from dividend payments, μ represents the expected total rate of return on the asset minus the rate of income payment, or dividend yield in the case of stocks.

Example: A Stock Price Process

Consider a stock that pays no dividends, has an expected return of 10% per annum, and has volatility of 20% per annum. If the current price is \$100, what is the process for the change in the stock price over the next week? What if the current price is \$10?

The process for the stock price is

$$\Delta S = S(\mu\Delta t + \sigma\sqrt{\Delta t} \times \epsilon)$$

where ϵ is a random draw from a standard normal distribution. If the interval is one week, or $\Delta t = 1/52 = 0.01923$, the mean is $\mu\Delta t = 0.10 \times 0.01923 = 0.001923$ and $\sigma\sqrt{\Delta t} = 0.20 \times \sqrt{0.01923} = 0.027735$. The process is $\Delta S = \$100(0.001923 + 0.027735 \times \epsilon)$. With an initial stock price at \$100, this gives $\Delta S = 0.1923 + 2.7735\epsilon$. With an initial stock price at \$10, this gives $\Delta S = 0.01923 + 0.27735\epsilon$. The trend and volatility are scaled down by a factor of 10.

This model is particularly important because it is the underlying process for the Black-Scholes formula. The key feature of this distribution is the fact that the volatility is proportional to S. This ensures that the stock price will stay positive. Indeed, as the stock price falls, its variance decreases, which makes it unlikely to experience a large down move that would push the price into negative values. As the limit of this model is a normal distribution for $dS/S = d\ln(S)$, S follows a **lognormal distribution.**

This process implies that, over an interval $T - t = \tau$, the logarithm of the ending price is distributed as

$$\ln(S_T) = \ln(S_t) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{4.6}$$

where ϵ is a standardized normal variable.

Example: A Stock Price Process (*Continued*)

Assume the price in one week is given by $S = \$100\exp(R)$, where R has annual expected value of 10% and volatility of 20%. Construct a two-tailed 95% confidence interval for S.

The standard normal deviates that corresponds to a 95% confidence interval are $\alpha_{MIN} = -1.96$ and $\alpha_{MAX} = 1.96$. In other words, we have 2.5% in each tail. The 95% confidence band for R is then $R_{MIN} = \mu\Delta t - 1.96\sigma\sqrt{\Delta t} = 0.001923 - 1.96 \times 0.027735 = -0.0524$ and $R_{MAX} = \mu\Delta t + 1.96\sigma\sqrt{\Delta t} = 0.001923 + 1.96 \times 0.027735 = 0.0563$. This gives $S_{MIN} = \$100\exp(-0.0524) = \$94.89$, and $S_{MAX} = \$100\exp(0.0563) = \$105.79$.

Whether a lognormal distribution is much better than the normal distribution depends on the horizon considered. If the horizon is one day only, the choice of the lognormal versus normal assumption does not really matter. It is highly unlikely that the stock price would drop below zero in one day, given typical volatilities. However, if the horizon is measured in years, the two assumptions do lead to different results. The lognormal distribution is more realistic as it prevents prices from turning negative.

In simulations, this process is approximated by small steps with a normal distribution with mean and variance given by

$$\frac{\Delta S}{S} \sim N(\mu\Delta t, \sigma^2 \Delta t) \tag{4.7}$$

To simulate the future price path for S, we start from the current price S_t and generate a sequence of independent standard normal variables ϵ, for $i = 1, 2, \ldots, n$. The next price, S_{t+1}, is built as $S_{t+1} = S_t + S_t(\mu\Delta t + \sigma\epsilon_1\sqrt{\Delta t})$. The following price, S_{t+2}, is taken as $S_{t+1} + S_{t+1}(\mu\Delta t + \sigma\epsilon_2\sqrt{\Delta t})$, and so on until we reach the target horizon, at which point the price $S_{t+n} = S_T$ should have a distribution close to the lognormal.

Table 4.1 illustrates a simulation of a process with a drift (μ) of 0% and volatility (σ) of 20% over the total interval, which is divided into 100 steps.

The initial price is \$100. The local expected return is $\mu\Delta t = 0.0/100 = 0.0$, and the volatility is $0.20 \times \sqrt{1/100} = 0.02$. The second column shows the realization of a uniform $U(0, 1)$ variable. The value for the first step is $u_1 = 0.0430$. The next column transforms this variable into a normal variable with mean 0.0 and volatility of 0.02, which gives -0.0343. The price increment is then obtained by multiplying the random variable by the previous price, which gives $-\$3.433$. This generates a new value of $S_1 = \$100 - \$3.43 = \$96.57$. The process is repeated until the final price of \$125.31 is reached at the 100th step.

This experiment can be repeated as often as needed. Define K as the number of **replications**, or random trials. Figure 4.1 displays the first three trials. Each leads

TABLE 4.1 Simulating a Price Path

	Random Variable		Price	
Step	Uniform	Normal	Increment	Price
i	u_i	$\mu\Delta t + \sigma\Delta z$	ΔS_i	S_{t+i}
0				100.00
1	0.0430	−0.0343	−3.433	96.57
2	0.8338	0.0194	1.872	98.44
3	0.6522	0.0078	0.771	99.21
4	0.9219	0.0284	2.813	102.02
...				
99				124.95
100	0.5563	0.0028	0.354	125.31

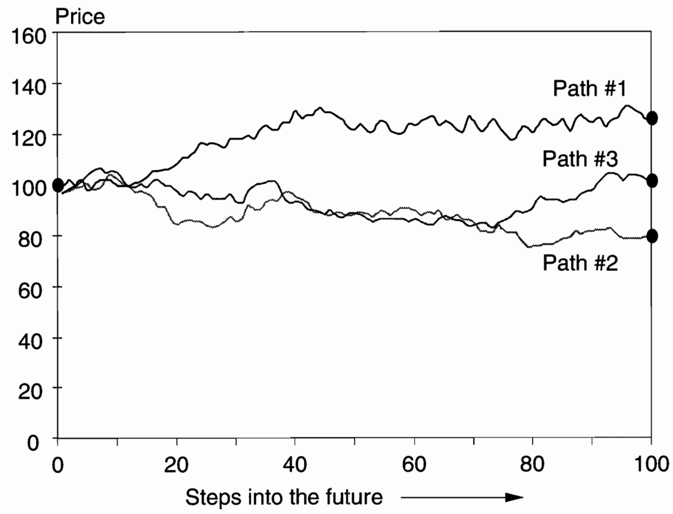

FIGURE 4.1 Simulating Price Paths

to a simulated final value S_T^k. This generates a distribution of simulated prices S_T. With just one step $n = 1$, the distribution must be normal. As the number of steps n grows large, the distribution tends to a lognormal distribution.

While very useful for modeling stock prices, this model has shortcomings. Price increments are assumed to have a normal distribution. In practice, we observe that price changes for most financial assets typically have fatter tails than the normal distribution. Returns may also experience changing variances.

In addition, as the time interval Δt shrinks, the volatility shrinks as well. This implies that large discontinuities cannot occur over short intervals. In reality, some assets experience discrete jumps, such as commodities, or securities issued by firms that go bankrupt. In such cases, the stochastic process should be changed to accommodate these observations.

EXAMPLE 4.1: FRM EXAM 2009—QUESTION 14

Suppose you simulate the price path of stock HHF using a geometric Brownian motion model with drift $\mu = 0$, volatility $\sigma = 0.14$, and time step $\Delta t = 0.01$. Let S_t be the price of the stock at time t. If $S_0 = 100$, and the first two simulated (randomly selected) standard normal variables are $\epsilon_1 = 0.263$ and $\epsilon_2 = -0.475$, what is the simulated stock price after the second step?

a. 96.79
b. 99.79
c. 99.97
d. 99.70

EXAMPLE 4.2: FRM EXAM 2003—QUESTION 40

In the geometric Brown motion process for a variable S,

I. S is normally distributed.
II. $d\ln(S)$ is normally distributed.
III. dS/S is normally distributed.
IV. S is lognormally distributed.

a. I only
b. II, III, and IV
c. IV only
d. III and IV

EXAMPLE 4.3: FRM EXAM 2002—QUESTION 126

Consider that a stock price S that follows a geometric Brownian motion $dS = aSdt + bSdz$, with b strictly positive. Which of the following statements is *false*?

a. If the drift a is positive, the price one year from now will be above today's price.
b. The instantaneous rate of return on the stock follows a normal distribution.
c. The stock price S follows a lognormal distribution.
d. This model does not impose mean reversion.

4.1.3 Drawing Random Variables

Most spreadsheets or statistical packages have functions that can generate uniform or standard normal random variables. This can be easily extended to distributions that better reflect the data (e.g., with fatter tails or nonzero skewness).

The methodology involves the inverse cumulative probability distribution function (p.d.f.). Take the normal distribution as an example. By definition, the cumulative p.d.f. $N(x)$ is always between 0 and 1. Because we have an analytical formula for this function, it can be easily inverted.

First, we generate a uniform random variable u drawn from $U(0, 1)$. Next, we compute x such that $u = N(x)$, or $x = N^{-1}(u)$. For example, set $u = 0.0430$, as in the first line of Table 4.1. This gives $x = -1.717$.[1] Because u is less than 0.5, we

[1] In Excel, a uniform random variable can be generated with the function u_i = RAND(). From this, a standard normal random variable can be computed with NORMSINV(u_i).

verify that x is negative. The variable can be transformed into any normal variable by multiplying by the standard deviation and adding the mean. More generally, any distribution function can be generated as long as the cumulative distribution function can be inverted.

4.1.4 Simulating Yields

The GBM process is widely used for stock prices and currencies. Fixed-income products are another matter, however.

Bond prices display long-term reversion to the face value, which represents the repayment of principal at maturity (assuming there is no default). Such a process is inconsistent with the GBM process, which displays no such mean reversion. The volatility of bond prices also changes in a predictable fashion, as duration shrinks to zero. Similarly, commodities often display mean reversion.

These features can be taken into account by modeling bond yields directly in a first step. In the next step, bond prices are constructed from the value of yields and a pricing function. The dynamics of interest rates r_t can be modeled by

$$\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma {r_t}^{\gamma} \Delta z_t \tag{4.8}$$

where Δz_t is the usual Wiener process. Here, we assume that $0 \leq \kappa < 1, \theta \geq 0$, $\sigma \geq 0$. Because there is only one stochastic variable for yields, the model is called a **one-factor model**.

This Markov process has a number of interesting features. First, it displays mean reversion to a long-run value of θ. The parameter κ governs the speed of mean reversion. When the current interest rate is high (i.e., $r_t > \theta$), the model creates a negative drift $\kappa(\theta - r_t)$ toward θ. Conversely, low current rates create a positive drift toward θ.

The second feature is the volatility process. This model includes the **Vasicek model** when $\gamma = 0$. Changes in yields are normally distributed because Δr is then a linear function of Δz, which is itself normal. The Vasicek model is particularly convenient because it leads to closed-form solutions for many fixed-income products. The problem, however, is that it could potentially lead to negative interest rates when the initial rate starts from a low value. This is because the volatility of the change in rates does not depend on the level, unlike that in the geometric Brownian motion.

Equation (4.8) is more general, however, because it includes a power of the yield in the variance function. With $\gamma = 1$, this is the **lognormal model**. Ignoring the trend, this gives $\Delta r_t = \sigma r_t \Delta z_t$, or $\Delta r_t / r_t = \sigma \Delta z_t$. This implies that the *rate of change* in the yield dr/r has a fixed variance. Thus, as with the GBM model, smaller yields lead to smaller movements, which makes it unlikely the yield will drop below zero. This model is more appropriate than the normal model when the initial yield is close to zero.

With $\gamma = 0.5$, this is the **Cox, Ingersoll, and Ross (CIR) model**. Ultimately, the choice of the exponent γ is an empirical issue. Recent research has shown that $\gamma = 0.5$ provides a good fit to the data.

This class of models is known as **equilibrium models**. They start with some assumptions about economic variables and imply a process for the short-term interest rate r. These models generate a predicted term structure, whose shape depends on the model parameters and the initial short rate. The problem with these models, however, is that they are not flexible enough to provide a good fit to today's term structure. This can be viewed as unsatisfactory, especially by practitioners who argue they cannot rely on a model that cannot be trusted to price today's bonds.

In contrast, **no-arbitrage models** are designed to be consistent with today's term structure. In this class of models, the term structure is an input into the parameter estimation. The earliest model of this type was the **Ho and Lee model:**

$$\Delta r_t = \theta(t)\Delta t + \sigma \Delta z_t \tag{4.9}$$

where $\theta(t)$ is a function of time chosen so that the model fits the initial term structure. This was extended to incorporate mean reversion in the **Hull and White model:**

$$\Delta r_t = [\theta(t) - a r_t]\Delta t + \sigma \Delta z_t \tag{4.10}$$

Finally, the **Heath, Jarrow, and Morton model** goes one step further and assumes that the volatility is a function of time.

The downside of these no-arbitrage models is that they do not impose any consistency between parameters estimated over different dates. The function $\theta(t)$ could be totally different from one day to the next, which is illogical. No-arbitrage models are also more sensitive to outliers, or data errors in bond prices used to fit the term structure.

4.1.5 Binomial Trees

Simulations are very useful to mimic the uncertainty in risk factors, especially with numerous risk factors. In some situations, however, it is also useful to describe the uncertainty in prices with discrete trees. When the price can take one of two steps, the tree is said to be **binomial.**

The binomial model can be viewed as a discrete equivalent to the geometric Brownian motion. As before, we subdivide the horizon T into n intervals $\Delta t = T/n$. At each node, the price is assumed to go either up with probability p or down with probability $1 - p$.

The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous process. One could choose

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu \Delta t} - d}{u - d} \tag{4.11}$$

TABLE 4.2 Binomial Tree

Step						
0		1		2		3
						u^3S
					↗	
				u^2S		
			↗		↘	
		uS				u^2dS
	↗		↘		↗	
S				udS		
	↘		↗		↘	
		dS				d^2uS
			↘		↗	
				d^2S		
					↘	
						d^3S

This matches the mean, for example,

$$E[\frac{S_1}{S_0}] = pu + (1-p)d = \frac{e^{\mu\Delta t} - d}{u-d}u + \frac{u - e^{\mu\Delta t}}{u-d}d = \frac{e^{\mu\Delta t}(u-d) - du + ud}{u-d} = e^{\mu\Delta t}$$

Table 4.2 shows how a binomial tree is constructed. As the number of steps increases, the discrete distribution of S_T converges to the lognormal distribution. This model will be used in a later chapter to price options.

EXAMPLE 4.4: INTEREST RATE MODEL

The Vasicek model defines a risk-neutral process for r that is $dr = a(b - r)dt + \sigma dz$, where a, b, and σ are constant, and r represents the rate of interest. From this equation we can conclude that the model is a

a. Monte Carlo type model
b. Single-factor term-structure model
c. Two-factor term-structure model
d. Decision tree model

EXAMPLE 4.5: INTEREST RATE MODEL INTERPRETATION

The term $a(b - r)$ in the previous question represents which term?

a. Gamma
b. Stochastic
c. Reversion
d. Vega

EXAMPLE 4.6: FRM EXAM 2000—QUESTION 118

Which group of term-structure models do the Ho-Lee, Hull-White, and Heath-Jarrow-Morton models belong to?

a. No-arbitrage models
b. Two-factor models
c. Lognormal models
d. Deterministic models

EXAMPLE 4.7: FRM EXAM 2000—QUESTION 119

A plausible stochastic process for the short-term rate is often considered to be one where the rate is pulled back to some long-run average level. Which one of the following term-structure models does *not* include this characteristic?

a. The Vasicek model
b. The Ho-Lee model
c. The Hull-White model
d. The Cox-Ingersoll-Ross model

4.2 IMPLEMENTING SIMULATIONS

4.2.1 Simulation for VAR

Implementing Monte Carlo (MC) methods for risk management follows these steps:

1. Choose a stochastic process for the risk factor price S (i.e., its distribution and parameters, starting from the current value S_t).
2. Generate pseudo-random variables representing the risk factor at the target horizon, S_T.
3. Calculate the value of the portfolio at the horizon, $F_T(S_T)$.
4. Repeat steps 2 and 3 as many times as necessary. Call K the number of replications.

These steps create a distribution of values, $F_T^1, \ldots, F_T^K$, which can be sorted to derive the VAR. We measure the cth quantile $Q(F_T, c)$ and the average value $\text{Ave}(F_T)$. If VAR is defined as the deviation from the expected value on the target date, we have

$$\text{VAR}(c) = \text{Ave}(F_T) - Q(F_T, c) \tag{4.12}$$

4.2.2 Simulation for Derivatives

Readers familiar with derivatives pricing will have recognized that this method is similar to the Monte Carlo method for valuing derivatives. In that case, we simply focus on the expected value on the target date discounted into the present:

$$F_t = e^{-r(T-t)}\,\mathrm{Ave}(F_T) \tag{4.13}$$

Thus derivatives valuation focuses on the discounted center of the distribution, while VAR focuses on the quantile on the target date.

Monte Carlo simulations have been long used to price derivatives. As will be seen in a later chapter, pricing derivatives can be done by assuming that the underlying asset grows at the risk-free rate r (assuming no income payment). For instance, pricing an option on a stock with expected return of 20% is done assuming that (1) the stock grows at the risk-free rate of 10% and (2) we discount at the same risk-free rate. This is called the **risk-neutral approach.**

In contrast, risk measurement deals with actual distributions, sometimes called **physical distributions.** For measuring VAR, the risk manager must simulate asset growth using the actual expected return μ of 20%. Therefore, risk management uses physical distributions, whereas pricing methods use risk-neutral distributions.

It should be noted that simulation methods are not applicable to all types of options. These methods assume that the value of the derivative instrument at expiration can be priced solely as a function of the end-of-period price S_T, and perhaps of its sample path. This is the case, for instance, with an Asian option, where the payoff is a function of the price *averaged* over the sample path. Such an option is said to be **path-dependent.**

Simulation methods, however, are inadequate to price American options, because such options can be exercised early. The optimal exercise decision, however, is complex to model because it should take into account *future* values of the option. This cannot be done with regular simulation methods, which only consider present and past information. Instead, valuing American options requires a **backward recursion**, for example with binomial trees. This method examines whether the option should be exercised, starting from the end and working backward in time until the starting time.

4.2.3 Accuracy

Finally, we should mention the effect of **sampling variability.** Unless K is extremely large, the empirical distribution of S_T will only be an approximation of the true distribution. There will be some natural variation in statistics measured from Monte Carlo simulations. Since Monte Carlo simulations involve *independent* draws, one can show that the standard error of statistics is inversely related to the square root of K. Thus more simulations will increase precision, but at a slow rate. For example, accuracy is increased by a factor of 10 going from $K = 10$ to $K = 1{,}000$, but then requires going from $K = 1{,}000$ to $K = 100{,}000$ for the same factor of 10.

This accuracy issue is worse for risk management than for pricing, because the quantiles are estimated less precisely than the average. For VAR measures, the precision is also a function of the selected confidence level. Higher confidence levels generate fewer observations in the left tail and hence less precise VAR measures. A 99% VAR using 1,000 replications should be expected to have only 10 observations in the left tail, which is not a large number. The VAR estimate is derived from the 10th and 11th sorted numbers. In contrast, a 95% VAR is measured from the 15th and 51st sorted numbers, which is more precise. In addition, the precision of the estimated quantile depends on the shape of the distribution. Relative to a symmetric distribution, a short option position has negative skewness, or a long left tail. The observations in the left tail therefore will be more dispersed, making is more difficult to estimate VAR precisely.

Various methods are available to speed up convergence:

- **Antithetic variable technique.** This technique uses twice the same sequence of random draws from t to T. It takes the original sequence and changes the sign of all their values. This creates twice the number of points in the final distribution of F_T without running twice the number of simulations.
- **Control variate technique.** This technique is used to price options with trees when a similar option has an analytical solution. Say that f_E is a European option with an analytical solution. Going through the tree yields the values of an American option and a European option, F_A and F_E. We then assume that the error in F_A is the same as that in F_E, which is known. The adjusted value is $F_A - (F_E - f_E)$.
- **Quasi-random sequences.** These techniques, also called quasi Monte Carlo (QMC), create draws that are not independent but instead are designed to fill the sample space more uniformly. Simulations have shown that QMC methods converge faster than Monte Carlo methods. In other words, for a fixed number of replications K, QMC values will be on average closer to the true value.

  The advantage of traditional MC, however, is that it also provides a standard error, which is on the order of $1/\sqrt{K}$ because draws are independent. So, we have an idea of how far the estimate might be from the true value, which is useful in deciding on the number of replications. In contrast, QMC methods give no measure of precision.

EXAMPLE 4.8: FRM EXAM 2005—QUESTION 67

Which one of the following statements about Monte Carlo simulation is *false*?

a. Monte Carlo simulation can be used with a lognormal distribution.
b. Monte Carlo simulation can generate distributions for portfolios that contain only linear positions.
c. One drawback of Monte Carlo simulation is that it is computationally very intensive.
d. Assuming the underlying process is normal, the standard error resulting from Monte Carlo simulation is inversely related to the square root of the number of trials.

EXAMPLE 4.9: FRM EXAM 2007—QUESTION 66

A risk manager has been requested to provide some indication of the accuracy of a Monte Carlo simulation. Using 1,000 replications of a normally distributed variable S, the relative error in the one-day 99% VAR is 5%. Under these conditions,

a. Using 1,000 replications of a long option position on S should create a larger relative error.
b. Using 10,000 replications should create a larger relative error.
c. Using another set of 1,000 replications will create an exact measure of 5.0% for relative error.
d. Using 1,000 replications of a short option position on S should create a larger relative error.

EXAMPLE 4.10: SAMPLING VARIATION

The measurement error in VAR, due to sampling variation, should be greater with

a. More observations and a high confidence level (e.g., 99%)
b. Fewer observations and a high confidence level
c. More observations and a low confidence level (e.g., 95%)
d. Fewer observations and a low confidence level

4.3 MULTIPLE SOURCES OF RISK

We now turn to the more general case of simulations with many sources of financial risk. Define N as the number of risk factors. If the factors S_j are independent, the randomization can be performed independently for each variable. For the GBM model,

$$\Delta S_{j,t} = S_{j,t-1}\mu_j \Delta t + S_{j,t-1}\sigma_j \epsilon_{j,t}\sqrt{\Delta t} \tag{4.14}$$

where the standard normal variables ϵ are independent across time and factor $j = 1, \ldots, N$.

In general, however, risk factors are correlated. The simulation can be adapted by, first, drawing a set of independent variables η, and, second, transforming them into correlated variables ϵ. As an example, with two factors only, we write

$$\begin{aligned} \epsilon_1 &= \eta_1 \\ \epsilon_2 &= \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2 \end{aligned} \tag{4.15}$$

Here, ρ is the correlation coefficient between the variables ϵ. Because the η's have unit variance and are uncorrelated, we verify that the variance of ϵ_2 is one, as required

$$V(\epsilon_2) = \rho^2 V(\eta_1) + [(1-\rho^2)^{1/2}]^2 V(\eta_2) = \rho^2 + (1-\rho^2) = 1$$

Furthermore, the correlation between ϵ_1 and ϵ_2 is given by

$$\mathrm{Cov}(\epsilon_1, \epsilon_2) = \mathrm{Cov}(\eta_1, \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2) = \rho\,\mathrm{Cov}(\eta_1, \eta_1) = \rho$$

Defining ϵ as the *vector* of values, we verified that the covariance matrix of ϵ is

$$V(\epsilon) = \begin{bmatrix} \sigma^2(\epsilon_1) & \mathrm{Cov}(\epsilon_1, \epsilon_2) \\ \mathrm{Cov}(\epsilon_1, \epsilon_2) & \sigma^2(\epsilon_2) \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = R$$

Note that this covariance matrix, which is the expectation of squared deviations from the mean, can also be written as

$$V(\epsilon) = E[(\epsilon - E(\epsilon)) \times (\epsilon - E(\epsilon))'] = E(\epsilon \times \epsilon')$$

because the expectation of ϵ is zero. To generalize this approach to many more risk factors, however, we need a systematic way to derive the transformation in Equation (4.15).

4.3.1 The Cholesky Factorization

We would like to generate N joint values of ϵ that display the correlation structure $V(\epsilon) = E(\epsilon\epsilon') = R$. Because the matrix R is a symmetric real matrix, it can be decomposed into its so-called Cholesky factors:

$$R = TT' \tag{4.16}$$

where T is a lower triangular matrix with zeros on the upper right corners (above the diagonal). This is known as the **Cholesky factorization.**

As in the previous section, we first generate a vector of independent η. Thus, their covariance matrix is $V(\eta) = I$, where I is the identity matrix with zeros everywhere except for the diagonal.

We then construct the transformed variable $\epsilon = T\eta$. The covariance matrix is now $V(\epsilon) = E(\epsilon\epsilon') = E((T\eta)(T\eta)') = E(T\eta\eta'T') = TE(\eta\eta')T' = TV(\eta)T' = TIT' = TT' = R$. This transformation therefore generates ϵ variables with the desired correlations.

To illustrate, let us go back to our two-variable case. The correlation matrix can be decomposed into its Cholesky factors:

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{21}a_{11} & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

To find the entries a_{11}, a_{21}, a_{22}, we solve each of the three equations

$$a_{11}^2 = 1$$
$$a_{11}a_{21} = \rho$$
$$a_{21}^2 + a_{22}^2 = 1$$

This gives $a_{11} = 1, a_{21} = \rho$, and $a_{22} = (1 - \rho^2)^{1/2}$. The Cholesky factorization is then

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & (1-\rho^2)^{1/2} \end{bmatrix}$$

Note that this conforms to Equation (4.15):

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

In practice, this decomposition yields a number of useful insights. The decomposition will fail if the number of independent factors implied in the correlation matrix is less than N. For instance, if $\rho = 1$, the two assets are perfectly correlated, meaning that we have twice the same factor. This could be, for instance, the case of two currencies fixed to each other. The decomposition gives $a_{11} = 1, a_{21} = 1, a_{22} = 0$. The new variables are then $\epsilon_1 = \eta_1$ and $\epsilon_2 = \eta_1$. In this case, the second variable η_2 is totally superfluous and simulations can be performed with one variable only.

4.3.2 The Curse of Dimensionality

Modern risk management is about measuring the risk of large portfolios, typically exposed to a large number of risk factors. The problem is that the number of computations increases geometrically with the number of factors N. The covariance matrix, for instance, has dimensions $N(N+1)/2$. A portfolio with 500 variables requires a matrix with 125,250 entries.

In practice, the risk manager should simplify the number of risk factors, discarding those that do not contribute significantly to the risk of the portfolio. Simulations based on the full set of variables would be inordinately time-consuming. The art of simulation is to design parsimonious experiments that represent the breadth of movements in risk factors.

This can be done by an economic analysis of the risk factors and portfolio strategies, as done in Part Three of this handbook. Alternatively, the risk manager can perform a statistical decomposition of the covariance matrix. A widely used method for this is the **principal component analysis** (PCA), which finds linear combinations of the risk factors that have maximal explanatory power. This type of analysis, which is as much an art as it is a science, can be used to reduce the dimensionality of the risk space.

EXAMPLE 4.11: FRM EXAM 2007—QUESTION 28

Let N be a $1 \times n$ vector of independent draws from a standard normal distribution, and let V be a covariance matrix of market time-series data. Then, if L is a diagonal matrix of the eigenvalues of V, E is a matrix of the eigenvectors of V, and $C'C$ is the Cholesky factorization of V, which of the following would generate a normally distributed random vector with mean zero and covariance matrix V to be used in a Monte Carlo simulation?

a. $NC'CN'$
b. NC'
c. $E'LE$
d. Cannot be determined from data given

EXAMPLE 4.12: FRM EXAM 2006—QUESTION 82

Consider a stock that pays no dividends, has a volatility of 25% pa, and has an expected return of 13% pa. The current stock price is $S_0 = \$30$. This implies the model $S_{t+1} = S_t(1 + 0.13\Delta t + 0.25\sqrt{\Delta t}\,\epsilon)$, where ϵ is a standard normal random variable. To implement this simulation, you generate a path of the stock price by starting at $t = 0$, generating a sample for ϵ, updating the stock price according to the model, incrementing t by 1, and repeating this process until the end of the horizon is reached. Which of the following strategies for generating a sample for ϵ will implement this simulation properly?

a. Generate a sample for ϵ by using the inverse of the standard normal cumulative distribution of a sample value drawn from a uniform distribution between 0 and 1.
b. Generate a sample for ϵ by sampling from a normal distribution with mean 0.13 and standard deviation 0.25.
c. Generate a sample for ϵ by using the inverse of the standard normal cumulative distribution of a sample value drawn from a uniform distribution between 0 and 1. Use Cholesky decomposition to correlate this sample with the sample from the previous time interval.
d. Generate a sample for ϵ by sampling from a normal distribution with mean 0.13 and standard deviation 0.25. Use Cholesky decomposition to correlate this sample with the sample from the previous time interval.

EXAMPLE 4.13: FRM EXAM 2006—QUESTION 83

Continuing with the previous question, you have implemented the simulation process discussed earlier using a time interval $\Delta t = 0.001$, and you are analyzing the following stock price path generated by your implementation.

t	S_{t-1}	ϵ	ΔS
0	30.00	0.0930	0.03
1	30.03	0.8493	0.21
2	30.23	0.9617	0.23
3	30.47	0.2460	0.06
4	30.53	0.4769	0.12
5	30.65	0.7141	0.18

Given this sample, which of the following simulation steps most likely contains an error?

a. Calculation to update the stock price
b. Generation of random sample value for ϵ
c. Calculation of the change in stock price during each period
d. None of the above

4.4 IMPORTANT FORMULAS

Wiener process: $\Delta z \sim N(0, \Delta t)$

Generalized Wiener process: $\Delta x = a\Delta t + b\Delta z$

Ito process: $\Delta x = a(x, t)\Delta t + b(x, t)\Delta z$

Geometric Brownian motion: $\Delta S = \mu S\Delta t + \sigma S\Delta z$

One-factor equilibrium model for yields: $\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma r_t{}^{\gamma}\Delta z_t$

Vasicek model, $\gamma = 0$

Lognormal model, $\gamma = 1$

CIR model, $\gamma = 0.5$

No-arbitrage models:

Ho and Lee model, $\Delta r_t = \theta(t)\Delta t + \sigma\Delta z_t$

Hull and White model, $\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma\Delta z_t$

Heath, Jarrow, and Morton model, $\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma(t)\Delta z_t$

Binomial trees: $u = e^{\sigma\sqrt{\Delta t}}$, $d = (1/u)$, $p = \frac{e^{\mu\Delta t} - d}{u - d}$

Cholesky factorization: $R = TT'$, $\epsilon = T\eta$

4.5 ANSWERS TO CHAPTER EXAMPLES

Example 4.1: FRM Exam 2009—Question 14

d. The process for the stock prices has mean of zero and volatility of $\sigma\sqrt{\Delta t} = 0.14\sqrt{0.01} = 0.014$. Hence the first step is $S_1 = S_0(1 + 0.014 \times 0.263) = 100.37$. The second step is $S_2 = S_1(1 + 0.014 \times -0.475) = 99.70$.

Example 4.2: FRM Exam 2003—Question 40

b. Both dS/S or $d\ln(S)$ are normally distributed. As a result, S is lognormally distributed. The only incorrect answer is I.

Example 4.3: FRM Exam 2002—Question 126

a. All the statements are correct except a., which is too strong. The expected price is higher than today's price but certainly not the price in all states of the world.

Example 4.4: Interest Rate Model

b. This model postulates only one source of risk in the fixed-income market. This is a single-factor term-structure model.

Example 4.5: Interest Rate Model Interpretation

c. This represents the expected return with mean reversion.

Example 4.6: FRM Exam 2000—Question 118

a. These are no-arbitrage models of the term structure, implemented as either one-factor or two-factor models.

Example 4.7: FRM Exam 2000—Question 119

b. Both the Vasicek and CIR models are one-factor equilibrium models with mean reversion. The Hull-White model is a no-arbitrage model with mean reversion. The Ho-Lee model is an early no-arbitrage model without mean reversion.

Example 4.8: FRM Exam 2005—Question 67

b. MC simulations do account for options. The first step is to simulate the process of the risk factor. The second step prices the option, which properly accounts for nonlinearity.

Example 4.9: FRM Exam 2007—Question 66

d. Short option positions have long left tails, which makes it more difficult to estimate a left-tailed quantile precisely. Accuracy with independent draws increases with the square root of K. Thus increasing the number of replications should shrink the standard error, so answer b. is incorrect.

Example 4.10: Sampling Variation

b. Sampling variability (or imprecision) increases with (1) fewer observations and (2) greater confidence levels. To show (1), we can refer to the formula for the precision of the sample mean, which varies inversely with the square root of the number of data points. A similar reasoning applies to (2). A greater confidence level involves fewer observations in the left tails, from which VAR is computed.

Example 4.11: FRM Exam 2007—Question 28

b. In the notation of the text, N is the vector of i.i.d. random variables η and $C'C = TT'$. The transformed variable is $T\eta$, or $C'N$, or its transpose.

Example 4.12: FRM Exam 2006—Question 82

a. The variable ϵ should have a standard normal distribution (i.e., with mean zero and unit standard deviation). Answer b. is incorrect because ϵ is transformed afterward to the desired mean and standard deviation. The Cholesky decomposition is not applied here because the sequence of random variables has no serial correlation.

Example 4.13: FRM Exam 2006—Question 83

b. The random variable ϵ should have a standard normal distribution, which means that it should have negative as well as positive values, which should average close to zero. This is not the case here. This is probably a uniform variable instead.

CHAPTER 5

Modeling Risk Factors

We now turn to an analysis of the distribution of risk factors used in financial risk management. A common practice is to use the volatility as a single measure of dispersion. More generally, risk managers need to consider the entire shape of the distribution as well as potential variation in time of this distribution.

The normal distribution is a useful starting point due to its attractive properties. Unfortunately, most financial time series are characterized by fatter tails than the normal distribution. In addition, there is ample empirical evidence that risk changes in a predictable fashion. This phenomenon, called **volatility clustering**, could also explain the appearance of fat tails. Extreme observations could be drawn from periods with high volatility. This could cause the appearance of fat tails when combining periods of low and high volatility.

Section 5.1 discusses the sampling of real data and the construction of returns. It shows how returns can be aggregated across time or, for a portfolio, across assets. Section 5.2 then describes the normal and lognormal distributions and explains why these choices are so popular, whereas Section 5.3 discusses distributions that have fatter tails than the normal distribution.

Section 5.4 then turns to time variation in risk. We describe the generalized autoregressive conditional heteroskedastic (GARCH) model and a special case, which is RiskMetrics' exponentially weighted moving average (EWMA). These models place more weight on more recent data and have proved successful in explaining volatility clustering. They should be part of the tool kit of risk managers.

5.1 REAL DATA

5.1.1 Measuring Returns

To start with an example, let us say that we observe movements in the daily yen/dollar exchange rate and wish to characterize the distribution of tomorrow's exchange rate. The risk manager's job is to assess the range of potential gains and losses on a trader's position. He or she observes a sequence of past prices $P_0, P_1, \ldots, P_t$, from which the distribution of tomorrow's price, P_{t+1}, should be inferred.

FRM Exam Part 1 topic.

The truly random component in tomorrow's price is not its level, but rather its change relative to today's price. We measure the *relative rate of change* in the spot price:

$$r_t = (P_t - P_{t-1})/P_{t-1} \tag{5.1}$$

Alternatively, we could construct the logarithm of the price ratio:

$$R_t = \ln[P_t/P_{t-1}] \tag{5.2}$$

which is equivalent to using continuous instead of discrete compounding. This is also

$$R_t = \ln[1 + (P_t - P_{t-1})/P_{t-1}] = \ln[1 + r_t]$$

Because $\ln(1 + x)$ is close to x if x is small, R_t should be close to r_t provided the return is small. For daily data, there is typically little difference between R_t and r_t.

The next question is whether the sequence of variables r_t can be viewed as independent observations. Independent observations have the very nice property that their joint distribution is the product of their marginal distribution, which considerably simplifies the analysis. The obvious question is whether this assumption is a workable approximation. In fact, there are good economic reasons to believe that rates of change on financial prices are close to independent.

The hypothesis of **efficient markets** postulates that current prices convey all relevant information about the asset. If so, any change in the asset price must be due to news, or events that are by definition impossible to forecast (otherwise, the event would not be news). This implies that changes in prices are unpredictable and, hence, satisfy our definition of independent random variables.

This hypothesis, also known as the **random walk** theory, implies that the conditional distribution of returns depends on only current prices, and not on the previous history of prices. If so, technical analysis must be a fruitless exercise. Technical analysts try to forecast price movements from past price patterns. If in addition the distribution of returns is constant over time, the variables are said to be **independent and identically distributed** (i.i.d.).

5.1.2 Time Aggregation

It is often necessary to translate parameters over a given horizon to another horizon. For example, we have data for daily returns, from which we compute a daily volatility that we want to extend to a monthly volatility. This is a **time aggregation** problem.

Returns can be easily aggregated when we use the log of the price ratio, because the log of a product is the sum of the logs of the individual terms. Over two periods, for instance, the price movement can be described as the sum of the price movements over each day:

$$R_{t,2} = \ln(P_t/P_{t-2}) = \ln(P_t/P_{t-1}) + \ln(P_{t-1}/P_{t-2}) = R_{t-1} + R_t \tag{5.3}$$

The expected return and variance are then $E(R_{t,2}) = E(R_{t-1}) + E(R_t)$ and $V(R_{t,2}) = V(R_{t-1}) + V(R_t) + 2\text{Cov}(R_{t-1}, R_t)$. Assuming returns are uncorrelated (i.e., that the covariance term is zero) and have identical distributions across days, we have $E(R_{t,2}) = 2E(R_t)$ and $V(R_{t,2}) = 2V(R_t)$.

More generally, define T as the number of steps. The multiple-period expected return and volatility are

$$\mu_T = \mu T \tag{5.4}$$

$$\sigma_T = \sigma\sqrt{T} \tag{5.5}$$

KEY CONCEPT

When successive returns are uncorrelated, the volatility increases as the horizon extends following the square root of time.

Assume now that the distribution is stable under addition, which means that it stays the same whether over one period or over multiple periods. This is the case for the normal distribution. If so, we can use the same multiplier α that corresponds to a selected confidence level for a one-period and T-period return. The multiple-period VAR is

$$\text{VAR}_T = \alpha(\sigma\sqrt{T})W = \text{VAR}_1\sqrt{T} \tag{5.6}$$

In other words, extension to a multiple period follows a square root of time rule. Figure 5.1 shows how VAR grows with the length of the horizon for various confidence levels. The figure shows that VAR increases more slowly than time. The one-month 99% VAR is 0.67, but increases to only 2.33 at a one-year horizon.

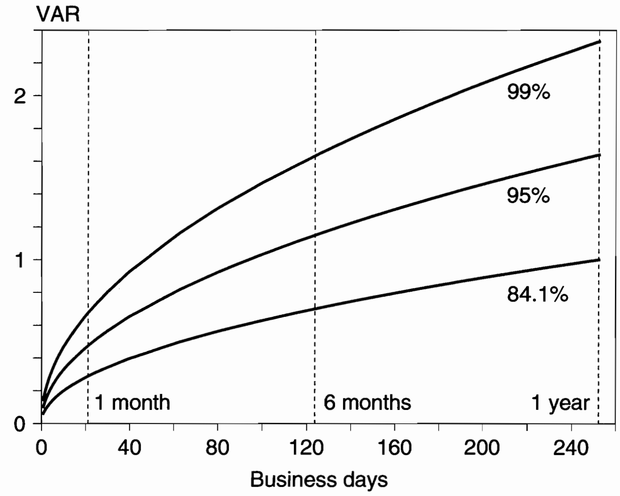

FIGURE 5.1 VAR at Increasing Horizons

In summary, the square root of time rule applies to parametric VAR under the following conditions:

- The distribution is the same at each period (i.e., there is no predictable time variation in expected return nor in risk).
- Returns are uncorrelated across each period.
- The distribution is the same for one- or T-period, or is stable under addition, such as the normal distribution.

If returns are not independent, we may be able to characterize longer-term risks. For instance, when returns follow a first-order autoregressive process,

$$R_t = \rho R_{t-1} + u_t \tag{5.7}$$

we can write the variance of two-day returns as

$$V[R_t + R_{t-1}] = V[R_t] + V[R_{t-1}] + 2\mathrm{Cov}[R_t, R_{t-1}] = \sigma^2 + \sigma^2 + 2\rho\sigma^2 \tag{5.8}$$

or

$$V[R_t + R_{t-1}] = \sigma^2 \times 2[1 + \rho] \tag{5.9}$$

In this case,

$$\mathrm{VAR}_2 = \alpha(\sigma\sqrt{2(1+\rho)})W = [\mathrm{VAR}_1\sqrt{2}]\sqrt{(1+\rho)} \tag{5.10}$$

Because we are considering correlations in the time series of the same variable, ρ is called the **autocorrelation coefficient**, or the **serial autocorrelation coefficient.** A positive value for ρ describes a situation where a movement in one direction is likely to be followed by another in the same direction. This implies that markets display **trends**, or **momentum.** In this case, the longer-term volatility increases faster than with the usual square root of time rule.

A negative value for ρ, by contrast, describes a situation where a movement in one direction is likely to be reversed later. This is an example of **mean reversion.** In this case, the longer-term volatility increases more slowly than with the usual square root of time rule.

EXAMPLE 5.1: TIME SCALING

Consider a portfolio with a one-day VAR of \$1 million. Assume that the market is trending with an autocorrelation of 0.1. Under this scenario, what would you expect the two-day VAR to be?

a. \$2 million
b. \$1.414 million
c. \$1.483 million
d. \$1.449 million

EXAMPLE 5.2: INDEPENDENCE

A fundamental assumption of the random walk hypothesis of market returns is that returns from one time period to the next are statistically independent. This assumption implies

- **a.** Returns from one time period to the next can never be equal.
- **b.** Returns from one time period to the next are uncorrelated.
- **c.** Knowledge of the returns from one time period does not help in predicting returns from the next time period.
- **d.** Both b. and c. are true.

EXAMPLE 5.3: FRM EXAM 2002—QUESTION 3

Consider a stock with daily returns that follow a random walk. The annualized volatility is 34%. Estimate the weekly volatility of this stock assuming that the year has 52 weeks.

- **a.** 6.80%
- **b.** 5.83%
- **c.** 4.85%
- **d.** 4.71%

EXAMPLE 5.4: FRM EXAM 2002—QUESTION 2

Assume we calculate a one-week VAR for a natural gas position by rescaling the daily VAR using the square root of time rule. Let us now assume that we determine the *true* gas price process to be mean reverting and recalculate the VAR. Which of the following statements is true?

- **a.** The recalculated VAR will be less than the original VAR.
- **b.** The recalculated VAR will be equal to the original VAR.
- **c.** The recalculated VAR will be greater than the original VAR.
- **d.** There is no necessary relationship between the recalculated VAR and the original VAR.

5.1.3 Portfolio Aggregation

Let us now turn to aggregation of returns across assets. Consider, for example, an equity portfolio consisting of investments in N shares. Define the number of each share held as q_i with unit price S_i. The portfolio value at time t is then

$$W_t = \sum_{i=1}^{N} q_i S_{i,t} \tag{5.11}$$

We can write the weight assigned to asset i as

$$w_{i,t} = \frac{q_i S_{i,t}}{W_t} \tag{5.12}$$

which by construction sum to unity. Using weights, however, rules out situations with zero net investment, $W_t = 0$, such as some derivatives positions. But we could have positive and negative weights if short selling is allowed, or weights greater than one if the portfolio can be leveraged.

The next period, the portfolio value is

$$W_{t+1} = \sum_{i=1}^{N} q_i S_{i,t+1} \tag{5.13}$$

assuming that the unit price incorporates any income payment. The gross, or dollar, return is then

$$W_{t+1} - W_t = \sum_{i=1}^{N} q_i (S_{i,t+1} - S_{i,t}) \tag{5.14}$$

and the *rate* of return is

$$\frac{W_{t+1} - W_t}{W_t} = \sum_{i=1}^{N} \frac{q_i S_{i,t}}{W_t} \frac{(S_{i,t+1} - S_{i,t})}{S_{i,t}} = \sum_{i=1}^{N} w_{i,t} \frac{(S_{i,t+1} - S_{i,t})}{S_{i,t}} \tag{5.15}$$

So, the portfolio rate of return is a linear combination of the asset returns

$$r_{p,t+1} = \sum_{i=1}^{N} w_{i,t}\, r_{i,t+1} \tag{5.16}$$

The dollar return is then

$$W_{t+1} - W_t = \left[\sum_{i=1}^{N} w_{i,t} r_{i,t+1} \right] W_t \tag{5.17}$$

and has a normal distribution if the individual returns are also normally distributed.

Alternatively, we could express the individual positions in dollar terms,

$$x_{i,t} = w_{i,t} W_t = q_i S_{i,t} \tag{5.18}$$

The dollar return is also, using dollar amounts,

$$W_{t+1} - W_t = \left[\sum_{i=1}^{N} x_{i,t} r_{i,t+1} \right] \tag{5.19}$$

As we have seen in the previous chapter, the variance of the portfolio dollar return is

$$V[W_{t+1} - W_t] = x' \Sigma x \tag{5.20}$$

Because the portfolio follows a normal distribution, it is fully characterized by its expected return and variance. The portfolio VAR is then

$$\text{VAR} = \alpha \sqrt{x' \Sigma x} \tag{5.21}$$

where α depends on the confidence level and the selected density function.

EXAMPLE 5.5: FRM EXAM 2004—QUESTION 39

Consider a portfolio with 40% invested in asset X and 60% invested in asset Y. The mean and variance of return on X are 0 and 25 respectively. The mean and variance of return on Y are 1 and 121 respectively. The correlation coefficient between X and Y is 0.3. What is the nearest value for portfolio volatility?

a. 9.51
b. 8.60
c. 13.38
d. 7.45

5.2 NORMAL AND LOGNORMAL DISTRIBUTIONS

5.2.1 Why the Normal?

The normal, or Gaussian, distribution is usually the first choice when modeling asset returns. This distribution plays a special role in statistics, as it is easy to handle and is stable under addition, meaning that a combination of jointly normal variables is itself normal. It also provides the limiting distribution of the average of *independent* random variables (through the central limit theorem).

Empirically, the normal distribution provides a rough, first-order approximation to the distribution of many random variables: rates of changes in currency prices, rates of changes in stock prices, rates of changes in bond prices, changes in yields, and rates of changes in commodity prices. All of these are characterized by many occurrences of small moves and fewer occurrences of large moves. This provides a rationale for a distribution with more weight in the center, such as the bell-shaped normal distribution. For many applications, this is a sufficient approximation. This may not be appropriate for measuring tail risk, however.

5.2.2 Computing Returns

In what follows, the random variable is the new price P_1, given the current price P_0. Defining $r = (P_1 - P_0)/P_0$ as the rate of return in the price, we can start with the assumption that this random variable is drawn from a normal distribution,

$$r \sim \Phi(\mu, \sigma) \tag{5.22}$$

with some mean μ and standard deviation σ. Turning to prices, we have $P_1 = P_0(1 + r)$ and

$$P_1 \sim P_0 + \Phi(P_0\mu, P_0\sigma) \tag{5.23}$$

For instance, starting from a stock price of \$100, if $\mu = 0\%$ and $\sigma = 15\%$, we have $P_1 \sim \$100 + \Phi(\$0, \$15)$.

In this case, however, the normal distribution cannot be even theoretically correct. Because of limited liability, stock prices cannot go below zero. Similarly, commodity prices and yields cannot turn negative. This is why another popular distribution is the **lognormal distribution**, which is such that

$$R = \ln(P_1/P_0) \sim \Phi(\mu, \sigma) \tag{5.24}$$

By taking the logarithm, the price is given by $P_1 = P_0 \exp(R)$, which precludes prices from turning negative, as the exponential function is always positive. Figure 5.2 compares the normal and lognormal distributions over a one-year horizon with $\sigma = 15\%$ annually. The distributions are very similar, except for the tails. The lognormal is skewed to the right.

The difference between the two distributions is driven by the size of the volatility parameter over the horizon. Small values of this parameter imply that the distributions are virtually identical. This can happen either when the asset is not very risky, that is, when the annual volatility is small, or when the horizon is very short. In this situation, there is very little chance of prices turning negative. The limited liability constraint is not important.

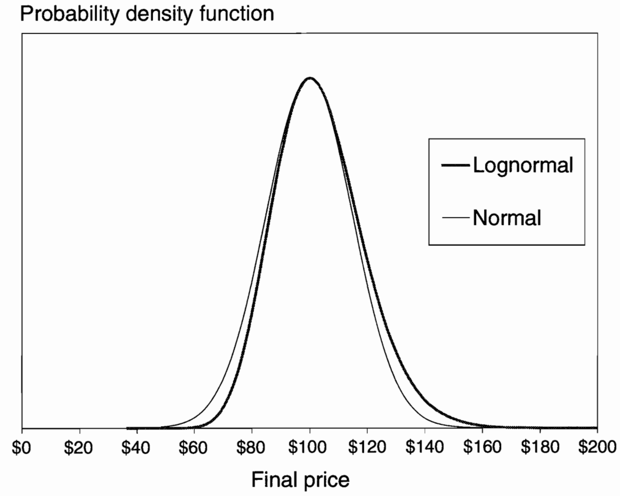

FIGURE 5.2 Normal and Lognormal Distributions—Annual Horizon

TABLE 5.1 Comparison between Discrete and Log Returns

	Daily	Annual
Initial price	100	100
Ending price	101	115
Discrete return	1.0000	15.0000
Log return	0.9950	13.9762
Relative difference	0.50%	7.33%

KEY CONCEPT

The normal and lognormal distributions are very similar for short horizons or low volatilities.

As an example, Table 5.1 compares the computation of returns over a one-day horizon and a one-year horizon. The one-day returns are 1.000% and 0.995% for discrete and log returns, respectively, which translates into a relative difference of 0.5%, which is minor. In contrast, the difference is more significant over longer horizons.

5.3 DISTRIBUTIONS WITH FAT TAILS

Perhaps the most serious problem with the normal distribution is the fact that its tails disappear too fast, at least faster than what is empirically observed in financial data. We typically observe that every market experiences one or more daily moves of four standard deviations or more per year. Such frequency is incompatible with a normal distribution. With a normal distribution, the probability of this happening is 0.0032% for one day, which implies a frequency of once every 125 years.

KEY CONCEPT

Every financial market experiences one or more daily price moves of four standard deviations or more each year. And in any year, there is usually at least one market that has a daily move greater than 10 standard deviations.

This empirical observation can be explained in a number of ways: (1) The true distribution has fatter tails (e.g., the Student's t), or (2) the observations are drawn from a mix of distributions (e.g., a mix of two normals, one with low risk, the other with high risk), or (3) the distribution is nonstationary.

The first explanation is certainly a possibility. Figure 5.3 displays the density function of the normal and Student's t distribution, with 4 and 6 degrees of freedom (df). The Student's t density has fatter tails, which better reflect the occurrences of extreme observations in empirical financial data.

The distributions are further compared in Table 5.2. The left-side panel reports the tail probability of an observation lower than the deviate. For instance, the probability of observing a draw less than -3 is 0.001, or 0.1% for the normal, 0.012 for the Student's t with 6 degrees of freedom, and 0.020 for the Student's t with 4 degrees of freedom. There is a greater probability of observing an extreme move when the data is drawn from a Student's t rather than from a normal distribution.

We can transform these into an expected number of occurrences in one year, or 250 business days. The right-side panel shows that the corresponding numbers are 0.34, 3.00, and 4.99 for the respective distributions. In other words, with a

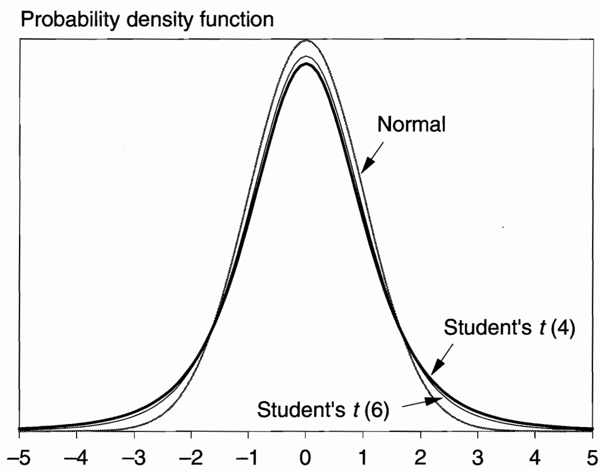

FIGURE 5.3 Normal and Student's t Distributions

TABLE 5.2 Comparison of the Normal and Student's t Distributions

	Tail Probability			Expected Number in 250 Days		
Deviate	Normal	t df = 6	t df = 4	Normal	t df = 6	t df = 4
−5	0.00000	0.00123	0.00375	0.00	0.31	0.94
−4	0.00003	0.00356	0.00807	0.01	0.89	2.02
−3	0.00135	0.01200	0.01997	0.34	3.00	4.99
−2	0.02275	0.04621	0.05806	5.69	11.55	14.51
−1	0.15866	0.17796	0.18695	39.66	44.49	46.74
				Deviate (Alpha)		
Probability = 1%				2.33	3.14	3.75
Ratio to normal				1.00	1.35	1.61

normal distribution, we should expect that this extreme movement below $z = -3$ will occur one day or less on average. With a Student's t with df = 4, the expected number is five in a year, which is closer to reality.

The bottom panel reports the deviate that corresponds to a 99% right-tail confidence level, or 1% left tail. For the normal distribution, this is the usual 2.33. For the Student's t with df = 4, α is 3.75, much higher. The ratio of the two is 1.61. Thus a rule of thumb would be to correct the VAR measure from a normal distribution by a ratio of 1.61 to achieve the desired coverage in the presence of fat tails. More generally, this explains why safety factors are used to multiply VAR measures, such as the Basel multiplicative factor of 3.

5.4 TIME VARIATION IN RISK

Fat tails can also occur when risk factors are drawn from a distribution with time-varying volatility. To be practical, this time variation must have some predictability.

5.4.1 Moving Average

Consider a traditional problem where a risk manager observes a sequence of T returns r_t, from which the variance must be estimated. To simplify, ignore the mean return. At time t, the traditional variance estimate is

$$\sigma_t^2 = (1/T)\sum_{i=1}^{T} r_{t-i}^2 \tag{5.25}$$

This is a simple average where the weight on each past observation is $w_i = 1/T$. This may not be the best use of the data, however, especially if more recent observations are more relevant for the next day.

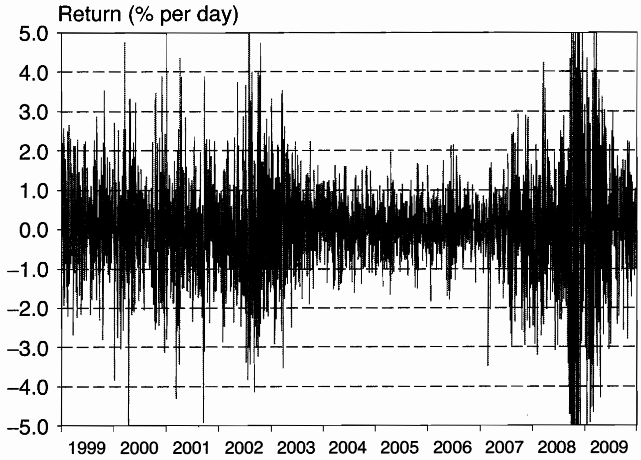

FIGURE 5.4 Daily Return for U.S. Equities

This is illustrated in Figure 5.4, which plots daily returns on the S&P 500 index. We observe **clustering** in volatility. Some periods are particularly hectic. After the Lehman bankruptcy in September 2008, there was a marked increase in the number of large returns, both positive and negative. Other periods, such as 2004 to 2006, were much more quiet. Simply taking the average over the entire period will underestimate risk during 2008 and overestimate risk during 2004 to 2006.

5.4.2 GARCH

A practical model for volatility clustering is the **generalized autoregressive conditional heteroskedastic** (**GARCH**) model developed by Engle (1982) and Bollerslev (1986). This class of models assumes that the return at time t has a particular distribution such as the normal, conditional on parameters μ_t and σ_t:

$$r_t \sim \Phi(\mu_t, \sigma_t) \tag{5.26}$$

The important point is that σ is indexed by time. We define the **conditional variance** as that conditional on current information. This may differ from the **unconditional variance**, which is the same for the whole sample. Thus the average variance is unconditional, whereas a time-varying variance is conditional.

There is substantial empirical evidence that conditional volatility models successfully forecast risk. The general assumption is that the conditional returns have a normal distribution, although this could be extended to other distributions such as the Student's t.

The GARCH model assumes that the conditional variance depends on the latest innovation, and on the previous conditional variance. Define $h_t = \sigma_t^2$ as the conditional variance, using information up to time $t-1$, and r_{t-1} as the previous day's return, also called innovation. The simplest such model is the GARCH(1,1) process,

$$h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1} \tag{5.27}$$

which involves one lag of the innovation and one lag of the previous forecast. The β term is important because it allows persistence in the variance, which is a realistic feature of the data. Here, we ignored the mean μ_t, which is generally small if the horizon is short. More generally, the GARCH(p, q) model has p lagged terms on historical returns and q lagged terms on previous variances.

The average unconditional variance is found by setting $E[r_{t-1}^2] = h_t = h_{t-1} = h$. Solving for h, we find

$$h = \frac{\alpha_0}{1 - \alpha_1 - \beta} \tag{5.28}$$

This model is stationary when the sum of parameters $\gamma = \alpha_1 + \beta$ are less than unity. This sum is also called the **persistence**, as it defines the speed at which shocks to the variance revert to their long-run values.

Figure 5.5 displays the one-day GARCH forecast for the S&P 500 index. The GARCH long-run volatility has been around 1.1% per day. Volatility peaked in September 2008, at the time of the Lehman Brothers bankruptcy, when it reached 5%. This reverted slowly to the long-run average later, which is a typical pattern of this forecast. Also note that the GARCH model identifies extended periods of low volatility, from 2004 to 2006.

To understand how the process works, consider Table 5.3. The parameters are $\alpha_0 = 0.01$, $\alpha_1 = 0.03$, $\beta = 0.95$. The unconditional variance is $0.01/(1 - 0.03 - 0.95) = 0.5$, or 0.7 daily volatility, which is typical of a currency series, as it translates into an annualized volatility of 11%. The process is stationary because $\alpha_1 + \beta = 0.98 < 1$.

At time 0, we start with the variance at $h_0 = 1.1$ (expressed in percent squared). The conditional volatility is $\sqrt{h_0} = 1.05\%$. The next day, there is a large return of 3%. The new variance forecast is then $h_1 = 0.01 + 0.03 \times 3^2 + 0.95 \times 1.1 = 1.32$. The conditional volatility just went up to 1.15%. If nothing happens the following days, the next variance forecast is $h_2 = 0.01 + 0.03 \times 0^2 + 0.95 \times 1.32 = 1.27$. And so on.

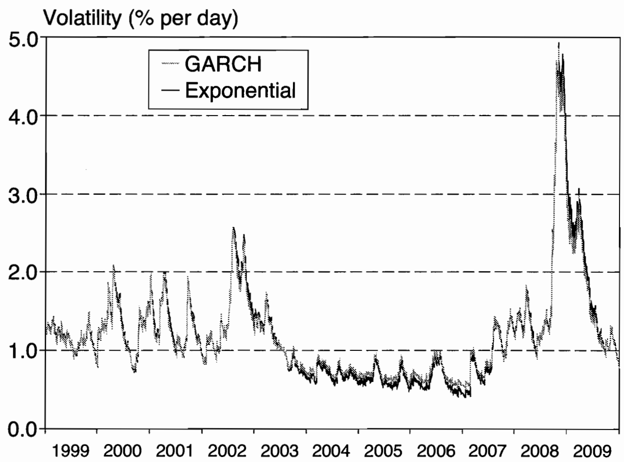

FIGURE 5.5 GARCH and EWMA Volatility Forecast for U.S. Equities

TABLE 5.3 Building a GARCH Forecast

Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.10
1	3.0	1.32	1.15	±2.30
2	0.0	1.27	1.13	±2.25
3	0.0	1.22	1.10	±2.20

How are the GARCH parameters derived? The parameters are estimated by a **maximum likelihood** method. This involves a numerical optimization of the likelihood of the observations. Typically, the scaled residuals $\epsilon_t = r_t/\sqrt{h_t}$ are assumed to have a normal distribution and to be independent. For each observation, the density is

$$f(r_t|\alpha_0, \alpha_1, \beta) = \frac{1}{\sqrt{2\pi h_t}} \exp[-\frac{1}{2h_t}(r_t - \mu_t)^2]$$

If we have T observations, their joint density is the product of the densities for each time period t. The likelihood function then depends on the three GARCH parameters, again ignoring the mean.

The optimization maximizes the logarithm of the likelihood function

$$\max F(\alpha_0, \alpha_1, \beta \mid r) = \sum_{t=1}^{T} \ln f(r_t \mid h_t) = \sum_{t=1}^{T} [\ln \frac{1}{\sqrt{2\pi h_t}} - \frac{r_t^2}{2h_t}] \tag{5.29}$$

where f is the normal density function. The optimization must be performed recursively. We fix some values for the three parameters, then start with h_0 and recursively compute h_1 to h_T. We then compute the likelihood function and let the optimizer converge to a maximum.

Finally, the GARCH process can be extrapolated to later days. This creates a **volatility term structure**. For the next-day forecast,

$$E_{t-1}(r_{t+1}^2) = \alpha_0 + \alpha_1 E_{t-1}(r_t^2) + \beta h_t = \alpha_0 + \alpha_1 h_t + \beta h_t = \alpha_0 + \gamma h_t$$

For the following day,

$$E_{t-1}(r_{t+2}^2) = \alpha_0 + \alpha_1 E_{t-1}(r_{t+1}^2) + \beta E_{t-1}(h_{t+1}) = \alpha_0 + (\alpha_1 + \beta) E_{t-1}(r_{t+1}^2)$$

$$E_{t-1}(r_{t+2}^2) = \alpha_0 + \gamma(\alpha_0 + \gamma h_t)$$

Generally,

$$E_{t-1}(r_{t+n}^2) = \alpha_0(1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}) + \gamma^n h_t$$

Figure 5.6 illustrates the dynamics of shocks to a GARCH process for various values of the persistence parameter. As the conditional variance deviates from the

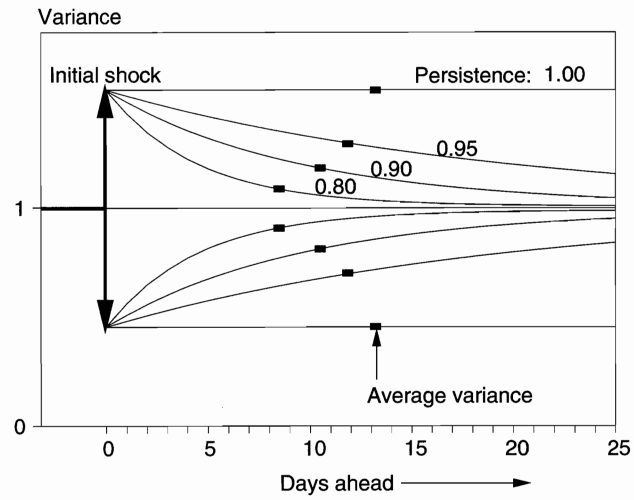

FIGURE 5.6 Shocks to a GARCH Process

starting value, it slowly reverts to the long-run value at a speed determined by $\alpha_1 + \beta$.

Note that these are forecasts of one-day variances at forward points in time. The total variance over the horizon is the sum of one-day variances. The *average* variance is marked with a black rectangle on the graph.

The graph also shows why the square root of time rule for extrapolating returns does not apply when risk is time-varying. If the initial value of the variance is greater than the long-run average, simply extrapolating the one-day variance to a longer horizon will overstate the average variance. Conversely, starting from a lower value and applying the square root of time rule will understate risk.

KEY CONCEPT

The square root of time rule used to scale one-day returns into longer horizons is generally inappropriate when risk is time-varying.

EXAMPLE 5.6: FRM EXAM 2009—QUESTION 2-13

Suppose σ_t^2 is the estimated variance at time t and u_t is the realized return at t. Which of the following GARCH(1,1) models will take the longest time to revert to its mean?

a. $\sigma_t^2 = 0.04 + 0.02u_{t-1}^2 + 0.92\sigma_{t-1}^2$
b. $\sigma_t^2 = 0.02 + 0.04u_{t-1}^2 + 0.94\sigma_{t-1}^2$
c. $\sigma_t^2 = 0.03 + 0.02u_{t-1}^2 + 0.95\sigma_{t-1}^2$
d. $\sigma_t^2 = 0.03 + 0.03u_{t-1}^2 + 0.93\sigma_{t-1}^2$

EXAMPLE 5.7: FRM EXAM 2006—QUESTION 132

Assume you are using a GARCH model to forecast volatility that you use to calculate the one-day VAR. If volatility is mean reverting, what can you say about the T-day VAR?

a. It is less than the $\sqrt{T}\times$ one-day VAR.
b. It is equal to $\sqrt{T}\times$ one-day VAR.
c. It is greater than the $\sqrt{T}\times$ one-day VAR.
d. It could be greater or less than the $\sqrt{T}\times$ one-day VAR.

EXAMPLE 5.8: FRM EXAM 2007—QUESTION 34

A risk manager estimates daily variance h_t using a GARCH model on daily returns r_t: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}$, with $\alpha_0 = 0.005$, $\alpha_1 = 0.04$, $\beta = 0.94$. The long-run *annualized* volatility is approximately

a. 13.54%
b. 7.94%
c. 72.72%
d. 25.00%

EXAMPLE 5.9: FRM EXAM 2009—QUESTION 2-17

Which of the following statements is *incorrect* regarding the volatility term structure predicted by a GARCH(1,1) model: $\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2$, where $\alpha + \beta < 1$?

a. When the current volatility estimate is below the long-run average volatility, this GARCH model estimates an upward-sloping volatility term structure.
b. When the current volatility estimate is above the long-run average volatility, this GARCH model estimates a downward-sloping volatility term structure.
c. Assuming the long-run estimated variance remains unchanged as the GARCH parameters α and β increase, the volatility term structure predicted by this GARCH model reverts to the long-run estimated variance more slowly.
d. Assuming the long-run estimated variance remains unchanged as the GARCH parameters α and β increase, the volatility term structure predicted by this GARCH model reverts to the long-run estimated variance faster.

5.4.3 EWMA

The RiskMetrics approach is a specific case of the GARCH process and is particularly simple and convenient to use. Variances are modeled using an **exponentially weighted moving average (EWMA)** forecast. The forecast is a weighted average of the previous forecast, with weight λ, and of the latest squared innovation, with weight $(1-\lambda)$:

$$h_t = \lambda h_{t-1} + (1-\lambda) r_{t-1}^2 \tag{5.30}$$

The λ parameter, with $0 < \lambda < 1$, is also called the **decay factor**. It determines the relative weights placed on previous observations. The EWMA model places geometrically declining weights on past observations, assigning greater importance to recent observations. By recursively replacing h_{t-1} in Equation (5.30), we have

$$h_t = (1-\lambda)[r_{t-1}^2 + \lambda r_{t-2}^2 + \lambda^2 r_{t-3}^2 + \cdots] \tag{5.31}$$

The weights therefore decrease at a geometric, or exponential, rate. The lower λ, the more quickly older observations are forgotten. RiskMetrics has chosen $\lambda = 0.94$ for daily data and $\lambda = 0.97$ for monthly data.

Table 5.4 shows how to build the EWMA forecast using a parameter of $\lambda = 0.95$, which is consistent with the previous GARCH example. At time 0, we start with the variance at $h_0 = 1.1$, as before. The next day, we have a return of 3%. The new variance forecast is then $h_1 = 0.05 \times 3^2 + 0.95 \times 1.1 = 1.50$. The next day, this moves to $h_2 = 0.05 \times 0^2 + 0.95 \times 1.50 = 1.42$. And so on.

This model is a special case of the GARCH process, where α_0 is set to 0, and α_1 and β sum to unity. The model therefore has permanent persistence. Shocks to the volatility do not decay, as shown in Figure 5.6 when the persistence is 1.00. Thus longer-term extrapolation from the GARCH and EWMA models may give quite different forecasts. Indeed, Equation (5.28) shows that the unconditional variance is not defined. Over a one-day horizon, however, the two models are quite similar and often indistinguishable from each other.

TABLE 5.4 Building a EWMA Forecast

Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.1
1	3.0	1.50	1.22	±2.4
2	0.0	1.42	1.19	±2.4
3	0.0	1.35	1.16	±2.3

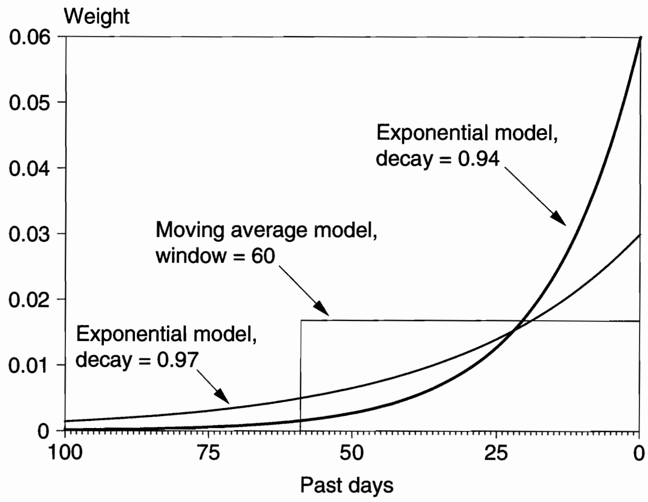

FIGURE 5.7 Weights on Past Observations

Figure 5.5 also displays the EWMA forecast, which follows a similar pattern to the GARCH forecast, reflecting the fact that it is a special case of the other. The EWMA forecast, however, has less mean reversion. For instance, it stays lower in 2004 to 2006.

Figure 5.7 displays the pattern of weights for previous observations. With $\lambda = 0.94$, the weights decay quickly. The weight on the last day is $(1 - \lambda) = (1 - 0.94) = 0.06$. The weight on the previous day is $(1 - \lambda)\lambda = 0.0564$, and so on. The weight drops below 0.00012 for data more than 100 days old. With $\lambda = 0.97$, the weights start at a lower level but decay more slowly. In comparison, moving average (MA) models have a fixed window, with equal weights within the window but otherwise zero. MA models with shorter windows give a greater weight to recent observations. As a result, they are more responsive to current events and more volatile.

EXAMPLE 5.10: FRM EXAM 2007—QUESTION 46

A bank uses the exponentially weighted moving average (EWMA) technique with λ of 0.9 to model the daily volatility of a security. The current estimate of the daily volatility is 1.5%. The closing price of the security is USD 20 yesterday and USD 18 today. Using continuously compounded returns, what is the updated estimate of the volatility?

- **a.** 3.62%
- **b.** 1.31%
- **c.** 2.96%
- **d.** 5.44%

EXAMPLE 5.11: FRM EXAM 2006—QUESTION 40

Using a daily RiskMetrics EWMA model with a decay factor $\lambda = 0.95$ to develop a forecast of the conditional variance, which weight will be applied to the return that is four days old?

a. 0.000
b. 0.043
c. 0.048
d. 0.950

EXAMPLE 5.12: EFFECT OF WEIGHTS ON OBSERVATIONS

Until January 1999 the historical volatility for the Brazilian real versus the U.S. dollar had been very small for several years. On January 13, Brazil abandoned the defense of the currency peg. Using the data from the close of business on January 13, which of the following methods for calculating volatility would have shown the greatest jump in measured historical volatility?

a. 250-day equal weight
b. Exponentially weighted with a daily decay factor of 0.94
c. 60-day equal weight
d. All of the above

EXAMPLE 5.13: FRM EXAM 2008—QUESTION 1-8

Which of the following four statements on models for estimating volatility is *incorrect*?

a. In the EWMA model, some positive weight is assigned to the long-run average variance rate.
b. In the EWMA model, the weights assigned to observations decrease exponentially as the observations become older.
c. In the GARCH(1,1) model, a positive weight is estimated for the long-run average variance rate.
d. In the GARCH(1,1) model, the weights estimated for observations decrease exponentially as the observations become older.

EXAMPLE 5.14: FRM EXAM 2009—QUESTION 2-16

Assume that an asset's daily return is normally distributed with zero mean. Suppose you have historical return data, $u_1, u_2, \ldots, u_m$ and that you want to use the maximum likelihood method to estimate the parameters of a EWMA volatility model. To do this, you define $v_i = \sigma_i^2$ as the variance estimated by the EWMA model on day i, so that the likelihood that these m observations occurred is given by: $\Pi_{i=1}^{m}[\frac{1}{\sqrt{2\pi v_i}}\exp[-u_i^2/(2v_i)]$. To maximize the likelihood that these m observations occurred, you must:

a. Find the value of λ that minimizes: $\sum_{i=1}^{m}[-\ln(v_i) - u_i^2/(2v_i)]$
b. Find the value of λ that maximizes: $\sum_{i=1}^{m}[-\ln(v_i) - u_i^2/(2v_i)]$
c. Find the value of λ that minimizes: $-m\ln(v_i) - \sum_{i=1}^{m}[u_i^2/(2v_i)]$
d. Find the value of λ that maximizes: $-m\ln(v_i) - \sum_{i=1}^{m}[u_i^2/(2v_i)]$

5.5 IMPORTANT FORMULAS

Multiperiod expected return and volatility for i.i.d. returns: $\mu_T = \mu T$, $\sigma_T = \sigma\sqrt{T}$

Two-period variance with nonzero autocorrelation: $V[R_t + R_{t-1}] = \sigma^2 \times 2[1+\rho]$

VAR assuming i.i.d. returns: $\text{VAR}_T = \alpha(\sigma\sqrt{T})W = \text{VAR}_1\sqrt{T}$

Portfolio VAR, from covariance matrix Σ and dollar exposures x: $\text{VAR} = \alpha\sqrt{x'\Sigma x}$

GARCH process: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}$

GARCH long-run mean: $h = \alpha_0/(1 - \alpha_1 - \beta)$

EWMA process: $h_t = \lambda h_{t-1} + (1-\lambda)r_{t-1}^2$

5.6 ANSWERS TO CHAPTER EXAMPLES

Example 5.1: Time Scaling

c. Knowing that the variance is V(2-day) = V(1-day) $[2+2\rho]$, we find VAR(2-day) = VAR(1-day) $\sqrt{2+2\rho} = \$1\sqrt{2+0.2} = \$1.483$, assuming the same distribution for the different horizons.

Example 5.2: Independence

d. The term *efficient markets* implies that the distribution of future returns does not depend on past returns. Hence, returns cannot be correlated. It could happen, however, that return distributions are independent but that, just by chance, two successive returns are equal.

Example 5.3: FRM Exam 2002—Question 3

d. Assuming a random walk, we can use the square root of time rule. The weekly volatility is then $34\% \times 1/\sqrt{52} = 4.71\%$.

Example 5.4: FRM Exam 2002—Question 2

a. With mean reversion, the volatility grows more slowly than the square root of time.

Example 5.5: FRM Exam 2004—Question 39

d. The variance of the portfolio is given by $\sigma_p^2 = (0.4)^2 25 + (0.6)^2 121 + 2(0.4)(0.6)0.3\sqrt{25 \times 121} = 55.48$. Hence the volatility is 7.45.

Example 5.6: FRM Exam 2009—Question 2-13

b. The persistence $(\alpha_1 + \beta)$ is, respectively, 0.94, 0.98, 0.97, and 0.96. Hence the model with the highest persistence will take the longest time to revert to the mean.

Example 5.7: FRM Exam 2006—Question 132

d. If the initial volatility were equal to the long-run volatility, then the T-day VAR could be computed using the square root of time rule, assuming normal distributions. If the starting volatility were higher, then the T-day VAR should be less than the $\sqrt{T}\times$ one-day VAR. Conversely, if the starting volatility were lower, then the T-day VAR should be more than the long-run value. However, the question does not indicate the starting point. Hence, answer d. is correct.

Example 5.8: FRM Exam 2007—Question 34

b. The long-run mean variance is $h = \alpha_0/(1 - \alpha_1 - \beta) = 0.005/(1 - 0.04 - 0.94) = 0.25$. Taking the square root, this gives 0.5 for daily volatility. Multiplying by $\sqrt{252}$, we have an annualized volatility of 7.937%.

Example 5.9: FRM Exam 2009—Question 2-17

d. The GARCH model has mean reversion in the conditional volatility, so statements a. and b. are correct. When σ_t is lower than the long-run average, the volatility structure goes up. Higher persistence $\alpha + \beta$ means that mean reversion is slower, so statement c. is correct.

Example 5.10: FRM Exam 2007—Question 46

a. The log return is $\ln(18/20) = -10.54\%$. The new variance forecasts is $h = 0.90 \times (1.5^2) + (1 - 0.90) \times 10.54^2 = 0.001313$, or taking the square root, 3.62%.

Example 5.11: FRM Exam 2006—Question 40

b. The weight of the last day is $(1 - 0.95) = 0.050$. For the day before, this is 0.05×0.95, and for four days ago, $0.05 \times 0.95^3 = 0.04287$.

Example 5.12: Effect of Weights on Observations

b. The EWMA model puts a weight of 0.06 on the latest observation, which is higher than the weight of $(1/60) = 0.0167$ for the 60-day MA and $(1/250) = 0.004$ for the 250-day MA.

Example 5.13: FRM Exam 2008—Question 1-8

a. The GARCH model has a finite unconditional variance, so statement c. is correct. In contrast, because $\alpha_1 + \beta$ sum to 1, the EWMA model has undefined long-run average variance. In both models weights decline exponentially with time.

Example 5.14: FRM Exam 2009—Question 2-16

b. The optimal parameter must maximize (not minimize) the likelihood function. Otherwise, the log-likelihood function is the log of the product, which is the sum of the logs. This gives, up to a constant, $\sum_{i=1}^{m}[-\ln(v_i) - u_i^2/(2v_i)]$, and there is no way to take the first term outside the summation because it depends on i. So, answers c. and d. are incorrect.

PART Three

Financial Markets and Products

CHAPTER 6

Bond Fundamentals

Risk management starts with the pricing of assets. The simplest assets to study are regular, fixed-coupon bonds. Because their cash flows are predetermined, we can translate their stream of cash flows into a present value by discounting at a fixed interest rate. Thus the valuation of bonds involves understanding compounded interest and discounting, as well as the relationship between present values and interest rates.

Risk management goes one step further than pricing, however. It examines potential changes in the price of assets as the interest rate changes. In this chapter, we assume that there is a single interest rate, or yield, that is used to price the bond. This will be our fundamental risk factor. This chapter describes the relationship between bond prices and yields and presents indispensable tools for the management of fixed-income portfolios.

This chapter starts our coverage of financial markets by discussing bond fundamentals. Section 6.1 reviews the concepts of discounting, present values, and future values. These concepts are fundamental to understand the valuation of financial assets. Section 6.2 then plunges into the price-yield relationship. It shows how the Taylor expansion rule can be used to relate movements in bond prices to those in yields. This Taylor expansion rule, however, covers much more than bonds. It is a building block of risk measurement methods based on local valuation, as we shall see later. Section 6.3 applies this expansion rule to the computation of partial derivatives for bonds. Section 6.4 then presents an economic interpretation of duration and convexity. Fixed-income managers routinely use these measures to help navigate their portfolios.

6.1 DISCOUNTING, PRESENT VALUE, AND FUTURE VALUE

An investor buys a zero-coupon bond that pays \$100 in 10 years. Because the investment is guaranteed by the U.S. government, we assume that there is no credit risk. So, this is a default-free bond, which is exposed to market risk only. Market risk arises because of possible fluctuations in the market price of this bond.

FRM Exam Part 1 topic. This chapter also covers basic risk models for bonds.

To value this \$100 payment, we need a **discounting factor**. This is also the **interest rate**, or more simply the **yield**. Define C_t as the cash flow at time t and the discounting factor as y. We define T as the number of periods until maturity (e.g., number of years), also known as **tenor**. The **present value** (PV) of the bond can be computed as

$$\text{PV} = \frac{C_T}{(1+y)^T} \tag{6.1}$$

For instance, a payment of $C_T = \$100$ in 10 years discounted at 6% is worth only \$55.84 now. So, all else fixed, the market value of zero-coupon bonds decreases with longer maturities. Also, keeping T fixed, the value of the bond decreases as the yield increases.

Conversely, we can compute the **future value** (FV) of the bond as

$$\text{FV} = \text{PV} \times (1+y)^T \tag{6.2}$$

For instance, an investment now worth PV = \$100 growing at 6% will have a future value of FV = \$179.08 in 10 years.

Here, the yield has a useful interpretation, which is that of an **internal rate of return** on the bond, or annual growth rate. It is easier to deal with rates of return than with dollar values. Rates of return, when expressed in percentage terms and on an annual basis, are directly comparable across assets. An annualized yield is sometimes defined as the **effective annual rate (EAR)**.

It is important to note that the interest rate should be stated along with the method used for compounding. Annual compounding is very common. Other conventions exist, however. For instance, the U.S. Treasury market uses semi-annual compounding. Define in this case y^S as the rate based on semiannual compounding. To maintain comparability, it is expressed in annualized form (i.e., after multiplication by 2). The number of periods, or semesters, is now $2T$. The formula for finding y^S is

$$\text{PV} = \frac{C_T}{(1+y^S/2)^{2T}} \tag{6.3}$$

For instance, a Treasury zero-coupon bond with a maturity of $T = 10$ years would have $2T = 20$ semiannual compounding periods. Comparing with Equation (6.1), we see that

$$(1+y) = (1+y^S/2)^2 \tag{6.4}$$

Continuous compounding is often used when modeling derivatives. It is the limit of the case where the number of compounding periods per year increases to infinity. The continuously compounded interest rate y^C is derived from

$$PV = C_T \times e^{-y^C T} \tag{6.5}$$

where $e^{(\cdot)}$, sometimes noted as $\exp(\cdot)$, represents the exponential function.

Note that in all of these Equations (6.1), (6.3), and (6.5), the present value and future cash flows are identical. Because of different compounding periods, however, the yields will differ. Hence, the compounding period should always be stated.

Example: Using Different Discounting Methods

Consider a bond that pays \$100 in 10 years and has a present value of \$55.8395. This corresponds to an annually compounded rate of 6% using $PV = C_T/(1+y)^{10}$, or $(1+y) = (C_T/PV)^{1/10}$.

This rate can be transformed into a semiannual compounded rate, using $(1 + y^S/2)^2 = (1 + y)$, or $y^S/2 = (1 + y)^{1/2} - 1$, or $y^S = [(1 + 0.06)^{(1/2)} - 1] \times 2 = 0.0591 = 5.91\%$. It can be also transformed into a continuously compounded rate, using $\exp(y^C) = (1 + y)$, or $y^C = \ln(1 + 0.06) = 0.0583 = 5.83\%$.

Note that as we increase the frequency of the compounding, the resulting rate decreases. Intuitively, because our money works harder with more frequent compounding, a lower investment rate will achieve the same payoff at the end.

KEY CONCEPT

For fixed present value and cash flows, increasing the frequency of the compounding will decrease the associated yield.

EXAMPLE 6.1: FRM EXAM 2002—QUESTION 48

An investor buys a Treasury bill maturing in one month for \$987. On the maturity date the investor collects \$1,000. Calculate effective annual rate (EAR).

a. 17.0%
b. 15.8%
c. 13.0%
d. 11.6%

EXAMPLE 6.2: FRM EXAM 2009—QUESTION 4-9

Lisa Smith, the treasurer of Bank AAA, has $100 million to invest for one year. She has identified three alternative one-year certificates of deposit (CDs), with different compounding periods and annual rates. CD1: monthly, 7.82%; CD2: quarterly, 8.00%; CD3: semiannually, 8.05%; and CD4: continuous, 7.95%. Which CD has the highest effective annual rate (EAR)?

a. CD1
b. CD2
c. CD3
d. CD4

EXAMPLE 6.3: FRM EXAM 2002—QUESTION 51

Consider a savings account that pays an annual interest rate of 8%. Calculate the amount of time it would take to double your money. Round to the nearest year.

a. 7 years
b. 8 years
c. 9 years
d. 10 years

6.2 PRICE-YIELD RELATIONSHIP

6.2.1 Valuation

The fundamental discounting relationship from Equation (6.1) can be extended to any bond with a fixed cash-flow pattern. We can write the present value of a bond P as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{6.6}$$

where: C_t = the cash flow (coupon or principal) in period t
t = the number of periods (e.g., half-years) to each payment
T = the number of periods to final maturity
y = the discounting factor per period (e.g., $y^S/2$)

A typical cash-flow pattern consists of a fixed coupon payment plus the repayment of the principal, or **face value** at expiration. Define c as the coupon *rate* and F as the face value. We have $C_t = cF$ prior to expiration, and at expiration, we have $C_T = cF + F$. The appendix reviews useful formulas that provide closed-form solutions for such bonds.

When the coupon rate c precisely matches the yield y, using the same compounding frequency, the present value of the bond must be equal to the face value. The bond is said to be a **par bond**. If the coupon is greater than the yield, the price must be greater than the face value, which means that this is a **premium bond**. Conversely, if the coupon is lower, or even zero for a zero-coupon bond, the price must be less than the face value, which means that this is a **discount bond**.

Equation (6.6) describes the relationship between the yield y and the value of the bond P, given its cash-flow characteristics. In other words, the value P can also be written as a nonlinear function of the yield y:

$$P = f(y) \tag{6.7}$$

Conversely, we can set P to the current market price of the bond, including any accrued interest. From this, we can compute the implied yield that will solve this equation.

Figure 6.1 describes the price-yield function for a 10-year bond with a 6% annual coupon. In risk management terms, this is also the relationship between the payoff on the asset and the risk factor. At a yield of 6%, the price is at par, $P = \$100$. Higher yields imply lower prices. This is an example of a **payoff function**, which links the price to the underlying risk factor.

Over a wide range of yield values, this is a highly nonlinear relationship. For instance, when the yield is zero, the value of the bond is simply the sum of cash flows, or \$160 in this case. When the yield tends to very large values, the bond price tends to zero. For small movements around the initial yield of 6%, however, the relationship is quasilinear.

There is a particularly simple relationship for **consols**, or **perpetual bonds**, which are bonds making regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed

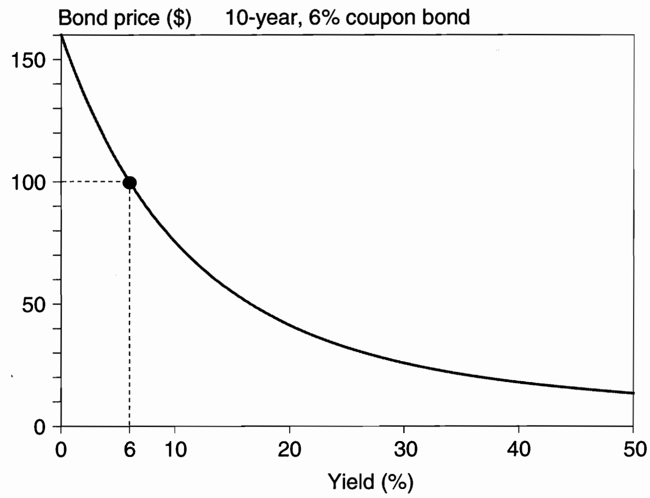

FIGURE 6.1 Price-Yield Relationship

percentage of the face value, $C_t = C = cF$. As a result, the price can be simplified from Equation (6.6) to

$$P = cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] = \frac{c}{y}F \tag{6.8}$$

as shown in the appendix. In this case, the price is simply proportional to the inverse of the yield. Higher yields lead to lower bond prices, and vice versa.

Example: Valuing a Bond

Consider a bond that pays \$100 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 6%? If it falls to 5%?

The bond cash flows are $C_1 = \$6, C_2 = \$6, \ldots, C_{10} = \$106$. Using Equation (6.6) and discounting at 6%, this gives the present value of cash flows of \$5.66, \$5.34, ..., \$59.19, for a total of \$100.00. The bond is selling at par. This is logical because the coupon is equal to the yield, which is also annually compounded. Alternatively, discounting at 5% leads to a price of \$107.72.

6.2.2 Taylor Expansion

Let us say that we want to see what happens to the price if the yield changes from its initial value, called y_0, to a new value, $y_1 = y_0 + \Delta y$. Risk management is all about assessing the effect of changes in risk factors such as yields on asset values. Are there shortcuts to help us with this?

We could recompute the new value of the bond as $P_1 = f(y_1)$. If the change is not too large, however, we can apply a very useful shortcut. The nonlinear relationship can be approximated by a **Taylor expansion** around its initial value[1]

$$P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2}f''(y_0)(\Delta y)^2 + \cdots \tag{6.9}$$

where $f'(\cdot) = \frac{dP}{dy}$ is the first derivative and $f''(\cdot) = \frac{d^2P}{dy^2}$ is the second derivative of the function $f(\cdot)$ valued at the starting point.[2] This expansion can be generalized to situations where the function depends on two or more variables. For bonds, the first derivative is related to the *duration* measure, and the second to *convexity*.

Equation (6.9) represents an infinite expansion with increasing powers of Δy. Only the first two terms (linear and quadratic) are ever used by finance

[1] This is named after the English mathematician Brook Taylor (1685–1731), who published this result in 1715. The full recognition of the importance of this result came only in 1755 when Euler applied it to differential calculus.

[2] This first assumes that the function can be written in polynomial form as $P(y + \Delta y) = a_0 + a_1\Delta y + a_2(\Delta y)^2 + \cdots$, with unknown coefficients a_0, a_1, a_2. To solve for the first, we set $\Delta y = 0$. This gives $a_0 = P_0$. Next, we take the derivative of both sides and set $\Delta y = 0$. This gives $a_1 = f'(y_0)$. The next step gives $2a_2 = f''(y_0)$. Here, the term *derivatives* takes the usual mathematical interpretation, and has nothing to do with *derivatives products* such as options.

practitioners. They provide a good approximation to changes in prices relative to other assumptions we have to make about pricing assets. If the increment is very small, even the quadratic term will be negligible.

Equation (6.9) is fundamental for risk management. It is used, sometimes in different guises, across a variety of financial markets. We will see later that this Taylor expansion is also used to approximate the movement in the value of a derivatives contract, such as an option on a stock. In this case, Equation (6.9) is

$$\Delta P = f'(S)\Delta S + \frac{1}{2} f''(S)(\Delta S)^2 + \cdots \tag{6.10}$$

where S is now the price of the underlying asset, such as the stock. Here, the first derivative $f'(S)$ is called *delta*, and the second $f''(S)$, *gamma*.

The Taylor expansion allows easy aggregation across financial instruments. If we have x_i units (numbers) of bond i and a total of N different bonds in the portfolio, the portfolio derivatives are given by

$$f'(y) = \sum_{i=1}^{N} x_i f_i'(y) \tag{6.11}$$

EXAMPLE 6.4: FRM EXAM 2009—QUESTION 4-8

A five-year corporate bond paying an annual coupon of 8% is sold at a price reflecting a yield to maturity of 6%. One year passes and the interest rates remain unchanged. Assuming a flat term structure and holding all other factors constant, the bond's price during this period will have

a. Increased
b. Decreased
c. Remained constant
d. Cannot be determined with the data given

6.3 BOND PRICE DERIVATIVES

For fixed-income instruments, the derivatives are so important that they have been given a special name.[3] The negative of the first derivative is the **dollar duration (DD)**:

$$f'(y_0) = \frac{dP}{dy} = -D^* \times P_0 \tag{6.12}$$

[3] Note that this chapter does not present duration in the traditional textbook order. In line with the advanced focus on risk management, we first analyze the properties of duration as a sensitivity measure. This applies to any type of fixed-income instrument. Later, we will illustrate the usual definition of duration as a weighted average maturity, which applies for fixed-coupon bonds only.

where D^* is called the **modified duration.** Thus, dollar duration is

$$\mathrm{DD} = D^* \times P_0 \tag{6.13}$$

where the price P_0 represent the *market* price, including any accrued interest. Sometimes, risk is measured as the **dollar value of a basis point (DVBP),**

$$\mathrm{DVBP} = \mathrm{DD} \times \Delta y = [D^* \times P_0] \times 0.0001 \tag{6.14}$$

with 0.0001 representing an interest rate change of one basis point (bp) or one-hundredth of a percent. The **DVBP,** sometimes called the **DV01,** measures can be easily added up across the portfolio.

KEY CONCEPT

The dollar value of a basis point is the dollar exposure of a bond price for a change in yield of 0.01%. It is also the duration times the value of the bond and is additive across the entire portfolio.

The second derivative is the **dollar convexity** (DC):

$$f''(y_0) = \frac{d^2 P}{dy^2} = C \times P_0 \tag{6.15}$$

where C is called the **convexity.**

For fixed-income instruments with known cash flows, the price-yield function is known, and we can compute analytical first and second derivatives. Consider, for example, our simple zero-coupon bond in Equation (6.1) where the only payment is the face value, $C_T = F$. We take the first derivative, which is

$$\frac{dP}{dy} = \frac{d}{dy}\left[\frac{F}{(1+y)^T}\right] = (-T)\frac{F}{(1+y)^{T+1}} = -\frac{T}{(1+y)}P \tag{6.16}$$

Comparing with Equation (6.12), we see that the modified duration must be given by $D^* = T/(1+y)$. The conventional measure of **duration** is $D = T$, which does not include division by $(1+y)$ in the denominator. This is also called **Macaulay duration.** Note that duration is expressed in periods, like T. With annual compounding, duration is in years. With semiannual compounding, duration is in semesters. It then has to be divided by 2 for conversion to years. Modified duration D^* is related to Macaulay duration D:

$$D^* = \frac{D}{(1+y)} \tag{6.17}$$

Modified duration is the appropriate measure of interest rate exposure. The quantity $(1 + y)$ appears in the denominator because we took the derivative of the present value term with discrete compounding. If we use continuous compounding, modified duration is identical to the conventional duration measure. In practice, the difference between Macaulay and modified duration is usually small.

Let us now go back to Equation (6.16) and consider the second derivative, or

$$\frac{d^2P}{dy^2} = -(T+1)(-T)\frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \times P \tag{6.18}$$

Comparing with Equation (6.15), we see that the convexity is $C = (T+1)T/(1+y)^2$. Note that its dimension is expressed in period squared. With semiannual compounding, convexity is measured in semesters squared. It then has to be divided by 4 for conversion to years squared.[4] So, convexity must be positive for bonds with fixed coupons.

Putting together all these equations, we get the Taylor expansion for the change in the price of a bond, which is

$$\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2 + \cdots \tag{6.19}$$

Therefore duration measures the first-order (linear) effect of changes in yield, and convexity measures the second-order (quadratic) term.

Example: Computing the Price Approximation[5]

Consider a 10-year zero-coupon Treasury bond trading at a yield of 6%. The present value is obtained as $P = 100/(1 + 6/200)^{20} = 55.368$. As is the practice in the Treasury market, yields are semiannually compounded. Thus all computations should be carried out using semesters, after which final results can be converted into annual units.

Here, Macaulay duration is exactly 10 years, as $D = T$ for a zero-coupon bond. Its modified duration is $D^* = 20/(1 + 6/200) = 19.42$ semesters, which is 9.71 years. Its convexity is $C = 21 \times 20/(1 + 6/200)^2 = 395.89$ semesters squared, which is 98.97 in years squared. Dollar duration is $\text{DD} = D^* \times P = 9.71 \times \$55.37 = \$537.55$. The DVBP is $\text{DVBP} = \text{DD} \times 0.0001 = \$0.0538$.

[4] This is because the conversion to annual terms is obtained by multiplying the semiannual yield Δy by 2. As a result, the duration term must be divided by 2 and the convexity term by 2^2, or 4, for conversion to annual units.

[5] For such examples in this handbook, please note that intermediate numbers are reported with fewer significant digits than actually used in the computations. As a result, using rounded-off numbers may give results that differ slightly from the final numbers shown here.

We want to approximate the change in the value of the bond if the yield goes to 7%. Using Equation (6.19), we have $\Delta P = -[9.71 \times \$55.37](0.01) + 0.5[98.97 \times \$55.37](0.01)^2 = -\$5.375 + \$0.274 = -\$5.101$. Using the linear term only, the new price is $\$55.368 - \$5.375 = \$49.992$. Using the two terms in the expansion, the predicted price is slightly higher, at $\$55.368 - \$5.375 + \$0.274 = \$50.266$.

These numbers can be compared with the exact value, which is \$50.257. The linear approximation has a relative pricing error of −0.53%, which is not bad. Adding a quadratic term reduces this to an error of only 0.02%, which is very small, given typical bid-ask spreads.

More generally, Figure 6.2 compares the quality of the Taylor series approximation. We consider a 10-year bond paying a 6% coupon semiannually. Initially, the yield is also at 6% and, as a result, the price of the bond is at par, at \$100. The graph compares three lines representing

1. The actual, exact price	$P = f(y_0 + \Delta y)$
2. The duration estimate	$P = P_0 - D^* P_0 \Delta y$
3. The duration and convexity estimate	$P = P_0 - D^* P_0 \Delta y + (1/2) C P_0 (\Delta y)^2$

The actual price curve shows an increase in the bond price if the yield falls and, conversely, a depreciation if the yield increases. This effect is captured by the tangent to the true price curve, which represents the linear approximation based on duration. For small movements in the yield, this linear approximation provides a reasonable fit to the exact price. For large movements in price, however, the price-yield function becomes more curved and the linear fit deteriorates. Under these conditions, the quadratic approximation is noticeably better.

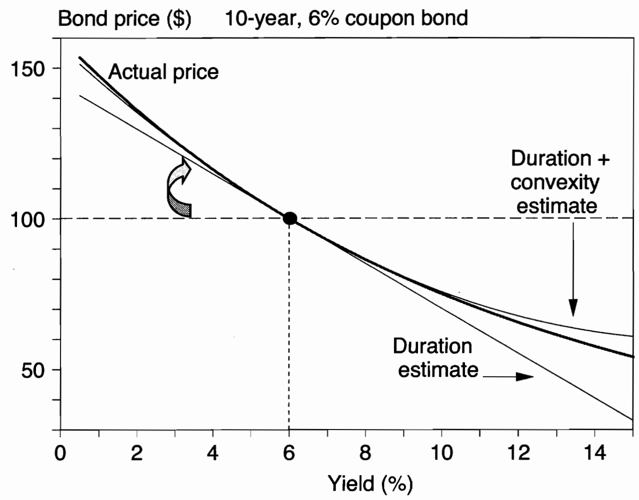

FIGURE 6.2 Price Approximation

KEY CONCEPT

Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point.

We should also note that the curvature is away from the origin, which explains the term *convexity* (as opposed to concavity). This curvature is beneficial since the second-order effect $0.5[C \times P](\Delta y)^2$ *must* be positive when convexity is positive. Some types of bonds, which involve the granting of an option to the investor, have negative convexity instead. In this case, the quadratic approximation is below the straight line instead of above as with positive convexity in Figure 6.2.

Figure 6.3 compares curves with different values for convexity. As the figure shows, when the yield rises, the price drops but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than along the tangent. In other words, the quadratic term is always beneficial.

KEY CONCEPT

Convexity is always positive for regular coupon-paying bonds. Greater convexity is beneficial for both falling and rising yields.

The bond's modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity cannot be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are

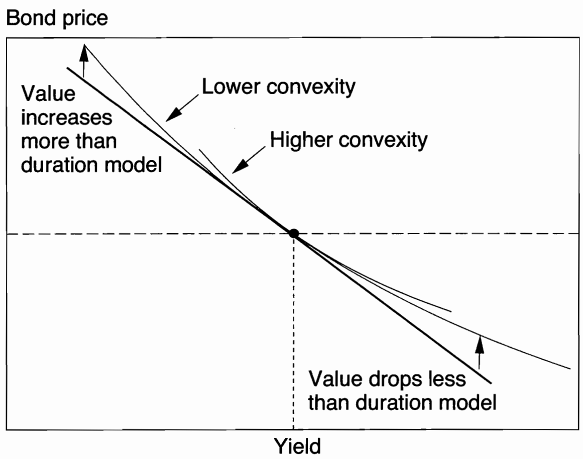

FIGURE 6.3 Effect of Convexity

uncertain. Instead, the portfolio manager has access to pricing models that can be used to reprice the securities under various yield environments.

As shown in Figure 6.4, we choose a change in the yield, Δy, and reprice the bond under an up move scenario, $P_+ = P(y_0 + \Delta y)$, and down move scenario, $P_- = P(y_0 - \Delta y)$. **Effective duration** is measured by the numerical derivative. Using $D^* = -(1/P)dP/dy$, it is estimated as

$$D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2\Delta y)P_0} \tag{6.20}$$

Using $C = (1/P)d^2P/dy^2$, **effective convexity** is estimated as

$$C^E = [D_- - D_+]/\Delta y = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0\Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0\Delta y)}\right]/\Delta y \tag{6.21}$$

To illustrate, consider a 30-year zero-coupon bond with a yield of 6%, semi-annually compounded. The initial price is \$16.9733. We revalue the bond at 5% and 7%, with prices shown in Table 6.1. The effective duration in Equation (6.20) uses the two extreme points. The effective convexity in Equation (6.21) uses the difference between the dollar durations for the up move and down move. Note that convexity is positive if duration increases as yields fall, or if $D_- > D_+$.

The computations are detailed in Table 6.1, which shows an effective duration of 29.56. This is very close to the true value of 29.13, and would be even closer if the step Δy was smaller. Similarly, the effective convexity is 869.11, which is close to the true value of 862.48.

Finally, this numerical approach can be applied to get an estimate of the duration of a bond by considering bonds with the same maturity but different coupons. If interest rates decrease by 1%, the market price of a 6% bond should go up to a value close to that of a 7% bond. Thus we replace a drop in yield of

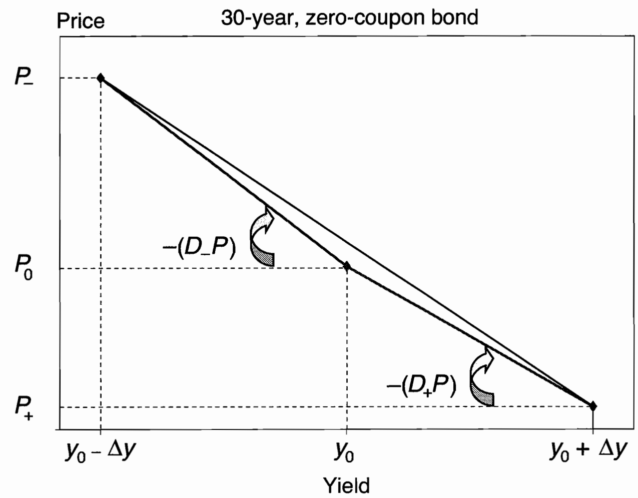

FIGURE 6.4 Effective Duration and Convexity

TABLE 6.1 Effective Duration and Convexity

State	Yield (%)	Bond Value	Duration Computation	Convexity Computation
Initial y_0	6.00	16.9733		
Up $y_0 + \Delta y$	7.00	12.6934		Duration up: 25.22
Down $y_0 - \Delta y$	5.00	22.7284		Duration down: 33.91
Difference in values			−10.0349	8.69
Difference in yields			0.02	0.01
Effective measure			29.56	869.11
Exact measure			29.13	862.48

Δy with an increase in coupon Δc and use the effective duration method to find the **coupon curve duration:**[6]

$$D^{CC} = \frac{[P_+ - P_-]}{(2P_0\Delta c)} = \frac{P(y_0; c + \Delta c) - P(y_0; c - \Delta c)}{(2\Delta c)P_0} \tag{6.22}$$

This approach is useful for securities that are difficult to price under various yield scenarios. It only requires the market prices of securities with different coupons.

Example: Computation of Coupon Curve Duration

Consider a 10-year bond that pays a 7% coupon semiannually. In a 7% yield environment, the bond is selling at par and has modified duration of 7.11 years. The prices of 6% and 8% coupon bonds are \$92.89 and \$107.11, respectively. This gives a coupon curve duration of $(107.11 - 92.89)/(0.02 \times 100) = 7.11$, which in this case is the same as modified duration.

EXAMPLE 6.5: FRM EXAM 2006—QUESTION 75

A zero-coupon bond with a maturity of 10 years has an annual effective yield of 10%. What is the closest value for its modified duration?

a. 9 years
b. 10 years
c. 99 years
d. 100 years

[6] For a more formal proof, we could take the pricing formula for a consol at par and compute the derivatives with respect to y and c. Apart from the sign, these derivatives are identical when $y = c$.

EXAMPLE 6.6: FRM EXAM 2007—QUESTION 115

A portfolio manager has a bond position worth USD 100 million. The position has a modified duration of eight years and a convexity of 150 years. Assume that the term structure is flat. By how much does the value of the position change if interest rates increase by 25 basis points?

a. USD −2,046,875
b. USD −2,187,500
c. USD −1,953,125
d. USD −1,906,250

EXAMPLE 6.7: FRM EXAM 2009—QUESTION 4-15

A portfolio manager uses her valuation model to estimate the value of a bond portfolio at USD 125.482 million. The term structure is flat. Using the same model, she estimates that the value of the portfolio would increase to USD 127.723 million if all interest rates fell by 30bp and would decrease to USD 122.164 million if all interest rates rose by 30bp. Using these estimates, the effective duration of the bond portfolio is closest to:

a. 8.38
b. 16.76
c. 7.38
d. 14.77

6.4 DURATION AND CONVEXITY

6.4.1 Economic Interpretation

The preceding section has shown how to compute analytical formulas for duration and convexity in the case of a simple zero-coupon bond. We can use the same approach for coupon-paying bonds. Going back to Equation (6.6), we have

$$\frac{dP}{dy} = \sum_{t=1}^{T} \frac{-tC_t}{(1+y)^{t+1}} = -\left[\sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}\right] / P \times \frac{P}{(1+y)} = -\frac{D}{(1+y)} P \qquad (6.23)$$

which defines duration as

$$D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} / P \tag{6.24}$$

The economic interpretation of duration is that it represents the average time to wait for each payment, weighted by the present value of the associated cash flow. Indeed, replacing P, we can write

$$D = \sum_{t=1}^{T} t \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} t \times w_t \tag{6.25}$$

where the weights w_t represent the ratio of the present value of each cash flow C_t relative to the total, and sum to unity. This explains why the duration of a zero-coupon bond is equal to the maturity. There is only one cash flow and its weight is one.

KEY CONCEPT

(Macaulay) duration represents an average of the time to wait for all cash flows.

Figure 6.5 lays out the present value of the cash flows of a 6% coupon, 10-year bond. Given a duration of 7.80 years, this coupon-paying bond is equivalent to a zero-coupon bond maturing in exactly 7.80 years.

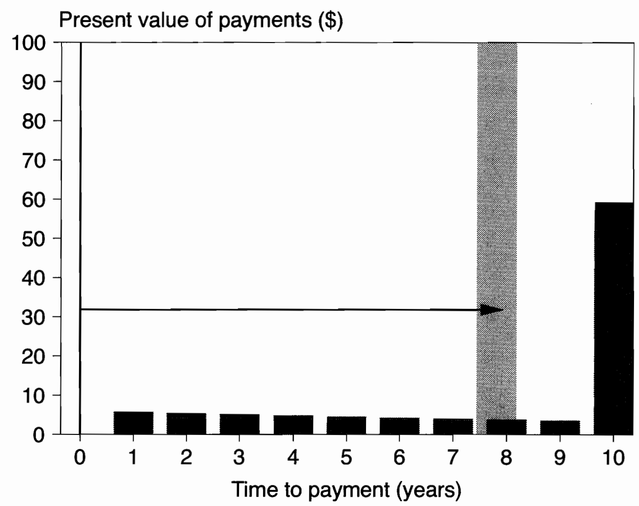

FIGURE 6.5 Duration as the Maturity of a Zero-Coupon Bond

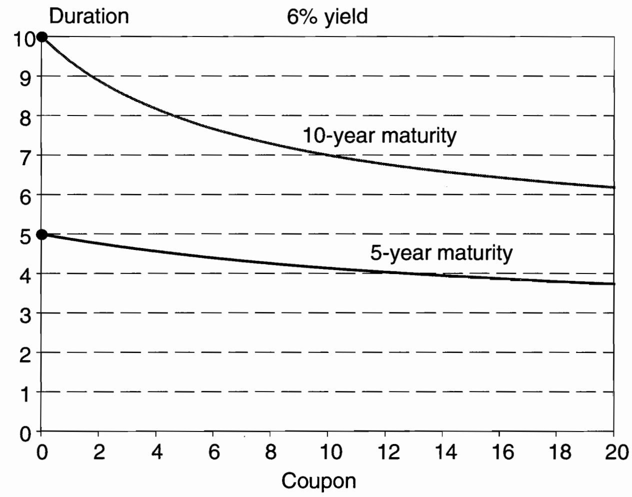

FIGURE 6.6 Duration and Coupon

For bonds with fixed coupons, duration is less than maturity. For instance, Figure 6.6 shows how the duration of a 10-year bond varies with its coupon, in a fixed 6% yield environment. With a zero coupon, Macaulay duration is equal to maturity. So, the curve starts at $D = 10$ years. Higher coupons place more weight on prior payments and therefore reduce duration.

Next, duration also varies with the current level of yield. Figure 6.7 shows that the duration of a 10-year bond decreases as the yield increases. This is because higher yields reduce the weight of distant cash flows in the current bond price, thus reducing its duration. The relationship is stronger for bonds with longer maturities. In addition, yield has even more effect on modified duration because it also appears in the denominator.

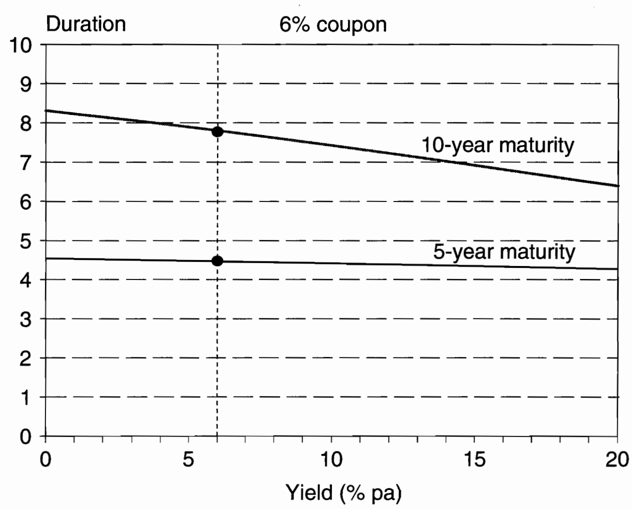

FIGURE 6.7 Duration-Yield Relationship

Duration can be expressed in a simple form for **consols**. From Equation (6.8), we have $P = (c/y)F$. Taking the derivative, we find

$$\frac{dP}{dy} = cF\frac{(-1)}{y^2} = (-1)\frac{1}{y}\left[\frac{c}{y}F\right] = (-1)\frac{1}{y}P = -\frac{D_C}{(1+y)}P \qquad (6.26)$$

Hence the Macaulay duration for the consol D_C is

$$D_C = \frac{(1+y)}{y} \qquad (6.27)$$

This shows that the duration of a consol is finite even if its maturity is infinite. Also, this duration does not depend on the coupon.

This formula provides a useful rule of thumb. For a long-term coupon-paying bond, duration should be lower than $(1+y)/y$ in most cases. For instance, when $y = 6\%$, the upper limit on duration is $D_C = 1.06/0.06$, or 17.7 years. In this environment, the duration of a par 30-year bond is 14.25, which is indeed lower than 17.7 years.

KEY CONCEPT

The duration of a long-term bond can be approximated by an upper bound, which is that of a consol with the same yield, $D_C = (1+y)/y$.

Figure 6.8 describes the relationship between duration, maturity, and coupon for regular bonds in a 6% yield environment. For the zero-coupon bond, $D = T$, which is a straight line going through the origin. For the par 6% bond, duration

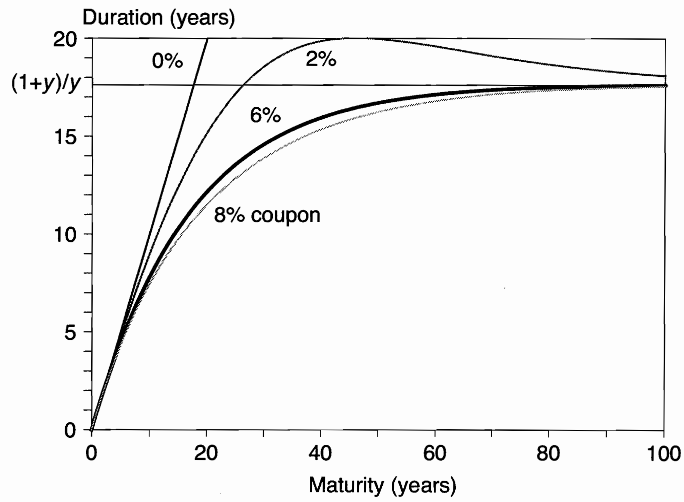

FIGURE 6.8 Duration and Maturity

increases monotonically with maturity until it reaches the asymptote of D_C. The 8% bond has lower duration than the 6% bond for fixed T. Greater coupons, for a fixed maturity, decrease duration, as more of the payments come early.

Finally, the 2% bond displays a pattern intermediate between the zero-coupon and 6% bonds. It initially behaves like the zero, exceeding D_C, and then falls back to the asymptote, which is the same for all coupon-paying bonds.

Taking now the second derivative in Equation (6.23), we have

$$\frac{d^2P}{dy^2} = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} = \left[\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P\right] \times P \tag{6.28}$$

which defines convexity as

$$C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P \tag{6.29}$$

Convexity can also be written as

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times w_t \tag{6.30}$$

Because the squared t term dominates in the fraction, this basically involves a weighted average of the square of time. Therefore, convexity is much greater for long-maturity bonds because they have payoffs associated with large values of t. The formula also shows that convexity is always positive for such bonds, implying that the curvature effect is beneficial. As we will see later, convexity can be negative for bonds that have uncertain cash flows, such as **mortgage-backed securities** (MBSs) or callable bonds.

Figure 6.9 displays the behavior of convexity, comparing a zero-coupon bond and a 6% coupon bond with identical maturities. The zero-coupon bond always has greater convexity, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity, for example about 900 for the 30-year zero. In contrast, the 30-year coupon bond has a convexity of about 300 only.

KEY CONCEPT

All else equal, duration and convexity both increase for longer maturities, lower coupons, and lower yields.

As an illustration, Table 6.2 details the steps of the computation of duration and convexity for a two-year, 6% semiannual coupon-paying bond. We first

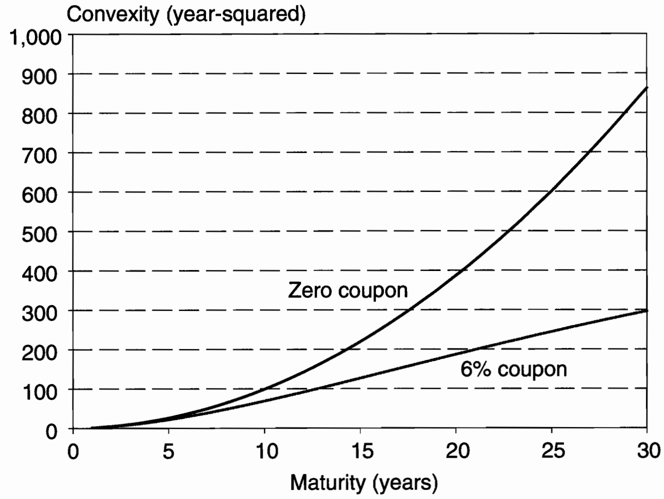

FIGURE 6.9 Convexity and Maturity

convert the annual coupon and yield into semiannual equivalent, \$3 and 3% each. The PV column then reports the present value of each cash flow. We verify that these add up to \$100, since the bond must be selling at par.

Next, the duration term column multiplies each PV term by time, or more precisely the number of half years until payment. This adds up to \$382.86, which divided by the price gives $D = 3.83$. This number is measured in half years, and we need to divide by 2 to convert to years. Macaulay duration is 1.91 years, and modified duration $D^* = 1.91/1.03 = 1.86$ years. Note that, to be consistent, the adjustment in the denominator involves the semiannual yield of 3%.

Finally, the rightmost column shows how to compute the bond's convexity. Each term involves PV_t times $t(t+1)/(1+y)^2$. These terms sum to 1,777.755, or divided by the price, 17.78. This number is expressed in units of time squared

TABLE 6.2 Computing Duration and Convexity

Period (Half Year) t	Payment C_t	Yield (%) (6 Mo.)	PV of Payment $C_t/(1+y)^t$	Duration Term tPV_t	Convexity Term $t(t+1)PV_t$ $\times[1/(1+y)^2]$
1	3	3.00	2.913	2.913	5.491
2	3	3.00	2.828	5.656	15.993
3	3	3.00	2.745	8.236	31.054
4	103	3.00	91.514	366.057	1,725.218
Sum:			100.00	382.861	1,777.755
(Half years)				3.83	17.78
(Years)				1.91	
Modified duration				1.86	
Convexity					4.44

and must be divided by 4 to be converted in annual terms. We find a convexity of $C = 4.44$, in year-squared.

EXAMPLE 6.8: FRM EXAM 2003—QUESTION 13

Suppose the face value of a three-year option-free bond is USD 1,000 and the annual coupon is 10%. The current yield to maturity is 5%. What is the modified duration of this bond?

a. 2.62
b. 2.85
c. 3.00
d. 2.75

EXAMPLE 6.9: FRM EXAM 2002—QUESTION 118

A Treasury bond has a coupon rate of 6% per annum (the coupons are paid semiannually) and a semiannually compounded yield of 4% per annum. The bond matures in 18 months and the next coupon will be paid 6 months from now. Which number of years is closest to the bond's Macaulay duration?

a. 1.023 years
b. 1.457 years
c. 1.500 years
d. 2.915 years

EXAMPLE 6.10: DURATION AND COUPON

A and B are two perpetual bonds; that is, their maturities are infinite. A has a coupon of 4% and B has a coupon of 8%. Assuming that both are trading at the same yield, what can be said about the duration of these bonds?

a. The duration of A is greater than the duration of B.
b. The duration of A is less than the duration of B.
c. A and B both have the same duration.
d. None of the above.

EXAMPLE 6.11: FRM EXAM 2004—QUESTION 16

A manager wants to swap a bond for a bond with the same price but higher duration. Which of the following bond characteristics would be associated with a higher duration?

I. A higher coupon rate
II. More frequent coupon payments
III. A longer term to maturity
IV. A lower yield

a. I, II, and III
b. II, III, and IV
c. III and IV
d. I and II

EXAMPLE 6.12: FRM EXAM 2001—QUESTION 104

When the maturity of a plain coupon bond increases, its duration increases

a. Indefinitely and regularly
b. Up to a certain level
c. Indefinitely and progressively
d. In a way dependent on the bond being priced above or below par

EXAMPLE 6.13: FRM EXAM 2000—QUESTION 106

Consider the following bonds:

Bond Number	Maturity (Years)	Coupon Rate	Frequency	Yield (Annual)
1	10	6%	1	6%
2	10	6%	2	6%
3	10	0%	1	6%
4	10	6%	1	5%
5	9	6%	1	6%

How would you rank the bonds from the shortest to longest duration?

a. 5-2-1-4-3
b. 6-2-3-4-5
c. 5-4-3-1-2
d. 2-4-5-1-3

EXAMPLE 6.14: FRM EXAM 2000—QUESTION 110

Which of the following statements is/are *true*?

I. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 10-year 6% bond.
II. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 6% bond with a duration of 10 years.
III. Convexity grows proportionately with the maturity of the bond.
IV. Convexity is always positive for all types of bonds.
V. Convexity is always positive for straight bonds.

a. I only
b. I and II only
c. I and V only
d. II, III, and V only

6.4.2 Portfolio Duration and Convexity

Fixed-income portfolios often involve very large numbers of securities. It would be impractical to consider the movements of each security individually. Instead, portfolio managers aggregate the duration and convexity across the portfolio. A manager who believes that rates will increase should shorten the portfolio duration relative to that of the benchmark. Say, for instance, that the benchmark has a duration of five years. The manager shortens the portfolio duration to one year only. If rates increase by 2%, the benchmark will lose approximately $5y \times 2\% = 10\%$. The portfolio, however, will lose only $1y \times 2\% = 2\%$, hence beating the benchmark by 8%.

Because the Taylor expansion involves a summation, the portfolio duration is easily obtained from the individual components. Say we have N components indexed by i. Defining D_p^* and P_p as the portfolio modified duration and value, the portfolio dollar duration (DD) is

$$D_p^* P_p = \sum_{i=1}^{N} D_i^* x_i P_i \tag{6.31}$$

where x_i is the number of units of bond i in the portfolio. A similar relationship holds for the portfolio dollar convexity (DC). If yields are the same for all components, this equation also holds for the Macaulay duration.

Because the portfolio's total market value is simply the summation of the component market values,

$$P_p = \sum_{i=1}^{N} x_i P_i \tag{6.32}$$

we can define the **portfolio weight** w_i as $w_i = x_i P_i / P_p$, provided that the portfolio market value is nonzero. We can then write the portfolio duration as a weighted average of individual durations:

$$D_p^* = \sum_{i=1}^{N} D_i^* w_i \tag{6.33}$$

Similarly, the portfolio convexity is a weighted average of convexity numbers:

$$C_p = \sum_{i=1}^{N} C_i w_i \tag{6.34}$$

As an example, consider a portfolio invested in three bonds, described in Table 6.3. The portfolio is long a 10-year and 1-year bond, and short a 30-year zero-coupon bond. Its market value is \$1,301,600. Summing the duration for each component, the portfolio dollar duration is \$2,953,800, which translates into a duration of 2.27 years. The portfolio convexity is $-76{,}918{,}323/1{,}301{,}600 = -59.10$, which is negative due to the short position in the 30-year zero, which has very high convexity.

Alternatively, assume the portfolio manager is given a benchmark, which is the first bond. The manager wants to invest in bonds 2 and 3, keeping the portfolio duration equal to that of the target, or 7.44 years. To achieve the target value and dollar duration, the manager needs to solve a system of two equations in the numbers x_1 and x_2:

$$\text{Value:} \qquad \$100 = x_1\$94.26 + x_2\$16.97$$

$$\text{Dollar duration:} \quad 7.44 \times \$100 = 0.97 \times x_1\$94.26 + 29.13 \times x_2\$16.97$$

TABLE 6.3 Portfolio Dollar Duration and Convexity

	Bond 1	Bond 2	Bond 3	Portfolio
Maturity (years)	10	1	30	
Coupon	6%	0%	0%	
Yield	6%	6%	6%	
Price P_i	\$100.00	\$94.26	\$16.97	
Modified duration D_i^*	7.44	0.97	29.13	
Convexity C_i	68.78	1.41	862.48	
Number of bonds x_i	10,000	5,000	−10,000	
Dollar amounts $x_i P_i$	\$1,000,000	\$471,300	−\$169,700	\$1,301,600
Weight w_i	76.83%	36.21%	−13.04%	100.00%
Dollar duration $D_i^* P_i$	\$744.00	\$91.43	\$494.34	
Portfolio DD: $x_i D_i^* P_i$	\$7,440,000	\$457,161	−\$4,943,361	\$2,953,800
Portfolio DC: $x_i C_i P_i$	68,780,000	664,533	−146,362,856	−76,918,323

The solution is $x_1 = 0.817$ and $x_2 = 1.354$, which gives a portfolio value of \$100 and modified duration of 7.44 years.[7] The portfolio convexity is 199.25, higher than the index. Such a portfolio consisting of very short and very long maturities is called a **barbell portfolio**. In contrast, a portfolio with maturities in

EXAMPLE 6.15: FRM EXAM 2002—QUESTION 57

A bond portfolio has the following composition:

1. Portfolio A: price \$90,000, modified duration 2.5, long position in 8 bonds
2. Portfolio B: price \$110,000, modified duration 3, short position in 6 bonds
3. Portfolio C: price \$120,000, modified duration 3.3, long position in 12 bonds

All interest rates are 10%. If the rates rise by 25 basis points, then the bond portfolio value will decrease by

a. \$11,430
b. \$21,330
c. \$12,573
d. \$23,463

EXAMPLE 6.16: FRM EXAM 2006—QUESTION 61

Consider the following portfolio of bonds (par amounts are in millions of USD).

Bond	Price	Par Amount Held	Modified Duration
A	101.43	3	2.36
B	84.89	5	4.13
C	121.87	8	6.27

What is the value of the portfolio's DV01 (dollar value of 1 basis point)?

a. \$8,019
b. \$8,294
c. \$8,584
d. \$8,813

[7] This can be obtained by first expressing x_2 in the first equation as a function of x_1 and then substituting back into the second equation. This gives $x_2 = (100 - 94.26x_1)/16.97$, and $744 = 91.43x_1 + 494.34x_2 = 91.43x_1 + 494.34(100 - 94.26x_1)/16.97 = 91.43x_1 + 2{,}913.00 - 2{,}745.79x_1$. Solving, we find $x_1 = (-2{,}169.00)/(-2{,}654.36) = 0.817$ and $x_2 = (100 - 94.26 \times 0.817)/16.97 = 1.354$.

EXAMPLE 6.17: FRM EXAM 2008—QUESTION 2-33

Which of the following statements is *correct* regarding the effects of interest rate shift on fixed-income portfolios with similar durations?

a. A barbell portfolio has greater convexity than a bullet portfolio because convexity increases linearly with maturity.
b. A barbell portfolio has greater convexity than a bullet portfolio because convexity increases with the square of maturity.
c. A barbell portfolio has lower convexity than a bullet portfolio because convexity increases linearly with maturity.
d. A barbell portfolio has lower convexity than a bullet portfolio because convexity increases with the square of maturity.

the same range is called a **bullet portfolio.** Note that the barbell portfolio has a much greater convexity than the bullet portfolio because of the payment in 30 years. Such a portfolio would be expected to outperform the bullet portfolio if yields moved by a large amount.

In sum, duration and convexity are key measures of fixed-income portfolios. They summarize the linear and quadratic exposure to movements in yields. This explains why they are essential tools for fixed-income portfolio managers.

6.5 IMPORTANT FORMULAS

Compounding: $(1 + y)^T = (1 + y^S/2)^{2T} = e^{y^C T}$

Fixed-coupon bond valuation: $P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$

Taylor expansion: $P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2} f''(y_0)(\Delta y)^2 + \cdots$

Duration as exposure: $\frac{dP}{dy} = -D^* \times P$, $DD = D^* \times P$, $DVBP = DD \times 0.0001$

Modified and conventional duration: $D^* = \frac{D}{(1+y)}$, $D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}/P$

Convexity: $\frac{d^2P}{dy^2} = C \times P$, $C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}}/P$

Price change: $\Delta P = -[D^* \times P](\Delta y) + 0.5[C \times P](\Delta y)^2 + \cdots$

Consol: $P = \frac{c}{y}F$, $D = \frac{(1+y)}{y}$

Portfolio duration and convexity: $D_p^* = \sum_{i=1}^{N} D_i^* w_i$, $C_p = \sum_{i=1}^{N} C_i w_i$

6.6 ANSWERS TO CHAPTER EXAMPLES

Example 6.1: FRM Exam 2002—Question 48

a. The EAR is defined by $FV/PV = (1 + EAR)^T$. So $EAR = (FV/PV)^{1/T} - 1$. Here, $T = 1/12$. So, $EAR = (1{,}000/987)^{12} - 1 = 17.0\%$.

Example 6.2: FRM Exam 2009—Question 4-9

d. A dollar initially invested will grow to (CD1) $(1+7.82\%/12)^{12} = 1.08107$, (CD2) $(1+8.00\%/4)^4 = 1.08243$, (CD3) $(1+8.05\%/2)^2 = 1.08212$, (CD4) $\exp(7.95\%) = 1.08275$. Hence, CD4 gives the highest final amount and EAR.

Example 6.3: FRM Exam 2002—Question 51

c. The time T relates the current and future values such that $\mathrm{FV}/\mathrm{PV} = 2 = (1 + 8\%)^T$. Taking logs of both sides, this gives $T = \ln(2)/\ln(1.08) = 9.006$.

Example 6.4: FRM Exam 2009—Question 4-8

b. Because the coupon is greater than the yield, the bond must be selling at a premium, or current price greater than the face value. If yields do not change, the bond price will converge to the face value. Given that it starts higher, it must decrease.

Example 6.5: FRM Exam 2006—Question 75

a. Without doing any computation, the Macaulay duration must be 10 years because this is a zero-coupon bond. With annual compounding, modified duration is $D^* = 10/(1+10\%)$, or close to 9 years.

Example 6.6: FRM Exam 2007—Question 115

c. The change in price is given by $\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2$ $= -[8 \times 100](0.0025) + 0.5[150 \times 100](0.0025)^2 = -2.000000 + 0.046875 = -1.953125$.

Example 6.7: FRM Exam 2009—Question 4-15

c. By Equation (6.20), effective duration is $D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{[127.723-122.164]}{(125.482\times 0.6\%)} = 7.38$.

Example 6.8: FRM Exam 2003—Question 13

a. As in Table 6.2, we lay out the cash flows and find

Period t	Payment C_t	Yield y	$\mathrm{PV}_t = C_t/(1+y)^t$	$t\mathrm{PV}_t$
1	100	5.00	95.24	95.24
2	100	5.00	90.71	181.41
3	1,100	5.00	950.22	2,850.66
Sum:			1,136.16	3,127.31

Duration is then 2.75, and modified duration 2.62.

Example 6.9: FRM Exam 2002—Question 118

b. For coupon-paying bonds, Macaulay duration is slightly less than the maturity, which is 1.5 years here. So, b. would be a good guess. Otherwise, we can compute duration exactly.

Example 6.10: Duration and Coupon

c. Going back to the duration equation for the consol, Equation (6.27), we see that it does not depend on the coupon but only on the yield. Hence, the durations must be the same. The price of bond A, however, must be half that of bond B.

Example 6.11: FRM Exam 2004—Question 16

c. Higher duration is associated with physical characteristics that push payments into the future, that is, longer term, lower coupons, and less frequent coupon payments, as well as lower yields, which increase the relative weight of payments in the future.

Example 6.12: FRM Exam 2001—Question 104

b. With a fixed coupon, the duration goes up to the level of a consol with the same coupon. See Figure 6.8.

Example 6.13: FRM Exam 2000—Question 106

a. The nine-year bond (number 5) has shorter duration because the maturity is shortest, at nine years, among comparable bonds. Next, we have to decide between bonds 1 and 2, which differ only in the payment frequency. The semiannual bond (number 2) has a first payment in six months and has shorter duration than the annual bond. Next, we have to decide between bonds 1 and 4, which differ only in the yield. With lower yield, the cash flows further in the future have a higher weight, so that bond 4 has greater duration. Finally, the zero-coupon bond has the longest duration. So, the order is 5-2-1-4-3.

Example 6.14: FRM Exam 2000—Question 110

c. Because convexity is proportional to the square of time to payment, the convexity of a bond is mainly driven by the cash flows far into the future. Answer I. is correct because the 10-year zero has only one cash flow, whereas the coupon bond has several others that reduce convexity. Answer II. is false because the 6% bond with 10-year duration must have cash flows much further into the future, say in 30 years, which will create greater convexity. Answer III. is false because convexity grows with the square of time. Answer IV. is false because some bonds, for example MBSs or callable bonds, can have negative convexity. Answer V. is correct because convexity must be positive for coupon-paying bonds.

Example 6.15: FRM Exam 2002—Question 57

a. The portfolio dollar duration is $D^* P = \sum x_i D_i^* P_i = +8 \times 2.5 \times \$90{,}000 - 6 \times 3.0 \times \$110{,}000 + 12 \times 3.3 \times \$120{,}000 = \$4{,}572{,}000$. The change in portfolio value is then $-(D * P)(\Delta y) = -\$4{,}572{,}000 \times 0.0025 = -\$11{,}430$.

Example 6.16: FRM Exam 2006—Question 61

c. First, the market value of each bond is obtained by multiplying the par amount by the ratio of the market price divided by 100. Next, this is multiplied by D^* to get the dollar duration DD. Summing, this gives \$85.841 million. We multiply by 1,000,000 to get dollar amounts and by 0.0001 to get the DV01, which gives \$8,584.

Bond	Price	Par	Market Value	D^*	DD
A	101.43	3	3.043	2.36	7.181
B	84.89	5	4.245	4.13	15.530
C	121.87	8	9.750	6.27	61.130
Sum					85.841

Example 6.17: FRM Exam 2008—Question 2-33

b. The statement compares two portfolios with the same duration. A barbell portfolio consists of a combination of short-term and long-term bonds. A bullet portfolio has only medium-term bonds. Because convexity is a quadratic function of time to wait for the payments, the long-term bonds create a large contribution to the convexity of the barbell portfolio, which must be higher than that of the bullet portfolio.

APPENDIX: APPLICATIONS OF INFINITE SERIES

When bonds have fixed coupons, the bond valuation problem often can be interpreted in terms of combinations of infinite series. The most important infinite series result is for a sum of terms that increase at a geometric rate:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{6.a} \tag{6.35}$$

This can be proved, for instance, by multiplying both sides by $(1 - a)$ and canceling out terms.

Equally important, consider a geometric series with a finite number of terms, say N. We can write this as the difference between two infinite series:

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = (1 + a + a^2 + a^3 + \cdots) - a^N(1 + a + a^2 + a^3 + \cdots) \tag{6.36}$$

such that all terms with order N or higher will cancel each other.

We can then write

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = \frac{1}{6.a} - a^N \frac{1}{6.a} \tag{6.37}$$

These formulas are essential to value bonds. Consider first a consol with an infinite number of coupon payments with a fixed coupon rate c. If the yield is y and the face value F, the value of the bond is

$$\begin{aligned}
P &= cF\left[\tfrac{1}{(1+y)} + \tfrac{1}{(1+y)^2} + \tfrac{1}{(1+y)^3} + \cdots\right] \\
&= cF\tfrac{1}{(1+y)}[1 + a^2 + a^3 + \cdots] \\
&= cF\tfrac{1}{(1+y)}\left[\tfrac{1}{6.a}\right] \\
&= cF\tfrac{1}{(1+y)}\left[\tfrac{1}{(1-1/(1+y))}\right] \\
&= cF\tfrac{1}{(1+y)}\left[\tfrac{(1+y)}{y}\right] \\
&= \tfrac{c}{y}F
\end{aligned}$$

Similarly, we can value a bond with a *finite* number of coupons over T periods at which time the principal is repaid. This is really a portfolio with three parts:

(1) a long position in a consol with coupon rate c
(2) a short position in a consol with coupon rate c that starts in T periods
(3) a long position in a zero-coupon bond that pays F in T periods

Note that the combination of (1) and (2) ensures that we have a finite number of coupons. Hence, the bond price should be:

$$P = \frac{c}{y}F - \frac{1}{(1+y)^T}\frac{c}{y}F + \frac{1}{(1+y)^T}F = \frac{c}{y}F\left[1 - \frac{1}{(1+y)^T}\right] + \frac{1}{(1+y)^T}F \tag{6.38}$$

where again the formula can be adjusted for different compounding methods.

This is useful for a number of purposes. For instance, when $c = y$, it is immediately obvious that the price must be at par, $P = F$. This formula also can be used to find closed-form solutions for duration and convexity.

CHAPTER 7

Introduction to Derivatives

This chapter provides an overview of derivatives markets. Derivatives are financial contracts traded in private **over-the-counter** (OTC) markets or on **organized exchanges**. As the term implies, derivatives derive their value from some underlying index, typically the price of an asset. Depending on the type of relationship, they can be broadly classified into two categories: linear and nonlinear instruments.

To the first category belong forward rate agreements (FRAs), futures, and swaps. Their value is a linear function of the underlying index. These are *obligations* to exchange payments according to a specified schedule. Forward contracts involve one payment and are relatively simple to evaluate. So are futures, which are traded on exchanges. Swaps involve a series of payments and generally can be reduced to portfolios of forward contracts. To the second category belong options, whose value is a nonlinear function of the underlying index. These are much more complex to evaluate and will be covered in the next chapter.

This chapter describes the general characteristics as well as the pricing of linear derivatives. Pricing is the first step toward risk measurement. The second step consists of combining the valuation formula with the distribution of underlying risk factors to derive the distribution of contract values. This will be done later, in the market risk section (Part Five of this book).

Section 7.1 provides an overview of the size of the derivatives markets as well as trading mechanisms. Section 7.2 then presents the valuation and pricing of forwards. Sections 7.3 and 7.4 introduce futures and swap contracts, respectively.

7.1 DERIVATIVES MARKETS

7.1.1 Definitions

A **derivative instrument** can be generally defined as a private contract whose value derives from some underlying asset price, reference rate, or index—such as a stock, bond, currency, or commodity. In addition, the contract must also specify a principal, or **notional** amount, which is defined in terms of currency, shares,

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bushels, or some other unit. Movements in the value of the derivative depend on the notional and on the underlying price or index.

In contrast with **securities**, such as stocks and bonds, which are issued to raise capital, derivatives are **contracts**, or private agreements between two parties. Thus the sum of gains and losses on derivatives contracts must be zero. For any gain made by one party, the other party must have suffered a loss of equal magnitude.

7.1.2 Size of Derivatives Markets

At the broadest level, derivatives markets can be classified by the underlying instrument, as well as by the type of trading. Table 7.1 describes the size and growth of the global derivatives markets. As of 2009, the total notional amounts add up to almost $688 trillion, of which $615 trillion is on OTC markets and $73 trillion on organized exchanges. These markets have grown exponentially, from $56 trillion in 1995.

TABLE 7.1 Global Derivatives Markets, 1995–2009 (Billions of U.S. Dollars)

	Notional Amounts	
	March 1995	Dec. 2009
OTC Instruments	47,530	614,674
Interest rate contracts	**26,645**	**449,793**
Forwards (FRAs)	4,597	51,749
Swaps	18,283	349,236
Options	3,548	48,808
Foreign exchange contracts	**13,095**	**49,196**
Forwards and forex swaps	8,699	23,129
Swaps	1,957	16,509
Options	2,379	9,558
Equity-linked contracts	**579**	**6,591**
Forwards and swaps	52	1,830
Options	527	4,762
Commodity contracts	**318**	**2,944**
Credit default swaps	**0**	**32,693**
Others	**6,893**	**73,456**
Exchange-Traded Instruments	8,838	73,140
Interest rate contracts	**8,380**	**67,057**
Futures	5,757	20,628
Options	2,623	46,429
Foreign exchange contracts	**88**	**311**
Futures	33	164
Options	55	147
Stock index contracts	**370**	**5,772**
Futures	128	965
Options	242	4,807
Total	**55,910**	**687,814**

Source: Bank for International Settlements.

The table shows that interest rate contracts, especially swaps, are the most widespread type of derivatives. On the OTC market, currency contracts are also widely used, especially outright forwards and **forex swaps**, which are a combination of spot and short-term forward transactions. Among exchange-traded instruments, interest rate futures and options are the most common.

The magnitude of the notional amount of $688 trillion is difficult to grasp. This number is several times the world **gross domestic product (GDP)**, which amounted to approximately $61 trillion in 2008. It is also greater than the total outstanding value of stocks ($34 trillion) and of debt securities ($83 trillion) at that time.

Notional amounts give an indication of equivalent positions in cash markets. For example, a long futures contract on a stock index with a notional of $1 million is equivalent to a cash position in the stock market of the same magnitude. They are also important because they drive the fees to the financial industry, which are typically set as a fraction of the notionals.

Notional amounts, however, do not give much information about the risks of the positions. The current (positive) market value of OTC derivatives contracts, for instance, is estimated at $22 trillion. This is only 3% of the notional. More generally, the risk of these derivatives is best measured by the potential change in mark-to-market values over the horizon—in other words, by a value at risk measure.

7.1.3 Trading Mechanisms

Derivatives can be traded in private decentralized markets, called **over-the-counter** (OTC) markets, or on **organized exchanges**. Trading OTC is generally done with a **derivatives dealer**, which is a specially organized firm, usually associated with a major financial institution, that buys and sells derivatives.

A major issue when dealing with derivatives is **counterparty risk**. A counterparty is defined as the opposite side of a financial transaction. Suppose that Bank A entered a derivative contract with Hedge Fund B, where A agrees to purchase the euro by selling dollars at a fixed price of $1.3. If the euro goes up, the contract moves in-the-money for A. This means, however, that B suffers a loss. If this loss is large enough, B could default. Hence, this contract creates credit risk. Suffice to say, this topic will be developed at length in another section of this book.

Derivatives dealers manage their counterparty risk through a variety of means. Even so, these potential exposures are a major concern. This explains why American International Group (AIG) was rescued by the U.S. government. AIG had sold credit default swaps (CDSs) to many other institutions, and its failure could have caused domino effects for other banks.

Counterparty risk can be addressed with a **clearinghouse**. A clearinghouse is a financial institution that provides settlement and clearing services for financial transactions that may have taken place on an organized exchange or OTC.

The purpose of a clearinghouse is to reduce counterparty risk by interposing itself between the buyer and the seller, thereby ensuring payments on the contract. This is why clearinghouses are said to provide **central counterparty** (CCP) clearing. This is done through a process called **novation**, which refers to the replacement of a contract between two counterparties with a contract between the remaining party and a third party.[1] Clearinghouses, however, may also offer other services than CCP clearing.

Clearinghouses reduce counterparty risk by a variety of means. First, they allow netting, or offsetting, transactions between counterparties due to the fungibility of the contracts.[2] Suppose, for instance, that Bank A bought $100 million worth of euros from B. At the same time, it sold an otherwise identical contract to Bank C in the amount of $80 million. If the two trades had gone through the same clearinghouse, the net exposure of Bank A would be $20 million only.

Second, clearinghouses have in place procedures to manage their credit risk. They require **collateral deposits**, also called margins, which must be adjusted on a daily basis or more frequently. They mark contracts to market based on independent valuation of trades. Should the **clearing member** fail to add to the collateral when needed, the clearinghouse has the right to liquidate the position. In addition, clearinghouses generally rely on a guarantee fund provided by clearing members that can be used to cover losses in a default event.

Turning now to the trading aspects, an **organized exchange**, or **bourse**, is a highly organized market where financial instruments are bought and sold. Trading can occur either in a physical location or electronically. The most active stock market in the world is the **New York Stock Exchange** (NYSE). Another example is the **CME Group**, which is the largest derivatives market in the world. Trading at the CME is conducted in open outcry format, or electronically through Globex. In 2009, Globex accounted for 80% of total trading at the CME.

Most exchanges are **public**, trade public securities, and require registration with a local regulator, such as the Securities and Exchange Commission (SEC). **Private** exchanges, in contrast, are not registered and allow for the trading of unregistered securities, such as private placements.

All exchanges have a clearinghouse. A clearinghouse may be dedicated to one or several exchanges. As an example of the first case, which is also called **vertical integration**, CME Clearing is the in-house clearer for the CME Group. In contrast, the **Options Clearing Corporation** (OCC) supports 14 exchanges and platforms, including the Chicago Board Options Exchange and NYSE Amex. LCH.Clearnet Group handles many of the trades in major European financial exchanges. This model is called **horizontal integration**.

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[1] Novation requires the consent of all parties. This is in contrast to an assignment, which is valid as long as the other party is given notice.

[2] An important example, covered in a credit risk chapter in Part Six, is the netting of currency payments created by the CLS Bank.

OTC trading does not require a clearinghouse. This is changing slowly, however. About 25% of interest rate swaps are now routed through clearinghouses. Similarly, ICE Trust, the U.S.-based electronic futures exchange group, has started to clear CDS index contracts. Legislation will surely push more trading toward CCPs. OTC derivatives dealers, however, are generally opposed to CCPs because these would cut into their business and profits.

CCPs have other advantages. They create **transparency** in financial markets. Because the trading information is now stored centrally, CCPs can disseminate information about transactions, prices, and positions. This was a major issue with AIG because apparently regulators were not aware of the extent of AIG's short CDS positions. Thus CCPs could be used for trade **reporting**. Currently, most electronic trades of credit derivatives are reported to the **Depository Trust & Clearing Corporation** (DTCC), which is a financial institution that settles the vast majority of securities transactions in the United States.

Clearinghouses, however, have disadvantages as well. They do not eliminate counterparty risk in the financial system but rather concentrate it among themselves. Even if CCPs are more creditworthy than most other parties, a CCP failure could create systemic risk. CCPs generally have very high credit ratings, for example AAA for the OCC. CCP failures have been extremely rare, but could happen again.[3]

Regulators are contemplating whether CCPs should have access to the lending facilities of a central bank. Of course, this creates a problem of moral hazard, which is a situation where clearers would have less incentive to create a robust structure because they know that central banks would ride to their rescue anyway.

Second, the risk management aspect of positions is very important. CCPs have been able to handle simple, standardized contracts, which are easy to price. Forcing more complex instruments to clearinghouses increases the probability of operational problems. Even standardized instruments may not be very liquid, which could cause large losses in case of a forced liquidation.

Third, CCPs' netting benefits would be defeated by having too many existing CCPs. In our example of Bank A with clients B and C, there is no netting if the two trades are routed to two different CCPs. Further, having CCPs for different products would reduce the effectiveness of International Swaps and Derivatives Association (ISDA) agreements because these allow netting across many different products. Such agreements are called **bilateral** because they cover a variety of trades between only two parties.

The establishment of CCPs creates a global coordination problem. The United States, United Kingdom, and continental Europe all want their own CCP because their CCP might need to be backstopped by the local central bank. Regulation might also be needed to avoid CCPs competing against each other by setting low margins, which would make them less safe.

[3] Recorded failures include one in France in 1974, in Malaysia in 1983, and in Hong Kong in 1987. There have been near failures as well. In the wake of the October 1987 crash, both the CME and OCC encountered difficulty in receiving margins.

EXAMPLE 7.1: NOVATION

Novation is the process of:

a. Creating a new trade between two counterparties
b. Terminating an existing trade between two counterparties
c. Discharging a contract between the original counterparties and creating two new contracts, each with a central counterparty
d. Assigning a trade to another party

EXAMPLE 7.2: CENTRAL COUNTERPARTIES

Which of the following is *not* an advantage of establishing CCPs?

a. CCPs allow netting of contracts.
b. CCPs can be applied to some types of OTC trades.
c. CCPs can create more transparency in trading.
d. CCPs eliminate all counterparty risk in the financial system.

7.2 FORWARD CONTRACTS

7.2.1 Overview

The most common transactions in financial instruments are **spot transactions**, that is, for physical delivery as soon as practical (perhaps in two business days or in a week). Historically, grain farmers went to a centralized location to meet buyers for their product. As markets developed, the farmers realized that it would be beneficial to trade for delivery at some future date. This allowed them to hedge out price fluctuations for the sale of their anticipated production.

This gave rise to **forward contracts**, which are private agreements to exchange a given asset against cash (or sometimes another asset) at a fixed point in the future. The terms of the contract are the quantity (number of units or shares), date, and price at which the exchange will be done.

A position that implies buying the asset is said to be **long**. A position to sell is said to be **short**. Any gain to one party must be a loss to the other.

These instruments represent contractual obligations, as the exchange must occur whatever happens to the intervening price, unless default occurs. Unlike an option contract, there is no choice to take delivery or not.

To avoid the possibility of losses, the farmer could enter a forward sale of grain for dollars. By so doing, the farmer locks up a price now for delivery in the future. We then say that the farmer is **hedged** against movements in the price.

We use the notations

$$\begin{aligned}
t &= \text{current time} \\
T &= \text{time of delivery} \\
\tau = T - t &= \text{time to maturity} \\
S_t &= \text{current spot price of the asset in dollars} \\
F_t(T) &= \text{current forward price of the asset for delivery at } T \\
&\quad \text{(also written as } F_t \text{ or } F \text{ to avoid clutter)} \\
V_t &= \text{current value of contract} \\
r &= \text{current domestic risk-free rate for delivery at } T \\
n &= \text{quantity, or number of units in contract}
\end{aligned}$$

The **face amount**, or **principal value**, of the contract is defined as the amount nF to pay at maturity, like a bond. This is also called the **notional amount.** We will assume that interest rates are continuously compounded so that the present value of a dollar paid at expiration is $\text{PV}(\$1) = e^{-r\tau}$.

Say that the initial forward price is $F_t = \$100$. A speculator agrees to buy $n = 500$ units for F_t at T. At expiration, the payoff on the forward contract is determined in two steps as follows:

1. The speculator pays $nF = \$50{,}000$ in cash and receives 500 units of the underlying.
2. The speculator could then sell the underlying at the prevailing spot price S_T, for a profit of $n(S_T - F)$. For example, if the spot price is at $S_T = \$120$, the profit is $500 \times (\$120 - \$100) = \$10{,}000$. This is also the mark-to-market value of the contract at expiration.

In summary, the value of the forward contract at expiration, for one unit of the underlying asset, is:

$$V_T = S_T - F \tag{7.1}$$

Here, the value of the contract at expiration is derived from the purchase and **physical delivery** of the underlying asset. There is a payment of cash in exchange for the actual asset.

Another mode of settlement is **cash settlement**. This involves simply measuring the market value of the asset upon maturity, S_T, and agreeing for the long side to receive $nV_T = n(S_T - F)$. This amount can be positive or negative, involving a profit or loss.

Figures 7.1 and 7.2 present the payoff patterns on long and short positions in a forward contract, respectively. It is important to note that the payoffs are *linear* in the underlying spot price. Also, the positions in the two figures are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and the short is zero, because these are private contracts.

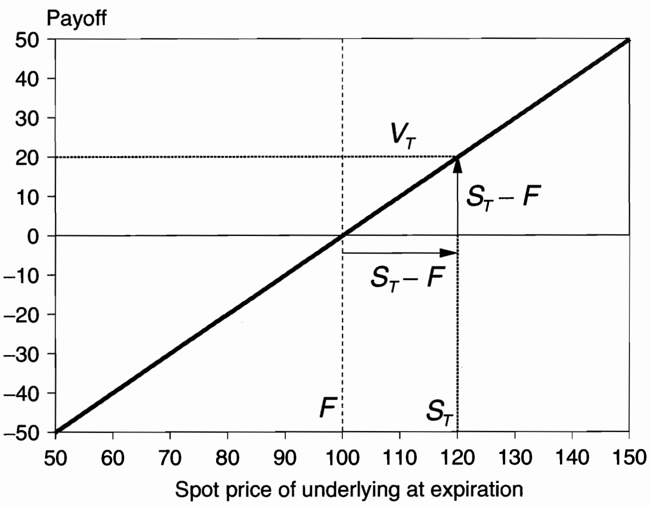

FIGURE 7.1 Payoff of Profits on Long Forward Contract

7.2.2 Valuing Forward Contracts

When evaluating forward contracts, two important questions arise. First, how is the current forward price F_t determined? Second, what is the current value V_t of an outstanding forward contract?

Initially, we assume that the underlying asset pays no income. This will be generalized in the next section. We also assume no transaction costs, that is, zero bid-ask spread on spot and forward quotations, as well as the ability to lend and borrow at the same risk-free rate.

Generally, forward contracts are established so that their initial value is zero. This is achieved by setting the forward price F_t appropriately by a **no-arbitrage relationship** between the cash and forward markets. No-arbitrage is a situation where positions with the same payoffs have the same price. This rules out situations

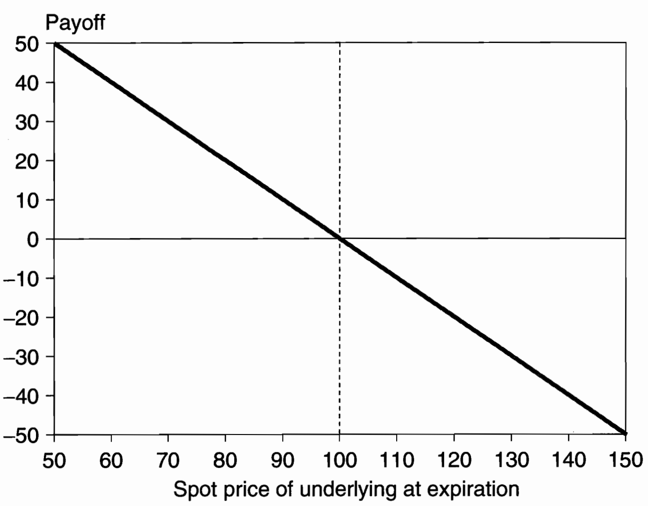

FIGURE 7.2 Payoff of Profits on Short Forward Contract

where **arbitrage profits** can exist. Arbitrage is a zero-risk, zero-net investment strategy that still generates profits.

Consider these strategies:

- Buy one share/unit of the underlying asset at the spot price S_t and hold to time T.
- Enter a forward contract to buy one share/unit of same underlying asset at the forward price F_t. In order to have sufficient funds at maturity to pay F_t, we invest the present value of F_t in an interest-bearing account. This is the present value $F_t e^{-r\tau}$. The forward price F_t is set so that the initial cost of the forward contract, V_t, is zero.

The two portfolios are economically equivalent because they will be identical at maturity. Each will contain one share of the asset. Hence their up-front cost must be the same. To avoid arbitrage, we must have:

$$S_t = F_t e^{-r\tau} \tag{7.2}$$

This equation defines the fair forward price F_t such that the initial value of the contract is zero. More generally, the term multiplying F_t is the present value factor for maturity τ, or PV($1). For instance, assuming $S_t = \$100$, $r = 5\%$, $\tau = 1$, we have $F_t = S_t e^{r\tau} = \$100 \times \exp(0.05 \times 1) = \$105.13$.

We see that the forward rate is higher than the spot rate. This reflects the fact that there is no down payment to enter the forward contract, unlike for the cash position. As a result, the forward price must be higher than the spot price to reflect the time value of money.

Abstracting from transaction costs, any deviation creates an arbitrage opportunity. This can be taken advantage of by buying the cheap asset and selling the expensive one. Assume, for instance, that $F = \$110$. We determined that the fair value is $S_t e^{r\tau} = \$105.13$, based on the cash price. We apply the principle of buying low at $105.13 and selling high at $110. We can lock in a sure profit in two steps by:

1. Buying now the asset spot at $100.
2. Selling now the asset forward at $110.

This can be done by borrowing the $100 to buy the asset now. At expiration, we will owe principal plus interest, or $105.13, but receive $110, for a profit of $4.87. This would be a blatant arbitrage opportunity, or "money machine."

Now consider a mispricing where $F = \$102$. We apply the principle of buying low at $102 and selling high at $105.13. We can lock in a sure profit in two steps by:

1. Short-selling now the asset spot at $100.
2. Buying now the asset forward at $102.

From the short sale, we invest the cash, which will grow to \$105.13. At expiration, we will have to deliver the stock, but this will be acquired through the forward purchase. We pay \$102 for this and are left with a profit of \$3.13.

This transaction involves the **short sale** of the asset, which is more involved than an outright purchase. When purchasing, we pay \$100 and receive one share of the asset. When short-selling, we borrow one share of the asset and promise to give it back at a future date; in the meantime, we sell it at \$100.[4]

7.2.3 Valuing an Off-Market Forward Contract

We can use the same reasoning to evaluate an outstanding forward contract with a locked-in delivery price of K. In general, such a contract will have nonzero value because K differs from the prevailing forward rate. Such a contract is said to be **off-market**.

Consider these strategies:

- Buy one share/unit of the underlying asset at the spot price S_t and hold it until time T.
- Enter a forward contract to buy one share/unit of same underlying asset at the price K; in order to have sufficient funds at maturity to pay K, we invest the present value of K in an interest-bearing account. This present value is also $Ke^{-r\tau}$. In addition, we have to pay the market value of the forward contract, or V_t.

The up-front cost of the two portfolios must be identical. Hence, we must have $V_t + Ke^{-r\tau} = S_t$, or

$$V_t = S_t - Ke^{-r\tau} \tag{7.3}$$

which defines the market value of an outstanding long position.[5] This gains value when the underlying S increases in value. A short position would have the reverse sign. Later, we will extend this relationship to the measurement of risk by considering the distribution of the underlying risk factors, S_t and r.

For instance, assume we still hold the previous forward contract with $F_t = \$105.13$ and after one month the spot price moves to $S_t = \$110$. The fixed rate is $K = \$105.13$ throughout the life of the contract. The interest has not changed at $r = 5\%$, but the maturity is now shorter by one month, $\tau = 11/12$. The new value of the contract is $V_t = S_t - Ke^{-r\tau} = \$110 - \$105.13\exp(-0.05 \times 11/12) = \$110 - \$100.42 = \$9.58$. The contract is now more valuable than before because the spot price has moved up.

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[4] In practice, we may not get full access to the proceeds of the sale when it involves individual stocks. The broker will typically allow us to withdraw only 50% of the cash. The rest is kept as a performance bond should the transaction lose money.

[5] Note that V_t is not the same as the forward price F_t. The former is the value of the contract; the latter refers to a specification of the contract.

7.2.4 Valuing Forward Contracts with Income Payments

We previously considered a situation where the asset produces no income payment. In practice, the asset may be

- A stock that pays a regular dividend
- A bond that pays a regular coupon
- A stock index that pays a dividend stream approximated by a continuous yield
- A foreign currency that pays a foreign-currency-denominated interest rate

Whichever income is paid on the asset, we can usefully classify the payment into **discrete**, that is, fixed dollar amounts at regular points in time, or on a **continuous** basis, that is, accrued in proportion to the time the asset is held. We must assume that the income payment is fixed or is certain. More generally, a storage cost is equivalent to a negative dividend.

We use these definitions:

$D =$ discrete (dollar) dividend or coupon payment
$r_t^*(T) =$ foreign risk-free rate for delivery at T
$q_t(T) =$ dividend yield

Whether the payment is a dividend or a foreign interest rate, the principle is the same. We can afford to invest less in the asset up front to get one unit at expiration. This is because the income payment can be reinvested into the asset. Alternatively, we can borrow against the value of the income payment to increase our holding of the asset.

It is also important to note that all prices (S, F) are measured in the domestic currency. For example, S could be expressed in terms of the U.S. dollar price of the euro, in which case r is the U.S. interest rate and r^* is the euro interest rate. Conversely, if S is the Japanese yen price of the U.S. dollar, r will represent the Japanese interest rate, and r^* the U.S. interest rate.

Continuing our example, consider a stock priced at \$100 that pays a dividend of $D = \$1$ in three months. The present value of this payment discounted over three months is $De^{-r\tau} = \$1 \exp(-0.05 \times 3/12) = \$0.99$. We only need to put up $S_t - \text{PV}(D) = \$100.00 - 0.99 = \$99.01$ to get one share in one year. Put differently, we buy 0.9901 fractional shares now and borrow against the (sure) dividend payment of \$1 to buy an additional 0.0099 fractional share, for a total of 1 share.

The pricing formula in Equation (7.2) is extended to

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{7.4}$$

where PV(D) is the present value of the dividend/coupon payments. If there is more than one payment, PV(D) represents the sum of the present values of each individual payment, discounted at the appropriate risk-free rate. With storage costs, we need to *add* the present value of storage costs PV(C) to the right side of Equation (7.4).

The approach is similar for an asset that pays a continuous income, defined per unit of time instead of discrete amounts. Holding a foreign currency, for instance, should be done through an interest-bearing account paying interest that accrues with time. Over the horizon τ, we can afford to invest less up front, $S_t e^{-r^*\tau}$, in order to receive one unit at maturity. The right-hand side of Equation (7.4) is now

$$F_t e^{-r\tau} = S_t e^{-r^*\tau} \tag{7.5}$$

Hence the forward price should be

$$F_t = S_t e^{-r^*\tau} / e^{-r\tau} \tag{7.6}$$

If instead interest rates are annually compounded, this gives

$$F_t = S_t (1 + r)^\tau / (1 + r^*)^\tau \tag{7.7}$$

Equation (7.6) can be also written in terms of the forward premium or discount, which is

$$\frac{(F_t - S_t)}{S_t} = e^{-r^*\tau} / e^{-r\tau} - 1 = \exp(r - r^*)\tau \approx (r - r^*)\tau \tag{7.8}$$

If $r^* < r$, we have $F_t > S_t$ and the asset trades at a **forward premium**. Conversely, if $r^* > r$, $F_t < S_t$ and the asset trades at a **forward discount**. Thus the forward price is higher or lower than the spot price, depending on whether the yield on the asset is lower than or higher than the domestic risk-free interest rate.

Equation (7.6) is also known as **interest rate parity** when dealing with currencies. Also note that both the spot and forward prices must be expressed in dollars per unit of the foreign currency when the domestic currency interest rate is r. This is the case, for example, for the dollar/euro or dollar/pound exchange rate. If, however, the exchange rate is expressed in foreign currency per dollar, then r must be the rate on the foreign currency. For the yen/dollar rate, for example, S is in yen per dollar, r is the yen interest rate, and r^* is the dollar interest rate.

KEY CONCEPT

The forward price differs from the spot price to reflect the time value of money and the income yield on the underlying asset. It is higher than the spot price if the yield on the asset is lower than the domestic risk-free interest rate, and vice versa.

With income payments, the value of an outstanding forward contract is

$$V_t = S_t e^{-r^*\tau} - K e^{-r\tau} \tag{7.9}$$

If F_t is the new, current forward price, we can also write

$$V_t = F_t e^{-r\tau} - Ke^{-r\tau} = (F_t - K)e^{-r\tau} \tag{7.10}$$

This provides a useful alternative formula for the valuation of a forward contract. The intuition here is that we could liquidate the outstanding forward contract by entering a reverse position at the current forward rate. The payoff at expiration is $(F - K)$, which, discounted back to the present, gives Equation (7.10).

KEY CONCEPT

The current value of an outstanding forward contract can be found by entering an offsetting forward position and discounting the net cash flow at expiration.

EXAMPLE 7.3: FRM EXAM 2008—QUESTION 2-15

The one-year U.S. dollar interest rate is 2.75% and one-year Canadian dollar interest rate is 4.25%. The current USD/CAD spot exchange rate is 1.0221–1.0225. Calculate the one-year USD/CAD forward rate. Assume annual compounding.

a. 1.0076
b. 1.0074
c. 1.0075
d. 1.03722

EXAMPLE 7.4: FRM EXAM 2005—QUESTION 16

Suppose that U.S. interest rates rise from 3% to 4% this year. The spot exchange rate quotes at 112.5 JPY/USD and the forward rate for a one-year contract is at 110.5. What is the Japanese interest rate?

a. 1.81%
b. 2.15%
c. 3.84%
d. 5.88%

EXAMPLE 7.5: FRM EXAM 2002—QUESTION 56

Consider a forward contract on a stock market index. Identify the *false* statement. Everything else being constant,

a. The forward price depends directly on the level of the stock market index.
b. The forward price will fall if underlying stocks increase the level of dividend payments over the life of the contract.
c. The forward price will rise if time to maturity is increased.
d. The forward price will fall if the interest rate is raised.

EXAMPLE 7.6: FRM EXAM 2007—QUESTION 119

A three-month futures contract on an equity index is currently priced at USD 1,000. The underlying index stocks are valued at USD 990 and pay dividends at a continuously compounded rate of 2%. The current continuously compounded risk-free rate is 4%. The potential arbitrage profit per contract, given this set of data, is closest to

a. USD 10.00
b. USD 7.50
c. USD 5.00
d. USD 1.50

7.3 FUTURES CONTRACTS

7.3.1 Overview

Forward contracts allow users to take positions that are economically equivalent to those in the underlying cash markets. Unlike cash markets, however, they do not involve substantial up-front payments. Thus, forward contracts can be interpreted as having *leverage*. Leverage is efficient, as it makes our money work harder.

Leverage creates credit risk for the counterparty, however. For a cash trade, there is no leverage. When a speculator buys a stock at the price of $100, the counterparty receives the cash and has no credit risk. Instead, when a speculator enters a forward contract to buy an asset at the price of $105, there is no up-front payment. In effect, the speculator borrows from the counterparty to invest in the asset. There is a risk of default should the value of the contract to the speculator fall sufficiently. In response, futures contracts have been structured so as to minimize credit risk for all counterparties. Otherwise, from a market risk standpoint, futures contracts are basically identical to forward contracts.

Futures contracts are standardized, negotiable, and exchange-traded contracts to buy or sell an underlying asset. They differ from forward contracts as follows.

- **Trading on organized exchanges.** In contrast to forwards, which are OTC contracts tailored to customers' needs, futures are traded on organized exchanges.
- **Standardization.** Futures contracts are offered with a limited choice of expiration dates. They trade in fixed contract sizes. This standardization ensures an active secondary market for many futures contracts, which can be easily traded, purchased, or resold. In other words, most futures contracts have good liquidity. The trade-off is that futures are less precisely suited to the needs of some hedgers, which creates basis risk (to be defined later).
- **Clearinghouse.** Futures contracts are also standardized in terms of the counterparty. After each transaction is confirmed, the clearinghouse basically interposes itself between the buyer and the seller, ensuring the performance of the contract. Thus, unlike forward contracts, counterparties do not have to worry about the credit risk of the other side of the trade.
- **Marking to market.** As the clearinghouse now has to deal with the credit risk of the two original counterparties, it has to monitor credit risk closely. This is achieved by daily marking to market, which involves settlement of the gains and losses on the contract every day. This will avoid the accumulation of large losses over time, potentially leading to an expensive default.
- **Margins.** Although daily settlement accounts for past losses, it does not provide a buffer against future losses. This is the goal of **margins**, which represent posting of collateral that can be seized should the other party default. The **initial margin** must be posted when initiating the position. If the equity in the account falls below the **maintenance margin**, the customer is required to provide additional funds to cover the initial margin. Note that the amount to fill up is not to the maintenance margin, but to the initial margin. The level of margin depends on the instrument and the type of position; in general, less volatile instruments or hedged positions require lower margins.

Example: Margins for a Futures Contract

Consider a futures contract on 1,000 units of an asset worth \$100. A long futures position is economically equivalent to holding \$100,000 worth of the asset directly. To enter the futures position, a speculator has to post only \$5,000 in margin, for example. This amount is placed in an equity account with the broker.

The next day, the futures price moves down by \$3, leading to a loss of \$3,000 for the speculator. The loss is subtracted from the equity account, bringing it down to \$5,000 − \$3,000 = \$2,000. The speculator would then receive a **margin call** from the broker, asking to have an additional \$3,000 of capital posted to the account. If he or she fails to meet the margin call, the broker has the right to liquidate the position.

Since futures trading is centralized on an exchange, it is easy to collect and report aggregate trading data. **Volume** is the number of contracts traded during the day, which is a flow item. **Open interest** represents the outstanding number of contracts at the close of the day, which is a stock item.

7.3.2 Valuing Futures Contracts

Valuation principles for futures contracts are very similar to those for forward contracts. The main difference between the two types of contracts is that any profit or loss accrues *during* the life of the futures contract instead of all at once, at expiration.

When interest rates are assumed constant or deterministic, forward and futures prices must be equal. With stochastic interest rates, there may be a small difference, depending on the correlation between the value of the asset and interest rates.

If the correlation is zero, then it makes no difference whether payments are received earlier or later. The futures price must be the same as the forward price. In contrast, consider a contract whose price is positively correlated with the interest rate. If the value of the contract goes up, it is more likely that interest rates will go up as well. This implies that profits can be withdrawn and reinvested at a higher rate. Relative to forward contracts, this marking-to-market feature is beneficial to a long futures position. As a result, the futures price must be higher in equilibrium.

In practice, this effect is observable only for interest-rate futures contracts, whose value is *negatively* correlated with interest rates. Because this feature is unattractive for the long position, the futures price must be *lower* than the forward price. Chapter 8 explains how to compute the adjustment, called the **convexity effect.**

EXAMPLE 7.7: FRM EXAM 2004—QUESTION 38

An investor enters into a short position in a gold futures contract at USD 294.20. Each futures contract controls 100 troy ounces. The initial margin is USD 3,200, and the maintenance margin is USD 2,900. At the end of the first day, the futures price drops to USD 286.6. Which of the following is the amount of the variation margin at the end of the first day?

a. 0
b. USD 34
c. USD 334
d. USD 760

EXAMPLE 7.8: FRM EXAM 2004—QUESTION 66

Which one of the following statements is *incorrect* regarding the margining of exchange-traded futures contracts?

a. Day trades and spread transactions require lower margin levels.
b. If an investor fails to deposit variation margin in a timely manner, the positions may be liquidated by the carrying broker.
c. Initial margin is the amount of money that must be deposited when a futures contract is opened.
d. A margin call will be issued only if the investor's margin account balance becomes negative.

7.4 SWAP CONTRACTS

Swap contracts are OTC agreements to exchange a *series* of cash flows according to prespecified terms. The underlying asset can be an interest rate, an exchange rate, an equity, a commodity price, or any other index. Typically, swaps are established for longer periods than forwards and futures.

For example, a 10-year currency swap could involve an agreement to exchange every year 5 million dollars against 3 million pounds over the next 10 years, in addition to a principal amount of 100 million dollars against 50 million pounds at expiration. The principal is also called **notional principal**.

Another example is that of a five-year interest rate swap in which one party pays 8% of the principal amount of 100 million dollars in exchange for receiving an interest payment indexed to a floating interest rate. In this case, since both payments are the same amount in the same currency, there is no need to exchange principal at maturity.

Swaps can be viewed as a portfolio of forward contracts. They can be priced using valuation formulas for forwards. Our currency swap, for instance, can be viewed as a combination of 10 forward contracts with various face values, maturity dates, and rates of exchange. We will give detailed examples in later chapters.

7.5 IMPORTANT FORMULAS

Forward price, no income on the asset: $F_t e^{-r\tau} = S_t$

Forward price, income on the asset:

Discrete dividend, $F_t e^{-r\tau} = S_t - \mathrm{PV}(D)$

Continuous dividend, $F_t e^{-r\tau} = S_t e^{-r^*\tau}$

Forward premium or discount: $\frac{(F_t - S_t)}{S_t} \approx (r - r^*)\tau$

Valuation of outstanding forward contract: $V_t = S_t e^{-r^*\tau} - Ke^{-r\tau} = F_t e^{-r\tau} - Ke^{-r\tau} = (F_t - K)e^{-r\tau}$

7.6 ANSWERS TO CHAPTER EXAMPLES

Example 7.1: Novation

c. Novation involves the substitution of counterparties. Clearinghouses use this process to interpose themselves between buyers and sellers. This requires consent from all parties, unlike an assignment.

Example 7.2: Central Counterparties

d. CCPs generally reduce counterparty risk but can be a source of systemic risk if they fail.

Example 7.3: FRM Exam 2008—Question 2-15

a. The spot price is the middle rate of \$1.0223. Using annual (not continuous) compounding, the forward price is $F = S(1 + r)/(1 + R^*) = 1.0223(1.0275)/(1.0425) = 1.0076$.

Example 7.4: FRM Exam 2005—Question 16

b. As is the convention in the currency markets, the exchange rate is defined as the yen price of the dollar, which is the foreign currency. The foreign currency interest rate is the latest U.S. dollar rate, or 4%. Assuming discrete compounding, the pricing formula for forward contracts is $F(JPY/USD)/(1 + rT) = S(JPY/USD)/(1 + r^*T)$. Therefore, $(1 + rT) = (F/S)(1 + r^*T) = (110.5/112.5)(1.04) = 1.0215$, and $r = 2.15\%$. Using continuous compounding gives a similar result. Another approach would consider the forward discount on the dollar, which is $(F - S)/S = -1.8\%$. Thus the dollar is 1.8% cheaper forward than spot, which must mean that the Japanese interest rate must be approximately 1.8% lower than the U.S. interest rate.

Example 7.5: FRM Exam 2002—Question 56

d. Defining the dividend yield as q, the forward price depends on the cash price according to $F \exp(-rT) = S \exp(-qT)$. This can also be written as $F = S \exp[+(r - q)T]$. Generally, $r > q$. Statement a. is correct: F depends directly on S. Statement b. is also correct, as higher q decreases the term between brackets and hence F. Statement c. is correct because the term $r - q$ is positive, leading to a larger term in brackets as the time to maturity T increases. Statement d. is false, as increasing r makes the forward contract more attractive, or increases F.

Example 7.6: FRM Exam 2007—Question 119

c. The fair value of the futures contract is given by $F = S\exp(-r^*T)/\exp(-rT) = 990\exp(-0.02 \times 3/12)/\exp(-0.04 \times 3/12) = 994.96$. Hence the actual futures price is too high by $(1{,}000 - 995) = 5$.

Example 7.7: FRM Exam 2004—Question 38

a. This is a tricky question. Because the investor is short and the price fell, the position creates a profit and there is no variation margin. However, for the long the loss is \$760, which would bring the equity to \$3,200 – \$760 = \$2,440. Because this is below the maintenance margin of \$2,900, an additional payment of \$760 is required to bring back the equity to the initial margin.

Example 7.8: FRM Exam 2004—Question 66

d. All the statements are correct, except d. If the margin account balance falls below the maintenance margin (not zero), a margin call will be issued.

CHAPTER 8

Option Markets

This chapter now turns to nonlinear derivatives, or options. As described in the previous chapter, options account for a large part of the derivatives markets. On organized exchanges, options represent more than \$50 trillion in derivatives outstanding. Over-the-counter options add up to more than \$60 trillion in notional amounts.

Although the concept behind these instruments is not new, option markets have blossomed since the early 1970s, because of a breakthrough in pricing options (the Black-Scholes formula) and advances in computing power. We start with **plain-vanilla** options: calls and puts. These are the basic building blocks of many financial instruments. They are also more common than complicated, **exotic** options.

The purpose of this chapter is to present a compact overview of important concepts for options, including their pricing. We will cover option sensitivities (the "Greeks") in a future chapter. Section 8.1 presents the payoff functions on basic options and combinations thereof. We then discuss option prices, or premiums, in Section 8.2. The Black-Scholes pricing approach is presented in Section 8.3. Next, Section 8.4 briefly summarizes more complex options. Finally, Section 8.5 shows how to value options using a numerical, binomial tree model.

8.1 OPTION PAYOFFS

8.1.1 Basic Options

Options are instruments that give their holder the *right* to buy or sell an asset at a specified price until a specified expiration date. The specified delivery price is known as the **delivery price**, or **exercise price**, or **strike price**, and is denoted by K.

Options to buy are **call options**. Options to sell are **put options**. As options confer a right to the purchaser of the option, but not an obligation, they will be exercised only if they generate profits. In contrast, forwards involve an obligation to either buy or sell and can generate profits or losses. Like forward contracts, options can be purchased or sold. In the latter case, the seller is said to **write** the option.

Depending on the timing of exercise, options can be classified into European or American options. **European options** can be exercised at maturity only. **American**

FRM Exam Part 1 topic. This chapter also covers the valuation of options.

options can be exercised at any time, before or at maturity. Because American options include the right to exercise at maturity, they must be at least as valuable as European options. In practice, however, the value of this early exercise feature is small, as an investor can generally receive better value by reselling the option on the open market instead of exercising it.

We use these notations, in addition to those in the previous chapter:

K = exercise price
c = value of European call option
C = value of American call option
p = value of European put option
P = value of American put option

To illustrate, take an option on an asset that currently trades at \$85 with a delivery price of \$100 in one year. If the spot price stays at \$85 at expiration, the holder of the call will not **exercise** the option, because the option is not profitable with a stock price less than \$100. In contrast, if the price goes to \$120, the holder will exercise the right to buy at \$100, will acquire the stock now worth \$120, and will enjoy a paper profit of \$20. This profit can be realized by selling the stock. For put options, a profit accrues if the spot price ends up below the exercise price $K = \$100$.

Thus the payoff profile of a long position in a call option at expiration is

$$C_T = \text{Max}(S_T - K, 0) \tag{8.1}$$

The payoff profile of a long position in a put option is

$$P_T = \text{Max}(K - S_T, 0) \tag{8.2}$$

If the current asset price S_t is close to the strike price K, the option is said to be **at-the-money**. If the current asset price S_t is such that the option could be exercised now at a profit, the option is said to be **in-the-money**. If the remaining situation is present, the option is said to be **out-of-the-money**. A call will be in-the-money if $S_t > K$. A put will be in-the-money if $S_t < K$.

As in the case of forward contracts, the payoff at expiration can be cash settled. Instead of actually buying the asset, the contract could simply pay \$20 if the price of the asset is \$120.

Because buying options can generate only profits (at worst zero) at expiration, an option contract must be a valuable asset (or at worst have zero value). This means that a payment is needed to acquire the contract. This up-front payment, which is much like an insurance premium, is called the option **premium.** This premium cannot be negative. An option becomes more expensive as it moves in-the-money.

Thus the payoffs on options must take into account this cost (for long positions) or benefit (for short positions). To compute the total payoff, we should

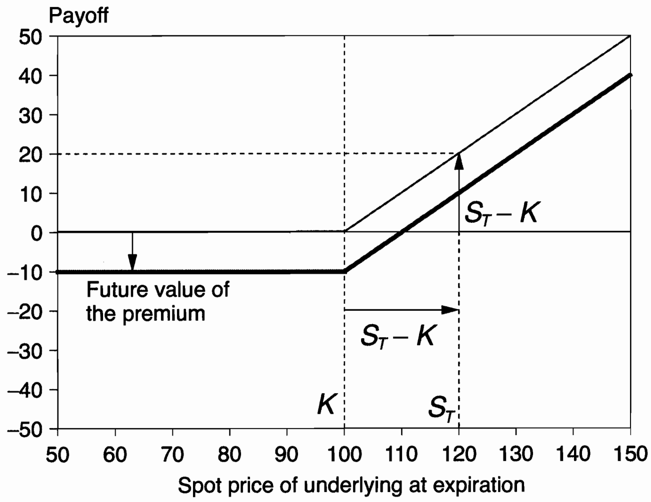

FIGURE 8.1 Profit Payoffs on Long Call

translate all option payoffs by the *future* value of the premium, that is, $ce^{r\tau}$, for European call options.

Figure 8.1 displays the total profit payoff on a call option as a function of the asset price at expiration. Assuming that $S_T = \$120$, the proceeds from exercise are $\$120 - \$100 = \$20$, from which we have to subtract the future value of the premium, say \$10. In the graphs that follow, we always take into account the cost of the option.

Figure 8.2 summarizes the payoff patterns on long and short positions in a call and a put contract. Unlike those of forwards, these payoffs are **nonlinear** in the underlying spot price. Sometimes they are referred to as the "hockey stick" diagrams. This is because forwards are obligations, whereas options are rights. Note that the positions for the same contract are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and for the short is zero.

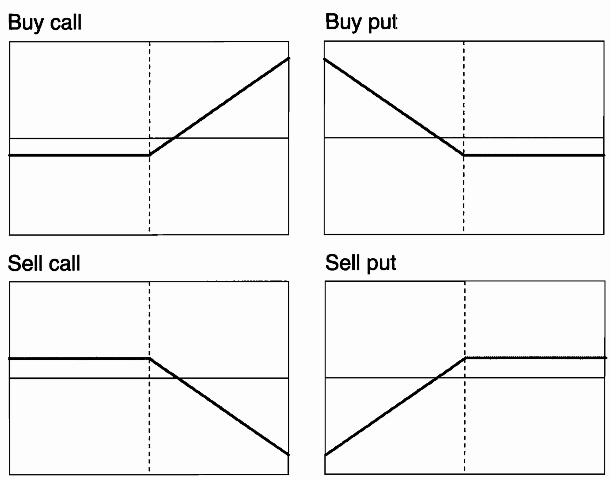

FIGURE 8.2 Profit Payoffs on Long and Short Calls and Puts

In the market risk section of this handbook (Part Five), we combine these payoffs with the distribution of the risk factors. Even so, it is immediately obvious that long option positions have limited downside risk, which is the loss of the premium. Short call option positions have unlimited downside risk because there is no upper limit on S. The worst loss on short put positions occurs if S goes to zero.

So far, we have covered options on cash instruments. Options can also be struck on futures. When exercising a call, the investor becomes long the futures contract. Conversely, exercising a put creates a short position in the futures contract. Because positions in futures are equivalent to leveraged positions in the underlying cash instrument, options on cash instruments and on futures are economically equivalent.

8.1.2 Put-Call Parity

These option payoffs can be used as the basic building blocks for more complex positions. A long position in the underlying asset can be decomposed into a long call plus a short put with the same strike prices and maturities, as shown in Figure 8.3.

The figure shows that the long call provides the equivalent of the upside while the short put generates the same downside risk as holding the asset. This link creates a relationship between the value of the call and that of the put, also known as **put-call parity**. The relationship is illustrated in Table 8.1, which examines the payoff at initiation and at expiration under the two possible states of the world. We consider only European options with the same maturity and exercise price. Also, we assume that there is no income payment on the underlying asset.

The portfolio consists of a long position in the call, a short position in the put, and an investment to ensure that we will be able to pay the exercise price at maturity. Long positions are represented by negative values, as they represent outflows, or costs.

The table shows that the final payoffs to portfolio (1) add up to S_T in the two states of the world, which is the same as a long position in the asset itself. Hence, to avoid arbitrage, the initial payoff must be equal to the current cost of the asset,

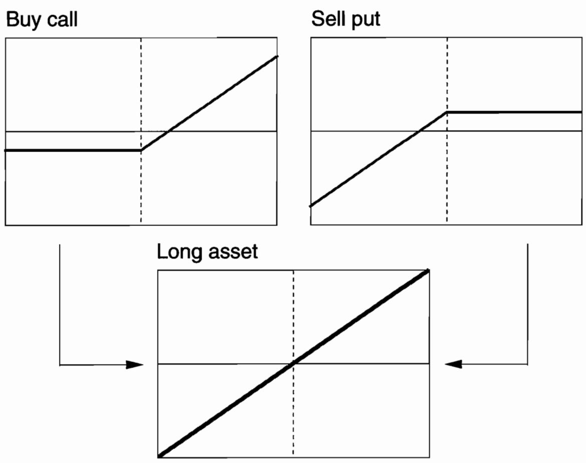

FIGURE 8.3 Decomposing a Long Position in the Asset

TABLE 8.1 Put-Call Parity

Portfolio	Position	Initial Payoff	Final Payoff $S_T < K$	Final Payoff $S_T \geq K$
(1)	Buy call	$-c$	0	$S_T - K$
	Sell put	$+p$	$-(K - S_T)$	0
	Invest	$-Ke^{-r\tau}$	K	K
	Total	$-c + p - Ke^{-r\tau}$	S_T	S_T
(2)	Buy asset	$-S$	S_T	S_T

which is $S_t = S$. So, we must have $-c + p - Ke^{-r\tau} = -S$. More generally, with income paid at the rate of r^*, put-call parity can be written as

$$c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau} \tag{8.3}$$

Because $c \geq 0$ and $p \geq 0$, this relationship can also be used to determine lower bounds for European calls and puts. Note that the relationship does not hold exactly for American options since there is a likelihood of early exercise, which could lead to mismatched payoffs.

Finally, this relationship can be used to determine the **implied dividend yield** from market prices. We observe c, p, S, and r and can solve for y or r^*. This yield is used for determining the forward rate in **dividend swaps**, which are contracts where the payoff is indexed to the actual dividends paid over the horizon, minus the implied dividends.

KEY CONCEPT

A long position in an asset is equivalent to a long position in a European call with a short position in an otherwise identical put, combined with a risk-free position.

EXAMPLE 8.1: FRM EXAM 2007—QUESTION 84

According to put-call parity, buying a put option on a stock is equivalent to

- **a.** Buying a call option and buying the stock with funds borrowed at the risk-free rate
- **b.** Selling a call option and buying the stock with funds borrowed at the risk-free rate
- **c.** Buying a call option, selling the stock, and investing the proceeds at the risk-free rate
- **d.** Selling a call option, selling the stock, and investing the proceeds at the risk-free rate

EXAMPLE 8.2: FRM EXAM 2005—QUESTION 72

A one-year European put option on a non-dividend-paying stock with strike at EUR 25 currently trades at EUR 3.19. The current stock price is EUR 23 and its annual volatility is 30%. The annual risk-free interest rate is 5%. What is the price of a European call option on the same stock with the same parameters as those of this put option? Assume continuous compounding.

a. EUR 1.19
b. EUR 3.97
c. EUR 2.41
d. Cannot be determined with the data provided

EXAMPLE 8.3: FRM EXAM 2008—QUESTION 2-10

The current price of stock ABC is \$42 and the call option with a strike at \$44 is trading at \$3. Expiration is in one year. The corresponding put is priced at \$2. Which of the following trading strategies will result in arbitrage profits? Assume that the risk-free rate is 10% and that the risk-free bond can be shorted costlessly. There are no transaction costs.

a. Long position in both the call option and the stock, and short position in the put option and risk-free bond
b. Long position in both the call option and the put option, and short position in the stock and risk-free bond
c. Long position in both the call option and the risk-free bond, and short position in the stock and the put option
d. Long position in both the put option and the risk-free bond, and short position in the stock and the call option

EXAMPLE 8.4: FRM EXAM 2006—QUESTION 74

Jeff is an arbitrage trader, who wants to calculate the implied dividend yield on a stock while looking at the over-the-counter price of a five-year European put and call on that stock. He has the following data: $S = \$85$, $K = \$90$, $r = 5\%$, $c = \$10$, $p = \$15$. What is the continuous implied dividend yield of that stock?

a. 2.48%
b. 4.69%
c. 5.34%
d. 7.71%

8.1.3 Combination of Options

Options can be combined in different ways, either with each other or with the underlying asset. Consider first combinations of the underlying asset and an option. A long position in the stock can be accompanied by a short sale of a call to collect the option premium. This operation, called a **covered call**, is described in Figure 8.4. Likewise, a long position in the stock can be accompanied by a purchase of a put to protect the downside. This operation is called a **protective put.**

Options can also be combined with an underlying position to limit the range of potential gains and losses. Suppose an investor is long a stock, currently trading at \$10. The investor can buy a put with a low strike price (e.g., \$7), partially financed by the sale of a call with high strike (e.g., \$12). Ignoring the net premium, the highest potential gain is \$2 and the worst loss is \$3. Such a strategy is called a **collar.** If the strike prices were the same, this would be equivalent to a short stock position, which creates a net payoff of exactly zero.

We can also combine a call and a put with the same or different strike prices and maturities. When the strike prices of the call and the put and their maturities are the same, the combination is referred to as a **straddle.** Figure 8.5 shows how to construct a long straddle (i.e., buying a call and a put with the same maturity and strike price). This position is expected to benefit from a large price move, whether up or down. The reverse position is a short straddle. When the strike prices are different, the combination is referred to as a **strangle.** Since strangles are out-of-the-money, they are cheaper to buy than straddles.

Thus far, we have concentrated on positions involving two classes of options. One can, however, establish positions with one class of options, called **spreads.** Calendar or **horizontal spreads** correspond to different maturities. **Vertical spreads** correspond to different strike prices. The names of the spreads are derived from the manner in which they are listed in newspapers: time is listed horizontally and strike prices are listed vertically. **Diagonal spreads** move across maturities and strike prices.

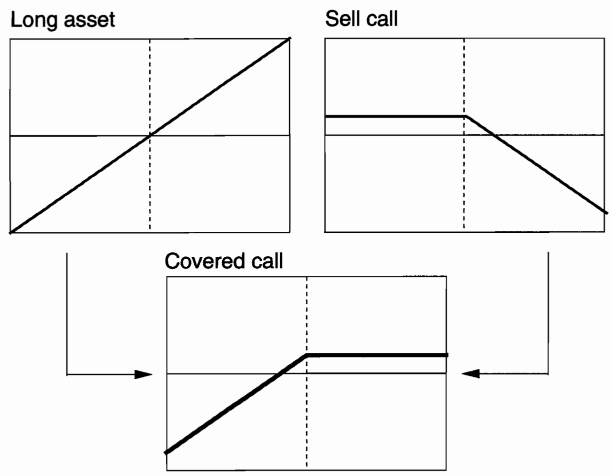

FIGURE 8.4 Creating a Covered Call

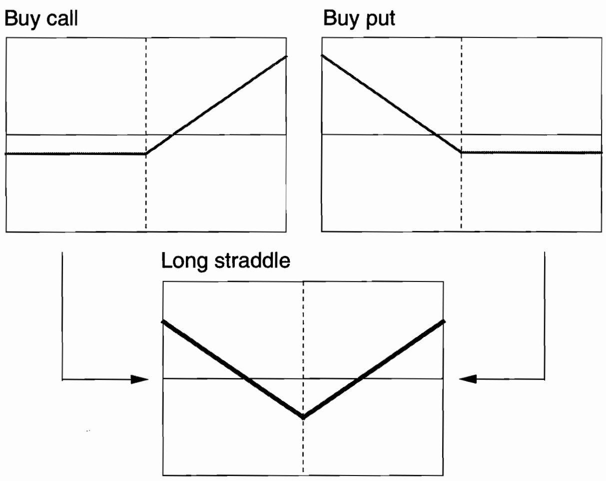

FIGURE 8.5 Creating a Long Straddle

For instance, a **bull spread** is positioned to take advantage of an increase in the price of the underlying asset. Conversely, a **bear spread** represents a bet on a falling price. Figure 8.6 shows how to construct a bull(ish) vertical spread with two calls with the same maturity. This could also be constructed with puts, however. Here, the spread is formed by buying a call option with a low exercise price K_1 and selling another call with a higher exercise price K_2. Note that the cost of the first call $c(S, K_1)$ must exceed the cost of the second call $c(S, K_2)$, because the first option is more in-the-money than the second. Hence, the sum of the two premiums represents a net cost. At expiration, when $S_T > K_2$, the payoff is $\text{Max}(S_T - K_1, 0) - \text{Max}(S_T - K_2, 0) = (S_T - K_1) - (S_T - K_2) = K_2 - K_1$, which is positive. Thus this position is expected to benefit from an up move, while incurring only limited downside risk.

Spreads involving more than two positions are referred to as butterfly or sandwich spreads. A **butterfly spread** involves three types of options with the same maturity: for example, a long call at a strike price K_1, two short calls at a higher strike price K_2, and a long call position at a higher strike price K_3, with the

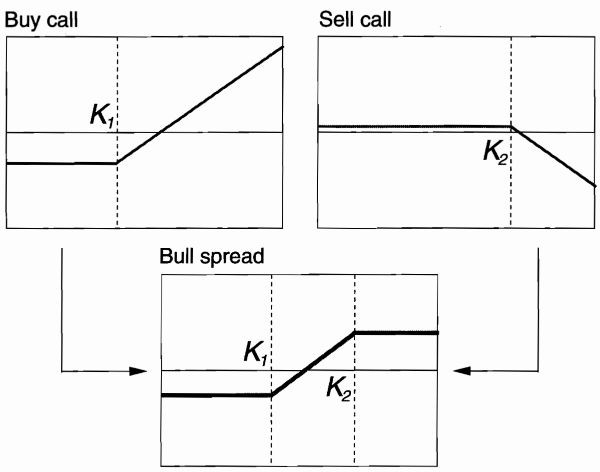

FIGURE 8.6 Creating a Bull Spread