q, I) with dimension vector $(1, \dots, 1)$. Following [BP], we note T_0 induce a tautological rational map $T : X \dashrightarrow M_\theta$ where M_θ is the moduli space of semistable representation of (Q, I) with dimension vector $(1, \dots, 1)$. We note M_θ is a projective scheme and

$$T(x) = \mathrm{Hom}(\oplus_{i=1}^{n}\mathcal{E}_i, \mathcal{O}_x) \tag{1.1}$$
$$= \oplus_{i=1}^{n}(\mathcal{E}_i^{\vee})_x \tag{1.2}$$

if $x \in X$ is in the domain of T. It is also important to notice that T can be similarly defined even if the collection of line bundles is not full.

It is natural to wonder when T is a morphism and the relation between M_θ and X for various θ. [QZ] provided some answers for this problem for rational surfaces. On the other hand, one can also consider the $(n+1)$-Kronecker quiver, which can be thought of as the quiver associated to the (not full) strong exceptional collection $\{\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)\}$ on $\mathbb{P}^n$. It is well know that for appropriate stability condition, T is an isomorphism. In this paper, we prove a similar result on P_m^n:

Theorem 1.3. *Let P_m^n be the blow up of $\mathbb{P}^n$ with center a linear subspace of dimension m. Let E be the exceptional divisor and H the pull back of hyperplane section. Then $\{\mathcal{O}_{P_m^n}, \mathcal{O}_{P_m^n}(H-E), \mathcal{O}_{P_m^n}(H)\}$ is a strong exceptional collection of line bundles that is not full. Let $\mathcal{A} = \mathrm{End}\Big(\mathcal{O}_{P_m^n} \oplus \mathcal{O}_{P_m^n}(H-E) \oplus \mathcal{O}_{P_m^n}(H)\Big)$ be the endomorphism algebra. Let Q be the quiver for $\mathcal{A}^{op}$. Then one can choose (many) stability conditions so that the moduli space of semistable representations of Q with dimension vector $(1,1,1)$ is a fine moduli space and the tautological rational map*

$$T : P_m^n \dashrightarrow M_\theta$$

is an isomorphism.

The proof of the theorem follows the idea in [QZ] by thinking of $\{\mathcal{O}_{P_m^n}, \mathcal{O}_{P_m^n}(H-E), \mathcal{O}_{P_m^n}(H)\}$ as an 'augmentation' of $\{\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)\}$ on $\mathbb{P}^n$. However, many arguments were simplified because the quiver in the present case is nicer and has no relations.

Corollary 1.4. *There are many stability conditions θ such that P_m^n is the moduli space of stable representations with dimension vector $(1,1,1)$ of the following quiver Q*

$$\begin{tikzcd}
1 \arrow[rr, shift left=2, "x_0"] \arrow[rr, shift right=2, "x_m"'] \arrow[dr, "e"'] & & 3 \\
& 2 \arrow[ur, shift left=2, "x_{m+1}"] \arrow[ur, shift right=2, "x_n"'] &
\end{tikzcd}$$

The fact that P^n_m is a quiver moduli is well-known, see [CS]. [CS] showed every projective toric variety is a fine moduli space for stable representations for a suitable quiver coming from a collection of globally generated line bundles. In particular, their method shows P^n_m is a quiver moduli for Q^{op}. On the other hand, our method does not require the global generatedness of the line bundles (note the above collection corresponds to $\{\mathcal{O}_{P^n_m}, \mathcal{O}_{P^n_m}(E), \mathcal{O}_{P^n_m}(H)\}$ in [CS]'s convention as we are using the opposite algebra). This allows our method to be applied more collections of line bundles, see [QZ].

Our second theorem shows that by varying the stability condition, we can also obtain $\mathbb{P}^n$ as a moduli space of representations of Q, and realize the natural contraction $\pi : P^n_m \to \mathbb{P}^n$ as a morphism between moduli of representations of the same quiver.

Theorem 1.5. *Let $\theta' = (-p, 0, p)$ $\theta = (-p, p-q, q)$ where $p, q \in \mathbb{Z}_{>0}, p < q$. Then the moduli space $M_{\theta'}$ of θ'-semistable representation of Q with dimension vector $(1,1,1)$ is isomorphic to $\mathbb{P}^n$ via the tautological map. Moreover, the morphism id induced by the identity map $\iota : \mathbf{k}[x_0, \ldots, x_n, e] \to \mathbf{k}[x_0, \ldots, x_n, e]$ on the coordinate ring of quiver variety of Q descents to the blow-up $\pi' : M_\theta \to M_{\theta'}$*

Notations and Conventions.

- P^n_m is the blow up of $\mathbb{P}^n$ centered at a linear subspace of dimension m. We require $n \geq 2$ and $0 \leq m \leq n-2$.
- For a sheaf $\mathcal{F}$ on a scheme X, we use $h^i(X, \mathcal{F})$ to denote the dimension of $H^i(X, \mathcal{F})$.

2. Preliminaries

2.1. Geometry of P^n_m.

In this section we provide basic facts on geometry of P^n_m. The main references for this section are [Har] [LYY].

Let $\pi : P^n_m \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ with center $\mathbb{L} = \{X_{m+1} = \ldots = X_n = 0\}$ being a linear subspace of dimension m. Then

$$\mathcal{N}_{\mathbb{L}/\mathbb{P}^n} = \mathcal{O}_{\mathbb{L}}(-1)^{\oplus(n-m)}$$

Thus the exceptional divisor $E \cong \mathbb{L} \times \mathbb{P}^{n-m-1} \cong \mathbb{P}^m \times \mathbb{P}^{n-m-1}$. The P^n_m is a smooth toric variety whose Picard group is

$$Pic(X) \cong \mathbb{Z}[H] \oplus \mathbb{Z}[E]$$

where H is the pull back of a hyperplane section of $\mathbb{P}^n$. The canonical divisor of P^n_m is given by

$$K_{P^n_m} = -(n+1)H + (n-m-1)E$$

The Chow ring of P^n_m has the following presentation

$$A(P^n_m) = \mathbb{Z}[H, E]/\big((H-E)^{n-m}, H^m - (H-E)H^{m-1}\big)$$

Recall a divisor D on a smooth projective variety is called strong left orthogonal if

$$h^i(\mathcal{O}(D)) = 0$$

for all $i > 0$ and

$$h^i(\mathcal{O}(-D)) = 0$$

for all $i \geq 0$.

Lemma 2.1. *The divisors $H, E, H-E$ on P^n_m are strong left orthogonal.*

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_{P^n_m}(-E) \to \mathcal{O}_{P^n_m} \to \mathcal{O}_E \to 0$$

Taking cohomology we obtain long exact sequence:

$$\cdots \to H^i(\mathcal{O}_{P^n_m}(-E)) \to H^i(\mathcal{O}_{P^n_m}) \to H^i(\mathcal{O}_E) \to \cdots$$

Since $E \cong \mathbb{P}^m \times \mathbb{P}^{n-m-1}$, by Kunneth formula, we get $H^0(\mathcal{O}_E) = k$ and $H^i(\mathcal{O}_E) = 0$ for $i > 0$. Moreover, it is clear that the map $H^0(\mathcal{O}_{P^n_m}) \to H^0(\mathcal{O}_E)$ is an isomorphism, hence $h^i(\mathcal{O}_{P^n_m}(-E)) = 0$ for all $i > 0$. Since E is effective, $h^0(\mathcal{O}_{P^n_m}(-E)) = 0$.
Using the same argument, we can show $h^i(\mathcal{O}_{P^n_m}(-H)) = h^i(\mathcal{O}_{P^n_m}(-H+E)) = 0$ for all $i \geq 0$.

Twist the short exact sequence

$$0 \to \mathcal{O}_{P^n_m}(-E) \to \mathcal{O}_{P^n_m} \to \mathcal{O}_E \to 0 \tag{2.2}$$

by $\mathcal{O}_{P^n_m}(E)$, we obtain

$$0 \to \mathcal{O}_{P^n_m} \to \mathcal{O}_{P^n_m}(E) \to \mathcal{O}_E(E) \to 0$$

Taking cohomology we obtain long exact sequence:

$$\cdots \to H^i(\mathcal{O}_{P^n_m}) \to H^i(\mathcal{O}_{P^n_m}(E)) \to H^i(\mathcal{O}_E(E)) \to \cdots$$

Denoting $Pic(E) \cong Pic(\mathbb{L}) \times Pic(\mathbb{P}^{n-m-1})$. Then $\mathcal{O}_E(E) = \mathcal{O}_E(1,-1)$. Using Kunneth formula, we see $h^i(\mathcal{O}_E(E)) = 0$ for all $i \geq 0$. From this one easily see $h^i(\mathcal{O}_{P^n_m}(E)) = 0$ for all $i > 0$.
Similarly, we can get $h^i(\mathcal{O}_{P^n_m}(H)) = 0$ for all $i > 0$.

Twist (2.2) by $O(H)$ we obtain short exact sequence

$$0 \to \mathcal{O}_{P^n_m}(H-E) \to \mathcal{O}_{P^n_m}(H) \to \mathcal{O}_E(H) \to 0$$

Note $\mathcal{O}_E(H) = \mathcal{O}_E(1,0)$. Again using Knneth formula and noting $H^0(\mathcal{O}_{P^n_m}(H)) \to H^0(\mathcal{O}_E(H))$ is surjective, we see $h^i(\mathcal{O}_{P^n_m}(H-E)) = h^i(\mathcal{O}_{P^n_m}(H)) = 0$ for all $i > 0$. This finishes the proof. □

Note a collection of line bundles $\{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ is strong exceptional if $\mathcal{O}(D_j - D_i)$ are strong left orthogonal for all $j > i$. As a result,

Corollary 2.3. *$\{\mathcal{O}_{P^n_m}, \mathcal{O}_{P^n_m}(H-E), \mathcal{O}_{P^n_m}(H)\}$ is a strong exceptional collection of line bundles.*

2.2. **Quivers and quiver representations.** See also [Br].
A quiver Q is given by two sets Q_{vx} and Q_{ar}, where the first set is the set of vertices and the second is the set of arrows, along with two functions $s, t : Q_{ar} \to Q_{vx}$ specifying the source and target of an arrow. The path algebra $\mathbf{k}Q$ is the associative $\mathbf{k}$-algebra whose underlying vector space has a basis consists of elements of Q_{ar}. The product of two basis elements is defined by concatenation of paths if possible, otherwise 0. The product of two general elements is defined by extending the above linearly. A bound quivers is a pair (Q, I). Here Q is a quiver and I is a two sided ideal of $\mathbf{k}Q$ generated by elements of the form $\sum_{i=1}^n k_i p_i$, where $k_i \in \mathbf{k}^*$ and p_i are paths with same heads and same tails for $i \in \{1, \ldots, n\}$. We simply use Q to denote this pair when the existence of I is understood.

Let Q be a quiver. A quiver representation $R = (R_v, r_a)$ consists of a vector space R_v for each $v \in Q_{vx}$ and a morphism of vector spaces $r_a : R_{s(a)} \to R_{t(a)}$ for each $a \in Q_{ar}$. For a bound quivers (Q, I), a representation $R = (R_v, r_a)$ is same as above, with the additional condition that

$$\sum_{i=1}^{n} k_i r_{p_i} = 0$$

if $\sum_{i=1}^{n} k_i p_i$ is a generator of I. A subrepresentation of R is a pair $R' = (R'_v, r'_a)$ where R'_v is a subspace of R_v for each $v \in Q_{vx}$ and $r'_a : R'_{s(a)} \to R'_{t(a)}$ a morphism of vector spaces for each $a \in Q_{ar}$ such that

$$r'_a = r_a|_{R'_{s(a)}}$$

and

$$r_a(R'_{s(a)}) \subset R'_{t(a)} \tag{2.4}$$

Thus we have the commutative diagram

$$\begin{array}{ccc} R'_i & \xrightarrow{r'_a} & R'_j \\ \downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \iota_j} \\ R_i & \xrightarrow{r_a} & R_j \end{array}$$

for any arrow a from i to j. We use $R' \subset R$ to denote that R' is a subrepresentation of R.

If the vertices of a quiver has a natural ordering, as it will be the case when we discuss quiver of sections of an exceptional collection of line bundles, we define dimension vector $\vec{d}$ so d_i is the dimension of the vector space R_i at that vertex. We call the set of vertices where R_v has positive dimension the support of R.

In this paper, we are particularly interested in representations with dimension vector $\mathbf{1} = (1, \ldots, 1)$. Notice when R is a representation with dimension vector $\mathbf{1}$, and $R' \subset R$, all the inclusion maps

$$\iota_k : R'_k \to R_k$$

are either zero map or identity. We prove the following easy lemma:

Lemma 2.5. *Let (Q, I) be a bound quivers whose vertices are label by $\{1, 2, \ldots, n\}$ and R be a representation of Q with dimension vector $\mathbf{1}$.Then any subrepresentation R' is determined by its dimension vector $\vec{d}$. Moreover, a vector $\vec{d}$ of size n with entries 0 and 1 is the dimension vector of a subrepresentation of R if and only if $r_a = 0$ for all $a \in Q_{ar}$ with $d_{s(a)} = 1$ and $d_{t(a)} = 0$.*

Proof. Since $\dim R_i = 1$, its subspaces are determined by dimensions. Moreover, we see the morphism of subspaces r'_a are restrictions of r_a, hence the dimension vector $\vec{d}$ determines R'_i for all $i \in Q_{vx}$.
Given any vector $\vec{d}$ as in the second part of the lemma, it is the dimension of a vector subspace if (2.4) is satisfied. Note (2.4) is always true unless for arrows with $d_{s(a)} = 1$ and $d_{t(a)} = 0$, in which case we must have $r_a = 0$. $\square$

2.3. **Moduli space of semistable representations of a quiver.** See also [King],[Re]. Given a bound quivers (Q, I),a weight is an element $\theta \in \mathbb{Z}^N$ where $N = |Q_{vx}|$ such that $\sum_{i=1}^{N} \theta_i = 0$. Let $\theta = (\theta_1, \ldots, \theta_N)$ be a weight, we defined its toric form to be

$$(-\theta_1, -\theta_1 - \theta_2, \ldots, -\theta_1 - \theta_2 - \ldots - \theta_{N-1}) \in \mathbb{Z}^{n-1}$$

It is an easy exercise to see that one can recover a weight from its toric form.

Definition 2.6. A weight is admissible if every entry of its toric form is a positive integer.

For a weight θ, the weight function is defined by by:

$$\theta(S) = \sum_{i=1}^{N} d_i\theta_i$$

where S is a representation of Q and d_i and θ_i are the i-th entries of $\vec{d}$ and θ respectively. We recall the definition of semi-stability:

Definition 2.7. A representation R is θ-semistabe if for any subrepresentation $R' \subset R$

$$\theta(R') \geq 0$$

R is θ-stable if all the above inequalities are strict.

We restrict our attention to R with dimension vector $\mathbf{1}$. Given a bound quivers (Q, I), we can associate to it an affine shceme Rep(Q) called the representation scheme of (Q, I). The coordinate ring of this affine shceme is the quotient of $\mathbf{k}[a \in Q_{ar}]$ by the ideal J which is generated by generators $\sum_{i=1}^{n} k_i p_i$ of I treated as elements in the above polynomial ring. It is obvious from the definition that closed points of representation scheme are in 1-to-1 correspondence with representations of Q with dimension vector $\mathbf{1}$. For a weight θ, the set of θ-semistable representations forms an open subscheme $\mathrm{Rep(Q)}_\theta^{SS}$ of Rep(Q), the set of θ-stable representations forms an open subscheme $\mathrm{Rep(Q)}_\theta^{S}$ of $\mathrm{Rep(Q)}_\theta^{SS}$.

The group $(\mathbf{k}^*)^{Q_{vx}}$ acts by incidence on Rep(Q), in other words, it acts by $(g \cdot a) = g_{t(a)} r_a g_{s(a)}^{-1}$. Apparently, the diagonal subgroup $\mathbf{k}^*_{\mathrm{diag}}$ of $(\mathbf{k}^*)^{Q_{vx}}$ consisting of elements of the form $(k, k, \ldots, k)$ for $k \in \mathbf{k}^*$ acts trivially on Rep(Q). So it is natural to only consider the action of $\mathrm{PGL}(\mathbf{1}) := (\mathbf{k}^*)^{Q_{vx}}/\mathbf{k}^*_{\mathrm{diag}}$.

Definition 2.8. Two representations of dimension vector $\mathbf{1}$ are isomorphic if they are in the same orbit under the action of $\mathrm{PGL}(\mathbf{1})$.

Give a weight θ, the moduli space of θ-semistable representation with dimension vector $\mathbf{1}$ is the GIT quotient

$$\begin{aligned} M_\theta :&= \mathrm{Rep(Q)}//_\theta \mathrm{PGL}(\mathbf{1}) \\ &= \mathrm{Rep(Q)}_\theta^{SS}//\mathrm{PGL}(\mathbf{1}) \end{aligned}$$

We mention a few facts about M_θ. For details, the readers are referred to [King]. An equivalent definition of M_θ is to consider the graded ring

$$B_\theta = \bigoplus_{r \geq 0} B(r\theta)$$

where $B(r\theta)$ is $r\theta$-semi-invariant functions in the coordinate ring of Rep(Q). Then the GIT quotient is defined as

$$M_\theta = \mathrm{Proj}(B_\theta)$$

From this definition, it is easy to see that M_θ is a reduced projective scheme. Note if all θ-semistable representations are θ-stable, i.e. $\mathrm{Rep(Q)}_\theta^{SS} = \mathrm{Rep(Q)}_\theta^{S}$, then M_θ is the fine moduli space of θ-stable representations, in particular, the closed points of M_θ are in 1-to-1 correspondence with the isomorphism classes of θ-stable representations. We now give an easy criterion for obtaining fine moduli spaces as above.

Lemma 2.9. *With the notions above, if for any proper nonempty subset P of Q_{vx}, we have $\sum_{i\in P}\theta_i \neq 0$, then any semistable representation R is in fact stable. In particular, M_θ is a fine moduli space.*

Proof. If R is strictly semistable, then there exist a proper nonzero subrepresentation R' such that $\theta(R') = \sum_{i\in supp(R')}\theta_i = 0$, but this cannot happen given the conditions in the statement. $\square$

2.4. **Quivers of Sections.** The main reference for this section is Craw-Smith[CS] and Craw-Winn[CW]. We mention that our indexing is different since we are concerned with the quiver with path algebra $\mathcal{A}^{op}$ instead of $\mathcal{A}$ as in the introduction.
Let $\mathcal{L} = \{L_1, L_2, \dots, L_n\}$ be a collection of line bundles on a projective variety X. For $1 \leq i, j \leq n$, we call a section $s \in H^0(X, L_j^\vee \otimes L_i)$ irreducible if s does not lie in the images of the multiplication map

$$H^0(X, L_j^\vee \otimes L_k) \otimes_{\mathbf{k}} H^0(X, L_k^\vee \otimes L_i) \to H^0(X, L_j^\vee \otimes L_i)$$

for $k \neq i, j$.

Definition 2.10. The quiver of sections of the collection $\mathcal{L}$ on X is defined to be a quiver with vertex set $Q_{vx} = \{1, \dots, n\}$ and where the arrows from i to j corresponds to a basis of irreducible sections of $H^0(X, L^\vee_{(n+1)-j} \otimes L_{(n+1)-i})$.

We mention one of the basic properties of a quiver of sections.

Lemma 2.11. [CW] *The quiver of sections Q is connected, acyclic and $1 \in Q_{vx}$ is the unique source.*

The quiver of sections only include information about the sections in $H^0(X, L^\vee_{(n+1)-j} \otimes L_{(n+1)-i})$, but left relations between them behind. We now define a two sided ideal

Definition 2.12. Let $I_{\mathcal{L}}$ be a two sided ideal in $\mathbf{k}Q$

$$I_{\mathcal{L}} = \big(\sum_{k=1}^{N} a_k p_k | p_k \text{ are paths from } i \text{ to } j \text{ and } \sum_{k=1}^{N} a_k p_k \text{ represents } 0 \text{ in } H^0(X, L^\vee_{(n+1)-j} \otimes L_{(n+1)-i})\big)$$

We call the pair $(Q, I_{\mathcal{L}})$ the bound quiver of sections of the collection $\mathcal{L}$.

Proposition 2.13. [CS][CW] *The quotient algebra $\mathbf{k}Q/I_{\mathcal{L}}$ is isomorphic to $\mathcal{A}^{op} = \mathrm{End}_{\mathcal{O}_X}(\oplus_{i=1}^n L_i^\vee)$ and for $1 \leq i, j \leq n$, we have $e_j(\mathbf{k}Q/I_{\mathcal{L}})e_i \cong H^0(X, L^\vee_{(n+1)-j} \otimes L_{(n+1)-i})$.*

Given any weight θ for $(Q, I_{\mathcal{L}})$, we can consider the moduli space of semistable representations M_θ. There is a tautological rational map

$$T : X \dashrightarrow M_\theta$$

so that if T is defined at x, then

$$T(x) = \bigoplus_{i=0}^{n} (L_i^\vee)_x$$

Moreover, T is defined at x if $\bigoplus_{i=0}^n (L_i^\vee)_x$ can be represented by a θ-semistable representation.

2.5. **Kronecker Quiver.** Fix $n \geq 2$. Let Q_0 be the $(n+1)$-Kronecker quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left=1.5, "x_0"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=1.5, "x_n"'] & 2 \end{tikzcd}$$

For $p \in \mathbb{Z}_{>0}$, let $\theta_0 = (-p, p)$.Let M_{θ_0} be the moduli space of semistable representations of Q_0 with dimension vector $(1,1)$. The following fact is well known:

Proposition 2.14. *The tautological rational map:*

$$T_0 : \mathbb{P}^n \dashrightarrow M_{\theta_0}$$

is an isomorphism.

Proof. By Lemma 2.5, a representation R of Q_0 is unstable if and only if $r_{x_i} = 0$ for all i. Note for $[a_0 : \ldots : a_n] \in \mathbb{P}^n$,

$$T_0([a_0 : \ldots : a_n]) = [(a_0, \ldots, a_m)]$$

where $[(a_0, \ldots, a_m)]$ is the orbit of the representation with a_i as the value of x_i. Since at least one of a_i is nonzero, T_0 is defined on all of $\mathbb{P}^n$. Moreover, it is clear that T_0 is one-to-one and onto, so it is an isomorphism. □

Remark 2.15. Note we can identify

$$B(k\theta_0) = \mathbf{k}[x_0^{d_0} \ldots x_n^{d_n}, d_0 + \ldots d_n = kp]$$

with a subspace vector space of the graded algebra $k[x_0, \ldots, x_n]$. Then

$$\bigoplus_{k \geq 0} B(k\theta)$$

is the algebra that corresponds to the p-uple embedding of $\mathbb{P}^n$.

3. Proof of Theorem 1.3

We consider the following collection of line bundles on P^n_m of length 3

$$\{\mathcal{O}_{P^n_m}, \mathcal{O}_{P^n_m}(H - E), \mathcal{O}_{P^n_m}(H)\}$$

Using Lemma 2.1, we see this is a strong exceptional collection of line bundles. It is clearly not full due to its small length. Its quiver of sections Q has no relations and is given by

$$\begin{tikzcd} 1 \arrow[rr, shift left=1.5, "x_0"] \arrow[rr, phantom, "\vdots"] \arrow[rr, shift right=1.5, "x_m"'] \arrow[dr, "e"'] & & 3 \\ & 2 \arrow[ur, shift left=1.5, "x_{m+1}"] \arrow[ur, phantom, "\ddots"] \arrow[ur, shift right=1.5, "x_n"'] & \end{tikzcd}$$

Let $p, q \in \mathbb{Z}_{>0}$ and $p < q$. We define $\theta = (-p, p-q, q)$ and $\theta_0 = (-p, p)$. Let M_θ be the moduli space of semistable representation of Q with dimension vector $(1,1,1)$.

Lemma 3.1. *M_θ is in fact the fine moduli space of stable representation of Q with dimension vector $(1,1,1)$. It is a smooth projective variety.*

Proof. By Lemma 2.9, M_θ is the same as the moduli space of stable representations of Q with dimension vector $(1,1,1)$.
The second part of the lemma follows from [King] □

Theorem 3.2. *There is a natural surjective morphism*

$$F : \mathrm{Rep(Q)} \to \mathrm{Rep(Q_0)}$$

Proof. We define a $\mathbf{k}$-algebra homomorphism $\phi : \mathbf{k}[X_0, \ldots, X_n] \to \mathbf{k}[x_0, \ldots, x_n, e]$ between the coordinate rings of the two affine scheme as follows:

$$\phi(X_i) = \begin{cases} x_i & \text{if } i \leq m \\ ex_i & \text{if } i > m \end{cases}$$

We let F be the corresponding morphism between affine schemes.

To see F is surjective, we note for $(a_0, \ldots, a_n) \in Rep(Q_0)$, we have

$$F(a_0, \ldots, a_n, 1) = (a_0, \ldots, a_n)$$

□

The next proposition shows F respects the PGL($\mathbf{1}$)- action.

Remark 3.3. From now on, if $(a_1, \ldots, a_n)$ is a semistable point in the affine representation scheme, we will use $[(a_1, \ldots, a_n)]$ to denote its orbit in the moduli space.

Proposition 3.4. *Let R_1,R_2 be two representations of Q with dimension vector* $\mathbf{1}$. *Suppose $R_1 \sim R_2$, via the element (g_1, g_2, g_3), then $F(R_1) \sim F(R_2)$.*

Proof. From the construction of F, one directly check the element

$$(g_1, g_3)$$

provides the equivalence. □

Let U consists of representations of Q so that $r_e \neq 0$.

Lemma 3.5. *Suppose $R \in U$, then R is θ-stable if and only if*

$$r_{x_i} \neq 0$$

for at least one $m + 1 \leq i \leq n$

Proof. Suppose

$$r_{x_i} = 0$$

for all $m + 1 \leq i \leq n$, there exist a subrepresentation $S \subset R$ with dimension vector $(0, 1, 0)$. Then

$$\theta(S) = p - q < 0$$

For the other direction, if

$$r_{x_i} \neq 0$$

for at least one $m+1 \leq i \leq n$, a nontrivial proper subrepresentation of S can only have one of the following dimension vectors

- $(0, 1, 1)$
- $(0, 0, 1)$

one easily check then R is stable. □

Proposition 3.6. *Suppose $R_1, R_2 \in U$, and $F(R_1) \sim F(R_2)$ under the action of (g_1, g_3), then $R_1 \sim R_2$.*

Proof. Let e_i denote the value of e in R_i for $i = 1, 2$, then $e_1 e_2 \neq 0$. Again by the construction of F, one directly checks that

$$\Big(g_1 e_2, , g_3 e_1, g_3 e_2\Big)$$

provides the equivalence. □

Lemma 3.7. *Suppose $R \in \mathbf{V}(e)$, then R is θ-stable if and only if*

$$r_{x_i} \neq 0$$

for at least one $0 \leq i \leq m$ and at least one $m+1 \leq i \leq n$.

Proof. Suppose $r_{x_i} = 0$ for all $0 \leq i \leq m$, there exist a subrepresentation $S \subset R$ with dimension vector $(1, 0, 0)$, then

$$\theta(S) = -p < 0$$

Suppose $r_{x_i} = 0$ for all $m+1 \leq i \leq n$, there exist a subrepresentation $S \subset R$ with dimension vector $(0, 1, 0)$, then

$$\theta(S) = p - q < 0$$

For the other direction, suppose

$$r_{x_i} \neq 0$$

for at least one $0 \leq i \leq m$ and at least one $m+1 \leq i \leq n$, a nontrivial proper subrepresentation of S can only have one of the following dimension vectors

- $(0, 1, 1)$
- $(1, 0, 1)$
- $(0, 0, 1)$

one easily check then R is stable. □

Corollary 3.8. *The tautological rational map T is a morphism.*

Proof. Let $x \in P^n_m$. If $x \in E \cong \mathbb{P}^m \times \mathbb{P}^{n-m-1}$, suppose $x = ([a_0, \ldots, a_m], [b_{m+1}, \ldots, b_n])$, then $T(x) = [(a_0, \ldots, a_m, b_{m+1}, \ldots, b_n, 0)]$. Since at least one of a_i and at least one of b_i is nonzero, by Lemma 3.7, $T(x)$ is stable.
If $x \in P^n_m \backslash E$, then we can write $x = \pi^{-1}([a_0, \ldots, a_n])$, where $a_i \neq 0$ for at least one $m+1 \leq i \leq n$ since $x \notin E$. Then $T(x) = [(a_0, \ldots, a_n, 1)]$. By Lemma 3.5, $T(x)$ is stable. □

Proposition 3.9. *The natural morphism*

$$F : \mathrm{Rep(Q)} \to \mathrm{Rep(Q_0)}$$

descends to a projective morphism

$$f : M_\theta \to M_{\theta_0}$$

which fits into a commutative diagram

$$\begin{array}{ccc} P^n_m & \xrightarrow{T} & M_\theta \\ \Big\downarrow \pi & & \Big\downarrow f \\ \mathbb{P}^n & \xrightarrow{T_0} & M_{\theta_0} \end{array}$$

Proof. By Lemma 3.4, Lemma 3.5 and Lemma 3.7, F descends to

$$f : M_\theta \to M_{\theta_0}$$

f is projective since it is a morphism between projective schemes.

Let $x \in P^n_m$. If $x \in E \cong \mathbb{P}^m \times \mathbb{P}^{n-m-1}$, suppose $x = ([a_0, \dots, a_m], [b_{m+1}, \dots, b_n])$, then $T(x) = [(a_0, \dots, a_m, b_{m+1}, \dots, b_n, 0)]$. Hence

$$F \circ T(x) = [(a_0, \dots, a_m, 0, \dots, 0)]$$

Now $\pi(x) = [a_0, \dots, a_m, 0 \dots, 0]$

$$\begin{aligned} T_0 \circ \pi(x) &= [(a_0, \dots, a_m, 0 \dots, 0)] \\ &= F \circ T(x) \end{aligned}$$

If $x \in P^n_m \backslash E$, then we can write $x = \pi^{-1}([a_0, \dots, a_n]$. Then $T(x) = [(a_0, \dots, a_n, 1)]$ and

$$\begin{aligned} F \circ T(x) &= [(a_0, \dots, a_n)] \\ &= T_0 \circ \pi(x) \end{aligned}$$

□

Let C denote the closed subscheme of M_θ containing stable orbits of representations with $r_e = 0$, i.e.

$$C = \mathbf{V}(e)^S // \mathrm{PGL}(\mathbf{1})$$

where $\mathbf{V}(e)^S$ is the open subscheme of $\mathbf{V}(e)$ consisting of stable representations.

Proposition 3.10. *We have $C \cong \mathbb{P}^m \times \mathbb{P}^{n-m-1} \cong \mathbb{L} \times \mathbb{P}^{n-m-1}$ and $f|C$ is the projection to $\mathbb{L}$*

Proof. By Lemma 3.7,

$$\mathbf{V}(e)^S = (\mathbb{A}^{m+1} \backslash 0) \times (\mathbb{A}^{n-m} \backslash 0)$$

For $(1, g_2, g_3) \in \mathrm{PGL}(\mathbf{1})$, it acts on $\mathbf{V}(e)^S$ by letting g_2 acts on the first component via scalar multiplication and g_3 acts on the second in the same way. Thus

$$\begin{aligned} C &= \mathbf{V}(e)^S // \mathrm{PGL}(\mathbf{1}) \\ &= (\mathbb{A}^{m+1} \backslash 0) // k^* \times (\mathbb{A}^{n-m} \backslash 0) // k^* \\ &= \mathbb{P}^m \times \mathbb{P}^{n-m-1} \end{aligned}$$

The fact that $f|C$ is the projection to the first component follows directly from the definition of F. □

Corollary 3.11. *f induces an isomorphism between $M_\theta \backslash C$ and $M_{\theta_0} \backslash T_0(\mathbb{L})$, thus f is a birational morphism.*

Proof. By Lemma 3.6, $f|_{M_\theta \backslash C}$ is injective. By Proposition 3.9, f is surjective. By Proposition 3.10, $f|_{M_\theta \backslash C} : M_\theta \backslash C \to M_{\theta_0} \backslash T_0(\mathbb{L})$ is also surjective. Since both $M_\theta \backslash C$ and $M_{\theta_0} \backslash T_0(\mathbb{L})$ are smooth varieties, $f|_{M_\theta \backslash C}$ is an isomorphism by Zariski Main Theorem.

□

Theorem 3.12. *$T : P^n_m \to M_\theta$ is an isomorphism and $f : M_\theta \to M_{\theta_0}$ is the blow down along $T_0(\mathbb{L})$.*

Proof. By Corollary 3.11, $f : M_\theta \to M_{\theta_0} \cong \mathbb{P}^n$ is a proper birational morphism between smooth projective varieties. So by weak factorization theorem, f can be factored into a sequence of blow ups with centers disjoint from $\mathbb{P}^n \backslash \mathbb{L}$. But since $C = \mathbb{L} \times \mathbb{P}^{n-m-1}$. f is the blow up with center $\mathbb{L}$.
Now the fact that T is an isomorphism follows immediately from universal property of blow ups. □

Remark 3.13. We would like to mention that [Fei] constructed similar birational maps between quiver moduli in more general setting using categorical techniques.

Proof of Theorem 1.3. We collection what we have proved. By Lemma 2.1, the collection $\{\mathcal{O}_{P^n_m}, \mathcal{O}_{P^n_m}(H-E), \mathcal{O}_{P^n_m}(H)\}$ is strong exceptional. Taking $\theta = (-p, p-q, q)$ for $p, q \in \mathbb{Z}_{>0}$, $p < q$ as in this section, we can apply Lemma 2.9 to show M_θ is a fine moduli space. Finally Theorem 3.12 shows T is an isomorphism. □

4. Proof of Theorem 1.5

Let $\theta' = (-p, 0, p)$.

Lemma 4.1. *R is θ'-semistable if and only if*

$$r_{x_i} \neq 0$$

for at least one $0 \le i \le m$ OR

$$r_{x_i} r_e \neq 0$$

for at least one $m+1 \le i \le n$.

Proof. By Lemma 2.5, a representation R of Q is θ'-unstable if and only if it has subrepresentation with dimension either $(1,0,0)$ or $(1,1,0)$, which in turn is equivalent to the conditions in the statement of the lemma. □

Remark 4.2. We note $M_{\theta'}$ is a coarse moduli space with strictly semistable representations. For example $(x_1, \ldots, x_m, 0, \ldots, 0, e)$ with $x_1 \neq 0$ has a subrepresentation with dimension vector $(0,1,0)$, which makes it not stable.

Proof of Theorem 1.5. To show $M_{\theta_0} \cong \mathbb{P}^n$, simply notice $B(\lambda\theta_0)$ is the $\mathbf{k}$ span of

$$\{x_1^{a_1} \ldots x_m^{a_m} (x_{m+1}e)^{a_{m+1}} \ldots (x_n e)^{a_n}\}$$

for $a_1 + \ldots + a_n = p$. Thus

$$M_{\theta_0} \cong \operatorname{Proj} \bigoplus_{\lambda \ge 0} B(\lambda\theta_0)$$

corresponds to the p-upple embedding of $\mathbb{P}^n$ having $x_1, \ldots, x_m, (x_{m+1}e), \ldots, (x_n e)$ as projective coordinates.
To show $id : Rep(Q) \to Rep(Q)$ descents to a morphism, we need to check if R is θ-stable, then R is θ_0-semistable. If $R \subset U$, then R is θ-stable implies $r_{x_i} \neq 0$ for at least one $m+1 \le i \le n$ by Lemma 3.5, which implies $r_e r_{x_i} \neq 0$ for at least one $m+1 \le i \le n$, thus R is θ_0-semistable by Lemma 4.1.

If $r_e = 0$, then R is θ-stable implies $r_{x_i} \neq 0$ for at least one $1 \le i \le m$ by Lemma 3.7, which in turn implies R is θ_0-semistable by Lemma 4.1. Thus id descends.

To see the induces morphism $\pi' : M_\theta \to M_{\theta_0}$ is the natural projection, we recall that $M_{\theta_0} \cong \mathbb{P}^n$ has $x_1, \ldots, x_m, (x_{m+1}e), \ldots, (x_n e)$ as projective coordinates. Then following the

definition of f and the proof of Corollary 3.11, we see $\pi'|_{M_\theta \setminus C}$ is an isomorphism.
It remains to observe that for $[a_1, \dots, a_m, b_{m+1} \dots, b_n, 0] \in C \cong \mathbb{P}^m \times \mathbb{P}^{n-m-1}$,

$$\pi'([a_1, \dots, a_m, b_{m+1} \dots, b_n, 0]) = [a_1, \dots, a_m, 0, \dots, 0]$$

where the right hand side is written in the coordinates of $x_1, \dots, x_m, (x_{m+1}e), \dots, (x_n e)$.
Thus π' is the blow down morphism. $\square$

Remark 4.3. The reason for *id* to descent to a contraction is the existence of strictly semistable representations in M_{θ_0}. There are two kinds of these representations:

- $[a_1, \dots, a_n, 0]$ where at least one of $a_i \neq 0$ for $1 \leq i \leq m$.
- $[b_1, \dots, b_m, 0, \dots, 0, e]$ where at least one of $b_i \neq 0$ for $1 \leq i \leq m$.

Notice the intersection of these two types are representations $[c_1, \dots, c_n, 0, \dots, 0, 0]$. Moreover, letting $(1, k, 1)$ acts on the first kind and $(1, 1/k, 1)$ acts on the second, and $k \to \infty$ we see there is a representation of form $[c_1, \dots, c_n, 0, \dots, 0, 0]$ lying in the orbit closure of any strictly semistable representation.
In particular, for all representation in $[a_1, \dots, a_m] \times \mathbb{P}^{n-m-1} \subset C \subset M_\theta$, their orbit closures in M_{θ_0} contain $[a_1, \dots, a_m, 0, \dots, 0, 0]$, thus they are represented by one single point in M_{θ_0}.

5. VGIT for full exceptional collection

In this section we apply the idea from the previous section to a full strong exceptional collection on P^n_m. This provides an example of VGIT to the work of [CG]. Recall from [Cr], P^n_m is a quiver flag variety by viewing it as the projective bundle $\mathbb{P}_{\mathbb{P}^{n-m-1}}(\mathcal{O}^{m+1}_{\mathbb{P}^{n-m-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-m-1}}(1))$. It has two tautological bundles $\mathcal{W}_1 = \mathcal{O}_{P^n_m}(H - E)$ and $\mathcal{W}_2 = \mathcal{O}_{P^n_m}(H)$. By [Cr, Theorem 4.5], the collection of line bundles

$$\left\{ \mathcal{W}_1^{\alpha_1} \otimes \mathcal{W}_2^{\alpha_2} : 0 \leq \alpha_1 < n - m, 0 \leq \alpha_2 < m + 2 \right\} \tag{5.1}$$

admits an order so that the resulting sequence is a full strong exceptional collection (c.f. [LYY, Theorem 4.4(4), 5.5(2)] for $n = 3$). It is clear that the sequence starts with $\{\mathcal{O}_{P^n_m}, \mathcal{O}_{P^n_m}(H - E), \mathcal{O}_{P^n_m}(H) \dots\}$. We again remind the reader of the different conventions between our notion of bound quiver and that in [Cr] and [CG]. For this section only, we will use the convention in [Cr] and [CG] since we are generalizing the results in those. Let Q' denote the bound quiver corresponding to (5.1). Note Q' has $N' + 1 = (n - m)(m + 2)$ vertices. We draw Q' near the its unique source:

$$\begin{tikzcd}
0 \arrow[rr, shift left, "x_0"] \arrow[rr, shift right, "x_m"'] \arrow[dr, shift left, "x_{m+1}"] \arrow[dr, shift right, "x_n"'] & & 2 \cdots\cdots \\
& 1 \arrow[ur, "e"'] &
\end{tikzcd}$$

Let $v' = (1, \dots, 1)$ and $\Theta = (-N', 1, \dots, 1)$, both having $N' + 1$ entries. Use $M(Q', v', \Theta)$ to denote the moduli space of Θ-semistable representations of Q' with dimension vector v'. Then by the definition of quiver flag varieties and their tautological bundles, there is a universal morphism [Cr]

$$U : P^n_m \to M(Q', v', \Theta)$$

. In fact, this construction generalizes our definition of the tautological map (1.1), except we need to remove all duals in (1.1). By [CG, Theorem 1.2], the universal morphism $U : P^n_m \to M(Q', v', \Theta)$ is an isomorphism.

Take $\Theta_\omega = (-N', \omega, N' - \omega, 0, \dots, 0)$ for $0 \le \omega \le N'$ and $\omega \in \mathbb{Z}$, which can be think of as a variation of Θ by pushing the positive weights to the second and third entry. Let $M(Q', v', \Theta_\omega)$ be denote the moduli space of Θ_ω-semistable representations of Q' with dimension vector v'.

Proposition 5.2. *There is a natural morphism:*

$$g_\omega : M(Q', v', \Theta) \to M(Q', v', \Theta_\omega)$$

for all integral $0 \le \omega \le N'$.

Proof. Let $\mathrm{Rep}(Q', \Theta)$ ($\mathrm{Rep}(Q', \Theta_\omega)$)denotes the set of $\Theta(\Theta_\omega)$-semistable representations of Q' with dimension v'. To construct g_ω, it suffices to construct a morphism $G : \mathrm{Rep}(Q', \Theta) \to \mathrm{Rep}(Q', \Theta_\omega)$. This amounts to showing any representation in $\mathrm{Rep}(Q', \Theta)$ is also Θ_ω-semistable.

Let R be a representation of Q' with dimension vector v'. Then R is Θ_ω-unstable if and only if R has a subrep S of dimension vector either $(1, 1, 0, 1, \dots, 1)$, $(1, 0, 1, 1, \dots, 1)$ or $(1, 0, 0, 1, \dots, 1)$ such that $\Theta_\omega(S) < 0$. In any case $\Theta(S) < 0$, which makes R Θ-unstable. Hence we have a natural morphism g_ω. □

We now analyze the two boundary case: $\omega = 0$ and $\omega = 5$.

Theorem 5.3. *$M(Q', v', \Theta_0)$ is isomorphic to $\mathbb{P}^n$. Moreover, the natural morphism*

$$g_0 : M(Q', v', \Theta) \to M(Q', v', \Theta_0)$$

coincides with the blow up morphism of $\mathbb{P}^n$ at the linear space $\mathbb{L}$.

Proof. The first statement follows from the same argument as in the beginning of proof of Theorem 1.5.

Identifying $\mathrm{Rep}(Q', \Theta_0)$ with $\mathbb{P}^n$ by the proof of Theorem 1.5, we see

$$\begin{aligned} g_0 : M(Q', v', \Theta) &\to M(Q', v', \Theta_\omega) \cong \mathbb{P}^n \\ [R] &\to [r_{x_0} : \dots : r_{x_m} : r_{x_{m+1}} r_e : \dots, r_{x_n} r_e] \end{aligned}$$

Just as in the previous two sections, if we identify $M(Q', v', \Theta)$ with P^n_m via u, then $[R] \in U$ if $r_e \neq 0$ and $[R] \in C$ in $r_e = 0$. Moreover if $\pi : P^n_m \to \mathbb{P}^n$ is the blow up morphism, then following the construction of the universal morphism and multigraded linear series (also compare Section 4), we have

$$\begin{aligned} \pi \circ u^{-1} : M(Q', v', \Theta) &\to \mathbb{P}^n \\ [R] &\to [r_{x_0} : \dots : r_{x_m} : r_{x_{m+1}} r_e : \dots, r_{x_n} r_e] \end{aligned}$$

Comparing g_0 with $\pi \circ u^{-1}$ finishes the proof. □

Remark 5.4 (c.f. Remark 4.3). To understand how contraction happens, consider a representation R of Q' with dimension vector v' such that $r_{x_j} = 0$ for all $m + 1 \le j \le n$ but $r_{x_i} \neq 0$ for some $0 \le i \le m$. Then R is Θ_0-semistable. It is clear that R has a subrep S of dimension vector $(1, 0, 1, \dots, 1)$. Since $\Theta(S) < 0$. Thus R is Θ-unstable. However $\Theta_0(S) = 0$, so R is strictly Θ_0-semistable. By Theorem 5.2, it is clear that $[R] \in M(Q', v', \Theta_0) \cong \mathbb{P}^n$ corresponds to a point in $\mathbb{L}$, the linear subspace we blow up. Let $g_\lambda = (1, \lambda^{-1}, 1, \dots, 1) \in \mathrm{PGL}(\mathbf{1})$, then

$$\lim_{\lambda \to 0} g_\lambda \cdot R$$

is the Θ_ω-polystable representation in the orbit closure of R.

Theorem 5.5. *$M(Q', v', \Theta_5)$ is isomorphic to $\mathbb{P}^{n-m-1}$. Moreover, the natural morphism*

$$g_5 : M(Q', v', \Theta) \to M(Q', v', \Theta_5)$$

coincides with the projection of $P^n_m = \mathbb{P}_{\mathbb{P}^{n-m-1}}(\mathcal{O}^{m+1}_{\mathbb{P}^{n-m-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-m-1}}(1))$ as a projective bundle over its base.

Proof. To show $M_{\Theta_5} \cong \mathbb{P}^{n-m-1}$, notice $B(\lambda\Theta_5)$ is the $\mathbf{k}$ span of

$$\{x_{m+1}^{a_{m+1}} \dots x_n^{a_n}\}$$

for $a_i \in \mathbb{N}$ and $a_{m+1} + \dots + a_n = 5$. Thus

$$M_{\Theta_5} \cong \operatorname{Proj} \bigoplus_{\lambda \geq 0} B(\lambda\Theta_5)$$

corresponds to the 5-upple embedding of $\mathbb{P}^{n-m-1}$ having $x_{m+1}, \dots, x_n$ as projective coordinates.
Now we see

$$\begin{aligned} g_5 : M(Q', v', \Theta) &\to M(Q', v', \Theta_5) \cong \mathbb{P}^{n-m-1} \\ [R] &\to [r_{x_{m+1}} : \dots, r_{x_n}] \end{aligned}$$

Moreover if $\pi' : P^n_m \to \mathbb{P}^{n-m-1}$ is the projection to the base, then following the construction of the universal morphism and multigraded linear series (also compare Section 4), we have

$$\begin{aligned} \pi' \circ u^{-1} : M(Q', v', \Theta) &\to \mathbb{P}^{n-m-1} \\ [R] &\to [r_{x_{m+1}} : \dots, r_{x_n}] \end{aligned}$$

Comparing g_5 with $\pi' \circ u^{-1}$ finishes the proof. □

Remark 5.6.

- For $0 < \omega < 5$, one can easily adapt arguments in Section 3 to see

$$g_\omega : M(Q', v', \Theta) \to M(Q', v', \Theta_\omega)$$

  is in fact an isomorphism.
- We can allow $\omega \in \mathbb{Q}$ (still insisting $0 \leq \omega \leq 5$) by defining the moduli space to be $M(Q', v', l\Theta_\omega)$ where l is the least positive integer so that $l\Theta_\omega$ has integral entries. The first remark extends and we hence have the partial wall and chamber structure:

ω	$\omega = 0$	$0 < \omega < 5$	$\omega = 5$
$M(Q', v', \Theta_\omega)$	$\mathbb{P}^n$	P^n_m	$\mathbb{P}^{n-m-1}$

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