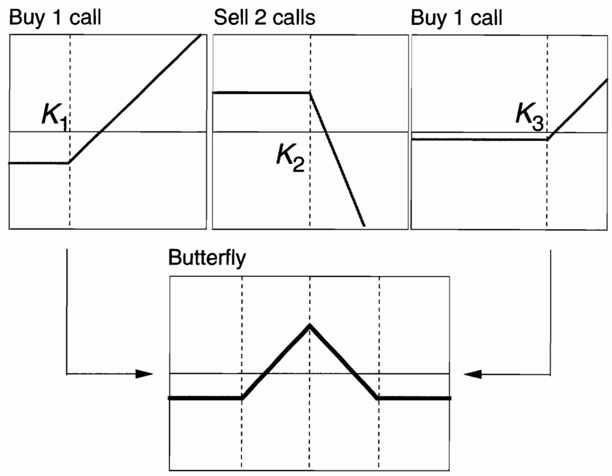

FIGURE 8.7 Creating a Butterfly Spread

same spacing. Figure 8.7 shows that this position is expected to benefit when the underlying asset price stays stable, close to K_2. The double position in the middle is called the body, and the others the wings. A sandwich spread is the opposite of a butterfly spread.

EXAMPLE 8.5: RISK OF OPTION CONTRACTS

Which of the following is the riskiest form of speculation using option contracts?

a. Setting up a spread using call options
b. Buying put options
c. Writing naked call options
d. Writing naked put options

EXAMPLE 8.6: FRM EXAM 2007—QUESTION 103

An investor sells a June 2008 call of ABC Limited with a strike price of USD 45 for USD 3 and buys a June 2008 call of ABC Limited with a strike price of USD 40 for USD 5. What is the name of this strategy and the maximum profit and loss the investor could incur?

a. Bear spread, maximum loss USD 2, maximum profit USD 3
b. Bull spread, maximum loss unlimited, maximum profit USD 3
c. Bear spread, maximum loss USD 2, maximum profit unlimited
d. Bull spread, maximum loss USD 2, maximum profit USD 3

EXAMPLE 8.7: FRM EXAM 2006—QUESTION 45

A portfolio manager wants to hedge his bond portfolio against changes in interest rates. He intends to buy a put option with a strike price below the portfolio's current price in order to protect against rising interest rates. He also wants to sell a call option with a strike price above the portfolio's current price in order to reduce the cost of buying the put option. What strategy is the manager using?

a. Bear spread
b. Strangle
c. Collar
d. Straddle

EXAMPLE 8.8: FRM EXAM 2002—QUESTION 42

Consider a bearish option strategy of buying one \$50 strike put for \$7, selling two \$42 strike puts for \$4 each, and buying one \$37 put for \$2. All options have the same maturity. Calculate the final profit per share of the strategy if the underlying is trading at \$33 at expiration.

a. \$1 per share
b. \$2 per share
c. \$3 per share
d. \$4 per share

EXAMPLE 8.9: FRM EXAM 2009—QUESTION 3-8

According to an in-house research report, it is expected that USDJPY (quoted as JPY/USD) will trade near 97 at the end of March. Frankie Shiller, the investment director of a house fund, decides to use an option strategy to capture this opportunity. The current level of the USDJPY exchange rate is 97 on February 28. Accordingly, which of the following strategies would be the most appropriate for the largest profit while the potential loss is limited?

a. Long a call option on USDJPY and long a put option on USDJPY with the same strike price of USDJPY 97 and expiration date
b. Long a call option on USDJPY with strike price of USDJPY 97 and short a call option on USDJPY with strike price of USDJPY 99 and the same expiration date
c. Short a call option on USDJPY and long a put option on USDJPY with the same strike price of USDJPY 97 and expiration date
d. Long a call option with strike price of USDJPY 96, long a call option with strike price of USDJPY 98, and sell two call options with strike price of USDJPY 97, all of them with the same expiration date

8.2 OPTION PREMIUMS

8.2.1 General Relationships

So far, we have examined the payoffs at expiration only. Also important is the instantaneous relationship between the option value and the current price S, which is displayed in Figures 8.8 and 8.9.

For a call, a higher price S increases the current value of the option, but in a nonlinear, convex fashion. For a put, lower values for S increase the value of the option, also in a convex fashion. As time goes by, the curved line approaches the hockey stick line.

Figures 8.8 and 8.9 decompose the current premium into

- An **intrinsic value**, which basically consists of the value of the option if exercised today, or $\mathrm{Max}(S_t - K, 0)$ for a call and $\mathrm{Max}(K - S_t, 0)$ for a put
- A **time value**, which consists of the remainder, reflecting the possibility that the option will create further gains in the future

Consider for example a one-year call with strike $K = \$100$. The current price is $S = \$120$ and interest rate $r = 5\%$. The asset pays no dividend. Say the call premium is \$26.17. This can be decomposed into an intrinsic value of $\$120 - \$100 = \$20$ and time value of \$6.17. The time value increases with the volatility of the underlying asset. It also generally increases with the maturity of the option.

As shown in the figures, options can be classified into:

- **At-the-money**, when the current spot price is close to the strike price
- **In-the-money**, when the intrinsic value is large

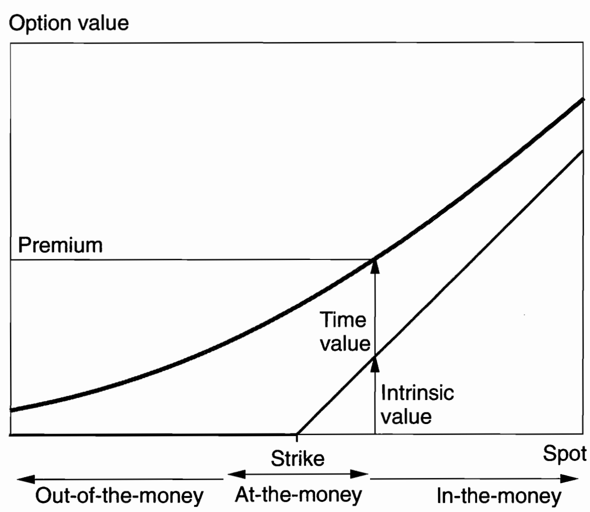

FIGURE 8.8 Relationship between Call Value and Spot Price

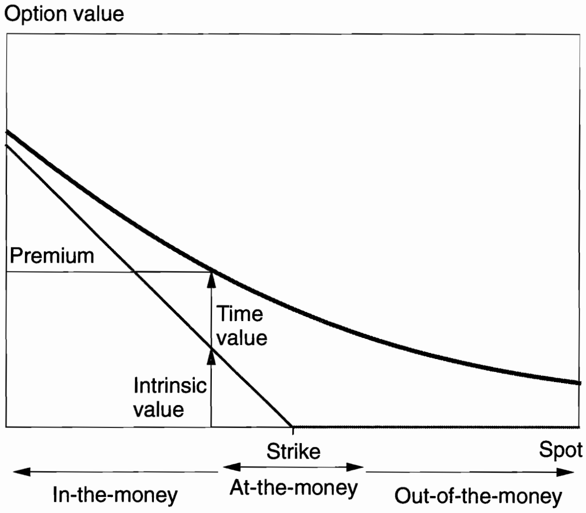

FIGURE 8.9 Relationship between Put Value and Spot Price

- **Out-of-the-money**, when the spot price is much below the strike price for calls and conversely for puts (out-of-the-money options have zero intrinsic value)

We can also identify some general bounds for European options that should always be satisfied; otherwise there would be an arbitrage opportunity (i.e., a money machine). For simplicity, assume there is no dividend. We know that a European option is worth less than an American option. First, the current value of a call must be less than, or equal to, the asset price:

$$c_t \leq C_t \leq S_t \tag{8.4}$$

This is because, in the limit, an option with zero exercise price is equivalent to holding the stock in this case. We are sure to exercise the option.

Second, the value of a European call must be greater than, or equal to, the price of the asset minus the present value of the strike price:

$$c_t \geq S_t - Ke^{-r\tau} \tag{8.5}$$

To prove this, we could simply use put-call parity, or Equation (8.3) with $r^* = 0$, imposing the condition that $p \geq 0$. Note that, since $e^{-r\tau} < 1$, we must have $S_t - Ke^{-r\tau} > S_t - K$ before expiration. Thus $S_t - Ke^{-r\tau}$ is a more informative lower bound than $S_t - K$. As an example, continue with our call option. The lower bound is $S_t - Ke^{-r\tau} = \$120 - \$100 \exp(-5\% \times 1) = \$24.88$. This is more informative than $S - K = \$20$.

We can also describe upper and lower bounds for put options. The value of a put cannot be worth more than K:

$$p_t \leq P_t \leq K \tag{8.6}$$

which is the upper bound if the price falls to zero. Using put-call parity, we can show that the value of a European put must satisfy the following lower bound:

$$p_t \geq Ke^{-r\tau} - S_t \tag{8.7}$$

8.2.2 Early Exercise of Options

These relationships can be used to assess the value of early exercise for American options. The basic trade-off arises between the value of the American option **dead**, that is, exercised, or **alive**, that is, nonexercised. Thus, the choice is between exercising the option and selling it on the open market.

Consider an American call on a non-dividend-paying stock. By exercising early, the holder gets exactly $S_t - K$. The value of the option alive, however, must be worth more than that of the equivalent European call. From Equation (8.5), this must satisfy $c_t \geq S_t - Ke^{-r\tau}$, which is strictly greater than $S_t - K$ because $e^{-r\tau} < 1$ with positive interest rates. Hence, an American call on a non-dividend-paying stock *should never* be exercised early.

In our example, the lower bound on the European call is \$24.88. If we exercise the American call, we get only $S - K = \$120 - \$100 = \$20$. Because this is less than the minimum value of the European call, the American call should not be exercised. As a result, the value of the American feature is zero and we always have $c_t = C_t$.

The only reason to exercise a call early is to capture a dividend payment. Intuitively, a high income payment makes holding the asset more attractive than holding the option. Thus American options on income-paying assets may be exercised early. Note that this applies also to options on futures, since the implied income stream on the underlying is the risk-free rate.

KEY CONCEPT

An American call option on a non-dividend-paying stock (or asset with no income) should never be exercised early. If the asset pays income, early exercise may occur, with a probability that increases with the size of the income payment.

For an American put, we must have

$$P_t \geq K - S_t \tag{8.8}$$

because it could be exercised now. Unlike the relationship for calls, this lower bound $K - S_t$ is strictly greater than the lower bound for European puts $Ke^{-r\tau} - S_t$. So, we could have early exercise.

To decide whether to exercise early, the holder of the option has to balance the benefit of exercising, which is to receive K now instead of later, against the loss of killing the time value of the option. Because it is better to receive money now than later, it may be worth exercising the put option early.

Thus, American puts on non-income-paying assets *could* be exercised early, unlike calls. The probability of early exercise decreases for lower interest rates and with higher income payments on the asset. In each case, it becomes less attractive to sell the asset.

KEY CONCEPT

An American put option on a non-dividend-paying stock (or asset with no income) may be exercised early. If the asset pays income, the possibility of early exercise decreases with the size of the income payments.

EXAMPLE 8.10: FRM EXAM 2002—QUESTION 50

Given strictly positive interest rates, the best way to close out a long American call option position early (on a stock that pays no dividends) would be to

a. Exercise the call
b. Sell the call
c. Deliver the call
d. Do none of the above

EXAMPLE 8.11: FRM EXAM 2005—QUESTION 15

You have been asked to verify the pricing of a two-year European call option with a strike price of USD 45. You know that the initial stock price is USD 50, and the continuous risk-free rate is 3%. To verify the possible price range of this call, you consider using price bounds. What is the difference between the upper and lower bounds for that European call?

a. 0.00
b. 7.62
c. 42.38
d. 45.00

EXAMPLE 8.12: FRM EXAM 2008—QUESTION 2-6

Which two of the following four statements are *correct* about the early exercise of American options on non-dividend-paying stocks?

I. It is never optimal to exercise an American call option early.
II. It can be optimal to exercise an American put option early.
III. It can be optimal to exercise an American call option early.
IV. It is never optimal to exercise an American put option early.

a. I and II
b. I and IV
c. II and III
d. III and IV

8.3 VALUING OPTIONS

8.3.1 Pricing by Replication

We now turn to the pricing of options. The philosophy of pricing models consists of replicating the payoff on the instrument by a portfolio of assets. To avoid arbitrage, the price of the instrument must then equal the price of the replicating portfolio.

Consider a call option on a stock whose price is represented by a binomial process. The initial price of $S_0 = \$100$ can only move up or down to two values (hence the name *binomial*), $S_1 = \$150$ or $S_2 = \$50$. The option is a call with $K = \$100$, and therefore can only take values of $c_1 = \$50$ or $c_2 = \$0$. We assume that the rate of interest is $r = 25\%$, so that a dollar invested now grows to \$1.25 at maturity.

$$
\begin{array}{lccc}
 & & S_1 = \$150 & c_1 = \$50 \\
 & \nearrow & & \\
S_0 = \$100 & & & \\
 & \searrow & & \\
 & & S_2 = \$50 & c_2 = \$0
\end{array}
$$

The key idea of derivatives pricing is that of **replication.** In other words, we replicate the payoff on the option by a suitable portfolio of the underlying asset plus a position, long or short, in a risk-free bill. This is feasible in this simple setup because we have two states of the world and two instruments, the stock and the bond. To prevent arbitrage, the current value of the derivative must be the same as that of the portfolio.

The portfolio consists of n shares and a risk-free investment currently valued at B (a negative value implies borrowing). We set $c_1 = nS_1 + B$, or $\$50 = n\$150 + B$ and $c_2 = nS_2 + B$, or $\$0 = n\$50 + B$ and solve the 2 by 2 system, which gives

$n = 0.5$ and $B = -\$25$. At time $t = 0$, the value of the loan is $B_0 = \$25/1.25 = \$20$. The current value of the portfolio is $nS_0 + B_0 = 0.5 \times \$100 - \$20 = \$30$. Hence the current value of the option must be $c_0 = \$30$. This derivation shows the essence of option pricing methods.

Note that we did not need the actual probability of an up move. Define this as p. To see how this can be derived, we write the current value of the stock as the discounted expected payoff assuming investors were risk-neutral:

$$S_0 = [p \times S_1 + (1 - p) \times S_2]/(1 + r) \tag{8.9}$$

where the term between brackets is the expectation of the future spot price, given by the probability times its value for each state. Solving for $100 = [p \times 150 + (1 - p) \times 50]/1.25$, we find a risk-neutral probability of $p = 0.75$. We now value the option in the same fashion:

$$c_0 = [p \times c_1 + (1 - p) \times c_2]/(1 + r) \tag{8.10}$$

which gives

$$c_0 = [0.75 \times \$50 + 0.25 \times \$0]/1.25 = \$30$$

This simple example illustrates a very important concept, which is that of **risk-neutral pricing.**

8.3.2 Black-Scholes Valuation

The Black-Scholes (BS) model is an application of these ideas that provides an elegant closed-form solution to the pricing of European calls. The derivation of the model is based on four assumptions:

Black-Scholes Model Assumptions

1. *The price of the underlying asset moves in a continuous fashion.*
2. *Interest rates are known and constant.*
3. *The variance of underlying asset returns is constant.*
4. *Capital markets are perfect* (i.e., short sales are allowed, there are no transaction costs or taxes, and markets operate continuously).

The most important assumption behind the model is that prices are continuous. This rules out discontinuities in the sample path, such as jumps, which cannot be hedged in this model.

The statistical process for the asset price is modeled by a geometric Brownian motion: Over a very short time interval, dt, the logarithmic return has a normal distribution with mean = μdt and variance = $\sigma^2 dt$. The total return can be modeled as

$$dS/S = \mu dt + \sigma dz \tag{8.11}$$

where the first term represents the drift component, and the second is the stochastic component, with dz distributed normally with mean zero and variance dt.

This process implies that the logarithm of the ending price is distributed as

$$\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{8.12}$$

where ϵ is a $N(0, 1)$ random variable. Hence, the price is lognormally distributed.

Based on these assumptions, Black and Scholes (1972) derived a closed-form formula for European options on a non-dividend-paying stock, called the **Black-Scholes model.** The key point of the analysis is that a position in the option can be replicated by a delta position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is locally risk-free, that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate.

As a result, we can directly compute the present value of the derivative as the discounted expected payoff

$$f_t = E_{\mathrm{RN}}[e^{-r\tau} F(S_T)] \tag{8.13}$$

where the underlying asset is assumed to grow at the risk-free rate, and the discounting is also done at the risk-free rate. Here, the subscript RN refers to the fact that the analysis assumes **risk neutrality**. In a risk-neutral world, the expected return on all securities must be the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a risk premium to induce them to take risks. The BS model value can be computed assuming that all payoffs grow at the risk-free rate and are discounted at the same risk-free rate.

This risk-neutral valuation approach is without a doubt the most important tool in derivatives pricing. Before the Black-Scholes breakthrough, Samuelson had derived a very similar model in 1965, but with the asset growing at the rate μ and discounting as some other rate μ^*.[1] Because μ and μ^* are unknown, the Samuelson model was not practical. The risk-neutral valuation is merely an artificial method to obtain the correct solution, however. It does not imply that investors are risk-neutral.

Furthermore, this approach has limited uses for risk management. The BS model can be used to derive the **risk-neutral probability** of exercising the option. For risk management, however, what matters is the actual probability of exercise, also called **physical probability**. This can differ from the RN probability.

We now turn to the formulation of the BS model. In the case of a European call, the final payoff is $F(S_T) = \mathrm{Max}(S_T - K, 0)$. Initially, we assume no dividend payment on the asset. The current value of the call is given by:

$$c = SN(d_1) - Ke^{-r\tau} N(d_2) \tag{8.14}$$

[1] Paul Samuelson, "Rational Theory of Warrant Price," *Industrial Management Review* 6 (1965): 13–39.

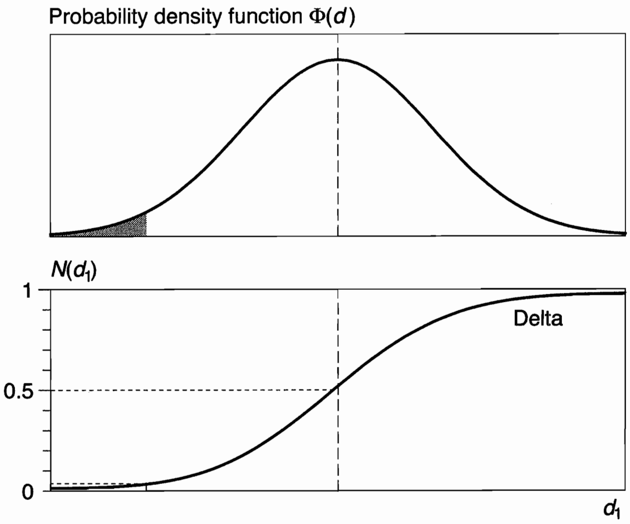

FIGURE 8.10 Cumulative Distribution Function

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d, as shown in Figure 8.10. Note that, since the normal density is symmetrical, $N(d) = 1 - N(-d)$, or the area to the left of d is the same as the area to the right of $-d$.

The values of d_1 and d_2 are:

$$d_1 = \frac{ln\,(S/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is:

$$p = S[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{8.15}$$

Example: Computing the Black-Scholes Value

Consider an at-the-money call on a stock worth $S = \$100$, with a strike price of $K = \$100$ and maturity of six months. The stock has annual volatility of $\sigma = 20\%$ and pays no dividend. The risk-free rate is $r = 5\%$.

First, we compute the present value factor, which is $e^{-r\tau} = \exp(-0.05 \times 6/12) = 0.9753$. We then compute the value of $d_1 = \ln[S/Ke^{-r\tau}]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = 0.2475$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.1061$. Using standard normal tables or the NORMSDIST Excel function, we find $N(d_1) = 0.5977$ and $N(d_2) = 0.5422$. Note that both values are greater than 0.5 since d_1 and d_2 are both positive. The option is at-the-money. As S is close to K, d_1 is close to zero and $N(d_1)$ close to 0.5.

The value of the call is $c = SN(d_1) - Ke^{-r\tau}N(d_2) = \$6.89$.

The value of the call can also be viewed as an equivalent position of $N(d_1) = 59.77\%$ in the stock and some borrowing: $c = \$59.77 - \$52.88 = \$6.89$. Thus this is a leveraged position in the stock.

The value of the put is \$4.42. Buying the call and selling the put costs \$6.89 − \$4.42 = \$2.47. This indeed equals $S - Ke^{-r\tau} = \$100 - \$97.53 = \$2.47$, which confirms put-call parity.

We should note that Equation (8.14) can be reinterpreted in view of the discounting formula in a risk-neutral world, Equation (8.13):

$$c = E_{\mathrm{RN}}[e^{-r\tau}\mathrm{Max}(S_T - K, 0)] = e^{-r\tau}[\int_K^\infty Sf(S)dS] - Ke^{-r\tau}[\int_K^\infty f(S)dS] \tag{8.16}$$

We see that the integral term multiplying K is the risk-neutral probability of exercising the call, or that the option will end up in-the-money $S > K$. Matching this up with Equation (8.14), this gives

$$\text{Risk} - \text{Neutral Probability of Exercise} = [\int_K^\infty f(S)dS] = N(d_2) \tag{8.17}$$

8.3.3 Extensions

Merton (1973) expanded the BS model to the case of a stock paying a continuous dividend yield q. Garman and Kohlhagen (1983) extended the formula to foreign currencies, reinterpreting the yield as the foreign rate of interest $q = r^*$, in what is called the **Garman-Kohlhagen model.**

The Merton model then replaces all occurrences of S by $Se^{-r^*\tau}$. The call is worth

$$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{8.18}$$

It is interesting to take the limit of Equation (8.14) as the option moves more in-the-money, that is, when the spot price S is much greater than K. In this case, d_1 and d_2 become very large and the functions $N(d_1)$ and $N(d_2)$ tend to unity. The value of the call then tends to

$$c(S \gg K) = Se^{-r^*\tau} - Ke^{-r\tau} \tag{8.19}$$

which is the valuation formula for a forward contract. A call that is deep in-the-money is equivalent to a long forward contract, because we are almost certain to exercise.

The **Black model** (1976) applies the same formula to options on futures. The only conceptual difference lies in the income payment to the underlying instrument. With an option on cash, the income is the dividend or interest on the cash instrument. In contrast, with a futures contract, the economically equivalent stream of income is the riskless interest rate. The intuition is that a futures contract can be viewed as equivalent to a position in the underlying asset with the investor setting aside an amount of cash equivalent to the present value of F.

KEY CONCEPT

With an option on futures, the implicit income is the risk-free rate of interest.

For the Black model, we simply replace S by F, the current futures quote, and replace r^* by r, the domestic risk-free rate. The Black model for the valuation of options on futures is:

$$c = [FN(d_1) - KN(d_2)]e^{-r\tau} \tag{8.20}$$

EXAMPLE 8.13: FRM EXAM 2001—QUESTION 91

Using the Black-Scholes model, calculate the value of a European call option given the following information: spot rate = 100; strike price = 110; risk-free rate = 10%; time to expiry = 0.5 years; $N(d_1) = 0.457185$; $N(d_2) = 0.374163$.

a. \$10.90
b. \$9.51
c. \$6.57
d. \$4.92

EXAMPLE 8.14: PROBABILITY OF EXERCISE

In the Black-Scholes expression for a European call option, the term used to compute option probability of exercise is

a. d_1
b. d_2
c. $N(d_1)$
d. $N(d_2)$

8.4 OTHER OPTION CONTRACTS

The options described so far are standard, plain-vanilla options. Many other types of options, however, have been developed.

Binary options, also called **digital options**, pay a fixed amount, say Q, if the asset price ends up above the strike price:

$$c_T = Q \times I(S_T - K) \tag{8.21}$$

where $I(x)$ is an indicator variable that takes the value of 1 if $x \geq 0$ and 0 otherwise. The payoff function is illustrated in Figure 8.11 when $K = \$100$.

Because the probability of ending in-the-money in a risk-neutral world is $N(d_2)$, the initial value of this option is simply

$$c = Qe^{-r\tau} N(d_2) \tag{8.22}$$

These options involve a sharp discontinuity around the strike price. Just below K, their value is zero. Just above, the value is the notional Q. Due to this discontinuity, these options are very difficult to hedge.

Another important class of options is barrier options. **Barrier options** are options where the payoff depends on the value of the asset hitting a barrier during a certain period of time. A **knock-out option** disappears if the price hits a certain barrier. A **knock-in option** comes into existence when the price hits a certain barrier.

An example of a knock-out option is the **down-and-out call**. This disappears if S hits a specified level H during its life. In this case, the knock-out price H must be lower than the initial price S_0. The option that appears at H is the **down-and-in call**. With identical parameters, the two options are perfectly complementary. When one disappears, the other appears. As a result, these two options must add

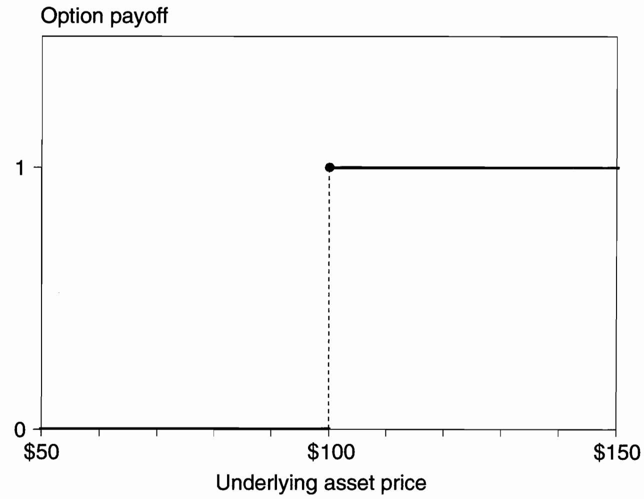

FIGURE 8.11 Payoff on a Binary Option

up to a regular call option. Similarly, an **up-and-out call** ceases to exist when S reaches $H > S_0$. The complementary option is the **up-and-in call.**

Figure 8.12 compares price paths for the four possible combinations of calls. In all figures, the dark line describes the relevant price path during which the option is alive; the gray line describes the remaining path.

The graphs illustrate that the down-and-out call and down-and-in call add up to the regular price path of a regular European call option. Thus, at initiation, the value of these two options must add up to

$$c = c_{DO} + c_{DI} \tag{8.23}$$

Because all these values are positive (or at worst zero), the value of each premium c_{DO} and c_{DI} must be no greater than that of c. A similar reasoning applies to the two options in the right panels. Sometimes the option offers a **rebate** if it is knocked out.

Similar combinations exist for put options. An **up-and-out put** ceases to exist when S reaches $H > S_0$. A **down-and-out put** ceases to exist when S reaches $H < S_0$. The only difference with Figure 8.12 is that the option is exercised at maturity if $S < K$.

Barrier options are attractive because they are cheaper than the equivalent European option. This, of course, reflects the fact that they are less likely to be exercised than other options.

In addition, these options are difficult to hedge because a discontinuity arises as the spot price get closer to the barrier. Just above the barrier, the option has positive value. For a very small movement in the asset price, going below the barrier, this value disappears.

Another widely used class of options is Asian options. **Asian options**, or **average rate options**, generate payoffs that depend on the average value of the

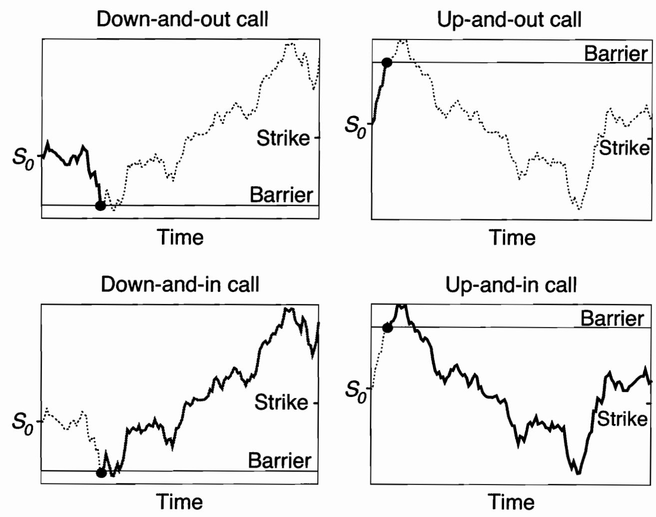

FIGURE 8.12 Paths for Knock-Out and Knock-In Call Options

underlying spot price during the life of the option, instead of the ending value. Define this as $S_{\text{AVE}}(t, T)$. The final payoff for a call is

$$c_T = \text{Max}(S_{\text{AVE}}(t, T) - K, 0) \tag{8.24}$$

Because an average is less variable than the final value at the end of the same period, such options are cheaper than regular options due to lower volatility. In fact, the price of the option can be treated like that of an ordinary option with the volatility set equal to $\sigma/\sqrt{3}$ and an adjustment to the dividend yield.[2] As a result of the averaging process, such options are easier to hedge than ordinary options.

Chooser options allow the holder to choose whether the option is a call or a put. At that point in time, the value of the option is

$$f_t = \text{Max}(c_t, p_t) \tag{8.25}$$

Thus it is a package of two options, a regular call plus an option to convert to a put. As a result, these options are more expensive than plain-vanilla options.

Compound options are options on options. A call on a call (cacall), for example, allows the holder to pay a fixed strike K_1 on the first exercise date T_1 to receive a call, which itself gives the right to buy the asset at a fixed strike K_2 on the second exercise date T_2. The first option will be exercised only if the value of the second option on that date $c(S, K_2, T_2)$ is greater than the strike price K_1.

These options are used, for example, to hedge bids for business projects that may or may not be accepted and that involve foreign currency exposure. If the project is accepted at date T_1, the option is more likely to be exercised. The compound option is cheaper than a regular call option at inception, at the expense of a higher total cost if both options are exercised. Compound options also include calls on puts, puts on puts, and puts on calls.

Finally, **lookback options** have payoffs that depend on the extreme values of S over the option's life. Define S_{MAX} as the maximum and S_{MIN} as the minimum. A fixed-strike lookback call option pays $\text{Max}(S_{MAX} - K, 0)$. A floating-strike lookback call option pays $\text{Max}(S_T - S_{MIN}, 0)$. These options are even more expensive than regular options.

EXAMPLE 8.15: FRM EXAM 2003—QUESTION 34

Which of the following options is strongly path-dependent?

a. An Asian option
b. A binary option
c. An American option
d. A European call option

[2] This is strictly true only when the averaging is a geometric average. In practice, average options involve an arithmetic average, for which there is no analytic solution; the lower volatility adjustment is just an approximation.

EXAMPLE 8.16: FRM EXAM 2006—QUESTION 59

All else being equal, which of the following options would cost more than plain-vanilla options that are currently at-the-money?

I. Lookback options
II. Barrier options
III. Asian options
IV. Chooser option

a. I only
b. I and IV
c. II and III
d. I, III, and IV

EXAMPLE 8.17: FRM EXAM 2002—QUESTION 19

Of the following options, which one does *not* benefit from an increase in the stock price when the current stock price is \$100 and the barrier has not yet been crossed?

a. A down-and-out call with barrier at \$90 and strike at \$110
b. A down-and-in call with barrier at \$90 and strike at \$110
c. An up-and-in put with barrier at \$110 and strike at \$100
d. An up-and-in call with barrier at \$110 and strike at \$100

8.5 VALUING OPTIONS BY NUMERICAL METHODS

Some options have analytical solutions, such as the Black-Scholes models for European vanilla options. For more general options, however, we need to use numerical methods.

The basic valuation formula for derivatives is Equation (8.13), which states that the current value is the discounted present value of expected cash flows, where all assets grow at the risk-free rate and are discounted at the same risk-free rate.

We can use the Monte Carlo simulation methods presented in Chapter 4 to generate sample paths, determine final option values, and discount them into the present. Such simulation methods can be used for European or even path-dependent options, such as Asian options.

Table 8.2 gives an example. Suppose we need to price a European call with parameters $S = 100$, $K = 100$, $T = 1$, $r = 5\%$, $r^* = 0$, $\sigma = 20\%$. We set up the simulation with, for instance, $n = 100$ steps over the horizon of one year.

TABLE 8.2 Example of Simulation for a European Call Option

Replication	Final Payoff S_T	c_T	Discounted Value
1	114.06	14.06	13.37
2	75.83	0.00	0.00
3	108.76	8.76	8.33
...			
Average			10.33

For each step, the trend is $r/n = 0.05/100$; the volatility is $\sigma/\sqrt(n) = 0.20/\sqrt{100}$. Each replication starts from a price of \$100 until the horizon. For instance, the first replication gives a final price of $S_T = \$114.06$. The option is in-the-money and is worth $c_T = \$14.06$. We then discount this number into the present and get \$13.37. In the second replication, $S_T = \$75.83$ and the option expires worthless: $c_T = 0$. Averaging across the K replications gives an average of \$10.33 in this case. The result is close to the actual Black-Scholes model price of \$10.45, obtained with Equation (8.14). The simulation, however, is much more general. The payoff at expiration could be a complicated function of the final price or even its intermediate values.

Simulation methods, however, cannot account for the possibility of early exercise, because they do not consider intermediate exercise choices. Instead, binomial trees must be used to value American options. As explained previously, the method consists of chopping up the time horizon into n intervals Δt and setting up the tree so that the characteristics of price movements fit the lognormal distribution.

At each node, the initial price S can go up to uS with probability p or down to dS with probability $(1 - p)$. The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous process. One could choose, for instance,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{8.26}$$

Since this a risk-neutral process, the total expected return must be equal to the risk-free rate r. Allowing for an income payment of r^*, this gives $\mu = r - r^*$.

The tree is built starting from the current time to maturity, from the left to the right. Next, the derivative is valued by starting at the end of the tree and working backward to the initial time, from the right to the left.

Consider first a European call option. At time T (maturity) and node j, the call option is worth $\text{Max}(S_{Tj} - K, 0)$. At time $T - 1$ and node j, the call option is the discounted expected value of the option at time T and nodes j and $j + 1$:

$$c_{T-1,j} = e^{-r\Delta t}[pc_{T,j+1} + (1 - p)c_{T,j}] \tag{8.27}$$

We then work backward through the tree until the current time.

For American options, the procedure is slightly different. At each point in time, the holder compares the value of the option *alive* (i.e., unexercised) and *dead* (i.e., exercised). The American call option value at node $T-1$, j is

$$C_{T-1,j} = \text{Max}[(S_{T-1,j} - K), c_{T-1,j}] \tag{8.28}$$

Example: Computing an American Option Value

Consider an at-the-money call on a foreign currency with a spot price of \$100, a strike price of $K = \$100$, and a maturity of six months. The annualized volatility is $\sigma = 20\%$. The domestic interest rate is $r = 5\%$; the foreign rate is $r^* = 8\%$. Note that we require an income payment for the American feature to be valuable. If $r^* = 0$, we know that the American option is worth the same as a European option, which can be priced with the Black-Scholes model. There would be no point in using a numerical method.

First, we divide the period into four intervals, for instance, so that $\Delta t = 0.50/4 = 0.125$. The discounting factor over one interval is $e^{-r\Delta t} = 0.9938$. We then compute:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.20\sqrt{0.125}} = 1.0733$$

$$d = (1/u) = 0.9317$$

$$a = e^{(r-r^*)\Delta t} = e^{(-0.03)0.125} = 0.9963$$

$$p = \frac{a-d}{u-d} = (0.9963 - 0.9317)/(1.0733 - 0.9317) = 0.4559$$

The procedure for pricing the option is detailed in Table 8.3. First, we lay out the tree for the spot price, starting with $S = 100$ at time $t = 0$, then $uS = 107.33$ and $dS = 93.17$ at time $t = 1$, and so on.

This allows us to value the European call. We start from the end, at time $t = 4$, and set the call price to $c = S - K = 132.69 - 100.00 = 32.69$ for the highest spot price, 15.19 for the next price, and so on, down to $c = 0$ if the spot price is below $K = 100.00$. At the previous step and highest node, the value of the call is

$$c = 0.9938[0.4559 \times 32.69 + (1 - 0.4559) \times 15.19] = 23.02$$

Continuing through the tree to time 0 yields a European call value of \$4.43. The Black-Scholes formula gives an exact value of \$4.76. Note how close the binomial approximation is, with just four steps. A finer partition would quickly improve the approximation.

Next, we examine the American call. At time $t = 4$, the values are the same as for the European call since the call expires. At time $t = 3$ and node $j = 4$, the option holder can either keep the call, in which case the value is still \$23.02, or exercise. When exercised, the option payoff is $S - K = 123.63 - 100.00 = 23.63$. Since this is greater than the value of the option alive, the holder should optimally

TABLE 8.3 Computation of American Option Value

	0	1	2	3	4
Spot price S_t	→	→	→	→	→
					132.69
				123.63	115.19
			115.19	107.33	100.00
		107.33	100.00	93.17	86.81
	100.00	93.17	86.81	80.89	75.36
European call c_t	←	←	←	←	←
					32.69
				23.02	15.19
			14.15	6.88	0.00
		8.10	3.12	0.00	0.00
	4.43	1.41	0.00	0.00	0.00
Exercised call $S_t - K$					
					32.69
				23.63	15.19
			15.19	7.33	0.00
		7.33	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
American call C_t	←	←	←	←	←
					32.69
				23.63	15.19
			15.19	7.33	0.00
		8.68	3.32	0.00	0.00
	4.74	1.50	0.00	0.00	0.00

exercise the option. We replace the European option value by \$23.63 at that node. Continuing through the tree in the same fashion, we find a starting value of \$4.74. The value of the American call is slightly greater than the European call price, as expected.

EXAMPLE 8.18: FRM EXAM 2006—QUESTION 86

Which one of the following statements about American options is *incorrect*?

a. American options can be exercised at any time until maturity.
b. American options are always worth at least as much as European options.
c. American options can easily be valued with Monte Carlo simulation.
d. American options can be valued with binomial trees.

8.6 IMPORTANT FORMULAS

Payoff on a long call and put: $C_T = \text{Max}(S_T - K, 0)$, $P_T = \text{Max}(K - S_T, 0)$

Put-call parity: $c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau}$

Bounds on call value (no dividends): $c_t \leq C_t \leq S_t$, $c_t \geq S_t - Ke^{-r\tau}$

Bounds on put value (no dividends): $p_t \leq P_t \leq K$, $p_t \geq Ke^{-r\tau} - S_t$

Geometric Brownian motion: $\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon$

Risk-neutral discounting formula: $f_t = E_{RN}[e^{-r\tau}F(S_T)]$

Black-Scholes call option pricing: $c = SN(d_1) - Ke^{-r\tau}N(d_2)$, $d_1 = \frac{\ln(S/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}$, $d_2 = d_1 - \sigma\sqrt{\tau}$

Black-Scholes put option pricing: $p = S[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1]$

Black-Scholes pricing with dividend, Garman-Kohlhagen model:
$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2)$

Black model, option on futures: $c = [FN(d_1) - KN(d_2)]e^{-r\tau}$

Pricing of binary option: $c = Qe^{-r\tau}N(d_2)$

Asian option: $c_T = \text{Max}(S_{AVE}(t, T) - K, 0)$

Binomial process: $u = e^{\sigma\sqrt{\Delta t}}$, $d = (1/u)$, $p = \frac{e^{\mu\Delta t} - d}{u - d}$

8.7 ANSWERS TO CHAPTER EXAMPLES

Example 8.1: FRM Exam 2007—Question 84

c. Buying a put creates a gain if the stock price falls, which is similar to selling the stock on the downside. On the upside, the loss is capped by buying a call.

Example 8.2: FRM Exam 2005—Question 72

c. By put-call parity, $c = p + Se^{-r^*\tau} - Ke^{-r\tau} = 3.19 + 23 - 25\exp(-0.05 \times 1) = 3.19 - 0.78 = 2.409$. Note that the volatility information is not useful.

Example 8.3: FRM Exam 2008—Question 2-10

c. Answers a. and b. have payoffs that depend on the stock price and therefore cannot create arbitrage profits. Put-call parity says that $c - p = 3 - 2 = \$1$ should equal $S - Ke^{-r\tau} = 42 - 44 \times 0.9048 = \$2.19$. The call option is cheap. Therefore buy the call and hedge it by selling the stock, for the upside. The benefit from selling the stock if S goes down is offset by selling a put.

Example 8.4: FRM Exam 2006—Question 74

c. By put-call parity, $c - p = Se^{-r^*\tau} - Ke^{-r\tau}$. Therefore, $Se^{-r^*\tau} = (c - p + Ke^{-r\tau}) = (10 - 15 + 90\exp(0.05 \times 5) = 65.09$. The dividend yield is then $y = -(1/T)\ln(65.09/85) = 5.337\%$.

Example 8.5: Risk of Option Contracts

c. Long positions in options can lose at worst the premium, so b. is wrong. Spreads involve long and short positions in options and have limited downside loss, so

a. is wrong. Writing options exposes the seller to very large losses. In the case of puts, the worst loss is the strike price K, if the asset price goes to zero. In the case of calls, however, the worst loss is in theory unlimited because there is a small probability of a huge increase in S. Between c. and d., c. is the better answer.

Example 8.6: FRM Exam 2007—Question 103

d. This position is graphed in Figure 8.6. It benefits from an increase in the price between 40 and 45, so is a bull spread. The worst loss occurs below $K_1 = 40$, when none of the options is exercised and the net lost premium is $5 - 3 = 2$. The maximum profit occurs above $K_2 = 45$, when the two options are exercised, for a net profit of \$5 minus the lost premium, which gives \$3.

Example 8.7: FRM Exam 2006—Question 45

c. The manager is long a portfolio, which is protected by buying a put with a low strike price and selling a call with a higher strike price. This locks in a range of profits and losses and is a collar. If the strike prices were the same, the hedge would be perfect.

Example 8.8: FRM Exam 2002—Question 42

b. Because the final price is below the lowest of the three strike prices, all the puts will be exercised. The final payoff is $(\$50 - \$33) - 2(\$42 - \$33) + (\$37 - \$33) = \$17 - \$18 + \$4 = \$3$. From this, we have to deduct the up-front cost, which is $-\$7 + 2(\$4) - \$2 = -\$1$. The total profit is then, ignoring the time value of money, $\$3 - \$1 = \$2$ per share.

Example 8.9: FRM Exam 2009—Question 3-8

d. The best strategy among these is a long butterfly, which benefits if the spot stays at the current level. Answer a. is a long straddle, which is incorrect because this will lose money if the spot rate does not move. Answer b. is a bull spread, which is incorrect because it assumes the spot price will go up. Answer c. is the same as a short spot position, which is also incorrect.

Example 8.10: FRM Exam 2002—Question 50

b. When there is no dividend, there is never any reason to exercise an American call early. Instead, the option should be sold to another party.

Example 8.11: FRM Exam 2005—Question 15

c. The upper bound is $S = 50$. The lower bound is $c \geq S - Ke^{-r\tau} = 50 - 45\exp(-0.03 \times 2) = 50 - 42.38 = 7.62$. Hence, the difference is 42.38.

Example 8.12: FRM Exam 2008—Question 2-6

a. If the stock does not pay a dividend, the value of the American call option alive is always higher than if exercised (basically because there is no dividend to capture).

Hence, it never pays to exercise a call early. On the other hand, exercising an American put early may be rational because it is better to receive the strike price now than later, with positive interest rates.

Example 8.13: FRM Exam 2001—Question 91

c. We use Equation (8.14), assuming there is no income payment on the asset. This gives $c = SN(d_1) - K\exp(-r\tau)N(d_2) = 100 \times 0.457185 - 110\exp(-0.1 \times 0.5) \times 0.374163 = \$6.568$.

Example 8.14: Probability of Exercise

d. This is the term multiplying the present value of the strike price, by Equation (8.17).

Example 8.15: FRM Exam 2003—Question 34

a. The payoff of an Asian option depends on the average value of S and therefore is path-dependent.

Example 8.16: FRM Exam 2006—Question 59

b. Lookback options use the maximum stock price over the period, which must be more than the value at the end. Hence they must be more valuable than regular European options. Chooser options involve an additional choice during the life of the option, and as a result are more valuable than regular options. Asian options involve the average, which is less volatile than the final price, so must be less expensive than regular options. Finally, barrier options can be structured so that the sum of two barrier options is equal to a regular option. Because each premium is positive, a barrier option must be less valuable than regular options.

Example 8.17: FRM Exam 2002—Question 19

b. A down-and-in call comes alive only when the barrier is touched; so an increase in S brings it away from the barrier. This is not favorable, so b. is the correct answer. A down-and-out call (answer a.) where the barrier has not been touched is still alive and hence benefits from an increase in S. An up-and-in put (c.) would benefit from an increase in S as this would bring it closer to the barrier of \$110. Finally, an up-and-in call (d.) would also benefit if S gets closer to the barrier.

Example 8.18: FRM Exam 2006—Question 86

c. This statement is incorrect because Monte Carlo simulations are strictly backward-looking, and cannot take into account optimal future exercise, which a binomial tree can do.

CHAPTER 9

Fixed-Income Securities

The next two chapters provide an overview of fixed-income markets, securities, and their derivatives. At the most basic level, **fixed-income securities** refer to bonds that promise to make fixed coupon payments. Over time, this narrow definition has evolved to include any security that obligates the borrower to make specific payments to the bondholder on specified dates. Thus, a **bond** is a security that is issued in connection with a borrowing arrangement. In exchange for receiving cash, the borrower becomes obligated to make a series of payments to the bondholder.

Section 9.1 provides an overview of the different segments of the bond market. Section 9.2 then introduces the various types of fixed-income securities. Section 9.3 reviews the basic tools for pricing fixed-income securities, including the determination of cash flows, discounting, and the term structure of interest rates, including yields, spot rates, and forward rates. Finally, Section 9.4 describes movements in risk factors in fixed-income markets.

Fixed-income derivatives are instruments whose value derives from some bond price, interest rate, or other bond market variable. Due to their complexity, these instruments are analyzed in the next chapter. Because of their importance, mortgage-backed securities (MBSs) and other securitized products will be covered in a later chapter.

9.1 OVERVIEW OF DEBT MARKETS

Fixed-income markets are truly global. They include domestic bonds, foreign bonds, and Eurobonds. Bonds issued by resident entities and denominated in the domestic currency are called **domestic bonds**. In contrast, **foreign bonds** are those floated by a foreign issuer in the domestic currency and subject to domestic country regulations (e.g., by the government of Sweden in dollars in the United States). **Eurobonds** are mainly placed outside the country of the currency in which they are denominated and are sold by an international syndicate of financial institutions (e.g., a dollar-denominated bond issued by IBM and marketed in London). The latter bond should not be confused with bonds denominated in the euro currency, which can be of any type.

FRM Exam Part 1 topic. This chapter also contains material on valuation.

The domestic bond market can be further decomposed into these categories:

- **Government bonds**, issued by central governments, or also called **sovereign bonds** (e.g., by the United States federal government in dollars)
- **Government agency and guaranteed bonds**, issued by agencies or guaranteed by the central government (e.g., by Fannie Mae, a U.S. government agency), which are public financial institutions
- **State and local bonds**, issued by local governments rather than the central government, also known as **municipal bonds** (e.g., by the state of California)
- Bonds issued by private **financial institutions**, including banks, insurance companies, or issuers of asset-backed securities (e.g., by Citibank in the U.S. market)
- **Corporate bonds**, issued by private nonfinancial corporations, including industrials and utilities (e.g., by IBM in the U.S. market)

Table 9.1 breaks down the world debt securities market, which was worth $90 trillion at the end of 2009. This includes the **bond markets**, traditionally defined as fixed-income securities with remaining maturities beyond one year, and the shorter-term **money markets**, with maturities below one year. The table includes all publicly tradable debt securities sorted by country of issuer and issuer type.

The table shows that U.S. entities have issued a total of $25.1 trillion in domestic debt and $6.6 trillion in international debt, for a total amount of $31.7 trillion, by far the biggest debt securities market. Next comes the Eurozone market, with a size of $24.9 trillion, and the Japanese market, with $11.9 trillion.

Focusing now on the type of borrower, domestic government debt is the largest sector. The domestic financial market is also important, especially for mortgage-backed securities. **Mortgage-backed securities** (MBSs) are securities issued in conjunction with mortgage loans, which are loans secured by the collateral of a specific real estate property. Payments on MBSs are repackaged cash flows supported by mortgage payments made by property owners. MBSs can be issued by government agencies as well as by private financial corporations. More generally, **asset-backed securities** (ABSs) are securities whose cash flows are supported by assets such as credit card receivables or car loan payments.

TABLE 9.1 Global Debt Securities Markets, 2009 (Billions of U.S. Dollars)

		Type				
Country of Issuer	Domestic	Gov't	Financial	Corporate	Int'l	Total
United States	25,065	9,475	12,805	2,785	6,646	31,711
Japan	11,522	9,654	1,085	783	380	11,902
Eurozone	14,043	6,872	5,032	2,139	10,879	24,922
United Kingdom	1,560	1,189	349	22	3,045	4,605
Others	12,032	6,914	3,792	1,326	5,128	17,160
Total	64,222	34,104	23,063	7,055	26,078	90,300

Source: Bank for International Settlements.

Finally, the remainder of the domestic market represents bonds raised by private, nonfinancial corporations. This sector is very large in the United States. In contrast, Japan and continental Europe rely more on bank debt to raise funds.

9.2 FIXED-INCOME SECURITIES

9.2.1 Instrument Types

Bonds pay interest on a regular basis, semiannually for U.S. Treasury and corporate bonds, annually for others such as Eurobonds, or quarterly for others. The most common types of bonds are

- **Fixed-coupon bonds**, which pay a fixed percentage of the principal every period and the principal as a **balloon** (one-time) payment at maturity.
- **Zero-coupon bonds**, which pay no coupons but only the principal; their return is derived from price appreciation only.
- **Annuities**, which pay a constant amount over time, which includes interest plus amortization, or gradual repayment, of the principal.
- **Perpetual bonds** or **consols**, which have no set redemption date and whose value derives from interest payments only.
- **Floating-coupon bonds**, which pay interest equal to a reference rate plus a margin, reset on a regular basis; these are usually called **floating-rate notes** (FRNs).
- **Structured notes**, which have more complex coupon patterns to satisfy the investor's needs.
- **Inflation-protected notes**, whose principal is indexed to the **Consumer Price Index** (CPI), hence providing protection against an increasing rate of inflation.[1]

There are many variations on these themes. For instance, **step-up bonds** have fixed coupons that start at a low rate and increase over time.

It is useful to consider floating-rate notes in more detail. Take, for instance, a 10-year \$100 million FRN paying semiannually six-month LIBOR in arrears.[2] **LIBOR**, the London Interbank Offered Rate, is a benchmark cost of borrowing for highly rated (AA) credits. Every semester, on the **reset date**, the value of six-month LIBOR is recorded. Say LIBOR is initially at 6%. At the next coupon date, the payment will be $(1/2) \times \$100 \times 6\% = \$3$ million. Simultaneously, we record a new value for LIBOR, say 8%. The next payment will then increase to

[1] In the United States, these government bonds are called **Treasury inflation-protected securities** (TIPS). The coupon payment is fixed in real terms, say 3%. If after six months, the cumulative inflation is 2%, the principal value of the bond increases from \$100 to $\$100 \times (1+2\%) = \$102$. The first semiannual coupon payment is then $(3\%/2) \times \$102 = \$1.53$.

[2] Note that the index could be defined differently. The floating payment could be tied to a Treasury rate, or LIBOR with a different maturity—say three-month LIBOR. The pricing of the FRN will depend on the index. Also, the coupon will typically be set to LIBOR plus some spread that depends on the creditworthiness of the issuer.

$4 million, and so on. At maturity, the issuer pays the last coupon plus the principal. Therefore, the coupon payment floats with the current interest rate (like a cork at the end of a fishing line).

Application: LIBOR and Other Benchmark Interest Rates

LIBOR, the London Interbank Offered Rate, is a reference rate based on interest rates at which banks borrow unsecured funds from each other in the London interbank market.

LIBOR is published daily by the **British Bankers' Association** (BBA) around 11:45 A.M. London time, and is computed from an average of the distribution of rates provided by reporting banks. LIBID, the London Interbank Bid Rate, represents the average deposit rate.

LIBOR is calculated for 10 different currencies and various maturities, from overnight to one year. LIBOR rates are the most widely used reference rates for short-term futures contracts, such as the Eurodollar futures.

For the euro, however, **EURIBOR**, or Euro Interbank Offered Rate, is most often used. It is sponsored by the European Banking Federation (EBF) and published by Reuters at 11 A.M. Central European Time (CET). In addition, **EONIA** (Euro Overnight Index Average) is an overnight unsecured lending rate that is published every day before 7 P.M. CET. The same panel of banks contributes to EURIBOR and EONIA. The equivalent for sterling is **SONIA** (Sterling Overnight Index Average).

Among structured notes, we should mention **inverse floaters**, also known as reverse floaters, which have coupon payments that vary inversely with the level of interest rates. A typical formula for the coupon is $c = 12\% - \text{LIBOR}$, if positive, payable semiannually. Assume the principal is $100 million. If LIBOR starts at 6%, the first coupon will be $(1/2) \times \$100 \times (12\% - 6\%) = \$3$ million. If after six months LIBOR moves to 8%, the second coupon will be $(1/2) \times \$100 \times (12\% - 8\%) = \$2$ million. The coupon will go to zero if LIBOR moves above 12%. Conversely, the coupon will increase if LIBOR drops. Hence, inverse floaters do best in a falling interest rate environment.

Bonds can also be issued with option features. The most important are:

- **Callable bonds**, where the issuer has the right to call back the bond at fixed prices on fixed dates, the purpose being to call back the bond when the cost of issuing new debt is lower than the current coupon paid on the bond.
- **Puttable bonds**, where the investor has the right to put the bond back to the issuer at fixed prices on fixed dates, the purpose being to dispose of the bond should its price deteriorate.
- **Convertible bonds**, where the bond can be converted into the common stock of the issuing company at a fixed price on a fixed date, the purpose being to partake in the good fortunes of the company (these will be covered in a separate chapter).

The key to analyzing these bonds is to identify and price the option feature. For instance, a callable bond can be decomposed into a long position in a straight bond minus a call option on the bond price. The call feature is unfavorable for investors who require a lower price to purchase the bond, thereby increasing its yield. Conversely, a put feature will make the bond more attractive, increasing its price and lowering its yield. Similarly, the convertible feature allows companies to issue bonds at a lower yield than otherwise.

EXAMPLE 9.1: FRM EXAM 2003—QUESTION 95

With any other factors remaining unchanged, which of the following statements regarding bonds is *not* valid?

a. The price of a callable bond increases when interest rates increase.
b. Issuance of a callable bond is equivalent to a short position in a straight bond plus a long call option on the bond price.
c. The put feature in a puttable bond lowers its yield compared with the yield of an equivalent straight bond.
d. The price of an inverse floater decreases as interest rates increase.

9.2.2 Methods of Quotation

Most bonds are quoted on a **clean price** basis, that is, without accounting for the accrued income from the last coupon. For U.S. bonds, this clean price is expressed as a percentage of the face value of the bond with fractions in 32nds, for instance as 104-12, which means $104 + 12/32$, for the 6.25% May 2030 Treasury bond. Transactions are expressed in number of units (e.g., $20 million face value).

Actual payments, however, must account for the accrual of interest. This is factored into the **gross price**, also known as the **dirty price**, which is equal to the clean price plus accrued interest. In the U.S. Treasury market, accrued interest (AI) is computed on an *actual/actual* basis:

$$\text{AI} = \text{Coupon} \times \frac{\text{Actual number of days since last coupon}}{\text{Actual number of days between last and next coupon}} \tag{9.1}$$

The fraction involves the actual number of days in both the numerator and denominator. For instance, say the 6.25% May 2030 Treasury bond paid the last coupon on November 15 and will pay the next coupon on May 15. The denominator is, counting the number of days in each month, $15 + 31 + 31 + 29 + 31 + 30 + 15 = 182$. If the trade settles on April 26, there are $15 + 31 + 31 + 29 + 31 + 26 = 163$ days into the period. The accrued interest is computed from the \$3.125 coupon as

$$\$3.125 \times \frac{163}{182} = \$2.798763$$

The total, gross price for this transaction is:

$$(\$20{,}000{,}000/100) \times [(104 + 12/32) + 2.798763] = \$21{,}434{,}753$$

Different markets have different day count conventions. A 30/360 convention, for example, assumes that all months count for 30 days exactly.

We should note that the accrued interest in the LIBOR market is based on *actual/360*. For instance, the interest accrued on a 6% \$1 million loan over 92 days is

$$\$1{,}000{,}000 \times 0.06 \times \frac{92}{360} = \$15{,}333.33$$

Another notable pricing convention is the discount basis for Treasury bills. These bills are quoted in terms of an annualized discount rate (DR) to the face value, defined as

$$\text{DR} = (\text{Face} - P)/\text{Face} \times (360/t) \tag{9.2}$$

where P is the price and t is the actual number of days. The dollar price can be recovered from

$$P = \text{Face} \times [1 - \text{DR} \times (t/360)] \tag{9.3}$$

For instance, a bill quoted at 5.19% discount with 91 days to maturity could be purchased for

$$\$100 \times [1 - 5.19\% \times (91/360)] = \$98.6881$$

This price can be transformed into a conventional yield to maturity, using

$$F/P = (1 + y \times t/365) \tag{9.4}$$

which gives 5.33% in this case. Note that the yield is greater than the discount rate because it is a rate of return based on the initial price. Because the price is lower than the face value, the yield must be greater than the discount rate.

9.2.3 Duration and Convexity

A previous chapter has explained the concept of duration, which is perhaps the most important risk measure in fixed-income markets. **Duration** represents the linear sensitivity of the bond rate of return to movements in yields. **Convexity** is the quadratic effect.

When cash flows are fixed, duration can be computed exactly as the weighted maturity of each payment, where the weights are proportional to the present value of the cash flows. In many other cases, we can infer duration from an economic analysis of the security. Consider a **floating-rate note** (FRN) with no credit risk. Just before the reset date, we know that the coupon will be set to the prevailing interest rate. The FRN is then similar to cash, or a money market instrument, which has no interest rate risk and hence is selling at par with zero duration. Just after the reset date, the investor is locked into a fixed coupon over the accrual period. The FRN is then economically equivalent to a zero-coupon bond with maturity equal to the time to the next reset date.

We have also seen that bonds with fixed or zero coupons have positive convexity. However, an investor in a callable bond has given the company the right to repurchase the bond at a fixed price. Suppose that rates fall, in which case a noncallable bond would see its price increase from \$100 to \$110. A company that has the right to call the bond at \$105 would then exercise this right, because it can buy the bond cheaply. This will create negative convexity in the bond. Conversely, bonds where the investor is long an option have greater positive convexity.

EXAMPLE 9.2: CALLABLE BOND DURATION

A 10-year zero-coupon bond is callable annually at par (its face value) starting at the beginning of year 6. Assume a flat yield curve of 10%. What is the bond duration?

a. 5 years
b. 7.5 years
c. 10 years
d. Cannot be determined based on the data given

EXAMPLE 9.3: DURATION OF FLOATERS

A money markets desk holds a floating-rate note with an eight-year maturity. The interest rate is floating at the three-month LIBOR rate, reset quarterly. The next reset is in one week. What is the approximate duration of the floating-rate note?

a. 8 years
b. 4 years
c. 3 months
d. 1 week

EXAMPLE 9.4: FRM EXAM 2009—QUESTION 4-16

From the time of issuance until the bond matures, which of the following bonds is most likely to exhibit negative convexity?

a. A puttable bond
b. A callable bond
c. An option-free bond selling at a discount
d. A zero-coupon bond

9.3 PRICING OF FIXED-INCOME SECURITIES

9.3.1 The NPV Approach

Fixed-income securities can be valued by, first, laying out their cash flows and, second, computing their net present value (NPV) using the appropriate discount rate. Let us write the market value of a bond P as the present value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{9.5}$$

where: C_t = cash flow (coupon and/or principal repayment) in period t
t = number of periods (e.g., half-years) to each payment
T = number of periods to final maturity
y = yield to maturity for this particular bond
P = price of the bond, including accrued interest

For a fixed-rate bond with face value F, the cash flow C_t is cF at each period, where c is the coupon rate, plus F upon maturity. Other cash flow patterns are possible, however. Figure 9.1 illustrates the time profile of the cash flows C_t for three bonds with initial market value of \$100, 10-year maturity, and 6% annual interest. The figure describes a straight coupon-paying bond, an annuity, and a zero-coupon bond. As long as the cash flows are predetermined, the valuation is straightforward.

Given the market price, solving for y gives the yield to maturity. This yield is another way to express the price of the bond and is more convenient when comparing various bonds. The yield is also the *expected* rate of return on the bond, provided all coupons are reinvested at the same rate. This interpretation fails, however, when the cash flows are random or when the life of the bond can change due to option-like features. Movements in interest rates also create **reinvestment risk.** This risk can be avoided only by investing in zero-coupon bonds, which do not make intermediate payments.

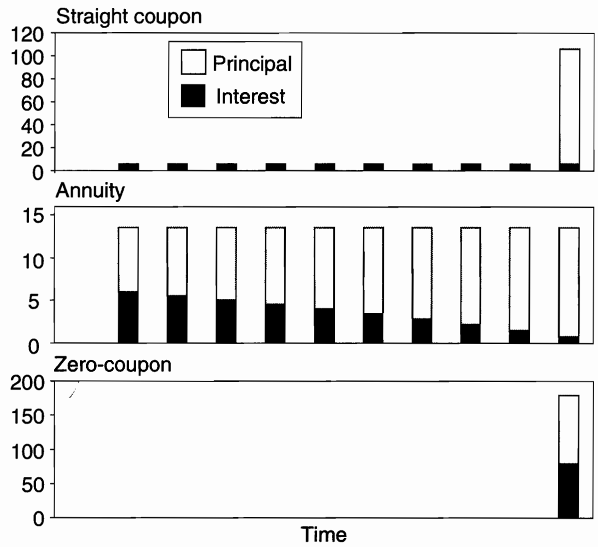

FIGURE 9.1 Time Profile of Cash Flows

> **EXAMPLE 9.5: FRM EXAM 2009—QUESTION 4-12**
>
> Your boss wants to devise a fixed-income strategy such that there is no reinvestment risk over five years. Reinvestment risk will not occur if:
>
> I. Interest rates remain constant over the time period the bonds are held.
> II. The bonds purchased are callable.
> III. The bonds purchased are issued at par.
> IV. Only zero-coupon bonds with a five-year maturity are purchased.
>
> **a.** I only
> **b.** I and II only
> **c.** III only
> **d.** I and IV

9.3.2 Pricing

We can also use information from the fixed-income market to assess the fair value of the bond. Say we observe that the yield to maturity for comparable bonds is y_T. We can then discount the cash flows using the same, market-determined yield. This gives a fair value for the bond:

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1 + y_T)^t} \tag{9.6}$$

Note that the discount rate y_T does not depend on t, but is fixed for all payments for this bond.

This approach, however, ignores the shape of the term structure of interest rates. Short maturities, for example, could have much lower rates, in which case it is inappropriate to use the same yield. We should really be discounting each cash flow at the zero-coupon rate that corresponds to each time period. This rate R_t is called the **spot interest rate** for maturity t. The fair value of the bond is then

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t)^t} \tag{9.7}$$

We can then check whether the market price is higher or lower. If the term structure is flat, the two approaches will be identical.

Note that the spot rates should be used to discount cash flows in the same risk class (i.e., for the same currency and credit risk). For instance, Treasury bonds should be priced using rates that have the same risk as the U.S. government. For high-quality corporate credits, the **swap curve** is often used. Swap rates correspond to the credit risk of AA-rated counterparties.

Another approach to assess whether a bond is rich or cheap is to add a fixed amount, called the **static spread** (SS), to the spot rates so that the NPV equals the current price:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t + \mathrm{SS})^t} \tag{9.8}$$

All else being equal, a bond with a large static spread is preferable to another with a lower spread. It means the bond is cheaper, or has a higher expected rate of return.

It is simpler, but less accurate, to compute a **yield spread** (YS), using yield to maturity, such that

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + y_T + \mathrm{YS})^t} \tag{9.9}$$

Table 9.2 gives an example of a 7% coupon, two-year bond. The term structure environment, consisting of spot rates and par yields, is described on the left side. The right side lays out the present value of the cash flows (PVCF) using different approaches. Let us first use Equation (9.7). Discounting the two cash flows at the spot rates gives a fair value of $\hat{P} = \$101.9604$. In fact, the bond is selling at a price of $P = \$101.5000$, so the bond is slightly cheap.

Next, we can use Equation (9.9), starting from the par yield of 5.9412%. The yield to maturity on this bond is 6.1798%, which implies a yield spread of $\mathrm{YS} = 6.1798 - 5.9412 = 0.2386\%$. Finally, we can use Equation (9.8). Using the static spread approach, we find that adding $\mathrm{SS} = 0.2482\%$ to the spot rates gives the current price. The last two approaches provide a plug-in so that the NPV exactly matches the observed market price.

TABLE 9.2 Bond Price and Term Structure

	Term Structure		7% Bond PVCF Discounted at		
Maturity (Year) i	Spot Rate R_i	Par Yield y_i	Spot SS = 0	Yield + YS $\Delta y = 0.2386$	Spot + SS SS = 0.2482
1	4.0000	4.0000	6.7308	6.5926	6.7147
2	6.0000	5.9412	95.2296	94.9074	94.7853
Sum			101.9604	101.5000	101.5000
Price			101.5000	101.5000	101.5000

For risk management purposes, it is better to consider the spot rates as the risk factors rather than the rate of return on price. This is more intuitive and uses variables that are more likely to be stationary. In contrast, using the history of bond prices is less useful because the maturity of a bond changes with the passage of time, which systematically alters its characteristics.

EXAMPLE 9.6: FRM EXAM 2009—QUESTION 4-11

Consider a bond with par value of EUR 1,000 and maturity in three years, and that pays a coupon of 5% annually. The spot rate curve is as follows: 1-year, 6%; 2-year, 7%; and 3-year, 8%. The value of the bond is closest to:

a. 904
b. 924
c. 930
d. 950

9.3.3 Spot and Forward Rates

Thus, fixed-income pricing relies heavily on **spot rates**, which are zero-coupon investment rates that start at the current time. From spot rates, we can infer **forward rates**, which are rates that start at a future date. Both are essential building blocks for the pricing of bonds. In addition, forward rates can be viewed as market-implied forecasts of future spot rates. Interest rate forecasting can add value only if the portfolio manager's forecast differs from the forward rate.

Consider, for instance, a one-year rate that starts in one year. This forward rate is defined as $F_{1,2}$ and can be inferred from the one-year and two-year spot rates, R_1 and R_2. The forward rate is the break-even future rate that equalizes the return on investments of different maturities.

To demonstrate this important concept, consider an investor who has two choices. The first is to lock in a two-year investment at the two-year rate. The second is to invest for a term of one year and roll over at the one-to-two-year

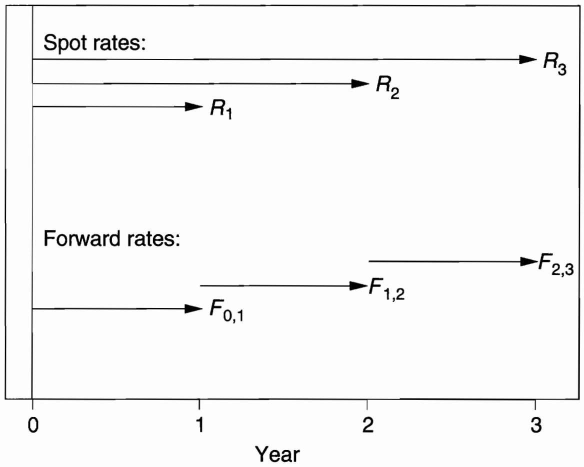

FIGURE 9.2 Spot and Forward Rates

forward rate. The two portfolios will have the same payoff when the future rate $F_{1,2}$ is such that

$$(1 + R_2)^2 = (1 + R_1)(1 + F_{1,2}) \tag{9.10}$$

For instance, if $R_1 = 4.00\%$ and $R_2 = 4.62\%$, we must have $F_{1,2} = 5.24\%$.

More generally, the T-period spot rate can be written as a geometric average of the spot and consecutive one-year forward rates:

$$(1 + R_T)^T = (1 + R_1)(1 + F_{1,2}) \ldots (1 + F_{T-1,T}) \tag{9.11}$$

where $F_{i,i+1}$ is the forward rate of interest prevailing now (at time t) over a horizon of i to $i + 1$. This sequence is shown in Figure 9.2. Table 9.3 displays a sequence of spot rates, forward rates, and par yields, using annual compounding. The last

TABLE 9.3 Spot Rates, Forward Rates, and Par Yields

Maturity (Year) i	Spot Rate R_i	Forward Rate $F_{i-1,i}$	Par Yield y_i	Discount Function $D(t_i)$
1	4.000	4.000	4.000	0.9615
2	4.618	5.240	4.604	0.9136
3	5.192	6.350	5.153	0.8591
4	5.716	7.303	5.640	0.8006
5	6.112	7.712	6.000	0.7433
6	6.396	7.830	6.254	0.6893
7	6.621	7.980	6.451	0.6383
8	6.808	8.130	6.611	0.5903
9	6.970	8.270	6.745	0.5452
10	7.112	8.400	6.860	0.5030

column is the **discount function**, which is simply the current price of a dollar paid at t.

Alternatively, one could infer a series of forward rates for various maturities, all starting in one year:

$$(1 + R_3)^3 = (1 + R_1)(1 + F_{1,3})^2, \ldots, \quad (1 + R_T)^T = (1 + R_1)(1 + F_{1,T})^{T-1} \tag{9.12}$$

This defines a term structure in one year, $F_{1,2}, F_{1,3}, \ldots, F_{1,T}$.

The forward rate can be interpreted as a measure of the slope of the term structure. To illustrate this point, expand both sides of Equation (9.10). After neglecting cross-product terms, we have

$$F_{1,2} \approx R_2 + (R_2 - R_1) \tag{9.13}$$

Thus, with an upward-sloping term structure, R_2 is above R_1, and $F_{1,2}$ will also be above R_2. In the preceding example, this gives $R_2 + (R_2 - R_1) = 4.62\% + (4.62\% - 4.00\%) = 4.62\% + 0.62\% = 5.24\%$, which is the correct number for $F_{1,2}$.

With an upward-sloping term structure, the spot rate curve is above the par yield curve. Consider a bond with two payments. The two-year par yield y_2 is implicitly defined from:

$$P = \frac{cF}{(1 + y_2)} + \frac{(cF + F)}{(1 + y_2)^2} = \frac{cF}{(1 + R_1)} + \frac{(cF + F)}{(1 + R_2)^2}$$

where P is set to par $P = F$. The par yield can be viewed as a weighted average of spot rates. In an upward-sloping environment, par yield curves involve coupons that are discounted at shorter and thus lower rates than the final payment. As a result, the par yield curve lies below the spot rate curve.[3] When the spot rate curve is flat, the spot curve is identical to the par yield curve and to the forward curve. In general, the curves differ. Figure 9.3 displays the case of an upward-sloping term structure. It shows that the yield curve is below the spot curve, while the forward curve is above the spot curve. With a downward-sloping term structure, as shown in Figure 9.4, the yield curve is above the spot curve, which is above the forward curve.

Note that, because interest rates have to be positive, forward rates have to be positive; otherwise there would be an arbitrage opportunity.[4]

---

[3] For a formal proof, consider a two-period par bond with a face value of \$1 and coupon of y_2. We can write the price of this bond as $1 = y_2/(1 + R_1) + (1 + y_2)/(1 + R_2)^2$. After simplification, this gives $y_2 = R_2(2 + R_2)/(2 + F_{1,2})$. In an upward-sloping environment, $F_{1,2} > R_2$ and thus $y_2 < R_2$.

[4] We abstract from transaction costs and assume we can invest and borrow at the same rate. For instance, $R_1 = 11.00\%$ and $R_2 = 4.62\%$ gives $F_{1,2} = -1.4\%$. This means that $(1 + R_1) = 1.11$ is greater than $(1 + R_2)^2 = 1.094534$. To take advantage of this discrepancy, we borrow \$1 million for two years and invest it for one year. After the first year, the proceeds are kept in cash, or under the proverbial mattress, for the second period. The investment gives \$1,110,000 and we have to pay back \$1,094,534 only. This would create an arbitrage profit of \$15,466 out of thin air, which is highly unlikely.

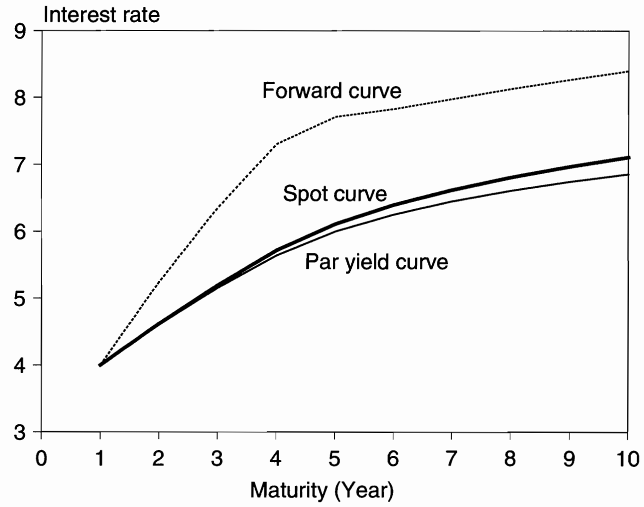

FIGURE 9.3 Upward-Sloping Term Structure

Forward rates allow us to project future cash flows that depend on future rates. The $F_{1,2}$ forward rate, for example, can be taken as the market's expectation of the second coupon payment on an FRN with annual payments and resets. We will also show later that positions in forward rates can be taken easily with derivative instruments.

As a result, the forward rate can be viewed as an expectation of the future spot rate. According to the **expectations hypothesis,**

$$F_{1,2}^{t} = E(R_1^{t+1}) \tag{9.14}$$

This assumes that there is no risk premium embedded in forward rates. An upward-sloping term structure implies that short-term rates will rise in the future.

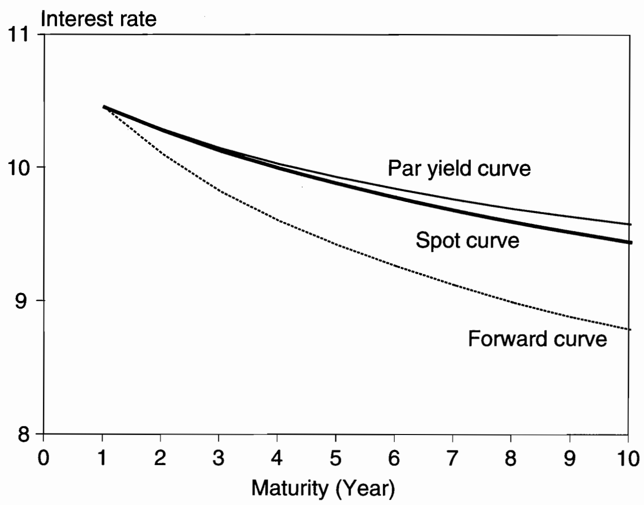

FIGURE 9.4 Downward-Sloping Term Structure

In Figure 9.3, the forward curve traces out the path of future one-year spot rates. Conversely, a downward-sloping curve would imply that future spot rates are expected to fall.

If this hypothesis is correct, then it does not matter which maturity should be selected for investment purposes. Longer maturities benefit from higher coupons but will suffer a capital loss, due to the increase in rates, that will offset this benefit exactly.

In practice, forward rates may contain a risk premium. Generally, investors prefer the safety of short-term instruments. They can be coaxed into buying longer maturities if the latter provide a positive premium. This explains why the spot or yield curves are upward-sloping most of the time.

KEY CONCEPT

In an upward-sloping term structure environment, the forward curve is above the spot curve, which is above the par yield curve. According to the expectations hypothesis, this implies a forecast for rising interest rates.

EXAMPLE 9.7: FRM EXAM 2007—QUESTION 32

The price of a three-year zero-coupon government bond is \$85.16. The price of a similar four-year bond is \$79.81. What is the one-year implied forward rate from year 3 to year 4?

a. 5.4%
b. 5.5%
c. 5.8%
d. 6.7%

EXAMPLE 9.8: FRM EXAM 2009—QUESTION 3-24

The term structure of swap rates is: 1-year, 2.50%; 2-year, 3.00%; 3-year, 3.50%; 4-year, 4.00%; 5-year, 4.50%. The two-year forward swap rate starting in three years is closest to

a. 3.50%
b. 4.50%
c. 5.51%
d. 6.02%

EXAMPLE 9.9: SHAPE OF TERM STRUCTURE

Suppose that the yield curve is upward sloping. Which of the following statements is *true*?

a. The forward rate yield curve is above the zero-coupon yield curve, which is above the coupon-bearing bond yield curve.
b. The forward rate yield curve is above the coupon-bearing bond yield curve, which is above the zero-coupon yield curve.
c. The coupon-bearing bond yield curve is above the zero-coupon yield curve, which is above the forward rate yield curve.
d. The coupon-bearing bond yield curve is above the forward rate yield curve, which is above the zero-coupon yield curve.

EXAMPLE 9.10: FRM EXAM 2004—QUESTION 61

According to the pure expectations hypothesis, which of the following statements is *correct* concerning the expectations of market participants in an upward-sloping yield curve environment?

a. Interest rates will increase and the yield curve will flatten.
b. Interest rates will increase and the yield curve will steepen.
c. Interest rates will decrease and the yield curve will flatten.
d. Interest rates will decrease and the yield curve will steepen.

9.4 FIXED-INCOME RISK

9.4.1 Price and Yield Volatility

Fixed-income risk arises from potential movements in the level and volatility of the risk factors, usually taken as bond yields.

Using the duration approximation, the volatility of the rate of return in the bond price can be related to the volatility of yield changes

$$\sigma\left(\frac{dP}{P}\right) \approx |\, D^* \,| \times \sigma(dy) \tag{9.15}$$

We now illustrate yield risk for a variety of markets.

9.4.2 Factors Affecting Yields

Movements in yields reflect economic fundamentals. The primary determinant of movements in the levels of interest rates is **inflationary expectations.** This is because investors care about the real (after-inflation) return on their investments. They are specially sensitive to inflation when they hold long-term bonds that pay fixed coupons. As a result, any perceived increase in the rate of inflation will make bonds with fixed nominal coupons less attractive, thereby increasing their yield.

Figure 9.5 compares the level of short-term U.S. interest rates with the concurrent level of inflation. This rate is the yield on U.S. government bills, so is risk-free. The graphs show that most of the long-term movements in nominal rates can be explained by inflation. In more recent years, however, inflation has been subdued. Rates have fallen accordingly.

The **real interest rate** is defined as the nominal rate minus the rate of inflation over the same period. This is generally positive but in recent years has been negative as the Federal Reserve has kept nominal rates to very low levels in order to stimulate economic activity.

Next, we consider a second effect, which is the shape of the term structure. Even though yields move largely in parallel across different maturities, the slope of the term structure changes as well. This can be measured by considering the difference in yields for two maturities. In practice, market observers focus on a long-term rate (e.g., from the 10-year Treasury note) and a short-term rate (e.g., from the three-month Treasury bill), as shown in Figure 9.6.

Generally, the two rates move in tandem, although the short-term rate displays more variability. The **term spread** is defined as the difference between the long rate and the short rate. Figure 9.7 relates the term spread to economic activity. Shaded areas indicate periods of U.S. economic recessions, as officially recorded by the National Bureau of Economic Research (NBER). As the graph shows, periods of recessions usually witness an increase in the term spread. Slow economic activity decreases the demand for capital, which in turn decreases short-term rates and

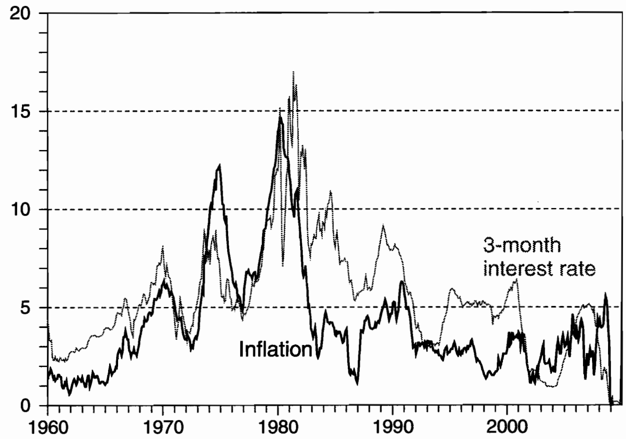

FIGURE 9.5 Inflation and Interest Rates

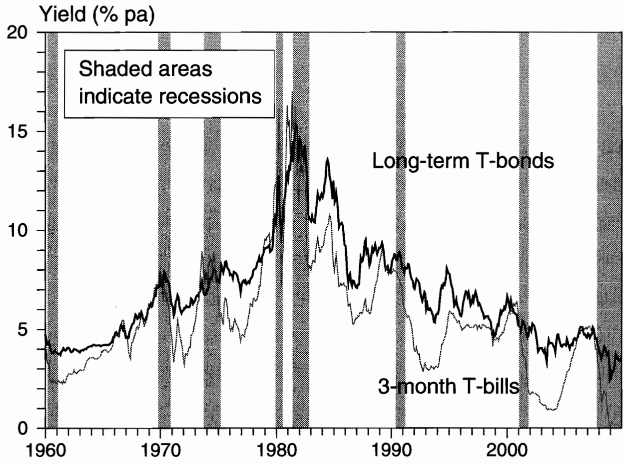

FIGURE 9.6 Movements in the Yields

increases the term spread. Equivalently, the central bank lowers short-term rates to stimulate the economy.

9.4.3 Bond Price and Yield Volatility

Table 9.4 compares the RiskMetrics volatility forecasts for U.S. bond prices as of December 2006. These numbers are exponentially weighted moving average (EWMA) daily and monthly forecasts. Monthly numbers are also shown as annualized, after multiplying by the square root of 12. The table includes Eurodeposits and zero-coupon Treasury rates for maturities ranging from 30 days to 30 years.

The table shows that short-term investments have very little price risk, as expected, due to their short maturities and durations. The price risk of 10-year bonds is around 6%, which is similar to that of floating currencies. The risk of 30-year bonds is higher, around 20%, which is similar to that of equities.

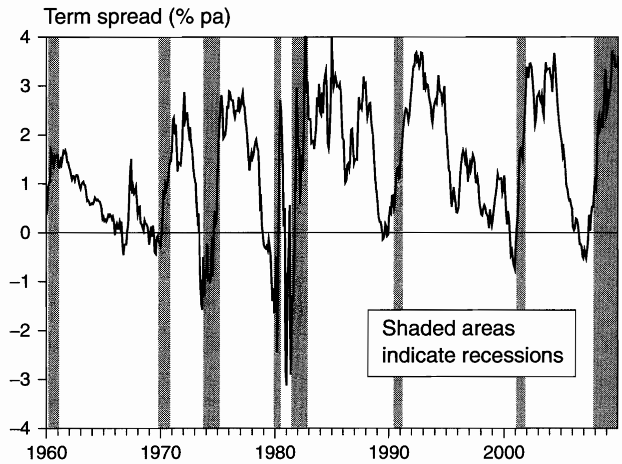

FIGURE 9.7 Term Structure Spread

TABLE 9.4 U.S. Fixed-Income Volatility (Percent), 2006

			$\sigma(dP/P)$			$\sigma(dy)$
Type/Maturity	Code	Yield Level	Daily	Monthly	Annual	Annual
Euro-30d	R030	5.325	0.001	0.004	0.01	0.17
Euro-360d	R360	5.338	0.028	0.148	0.51	0.54
Zero-2Y	Z02	4.811	0.088	0.444	1.54	0.22
Zero-5Y	Z05	4.688	0.216	1.084	3.76	0.44
Zero-10Y	Z10	4.698	0.375	1.874	6.49	0.48
Zero-20Y	Z20	4.810	0.690	3.441	11.92	0.49
Zero-30Y	Z30	4.847	1.014	5.049	17.49	0.63

The table shows yield volatilities in the last column. Yield volatility is around 0.50% for most maturities. This is also the case for bond markets in other currencies. There were, however, periods of higher inflation during which bond yields were much more volatile, as can be seen from Figure 9.6. For example, the early 1980s experienced wide swings in yields.

EXAMPLE 9.11: FRM EXAM 2007—QUESTION 50

A portfolio consists of two zero-coupon bonds, each with a current value of $10. The first bond has a modified duration of one year and the second has a modified duration of nine years. The yield curve is flat, and all yields are 5%. Assume all moves of the yield curve are parallel shifts. Given that the daily volatility of the yield is 1%, which of the following is the best estimate of the portfolio's daily value at risk (VAR) at the 95% confidence level?

a. USD 1.65
b. USD 2.33
c. USD 1.16
d. USD 0.82

9.4.4 Real Yield Risk

So far, the analysis has considered only **nominal interest rate risk**, as most bonds represent obligations in nominal terms (e.g., in dollars for the coupon and principal payment). Recently, however, many countries have issued inflation-protected bonds, which make payments that are fixed in real terms but indexed to the rate of inflation.

In this case, the source of risk is **real interest rate risk**. This real yield can be viewed as the internal rate of return that will make the discounted value of promised real bond payments equal to the current real price. This is a new source of risk, as movements in real interest rates may not correlate perfectly with movements in nominal yields. In addition, these two markets can be used to infer inflationary expectations. The implied rate of inflation can be measured as the nominal yield minus the real yield.

Example: Real and Nominal Yields

Consider, for example, the 10-year Treasury Inflation-Protected Security (TIPS) paying a 3% coupon in real terms. The actual coupon and principal payments are indexed to the increase in the Consumer Price Index (CPI).

The TIPS is now trading at a clean real price of 108-23+. Discounting the coupon payments and the principal gives a real yield of $r = 1.98\%$. Note that since the security is trading at a premium, the real yield must be lower than the coupon.

Projecting the rate of inflation at $\pi = 2\%$, semiannually compounded, we infer the projected nominal yield as $(1 + y/200) = (1 + r/200)(1 + \pi/200)$, which gives 4.00%. This is the same order of magnitude as the current nominal yield on the 10-year Treasury note, which is 3.95%. The two securities have very different risk profiles, however. If the rate of inflation moves to 5%, payments on the TIPS will grow at 5% plus 2%, while the coupon on the regular note will stay fixed.

9.4.5 Credit Spread Risk

Credit spread risk is the risk that yields on duration-matched credit-sensitive bonds and risk-free bonds could move differently. The topic of credit risk will be analyzed in more detail in Part Six of this book. Suffice to say that the credit spread represents a compensation for the loss due to default, plus perhaps a risk premium that reflects investor risk aversion.

A position in a credit spread can be established by investing in credit-sensitive bonds, such as corporates, agencies, and mortgage-backed securities (MBSs), and shorting Treasuries with the appropriate duration. This type of position benefits from a stable or shrinking credit spread, but loses from a widening of spreads. Because credit spreads cannot turn negative, their distribution is asymmetric, however. When spreads are tight, large moves imply increases in spreads rather than decreases. Thus positions in credit spreads can be exposed to large losses.

Figure 9.8 displays the time series of credit spreads since 1960, taken from the Baa-Treasury spread. The graph shows that credit spreads display cyclical patterns, increasing during recessions and decreasing during economic expansions. Greater spreads during recessions reflect the greater number of defaults during difficult times. At the time of the 2008 credit crisis, the spread exceeded 5%, which was a record high. Investors were worried that many companies were going to default, which would have decreased the value of their bonds.

As with the term structure of risk-free rates, there is a term structure of credit spreads. For good credit firms, this slopes upward, reflecting the low probability of default in the near term but the inevitable greater uncertainty at longer horizons. In 2009, for example, there were only four U.S. nonfinancials with a perfect credit

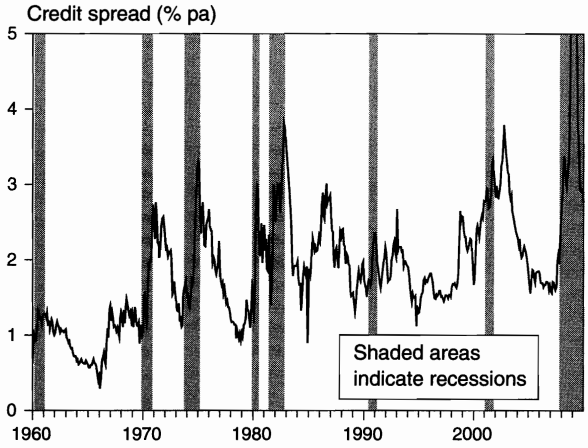

FIGURE 9.8 Credit Spreads

rating of AAA. There were 14 such companies in 1994. Over long horizons, credit tends to deteriorate, which explains why credit spreads widen for longer maturities.

EXAMPLE 9.12: FRM EXAM 2002—QUESTION 128

During 2002, an Argentinean pension fund with 80% of its assets in dollar-denominated debt lost more than 40% of its value. Which of the following reasons could explain all of the 40% loss?

a. The assets were invested in a diversified portfolio of AAA firms in the United States.
b. The assets invested in local currency in Argentina lost all of their value, while the value of the dollar-denominated assets stayed constant.
c. The dollar-denominated assets were invested in U.S. Treasury debt, but the fund had bought credit protection on sovereign debt from Argentina.
d. The fund had invested 80% of its funds in dollar-denominated sovereign debt from Argentina.

EXAMPLE 9.13: FRM EXAM 2008—QUESTION 2-41

Which of the following would *not* cause an upward-sloping yield curve?

a. An investor preference for short-term instruments
b. An expected decline in interest rates
c. An improving credit risk outlook
d. An expected increase in the inflation rate

9.5 ANSWERS TO CHAPTER EXAMPLES

Example 9.1: FRM Exam 2003—Question 95

a. Answer b. is valid because a short position in a callable bond is the same as a short position in a straight bond plus a long position in a call (the issuer can call the bond back). Answer c. is valid because a put is favorable for the investor, so it lowers the yield. Answer d. is valid because an inverse floater has high duration.

Example 9.2: Callable Bond Duration

c. Because this is a zero-coupon bond, it will always trade below par, and the call should never be exercised. Hence its duration is the maturity, 10 years.

Example 9.3: Duration of Floaters

d. Duration is not related to maturity when coupons are not fixed over the life of the investment. We know that at the next reset, the coupon on the FRN will be set at the prevailing rate. Hence, the market value of the note will be equal to par at that time. The duration or price risk is only related to the time to the next reset, which is one week here.

Example 9.4: FRM Exam 2009—Question 4-16

b. A callable bond is short an option, which creates negative convexity for some levels of interest rates. Regular bonds, as in answers c. and d., have positive convexity, as well as puttable bonds.

Example 9.5: FRM Exam 2009—Question 4-12

d. Reinvestment risk occurs when the intermediate coupon payments have to be reinvested at a rate that differs from the initial rate. This does not happen if interest rates stay constant, or with zero-coupon bonds. Callable bonds can be called early, which creates even more reinvestment risk for the principal.

Example 9.6: FRM Exam 2009—Question 4-11

b. The price is $50/(1+6\%) + 50/(1+7\%)^2 + 1{,}050/(1+8\%)^3 = 924.36$.

Example 9.7: FRM Exam 2007—Question 32

d. The forward rate can be inferred from $P_4 = P_3/(1+F_{3,4})$, or $(1+R_4)^4 = (1+R_3)^3(1+F_{3,4})$. Solving, this gives $F_{3,4} = (85.16/79.81) - 1 = 0.067$.

Example 9.8: FRM Exam 2009—Question 3-24

d. We compute first the accrual of a dollar over three and five years. For $T = 3$, this is $(1+3.50\%)^3 = 1.10872$. For $T = 5$, this is $(1+4.50\%)^5 = 1.24618$. This gives $1.24618 = (1 + F_{3,5})^2 \times 1.10872$. Solving, we find 6.018%. Note that we can use Equation (9.13) for an approximation. Here, this is $5R_5 = 3R_3 + 2F_{3,5}$, or $F_{3,5} = R_5 + (3/2)(R_5 - R_3) = 4.50\% + 1.5(4.50\% - 3.50\%) = 6\%$.

Example 9.9: Shape of Term Structure

a. See Figures 9.3 and 9.4. The coupon yield curve is an average of the spot, zero-coupon curve; hence it has to lie below the spot curve when it is upward sloping. The forward curve can be interpreted as the spot curve plus the slope of the spot curve. If the latter is upward sloping, the forward curve has to be above the spot curve.

Example 9.10: FRM Exam 2004—Question 61

a. An upward-sloping term structure implies forward rates higher than spot rates, or that short-term rates will increase. Because short-term rates increase more than long-term rates, this implies a flattening of the yield curve.

Example 9.11: FRM Exam 2007—Question 50

a. The dollar duration of the portfolio is $1 \times \$10 + 9 \times \$10 = \$100$. Multiplied by 0.01 and 1.65, this gives \$1.65.

Example 9.12: FRM Exam 2002—Question 128

d. In 2001, Argentina defaulted on its debt, both in the local currency and in dollars. Answer a. is wrong because a diversified portfolio could not have lost so much. The funds were invested at 80% in dollar-denominated assets, so b. is wrong; even a total wipeout of the local-currency portion could not explain a loss of 40% on the portfolio. If the fund had bought credit protection, it would not have lost as much, so c. is wrong. The fund must have had credit exposure to Argentina, so answer d. is correct.

Example 9.13: FRM Exam 2008—Question 2-41

b. An upward-sloping yield curve could be explained by a preference for short-term maturities (answer a.), which requires a higher long-term yield, so answer a. is not the correct choice. An upward-sloping yield curve could also be explained by expectations of higher interest rates or higher inflation (d.). Finally, improving credit conditions (c.) would reduce the cumulative probability of default and thus flatten the term structure. Only an expected decline in interest rates (b.) would *not* cause an upward-sloping yield curve.

CHAPTER 10

Fixed-Income Derivatives

This chapter turns to the analysis of fixed-income derivatives. These are instruments whose value derives from a bond price, interest rate, or other bond market variable. As shown in Table 7.1, fixed-income derivatives account for the largest proportion of the global derivatives markets. Understanding fixed-income derivatives is also important because many fixed-income securities have derivative-like characteristics.

This chapter focuses on the use of fixed-income derivatives, as well as their valuation. Pricing involves finding the fair market value of the contract. For risk management purposes, however, we also need to assess the range of possible movements in contract values. This will be further examined in the chapters on market risk.

This chapter presents the most important interest rate derivatives and discusses fundamentals of pricing. Section 10.1 discusses interest rate forward contracts, also known as forward rate agreements (FRAs). Section 10.2 then turns to the discussion of interest rate futures, covering Eurodollar and Treasury bond futures. Although these products are dollar-based, similar products exist on other capital markets. Interest rate swaps are analyzed in Section 10.3. Swaps are very important instruments due to their widespread use. Finally, interest rate options are covered in Section 10.4, including caps and floors, swaptions, and exchange-traded options.

10.1 FORWARD CONTRACTS

Forward rate agreements (FRAs) are over-the-counter (OTC) financial contracts that allow counterparties to lock in an interest rate starting at a future time. The buyer of an FRA locks in a borrowing rate, and the seller locks in a lending rate. In other words, the long benefits from an increase in rates, and the short benefits from a fall in rates.

As an example, consider an FRA that settles in one month on three-month LIBOR. Such FRA is called 1×4. The first number corresponds to the first settlement date, the second to the time to final maturity. Call τ the period to which LIBOR applies, three months in this case. On the settlement date, in one month,

FRM Exam Part 1 topic.

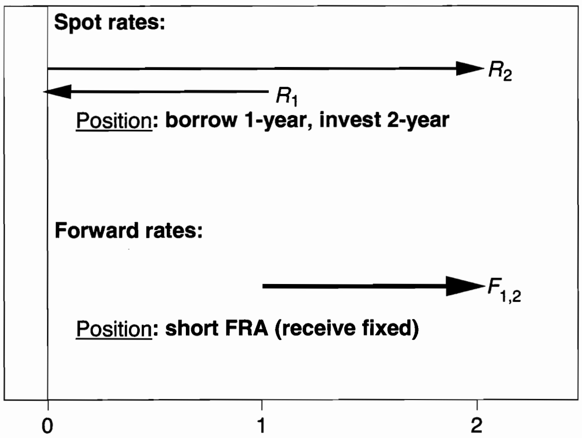

FIGURE 10.1 Decomposition of a Short FRA Position

the payment to the long involves the net value of the difference between the spot rate S_T (the prevailing three-month LIBOR rate) and the locked-in forward rate F. The payoff is $S_T - F$, as with other forward contracts, present valued to the first settlement date. This gives

$$V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1) \tag{10.1}$$

where $\text{PV}(\$1) = \$1/(1 + S_T\tau)$. The amount is settled in cash.

Figure 10.1 shows that a *short* position in an FRA is equivalent to borrowing short-term to finance a long-term investment. In both cases, there is no up-front investment. The duration is equal to the difference between the durations of the two legs, and can be inferred from the derivative of Equation (10.1). The duration of a short FRA is τ. Its dollar duration is $\text{DD} = \tau \times \text{Notional} \times \text{PV}(\$1)$.

Example: Using an FRA

A company will receive \$100 million in six months to be invested for a six-month period. The Treasurer is afraid rates will fall, in which case the investment return will be lower. The company needs to take a position that will offset this loss by generating a gain when rates fall. Because a short FRA gains when rates fall, the Treasurer needs to *sell* a 6×12 FRA on \$100 million at the rate of, say, $F = 5\%$. This locks in an investment rate of 5% starting in six months.

When the FRA expires in six months, assume that the prevailing six-month spot rate is $S_T = 3\%$. This will lower the investment return on the cash received, which is the scenario the Treasurer feared. Using Equation (10.1), the FRA has a payoff of $V_T = -(3\% - 5\%) \times (6/12) \times \$100$ million $= \$1{,}000{,}000$, which multiplied by the 3% present value factor gives \$985,222. In effect, this payment offsets the lower return that the company received on a floating investment, guaranteeing a return equal to the forward rate. This contract is also equivalent to borrowing the present value of \$100 million for six months and investing the proceeds for 12 months.

KEY CONCEPT

A long FRA position benefits from an increase in rates. A short FRA position is similar to a long position in a bond. Its duration is positive and equal to the difference between the two maturities.

EXAMPLE 10.1: FRM EXAM 2002—QUESTION 27

A long position in a 2×5 FRA is equivalent to the following positions in the spot market:

a. Borrowing in two months to finance a five-month investment
b. Borrowing in five months to finance a two-month investment
c. Borrowing half a loan amount at two months and the remainder at five months
d. Borrowing in two months to finance a three-month investment

EXAMPLE 10.2: FRM EXAM 2005—QUESTION 57

ABC, Inc., entered a forward rate agreement (FRA) to receive a rate of 3.75% with continuous compounding on a principal of USD 1 million between the end of year 1 and the end of year 2. The zero rates are 3.25% and 3.50% for one and two years. What is the value of the FRA when the deal is just entered?

a. USD 35,629
b. USD 34,965
c. USD 664
d. USD 0

EXAMPLE 10.3: FRM EXAM 2001—QUESTION 70

Consider the buyer of a 6×9 FRA. The contract rate is 6.35% on a notional amount of \$10 million. Calculate the settlement amount of the *seller* if the settlement rate is 6.85%. Assume a 30/360-day count basis.

a. −12,500
b. −12,290
c. +12,500
d. +12,290

10.2 FUTURES

Whereas FRAs are over-the-counter contracts, futures are traded on organized exchanges. We will cover the most important types of futures contracts, Eurodollar and T-bond futures.

10.2.1 Eurodollar Futures

Eurodollar futures are futures contracts tied to a forward LIBOR rate. Since their creation on the Chicago Mercantile Exchange, Eurodollar futures have spread to equivalent contracts such as EURIBOR futures (denominated in euros),[1] Euroyen futures (denominated in Japanese yen), and so on. These contracts are akin to FRAs involving three-month forward rates starting on a wide range of dates, up to 10 years into the future.

The formula for calculating the value of one contract is

$$P_t = 10{,}000 \times [100 - 0.25(100 - \mathrm{FQ}_t)] = 10{,}000 \times [100 - 0.25F_t] \qquad (10.2)$$

where FQ_t is the quoted Eurodollar futures price. This is quoted as 100.00 minus the interest rate F_t, expressed in percent, that is, $\mathrm{FQ}_t = 100 - F_t$. The 0.25 factor represents the three-month maturity, or 0.25 years. For instance, if the market quotes $\mathrm{FQ}_t = 94.47$, we have $F_t = 100 - 94.47 = 5.53$, and the contract value is $P = 10{,}000[100 - 0.25 \times 5.53] = \$986{,}175$. At expiration, the contract value settles to

$$P_T = 10{,}000 \times [100 - 0.25S_T] \qquad (10.3)$$

where S_T is the three-month Eurodollar spot rate prevailing at T. Payments are cash settled.

As a result, F_t can be viewed as a three-month forward rate that starts at the maturity of the futures contract. The formula for the contract price may look complicated but in fact is structured so that an increase in the interest rate leads to a decrease in the price of the contract, as is usual for fixed-income instruments. Also, because the change in the price is related to the interest rate by a factor of 0.25, this contract has a constant duration of three months. It is useful to remember that the DV01 is $\$10{,}000 \times 0.25 \times 0.01 = \$25$. This is so even though the notional amount is \$1 million. In this case, the notional amount is nowhere close to what could be lost on the contract even in the worst-case scenario. Even if the rate changes by 100bp, the loss would be only \$2,500.

Example: Using Eurodollar Futures

As in the previous section, the Treasurer wants to hedge a future investment of \$100 million in six months for a six-month period. The company needs to take a position that will offset the earnings loss by generating a gain when rates fall.

[1] EURIBOR futures are based on the European Banking Federation's Euro Interbank Offered Rate (EBF EURIBOR).

Because a long Eurodollar futures position gains when rates fall, the Treasurer should *buy* Eurodollar futures.

If the futures contract trades at $FQ_t = 95.00$, the dollar value of one contract is $P = 10{,}000 \times [100 - 0.25(100 - 95)] = \$987{,}500$. The Treasurer needs to buy a suitable number of contracts that will provide the best hedge against the loss of earnings. The computation of this number will be detailed in a future chapter.

A previous chapter has explained that the pricing of forwards is similar to that of futures, except when the value of the futures contract is strongly correlated with the reinvestment rate. This is the case with Eurodollar futures.

Interest rate futures contracts are designed to move like a bond, that is, to lose value when interest rates increase. The correlation is negative. This implies that when interest rates rise, the futures contract loses value and in addition funds have to be provided precisely when the borrowing cost or reinvestment rate is higher. Conversely, when rates drop, the contract gains value and the profits can be withdrawn but are now reinvested at a lower rate. Relative to forward contracts, this marking-to-market feature is *disadvantageous* to long futures positions. This has to be offset by a *lower* futures contract value. Given that the value is negatively related to the futures rate, by $P_t = 10{,}000 \times [100 - 0.25 \times F_t]$, this implies a *higher* Eurodollar futures rate F_t.

The difference is called the **convexity adjustment** and can be described as[2]

$$\textbf{Futures Rate} = \textbf{Forward Rate} + (1/2)\sigma^2 t_1 t_2 \tag{10.4}$$

where σ is the volatility of the change in the short-term rate, t_1 is the time to maturity of the futures contract, and t_2 is the maturity of the rate underlying the futures contract.

Example: Convexity Adjustment

Consider a 10-year Eurodollar contract, for which $t_1 = 10$, $t_2 = 10.25$. The maturity of the futures contract itself is 10 years and that of the underlying rate is 10 years plus three months.

Typically, $\sigma = 1\%$, so that the adjustment is $(1/2)0.01^2 \times 10 \times 10.25 = 0.51\%$. So, if the forward price is 6%, the equivalent futures rate would be 6.51%. Note that the effect is significant for long maturities only. Changing t_1 to one year and t_2 to 1.25, for instance, reduces the adjustment to 0.006%, which is negligible.

10.2.2 T-Bond Futures

T-bond futures are futures contracts tied to a pool of Treasury bonds that consists of all bonds with a remaining maturity greater than 15 years (and noncallable

[2] This formula is derived from the Ho-Lee model. See, for instance, John C. Hull, *Options, Futures, and Other Derivatives*, 7th ed. (Upper Saddle River, NJ: Prentice Hall, 2008).

within 15 years). Similar contracts exist on shorter rates, including 2-, 5-, and 10-year Treasury notes. Government bond futures also exist in other markets, including Canada, the United Kingdom, the Eurozone, and Japan.

Futures contracts are quoted like T-bonds, for example 97-02, in percent plus 32nds, with a notional of \$100,000. Thus the price of the contract is $P =$ $\$100{,}000 \times (97 + 2/32)/100 = \$97{,}062.50$. The next day, if yields go up and the quoted price falls to 96-0, the new value is \$96,000, and the loss on the long position is $P_2 - P_1 = -\$1{,}062.50$.

It is important to note that the T-bond futures contract is settled by physical delivery. To ensure interchangeability between the deliverable bonds, the futures contract uses a **conversion factor** (CF) for delivery. This factor multiplies the futures price for payment to the short. The goal of the CF is to attempt to equalize the net cost of delivering the various eligible bonds.

The conversion factor is needed because bonds trade at widely different prices. High-coupon bonds trade at a premium, low-coupon bonds at a discount. Without this adjustment, the party with the short position (the short) would always deliver the same cheap bond and there would be little exchangeability between bonds. Exchangeability is an important feature, however, as it minimizes the possibility of market squeezes. A **squeeze** occurs when holders of the short position cannot acquire or borrow the securities required for delivery under the terms of the contract.

So, the short buys a bond, delivers it, and receives the quoted futures price times a conversion factor that is specific to the delivered bond (plus accrued interest). The short should rationally pick the bond that minimizes the net cost:

$$\text{Cost} = \text{Price} - \text{Futures Quote} \times \text{CF} \tag{10.5}$$

The bond with the lowest net cost is called **cheapest to deliver** (CTD).

In practice, the CF is set by the exchange at initiation of the contract for each bond. It is computed by discounting the bond's cash flows at a notional 6% rate, assuming a flat term structure. Take, for instance, the $7\frac{5}{8}\%$ of 2025. The CF is computed as

$$\text{CF} = \frac{(7.625\%/2)}{(1 + 6\%/2)^1} + \cdots + \frac{(1 + 7.625\%/2)}{(1 + 6\%/2)^T} \tag{10.6}$$

which gives CF = 1.1717. High-coupon bonds have higher CFs. Also, because the coupon is greater than 6%, the CF is greater than one.

The net cost calculations are illustrated in Table 10.1 for three bonds. The net cost for the first bond in the table is $\$104.375 - 110.8438 \times 0.9116 = \$3.330$. For the 6% coupon bond, the CF is exactly unity. The net cost for the third bond in the table is \$1.874. Because this is the lowest entry, this bond is the CTD for this group. Note how the CF adjustment brings the cost of all bonds much closer to each other than their original prices.

The adjustment is not perfect when current yields are far from 6%, or when the term structure is not flat, or when bonds do not trade at their theoretical prices. Assume, for instance, that we operate in an environment where yields are flat at

TABLE 10.1 Calculation of CTD

Bond	Price	Futures	CF	Cost
$5\frac{1}{4}$% Nov. 2028	104.3750	110.8438	0.9116	3.330
6% Feb. 2026	112.9063	110.8438	1.0000	2.063
$7\frac{5}{8}$% Feb. 2025	131.7500	110.8438	1.1717	1.874

5% and all bonds are priced at par. Discounting at 6% will create CF factors that are lower than one; the longer the maturity of the bond, the greater the difference. The net cost $P - F \times \text{CF}$ will then be greater for longer-term bonds. This tends to favor short-term bonds for delivery. When the term structure is upward sloping, the opposite occurs, and there is a tendency for long-term bonds to be delivered.

As a first approximation, this CTD bond drives the characteristics of the futures contract. As before, the equilibrium futures price is given by

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{10.7}$$

where S_t is the gross price of the CTD and PV(D) is the present value of the coupon payments. This has to be further divided by the conversion factor for this bond. The duration of the futures contract is also given by that of the CTD.

In fact, this relationship is only approximate, because the short has an *option* to deliver the cheapest of a group of bonds. The value of this delivery option should depress the futures price because the party who is long the futures is also short the option. As a result, this requires a lower acquisition price. In addition, the short has a **timing option**, which allows delivery on any day of the delivery month. Unfortunately, these complex options are not easy to evaluate.

EXAMPLE 10.4: FRM EXAM 2009—QUESTION 3-11

The yield curve is upward sloping. You have a short T-bond futures position. The following bonds are eligible for delivery:

Bond	A	B	C
Spot price	102-14/32	106-19/32	98-12/32
Coupon	4%	5%	3%
Conversion factor	0.98	1.03	0.952

The futures price is 103-17/32 and the maturity date of the contract is September 1. The bonds pay their coupon semiannually on June 30 and December 31. The cheapest to deliver bond is:

a. Bond A
b. Bond B
c. Bond C
d. Insufficient information

EXAMPLE 10.5: FRM EXAM 2009—QUESTION 3-23

Which of the following statements related to forward and futures prices is *true*?

a. If the forward price does not equal the futures price, arbitragers will exploit this arbitrage opportunity.
b. The level of interest rates determines whether the forward price is higher or lower than the futures price.
c. The volatility of interest rates determines whether the forward price is higher or lower than the futures price.
d. Whether the forward price will be higher or lower than the futures price depends on correlation between interest rate and futures price.

EXAMPLE 10.6: FRM EXAM 2007—QUESTION 80

Consider a forward rate agreement (FRA) with the same maturity and compounding frequency as a Eurodollar futures contract. The FRA has a LIBOR underlying. Which of the following statements is *true* about the relationship between the forward rate and the futures rate?

a. The forward rate is normally higher than the futures rate.
b. They have no fixed relationship.
c. The forward rate is normally lower than the futures rate.
d. They should be exactly the same.

10.3 SWAPS

Swaps are agreements by two parties to exchange cash flows in the future according to a prearranged formula. Interest rate swaps have payments tied to an interest rate. The most common type of swap is the **fixed-for-floating** swap, where one party commits to pay a fixed percentage of notional against a receipt that is indexed to a floating rate, typically LIBOR. The risk is that of a change in the level of rates.

Other types of swaps are **basis swaps**, where both payments are indexed to a floating rate. For instance, the swap can involve exchanging payments tied to three-month LIBOR against a three-month Treasury bill rate. The risk is that of a change in the spread between the reference rates.

10.3.1 Instruments

Consider two counterparties, A and B, that can raise funds at either fixed or floating rates, $100 million over 10 years. A wants to raise floating, and B wants to raise fixed.

Table 10.2a displays capital costs. Company A has an **absolute advantage** in the two markets, as it can raise funds at rates systematically lower than B can. Company A, however, has a **comparative advantage** in raising fixed, as the cost is 1.2% lower than for B. In contrast, the cost of raising floating is only 0.70% lower than for B. Conversely, company B must have a comparative advantage in raising floating.

This provides a rationale for a swap that will be to the mutual advantage of both parties. If both companies directly issue funds in their final desired markets, the total cost will be LIBOR + 0.30% (for A) and 11.20% (for B), for a total of LIBOR + 11.50%. In contrast, the total cost of raising capital where each has a comparative advantage is 10.00% (for A) and LIBOR + 1.00% (for B), for a total of LIBOR + 11.00%. The gain to both parties from entering a swap is

TABLE 10.2a Cost of Capital Comparison

Company	Fixed	Floating
A	10.00%	LIBOR + 0.30%
B	11.20%	LIBOR + 1.00%

TABLE 10.2b Swap to Company A

Operation	Fixed	Floating
Issue debt	Pay 10.00%	
Enter swap	Receive 10.00%	Pay LIBOR + 0.05%
Net		Pay LIBOR + 0.05%
Direct cost		Pay LIBOR + 0.30%
Savings		0.25%

TABLE 10.2c Swap to Company B

Operation	Floating	Fixed
Issue debt	Pay LIBOR + 1.00%	
Enter swap	Receive LIBOR + 0.05%	Pay 10.00%
Net		Pay 10.95%
Direct cost		Pay 11.20%
Savings		0.25%

$11.50\% - 11.00\% = 0.50\%$. For instance, the swap described in Tables 10.2b and 10.2c splits the benefit equally between the two parties.

Company A issues fixed debt at 10.00%, and then enters a swap whereby it promises to pay LIBOR + 0.05% in exchange for receiving 10.00% fixed payments. Its net, effective funding cost is therefore LIBOR + 0.05%, which is less than the direct cost by 25bp.

Similarly, company B issues floating debt at LIBOR + 1.00%, and then enters a swap whereby it receives LIBOR + 0.05% in exchange for paying 10.00% fixed. Its net, effective funding cost is therefore $11.00\% - 0.05\% = 10.95\%$, which is less than the direct cost by 25bp. Both parties benefit from the swap.

In terms of actual cash flows, swap payments are typically *netted* against each other. For instance, if the first LIBOR rate is at 9% assuming annual payments, company A would be owed $10\% \times \$100 = \$1$ million, and have to pay LIBOR + 0.05%, or $9.05\% \times \$100 = \$0.905$ million. This gives a net receipt of \$95,000. There is no need to exchange principals since both involve the same amount.

10.3.2 Quotations

Swaps can be quoted in terms of spreads relative to the yield of similar-maturity Treasury notes. For instance, a dealer may quote 10-year swap spreads as 31/34bp against LIBOR. If the current note yield is 6.72, this means that the dealer is willing to pay $6.72 + 0.31 = 7.03\%$ against receiving LIBOR, or that the dealer is willing to receive $6.72 + 0.34 = 7.06\%$ against paying LIBOR. Of course, the dealer makes a profit from the spread, which is rather small, at 3bp only. Equivalently, the outright quote is 7.03/7.06 for the swap.

Note that the swap should trade at a positive credit spread to Treasuries. This is because the other leg is quoted in relation to LIBOR, which also has credit risk. More precisely, swap rates correspond to the credit risk of AA-rated counterparties.

Table 7.1 has shown that the interest rate swap market is by far the largest derivative market in terms of notional. Because the market is very liquid, market quotations for the fixed-rate leg have become benchmark interest rates. Thus, swap rates form the basis for the **swap curve**, which is also called the par curve, because it is equivalent to yields on bonds selling at par. Because the floating-rate leg is indexed to LIBOR, which carries credit risk, the swap curve is normally higher than the par curve for government bonds in the same currency.

10.3.3 Pricing

We now discuss the pricing of interest rate swaps. Consider, for instance, a three-year \$100 million swap, where we receive a fixed coupon of 5.50% against LIBOR. Payments are annual and we ignore credit spreads. We can price the swap using either of two approaches: taking the difference between two bond prices or valuing a sequence of forward contracts. This is illustrated in Figure 10.2.

The top part of the figure shows that this swap is equivalent to a long position in a three-year fixed-rate 5.5% bond and a short position in a three-year

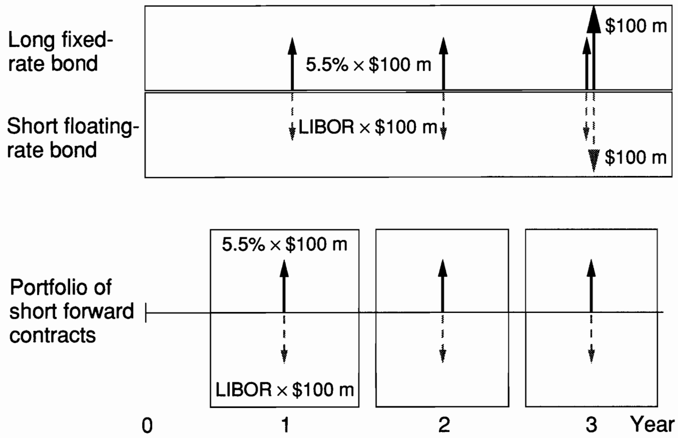

FIGURE 10.2 Alternative Decompositions for Swap Cash Flows

floating-rate note (FRN). If B_F is the value of the fixed-rate bond and B_f is the value of the FRN, the value of the swap is $V = B_F - B_f$.

The value of the FRN should be close to par. Just before a reset, B_f will behave exactly like a cash investment, as the coupon for the next period will be set to the prevailing interest rate. Therefore, its market value should be close to the face value. Just after a reset, the FRN will behave like a bond with a one-year maturity. But overall, fluctuations in the market value of B_f should be small.

Consider now the swap value. If at initiation the swap coupon is set to the prevailing par yield, B_F is equal to the face value, $B_F = 100$. Because $B_f = 100$ just before the reset on the floating leg, the value of the swap is zero: $V = B_F - B_f = 0$. This is like a forward contract at initiation.

After the swap is consummated, its value will be affected by interest rates. If rates fall, the swap will move in-the-money, since it receives higher coupons than prevailing market yields. B_F will increase, whereas B_f will barely change.

Thus the duration of a receive-fixed swap is similar to that of a fixed-rate bond, including the fixed coupons and principal at maturity. This is because the duration of the floating leg is close to zero. The fact that the principals are not exchanged does not mean that the duration computation should not include the principal. Duration should be viewed as an interest rate sensitivity.

KEY CONCEPT

A position in a receive-fixed swap is equivalent to a long position in a bond with similar coupon characteristics and maturity offset by a short position in a floating-rate note. Its duration is close to that of the fixed-rate note.

We now value the three-year swap using term-structure data from the preceding chapter. The time is just before a reset, so $B_f = \$100$ million. We compute

B_F (in millions) as

$$B_F = \frac{\$5.5}{(1+4.000\%)} + \frac{\$5.5}{(1+4.618\%)^2} + \frac{\$105.5}{(1+5.192\%)^3} = \$100.95$$

The outstanding value of the swap is therefore $V = \$100.95 - \$100 = \$0.95$ million.

Alternatively, the swap can be valued as a sequence of forward contracts, as shown in the bottom part of Figure 10.2. Recall from Chapter 7 that the value of a unit position in a long forward contract is given by

$$V_i = (F_i - K)\exp(-r_i\tau_i) \tag{10.8}$$

where F_i is the current forward rate, K the prespecified rate, and r_i the spot rate for time τ_i. Extending this to multiple maturities, and to discrete compounding using R_i, the swap can be valued as

$$V = \sum_i n_i(F_i - K)/(1 + R_i)^{\tau_i} \tag{10.9}$$

where n_i is the notional amount for maturity i.

A long forward rate agreement benefits if rates go up. Indeed, Equation (10.8) shows that the value increases if F_i goes up. In the case of our swap, we *receive* a fixed rate K. So, the position loses money if rates go up, as we could have received a higher rate. Hence, the sign on Equation (10.9) must be reversed.

Using the forward rates listed in Table 9.3, we find

$$V = -\frac{\$100(4.000\% - 5.50\%)}{(1+4.000\%)} - \frac{\$100(5.240\% - 5.50\%)}{(1+4.618\%)^2}$$
$$-\frac{\$100(6.350\% - 5.50\%)}{(1+5.192\%)^3}$$
$$V = +1.4423 + 0.2376 - 0.7302 = \$0.95 \text{ million}$$

This is identical to the previous result, as it should be. The swap is in-the-money primarily because of the first payment, which pays a rate of 5.5% whereas the forward rate is only 4.00%.

Thus, interest rate swaps can be priced and hedged using a sequence of forward rates, such as those implicit in Eurodollar contracts. The practice of daily marking to market of futures induces a slight convexity bias in futures rates, which have to be adjusted downward to get forward rates.

Figure 10.3 compares a sequence of quarterly forward rates with the five-year swap rate prevailing at the same time. Because short-term forward rates are less than the swap rate, the near payments are in-the-money. In contrast, the more distant payments are out-of-the-money. The current market value of this swap is zero, which implies that all the near-term positive values must be offset by distant negative values.

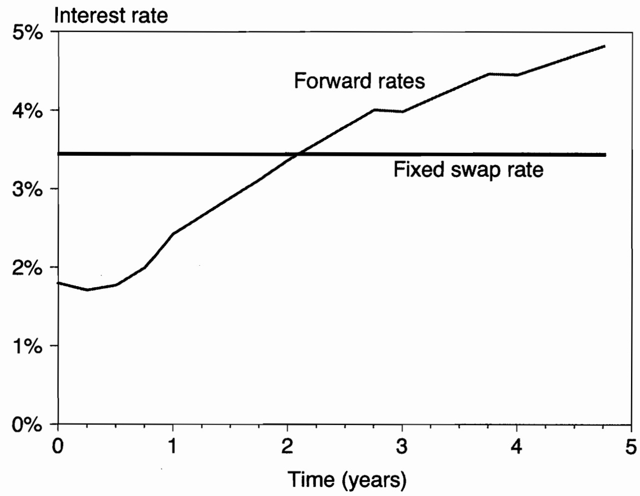

FIGURE 10.3 Sequence of Forward Rates and Swap Rate

EXAMPLE 10.7: FRM EXAM 2005—QUESTION 51

Consider the following information about an interest rate swap: two-year term, semiannual payment, fixed rate = 6%, floating rate = LIBOR + 50 basis points, notional USD 10 million. Calculate the net coupon exchange for the first period if LIBOR is 5% at the beginning of the period and 5.5% at the end of the period.

a. Fixed-rate payer pays USD 0.
b. Fixed-rate payer pays USD 25,000.
c. Fixed-rate payer pays USD 50,000.
d. Fixed-rate payer receives USD 25,000.

EXAMPLE 10.8: FRM EXAM 2000—QUESTION 55

Bank XYZ enters into a five-year swap contract with ABC Co. to pay LIBOR in return for a fixed 8% rate on a principal of $100 million. Two years from now, the market rate on three-year swaps at LIBOR is 7%. At this time ABC Co. declares bankruptcy and defaults on its swap obligation. Assume that the net payment is made only at the end of each year for the swap contract period. What is the market value of the loss incurred by Bank XYZ as a result of the default?

a. $1.927 million
b. $2.245 million
c. $2.624 million
d. $3.011 million

EXAMPLE 10.9: FRM EXAM 2009—QUESTION 3-4

A bank entered into a three-year interest rate swap for a notional amount of USD 250 million, paying a fixed rate of 7.5% and receiving LIBOR annually. Just after the payment was made at the end of the first year, the continuously compounded spot one-year and two-year LIBOR rates are 8% and 8.5%, respectively. The value of the swap at that time is closest to

a. USD 14 million
b. USD −6 million
c. USD −14 million
d. USD 6 million

10.4 OPTIONS

There is a large variety of fixed-income options. We briefly describe here caps and floors, swaptions, and exchange-traded options. In addition to these stand-alone instruments, fixed-income options are embedded in many securities. For instance, a callable bond can be viewed as a regular bond plus a short position in an option.

When considering fixed-income options, the underlying can be a yield or a price. Due to the negative price-yield relationship, a call option on a bond can also be viewed as a put option on the underlying yield.

10.4.1 Caps and Floors

A **cap** is a call option on interest rates with unit value

$$C_T = \text{Max}[i_T - K, 0] \tag{10.10}$$

where $K = i_C$ is the cap rate and i_T is the rate prevailing at maturity.

In practice, caps are purchased jointly with the issuance of floating-rate notes (FRNs) that pay LIBOR plus a spread on a periodic basis for the term of the note. By purchasing the cap, the issuer ensures that the cost of capital will not exceed the capped rate. Such caps are really a combination of individual options, called **caplets.**

The payment on each caplet is determined by C_T, the notional, and an accrual factor. Payments are made in **arrears**, that is, at the end of the period. For instance, take a one-year cap on a notional of $1 million and a six-month LIBOR cap rate of 5%. The agreement period is from January 15 to the next January with a reset on July 15. Suppose that on July 15, LIBOR is at 5.5%. On the following January, the payment is

$$\$1 \text{ Million} \times (0.055 - 0.05)(184/360) = \$2{,}555.56$$

using *Actual*/360 interest accrual. If the cap is used to hedge an FRN, this would help to offset the higher coupon payment, which is now 5.5%.

A **floor** is a put option on interest rates with value

$$P_T = \text{Max}[K - i_T, 0] \tag{10.11}$$

where $K = i_F$ is the floor rate. A **collar** is a combination of buying a cap and selling a floor. This combination decreases the net cost of purchasing the cap protection. Figure 10.4 shows an example of a price path, with a cap rate of 3.5% and a floor rate of 2%. There are three instances where the cap is exercised, leading to a receipt of payment. There is one instance where the rate is below the floor, requiring a payment.

When the cap and floor rates converge to the same value $K = i_C = i_F$, the overall debt cost becomes fixed instead of floating. The collar is then the same as a pay-fixed swap, which is the equivalent of put-call parity,

$$\text{Long Cap}(i_C = K) + \text{Short Floor}(i_F = K) = \text{Long Pay-Fixed Swap} \tag{10.12}$$

Caps are typically priced using a variant of the Black model, assuming that interest rate changes are lognormal. The value of the cap is set equal to a portfolio of N caplets, which are European-style individual options on different interest rates with regularly spaced maturities

$$c = \sum_{j=1}^{N} c_j \tag{10.13}$$

For each caplet, the unit price is

$$c_j = [F\,N(d_1) - K\,N(d_2)]\text{PV}(\$1) \tag{10.14}$$

where F is the current forward rate for the period t_j to t_{j+1}, K is the cap rate, and PV($1) is the discount factor to time t_{j+1}. To obtain a dollar amount, we must adjust for the notional amount as well as the length of the accrual period.

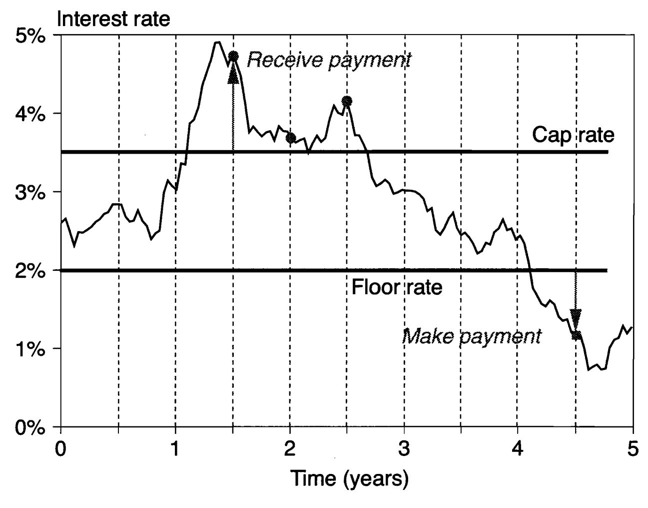

FIGURE 10.4 Exercise of Cap and Floor

The volatility entering the pricing model, σ, is that of the forward rate between now and the expiration of the option contract, that is, at t_j. Generally, volatilities are quoted as one number for all caplets within a cap; this is called **flat volatilities.**

$$\sigma_j = \sigma$$

Alternatively, volatilities can be quoted separately for each forward rate in the caplet; this is called **spot volatilities.**

Example: Computing the Value of a Cap

Consider the previous cap on \$1 million at the capped rate of 5%. Assume a flat term structure at 5.5% and a volatility of 20% pa. The reset is on July 15, in 181 days. The accrual period is 184 days.

Since the term structure is flat, the six-month forward rate starting in six months is also 5.5%. First, we compute the present value factor, which is $PV(\$1) = 1/(1 + 0.055 \times 365/360) = 0.9472$, and the volatility, which is $\sigma\sqrt{\tau} = 0.20\sqrt{181/360} = 0.1418$.

We then compute the value of $d_1 = \ln[F/K]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = \ln[0.055/0.05]/0.1418 + 0.1418/2 = 0.7430$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.7430 - 0.1418 = 0.6012$. We find $N(d_1) = 0.7713$ and $N(d_2) = 0.7261$. The unit value of the call is $c = [F N(d_1) - K N(d_2)]PV(\$1) = 0.5789\%$. Finally, the total price of the call is \$1 million $\times$ 0.5789% $\times$ (184/360) = \$2,959.

Figure 10.3 can be taken as an illustration of the sequence of forward rates. If the cap rate is the same as the prevailing swap rate, the cap is said to be *at-the-money*. In the figure, the near caplets are out-of-the-money because $F_i < K$. The distant caplets, however, are in-the-money.

EXAMPLE 10.10: FRM EXAM 2002—QUESTION 22

An interest rate cap runs for 12 months based on three-month LIBOR with a strike price of 4%. Which of the following is generally *true*?

a. The cap consists of three caplet options with maturities of three months, the first one starting today based on three-month LIBOR set in advance and paid in arrears.
b. The cap consists of four caplets starting today, based on LIBOR set in advance and paid in arrears.
c. The implied volatility of each caplet will be identical no matter how the yield curve moves.
d. Rate caps have only a single option based on the maturity of the structure.

EXAMPLE 10.11: FRM EXAM 2004—QUESTION 10

The payoff to a swap where the investor receives fixed and pays floating can be replicated by all of the following *except*

a. A short position in a portfolio of FRAs
b. A long position in a fixed-rate bond and a short position in a floating-rate bond
c. A short position in an interest rate cap and a long position in a floor
d. A long position in a floating-rate note and a short position in a floor

EXAMPLE 10.12: FRM EXAM 2003—QUESTION 27

A portfolio management firm manages the fixed-rate corporate bond portfolio owned by a defined-benefit pension fund. The duration of the bond portfolio is five years; the duration of the pension fund's liabilities is seven years. Assume that the fund sponsor strongly believes that rates will decline over the next six months and is concerned about the duration mismatch between portfolio assets and pension liabilities. Which of the following strategies would be the best way to eliminate the duration mismatch?

a. Enter into a swap transaction in which the firm pays fixed and receives floating.
b. Enter into a swap transaction in which the firm receives fixed and pays floating.
c. Purchase an interest rate cap expiring in six months.
d. Sell Eurodollar futures contracts.

10.4.2 Swaptions

Swaptions are over-the-counter (OTC) options that give the buyer the right to enter a swap at a fixed point in time at specified terms, including a fixed coupon rate.

These contracts take many forms. A **European swaption** is exercisable on a single date at some point in the future. On that date, the owner has the right to enter a swap with a specific rate and term. Consider, for example, a 1Y × 5Y swaption. This gives the owner the right to enter in one year a long or short position in a five-year swap.

A fixed-term **American swaption** is exercisable on any date during the exercise period. In our example, this would be during the next year. If, for instance, exercise occurs after six months, the swap would terminate in five years and six months from now. So, the termination date of the swap depends on the exercise date. In contrast, a **contingent American swaption** has a prespecified termination date, for instance exactly six years from now. Finally, a **Bermudan option** gives the holder the right to exercise on a specific set of dates during the life of the option.

As an example, consider a company that, in one year, will issue five-year floating-rate debt. The company wishes to have the option to swap the floating payments into fixed payments. The company can purchase a swaption that will give it the right to create a five-year pay-fixed swap at the rate of 8%. If the prevailing swap rate in one year is higher than 8%, the company will exercise the swaption, otherwise not. The value of the option at expiration will be

$$P_T = \text{Max}[V(i_T) - V(K), 0] \tag{10.15}$$

where $V(i)$ is the value of a swap to pay a fixed rate i, i_T is the prevailing swap rate for the swap maturity, and K is the locked-in swap rate. This contract is called a European 6/1 put swaption, or one into five-year payer option.

Such a swap is equivalent to an option on a bond. As this swaption creates a profit if rates rise, it is akin to a one-year put option on a six-year bond. A put option benefits when the bond value falls, which happens when rates rise. Conversely, a swaption that gives the right to receive fixed is akin to a call option on a bond. Table 10.3 summarizes the terminology for swaps, caps and floors, and swaptions.

Swaptions can be used for a variety of purposes. Consider an investor in a mortgage-backed security (MBS). If long-term rates fall, prepayment will increase, leading to a shortfall in the price appreciation in the bond. This risk can be hedged by buying receiver swaptions. If rates fall, the buyer will exercise the option, which creates a profit to offset the loss on the MBS. Alternatively, this risk can also be hedged by issuing callable debt. This creates a long position in an option that generates a profit if rates fall. As an example, Fannie Mae, a government-sponsored enterprise that invests heavily in mortgages, uses these techniques to hedge its prepayment risk.

TABLE 10.3 Summary of Terminology for OTC Swaps and Options

Product	Buy (Long)	Sell (Short)
Fixed/floating swap	Pay fixed Receive floating	Pay floating Receive fixed
Cap	Pay premium Receive Max($i - i_C$, 0)	Receive premium Pay Max($i - i_C$, 0)
Floor	Pay premium Receive Max($i_F - i$, 0)	Receive premium Pay Max($i_F - i$, 0)
Put swaption (payer option)	Pay premium Option to pay fixed and receive floating	Receive premium If exercised, receive fixed and pay floating
Call swaption (receiver option)	Pay premium Option to pay floating and receive fixed	Receive premium If exercised, receive floating and pay fixed

Finally, swaptions are typically priced using a variant of the Black model, assuming that interest rates are lognormal. The value of the swaption is then equal to a portfolio of options on different interest rates, all with the same maturity. In practice, swaptions are traded in terms of volatilities instead of option premiums. The applicable forward rate starts at the same time as the option, with a term equal to that of the option.

EXAMPLE 10.13: FRM EXAM 2003—QUESTION 56

As your company's risk manager, you are looking for protection against adverse interest rate changes in five years. Using Black's model for options on futures to price a European swap option (swaption) that gives the option holder the right to cancel a seven-year swap after five years, which of the following would you use in the model?

a. The two-year forward par swap rate starting in five years' time
b. The five-year forward par swap rate starting in two years' time
c. The two-year par swap rate
d. The five-year par swap rate

10.4.3 Exchange-Traded Options

Among exchange-traded fixed-income options, we describe options on Eurodollar futures and on T-bond futures.

Options on Eurodollar futures give the owner the right to enter a long or short position in Eurodollar futures at a fixed price. The payoff on a put option, for example, is

$$P_T = \text{Notional} \times \text{Max}[K - \text{FQ}_T, 0] \times (90/360) \qquad (10.16)$$

where K is the strike price and FQ_T the prevailing futures price quote at maturity. In addition to the cash payoff, the option holder enters a position in the underlying futures. Since this is a put, it creates a short position after exercise, with the counterparty taking the opposing position. Note that, since futures are settled daily, the value of the contract is zero.

Since the futures price can also be written as $\text{FQ}_T = 100 - i_T$ and the strike price as $K = 100 - i_C$, the payoff is also

$$P_T = \text{Notional} \times \text{Max}[i_T - i_C, 0] \times (90/360) \qquad (10.17)$$

which is equivalent to that of a cap on rates. Thus a put on Eurodollar futures is equivalent to a caplet on LIBOR.

In practice, there are minor differences in the contracts. Options on Eurodollar futures are American style instead of European style. Also, payments are made at the expiration date of Eurodollar futures options instead of in arrears.

Options on T-bond futures give the owner the right to enter a long or short position in futures at a fixed price. The payoff on a call option, for example, is

$$C_T = \text{Notional} \times \text{Max}[F_T - K, 0] \tag{10.18}$$

An investor who thinks that rates will fall, or that the bond market will rally, could buy a call on T-bond futures. In this manner, he or she will participate in the upside, without downside risk.

EXAMPLE 10.14: FRM EXAM 2007—QUESTION 95

To hedge against future, unanticipated, and significant increases in borrowing rates, which of the following alternatives offers the greatest flexibility for the borrower?

a. Interest rate collar
b. Fixed for floating swap
c. Call swaption
d. Interest rate floor

EXAMPLE 10.15: FRM EXAM 2009—QUESTION 2-24

The yield curve is upward sloping and a portfolio manager has a long position in 10-year Treasury notes funded through overnight repurchase agreements. The risk manager is concerned with the risk that market rates may increase further and reduce the market value of the position. What hedge could be put on to reduce the position's exposure to rising rates?

a. Enter into a 10-year pay-fixed and receive-floating interest rate swap.
b. Enter into a 10-year receive-fixed and pay-floating interest rate swap.
c. Establish a long position in 10-year Treasury note futures.
d. Buy a call option on 10-year Treasury note futures.

10.5 IMPORTANT FORMULAS

Long 1 × 4 FRA = Invest for one period, borrow for four

Payment on FRA: $V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1)$

Valuation of Eurodollar contract: $P_t = 10{,}000 \times [100 - 0.25(100 - FQ_t)] = 10{,}000 \times [100 - 0.25F_t]$

Eurodollar contract risk: DV01 = \$25

Futures convexity adjustment: Futures Rate = Forward Rate + $(1/2)\sigma^2 t_1 t_2$ (negative relationship between contract value and rates)

T-bond futures net delivery cost: Cost = Price − Futures Quote × CF

T-bond futures conversion factor: CF = NPV of bond at 6%

Valuation of interest rate swap: $V = B_F(\text{Fixed-Rate}) - B_f(\text{Floating-Rate})$ Long Receive-Fixed = Long Fixed-Coupon Bond + Short FRN

Valuation of interest rate swap as forward contracts: $V = \sum_i n_i(F_i - K)/(1 + R_i)^{\tau_i}$

Interest-rate cap: $C_T = \text{Max}[i_T - K, 0]$

Interest-rate floor: $P_T = \text{Max}[K - i_T, 0]$

Collar: Long cap plus short floor

Cap valuation: $c = \sum_{j=1}^{N} c_j, \quad c_j = [FN(d_1) - KN(d_2)]\text{PV}(\$1)$

Put swaption (1Y × 5Y) (right to pay fixed, starting in one year for five years): $P_T = \text{Max}[V(i_T) - V(K), 0]$

Put option on Eurodollar futures = cap on rates

10.6 ANSWERS TO CHAPTER EXAMPLES

Example 10.1: FRM Exam 2002—Question 27

b. An FRA defined as $t_1 \times t_2$ involves a forward rate starting at time t_1 and ending at time t_2. The buyer of this FRA locks in a borrowing rate for months 3 to 5. This is equivalent to borrowing for five months and reinvesting the funds for the first two months.

Example 10.2: FRM Exam 2005—Question 57

d. The market-implied forward rate is given by $\exp(-R_2 \times 2) = \exp(-R_1 \times 1 - F_{1,2} \times 1)$, or $F_{1,2} = 2 \times 3.50 - 1 \times 3.25 = 3.75\%$. Given that this is exactly equal to the quoted rate, the value must be zero. If instead this rate was 3.50%, for example, the value would be $V = \$1{,}000{,}000 \times (3.75\% - 3.50\%) \times (2 - 1)$ $\exp(-3.50\% \times 2) = 2{,}331$.

Example 10.3: FRM Exam 2001—Question 70

b. The seller of an FRA agrees to receive fixed. Since rates are now higher than the contract rate, this contract must show a loss for the seller. The loss is \$10,000,000 × (6.85% − 6.35%) × (90/360) = \$12,500 when paid in arrears (i.e., in nine months). On the settlement date (i.e., brought forward by three months), the loss is \$12,500/(1 + 6.85% × 0.25) = \$12,290.

Example 10.4: FRM Exam 2009—Question 3-11

b. The cost of delivering each bond is the price divided by the conversion factor. This gives, respectively, $(102 + 14/32)/0.98 = 104.53$, 103.49, and 103.55. Hence the CTD is bond B. All other information is superfluous.

Example 10.5: FRM Exam 2009—Question 3-23

d. Forward rates may not equal futures rates due to the correlation between the interest rate, or reinvestment rate, and the futures contract profit. As seen in Equation (10.4), the volatility determines the size of the bias but not the direction.

Example 10.6: FRM Exam 2007—Question 80

c. Equation (10.4) shows that the futures rate exceeds the forward rate.

Example 10.7: FRM Exam 2005—Question 51

b. The floating leg uses LIBOR at the beginning of the period, plus 50bp, or 5.5%. The payment is given by $\$10,000,000 \times (0.06 - 0.055) \times 0.5 = 25,000$.

Example 10.8: FRM Exam 2000—Question 55

c. Using Equation (10.9) for three remaining periods, we have the discounted value of the net interest payment, or $(8\% - 7\%)\$100\text{m} = \$1\text{m}$, discounted at 7%, which is $\$934,579 + \$873,439 + \$816,298 = \$2,624,316$.

Example 10.9: FRM Exam 2009—Question 3-4

d. This question differs from the previous one, which gave the swap rate. Here, we have the spot rates for maturities of one and two years. The coupon is 7.5. The net present value (NPV) of the payments is then $V = \$18.75\exp(-1 \times 8\%) + (\$250 + \$18.75)\exp(-2 \times 8.5\%) = \$244$ million. Right after the reset, the value of the FRN is \$250 million, leading to a gain of \$6 million. This is a gain because the bank must pay a fixed rate but current rates are higher.

Example 10.10: FRM Exam 2002—Question 22

a. Interest rate caps involve multiple options, or caplets. The first one has terms that are set in three months. It locks in $\text{Max}[R(t + 3) - 4\%, 0]$. Payment occurs in arrears in six months. The second one is a function of $\text{Max}[R(t + 6) - 4\%, 0]$. The third is a function of $\text{Max}[R(t + 9) - 4\%, 0]$ and is paid at $t + 12$. The sequence then stops because the cap has a term of 12 months only. This means there are three caplets.

Example 10.11: FRM Exam 2004—Question 10

d. A receive-fixed swap position is equivalent to being long a fixed-rate bond, or being short a portfolio of FRAs (which gain if rates go down), or selling a cap and buying a floor with the same strike price (which gains if rates go up). A short position in a floor does not generate a gain if rates drop. It is asymmetric anyway.

Example 10.12: FRM Exam 2003—Question 27

b. The manager should increase the duration of assets, or buy coupon-paying bonds. This can be achieved by entering a receive-fixed swap, so b. is correct and a. is wrong. Buying a cap will not provide protection if rates drop. Selling Eurodollar futures will lose money if rates drop.

Example 10.13: FRM Exam 2003—Question 56

a. The forward rate should start at the beginning of the option in five years, with a maturity equal to the duration of the option, or two years.

Example 10.14: FRM Exam 2007—Question 95

c. A swaption gives the borrower the flexibility to lock in a low rate. A regular swap does not offer flexibility as an option. A collar fixes a range of rates, but not much flexibility. A floor involves protection if rates go down, not up. (Note that buying a cap would have been another good choice.)

Example 10.15: FRM Exam 2009—Question 2-24

a. The bond position has positive duration. Entering a pay-fixed swap gains if rates go up; this negative duration can provide a hedge against the original position. Answer b. is thus incorrect. Answer c. is the same as the original position and is not a hedge. In answer d., a call on futures would not create a profit if rates go up, in which case the futures would go down. Buying a put would be a correct answer.

CHAPTER 11

Equity, Currency, and Commodity Markets

Having covered fixed-income instruments, we now turn to equity, currency, and commodity markets. Equities, or common stocks, represent ownership shares in a corporation. Due to the uncertainty in their cash flows, as well as in the appropriate discount rate, equities are much more difficult to value than fixed-income securities. They are also less amenable to the quantitative analysis that is used in fixed-income markets. Equity derivatives, however, can be priced reasonably precisely in relation to underlying stock prices.

Next, the foreign currency markets include spot, forward, options, futures, and swaps markets. The foreign exchange markets are by far the largest financial markets in the world, with daily turnover above $3 trillion.

Commodity markets consist of agricultural products, metals, energy, and other products. Commodities differ from financial assets, as their holding provides an implied benefit known as convenience yield but also incurs storage costs.

Section 11.1 introduces equity markets and presents valuation methods as well as some evidence on equity risk. Section 11.2 then provides an overview of important equity derivatives, including stock index contracts such as futures, options, and swaps as well as derivatives on single stocks. For most of these contracts, pricing methods have been developed in the previous chapters and do not require special treatment. Convertible bonds and warrants will be covered in a separate chapter.

Section 11.3 presents a brief introduction to currency markets. Currency derivatives are discussed in Section 11.4. We analyze currency swaps in some detail because of their unique features and importance. Finally, Sections 11.5 and 11.6 discuss commodity markets and commodity derivatives.

11.1 EQUITIES

11.1.1 Overview

Common stocks, also called **equities**, are securities that represent ownership in a corporation. Bonds are *senior* to equities; that is, they have a prior claim on

FRM Exam Part 1 topic.

the firm's assets in case of bankruptcy. Hence equities represent **residual claims** to what is left of the value of the firm after bonds, loans, and other contractual obligations have been paid off.

Another important feature of common stocks is their **limited liability**, which means that the most shareholders can lose is their original investment. This is unlike owners of unincorporated businesses, whose creditors have a claim on the personal assets of the owner should the business turn bad.

Table 11.1 describes the global equity markets. The total market value of common stocks was worth approximately $48 trillion at the end of 2009. The United States accounts for the largest share, followed by Japan, the Eurozone, and the United Kingdom. In 2008, global stocks fell by 42%, which implies a loss of market value of about $26 trillion. About $10 trillion of this was recovered in 2009. Thus, investing in equities involves substantial risks.

Preferred stocks differ from common stock because they promise to pay a specific stream of dividends. So, they behave like a perpetual bond, or consol. Unlike bonds, however, failure to pay these dividends does not result in default. Instead, the corporation must withhold dividends to holders of common stock until the preferred dividends have been paid out. In other words, preferred stocks are junior to bonds, but senior to common stocks.

With **cumulative preferred dividends**, all current and previously postponed dividends must be paid before any dividends on common stock shares can be paid. Preferred stocks usually have no voting rights.

Unlike interest payments, preferred stocks' dividends are not tax-deductible expenses. Preferred stocks, however, have an offsetting tax advantage. Corporations that receive preferred dividends pay taxes on only 30% of the amount received, which lowers their income tax burden. As a result, most preferred stocks are held by corporations. The market capitalization of preferred stocks is much lower than that of common stocks, as seen from the IBM example. Trading volumes are also much lower.

TABLE 11.1 Global Equity Markets, 2009 (Billions of U.S. Dollars)

United States	15,077
Japan	3,444
Eurozone	7,271
United Kingdom	2,796
Other Europe	2,109
Other Pacific	4,084
Canada	1,677
Developed	36,459
Emerging	7,239
World	47,783

Source: World Federation of Exchanges.

Example: IBM Preferred Stock

IBM issued 11.25 million preferred shares in June 1993. These are traded as 45 million depositary shares, each representing one-fourth of the preferred, under the ticker IBM-A on the New York Stock Exchange (NYSE). Dividends accrue at the rate of \$7.50 per annum, or \$1.875 per depositary share.

As of April 2001, the depositary shares were trading at \$25.40, within a narrow 52-week trading range between \$25.00 and \$26.25. Using the valuation formula for a consol, the shares were trading at an implied yield of 7.38%. The total market capitalization of the IBM-A shares amounts to approximately \$1,143 million. In comparison, the market value of the common stock is \$214,602 million, which is much larger.

11.1.2 Valuation

Common stocks are difficult to value. Like any other asset, their value derives from their future benefits, that is, from their stream of future cash flows (i.e., dividend payments) or future stock price.

We have seen that valuing Treasury bonds is relatively straightforward, as the stream of cash flows, coupon, and principal payments can be easily laid out and discounted into the present. It is an entirely different affair for common stocks. Consider for illustration a simple case where a firm pays out a dividend D over the next year that grows at the constant rate of g. We ignore the final stock value and discount at the constant rate of r, such that $r > g$. The firm's value, P, can be assessed using the net present value formula, like a bond:

$$
\begin{aligned}
P &= \textstyle\sum_{t=1}^{\infty} C_t/(1+r)^t \\
&= \textstyle\sum_{t=1}^{\infty} D(1+g)^{(t\;\;1)}/(1+r)^t \\
&= [D/(1+r)] \textstyle\sum_{t=0}^{\infty} [(1+g)/(1+r)]^t \\
&= [D/(1+r)] \times \left[\frac{1}{1-(1+g)/(1+r)}\right] \\
&= [D/(1+r)] \times [(1+r)/(r-g)]
\end{aligned}
$$

This is also the so-called Gordon growth model,

$$P = \frac{D}{r-g} \tag{11.1}$$

The problem with equities is that the growth rate of dividends is uncertain and, in addition, it is not clear what the required discount rate should be. To make things even harder, some companies simply do not pay any dividend and instead create value from the appreciation of their share price.

Still, this valuation formula indicates that large variations in equity prices can arise from small changes in the discount rate or in the growth rate of dividends, thus explaining the large volatility of equities. More generally, the value of the equity depends on the underlying business fundamentals as well as on the amount of leverage, or debt, in the capital structure.

11.1.3 Equity Risk

Equity risk arises from potential movements in the value of stock prices. We can usefully decompose the total risk of an equity portfolio into a marketwide risk and stock-specific risk. Focusing on volatility as a single measure of risk, stock index volatility typically ranges from 12% to 20% per annum.

Markets that are less diversified are typically more volatile. **Concentration** refers to the proportion of the index due to the biggest stocks. In Finland, for instance, half of the index represents one firm only, Nokia, which makes the index more volatile than it otherwise would be.

11.2 EQUITY DERIVATIVES

Equity derivatives can be traded in over-the-counter (OTC) markets as well as on organized exchanges. We consider only the most popular instruments.

11.2.1 Stock Index Contracts

Derivative contracts on stock indices are widely used due to their ability to hedge general stock market risks. Active contracts include stock index futures and their options as well as index swaps.

Stock index futures are the most active derivative contracts on stock indices and are traded all over the world. In fact, the turnover corresponding to the notional amount is sometimes greater than the total amount of trading in physical stocks in the same market. The success of these contracts can be explained by their versatility for risk management. Stock index futures allow investors to manage efficiently their exposure to broad stock market movements. Speculators can take easily directional bets with futures, on the upside or downside. Hedgers also find that futures provide a cost-efficient method to protect against price risk.

Perhaps the most active contract is the S&P 500 futures contract on the Chicago Mercantile Exchange (now CME Group). The contract notional is defined as $250 times the index level. Table 11.2 displays quotations as of December 31, 1999.

The table shows that most of the volume was concentrated in the near contract, that is, March in this case. Translating the trading volume in number of contracts into a dollar equivalent, we find $\$250 \times 1{,}484.2 \times 34{,}897$, which gives

TABLE 11.2 Sample S&P Futures Quotations

Maturity	Open	Settle	Change	Volume	Open Interest
March	1,480.80	1,484.20	+3.40	34,897	356,791
June	1,498.00	1,503.10	+3.60	410	8,431

$13 billion. More recently, in 2009, the average daily volume was $33 billion. This is nearly half the trading volume of $71 billion for stocks on the New York Stock Exchange (NYSE). So, these markets are very liquid.

We can also compute the daily profit on a long position, which would have been $250 × (+3.40), or $850 on that day. In relative terms, this daily move was +3.4/1,480.8, which is only 0.23%. The typical daily standard deviation is about 1%, which gives a typical profit or loss of $3,710.50.

These contracts are cash settled. They do not involve delivery of the underlying stocks at expiration. In terms of valuation, the futures contract is priced according to the usual cash-and-carry relationship,

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{11.2}$$

where y is the dividend yield defined per unit time. For instance, the yield on the S&P was $y = 0.94\%$ per annum on that day.

Here, we assume that the dividend yield is known in advance and paid on a continuous basis, which is a good approximation. With a large number of firms in the index, dividends will be spread reasonably evenly over the quarter.

To check whether the futures contract was fairly valued, we need the spot price, $S = 1{,}469.25$, the short-term interest rate, $r = 5.3\%$, and the number of days to maturity, which was 76 (to March 16). Note that rates are not continuously compounded. The present value factor is $\mathrm{PV}(\$1) = 1/(1 + r\tau) = 1/(1 + 5.3\%(76/365)) = 0.9891$. Similarly, the present value of the dividend stream is $1/(1 + y\tau) = 1/(1 + 0.94\%(76/365)) = 0.9980$. The fair price is then

$$F = [S/(1 + y\tau)]\,(1 + r\tau) = [1{,}469.25 \times 0.9980]/0.9891 = 1{,}482.6$$

This is rather close to the settlement value of $F = 1{,}484.2$. The discrepancy is probably because the quotes were not measured simultaneously. Because the yield is less than the interest rate, the forward price is greater than the spot price.

Figure 11.1 displays the convergence of futures and cash prices for the S&P 500 stock index futures contract traded on the CME. Note two major features. First, the futures price is always above the spot price as predicted. The difference, however, shrinks to zero as the contract goes to maturity. Second, the correlation between the two prices is very high, reflecting the cash-and-carry relationship in Equation (11.2).

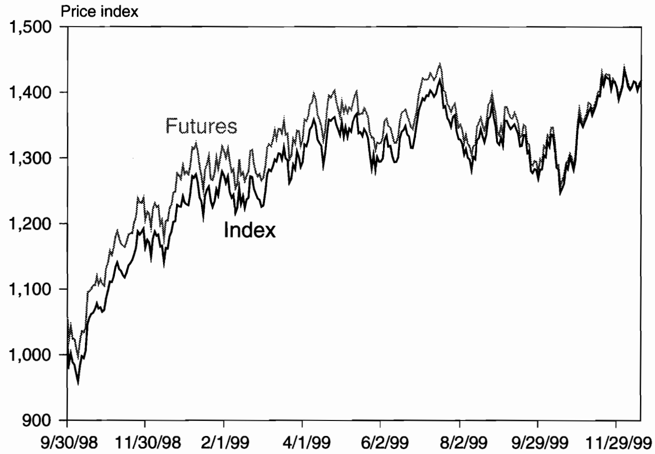

FIGURE 11.1 Futures and Cash Prices for S&P 500 Futures

Because financial institutions engage in stock index arbitrage, we would expect the cash-and-carry relationship to hold very well. One notable exception was during the market crash of October 19, 1987. The market lost more than 20% in a single day. Throughout the day, however, futures prices were more up-to-date than cash prices because of execution delays in cash markets. As a result, the S&P stock index futures value was very cheap compared with the underlying cash market. Arbitrage, however, was made difficult due to chaotic market conditions.

Next, **equity swaps** are agreements to exchange cash flows tied to the return on a stock market index in exchange for a fixed or floating rate of interest. An example is a swap that provides the return on the S&P 500 index every six months in exchange for payment of LIBOR plus a spread. The swap will be typically priced so as to have zero value at initiation. Equity swaps can be valued as portfolios of forward contracts, as in the case of interest rate swaps. These swaps are used by investment managers to acquire exposure to, for example, an emerging stock market without having to invest in the market itself. In some cases, these swaps can also be used to skirt restrictions on foreign investments.

EXAMPLE 11.1: FRM EXAM 2000—QUESTION 12

Suppose the price for a six-month S&P index futures contract is 552.3. If the risk-free interest rate is 7.5% per year and the dividend yield on the stock index is 4.2% per year, and the market is complete and there is no arbitrage, what is the price of the index today?

a. 543.26
b. 552.11
c. 555.78
d. 560.02

EXAMPLE 11.2: FRM EXAM 2009—QUESTION 3-1

A stock index is valued at USD 750 and pays a continuous dividend at the rate of 2% per annum. The six-month futures contract on that index is trading at USD 757. The risk-free rate is 3.50% continuously compounded. There are no transaction costs or taxes. Is the futures contract priced so that there is an arbitrage opportunity? If yes, which of the following numbers comes closest to the arbitrage profit you could realize by taking a position in one futures contract?

a. $4.18
b. $1.35
c. $12.60
d. There is no arbitrage opportunity.

11.2.2 Single Stock Contracts

Derivative contracts tied to single stocks are also widely used. These include futures and options as well as contracts for differences.

In late 2000, the United States passed legislation authorizing trading in **single stock futures**, which are futures contracts on individual stocks. Such contracts were already trading in Europe and elsewhere. In the United States, electronic trading started in November 2002 and now takes place on OneChicago, a joint venture of Chicago exchanges.

Each contract gives the obligation to buy or sell 100 shares of the underlying stock. Settlement usually involves physical delivery, that is, the exchange of the underlying stock. Relative to trading in the underlying stocks, single stock futures have many advantages. Positions can be established more efficiently due to their low margin requirements, which are generally 20% of the cash value. In contrast, margins for stocks are higher. Also, short selling eliminates the costs and inefficiencies associated with the stock loan process. Other than physical settlement, these contracts trade like stock index futures.

Contracts for differences (CFDs) are contracts whose payoff is tied to the value of the underlying stock. CFDs were originally developed in the early 1990s in London, in large part to avoid an expensive stamp duty, which is a UK tax on trades involving the physical trading of stocks. Like futures, CFDs are subject to margin requirements. Payoffs are tied to the change in the price of the stock and a financing charge. Dividends are passed on to the long CFD position. CFDs have no expiration and can be rolled over as needed, provided the margin requirements are met. CFDs are traded over-the-counter with a broker or market maker.

11.2.3 Equity Options

Options can be traded on individual stocks, on stock indices, or on stock index futures. In the United States, stock options trade, for example, on the Chicago

Board Options Exchange (CBOE). Each option gives the right to buy or sell a round lot of 100 shares. Settlement involves physical delivery.

Traded options are typically American-style, so their valuation should include the possibility of early exercise. In practice, however, their values do not differ much from those of European options, which can be priced by the Black-Scholes model. When the stock pays no dividend, the values are the same. For more precision, we can use numerical models such as binomial trees to take into account dividend payments.

The most active *index* options in the United States are options on the S&P 100 and S&P 500 index traded on the CBOE. The former are American-style, while the latter are European-style. These options are cash settled, as it would be too complicated to deliver a basket of 100 or 500 underlying stocks. Each contract is for $100 times the value of the index. European options on stock indices can be priced using the Black-Scholes formula. Finally, options on S&P 500 stock index futures are also popular. These give the right to enter a long or short futures position at a fixed price.

11.3 CURRENCIES

11.3.1 Overview

The foreign exchange (**forex**) or currency markets have enormous trading activity, with daily turnover estimated at $3,210 billion in 2007. Their size and growth are described in Table 11.3. This trading activity dwarfs that of bond or stock markets. In comparison, the daily trading volume on the New York Stock Exchange (NYSE) is approximately $80 billion. Even though the largest share of these transactions is between dealers or with other financial institutions, the volume of trading with other, nonfinancial institutions is still quite large, at $549 billion daily.

TABLE 11.3 Average Daily Trading Volume in Currency Markets (Billions of U.S. Dollars)

Year	Spot	Forwards, Forex Swaps	Total
1989	350	240	590
1992	416	404	820
1995	517	673	1,190
1998	592	898	1,490
2001	399	811	1,210
2004	656	1,224	1,880
2007	1,005	2,076	3,210
Of which, between:			
Dealers			1,374
Financials			1,287
Others			549

Source: Bank for International Settlements surveys.

Spot transactions are exchanges of two currencies for settlement as soon as it is practical, typically in two business days. They account for about 35% of trading volume. Other transactions are outright forward contracts and forex swaps. **Outright forward contracts** are agreements to exchange two currencies at a future date, and account for about 12% of the total market. **Forex swaps** involve two transactions, an exchange of currencies on a given date and a reversal at a later date, and account for 53% of the total market. Note that forex swaps are typically of a short-term nature and should not be confused with long-term currency swaps, which involve a stream of payments over longer horizons.

In addition to these contracts, the market also includes OTC forex options ($212 billion daily) and exchange-traded derivatives ($72 billion daily). The most active currency futures are traded on the Chicago Mercantile Exchange (now CME Group) and settled by physical delivery. The CME also trades options on currency futures.

Currencies are generally quoted in **European terms**, that is, in units of the foreign currency per dollar. The yen, for example, is quoted as 120 yen per U.S. dollar. Two notable exceptions are the British pound (sterling) and the euro, which are quoted in **American terms**, that is in dollars per unit of the foreign currency. The pound, for example, is quoted as 1.6 dollars per pound.

EXAMPLE 11.3: FRM EXAM 2003—QUESTION 2

The current spot CHF/USD rate is 1.3680 CHF. The three-month USD interest rate is 1.05%, and the three-month Swiss interest rate is 0.35%, both continuously compounded and per annum. A currency trader notices that the three-month forward price is USD 0.7350. In order to arbitrage, the trader should

a. Borrow CHF, buy USD spot, go long Swiss franc forward
b. Borrow CHF, sell Swiss franc spot, go short Swiss franc forward
c. Borrow USD, buy Swiss franc spot, go short Swiss franc forward
d. Borrow USD, sell USD spot, go long Swiss franc forward

11.3.2 Currency Risk

Currency risk arises from potential movements in the value of foreign exchange rates. Currency risk arises in the following environments.

In a *pure currency float*, the external value of a currency is free to move, to depreciate or appreciate, as pushed by market forces. An example is the dollar/euro exchange rate. In these cases, currency volatility typically ranges from 6% to 10% per annum. This is considerably less than the volatility of equities.

In a *fixed currency system*, a currency's external value is fixed (or pegged) to another currency. An example is the Hong Kong dollar, which is fixed against the U.S. dollar. This does not mean there is no risk, however, due to possible readjustments in the **parity value**, called devaluations or revaluations. Thus, they are subject to **devaluation risk**.

In a *change in currency regime*, a currency that was previously fixed becomes flexible, or vice versa. For instance, the Argentinian peso was fixed against the dollar until 2001, and floated thereafter. Changes in regime can also lower currency risk, as in the case of the euro.[1]

EXAMPLE 11.4: FRM EXAM 2009—QUESTION 3-19

Bonumeur SA is a French company that produces strollers for children and is specialized in strollers for twins and triplets for the EU market. The company buys the wheels of the strollers on the U.S. market. Invoices are paid in USD. What is Bonumeur's currency risk and how can the company hedge its exposure?

a. EUR depreciating against USD; selling EUR against buying USD forward
b. EUR depreciating against USD; selling USD against buying EUR forward
c. EUR appreciating against USD; selling EUR against buying USD forward
d. EUR appreciating against USD; selling USD against buying EUR forward

11.4 CURRENCY DERIVATIVES

Currency markets offer the full range of financial instruments, including futures, forwards, and options. These derivatives can be priced according to standard valuation models, specifying the income payment to be a continuous flow defined by the foreign interest rate. For currency forwards, for example, the relationship between forward and spot prices is very similar to that in Figure 11.1. The two prices are highly correlated and converge to each other at maturity.

Because of their importance, currency swaps are examined in more detail. **Currency swaps** are agreements by two parties to exchange a stream of cash flows in different currencies according to a prearranged formula.

11.4.1 Currency Swaps

Consider two counterparties, company A and company B, which can raise funds either in dollars or in yen, \$100 million or ¥10 billion at the current rate of

[1] As of 2009, the Eurozone includes a block of 16 countries. Early adopters in 1999 include Austria, Belgium, Luxembourg, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal, and Spain. Greece joined on January 1, 2001. Slovenia joined on January 1, 2007. Cyprus and Malta joined on January 1, 2008. Slovakia joined on January 1, 2009. Currency risk is not totally eliminated, however, as there is always a possibility that the currency union could dissolve.

TABLE 11.4a Cost of Capital Comparison

Company	Yen	Dollar
A	5.00%	9.50%
B	6.50%	10.00%

TABLE 11.4b Swap to Company A

Operation	Yen	Dollar
Issue debt	Pay yen 5.00%	
Enter swap	Receive yen 5.00%	Pay dollar 9.00%
Net		Pay dollar 9.00%
Direct cost		Pay dollar 9.50%
Savings		0.50%

TABLE 11.4c Swap to Company B

Operation	Dollar	Yen
Issue debt	Pay dollar 10.00%	
Enter swap	Receive dollar 9.00%	Pay yen 5.00%
Net		Pay yen 6.00%
Direct cost		Pay yen 6.50%
Savings		0.50%

100¥/$, over 10 years. Company A wants to raise dollars, and company B wants to raise yen. Table 11.4a displays borrowing costs. This example is similar to that of interest rate swaps, except that rates are now in different currencies.

Company A has an **absolute advantage** in the two markets as it can raise funds at rates systematically lower than company B. Company B, however, has a **comparative advantage** in raising dollars, as the cost is only 0.50% higher than for company A, compared to the cost difference of 1.50% in yen. Conversely, company A must have a comparative advantage in raising yen.

This provides the basis for a swap that will be to the mutual advantage of both parties. If both institutions directly issue funds in their final desired market, the total cost will be 9.50% (for A) and 6.50% (for B), for a total of 16.00%. In contrast, the total cost of raising capital where each has a comparative advantage is 5.00% (for A) and 10.00% (for B), for a total of 15.00%. The gain to both parties from entering a swap is $16.00 - 15.00 = 1.00\%$. For instance, the swap described in Tables 11.4b and 11.4c splits the benefit equally between the two parties.

Company A issues yen debt at 5.00%, then enters a swap whereby it promises to pay 9.00% in dollars in exchange for receiving 5.00% yen payments. Its effective funding cost is therefore 9.00%, which is less than the direct cost by 50bp.

Similarly, company B issues dollar debt at 10.00%, then enters a swap whereby it receives 9.00% dollars in exchange for paying 5.00% yen. If we add up the difference in dollar funding cost of 1.00% to the 5.00% yen funding costs, the effective funding cost is therefore 6.00%, which is less than the direct cost by 50bp.[2] Both parties benefit from the swap.

While payments are typically netted for an interest rate swap, because they are in the same currency, this is not the case for currency swaps. Full interest payments are made in different currencies. In addition, at initiation and termination, there is exchange of principal in different currencies. For instance, assuming annual payments, company A will receive 5.00% on a notional of ¥10 billion, which is ¥500 million in exchange for paying 9.00% on a notional of $100 million, or $9 million every year.

11.4.2 Swap Pricing

Consider now the pricing of the swap to company A. This involves receiving 5.00% yen in exchange for paying 9.00% dollars. As with interest rate swaps, we can price the swap using either of two approaches, taking the difference between two bond prices or valuing a sequence of forward contracts.

This swap is equivalent to a long position in a fixed-rate 5% 10-year yen-denominated bond and a short position in a 9% 10-year dollar-denominated bond. The value of the swap is that of a long yen bond minus a dollar bond. Defining S as the dollar price of the yen and P and P^* as the dollar and yen bond, respectively, we have:

$$V = S(\$/¥)P^*(¥) - P(\$) \tag{11.3}$$

Here, we indicate the value of the yen bond by an asterisk, P^*.

In general, the bond value can be written as $P(c, r, F)$ where the coupon is c, the yield is r, and the face value is F. Our swap is initially worth (in millions):

$$\begin{aligned} V &= \frac{1}{100}P^*(5\%, 5\%, ¥10{,}000) - P(9\%, 9\%, \$100) \\ &= \frac{\$1}{¥100}¥10{,}000 - \$100 = \$0 \end{aligned}$$

Thus, the initial value of the swap is zero, assuming a flat term structure for both countries and no credit risk.

We can identify three conditions under which the swap will be in-the-money. This will happen (1) if the value of the yen S appreciates, or (2) if the yen interest

[2] Note that B is somewhat exposed to currency risk, as funding costs cannot be simply added when they are denominated in different currencies. The error, however, is of a second-order magnitude.

rate r^* falls, or (3) if the dollar interest rate r goes up. Thus the swap is exposed to three risk factors: the spot rate and two interest rates. The latter exposures are given by the duration of the equivalent bond.

KEY CONCEPT

A position in a receive foreign currency swap is equivalent to a long position in a foreign currency bond offset by a short position in a dollar bond.

The swap can be alternatively valued as a sequence of forward contracts. Recall that the valuation of a forward contract on one yen is given by

$$V_i = (F_i - K)\exp(-r_i\tau_i) \tag{11.4}$$

using continuous compounding. Here, r_i is the dollar interest rate, F_i is the prevailing forward rate (in \$/yen), and K is the locked-in rate of exchange, defined as the ratio of the dollar-to-yen payment on this maturity. Extending this to multiple maturities, the swap is valued as

$$V = \sum_i n_i(F_i - K)\exp(-r_i\tau_i) \tag{11.5}$$

where $n_i F_i$ is the dollar value of the yen payments translated at the forward rate and the other term, $n_i K$, is the dollar payment in exchange.

Table 11.5 compares the two approaches for a three-year swap with annual payments. Market rates have now changed and are $r = 8\%$ for U.S. yields and $r^* = 4\%$ for yen yields. We assume annual compounding. The spot exchange rate has moved from 100¥/\$ to 95¥/\$, reflecting a depreciation of the dollar (or appreciation of the yen).

The middle panel shows the valuation using the difference between the two bonds. First, we discount the cash flows in each currency at the newly prevailing yield. This gives $P =$ \$102.58 for the dollar bond and ¥10,277.51 for the yen bond. Translating the latter at the new spot rate of ¥95, we get \$108.18. The swap is now valued at \$108.18 − \$102.58, which is a positive value of $V =$ \$5.61 million. The appreciation of the swap is principally driven by the appreciation of the yen.

The bottom panel shows how the swap can be valued by a sequence of forward contracts. First, we compute the forward rates for the three maturities. For example, the one-year rate is $95 \times (1 + 4\%)/(1 + 8\%) = 91.48$ ¥/\$, by interest rate parity. Next, we convert each yen receipt into dollars at the forward rate, for example ¥500 million in one year, which is \$5.47 million. This is offset against a payment of \$9 million, for a net planned cash outflow of −\$3.53 million. Discounting and adding up the planned cash flows, we get $V =$ \$5.61 million, which must be exactly equal to the value found using the alternative approach.

TABLE 11.5 Pricing a Currency Swap

	Specifications		Market Data
	Notional Amount (Millions)	Contract Rates	Market Rates
Dollar	$100	9%	8%
Yen	¥10,000	5%	4%
Exchange rate		100¥/$	95¥/$

Valuation Using Bond Approach (Millions)

	Dollar Bond			Yen Bond		
Time (Year)	Dollar Payment	PV($1)	PV(CF)	Yen Payment	PV(¥1)	PV(CF)
1	9	0.9259	8.333	500	0.9615	480.769
2	9	0.8573	7.716	500	0.9246	462.278
3	109	0.7938	86.528	10,500	0.8890	9,334.462
Total			$102.58			¥10,277.51
Swap ($)			−$102.58			$108.18
Value						$5.61

Valuation Using Forward Contract Approach (Millions)

Time (Year)	Forward Rate (¥/$)	Yen Receipt (¥)	Yen Receipt ($)	Dollar Payment ($)	Difference CF ($)	PV(CF) ($)
1	91.48	500	5.47	−9.00	−3.534	−3.273
2	88.09	500	5.68	−9.00	−3.324	−2.850
3	84.83	10,500	123.78	−109.00	14.776	11.730
Value						$5.61

EXAMPLE 11.5: FRM EXAM 2008—QUESTION 2-27

Which of the following statements is correct when comparing the differences between an interest rate swap and a currency swap?

a. At maturity, the counterparties to interest rate swaps and the counterparties to currency swaps both exchange the principal of the swap.
b. At maturity, the counterparties to interest rate swaps do not exchange the principal, but the counterparties to currency swaps exchange the value difference in principal determined by prevailing exchange rates.
c. At maturity, the counterparties to interest rate swaps do not exchange the principal, and counterparties to currency swaps do exchange the principal.
d. Counterparties to interest rate swaps are exposed to more counterparty credit risk due to the magnifying effect of currency, interest rate, and settlement risk embedded within the transaction.

EXAMPLE 11.6: FRM EXAM 2006—QUESTION 88

You have entered into a currency swap in which you receive 4%pa in yen and pay 6%pa in dollars once a year. The principals are 1,000 million yen and 10 million dollars. The swap will last for another two years, and the current exchange rate is 115 yen/$. The annualized spot rates (with continuous compounding) are 2.00% and 2.50% in yen for one- and two-year maturities, and 4.50% and 4.75% in dollars. What is the value of the swap to you in million dollars?

a. −1.270
b. −0.447
c. 0.447
d. 1.270

EXAMPLE 11.7: FRM EXAM 2007—QUESTION 87

Your company is expecting a major export order from a London-based client. The receivables under the contract are to be billed in GBP, while your reporting currency is USD. Since the order is a large sum, your company does not want to bear the exchange risk and wishes to hedge it using derivatives. To minimize the cost of hedging, which of the following is the most suitable contract?

a. A chooser option for GBP/USD pair
b. A currency swap where you pay fixed in USD and receive floating in GBP
c. A barrier put option to sell GBP against USD
d. An Asian call option on GBP against USD

11.5 COMMODITIES

11.5.1 Overview

Commodities are typically traded on exchanges. Contracts include spot, futures, and options on futures. There is also an OTC market for long-term commodity swaps, where payments are tied to the price of a commodity against a fixed or floating rate.

Commodity contracts can be classified into:

- **Agricultural products**, including grains and oilseeds (corn, wheat, soybeans), food and fiber (cocoa, coffee, sugar, orange juice)
- **Livestock and meat** (cattle, hogs)

- **Base metals** (aluminum, copper, nickel, zinc)
- **Precious metals** (gold, silver, platinum)
- **Energy products** (natural gas, heating oil, unleaded gasoline, crude oil)
- **Weather derivatives** (temperature, hurricanes, snow)
- **Environmental products** (CO_2 allowances)

The **S&P GSCI**, formerly the Goldman Sachs Commodity Index, is a broad production-weighted index of commodity price performance, which is composed of 24 liquid exchange-traded futures contracts. As of December 2009, the index contains 70% energy products, 8% industrial metals, 3% precious metals, 14% agricultural products, and 4% livestock products. The CME Group trades futures and options contracts on the S&P GSCI.

In the past few years, active markets have developed for **electricity products**, electricity futures for delivery at specific locations, for instance California/Oregon border (COB), Palo Verde, and so on. These markets have mushroomed following the deregulation of electricity prices, which has led to more variability in electricity prices.

More recently, OTC markets and exchanges have introduced **weather derivatives**, where the payout is indexed to temperature or precipitation. On the CME, for instance, contract payouts are based on the Degree Day Index over a calendar month. This index measures the extent to which the daily temperature deviates from the average. These contracts allow users to hedge situations where their income is negatively affected by extreme weather. Markets are constantly developing new products.

Such commodity markets allow participants to exchange risks. Farmers, for instance, can sell their crops at a fixed price on a future date, insuring themselves against variations in crop prices. Likewise, consumers can buy these crops at a fixed price.

11.6 COMMODITY DERIVATIVES

11.6.1 Valuation

Commodities differ from financial assets in two notable dimensions: they may be expensive, even impossible, to store and they may generate a flow of benefits that are not directly measurable.

The first dimension involves the cost of carrying a physical inventory of commodities. For most financial instruments, this cost is negligible. For bulky commodities, this cost may be high. Other commodities, like electricity, cannot be stored easily.

The second dimension involves the benefit from holding the physical commodity. For instance, a company that manufactures copper pipes benefits from an inventory of copper, which is used up in its production process. This flow is also called **convenience yield** for the holder. For a financial asset, this flow would be a monetary income payment for the investor. When an asset such as gold can be lent out for a profit, the yield represents the **lease rate**, which is the return to lending gold short-term.

Consider the first factor, storage cost only. The cash-and-carry relationship should be modified as follows. We compare two positions. In the first, we buy the commodity spot plus pay up front the present value of storage costs PV(C). In the second, we enter a forward contract and invest the present value of the forward price. Since the two positions are identical at expiration, they must have the same initial value:

$$F_t e^{-r\tau} = S_t + \mathrm{PV}(C) \tag{11.6}$$

where $e^{-r\tau}$ is the present value factor. Alternatively, if storage costs are incurred per unit time and defined as c, we can restate this relationship as

$$F_t e^{-r\tau} = S_t e^{c\tau} \tag{11.7}$$

Due to these costs, the forward rate should be much greater than the spot rate, as the holder of a forward contract benefits not only from the time value of money but also from avoiding storage costs.

Example: Computing the Forward Price of Gold

Let us use data from December 1999. The spot price of gold is $S = \$288$, the one-year interest rate is $r = 5.73\%$ (continuously compounded), and storage costs are \$2 per ounce per year, paid up front. The fair price for a one-year forward contract should be $F = [S + \mathrm{PV}(C)]e^{r\tau} = [\$288 + \$2]e^{5.73\%} = \$307.1$.

Let us now turn to the convenience yield, which can be expressed as y per unit time. In fact, y represents the net benefit from holding the commodity, after storage costs. Following the same reasoning as before, the forward price on a commodity should be given by

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{11.8}$$

where $e^{-y\tau}$ is an actualization factor. This factor may have an economically identifiable meaning, reflecting demand and supply conditions in the cash and futures markets. Alternatively, it can be viewed as a ***plug-in*** that, given F, S, and $e^{-r\tau}$, will make Equation (11.8) balance.

Let us focus, for example, on the one-year contract. Using $S = \$25.60$, $F = \$20.47$, $r = 5.73\%$ and solving for y,

$$y = r - \frac{1}{\tau}\ln(F/S) \tag{11.9}$$

we find $y = 28.10\%$, which is quite large. In fact, variations in y can be substantial. Just one year before, a similar calculation would have given $y = -9\%$, which

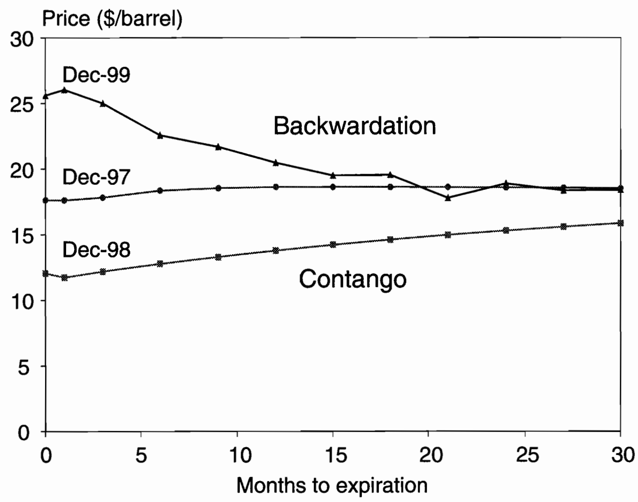

FIGURE 11.2 Term Structure of Futures Prices for Crude Oil

implies a negative convenience yield, or a storage cost. This yield depends on the maturity of the contract.

Figure 11.2, for example, displays the shape of the term structure of spot and futures prices for the New York Mercantile Exchange (NYMEX) crude oil contract. In December 1997, the term structure is relatively flat. In December 1998, the term structure becomes strongly upward sloping. The market is said to be in a **contango** when the futures price trades at a premium relative to the spot price. Using Equation (11.9), this implies that the convenience yield is smaller than the interest rate $y < r$.

In contrast, the term structure is downward sloping in December 1999. A market is said to be in **backwardation** (or inverted) when forward prices trade at a discount relative to spot prices. This implies that the convenience yield is greater than the interest rate $y > r$. In other words, a high convenience yields puts a higher price on the cash market, as there is great demand for immediate consumption of the commodity.

Table 11.6 displays futures prices for selected contracts. Futures prices are generally increasing with maturity, reflecting the time value of money, storage cost, and low convenience yields. Corn, for example, is in contango. There are some irregularities, however, reflecting anticipated imbalances between demand

TABLE 11.6 Futures Prices as of December 31, 2009

Maturity	Corn	Sugar	Copper	Gold	Natural Gas	Gasoline	Heating Oil
Jan.			333.8	1,095.2		205.3	211.9
March	414.5	26.95	334.7	1,096.9	5.532	207.2	212.2
July	433.0	23.02	337.1	1,099.5	5.695	219.9	215.3
Sept.	437.5	22.20	337.6	1,101.0	5.795	218.9	219.4
Dec.	440.8	21.50	338.2	1,104.1	6.548	209.4	226.6
Mar. 11	449.8	21.05	338.5	1,108.4	6.560	216.0	230.7
...							
Dec. 11	447.8	17.50	339.4	1,126.9	6.820	217.4	239.0

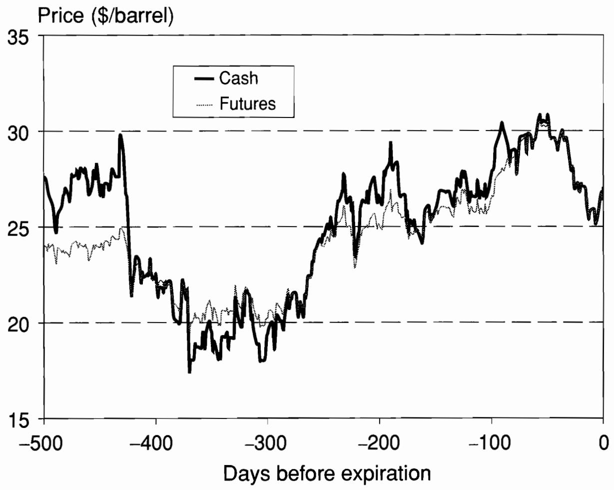

FIGURE 11.3 Futures and Spot Prices for Crude Oil

and supply. For instance, sugar is in backwardation. Also, gasoline futures prices tend to increase in the summer due to increased automobile driving. Heating oil displays the opposite pattern, where prices increase during the winter due to the demand for heating. Agricultural products can also be highly seasonal. In contrast, futures prices for gold are going up monotonically with time, since this is a perfectly storable good.

Finally, Figure 11.3 compares the spot and futures prices for crude oil. There is substantial variation in the basis between the spot and futures prices for crude oil. The market switches from backwardation ($S > F$) to contango ($S < F$).

KEY CONCEPT

Markets are in contango if spot prices are lower than forward prices. This occurs when the convenience yield is lower than the interest rate. Markets are in backwardation if spot prices are higher than forward prices. This occurs when there is high current demand for the commodity, which implies high convenience yields.

EXAMPLE 11.8: FRM EXAM 2008—QUESTION 2-30

If the lease rate of commodity A is less than the risk-free rate, what is the market structure of commodity A?

a. Backwardation
b. Contango
c. Flat
d. Inversion

11.6.2 Futures and Expected Spot Prices

An interesting issue is whether today's futures price gives the best forecast of the future spot price. If so, it satisfies the **expectations hypothesis**, which can be written as:

$$F_t = E_t[S_T] \tag{11.10}$$

The reason this relationship may hold is as follows. Say that the one-year oil futures price is $F = \$20.47$. If the market forecasts that oil prices in one year will be at \$25, one could make a profit by going long a futures contract at the cheap futures price of $F = \$20.47$, waiting a year, then buying oil at \$20.47, and reselling it at the higher price of \$25. In other words, deviations from this relationship imply **speculative profits**.

To be sure, these profits are not risk-free. Hence, they may represent some compensation for risk. For instance, if the market is dominated by producers that want to hedge by selling oil futures, F will be abnormally low compared with expectations. Thus the relationship between futures prices and expected spot prices can be complex.

For financial assets for which the arbitrage between cash and futures is easy, the futures or forward rate is solely determined by the cash-and-carry relationship (i.e., the interest rate and income on the asset). For commodities, however, the arbitrage may not be so easy. As a result, the futures price may deviate from the cash-and-carry relationship through this convenience yield factor. Such prices may reflect expectations of future spot prices, as well as speculative and hedging pressures.

A market trades in **contango** when the futures price trades at a premium relative to the spot price. Normally, the size of the premium should be limited by arbitrage opportunities. If this became too large, traders could buy the commodity spot, put it in storage, and simultaneously sell it for future delivery at the higher forward price. In December 2008, however, the premium for one-year oil contracts reached an all-time high of \$13 per barrel. This was explained by the credit crunch, which prevented oil traders from securing loans to finance oil storage.

With backwardation, the futures price tends to increase as the contract nears maturity. In such a situation, a **roll-over strategy** should be profitable, provided that prices do not move too much. This involves buying a long-maturity contract, waiting, and then selling it at a higher price in exchange for buying a cheaper, longer-term contract.

This strategy is comparable to **riding the yield curve** when upward sloping. This involves buying long maturities and waiting to have yields fall due to the passage of time. If the shape of the yield curve does not change too much, this will generate a capital gain from bond price appreciation. Because of the negative price-yield relationship, a positively sloped yield curve is equivalent to backwardation in bond prices.

This was basically the strategy followed by Metallgesellschaft Refining & Marketing (MGRM), the U.S. subsidiary of **Metallgesellschaft**, which had made large sales of long-term oil to clients on the OTC market. These were hedged by rolling over long positions in West Texas Intermediate (WTI) crude oil futures. This made money as long as the market was in backwardation. When the market turned to contango, however, the long positions started to lose money as they got closer to maturity. In addition, the positions were so large that they moved markets against MGRM. These losses caused cash-flow or liquidity problems. MGRM ended up liquidating the positions, which led to a realized loss of \$1.3 billion.

A similar problem afflicted **Amaranth**, a hedge fund that lost \$6.6 billion as a result of bad bets against natural gas futures. In September 2006, the price of natural gas fell sharply. In addition, the spread between prices in winter and summer months collapsed. As the size of the positions was huge, this led to large losses that worsened when the fund attempted to liquidate the contracts.

EXAMPLE 11.9: FRM EXAM 2007—QUESTION 29

On January 1, a risk manager observes that the one-year continuously compounded interest rate is 5% and the storage cost of a commodity product A is USD 0.05 per quarter (payable at each quarter end). The manager further observes the following forward prices for product A: March, 5.35; June, 5.90; September, 5.30; December, 5.22. Given the following explanation of supply and demand for this product, how would you best describe its forward price curve from June to December?

a. Backwardation as the supply of product A is expected to decline after summer
b. Contango as the supply of product A is expected to decline after summer
c. Contango as there is excess demand for product A in early summer
d. Backwardation as there is excess demand for product A in early summer

EXAMPLE 11.10: FRM EXAM 2007—QUESTION 30

Continuing with the previous question, what is the annualized rate of return earned on a cash-and-carry trade entered into in March and closed out in June?

a. 9.8%
b. 8.9%
c. 39.1%
d. 35.7%

EXAMPLE 11.11: FRM EXAM 2008—QUESTION 4-16

In late 1993, MGRM reported losses of about $1.3 billion in connection with the implementation of a hedging strategy in the oil futures market. In 1992, the company had begun a new strategy to sell petroleum to independent retailers at fixed prices above the prevailing market price for periods of up to 10 years. At the same time, MGRM implemented a hedging strategy using a large number of short-term derivative contracts such as swaps and futures on crude oil. This led to a timing (maturity) mismatch between the short-term hedges and the long-term liability. Unfortunately, the company suffered significant losses with its hedging strategy when oil market conditions abruptly changed to:

a. Contango, which occurs when the futures price is above the spot price
b. Contango, which occurs when the futures price is below the spot price
c. Normal backwardation, which occurs when the futures price is above the spot price
d. Normal backwardation, which occurs when the futures price is below the spot price

11.6.3 Commodity Risk

Commodity risk arises from potential movements in the value of commodity prices. Table 11.7 describes the risks of a sample of commodity contracts.[3] These can be grouped into *precious metals* (gold, platinum, silver); *base metals* (aluminum, copper, nickel, zinc); and *energy products* (natural gas, heating oil, unleaded gasoline, crude oil—West Texas Intermediate). The table reports the annualized volatility for spot or short-term contracts as well as for longer-term (typically 12- to 15-month) futures.

Precious and base metals have an annual volatility ranging from 20% to 30%, higher than for stock markets. Energy products, in contrast, are much more volatile with numbers ranging from 20% to 70%. This is because energy products are less storable than metals and, as a result, are much more affected by variations in demand and supply.

As Table 11.7 shows, futures prices are less volatile for longer maturities. This decreasing term structure of volatility is more marked for energy products and less so for base metals.

In terms of correlations, Figure 11.3 has shown that movements in futures prices are much less tightly related to spot prices than for financial contracts. Thus, the futures contract represents a separate risk factor. In addition, correlations

[3] These data are provided by RiskMetrics as of December 2006. Volatilities are derived from an exponentially weighted moving average (EWMA) model with a forecast horizon of one month, and annualized.

TABLE 11.7 Commodity Volatility, 2006 (Percent per Annum)

Commodity	Spot	Futures
Gold	17	
Platinum	29	
Silver	33	
Aluminium	28	20
Copper	30	27
Nickel	41	45
Zinc	36	28
Natural gas	72	41
Heating oil	33	20
Unleaded gas	36	25
Crude oil	28	19

across maturities are lower for energy products than for metals. This explains why trading energy requires risk measurement systems with numerous risk factors, across maturities, grades, and locations.

EXAMPLE 11.12: FRM EXAM 2006—QUESTION 115

Assume the risk-free rate is 5% per annum, the cost of storing oil for a year is 1% per annum, the convenience yield for owning oil for a year is 2% per annum, and the current price of oil is USD 50 per barrel. All rates are continuously compounded. What is the forward price of oil in a year?

a. USD 49.01
b. USD 52.04
c. USD 47.56
d. USD 49.50

EXAMPLE 11.13: FRM EXAM 2006—QUESTION 138

Imagine a stack-and-roll hedge of monthly commodity deliveries that you continue for the next five years. Assume the hedge ratio is adjusted to take into account the mistiming of cash flows but is not adjusted for the basis risk of the hedge. In which of the following situations is your calendar basis risk likely to be greatest?

a. Stack-and-roll in the front month in oil futures
b. Stack-and-roll in the 12-month contract in natural gas futures
c. Stack-and-roll in the three-year contract in gold futures
d. All three situations will have the same basis risk.

11.7 IMPORTANT FORMULAS

Gordon growth model for valuation of stocks: $P = \frac{D}{r-g}$

Stock index futures: $F_t e^{-r\tau} = S_t e^{-y\tau}$

Pricing a currency swap as two bond positions: $V = S(\$/Y)P^*(Y) - P(\$)$

Pricing a currency swap as a sequence of forwards: $V = \sum_i n_i(F_i - K)\exp(-r_i\tau_i)$

Pricing of commodity futures with storage costs: $F_t e^{-r\tau} = S_t + \text{PV}(C)$, or $F_t e^{-r\tau} = S_t e^{c\tau}$

Expectations hypothesis: $F_t = E_t[S_T]$

Contango: $F_t > S_t,\ y < r$

Backwardation: $F_t < S_t,\ y > r$

11.8 ANSWERS TO CHAPTER EXAMPLES

Example 11.1: FRM Exam 2000—Question 12

a. This is the cash-and-carry relationship, solved for S. We have $Se^{-y\tau} = Fe^{-r\tau}$, or $S = 552.3 \times \exp(-7.5/200)/\exp(-4.2/200) = 543.26$. We verify that the forward price is greater than the spot price since the dividend yield is less than the risk-free rate.

Example 11.2: FRM Exam 2009—Question 3-1

b. The fair forward price is $F = Se^{-y\tau}/e^{-r\tau} = 750\exp(-0.02 \times 6/12)/\exp(-0.035 \times 6/12) = 750 \times 0.9905/0.9827 = 755.65$. The actual price is 757.00. Hence buying at the cheap price and selling at the forward price gives a profit of \$1.35.

Example 11.3: FRM Exam 2003—Question 2

c. For consistency, translate the spot rate into dollars, $S = 0.7310$. The CHF interest rate is lower than the USD rate, so the CHF must be selling at a forward premium. The fair forward price is $F = S\exp((r - r^*)\tau) = 0.7310\exp((0.0105 - 0.0035)\,0.25) = 0.7323$. Because this is less than the observed price of 0.7350, we sell at the expensive forward price and borrow USD, buy CHF spot, and invest in CHF. At maturity, we liquidate the CHF investment to satisfy the forward sale into dollars, repay the loan, and make a tidy profit.

Example 11.4: FRM Exam 2009—Question 3-19

a. Because the company has revenues fixed in EUR and some costs in USD, it would be hurt if the USD appreciated. So, the risk is that of a depreciation of the

EUR against the USD. This can be hedged by buying the USD forward, which will lock in the EUR payment even if the USD appreciates.

Example 11.5: FRM Exam 2008—Question 2-27

c. Because principals on currency swaps are in different currencies, they need to be exchanged. In contrast, the principal amounts for interest rate swaps are in the same currency and are not exchanged.

Example 11.6: FRM Exam 2006—Question 88

a. The net present values of the payoffs in two currencies are described in the following table. As a result, the value of the currency swap is given by the dollar value of a long position in the yen bond minus a position in the dollar bond, or $(1/115)1{,}000(102.85/100) - 10(102.13/100) = \$8.943 - \$10.213 = -\$1.270$.

	Yen			USD		
T	Rate	CF	NPV	Rate	CF	NPV
1	2.00%	4	3.92	4.50%	6	5.74
2	2.50%	104	98.93	4.75%	106	96.39
Sum			102.85			102.13

Example 11.7: FRM Exam 2007—Question 87

c. A cross-currency swap is inappropriate because there is no stream of payment but just one. Also, one would want to pay GBP, not receive it. An Asian option is generally cheap, but this should be a put option, not a call. Among the two remaining choices, the chooser option is more expensive because it involves a call and a put.

Example 11.8: FRM Exam 2008—Question 2-30

b. If the lease rate is, for example, zero, the futures price must be greater than the spot price, which describes a contango.

Example 11.9: FRM Exam 2007—Question 29

d. From June to December, prices go down, which is backwardation. June prices are abnormally high because of excess demand, which pushes prices up.

Example 11.10: FRM Exam 2007—Question 30

d. The trade involves now going long a March contract and short a June contract. In practice, this means taking delivery of the commodity and holding it for three months until resale in June. The final payout is $5.90 - 0.05$ on a base of 5.35. This gives an annualized rate of return of $r = 4\ln(5.85/5.35) = 35.7\%$.

Example 11.11: FRM Exam 2008—Question 4-16

a. MGRM had purchased oil in short-term futures market as a hedge against the long-term sales. The long futures positions lost money due to the move into contango, which involves the spot price falling below longer-term prices.

Example 11.12: FRM Exam 2006—Question 115

b. Using $F_t e^{-r\tau} = S_t e^{-y\tau}$, we have $F = S\exp(-(y - c)\tau + r\tau) = 50\exp(-(0.02 - 0.01) + 0.05) = 52.04$.

Example 11.13: FRM Exam 2006—Question 138

a. For gold, forward rates closely follow spot rates, so there is little basis risk. For oil and natural gas, there is most movement at the short end of the term structure of futures prices. So using short maturities, or the front month, has the greatest basis risk.

PART

Four

Valuation and Risk Models

CHAPTER 12

Introduction to Risk Models

This chapter provides an introduction to risk models. Modern risk management is **position-based**. This is more forward-looking than **return-based** information. Position-based risk measures are more informative because they can be used to manage the portfolio, which involves changing the positions.

Part Four of this book focuses primarily on market risk models. Ideally, risk should be measured at the top level of the portfolio or institution. This has led to a push toward risk measures that are comparable across different types of risk. One such summary measure is **value at risk** (VAR). VAR is a statistical measure of *total* portfolio risk, taken as the worst loss at a specified confidence level over the horizon. More generally, risk managers should evaluate the entire distribution of profits and losses. In addition, the analysis should be complemented by **stress-testing**, which identifies potential losses under extreme market conditions that may not show up in the recent history.

Section 12.1 gives a brief overview of financial market risks. Section 12.2 describes the broad components of a VAR system. Section 12.3 then shows how to compute VAR for a simple portfolio exposed to one risk factor only. It also discusses caveats, or pitfalls to be aware of when interpreting VAR numbers. Section 12.4 turns to the choice of VAR parameters, that is, the confidence level and horizon. Next, Section 12.5 shows how to implement stress tests. Finally, Section 12.6 describes how risk models can be classified into local valuation and full valuation methods.

12.1 INTRODUCTION TO FINANCIAL MARKET RISKS

12.1.1 Types of Financial Risks

Financial risks include market risk, credit risk, and operational risk. **Market risk** is the risk of losses due to movements in financial market prices or volatilities. This usually includes **liquidity risk**, which is the risk of losses due to the need to liquidate positions to meet funding requirements. Liquidity risk, unfortunately, is not amenable to formal quantification. Because of its importance, it will be covered in Chapter 26. **Credit risk** is the risk of losses due to the fact that counterparties

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FRM Exam Part 1 topic.

may be unwilling or unable to fulfill their contractual obligations. Credit risk will be covered in Part Six. **Operational risk** is the risk of loss resulting from failed or inadequate internal processes, systems, and people, or from external events. This subject will be covered in Chapter 25. Oftentimes, however, these three categories interact with each other, so that any classification is, to some extent, arbitrary.

For example, credit risk can interact with other types of risks. At the most basic level, it involves the risk of default on the asset, such as a loan or bond. When the asset is traded, however, market risk also reflects credit risk. Take a corporate bond, for example. Some of the price movement may be due to movements in risk-free interest rates, which is pure market risk. The remainder will reflect the market's changing perception of the likelihood of default. Thus, for traded assets, there is no clear-cut delineation of market and credit risk. Some arbitrary classification must take place. Furthermore, operational risk is often involved as well.

Consider a simple transaction whereby a trader purchases 1 million worth of British pound (GBP) spot from Bank A. The current rate is $1.5/GBP, for settlement in two business days. So, our bank will have to deliver $1.5 million in two days in exchange for receiving GBP 1 million. This simple transaction involves a series of risks.

- *Market risk:* During the day, the spot rate could change. Say that after a few hours the rate moves to $1.4/GBP. The trader cuts the position and enters a spot sale with another bank, Bank B. The million pounds is now worth only $1.4 million. The loss of $100,000 is the change in the market value of the investment.
- *Credit risk:* The next day, Bank B goes bankrupt. The trader must now enter a new, replacement trade with Bank C. If the spot rate has dropped further from $1.4/GBP to $1.35/GBP, the gain of $50,000 on the spot sale with Bank B is now at risk. The loss is the change in the market value of the investment, if positive. Thus there is interaction between market and credit risk.
- *Settlement risk:* The next day, our bank wires the $1.5 million to Bank A in the morning, which defaults at noon and does not deliver the promised GBP 1 million. This is also known as **Herstatt risk** because this German bank defaulted on such obligations in 1974, potentially destabilizing the whole financial system. The loss is now potentially the whole principal in dollars.
- *Operational risk:* Suppose that our bank wired the $1.5 million to a wrong bank, Bank D. After two days, our back office gets the money back, which is then wired to Bank A plus compensatory interest. The loss is the interest on the amount due.

12.1.2 Risk Management Tools

In the past, risks were measured using a variety of ad hoc tools, which were not comparable across types of risk. These included **notional amounts** and **sensitivity measures**. While these measures provide a useful intuition of risk, they do not provide consistent estimates of the potential for downside loss across the