portfolio. They fail to take into account differences in volatilities across markets, correlations across risk factors, as well as the probability of adverse moves in the risk factors.

Consider, for instance, a five-year **inverse floater**, which pays a coupon equal to 16% minus twice current LIBOR, if positive, on a notional principal of $100 million. The initial market value of the note is $100 million. This type of investment is extremely sensitive to movements in interest rates. If rates go up, the coupon payments will drop sharply. In addition, the discount rate also increases. As with all bonds, this investment can be priced by discounting the future cash flows into the present. This combination of lower cash flows and higher discount rate will push the bond price down sharply.

The question is, how much could an investor lose on this investment over a specified horizon? The *notional amount* is only indirectly informative. The worst-case scenario is one where interest rates rise above 8%. In this situation, the coupon will drop to $16 - 2 \times 8 =$ zero. The bond becomes a zero-coupon bond, whose value is $68 million, discounted at 8%. This gives a loss of $\$100 - \$68 = \$32$ million. While sizable, this is still less than the notional.

A *sensitivity measure* such as duration is more helpful. In this case, the bond has three times the modified duration of a similar five-year note, which gives $D = 3 \times 4.5 = 13.5$ years. So, if interest rates go up by 1%, the bond would lose 13.5% of its value. This duration measure reveals the extreme sensitivity of the bond to interest rates but does not answer the question of whether such a disastrous movement in interest rates is likely. It also ignores the nonlinearity between the note price and yields.

Another general problem is that these sensitivity measures do not allow the investor to aggregate risk across different markets. Let us say that this investor also holds a position in a bond denominated in another currency, the euro. Do the risks add up, or diversify each other?

VAR provides a uniform answer to all these questions. One number aggregates the risks across the whole portfolio, taking into account leverage and diversification, and providing a risk measure with an associated probability.

If the worst increase in yield at the 95% level is 1.65% over the next year, we can compute VAR as

$$\text{VAR} = (\text{Market Value} \times \text{Modified Duration}) \times \text{Worst Yield Increase} \qquad (12.1)$$

In this case, $\text{VAR} = \$100 \times 13.5 \times 0.0165 = \$22$ million. The investor can now make a statement such as: The worst loss at the 95% confidence level is approximately $22 million. The risk manager can now explain the risk of the investment in a simple, intuitive fashion.

Measures such as notional amounts and exposures have been, and are still, used to set limits, in an attempt to control risk before it occurs, or *ex ante*. These measures should be supplemented by VAR, which is an *ex ante* measure of the potential dollar loss. Other risk management tools include **stop losses**, which are rules enforcing position cuts after losses occur, that is, *ex post*. While stop losses are useful, especially in trending markets, they provide only partial protection because they are applied *after* a loss. In other words, this is too late, except for preventing further losses.

12.1.3 Sources of Loss

Our example can help isolate the sources of a market loss. The bond's value change can be described as

$$dP = -(D^* P) \times dy \tag{12.2}$$

where $D^* P$ is the *dollar duration* and dy the change in the yield.

This illustrates the general principle that losses can occur because of a combination of two components:

1. The exposure to the factor, or dollar duration. This is a choice variable that represents the positions.
2. The movement in the risk factor itself. This is external to the portfolio.

This is a general decomposition. It also applies to *systematic risk*, or exposure to the stock market. We can generally decompose the return on stock i, R_i into a component due to the market R_M and some residual risk

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \approx \beta_i \times R_M \tag{12.3}$$

We ignore the constant α_i because it does not contribute to risk, as well as the residual ϵ_i, which is diversified. Note that R_i is expressed here in terms of **rate of return** and, hence, has no dimension. To get a change in a dollar price, we write

$$dP_i = R_i P_i \approx (\beta P_i) \times R_M \tag{12.4}$$

The term between parentheses is the exposure, a choice variable.

This concept of linear exposure also applies to an option *delta*, defined as Δ.[1] The change in the value of a derivative f can be expressed in terms of the change in the price of the underlying asset S:

$$df = (\Delta) \times dS \tag{12.5}$$

Equations (12.2), (12.4), and (12.5) all reveal that the change in value is linked to an **exposure** coefficient and a change in a market variable:

Market Loss = Exposure × Adverse Movement in Financial Variable

To have a loss, we need to have some exposure *and* an unfavorable move in the risk factor. Thus we can manage the portfolio risk by changing its exposure.

[1] To avoid confusion, we use the conventional notation of Δ for the first partial derivative of the option. Changes are expressed in infinitesimal amounts df and dS.

For instance, moving a bond portfolio into cash creates a dollar duration of zero, in which case interest rate movements have no effect on the value of the portfolio. More generally, the relationship between the portfolio value and the risk factor need not be linear.

EXAMPLE 12.1: FRM EXAM 2005—QUESTION 32

Which of the following statements about trader limits are *correct*?

I. Stop loss limits are useful if markets are trending.
II. Exposure limits do not allow for diversification.
III. VAR limits are not susceptible to arbitrage.
IV. Stop loss limits are effective in preventing losses.

a. I and II
b. III and IV
c. I and III
d. II and IV

12.2 COMPONENTS OF RISK MEASUREMENT SYSTEMS

As described in Figure 12.1, a risk measurement system combines the following three steps:

1. Collect the **portfolio positions** and map them onto the risk factors.
2. From market data, construct the distribution of **risk factors** (e.g., normal, empirical, or other).
3. Construct the **distribution of portfolio returns** using one of the three methods (parametric, historical, Monte Carlo), and summarize the downside risk with VAR.

Consider for instance a position of $4 billion short the yen, long the dollar. This position corresponds to a well-known hedge fund that took a bet that the yen would fall in value against the dollar. The portfolio manager rightfully asks: How much could this position lose over a day?

12.2.1 Portfolio Positions

We start with portfolio positions. In this example, the current position is short the Japanese yen in the dollar amount of $4 billion.

The assumption is that all positions are constant over the horizon. This, of course, cannot be true in an environment where traders turn over their portfolios actively. Rather, it is a simplification.

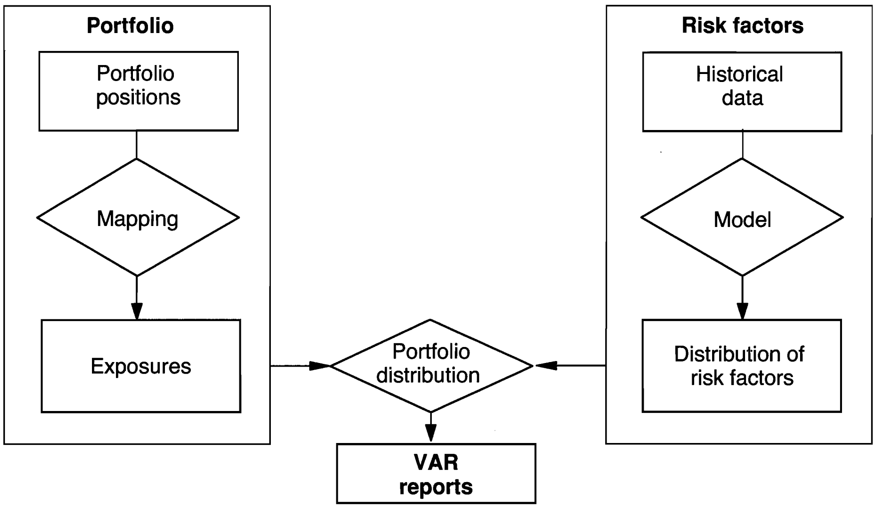

FIGURE 12.1 Components of a Risk System

The true risk can be greater or lower than the VAR measure. It can be greater if VAR is based on close-to-close positions that reflect lower trader limits and if traders take more risks during the day. Conversely, the true risk can be lower if management enforces loss limits, in other words, cuts down the risk that traders can take if losses develop.

12.2.2 Risk Factors

Next comes the choice of the risk factors. In this example of a single position, the main risk factor is obviously the change in the yen/dollar exchange rate. We start by collecting a relevant history of the exchange rate. This is an example where traditional risk models will give useful results because the historical data reveals a lot of movements in the risk factor, which are representative of future risks.

The **risk factors** represent a subset of all market variables that adequately span the risks of the current, or allowed, portfolio. For large portfolios, there are literally tens of thousands of securities available, but a much more restricted set of useful risk factors.

The key is to choose market factors that are adequate for the portfolio. For a simple fixed-income portfolio, one bond market risk factor may be enough. In contrast, for a highly leveraged portfolio, multiple risk factors are needed. For an option portfolio, volatilities should be added as risk factors. In general, the more complex the strategies, the greater the number of risk factors that should be used.

12.2.3 Portfolio Distribution

Finally, information about the portfolio positions and the movements in the risk factors should be combined to build the distribution of portfolio returns.

The next section illustrates different methods. The choice depends on the nature of the portfolio. A simple method may be sufficient for simple portfolios. For a fixed-income portfolio, a linear method may be adequate. In contrast, if the portfolio contains options, we need to include nonlinear effects. For simple, plain-vanilla options, we may be able to approximate their price behavior with a first and second derivative (delta and gamma). For more complex options, such as digital or barrier options, this may not be sufficient.

This is why risk management is as much an art as it is a science. Risk managers need to make reasonable approximations to come up with a cost-efficient measure of risk. They also need to be aware of the fact that traders could be induced to find holes in the risk management system.

Once this risk measurement system is in place, it can also be used to perform stress tests. The risk manager can easily submit the current portfolio to various scenarios, which are simply predefined movements in the risk factors. Therefore, stress tests are simple extensions of VAR systems.

EXAMPLE 12.2: POSITION-BASED RISK MEASURES

The standard VAR calculation for extension to multiple periods also assumes that positions are fixed. If risk management enforces loss limits, the true VAR will be

a. The same
b. Greater than calculated
c. Less than calculated
d. Unable to be determined

12.3 DOWNSIDE RISK MEASURES

12.3.1 VAR: Definition

VAR appeared as a risk measure in 1993, after its endorsement by the Group of Thirty (G-30).[2] The methodology behind VAR, however, is not new.

VAR is a summary measure of downside risk expressed in dollars, or in the reference currency. A general definition is:

VAR is the maximum loss over a target horizon such that there is a low, prespecified probability that the actual loss will be larger.

12.3.2 VAR: Historical Simulation

Let us go back to our position of $4 billion short the yen. To measure its risk, we could use 10 years of historical daily data on the yen/dollar rate, say from 2000

[2] The G-30 is a private, nonprofit association, consisting of senior representatives of the private and public sector and of academia. In the wake of the derivatives disasters of the early 1990s, the G-30 issued a report that has become a milestone document for risk management. Group of Thirty, *Derivatives: Practices and Principles* (New York: Group of Thirty, 1993).

through 2009. We then simulate a daily return in dollars as

$$R_t(\$) = Q_0(\$)[S_t - S_{t-1}]/S_{t-1} \tag{12.6}$$

where Q_0 is the current dollar value of the position and S is the spot rate in yen per dollar measured over two consecutive days.

For instance, for two hypothetical days $S_1 = 112.0$ and $S_2 = 111.8$. The simulated return is

$$R_2(\$) = \$4{,}000 \text{ million} \times [111.8 - 112.0]/112.0 = -\$7.2 \text{ million}$$

Repeating this operation over the entire sample, or 2,527 trading days, creates a time series of fictitious returns, which is plotted in Figure 12.2. The method is called **historical simulation** because it simulates the current portfolio using the recent history.

We can now construct a frequency distribution of daily returns. This is based on ordered losses from worst to best. For instance, there are two losses below \$150 million, eight losses between \$150 million and \$100 million, and so on. The histogram, or frequency distribution, is graphed in Figure 12.2.

We now wish to summarize the distribution by one number. We could describe the quantile, that is, the level of loss that will not be exceeded at some high **confidence level**. Select, for instance, this confidence level as $c = 95\%$. This corresponds to a **right-tail probability**. We could as well define VAR in terms of a **left-tail probability**, which we write as $p = 1 - c$.

Define x as the dollar profit or loss. VAR is typically reported as a positive number, even if it is a loss. It is defined implicitly by

$$c = \int_{-\text{VAR}}^{\infty} f(x)dx \tag{12.7}$$

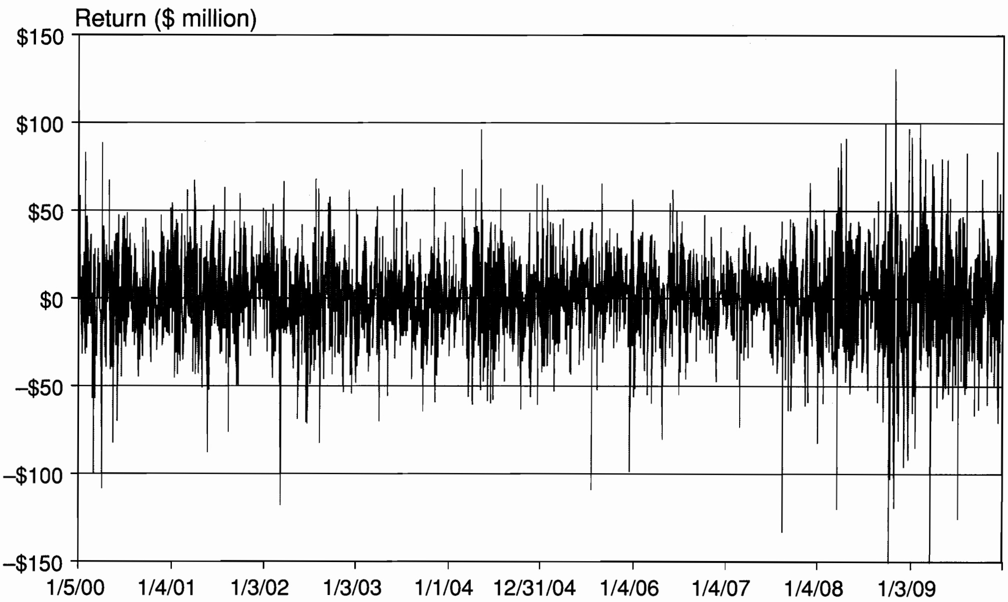

FIGURE 12.2 Simulated Daily Returns

When the outcomes are discrete, VAR is the smallest loss such that the right-tail probability is at least c.

Sometimes, VAR is reported as the deviation between the mean and the quantile. This second definition is more consistent than the usual one. Because it considers the deviation between two values on the target date, it takes into account the time value of money. In most applications, however, the time horizon is very short, in which case the average return on financial series is close to zero. As a result, the two definitions usually give similar values.

In this hedge fund example, we want to find the cutoff value $R^* > 0$ such that the probability of a loss worse than $-R^*$ is $p = 1 - c = 5\%$. With a total of $T = 2{,}527$ observations, this corresponds to a total of $pT = 0.05 \times 2{,}527 = 126$ observations in the left tail. We pick from the ordered distribution the cutoff value, which is $R^* = \$42$ million. We can now make a statement such as: The maximum loss over one day is about \$42 million at the 95% confidence level. This describes risk in a way that notional amounts or exposures cannot convey.

12.3.3 VAR: Parametric

Another approach to VAR measurement is to assume that the distribution of returns belongs to a particular density function, such as the normal distribution. Other distributions are possible, however. The dispersion parameter is measured by the usual standard deviation (SD), defined as

$$\mathrm{SD}(X) = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} [x_i - E(X)]^2} \tag{12.8}$$

The advantage of this measure is that it takes into account all observations, not just the few around the quantile. Any large negative value, for example, will affect the computation of the variance, increasing $\mathrm{SD}(X)$. If we are willing to take a stand on the shape of the distribution, say normal or Student's t, we do know that the standard deviation is the most efficient measure of dispersion. For example, for our yen position, this value is $\mathrm{SD} = \$26.8$ million.

We can translate this standard deviation into a VAR measure, using a multiplier $\alpha(c)$ that depends on the distribution and the selected confidence level c:

$$\mathrm{VAR} = \alpha\sigma W \tag{12.9}$$

where σ is the volatility of the rate of return, which is unitless, and W is the amount invested, measured in the reference currency. Here $\mathrm{SD} = \sigma W$, which is in dollars.

With a normal distribution and $c = 95\%$, we have $\alpha = 1.645$. This gives a VAR estimate of $1.645 \times 26.8 = \$44$ million, which is not far from the empirical quantile of \$42 million.

Note that Equation (12.9) measures VAR relative to the mean, because the standard deviation is a measure of dispersion around the mean. If it is important to measure the loss relative to the initial value, VAR is then

$$\mathrm{VAR} = (\alpha\sigma - \mu)W \tag{12.10}$$

where μ is the expected rate of return over the horizon. In this case, the mean is very small, at $-\$0.1$ million, which hardly affects VAR.

The disadvantage of the standard deviation is that it is symmetrical and cannot distinguish between large losses or gains. Also, computing VAR from SD requires a distributional assumption, which may not be valid.

Using the standard deviation to compute VAR is an example of the **parametric approach** (because it relies on a distribution with parameters). In the previous section, VAR was computed from the empirical distribution, which is an example of a **nonparametric approach.**

12.3.4 VAR: Monte Carlo

Finally, a third approach to risk measurement is to simulate returns using Monte Carlo simulations. This involves assuming a particular density for the distribution of risk factors and then drawing random samples from these distributions to generate returns on the portfolio.

EXAMPLE 12.3: FRM EXAM 2005—QUESTION 43

The 10-Q report of ABC Bank states that the monthly VAR of ABC Bank is USD 10 million at the 95% confidence level. What is the proper interpretation of this statement?

- **a.** If we collect 100 monthly gain/loss data of ABC Bank, we will always see five months with losses larger than \$10 million.
- **b.** There is a 95% probability that the bank will lose less than \$10 million over a month.
- **c.** There is a 5% probability that the bank will gain less than \$10 million each month.
- **d.** There is a 5% probability that the bank will lose less than \$10 million over a month.

12.3.5 VAR: Caveats

VAR is a useful summary measure of risk but is subject to caveats:

- *VAR does not describe the worst possible loss.* This is not what VAR is designed to measure. Indeed, we would expect the VAR number to be exceeded

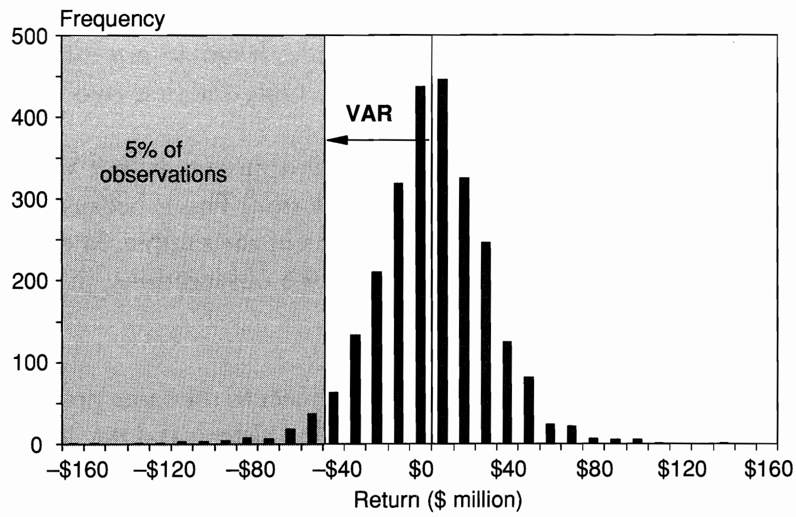

FIGURE 12.3 Distribution of Daily Returns

with a frequency of p, that is five days out of a hundred for a 95% confidence level. This is perfectly normal. In fact, backtesting procedures are designed to check whether the frequency of exceedences is in line with p. Backtesting will be covered in Chapter 16.

- *VAR does not describe the losses in the left tail.* VAR does not say anything about the distribution of losses in its left tail. It just indicates the probability of such a value occurring. For the same VAR number, however, we can have very different distribution shapes. In the case of Figure 12.3, the average value of the losses worse than $42 million is around $63 million, which is 50% worse than the VAR. So, it would be unusual to sustain many losses beyond $200 million.

  Other distributions are possible, however, while maintaining the same VAR. Figure 12.4 illustrates a distribution with 125 occurrences of large losses of $160 million. Because there is still one observation left just below $42 million, VAR is unchanged at $42 million. Yet this distribution implies a high probability of sustaining very large losses, unlike the original one.

  This can create other strange results. For instance, one can construct examples, albeit stretched, where the VAR of a portfolio is greater than the sum of the VARs for its components. In this case, the risk measure is said to fail the *subadditivity* property. The standard deviation, however, is subadditive: The SD of a portfolio must be smaller than, or at worst equal to, the sum of the SDs of subportfolios. As a result, VAR computed from the standard deviation is subadditive as well.

- *VAR is measured with some error.* The VAR number itself is subject to normal sampling variation. In our example, we used 10 years of daily data. Another sample period, or a period of different length, will lead to a different VAR number. Different statistical methodologies or simplifications can also lead to different VAR numbers. One can experiment with sample periods and methodologies to get a sense of the precision in VAR. Hence, it is useful to remember that there is limited precision in VAR numbers. What matters is the first-order magnitude. It would not make sense to report VAR as \$41.989 million, for example. Only the first two digits are meaningful in this case.

An advantage of the parametric approach is that VAR is more precisely estimated than with historical simulation. This is because the standard deviation estimator uses all the observations in the sample, in contrast with the sample quantile, which uses just one or two observations, in addition to the count in the left tail.

In addition, VAR measures are subject to the same problems that affect all risk measures based on a window of recent historical data. Ideally, the past window should reflect the range of future outcomes. If not, all risk measures based on recent historical data may be misleading.

12.3.6 Alternative Measures of Risk

The conventional VAR measure is the *quantile* of the distribution measured in dollars. This single number is a convenient summary, but its very simplicity can be dangerous. We see in Figure 12.4 that the same VAR can hide very different distribution patterns. Chapter 15 reviews desirable properties for risk measures and shows that VAR can display undesirable properties under some conditions. In particular, the VAR of a portfolio can be greater than the sum of subportfolio VARs. If so, merging portfolios can increase risk, which is an unexpected result. Alternative measures of risk are described next.

The Conditional VAR A related concept is the expected value of the loss when it exceeds VAR. This measures the average of the loss conditional on the fact that

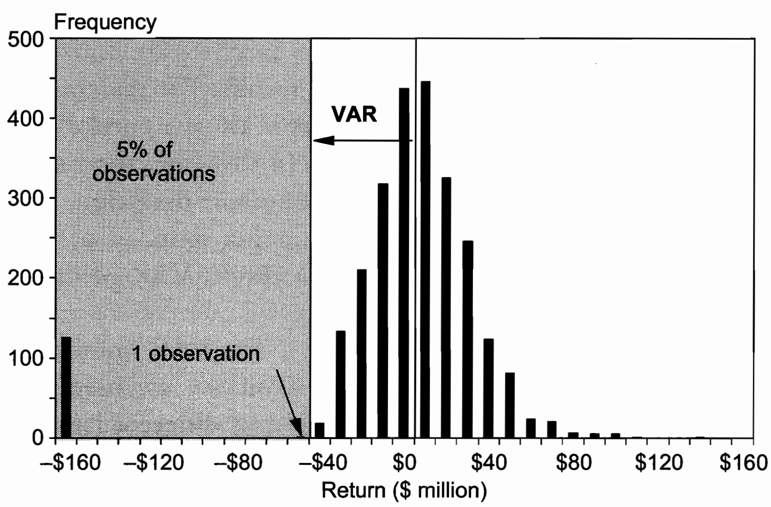

FIGURE 12.4 Altered Distribution with Same VAR

it is greater than VAR. Define the VAR number as $-q$. Formally, the **conditional VAR** (CVAR) is the negative of

$$E[X \mid X < q] = \int_{-\infty}^{q} x f(x) dx \Big/ \int_{-\infty}^{q} f(x) dx \tag{12.11}$$

Note that the denominator represents the probability of a loss exceeding VAR, which is also $p = 1 - c$. This ratio is also called **expected shortfall, tail conditional expectation, conditional loss**, or **expected tail loss.** CVAR indicates the potential loss if the portfolio is "hit" beyond VAR. Because CVAR is an average of the tail loss, one can show that it qualifies as a *subadditive* risk measure. For our yen position, the average loss beyond the \$42 million VAR is CVAR = \$63 million.

The Semistandard Deviation This is a simple extension of the usual standard deviation that considers only data points that represent a loss. Define N_L as the number of such points. The measure is

$$\mathrm{SD}_L(X) = \sqrt{\frac{1}{(N_L)} \sum_{i=1}^{N} [\mathrm{Min}(x_i, 0)]^2} \tag{12.12}$$

The advantage of this measure is that it accounts for asymmetries in the distribution (e.g., negative skewness, which is especially dangerous). The semistandard deviation is sometimes used to report downside risk, but is much less intuitive and less popular than VAR.

The Drawdown **Drawdown** is the decline from peak over a fixed time interval. Define x^{MAX} as the local maximum over this period $[0, T]$, which occurs at time $t_{MAX} \in [0, T]$. Relative to this value, the drawdown at time t is

$$\mathrm{DD}(X) = \frac{(x^{MAX} - x_t)}{x^{MAX}} \tag{12.13}$$

The maximum drawdown is the largest such value over the period, or decline from peak to trough (local maximum to local minimum).

This measure is useful if returns are not independent from period to period. When a market trends, for example, the cumulative loss over a longer period is greater than the loss extrapolated from a shorter period. Alternatively, drawdowns are useful measures of risk if the portfolio is actively managed. A portfolio insurance program, for example, should have lower drawdowns relative to a fixed position in the risky asset because it cuts the position as losses accumulate.

The disadvantage of this measure is that it is backward-looking. It cannot be constructed from the current position, as in the case of VAR. In addition, the maximum drawdown corresponds to different time intervals (i.e., $t_{MAX} - t_{MIN}$). As a result, maximum drawdown measures are not directly comparable across portfolios, in contrast with VAR or the standard deviation, which are defined over a fixed horizon or in annual terms.

EXAMPLE 12.4: FRM EXAM 2003—QUESTION 5

Given the following 30 ordered percentage returns of an asset, calculate the VAR and expected shortfall at a 90% confidence level: $-16, -14, -10, -7, -7, -5, -4, -4, -4, -3, -1, -1, 0, 0, 0, 1, 2, 2, 4, 6, 7, 8, 9, 11, 12, 12, 14, 18, 21, 23$.

a. VAR (90%) = 10, expected shortfall = 14
b. VAR (90%) = 10, expected shortfall = 15
c. VAR (90%) = 14, expected shortfall = 15
d. VAR (90%) = 18, expected shortfall = 22

EXAMPLE 12.5: FRM EXAM 2009—QUESTION 4-4

Worse-than-VAR scenarios are defined as scenarios that lead to losses in the extreme left tail of the return distribution equal to or exceeding VAR at a given level of confidence. Which of the following statements is an accurate description of VAR?

a. VAR is the average of the worse-than-VAR scenario returns.
b. VAR is the standard deviation of the worse-than-VAR scenario returns.
c. VAR is the most pessimistic scenario return (maximum loss) from the worse-than-VAR scenarios.
d. VAR is the most optimistic scenario return (minimum loss) from the worse-than-VAR scenarios.

12.4 VAR PARAMETERS

To measure VAR, we first need to define two quantitative parameters: the confidence level and the horizon.

12.4.1 Confidence Level

The higher the confidence level c, the greater the VAR measure. Varying the confidence level provides useful information about the return distribution and potential extreme losses. It is not clear, however, whether one should stop at 99%, 99.9%, 99.99%, or higher. Each of these values will create an increasingly larger loss, but a loss that is increasingly less likely.

Another problem is that as c increases, the number of occurrences below VAR shrinks, leading to poor measures of high quantiles. With 1,000 observations, for example, VAR can be taken as the 10th lowest observation for a 99% confidence level. If the confidence level increases to 99.9%, VAR is taken from the lowest observation only. Finally, there is no simple way to estimate a 99.99% VAR from this sample because it has too few observations.

The choice of the confidence level depends on the use of VAR. For most applications, VAR is simply a benchmark measure of downside risk. If so, what really matters is *consistency* of the VAR confidence level across trading desks or time.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a high confidence level is advisable. Obviously, institutions would prefer to go bankrupt very infrequently. This **capital adequacy** use, however, applies to the overall institution and not to trading desks.

Another important point is that VAR models are useful only insofar as they can be verified. This is the purpose of backtesting, which systematically checks whether the frequency of losses exceeding VAR is in line with $p = 1 - c$. For this purpose, the risk manager should choose a value of c that is not too high. Picking, for instance, $c = 99.99\%$ should lead, on average, to one exceedence out of 10,000 trading days, or 40 years. In other words, it is going to be impossible to verify if the true probability associated with VAR is indeed 99.99%. For all these reasons, the usual recommendation is to pick a confidence level that is not too high, such as 95% to 99%.

12.4.2 Horizon

The longer the horizon T, the greater the VAR measure. This extrapolation is driven by two factors: the behavior of the risk factors and the portfolio positions.

To extrapolate from a one-day horizon to a longer horizon, we need to assume that returns are independent and identically distributed (i.i.d.). If so, the daily volatility can be transformed into a multiple-day volatility by multiplication by the square root of time. We also need to assume that the distribution of daily returns is unchanged for longer horizons, which restricts the class of distribution to the so-called stable family, of which the normal is a member. If so, we have

$$\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T} \tag{12.14}$$

This requires (1) the distribution to be invariant to the horizon (i.e., the same α as for the normal), (2) the distribution to be the same for various horizons (i.e., no time decay in variances), and (3) innovations to be independent across days.

KEY CONCEPT

VAR can be extended from a one-day horizon to T days by multiplication by the square root of time. This adjustment is valid with independent and identically distributed (i.i.d.) returns that have a normal distribution.

The choice of the horizon also depends on the characteristics of the portfolio. If the positions change quickly, or if exposures (e.g., option deltas) change as underlying prices change, increasing the horizon will create slippage in the VAR measure.

Again, the choice of the horizon depends on the use of VAR. If the purpose is to provide an accurate benchmark measure of downside risk, the horizon should be relatively short, ideally less than the average period for major portfolio rebalancing.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a long horizon is advisable. For **capital adequacy** purposes, institutions will want to have enough time for corrective action as problems start to develop. The VAR horizon should also be long enough to allow for orderly liquidation of the positions. In other words, less liquid assets should be evaluated with a longer horizon.

In practice, the horizon cannot be less than the frequency of reporting of profits and losses (P&L). Typically, banks measure P&L on a daily basis, and corporates on a longer interval (ranging from daily to monthly). This interval is the minimum horizon for VAR.

Another criterion relates to the need for backtesting. Shorter time intervals create more data points that can be used to match VAR with the subsequent P&L. For statistical tests, having more data points means that the tests will be more powerful, or more likely to identify problems in the VAR model. So, for the purpose of backtesting, it is advisable to have a horizon as short as possible.

For all these reasons, the usual recommendation is to pick a horizon that is as short as feasible for trading desks, for instance one day. For institutions such as pension funds, for instance, a one-month horizon may be more appropriate.

In summary, the choice of the confidence level and horizon depend on the intended use for the risk measures. For backtesting purposes, we should select a low confidence level and a short horizon. For capital adequacy purposes, a high confidence level and a long horizon are required. In practice, these conflicting objectives can be accommodated by a more complex rule, as is the case for the Basel market risk charge.

12.4.3 Application: The Basel Rules

An important use of risk models is for capital adequacy purposes. The Basel Committee on Banking Supervision has laid out minimum capital requirements for commercial banks to cover the market risk of their trading portfolios. The rules define a **Market Risk Charge** (MRC) that is based on the bank's internal VAR measures. The original rules, as laid out in 1996, require the following parameters:

- A horizon of 10 trading days, or two calendar weeks
- A 99% confidence interval

- An observation period based on at least a year of historical data and updated at least once a quarter

Under the **Internal Models Approach** (IMA) as defined in 1996, the MRC includes a **general market risk charge** (GMRC) plus other components:

$$\text{GMRC}_t = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60}\text{VAR}_{t-i},\ \text{VAR}_{t-1}\right) \tag{12.15}$$

The GMRC involves the average of the trading VAR over the last 60 days, times a supervisor-determined multiplier k (with a minimum value of 3), as well as yesterday's VAR. The Basel Committee allows the 10-day VAR to be obtained from an extrapolation of one-day VAR figures. Thus VAR is really

$$\text{VAR}_t(10, 99\%) = \sqrt{10} \times \text{VAR}_t(1, 99\%)$$

Presumably, the 10-day period corresponds to the time required for corrective action by bank regulators, should an institution start to run into trouble. Presumably as well, the 99% confidence level corresponds to a low probability of bank failure due to market risk. Even so, one occurrence every 100 periods implies a high frequency of failure. There are $52/2 = 26$ two-week periods in one year. Thus, one failure should be expected to happen every $100/26 = 3.8$ years, which is still much too frequent. This explains why the Basel Committee has applied a multiplier factor, $k \geq 3$, to guarantee further safety. In addition, this factor is supposed to protect against fat tails, unstable parameters, changing positions, and, more generally, model risk.

In 2009, the rules were revised to require updating at least every month. In addition, the GMRC was expanded to include a stressed VAR measure, which is explained in Chapter 28.

EXAMPLE 12.6: FRM EXAM 2008—QUESTION 2-2

Assume that the P&L distribution of a liquid asset is i.i.d. normally distributed. The position has a one-day VAR at the 95% confidence level of $100,000. Estimate the 10-day VAR of the same position at the 99% confidence level.

a. $1,000,000
b. $450,000
c. $320,000
d. $220,000

EXAMPLE 12.7: FRM EXAM 2009—QUESTION 4-3

Assume that portfolio daily returns are independent and identically normally distributed. Sam Neil, a new quantitative analyst, has been asked by the portfolio manager to calculate portfolio VARs over 10, 15, 20, and 25 days. The portfolio manager notices something amiss with Sam's calculations, displayed here. Which one of the following VARs on this portfolio is inconsistent with the others?

a. VAR(10-day) = USD 316M
b. VAR(15-day) = USD 465M
c. VAR(20-day) = USD 537M
d. VAR(25-day) = USD 600M

EXAMPLE 12.8: MARKET RISK CHARGE

The 95%, one-day RiskMetrics VAR for a bank trading portfolio is \$1,000,000. What is the approximate general market risk charge, as defined in 1996?

a. \$3,000,000
b. \$9,500,000
c. \$4,200,000
d. \$13,400,000

12.5 STRESS-TESTING

12.5.1 Limitations of VAR Measures

We have seen in a previous section that VAR measures have inherent limitations. In addition, the traditional application of historical simulation creates special problems due to the choice of the **moving window**. This typically uses one to three years of historical data.

During the credit crisis that started in 2007, risk management systems failed at many banks. Some banks suffered losses that were much more frequent and much worse than they had anticipated. In 2007 alone, for example, UBS suffered 29 exceptions, or losses worse than VAR, instead of the expected number of two or three (i.e., 1% of 250 days).

This was in part due to the fact that 2007 followed an extended period of stability. Figure 12.5, for example, plots the daily volatility forecast for the S&P 500 stock index using an exponentially weighted moving average (EWMA) with

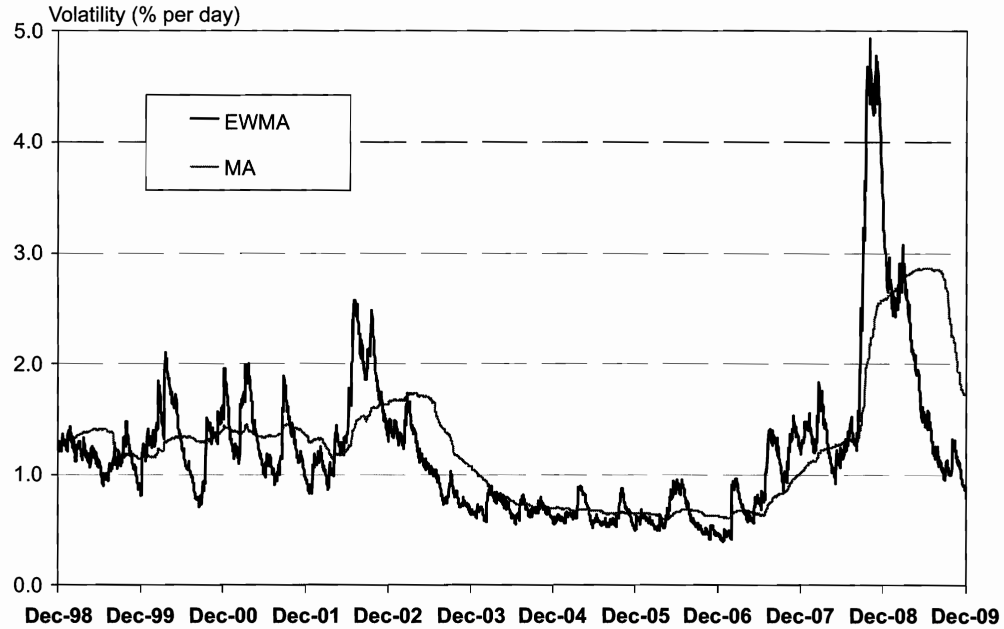

FIGURE 12.5 Volatility of S&P 500 Stock Index

decay of 0.94. This model shows that during 2004 to 2006, the volatility was very low, averaging 0.7% daily. As a result, many financial institutions entered 2007 with high levels of leverage.

Banks, however, do not model their risk using this EWMA forecast. Typically, they use a moving window with equal weight on each day, which is essentially a moving average (MA) model. Therefore, the graph also shows a volatility forecast coming from an MA model. The figure shows that the MA model systematically underestimated the EWMA volatility starting in mid-2007, which explains the high number of exceptions. The lesson from this episode is that relying on recent data may not be sufficient to assess risks. This is why traditional VAR models must be complemented by stress tests.

12.5.2 Principles of Stress Tests

VAR should be complemented by **stress-testing**, which aims at identifying situations that could create extraordinary losses but plausible losses. One drawback of stress tests is that they are more subjective than VAR measures. A VAR number reflects realized risk. An extreme scenario, by contrast, may be more difficult to accept by senior management if it does not reflect an actual observation and if it looks too excessive.

In the case of our hedge fund with a $4 billion position short the yen, we have seen that the daily VAR at the 95 percent level of confidence is on the order of $42 million. In addition, it would be informative to see how worse the loss could be. Going back over the last twenty years, for example, the worst movement in the exchange rate was a loss of −5.4% on October 7, 1998. This leads to a stress loss of $215 million. Such a loss is plausible.

Stress-testing is a key risk management process, which includes (1) scenario analysis; (2) stressing models, volatilities, and correlations; and (3) developing policy responses. **Scenario analysis** submits the portfolio to large movements in financial market variables. These scenarios can be created using a number of methods.

- *Moving key variables one at a time*, which is a simple and intuitive method. Unfortunately, it is difficult to assess realistic comovements in financial variables. It is unlikely that all variables will move in the worst possible direction at the same time.
- *Using historical scenarios*, for instance the 1987 stock market crash, the devaluation of the British pound in 1992, the bond market debacle of 1984, the Lehman bankruptcy, and so on.
- *Creating prospective scenarios*, for instance working through the effects, direct and indirect, of a U.S. stock market crash. Ideally, the scenario should be tailored to the portfolio at hand, assessing the worst thing that could happen to current positions.
- *Reverse stress tests* start from assuming a large loss and then explore the conditions that would lead to this loss. This type of analysis forces institutions to think of other scenarios and to address issues not normally covered in regular stress tests, such as financial contagion.

Stress-testing is useful to guard against **event risk**, which is the risk of loss due to an observable political or economic event. The problem (from the viewpoint of stress-testing) is that such events are relatively rare and may be difficult to anticipate. These include:

- *Changes in governments* leading to changes in economic policies
- *Changes in economic policies*, such as default, capital controls, inconvertibility, changes in tax laws, expropriations, and so on
- *Coups, civil wars, invasions*, or other signs of political instability
- *Currency devaluations*, which are usually accompanied by other drastic changes in market variables

Even so, designing stress tests is not an easy matter. Recent years have demonstrated that markets seem to be systematically taken by surprise. Few people seem to have anticipated the Russian default, for instance. The Argentinian default of 2001 was also unique in many respects.

Example: Turmoil in Argentina

Argentina is a good example of political risk in emerging markets. Up to 2001, the Argentine peso was fixed to the U.S. dollar at a one-to-one exchange rate. The government had promised it would defend the currency at all costs. Argentina, however, suffered from the worst economic crisis in decades, compounded by the cost of excessive borrowing.

In December 2001, Argentina announced it would stop paying interest on its \$135 billion foreign debt. This was the largest sovereign default recorded so far. Economy Minister Cavallo also announced sweeping restrictions on withdrawals from bank deposits to avoid capital flight. On December 20, President Fernando de la Rua resigned after 25 people died in street protests and rioting. President Duhalde took office on January 2 and devalued the currency on January 6. The exchange rate promptly moved from 1 peso/dollar to more than 3 pesos.

Such moves could have been factored into risk management systems by scenario analysis. What was totally unexpected, however, was the government's announcement that it would treat bank loans and deposits differentially. Dollar-denominated bank deposits were converted into devalued pesos, but dollar-denominated bank loans were converted into pesos at a one-to-one rate. This mismatch rendered much of the banking system technically insolvent, because loans (bank assets) overnight became less valuable than deposits (bank liabilities). Whereas risk managers had contemplated the market risk effect of a devaluation, few had considered this possibility of such political actions.

By 2005, the Argentinian government proposed to pay back about 30% of the face value of its debt. This recovery rate was very low by historical standards.

The goal of stress-testing is to identify areas of potential vulnerability. This is not to say that the institution should be totally protected against every possible contingency, as this would make it impossible to take any risk. Rather, the objective of stress-testing and management response should be to ensure that the institution can withstand likely scenarios without going bankrupt. Stress-testing can be easily implemented once the VAR structure is in place. In Figure 12.1, all that is needed is to enter the scenario values into the risk factor inputs.

EXAMPLE 12.9: FRM EXAM 2008—QUESTION 2-29

Which of the following statements about stress testing are *true*?

- **I.** Stress testing can complement VAR estimation in helping risk managers identify crucial vulnerabilities in a portfolio.
- **II.** Stress testing allows users to include scenarios that did not occur in the lookback horizon of the VAR data but are nonetheless possible.
- **III.** A drawback of stress testing is that it is highly subjective.
- **IV.** The inclusion of a large number of scenarios helps management better understand the risk exposure of a portfolio.

- **a.** I and II only.
- **b.** III and IV only.
- **c.** I, II, and III only.
- **d.** I, II, III, and IV.

EXAMPLE 12.10: FRM EXAM 2006—QUESTION 87

Which of the following is true about stress testing?

a. It is used to evaluate the potential impact on portfolio values of unlikely, although plausible, events or movements in a set of financial variables.
b. It is a risk management tool that directly compares predicted results to observed actual results. Predicted values are also compared with historical data.
c. Both a. and b. are true.
d. None of the above are true.

EXAMPLE 12.11: FRM EXAM 2008—QUESTION 2-18

John Flag, the manager of a \$150 million distressed bond portfolio, conducts stress tests on the portfolio. The portfolio's annualized return is 12%, with an annualized return volatility of 25%. In the past two years, the portfolio encountered several days when the daily value change of the portfolio was more than 3 standard deviations. If the portfolio would suffer a 4-sigma daily event, estimate the change in the value of this portfolio.

a. \$9.48 million
b. \$23.70 million
c. \$37.50 million
d. \$150 million

12.6 VAR: LOCAL VERSUS FULL VALUATION

This section turns to a general classification of risk models into local valuation and full valuation methods. **Local valuation methods** make use of the valuation of the instruments at the current point, along with the first and perhaps the second partial derivatives. **Full valuation methods**, in contrast, reprice the instruments over a broad range of values for the risk factors.

The various approaches to VAR are described in Figure 12.6. The left branch describes local valuation methods, also known as **analytical methods**. These include linear models and nonlinear models. Linear models are based on the covariance matrix approach. The covariance matrix can be simplified using factor models, or even a diagonal model, which is a one-factor model.

Nonlinear models take into account the first and second partial derivatives. The latter are called gamma or convexity. Next, the right branch describes full valuation methods and includes historical or Monte Carlo simulations.

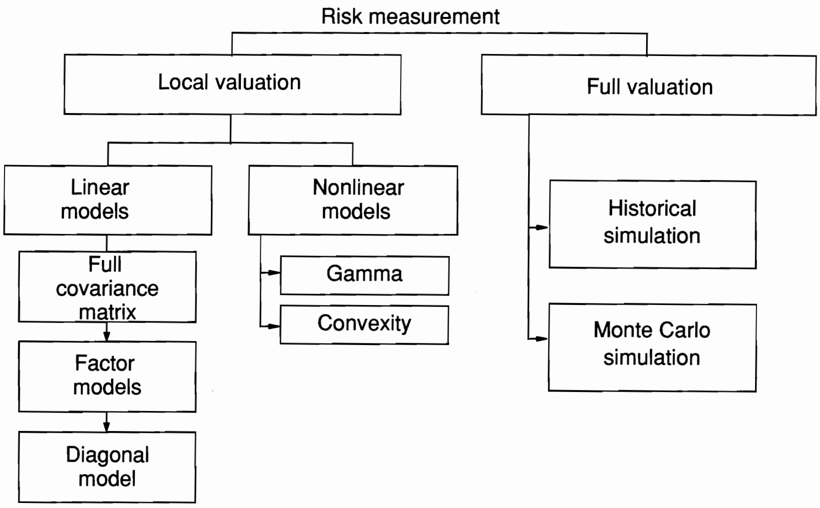

FIGURE 12.6 VAR Methods

12.6.1 Local Valuation

VAR was born from the recognition that we need an estimate that accounts for various sources of risk and expresses loss in terms of probability. Extending the duration equation to the worst change in yield at some confidence level dy, we have

$$(\text{Worst } dP) = (-D^* P) \times (\text{Worst } dy) \tag{12.16}$$

where D^* is modified duration. For a long position in the bond, the worst movement in yield is an increase at, say, the 95% confidence level. This will lead to a fall in the bond value at the same confidence level. We call this approach **local valuation**, because it uses information about the initial price and the exposure at the initial point. As a result, the VAR for the bond is given by

$$\text{VAR}(dP) = \mid -D^* P \mid \times \text{VAR}(dy) \tag{12.17}$$

More generally, the **delta-normal** method replaces all positions by their delta exposures and assumes that risk factors have multivariate normal distributions. In this case, VAR is

$$\text{VAR}(df) = \mid \Delta \mid \text{VAR}(dS) \tag{12.18}$$

The main advantage of this approach is its simplicity: The distribution of the price is the same as that of the change in yield. This is particularly convenient for portfolios with numerous sources of risks, because linear combinations of normal distributions are normally distributed. Figure 12.7, for example, shows how the linear exposure combined with the normal density (in the right panel) creates a normal density. This linear model can be extended to an approximation that accounts for quadratic terms, called delta-gamma, which are detailed in Chapter 14.

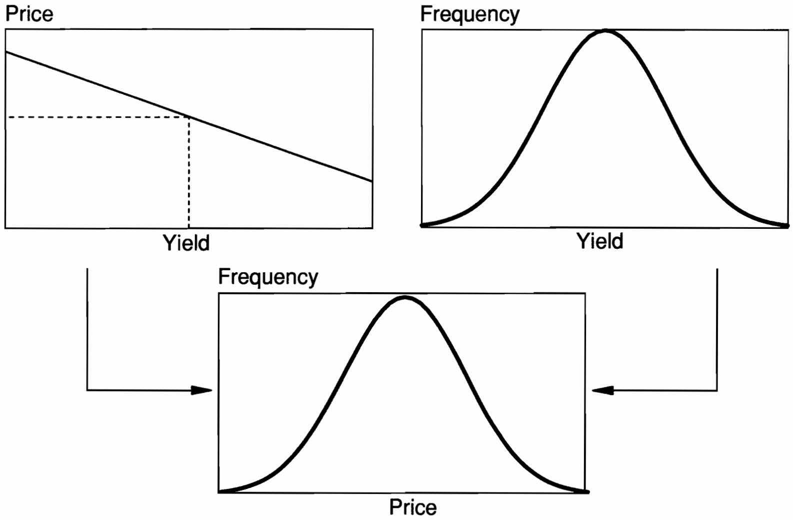

FIGURE 12.7 Distribution with Linear Exposures

12.6.2 Full Valuation

More generally, to take into account nonlinear relationships, one would have to reprice the bond under different scenarios for the yield. Defining y_0 as the initial yield,

$$(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0] \tag{12.19}$$

We call this approach **full valuation**, because it requires repricing the asset.

This approach is illustrated in Figure 12.8, where the nonlinear exposure combined with the normal density creates a distribution that is no longer symmetrical,

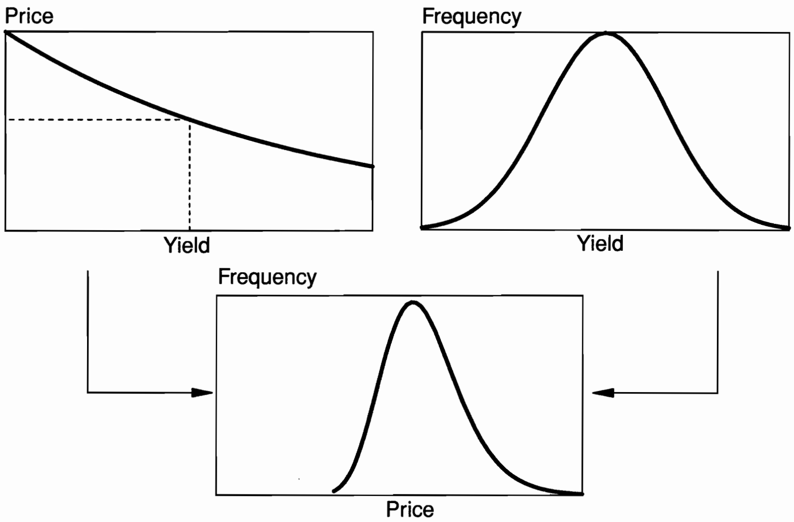

FIGURE 12.8 Distribution with Nonlinear Exposures

but skewed to the right. This is more precise but, unfortunately, more complex than a simple, linear valuation method.

Full valuation methods are needed when the portfolio has options, especially in situations where movements in risk factors are large. This explains why stress tests require full valuation.

EXAMPLE 12.12: FRM EXAM 2004—QUESTION II-60

Which of the following methodologies would be most appropriate for stress-testing your portfolio?

a. Delta-gamma valuation
b. Full revaluation
c. Marking to market
d. Delta-normal VAR

12.7 IMPORTANT FORMULAS

VAR: $c = \int_{-\text{VAR}}^{\infty} f(x)dx$

CVAR: $E[X \mid X < q] = \int_{-\infty}^{q} xf(x)dx \Big/ \int_{-\infty}^{q} f(x)dx$

Drawdown: $\text{DD}(X) = \frac{(x^{MAX} - x_t)}{x^{MAX}}$

Square root of time adjustment: $\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T}$

Market Risk Charge: $\text{GMRC}_t^{IMA} = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60} \text{VAR}_{t-i},\ \text{VAR}_{t-1}\right)$

Linear VAR, fixed-income: $\text{VAR}(dP) = \mid -D^* P \mid \times \text{VAR}(dy)$

Full-valuation VAR, fixed-income: $(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0]$

Delta VAR: $\text{VAR}(df) = \mid \Delta \mid \text{VAR}(dS)$

12.8 ANSWERS TO CHAPTER EXAMPLES

Example 12.1: FRM Exam 2005—Question 32

a. Stop loss limits cut down the positions after a loss is incurred, which is useful if markets are trending. Exposure limits do not allow for diversification, because correlations are not considered. VAR limits can be arbitraged, especially with weak VAR models. Finally, stop loss limits are put in place after losses are incurred, so cannot prevent all losses. As a result, statements I. and II. are correct.

Example 12.2: Position-Based Risk Measures

c. The true VAR will be less than calculated. Loss limits cut down the positions as losses accumulate. This is similar to a long position in an option, where the delta increases as the price increases, and vice versa. Long positions in options have shortened left tails, and hence involve less risk than unprotected positions.

Example 12.3: FRM Exam 2005—Question 43

b. VAR is the worst loss such that there is a 95% probability that the losses will be less severe. Alternatively, there is a 5% probability that the losses will be worse. So b. is correct. Answer d. says "lose less" and therefore is incorrect.

Example 12.4: FRM Exam 2003—Question 5

b. The 10% lower cutoff point is the third lowest observation, which is VAR $= 10$. The expected shortfall is then the average of the observations in the tails, which is 15.

Example 12.5: FRM Exam 2009—Question 4-4

d. CVAR is the average of losses worse than VAR, so answer a. is incorrect. Expressed in absolute value, VAR is lower than any other losses used for CVAR, so VAR must be the most optimistic loss.

Example 12.6: FRM Exam 2008—Question 2-2

b. We need to scale the VAR to a 99% level using $\$100,000 \times 2.326/1.645 = \$141,398$. Multiplying by $\sqrt{10}$ then gives \$447,140.

Example 12.7: FRM Exam 2009—Question 4-3

a. We compute the daily VAR by dividing each VAR by the square root of time. This gives $316/\sqrt{10} = 100$, then 120, 120, and 120. So, answer a. is out of line.

Example 12.8: Market Risk Charge

d. First, we have to convert the 95% VAR to a 99% measure, assuming a normal distribution in the absence of other information. The GMRC is then $3 \times VAR \times \sqrt{10} = 3 \times \$1,000,000(2.33)/1.65) \times \sqrt{10} = \$13,396,000$.

Example 12.9: FRM Exam 2008—Question 2-29

c. All the statements are correct except IV., because too many scenarios will make it more difficult to interpret the risk exposure.

Example 12.10: FRM Exam 2006—Question 87

a. Stress testing is indeed used to evaluate the effect of extreme events. Answer b. is about backtesting, not stress-testing.

Example 12.11: FRM Exam 2008—Question 2-18

a. First, we transform the volatility into a daily measure, which is $25\%/\sqrt{252} = 1.57\%$. Multiplying, we get $150 \times 1.57\% \times 4 = \$9.45$.

Example 12.12: FRM Exam 2004—Question II-60

b. By definition, stress-testing involves large movements in the risk factors. This requires a full revaluation of the portfolio.

CHAPTER 13

Managing Linear Risk

Risk that has been measured can be managed. This chapter turns to the active management of market risks. An important aspect of managing risk is **hedging**, which consists of taking positions that lower the risk profile of the portfolio.

Techniques for hedging have been developed in the futures markets, where farmers, for instance, use financial instruments to hedge the price risk of their products. In this case, the objective is to find the optimal position in a futures contract that minimizes the volatility of the total portfolio. This portfolio consists of two positions, a fixed inventory exposed to a risk factor and a hedging instrument.

In this chapter, we examine hedging in cases where the value of the hedging instrument is linearly related to the underlying risk factor. This involves futures, forwards, and swaps. The next chapter examines risk management using nonlinear instruments (i.e., options).

In general, hedging can create **hedge slippage**, or **basis risk**. Basis risk arises when changes in payoffs on the hedging instrument do not perfectly offset changes in the value of the inventory position. Hedging is effective when basis risk is much less than outright price risk.

This chapter discusses the management of risk with linear instruments. Section 13.1 presents unitary hedging, where the quantity hedged is the same as the quantity protected. Section 13.2 then turns to a general method for finding the optimal hedge ratio. This method is applied in Section 13.3 for hedging bonds and equities.

13.1 UNITARY HEDGING

13.1.1 Futures Hedging

Consider the situation of a U.S. exporter who has been promised a payment of 125 million Japanese yen in seven months. This defines the underlying position, which can be viewed as an anticipated inventory. The perfect hedge would be to enter a seven-month forward contract over-the-counter (OTC). Assume for this illustration that this OTC contract is not convenient. Instead, the exporter decides to turn to an exchange-traded futures contract, which can be bought or sold easily.

FRM Exam Part 1 topic.

TABLE 13.1 Example of a Futures Hedge

Item	Initial Time	Exit Time	Gain or Loss
Market Data:			
Maturity (months)	9	2	
U.S. interest rate	6%	6%	
Yen interest rate	5%	2%	
Spot (¥/$)	¥125.00	¥150.00	
Futures (¥/$)	¥124.07	¥149.00	
Contract Data:			
Spot ($/¥)	0.008000	0.006667	−$166,667
Futures ($/¥)	0.008060	0.006711	$168,621
Basis ($/¥)	−0.000060	−0.000045	$1,954

The Chicago Mercantile Exchange (now CME Group) lists yen contracts with face amount of ¥12,500,000 that expire in nine months. The exporter places an order to sell 10 contracts, with the intention of reversing the position in seven months, when the contract will still have two months to maturity.[1] Because the amount sold is the same as the underlying, this is called a **unitary hedge**.

In this case, the hedge stays in place until the end of the hedging horizon. More generally, we can distinguish between **static hedging**, which consists of putting on, and leaving, a position until the hedging horizon, and **dynamic hedging**, which consists of continuously rebalancing the portfolio to the horizon. Dynamic hedging is associated with options, which we examine in the next chapter.

Table 13.1 describes the initial and final conditions for the contract. At each date, the futures price is determined by interest parity. Suppose that the yen depreciates sharply, or that the dollar goes up from ¥125 to ¥150. This leads to a loss on the anticipated cash position of ¥125,000,000 $\times$ (0.006667 − 0.00800) = −$166,667. This loss, however, is offset by a gain on the futures, which is (−10) $\times$ ¥12,500,000 $\times$ (0.006711 − 0.00806) = $168,621. The net is a small gain of $1,954. Overall, the exporter has been hedged.

This example shows that futures hedging can be quite effective, removing the effect of fluctuations in the risk factor. Define Q as the amount of yen transacted and S and F as the spot and futures rates, indexed by 1 at the initial time and by 2 at the exit time. The P&L on the unhedged transaction is

$$Q[S_2 - S_1] \tag{13.1}$$

[1] In practice, if the liquidity of long-dated contracts is not adequate, the exporter could use nearby contracts and roll them over prior to expiration into the next contracts. When there are multiple exposures, this practice is known as a **stack hedge**. Another type of hedge is the **strip hedge**, which involves hedging the exposures with a number of different contracts. While a stack hedge has superior liquidity, it also entails greater basis risk than a strip hedge. Hedgers must decide whether the greater liquidity of a stack hedge is worth the additional basis risk.

With unit hedging, the total profit is

$$Q[(S_2 - S_1)] - Q[(F_2 - F_1)] = Q[(S_2 - F_2) - (S_1 - F_1)] = Q[b_2 - b_1] \quad (13.2)$$

where $b = S - F$ is the **basis**. The profit depends on only the movement in the basis. Hence the effect of hedging is to transform price risk into basis risk. A short hedge position is said to be *long the basis*, since it benefits from an increase in the basis.

KEY CONCEPT

A short hedge position is long the basis; that is, it benefits when the basis widens or strengthens. This is because the position is short the hedging instrument, which falls in value (relative to the spot price) when the basis widens.

In this case, the basis risk is minimal for a number of reasons. First, the cash and futures correspond to the same asset. Second, the cash-and-carry relationship holds very well for currencies. Third, the remaining maturity at exit is rather short. This is not always the case, however.

13.1.2 Basis Risk

Basis risk arises when the characteristics of the futures contract differ from those of the underlying position. Futures contracts are standardized to a particular grade, say West Texas Intermediate (WTI) for oil futures traded on the New York Mercantile Exchange (NYMEX). This defines the grade of crude oil deliverable against the contract. A hedger, however, may have a position in a different grade, which may not be perfectly correlated with WTI. Thus basis risk is the uncertainty whether the cash-futures spread will widen or narrow during the hedging period. Hedging can be effective, however, if movements in the basis are dominated by movements in cash markets.

For most commodities, basis risk is inevitable. Organized exchanges strive to create enough trading and liquidity in their listed contracts, which requires standardization. Speculators also help to increase trading volumes and provide market liquidity. Thus there is a trade-off between liquidity and basis risk.

Basis risk is higher with **cross-hedging**, which involves using a futures on a totally different asset or commodity than the cash position. For instance, a U.S. exporter who is due to receive a payment in Norwegian kroner (NK) could hedge using a futures contract on the \$/euro exchange rate. Relative to the dollar, the euro and the NK should behave similarly, but there is still some basis risk.

Basis risk is lowest when the underlying position and the futures correspond to the same asset. Even so, some basis risk remains because of differing maturities. As we have seen in the yen hedging example, the maturity of the futures contract is nine months instead of seven months. As a result, the liquidation price of the futures is uncertain.

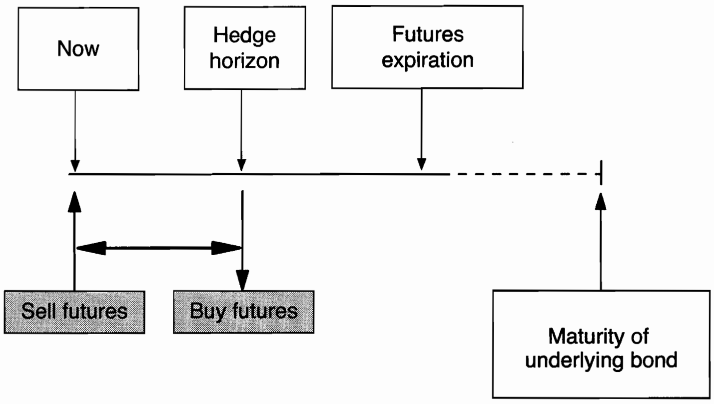

FIGURE 13.1 Hedging Horizon and Contract Maturity

Figure 13.1 describes the various time components for a hedge using T-bond futures. The first component is the *maturity of the underlying bond*, say 20 years. The second component is the *time to futures expiration*, say nine months. The third component is the *hedge horizon*, say seven months. Basis risk occurs when the hedge horizon does not match the time to futures expiration.

EXAMPLE 13.1: FRM EXAM 2000—QUESTION 79

Under which scenario is basis risk likely to exist?

a. A hedge (which was initially matched to the maturity of the underlying) is lifted before expiration.
b. The correlation of the underlying and the hedge vehicle is less than one and their volatilities are unequal.
c. The underlying instrument and the hedge vehicle are dissimilar.
d. All of the above are correct.

EXAMPLE 13.2: FRM EXAM 2009—QUESTION 3-14

Mary has IBM stock and will sell it two months from now at a specified date in the middle of the month. Mary would like to hedge the price of risk of IBM stock. How could she best hedge the IBM stock without incurring basis risk?

a. Short a two-month forward contract on IBM stock
b. Short a three-month futures contract on IBM stock
c. Short a two-month forward contract on the S&P 500 index
d. Answers a. and b. are correct.

EXAMPLE 13.3: FRM EXAM 2009—QUESTION 3-15

Which of the following statements is/are *true* with respect to basis risk?

I. Basis risk arises in cross-hedging strategies, but there is no basis risk when the underlying asset and hedge asset are identical.
II. A short hedge position benefits from unexpected strengthening of basis.
III. A long hedge position benefits from unexpected strengthening of basis.

a. I and II
b. I and III
c. II only
d. III only

EXAMPLE 13.4: FRM EXAM 2007—QUESTION 99

Which of the following trades contain mainly basis risk?

I. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Nov 07 NYMEX WTI Crude Oil contracts
II. Long 1,000 lots Nov 07 ICE Brent Oil contracts and long 2,000 lots Nov 07 ICE Brent Oil at-the-money put
III. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Dec 07 ICE Brent Oil contracts
IV. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Dec 07 NYMEX WTI Crude Oil contracts

a. II and IV only
b. I and III only
c. I, III, and IV only
d. III and IV only

13.2 OPTIMAL HEDGING

The previous section gave an example of a unit hedge, where the quantities transacted are identical in the two markets. In general, this is not appropriate. We have to decide how much of the hedging instrument to transact.

Consider a situation where a portfolio manager has an inventory of carefully selected corporate bonds that should do better than their benchmark. The manager wants to guard against interest rate increases, however, over the next three months.

In this situation, it would be too costly to sell the entire portfolio only to buy it back later. Instead, the manager can implement a temporary hedge using derivative contracts, for instance T-bond futures.

Here, we note that the only risk is **price risk**, as the quantity of the inventory is known. This may not always be the case, however. Farmers, for instance, have uncertainty over both the price and size of their crop. If so, the hedging problem is substantially more complex as it involves hedging *revenues*, which involves analyzing demand and supply conditions.

13.2.1 The Optimal Hedge Ratio

Define ΔS as the change in the dollar value of the inventory and ΔF as the change in the dollar value of one futures contract. The inventory, or position to be hedged, can be existing or **anticipatory**, that is, to be received in the future with a great degree of certainty. The manager is worried about potential movements in the value of the inventory ΔS.

If the manager goes long N futures contracts, the total change in the value of the portfolio is

$$\Delta V = \Delta S + N \Delta F \tag{13.3}$$

One should try to find the hedge that reduces risk to the minimum level. The variance of total profits is equal to

$$\sigma^2_{\Delta V} = \sigma^2_{\Delta S} + N^2 \sigma^2_{\Delta F} + 2N\sigma_{\Delta S, \Delta F} \tag{13.4}$$

Note that volatilities are initially expressed in dollars, not in rates of return, as we attempt to stabilize dollar values.

Taking the derivative with respect to N

$$\frac{\partial \sigma^2_{\Delta V}}{\partial N} = 2N\sigma^2_{\Delta F} + 2\sigma_{\Delta S, \Delta F} \tag{13.5}$$

For simplicity, drop the Δ in the subscripts. Setting Equation (13.5) equal to zero and solving for N, we get

$$N^* = -\frac{\sigma_{\Delta S, \Delta F}}{\sigma^2_{\Delta F}} = -\frac{\sigma_{SF}}{\sigma^2_F} = -\rho_{SF}\frac{\sigma_S}{\sigma_F} \tag{13.6}$$

where σ_{SF} is the covariance between futures and spot price changes. Here, N^* is the **minimum variance hedge ratio.**

In practice, there is often confusion about the definition of the portfolio value and unit prices. Here S consists of the number of units (shares, bonds, bushels, gallons) times the unit price (stock price, bond price, wheat price, fuel price).

It is sometimes easier to deal with unit prices and to express volatilities in terms of *rates of changes in unit prices*, which are unitless. Defining quantities Q

and unit prices s, we have $S = Qs$. Similarly, the notional amount of one futures contract is $F = Q_f f$. We can then write

$$\sigma_{\Delta S} = Q\sigma(\Delta s) = Qs\sigma(\Delta s/s)$$
$$\sigma_{\Delta F} = Q_f\sigma(\Delta f) = Q_f f\sigma(\Delta f/f)$$
$$\sigma_{\Delta S,\Delta F} = \rho_{sf}[Qs\sigma(\Delta s/s)][Q_f f\sigma(\Delta f/f)]$$

Using Equation (13.6), the optimal hedge ratio N^* can also be expressed as

$$N^* = -\rho_{SF}\frac{Qs\sigma(\Delta s/s)}{Q_f f\sigma(\Delta f/f)} = -\rho_{SF}\frac{\sigma(\Delta s/s)}{\sigma(\Delta f/f)}\frac{Qs}{Q_f f} = -\beta_{sf}\frac{Q\times s}{Q_f\times f} \tag{13.7}$$

where β_{sf} is the coefficient in the regression of $\Delta s/s$ over $\Delta f/f$. The second term represents an adjustment factor for the size of the cash position and of the futures contract.

The optimal amount N^* can be derived from the slope coefficient of a regression of $\Delta s/s$ on $\Delta f/f$:

$$\frac{\Delta s}{s} = \alpha + \beta_{sf}\frac{\Delta f}{f} + \epsilon \tag{13.8}$$

As seen in Chapter 3, standard regression theory shows that

$$\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf}\frac{\sigma_s}{\sigma_f} \tag{13.9}$$

Thus the **best hedge** is obtained from a regression of (change in) the value of the inventory on the value of the hedge instrument.

KEY CONCEPT

The optimal hedge is given by the negative of the beta coefficient of a regression of changes in the cash value on changes in the payoff on the hedging instrument.

We can do more than this, though. At the optimum, we can find the variance of profits by replacing N in Equation (13.4) by N^*, which gives

$$\sigma_V^{*2} = \sigma_S^2 + \left(\frac{\sigma_{SF}}{\sigma_F^2}\right)^2\sigma_F^2 + 2\left(\frac{-\sigma_{SF}}{\sigma_F^2}\right)\sigma_{SF} = \sigma_S^2 + \frac{\sigma_{SF}^2}{\sigma_F^2} + 2\frac{-\sigma_{SF}^2}{\sigma_F^2} = \sigma_S^2 - \frac{\sigma_{SF}^2}{\sigma_F^2} \tag{13.10}$$

We can measure the quality of the optimal hedge ratio in terms of the amount by which we decreased the variance of the original portfolio:

$$R^2 = \frac{(\sigma_S^2 - \sigma_V^{*2})}{\sigma_S^2} \tag{13.11}$$

After substitution of Equation (13.10), we find that $R^2 = (\sigma_S^2 - \sigma_S^2 + \sigma_{SF}^2/\sigma_F^2)/\sigma_S^2 = \sigma_{SF}^2/(\sigma_F^2\sigma_S^2) = \rho_{SF}^2$. This unitless number is also the coefficient of determination, or the percentage of variance in $\Delta s/s$ explained by the independent variable $\Delta f/f$ in Equation (13.8). Thus this regression also gives us the **effectiveness** of the hedge, which is measured by the proportion of variance eliminated.

We can also express the volatility of the hedged position from Equation (13.10) using the R^2 as

$$\sigma_V^* = \sigma_S\sqrt{(1 - R^2)} \tag{13.12}$$

This shows that if $R^2 = 1$, the regression fit is perfect, and the resulting portfolio has zero risk. In this situation, the portfolio has no basis risk. However, if the R^2 is very low, the hedge is not effective.

13.2.2 Example

An airline knows that it will need to purchase 10,000 metric tons of jet fuel in three months. It wants some protection against an upturn in prices using futures contracts.

The company can hedge using heating oil futures contracts traded on NYMEX. The notional for one contract is 42,000 gallons. As there is no futures contract on jet fuel, the risk manager wants to check if heating oil could provide an efficient hedge instead. The current price of jet fuel is \$277/metric ton. The futures price of heating oil is \$0.6903/gallon. The standard deviation of the rate of change in jet fuel prices over three months is 21.17%, that of futures is 18.59%, and the correlation is 0.8243.

Compute

1. The notional and standard deviation of the unhedged fuel cost in dollars
2. The optimal number of futures contract to buy/sell, rounded to the closest integer
3. The standard deviation of the hedged fuel cost in dollars

Answer

1. The position notional is Qs = \$2,770,000. The standard deviation in dollars is

$$\sigma(\Delta s/s)sQ = 0.2117 \times \$277 \times 10{,}000 = \$586{,}409$$

For reference, that of one futures contract is

$$\sigma(\Delta f/f) f Q_f = 0.1859 \times \$0.6903 \times 42{,}000 = \$5{,}389.72$$

with a futures notional of $f Q_f = \$0.6903 \times 42{,}000 = \$28{,}992.60$.

2. The cash position corresponds to a payment, or liability. Hence, the company will have to *buy* futures as protection. First, we compute beta, which is $\beta_{sf} = 0.8243(0.2117/0.1859) = 0.9387$. The corresponding covariance term is $\sigma_{sf} = 0.8243 \times 0.2117 \times 0.1859 = 0.03244$. Adjusting for the notionals, this is $\sigma_{SF} = 0.03244 \times \$2{,}770{,}000 \times \$28{,}993 = 2{,}605{,}268{,}452$. The optimal hedge ratio is, using Equation (13.7),

$$N^* = \beta_{sf} \frac{Q \times s}{Q_f \times f} = 0.9387 \frac{10{,}000 \times \$277}{42{,}000 \times \$0.69} = 89.7$$

or 90 contracts after rounding (which we ignore in what follows).

3. To find the risk of the hedged position, we use Equation (13.10). The volatility of the unhedged position is $\sigma_S = \$586{,}409$. The variance of the hedged position is

$$\begin{array}{rll} \sigma_S^2 = & (\$586{,}409)^2 & = +343{,}875{,}515{,}281 \\ -\sigma_{SF}^2/\sigma_F^2 = & -(2{,}605{,}268{,}452/5{,}390)^2 & = -233{,}653{,}264{,}867 \\ \hline \text{V(hedged)} & & = +110{,}222{,}250{,}414 \end{array}$$

Taking the square root, the volatility of the hedged position is $\sigma_V^* = \$331{,}997$. Thus the hedge has reduced the risk from \$586,409 to \$331,997. Computing the R^2, we find that one minus the ratio of the hedged and unhedged variances is $(1 - 110{,}222{,}250{,}414/343{,}875{,}515{,}281) = 67.95\%$. This is exactly the square of the correlation coefficient, $0.8243^2 = 0.6795$, or effectiveness of the hedge.

Figure 13.2 displays the relationship between the risk of the hedged position and the number of contracts. With no hedging, the volatility is \$586,409. As N increases, the risk decreases, reaching a minimum for $N^* = 90$ contracts. The graph also shows that the quadratic relationship is relatively flat for a range of values around the minimum. Choosing anywhere between 80 and 100 contracts will have little effect on the total risk. Given the substantial reduction in risk, the risk manager could choose to implement the hedge.

13.2.3 Liquidity Issues

Although futures hedging can be successful at mitigating market risk, it can create other risks. Futures contracts are marked to market daily. Hence they can involve large cash inflows or outflows. Cash outflows, in particular, can create liquidity problems, especially when they are not offset by cash inflows from the underlying position.

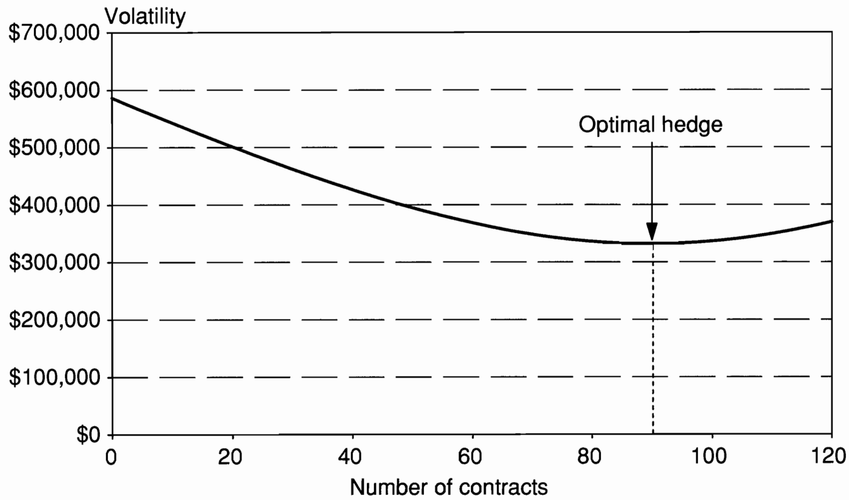

FIGURE 13.2 Risk of Hedged Position and Number of Contracts

EXAMPLE 13.5: FRM EXAM 2001—QUESTION 86

If two securities have the same volatility and a correlation equal to −0.5, their minimum variance hedge ratio is

a. 1:1
b. 2:1
c. 4:1
d. 16:1

EXAMPLE 13.6: FRM EXAM 2007—QUESTION 125

A firm is going to buy 10,000 barrels of West Texas Intermediate Crude Oil. It plans to hedge the purchase using the Brent Crude Oil futures contract. The correlation between the spot and futures prices is 0.72. The volatility of the spot price is 0.35 per year. The volatility of the Brent Crude Oil futures price is 0.27 per year. What is the hedge ratio for the firm?

a. 0.9333
b. 0.5554
c. 0.8198
d. 1.2099

EXAMPLE 13.7: FRM EXAM 2009—QUESTION 3-26

XYZ Co. is a gold producer and will sell 10,000 ounces of gold in three months at the prevailing market price at that time. The standard deviation of the change in the price of gold over a three-month period is 3.6%. In order to hedge its price exposure, XYZ Co. decides to use gold futures to hedge. The contract size of each gold futures contract is 10 ounces. The standard deviation of the gold futures price is 4.2%. The correlation between quarterly changes in the futures price and the spot price of gold is 0.86. To hedge its price exposure, how many futures contracts should XYZ Co. go long or short?

a. Short 632 contracts
b. Short 737 contracts
c. Long 632 contracts
d. Long 737 contracts

13.3 APPLICATIONS OF OPTIMAL HEDGING

The linear framework presented here is completely general. We now specialize it to two important cases, duration and beta hedging. The first applies to the bond market, the second to the stock market.

13.3.1 Duration Hedging

Modified duration can be viewed as a measure of the exposure of relative changes in prices to movements in yields. Using the definitions in Chapter 6, we can write

$$\Delta P = (-D^* P)\Delta y \tag{13.13}$$

where D^* is the modified duration. The **dollar duration** is defined as $(D^* P)$.

Assuming the duration model holds, which implies that the change in yield Δy does not depend on maturity, we can rewrite this expression for the cash and futures positions

$$\Delta S = (-D_S^* S)\Delta y \qquad \Delta F = (-D_F^* F)\Delta y$$

where D_S^* and D_F^* are the modified durations of S and F, respectively. Note that these relationships are supposed to be perfect, without an error term. The variances and covariance are then

$$\sigma_S^2 = (D_S^* S)^2 \sigma^2(\Delta y) \qquad \sigma_F^2 = (D_F^* F)^2 \sigma^2(\Delta y) \qquad \sigma_{SF} = (D_F^* F)(D_S^* S)\sigma^2(\Delta y)$$

We can replace these in Equation (13.6):

$$N^* = -\frac{\sigma_{SF}}{\sigma_F^2} = -\frac{(D_F^* F)(D_S^* S)}{(D_F^* F)^2} = -\frac{(D_S^* S)}{(D_F^* F)} \tag{13.14}$$

Alternatively, this can be derived as follows. Write the total portfolio payoff as

$$\begin{aligned}
\Delta V &= \Delta S + N\Delta F \\
&= (-D_S^* S)\Delta y + N(-D_F^* F)\Delta y \\
&= -[(D_S^* S) + N(D_F^* F)] \times \Delta y
\end{aligned}$$

which is zero when the net exposure, represented by the term between brackets, is zero. In other words, the optimal hedge ratio is simply minus the ratio of the dollar duration of cash relative to the dollar duration of the hedge. This ratio can also be expressed in dollar value of a basis point (DVBP). This gives

$$N^* = -\frac{\text{DVBP}_S}{\text{DVBP}_F} \tag{13.15}$$

More generally, we can use N as a tool to modify the total duration of the portfolio. If we have a target duration of D_V, this can be achieved by setting $[(D_S^* S) + N(D_F^* F)] = D_V^* V$, or

$$N = \frac{(D_V^* V - D_S^* S)}{(D_F^* F)} \tag{13.16}$$

of which Equation (13.14) is a special case.

KEY CONCEPT

The optimal duration hedge is given by the ratio of the dollar duration of the position to that of the hedging instrument.

Example: T-Bond Futures Hedging

A portfolio manager holds a bond portfolio worth \$10 million with a modified duration of 6.8 years, to be hedged for three months. The current futures price is 93-02, with a notional of \$100,000. We assume that its duration can be derived from that of the cheapest-to-deliver bond, which is 9.2 years.[2]

To be consistent, all of these values should be measured as of the hedge horizon date. So, the portfolio and futures durations are forecast to be 6.8 and 9.2 years

[2] T-bond futures are described in Chapter 10. Note that this hedging method ignores the effect of options held by the short, which include a delivery and timing option. Also, the duration of the futures should be the duration of the cheapest-to-deliver bond divided by the conversion factor.

in three months. The $10 million should be the forward value of the portfolio, although in practice it is often taken from the current value.

Compute

1. The dollar value of the futures contract notional
2. The number of contracts to buy/sell for optimal protection

Answer

1. The dollar notional is $[93 + (2/32)]/100 \times \$100{,}000 = \$93{,}062.5$.
2. The optimal number to *sell* is from Equation (13.14):

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{6.8 \times \$10{,}000{,}000}{9.2 \times \$93{,}062.5} = -79.4$$

or 79 contracts after rounding. Note that the DVBP of the futures is about $9.2 \times \$93{,}000 \times 0.01\% = \$85$.

Example: Eurodollar Futures Hedging

On February 2, a corporate Treasurer wants to hedge a July 17 issue of $5 million of commercial paper with a maturity of 180 days, leading to anticipated proceeds of $4.52 million. The September Eurodollar futures contract trades at 92, and has a notional amount of $1 million.[3]

Compute

1. The current dollar value of the futures contract
2. The number of contracts to buy/sell for optimal protection

Answer

1. The current dollar price is given by $\$10{,}000[100 - 0.25(100 - 92)] = \$980{,}000$. Note that the duration of the futures is always three months (90 days), since the contract refers to three-month LIBOR.
2. If rates increase, the cost of borrowing will be higher. We need to offset this by a gain, or a short position in the futures. The optimal number is from Equation (13.14):

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{180 \times \$4{,}520{,}000}{90 \times \$980{,}000} = -9.2$$

or 9 contracts after rounding. Note that the DVBP of the futures is about $0.25 \times \$1{,}000{,}000 \times 0.01\% = \$25$. In this case, the DVBP of the position to be hedged is $226. Dividing by $25 gives a hedge ratio of 9.

[3] Eurodollar futures are described in Chapter 10.

EXAMPLE 13.8: DURATION HEDGING

What assumptions does a duration-based hedging scheme make about the way in which interest rates move?

a. All interest rates change by the same amount.
b. A small parallel shift occurs in the yield curve.
c. Any parallel shift occurs in the term structure.
d. Interest rates' movements are highly correlated.

EXAMPLE 13.9: HEDGING WITH EURODOLLAR FUTURES

If all spot interest rates are increased by one basis point, a value of a portfolio of swaps will increase by $1,100. How many Eurodollar futures contracts are needed to hedge the portfolio?

a. 44
b. 22
c. 11
d. 1,100

EXAMPLE 13.10: FRM EXAM 2007—QUESTION 17

On June 2, a fund manager with USD 10 million invested in government bonds is concerned that interest rates will be highly volatile over the next three months. The manager decides to use the September Treasury bond futures contract to hedge the portfolio. The current futures price is USD 95.0625. Each contract is for the delivery of USD 100,000 face value of bonds. The duration of the manager's bond portfolio in three months will be 7.8 years. The cheapest-to-deliver (CTD) bond in the Treasury bond futures contract is expected to have a duration of 8.4 years at maturity of the contract. At the maturity of the Treasury bond futures contract, the duration of the underlying benchmark Treasury bond is nine years. What position should the fund manager undertake to mitigate his interest rate risk exposure?

a. Short 94 contracts
b. Short 98 contracts
c. Short 105 contracts
d. Short 113 contracts

EXAMPLE 13.11: FRM EXAM 2004—QUESTION 4

Albert Henri is the fixed income manager of a large Canadian pension fund. The present value of the pension fund's portfolio of assets is CAD 4 billion while the expected present value of the fund's liabilities is CAD 5 billion. The respective modified durations are 8.254 and 6.825 years. The fund currently has an actuarial deficit (assets < liabilities) and Albert must avoid widening this gap. There are currently two scenarios for the yield curve: The first scenario is an upward shift of 25bp, with the second scenario a downward shift of 25bp. The most liquid interest rate futures contract has a present value of CAD 68,336 and a duration of 2.1468 years. Analyzing both scenarios separately, what should Albert Henri do to avoid widening the pension fund gap? Choose the best option.

	First Scenario	Second Scenario
a.	Do nothing.	Buy 7,559 contracts
b.	Do nothing.	Sell 7,559 contracts.
c.	Buy 7,559 contracts.	Do nothing.
d.	Do nothing.	Do nothing.

13.3.2 Beta Hedging

We now turn to equity hedging using stock index futures. **Beta**, or **systematic risk**, can be viewed as a measure of the exposure of the rate of return on a portfolio i to movements in the market m:

$$R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it} \tag{13.17}$$

where β represents the systematic risk, α the intercept (which is not a source of risk and therefore is ignored for risk management purposes), and ϵ the residual component, which is uncorrelated with the market. We can also write, in line with the previous sections and ignoring the residual and intercept,

$$(\Delta S/S) \approx \beta(\Delta M/M) \tag{13.18}$$

Now, assume that we have at our disposal a stock index futures contract, which has a beta of unity $(\Delta F/F) = 1(\Delta M/M)$. For options, the beta is replaced by the net delta, $(\Delta C) = \delta(\Delta M)$.

As in the case of bond duration, we can write the total portfolio payoff as

$$\begin{aligned} \Delta V &= \Delta S + N\Delta F \\ &= (\beta S)(\Delta M/M) + NF(\Delta M/M) \\ &= [(\beta S) + NF] \times (\Delta M/M) \end{aligned}$$

which is set to zero when the net exposure, represented by the term between brackets, is zero. The optimal number of contracts to short is

$$N^* = -\frac{\beta S}{F} \tag{13.19}$$

KEY CONCEPT

The optimal hedge with stock index futures is given by the beta of the cash position times its value divided by the notional of the futures contract.

Example

A portfolio manager holds a stock portfolio worth \$10 million with a beta of 1.5 relative to the S&P 500. The current futures price is 1,400, with a multiplier of \$250.

Compute

1. The notional of the futures contract
2. The number of contracts to sell short for optimal protection

Answer

1. The notional amount of the futures contract is $\$250 \times 1{,}400 = \$350{,}000$.
2. The optimal number of contracts to short is, from Equation (13.19),

$$N^* = -\frac{\beta S}{F} = -\frac{1.5 \times \$10{,}000{,}000}{1 \times \$350{,}000} = -42.9$$

or 43 contracts after rounding.

13.3.3 General Considerations

The quality of the hedge depends on the size of the residual risk in the market model of Equation (13.17). For large portfolios, the approximation may be good because residual risk diversifies away. In contrast, hedging an individual stock with stock index futures may give poor results.

For instance, the correlation of a typical U.S. stock with the S&P 500 is 0.50. For an industry index, it is typically 0.75. Using the regression effectiveness in Equation (13.12), we find that the volatility of the hedged portfolio is still about $\sqrt{1-0.5^2} = 87\%$ of the unhedged volatility for a typical stock and about 66% of the unhedged volatility for a typical industry. Thus hedging a portfolio of stocks (an industry index) with a general market hedge is more effective.

A final note on hedging is in order. If the objective of hedging is to lower volatility, hedging will eliminate downside risk but also any upside potential. The objective of hedging is to lower risk, not to make profits, so this is a double-edged sword. Whether hedging is beneficial should be examined in the context of the trade-off between risk and return.

EXAMPLE 13.12: FRM EXAM 2009—QUESTION 3-10

You have a portfolio of USD 5 million to be hedged using index futures. The correlation coefficient between the portfolio and futures being used is 0.65. The standard deviation of the portfolio is 7% and that of the hedging instrument is 6%. The futures price of the index futures is USD 1,500 and one contract size is 100 futures. Among the following positions, which one reduces risk the most?

a. Long 33 futures contracts
b. Short 33 futures contracts
c. Long 25 futures contracts
d. Short 25 futures contracts

EXAMPLE 13.13: FRM EXAM 2007—QUESTION 107

The current value of the S&P 500 index is 1,457, and each S&P futures contract is for delivery of 250 times the index. A long-only equity portfolio with market value of USD 300,100,000 has a beta of 1.1. To reduce the portfolio beta to 0.75, how many S&P futures contracts should you sell?

a. 288 contracts
b. 618 contracts
c. 906 contracts
d. 574 contracts

13.4 IMPORTANT FORMULAS

Profit on position with unit hedge: $Q[(S_2 - S_1) - (F_2 - F_1)] = Q[b_2 - b_1]$

Short hedge position = long the basis, or benefits when the basis widens/strengthens

Optimal hedge ratio: $N^* = -\beta_{sf} \frac{Q \times s}{Q_f \times f}$

Optimal hedge ratio (unitless): $\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf} \frac{\sigma_s}{\sigma_f}$

Volatility of the hedged position: $\sigma_V^* = \sigma_S\sqrt{(1 - R^2)}$
Duration hedge: $N^* = -\frac{(D_S^* S)}{(D_F^* F)}$
Beta hedge: $N^* = -\beta\frac{S}{F}$

13.5 ANSWERS TO CHAPTER EXAMPLES

Example 13.1: FRM Exam 2000—Question 79

d. Basis risk occurs if movements in the value of the cash and hedged positions do not offset each other perfectly. This can happen if the instruments are dissimilar or if the correlation is not unity. Even with similar instruments, if the hedge is lifted before the maturity of the underlying, there is some basis risk.

Example 13.2: FRM Exam 2009—Question 3-14

a. Basis risk is minimized when the maturity of the hedging instrument coincides with the horizon of the hedge (i.e., two months) and when the hedging instrument is exposed to the same risk factor (i.e., IBM).

Example 13.3: FRM Exam 2009—Question 3-15

c. Basis risk can arise if the maturities are different, so answer I. is incorrect. A short hedge position is long the basis, which means that it benefits when the basis strengthens, because this means that the futures price drops relative to the spot price, which generates a profit.

Example 13.4: FRM Exam 2007—Question 99

c. There is mainly basis risk for positions that are both long and short either different months or contracts. Position II. is long twice the same contract and thus has no basis risk (but a lot of directional risk).

Example 13.5: FRM Exam 2001—Question 86

b. Set x as the amount to invest in the second security, relative to that in the first (or the hedge ratio). The variance is then proportional to $1 + x^2 + 2x\rho$. Taking the derivative and setting to zero, we have $x = -\rho = 0.5$. Thus, one security must have twice the amount in the other. Alternatively, the hedge ratio is given by $N^* = -\rho\frac{\sigma_S}{\sigma_F}$, which gives 0.5. Answer b. is the only one that is consistent with this number or its inverse.

Example 13.6: FRM Exam 2007—Question 125

a. The optimal hedge ratio is $\beta_{sf} = \rho_{sf}\frac{\sigma_s}{\sigma_f} = 0.72\ 0.35/0.27 = 0.933$.

Example 13.7: FRM Exam 2009—Question 3-26

b. XYZ will incur a loss if the price of gold falls, so should short futures as a hedge. The optimal hedge ratio is $\rho\sigma_s/\sigma_f = 0.86 \times 3.6/4.2 = 0.737$. Taking into account the size of the position, the number of contracts to sell is $0.737 \times 10{,}000/10 = 737$.

Example 13.8: Duration Hedging

b. The assumption is that of (1) parallel and (2) small moves in the yield curve. Answers a. and c. are the same, and omit the size of the move. Answer d. would require perfect, not high, correlation plus small moves.

Example 13.9: Hedging with Eurodollar Futures

a. The DVBP of the portfolio is \$1,100. That of the futures is \$25. Hence the ratio is $1{,}100/25 = 44$.

Example 13.10: FRM Exam 2007—Question 17

b. The number of contracts to short is $N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -(7.8 \times 10{,}000{,}000)/(8.4 \times (95.0625) \times 1{,}000) = -97.7$, or 98 contracts. Note that the relevant duration for the futures is that of the CTD; other numbers are irrelevant.

Example 13.11: FRM Exam 2004—Question 4

a. We first have to compute the dollar duration of assets and liabilities, which gives, in millions, $4{,}000 \times 8.254 = 33{,}016$ and $5{,}000 \times 6.825 = 34{,}125$, respectively. Because the DD of liabilities exceeds that of assets, a decrease in rates will increase the liabilities more than the assets, leading to a worsening deficit. Albert needs to buy interest rate futures as an offset. The number of contracts is $(34{,}125 - 33{,}016)/(68{,}336 \times 2.1468/1{,}000{,}000) = 7{,}559$.

Example 13.12: FRM Exam 2009—Question 3-10

d. To hedge, the portfolio manager should sell index futures, to create a profit if the portfolio loses value. The portfolio beta is $0.65 \times (7\%/6\%) = 0.758$. The number of contracts is $N^* = -\beta S/F = -(0.758 \times 5{,}000{,}000)/(1{,}500 \times 100) = -25.3$, or 25 contracts.

Example 13.13: FRM Exam 2007—Question 107

a. This is as in the previous question, but the hedge is partial (i.e., for a change of 1.10 to 0.75). So, $N^* = -\beta S/F = -(1.10 - 0.75)300{,}100{,}000/(1{,}457 \times 250) = -288.3$ contracts.

CHAPTER 14

Nonlinear (Option) Risk Models

The previous chapter focused on linear risk models (e.g., hedging using contracts such as forwards and futures whose values are linearly related to the underlying risk factors). Because linear combinations of normal random variables are also normally distributed, linear hedging maintains normal distributions, which considerably simplifies the risk analysis.

Nonlinear risk models, however, are much more complex. In particular, option values can have sharply asymmetrical distributions. Even so, it is essential for risk managers to develop an ability to evaluate this type of risk, because options are so widespread in financial markets. Since options can be replicated by dynamic trading, this also provides insights into the risks of active trading strategies.

In a previous chapter, we have seen that market losses can be ascribed to the combination of two factors: exposure and adverse movements in the risk factor. Thus a large loss could occur because of the risk factor, which is bad luck. Too often, however, losses occur because the exposure profile is similar to a short option position. This is less forgivable, because exposure is under the control of the portfolio manager.

The challenge is to develop measures that provide an intuitive understanding of the exposure profile. Section 14.1 introduces option pricing and the Taylor approximation. It starts from the Black-Scholes formula that was presented in Chapter 8. Partial derivatives, also known as Greeks, are analyzed in Section 14.2. Section 14.3 then turns to the distribution profile of option positions and the measurement of value at risk (VAR) using delta and gamma.

14.1 OPTION MODELS

14.1.1 Definitions

We consider a **derivative** instrument whose value depends on an underlying asset, which can be a price, an index, or a rate. As an example, consider a call option

FRM Exam Part 1 topic. Also note that FRM Exam Part 1 includes credit ratings, which are covered in Chapter 20.

where the underlying asset is a foreign currency. We use these definitions:

S_t = current spot price of the asset in dollars
F_t = current forward price of the asset
K = exercise price of option contract
f_t = current value of derivative instrument
r_t = domestic risk-free rate
r_t^* = foreign risk-free rate (also written as y)
σ_t = annual volatility of the rate of change in S
τ = time to maturity

More generally, r^* represents the income payment on the asset, which represents the *annual rate* of dividend or coupon payments on a stock index or bond.

For most options, we can write the value of the derivative as the function

$$f_t = f(S_t, r_t, r_t^*, \sigma_t, \; K, \tau) \tag{14.1}$$

The contract specifications are represented by K and the time to maturity by τ. The other factors are affected by market movements, creating volatility in the value of the derivative. For simplicity, we drop the time subscripts in what follows.

Derivatives pricing is all about finding the value of f, given the characteristics of the option at expiration and some assumptions about the behavior of markets. For a forward contract, for instance, the expression is very simple. It reduces to

$$f = Se^{-r^*\tau} - Ke^{-r\tau} \tag{14.2}$$

More generally, we may not be able to derive an analytical expression for the function f, requiring numerical methods.

14.1.2 Taylor Expansion

We are interested in describing the movements in f. The exposure profile of the derivative can be described *locally* by taking a Taylor expansion,

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots \tag{14.3}$$

Because the value depends on S in a nonlinear fashion, we added a quadratic term for S. The terms in Equation (14.3) approximate a nonlinear function by linear and quadratic polynomials.

Option pricing is about finding f. **Option hedging** uses the partial derivatives. **Risk management** is about combining those with the movements in the risk factors.

Figure 14.1 describes the relationship between the value of a European call and the underlying asset. The actual price is the solid thick line. The straight thin line is the linear (delta) estimate, which is the tangent at the initial point. The other line is the quadratic (delta plus gamma) estimates; it gives a much better fit because it has more parameters.

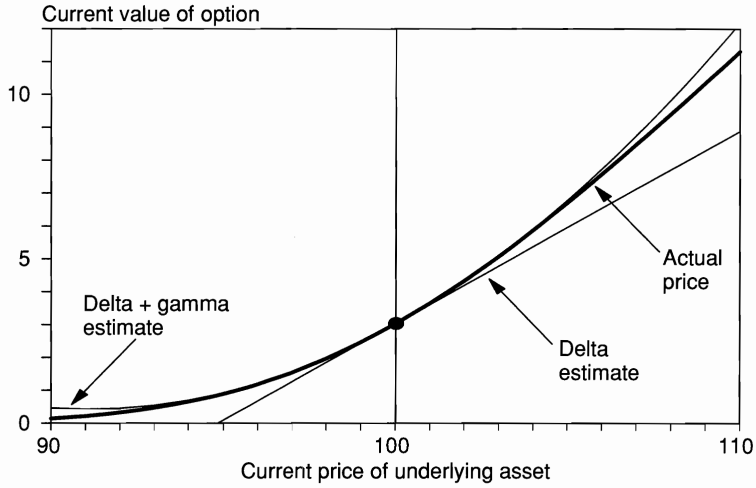

FIGURE 14.1 Delta-Gamma Approximation for a Long Call

Note that, because we are dealing with sums of local price movements, we can aggregate the sensitivities at the portfolio level. This is similar to computing the portfolio duration from the sum of durations of individual securities, appropriately weighted.

Defining $\Delta = \frac{\partial f}{\partial S}$, for example, we can summarize the portfolio or book Δ_P in terms of the total sensitivity,

$$\Delta_P = \sum_{i=1}^{N} x_i \Delta_i \tag{14.4}$$

where x_i is the number of options of type i in the portfolio. To hedge against first-order price risk, it is sufficient to hedge the *net* portfolio delta. This is more efficient than trying to hedge every single instrument individually.

The Taylor expansion will provide a bad approximation in a number of cases:

- *Large movements in the underlying risk factor*
- *Highly nonlinear exposures*, such as options near expiry or exotic options
- *Cross-partial effects*, such as σ changing in relation with S

If this is the case, we need to turn to a **full revaluation** of the instrument. Using the subscripts 0 and 1 as the initial and final values, the change in the option value is

$$f_1 - f_0 = f(S_1, r_1, r_1^*, \sigma_1,\ K, \tau_1) - f(S_0, r_0, r_0^*, \sigma_0,\ K, \tau_0) \tag{14.5}$$

14.1.3 Option Pricing

We now present the various partial derivatives for conventional European call and put options. As we have seen in Chapter 8, the **Black-Scholes** (BS) model provides a closed-form solution, from which these derivatives can be computed analytically.

The key point of the BS derivation is that a position in the option can be replicated by a delta position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is risk-free locally, that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate. The option value is the discounted expected payoff:

$$f_t = E_{\mathrm{RN}}[e^{-r\tau} F(S_T)] \tag{14.6}$$

where E_{RN} represents the expectation of the future payoff in a risk-neutral world, that is, assuming the underlying asset grows at the risk-free rate and the discounting also employs the risk-free rate.

In the case of a European call, the final payoff is $F(S_T) = \mathrm{Max}(S_T - K, 0)$, and the current value of the call is given by

$$c = Se^{-r^*\tau} N(d_1) - Ke^{-r\tau} N(d_2) \tag{14.7}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d. The values of d_1 and d_2 are

$$d_1 = \frac{ln(Se^{-r^*\tau}/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is

$$p = Se^{-r^*\tau}[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{14.8}$$

14.2 OPTION GREEKS

14.2.1 Option Sensitivities: Delta and Gamma

Given these closed-form solutions for European options, we can derive all partial derivatives. The most important sensitivity is the **delta**, which is the first partial derivative with respect to the price. For a call option, this can be written explicitly as:

$$\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau} N(d_1) \tag{14.9}$$

which is always positive and below unity.

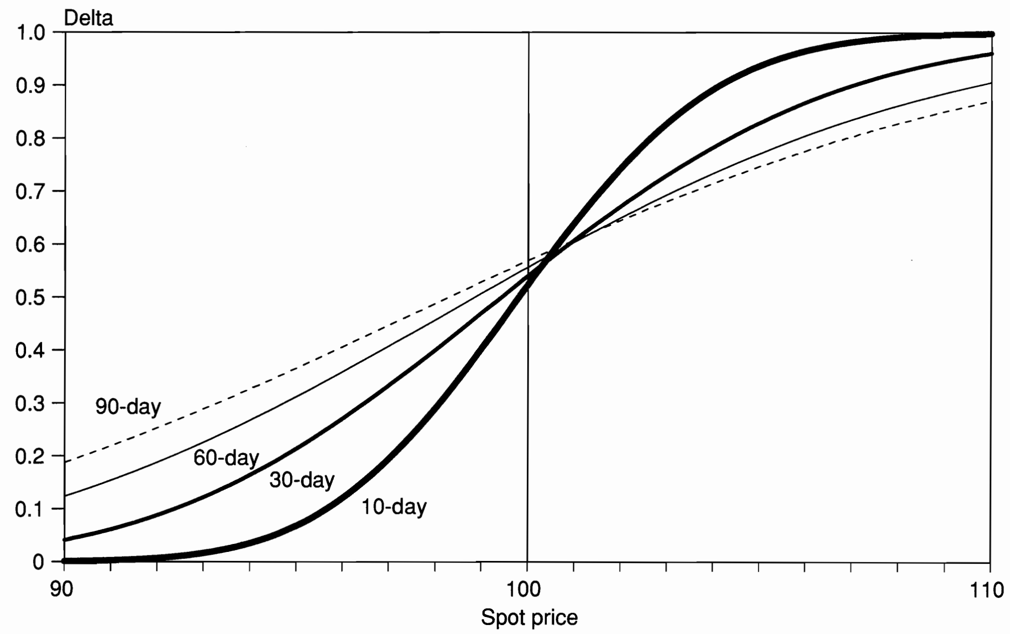

FIGURE 14.2 Option Delta

Figure 14.2 relates delta to the current value of S, for various maturities. The essential feature of this figure is that Δ varies substantially with the spot price and with time. As the spot price increases, d_1 and d_2 become very large, and Δ tends toward $e^{-r^*\tau}$, close to 1 for short maturities. In this situation, the option behaves like an outright position in the asset. Indeed, the limit of Equation (14.7) is $c = Se^{-r^*\tau} - Ke^{-r\tau}$, which is exactly the value of our forward contract, Equation (14.2).

At the other extreme, if S is very low, Δ is close to zero and the option is not very sensitive to S. When S is close to the strike price K, Δ is close to 0.5, and the option behaves like a position of 0.5 in the underlying asset.

KEY CONCEPT

The delta of an at-the-money call option is close to 0.5. Delta moves to 1 as the call goes deep in-the-money (ITM). It moves to zero as the call goes deep out-of-the-money (OTM).

The delta of a put option is

$$\Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau}[N(d_1) - 1] \tag{14.10}$$

which is always negative. It behaves similarly to the call Δ, except for the sign. The delta of an at-the-money (ATM) put is about -0.5.

KEY CONCEPT

The delta of an at-the-money put option is close to -0.5. Delta moves to -1 as the put goes deep in-the-money. It moves to zero as the put goes deep out-of-the-money.

The figure also shows that, as the option nears maturity, the Δ function becomes more curved. The function converges to a step function (i.e., 0 when $S < K$, and 1 otherwise). Close-to-maturity options have unstable deltas.

For a European call or put, gamma (Γ) is the second-order term,

$$\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^*\tau}\Phi(d_1)}{S\sigma\sqrt{\tau}} \tag{14.11}$$

which has the bell shape of the normal density function Φ. This is also the derivative of Δ with respect to S. Thus Γ measures the instability in Δ. Note that gamma is identical for a call and put with identical characteristics.

Figure 14.3 plots the call option gamma. At-the-money options have the highest gamma, which indicates that Δ changes very fast as S changes. In contrast, both in-the-money options and out-of-the-money options have low gammas because their deltas are constant, close to one or zero, respectively. The figure also shows that as the maturity nears, the option gamma increases. This leads to a useful rule (see box).

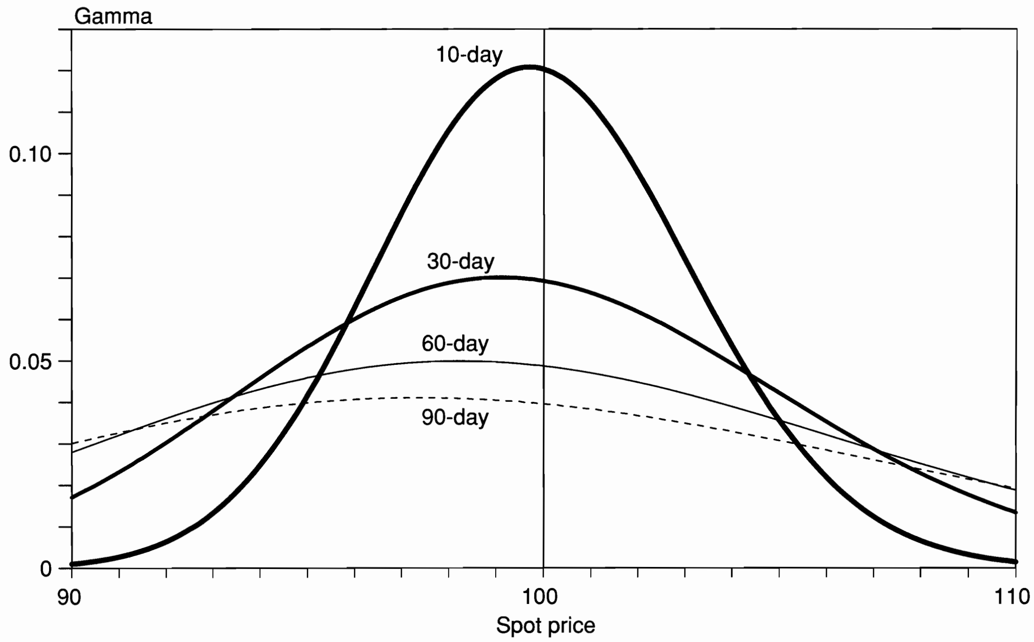

FIGURE 14.3 Option Gamma

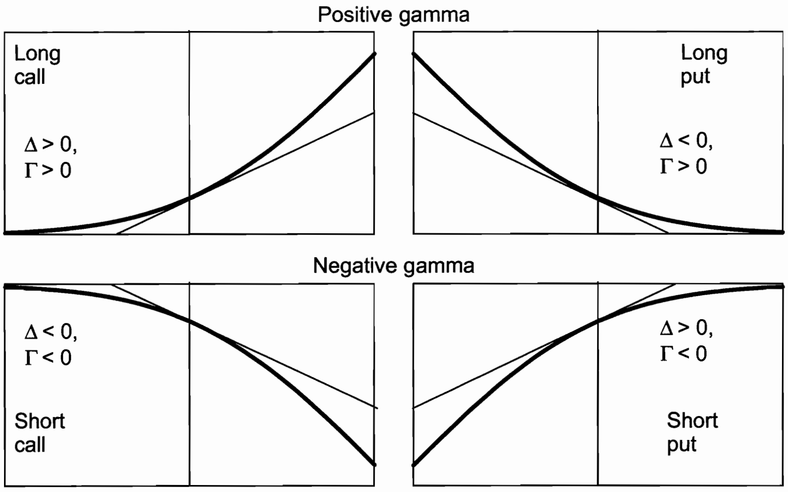

FIGURE 14.4 Delta and Gamma of Option Positions

KEY CONCEPT

For vanilla options, gamma is the highest, or nonlinearities are most pronounced, for short-term at-the-money options.

Thus, gamma is similar to the concept of convexity developed for bonds. Fixed-coupon bonds, however, always have positive convexity, whereas options can create positive or negative convexity. Positive convexity or gamma is beneficial, as it implies that the value of the asset drops more slowly and increases more quickly than otherwise. In contrast, negative convexity can be dangerous because it implies faster price falls and slower price increases.

Figure 14.4 summarizes the delta and gamma exposures of positions in options. Long positions in options, whether calls or puts, create positive convexity. Short positions create negative convexity. In exchange for assuming the harmful effect of this negative convexity, option sellers receive the premium.

EXAMPLE 14.1: FRM EXAM 2006—QUESTION 91

The dividend yield of an asset is 10% per annum. What is the delta of a long forward contract on the asset with six months to maturity?

a. 0.95
b. 1.00
c. 1.05
d. Cannot determine without additional information

EXAMPLE 14.2: FRM EXAM 2004—QUESTION 21

A 90-day European put option on Microsoft has an exercise price of \$30. The current market price for Microsoft is \$30. The delta for this option is close to

a. -1
b. -0.5
c. 0.5
d. 1

EXAMPLE 14.3: FRM EXAM 2006—QUESTION 80

You are given the following information about a European call option: Time to maturity = 2 years; continuous risk-free rate = 4%; continuous dividend yield = 1%; $N(d_1) = 0.64$. Calculate the delta of this option.

a. -0.64
b. 0.36
c. 0.63
d. 0.64

EXAMPLE 14.4: FRM EXAM 2009—QUESTION 4-27

An analyst is doing a study on the effect on option prices of changes in the price of the underlying asset. The analyst wants to find out when the deltas of calls and puts are most sensitive to changes in the price of the underlying. Assume that the options are European and that the Black-Scholes formula holds. An increase in the price of the underlying has the largest absolute value impact on delta for:

a. Calls deep in-the-money and puts deep out-of-the-money
b. Deep in-the-money puts and calls
c. Deep out-of-the-money puts and calls
d. At-the-money puts and calls

EXAMPLE 14.5: FRM EXAM 2001—QUESTION 79

A bank has sold USD 300,000 of call options on 100,000 equities. The equities trade at 50, the option strike price is 49, the maturity is in three months, volatility is 20%, and the interest rate is 5%. How does the bank delta-hedge?

a. Buy 65,000 shares
b. Buy 100,000 shares
c. Buy 21,000 shares
d. Sell 100,000 shares

EXAMPLE 14.6: FRM EXAM 2006—QUESTION 106

Suppose an existing short option position is delta-neutral, but has a gamma of −600. Also assume that there exists a traded option with a delta of 0.75 and a gamma of 1.50. In order to maintain the position gamma-neutral and delta-neutral, which of the following is the appropriate strategy to implement?

a. Buy 400 options and sell 300 shares of the underlying asset.
b. Buy 300 options and sell 400 shares of the underlying asset.
c. Sell 400 options and buy 300 shares of the underlying asset.
d. Sell 300 options and buy 400 shares of the underlying asset.

14.2.2 Option Sensitivities: Vega

Unlike linear contracts, options are exposed to movements not only in the direction of the spot price, but also in its volatility. Options therefore can be viewed as volatility bets.

The sensitivity of an option to volatility is called the option **vega** (sometimes also called lambda, or kappa). For European calls and puts, this is

$$\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r^{*}\tau}\sqrt{\tau}\,\Phi(d_1) \tag{14.12}$$

which also has the bell shape of the normal density function Φ. As with gamma, vega is identical for similar call and put positions. Vega must be positive for long option positions.

Figure 14.5 plots the call option vega. The graph shows that at-the-money options are the most sensitive to volatility. The time effect, however, is different from that for gamma, because the term $\sqrt{\tau}$ appears in the numerator instead of denominator. Thus, vega decreases with maturity, unlike gamma, which increases with maturity.

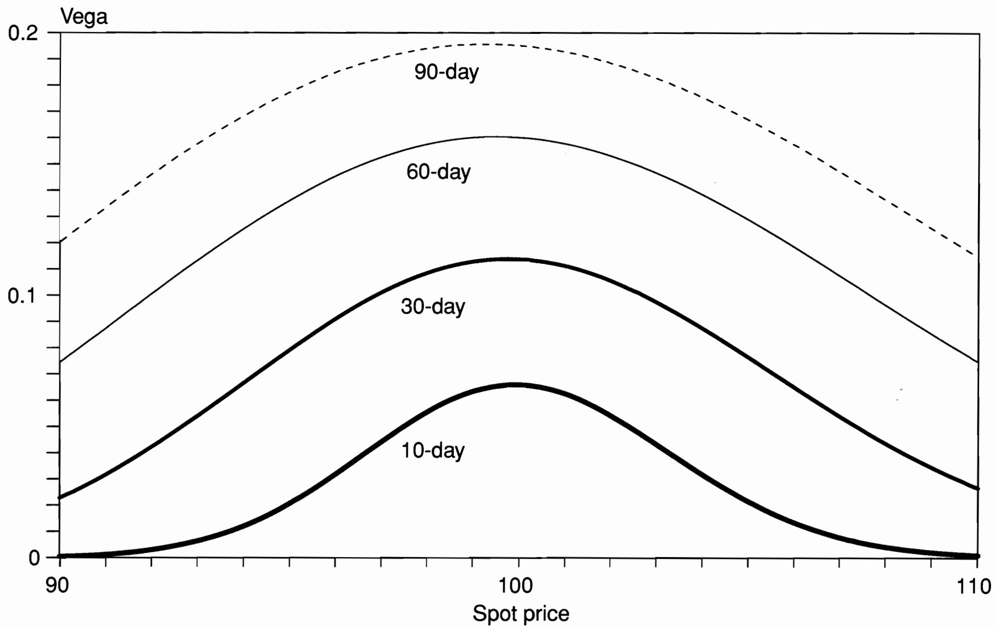

FIGURE 14.5 Option Vega

> **KEY CONCEPT**
>
> Vega is highest for long-term at-the-money options.

14.2.3 Option Sensitivities: Rho

The sensitivity to the domestic interest rate, also called **rho**, is, for a call,

$$\rho_c = \frac{\partial c}{\partial r} = K e^{-r\tau} \tau N(d_2) \tag{14.13}$$

For a put,

$$\rho_p = \frac{\partial p}{\partial r} = -K e^{-r\tau} \tau N(-d_2) \tag{14.14}$$

An increase in the rate of interest increases the value of the call, as the underlying asset grows at a higher rate, which increases the probability of exercising the call, with a fixed strike price K. In the limit, for an infinite interest rate, the probability of exercise is 1 and the call option is equivalent to the stock itself.

As shown in Figure 14.6, rho is positive for a call option and higher when the call is in-the-money. The reasoning is opposite for a put option, for which rho is negative. In each case, the sensitivity is roughly proportional to the remaining time to maturity.

The exposure to the yield on the asset is, for calls and puts, respectively,

$$\rho_C^* = \frac{\partial c}{\partial r^*} = -S e^{-r^*\tau} \tau N(d_1) \tag{14.15}$$

$$\rho_P^* = \frac{\partial p}{\partial r^*} = S e^{-r^*\tau} \tau N(-d_1) \tag{14.16}$$

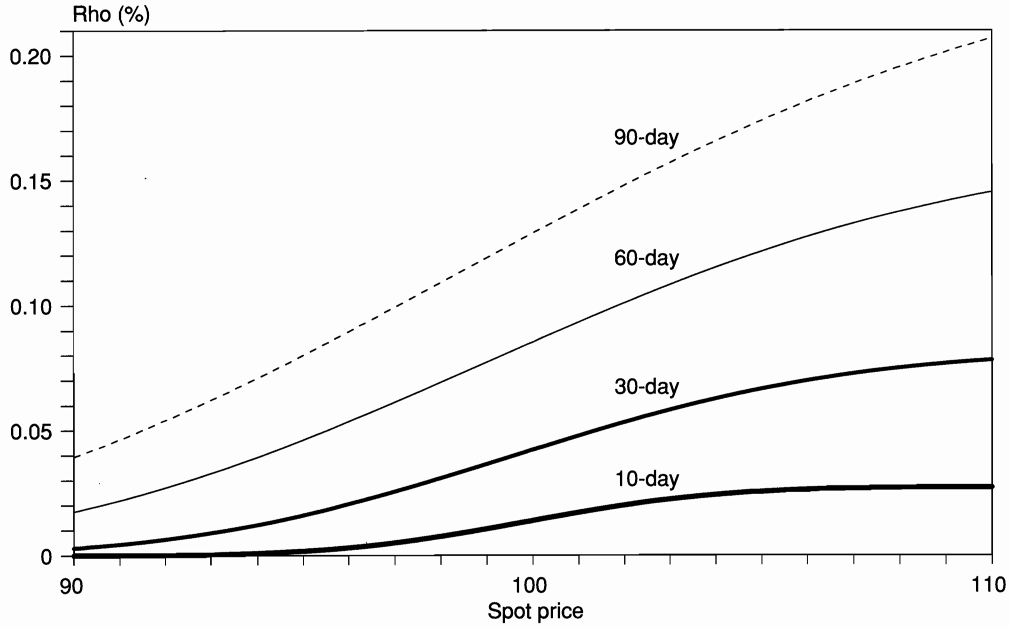

FIGURE 14.6 Call Option Rho

An increase in the dividend yield decreases the growth rate of the underlying asset, which is harmful to the value of the call but helpful to the value of a put.

As shown in Figure 14.7, rho is negative and also greater in absolute value when the call is in-the-money. Again, the reasoning is opposite for a put option. In each case, the sensitivity is proportional to the remaining time to maturity.

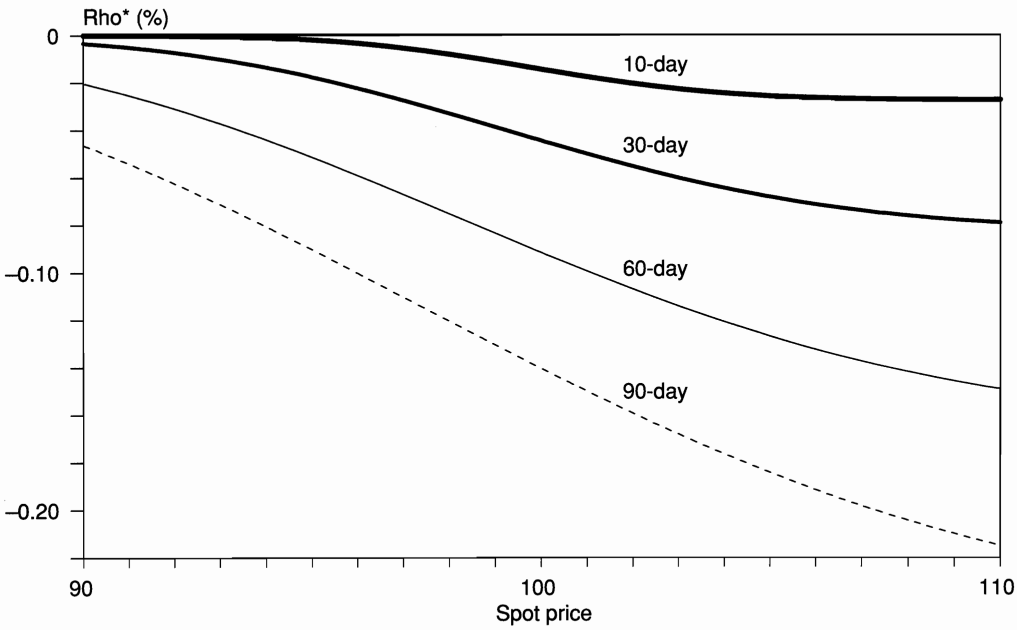

FIGURE 14.7 Call Option Rho*

EXAMPLE 14.7: FRM EXAM 2009—QUESTION 4-26

Ms. Zheng is responsible for the options desk in a London bank. She is concerned about the impact of dividends on the options held by the options desk. She asks you to assess which options are the most sensitive to dividend payments. What would be your answer if the value of the options is found by using the Black-Scholes model adjusted for dividends?

a. Everything else equal, out-of-the-money call options experience a larger decrease in value than in-the-money call options as expected dividends increase.
b. The increase in the value of in-the-money put options caused by an increase in expected dividends is always larger than the decrease in value of in-the-money call options.
c. Keeping the type of option constant, in-the-money options experience the largest absolute change in value and out-of-the-money options the smallest absolute change in value as expected dividends increase.
d. Keeping the type of option constant, at-the-money options experience the largest absolute change in value and out-of-the-money options the smallest absolute change in value as a result of dividend payment.

14.2.4 Option Sensitivities: Theta

Finally, the variation in option value due to the passage of time is called **theta.** This is also the **time decay**. Unlike other factors, however, the movement in remaining maturity is perfectly predictable. Time is not a risk factor.

For a European call, this is

$$\Theta_c = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} + r^*Se^{-r^*\tau}N(d_1) - rKe^{-r\tau}N(d_2) \qquad (14.17)$$

For a European put, this is

$$\Theta_p = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} - r^*Se^{-r^*\tau}N(-d_1) + rKe^{-r\tau}N(-d_2) \qquad (14.18)$$

Theta is generally negative for long positions in both calls and puts. This means that the option loses value as time goes by.

For American options, however, Θ is *always* negative. Because they give their holder the choice to exercise early, shorter-term American options are unambiguously less valuable than longer-term options. For European options, the positive

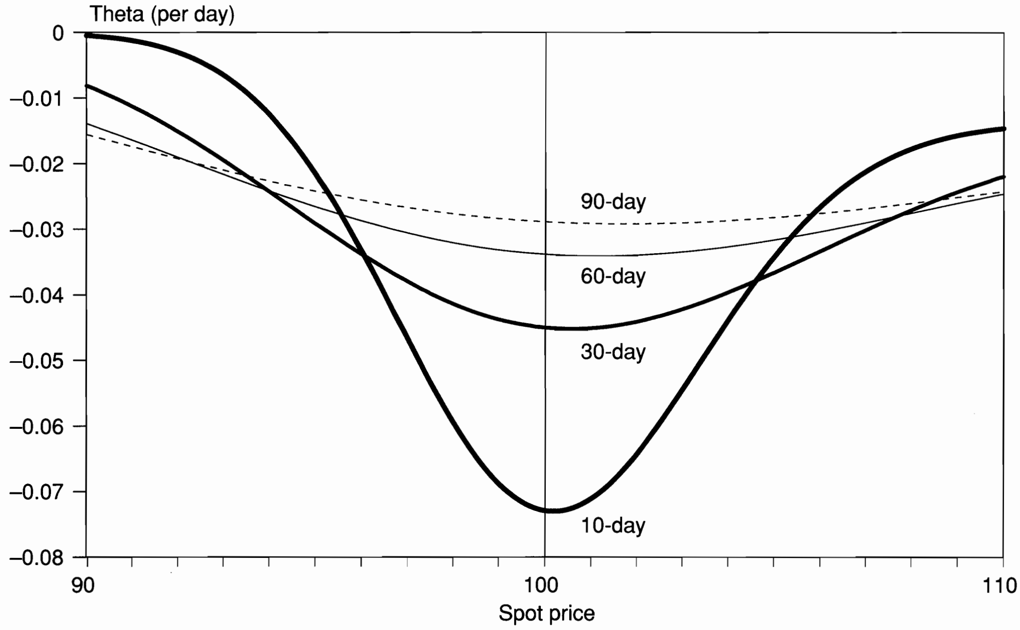

FIGURE 14.8 Option Theta

terms in Equations (14.17) and (14.18) indicate that theta could be positive for some parameter values, albeit this would be unusual.

Figure 14.8 displays the behavior of a call option theta for various prices of the underlying asset and maturities. For long positions in options, theta is negative, which reflects the fact that the option is a wasting asset. Like gamma, theta is greatest for short-term at-the-money options, when measured in absolute value. At-the-money options lose a great proportion of their value when the maturity is near.

14.2.5 Option Pricing and the Greeks

Having defined the option sensitivities, we can illustrate an alternative approach to the derivation of the Black-Scholes formula. Recall that the underlying process for the asset follows a stochastic process known as a **geometric Brownian motion** (GBM),

$$dS = \mu S dt + \sigma S dz \tag{14.19}$$

where dz has a normal distribution with mean zero and variance dt.

Considering only this *single* source of risk, we can return to the Taylor expansion in Equation (14.3). The value of the derivative is a function of S and time, which we can write as $f(S, t)$. The question is, how does f evolve over time?

We can relate the stochastic process of f to that of S using **Ito's lemma**, named after its creator. This can be viewed as an extension of the Taylor approximation to a stochastic environment. Applied to the GBM, this gives

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial \tau}\right)dt + \left(\frac{\partial f}{\partial S}\sigma S\right)dz \tag{14.20}$$

This is also

$$df = (\Delta\,\mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\,\sigma S)dz \tag{14.21}$$

The first term, including dt, is the trend. The second, including dz, is the stochastic component.

Next, we construct a portfolio delicately balanced between S and f that has no exposure to dz. Define this portfolio as

$$\Pi = f - \Delta\,S \tag{14.22}$$

Using Equations (14.19) and (14.21), its stochastic process is

$$\begin{aligned} d\Pi &= [(\Delta\,\mu S + \tfrac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\,\sigma S)dz] - \Delta\,[\mu S dt + \sigma S dz] \\ &= (\Delta\,\mu S)dt + (\tfrac{1}{2}\Gamma\sigma^2 S^2)dt + \Theta dt + (\Delta\,\sigma S)dz - (\Delta\,\mu S)dt - (\Delta\,\sigma S)dz \\ &= (\tfrac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt \end{aligned} \tag{14.23}$$

This simplification is extremely important. Note how the terms involving dz cancel each other out. The portfolio has been immunized against this source of risk. At the same time, the terms in μS also happened to cancel each other out. The fact that μ disappears from the trend in the portfolio is important because it explains why the trend of the underlying asset does not appear in the Black-Scholes formula.

Continuing, we note that the portfolio Π has no risk. To avoid arbitrage, it must return the risk-free rate:

$$d\Pi = [r\Pi]dt = r(f - \Delta\,S)dt \tag{14.24}$$

If the underlying asset has a dividend yield of y, this must be adjusted to

$$d\Pi = (r\Pi)dt + y\Delta\,Sdt = r(f - \Delta\,S)dt + y\Delta\,Sdt = [rf - (r - y)\Delta\,S]dt \tag{14.25}$$

Setting the trends in Equations (14.23) and (14.25) equal to each other, we must have

$$(r - y)\Delta\,S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{14.26}$$

This is the Black-Scholes **partial differential equation (PDE)**, which applies to any contract, or portfolio, that derives its value from S. The solution of this equation, with appropriate boundary conditions, leads to the BS formula for a European call, Equation (14.7).

We can use this relationship to understand how the sensitivities relate to each other. Consider a portfolio of derivatives, all on the same underlying asset, that is delta-hedged. Setting $\Delta = 0$ in Equation (14.26), we have

$$\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{14.27}$$

This shows that, for such portfolio, when Γ is large and positive, Θ must be negative if rf is small. In other words, a delta-hedged position with positive gamma, which is beneficial in terms of price risk, must have negative theta, or time decay. An example is the long straddle examined in Chapter 8. Such a position is delta-neutral and has large gamma or convexity. It would benefit from a large move in S, whether up or down. This portfolio, however, involves buying options whose values decay very quickly with time. Thus, there is an intrinsic trade-off between Γ and Θ.

KEY CONCEPT

For delta-hedged portfolios, Γ and Θ must have opposite signs. Portfolios with positive convexity, for example, must experience time decay.

14.2.6 Option Sensitivities: Summary

We now summarize the sensitivities of option positions with some illustrative data in Table 14.1. Three strike prices are considered, $K = 90$, 100, and 110. We verify that the Γ, Λ, Θ measures are all highest when the option is at-the-money ($K = 100$). Such options have the most nonlinear patterns.

The table also shows the loss for the worst daily movement in each risk factor at the 95% confidence level. For S, this is $dS = -1.645 \times 20\% \times \$100/\sqrt{252} = -\$2.08$. We combine this with delta, which gives a potential loss of $\Delta \times dS = -\$1.114$, or about a fourth of the option value.

Next, we examine the second-order term, S^2. The worst squared daily movement is $dS^2 = 2.08^2 = 4.33$ in the risk factor at the 95% confidence level. We combine this with gamma, which gives a potential gain of $\frac{1}{2}\Gamma \times dS^2 = 0.5 \times 0.039 \times 4.33 = \$0.084$. Note that this is a gain because gamma is positive, but much smaller than the first-order effect. Thus the worst loss due to S would be $-\$1.114 + \$0.084 = -\$1.030$ using the linear and quadratic effects.

For σ, we observe a volatility of daily changes in σ on the order of 1.5%. The worst daily move is therefore $-1.645 \times 1.5 = -2.5$, expressed in percent, which gives a worst loss of $-\$0.495$. Finally, for r, we assume an annual volatility of changes in rates of 1%. The worst daily move is then $-1.645 \times 1/\sqrt{252} = -0.10$, in percent, which gives a worst loss of $-\$0.013$. So, most of the risk originates

TABLE 14.1 Derivatives for a European Call
Parameters: $S = \$100, \sigma = 20\%, r = 5\%, y = 3\%, \tau = 3$ months

			Strike			Worst Loss	
	Variable	Unit	$K = 90$	$K = 100$	$K = 110$	Variable	Loss
c		Dollars	\$11.02	\$4.22	\$1.05		
		Change per:					
Δ	Spot price	Dollar	0.868	0.536	0.197	−\$2.08	−\$1.114
Γ	Spot price	Dollar	0.020	0.039	0.028	4.33	\$0.084
Λ	Volatility	(% pa)	0.103	0.198	0.139	−2.5	−\$0.495
ρ	Interest rate	(% pa)	0.191	0.124	0.047	−0.10	−\$0.013
ρ^*	Asset yield	(% pa)	−0.220	−0.135	−0.049	0.10	−\$0.014
Θ	Time	Day	−0.014	−0.024	−0.016		

from S. In this case, a linear approximation using Δ only would capture most of the downside risk. For near-term at-the-money options, however, the quadratic effect is more important.

EXAMPLE 14.8: FRM EXAM 2004—QUESTION 65

Which of the following statements is *true* regarding options Greeks?

a. Theta tends to be large and positive when buying at-the-money options.
b. Gamma is greatest for in-the-money options with long maturities.
c. Vega is greatest for at-the-money options with long maturities.
d. Delta of deep in-the-money put options tends toward +1.

EXAMPLE 14.9: FRM EXAM 2006—QUESTION 33

Steve, a market risk manager at Marcat Securities, is analyzing the risk of its S&P 500 index options trading desk. His risk report shows the desk is net long gamma and short vega. Which of the following portfolios of options shows exposures consistent with this report?

a. The desk has substantial long-expiry long call positions and substantial short-expiry short put positions.
b. The desk has substantial long-expiry long put positions and substantial long-expiry short call positions.
c. The desk has substantial long-expiry long call positions and substantial short-expiry short call positions.
d. The desk has substantial short-expiry long call positions and substantial long-expiry short call positions.

EXAMPLE 14.10: FRM EXAM 2006—QUESTION 54

Which of the following statements is *incorrect*?

a. The vega of a European-style call option is highest when the option is at-the-money.
b. The delta of a European-style put option moves toward zero as the price of the underlying stock rises.
c. The gamma of an at-the-money European-style option tends to increase as the remaining maturity of the option decreases.
d. Compared to an at-the-money European-style call option, an out-of-the-money European-style option with the same strike price and remaining maturity has a greater negative value for theta.

EXAMPLE 14.11: VEGA AND GAMMA

How can a trader produce a short vega, long gamma position?

a. Buy short-maturity options, sell long-maturity options.
b. Buy long-maturity options, sell short-maturity options.
c. Buy and sell options of long maturity.
d. Buy and sell options of short maturity.

EXAMPLE 14.12: VEGA AND THETA

An option portfolio exhibits high unfavorable sensitivity to increases in implied volatility and while experiencing significant daily losses with the passage of time. Which strategy would the trader most likely employ to hedge the portfolio?

a. Sell short-dated options and buy long-dated options.
b. Buy short-dated options and sell long-dated options.
c. Sell short-dated options and sell long-dated options.
d. Buy short-dated options and buy long-dated options.

14.3 OPTION RISKS

14.3.1 Distribution of Option Payoffs

Unlike linear derivatives such as forwards and futures, payoffs on options are intrinsically asymmetric. This is not necessarily because of the distribution of the underlying factor, which is often symmetrical, but rather is due to the exposure

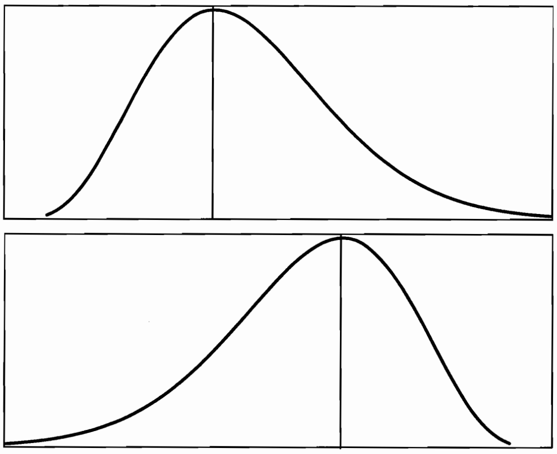

FIGURE 14.9 Distributions of Payoffs on Long and Short Options

profile. Long positions in options, whether calls or puts, have positive gamma, positive skewness, or long right tails. In contrast, short positions in options are short gamma and hence have negative skewness or long left tails. This is illustrated in Figure 14.9.

14.3.2 Option VAR: Linear

We now summarize VAR formulas for simple option positions. Assuming a normal distribution, the VAR of the underlying asset is

$$\mathrm{VAR}(dS) = \alpha S \sigma(dS/S) \tag{14.28}$$

where α corresponds to the desired confidence level (e.g., $\alpha = 1.645$ for a 95% confidence level).

Next, we relate the movement in the option to that in the asset value. Consider a long position in a call for which Δ is positive:

$$dc = \Delta \times dS \tag{14.29}$$

The option VAR is the positive number $\mathrm{VAR}_1(dc) = -dc = \Delta \times -dS = \Delta \times \mathrm{VAR}(dS)$. Generally, the linear VAR for an option is

$$\mathrm{VAR}_1(dc) = |\Delta| \times \mathrm{VAR}(dS) \tag{14.30}$$

14.3.3 Option VAR: Quadratic

Next, we want to take into account nonlinear effects using the Taylor approximation.

$$df \approx \frac{\partial f}{\partial S} dS + (1/2)\frac{\partial^2 f}{\partial S^2} dS^2 = \Delta\, dS + (1/2)\Gamma\, dS^2 \tag{14.31}$$

When the value of the instrument is a monotonic function of the underlying risk factor, we can use the Taylor expansion to find the worst move in the value f from the worst move in the risk factor S.

For a long call option, for example, the worst value is achieved as the underlying price moves down by VAR(dS). The quadratic VAR for this option is

$$\text{VAR}_2(dc) = |\Delta| \times \text{VAR}(dS) - (1/2)\Gamma \times \text{VAR}(dS)^2 \tag{14.32}$$

This method is called **delta-gamma** because it provides an analytical, second-order correction to the delta-normal VAR. This simple adjustment, unfortunately, works only when the payoff function is monotonic, that is, involves a one-to-one relationship between the option value f and S.

Equation (14.32) is fundamental. It explains why long positions in options with positive gamma have less risk than with a linear model. Conversely, short positions in options have negative gamma and thus greater risk than implied by a linear model.

This also applies to fixed-income positions, where the Taylor expansion uses modified duration D^* and convexity C:

$$dP \approx \frac{\partial P}{\partial y} dy + (1/2)\frac{\partial^2 P}{\partial y^2}(dy)^2 = (-D^* P)dy + (1/2)CP(dy)^2 \tag{14.33}$$

Because the price is a monotonic function of the underlying yield, we can use the Taylor expansion to find the worst down move in the bond price from the worst move in the yield. Calling this $dy^* = \text{VAR}(dy)$, we have

$$(\text{Worst}\, dP) = P(y_0 + dy^*) - P(y_0) \approx (-D^* P)(dy^*) + (1/2)(C\,P)(dy^*)^2 \tag{14.34}$$

Similar to Equation (14.32), this gives

$$\text{VAR}(dP) = |-D^* P| \times \text{VAR}(dy) - (1/2)(C\,P) \times \text{VAR}(dy)^2 \tag{14.35}$$

Lest we think that such options require sophisticated risk management methods, what matters is the *extent* of nonlinearity. Figure 14.10 illustrates the risk of a call option with a maturity of three months. It shows that the degree of nonlinearity also depends on the horizon. With a VAR horizon of two weeks, the range of possible values for S is quite narrow. If S follows a normal distribution, the option value will be approximately normal. However, if the VAR horizon is set at two months, the nonlinearities in the exposure combine with the greater range of price movements to create a heavily skewed distribution.

So, for plain-vanilla options, the linear approximation may be adequate as long as the VAR horizon is kept short. For more exotic options, or longer VAR horizons, risk managers must account for nonlinearities.

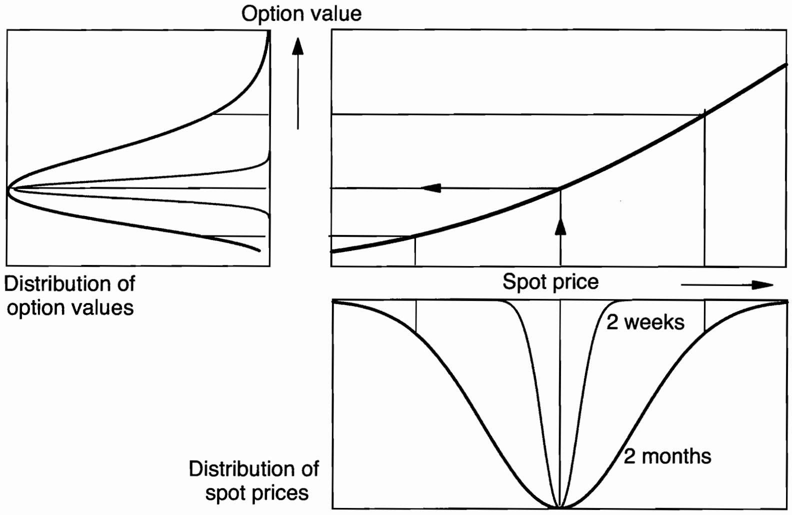

FIGURE 14.10 Skewness and VAR Horizon

EXAMPLE 14.13: FRM EXAM 2005—QUESTION 130

An option on the Bovespa stock index is struck on 3,000 Brazilian reais (BRL). The delta of the option is 0.6, and the annual volatility of the index is 24%. Using delta-normal assumptions, what is the 10-day VAR at the 95% confidence level? Assume 260 days per year.

- **a.** 44 BRL
- **b.** 139 BRL
- **c.** 2,240 BRL
- **d.** 278 BRL

EXAMPLE 14.14: FRM EXAM 2009—QUESTION 4-6

An investor is long a short-term at-the-money put option on an underlying portfolio of equities with a notional value of USD 100,000. If the 95% VAR of the underlying portfolio is 10.4%, which of the following statements about the VAR of the option position is *correct* when second-order terms are considered?

- **a.** The VAR of the option position is slightly more than USD 5,200.
- **b.** The VAR of the option position is slightly more than USD 10,400.
- **c.** The VAR of the option position is slightly less than USD 5,200.
- **d.** The VAR of the option position is slightly less than USD 10,400.

14.4 IMPORTANT FORMULAS

Black-Scholes option pricing model: $c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2)$

Taylor series expansion:

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots$$

$$df = \Delta dS + \frac{1}{2}\Gamma dS^2 + \rho dr + \rho^* dr^* + \Lambda d\sigma + \Theta d\tau + \cdots$$

Delta: $\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau}N(d_1)$, $\Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau}[N(d_1) - 1]$

Gamma (for calls and puts): $\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^*\tau}}{S\sigma\sqrt{\tau}}\Phi(d_1)$

Vega (for calls and puts): $\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r^*\tau}\sqrt{\tau}\ \Phi(d_1)$

	Long Call			Long Put		
	OTM	ATM	ITM	ITM	ATM	OTM
Δ	$\to 0$	0.5	$\to 1$	$\to -1$	-0.5	$\to 0$
Γ	Low	High, > 0 esp. short-term	Low	Low	High, > 0 esp. short-term	Low
Λ	Low	High, > 0 esp. long-term	Low	Low	High, > 0 esp. long-term	Low
Θ	Low	High, < 0 esp. short-term	Low	Low	High, < 0 esp. short-term	Low

Black-Scholes PDE: $(r - y)\Delta S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf$

Linear VAR for an option: $\text{VAR}_1(dc) = |\Delta| \times \text{VAR}(dS)$

Quadratic VAR for an option: $\text{VAR}_2(dc) = |\Delta| \times \text{VAR}(dS) - \frac{1}{2}\Gamma \times \text{VAR}(dS)^2$

14.5 ANSWERS TO CHAPTER EXAMPLES

Example 14.1: FRM Exam 2006—Question 91

a. The delta of a long forward contract is $e^{-r^*\tau} = \exp(-0.10 \times 0.5) = 0.95$.

Example 14.2: FRM Exam 2004—Question 21

b. The option is ATM because the strike price is close to the spot price. This is a put, so the delta must be close to -0.5.

Example 14.3: FRM Exam 2006—Question 80

c. This is a call option, so delta must be positive. This is given by $\Delta = \exp(-r^*\tau)$ $N(d_1) = \exp(-0.01 \times 2) \times 0.64 = 0.63$.

Example 14.4: FRM Exam 2009—Question 4-27

d. From Figure 14.3, the delta is most sensitive, or gamma the highest, for ATM short-term options. Under the BS model, gamma is the same for calls and puts.

Example 14.5: FRM Exam 2001—Question 79

a. This is an at-the-money option with a delta of about 0.5. Since the bank sold calls, it needs to delta-hedge by buying the shares. With a delta of 0.54, it would need to buy approximately 50,000 shares. Answer a. is the closest. Note that most other information is superfluous.

Example 14.6: FRM Exam 2006—Question 106

a. Because gamma is negative, we need to buy a call to increase the portfolio gamma back to zero. The number is $600/1.5 = 400$ calls. This, however, will increase the delta from zero to $400 \times 0.75 = 300$. Hence, we must sell 300 shares to bring the delta back to zero. Note that positions in shares have zero gamma.

Example 14.7: FRM Exam 2009—Question 4-26

c. OTM call options are not very sensitive to dividends, as indicated in Figure 14.7, so answer a. is incorrect. This also shows that ITM options have the highest ρ^* in absolute value.

Example 14.8: FRM Exam 2004—Question 65

c. Theta is negative for long positions in ATM options, so a. is incorrect. Gamma is small for ITM options, so b. is incorrect. Delta of ITM puts tends to -1, so d. is incorrect.

Example 14.9: FRM Exam 2006—Question 33

d. Long gamma means that the portfolio is long options with high gamma, typically short-term (short-expiry) ATM options. Short vega means that the portfolio is short options with high vega, typically long-term (long-expiry) ATM options.

Example 14.10: FRM Exam 2006—Question 54

d. Vega is highest for ATM European options, so statement a. is correct. Delta is negative and moves to zero as S increases, so statement b. is correct. Gamma increases as the maturity of an ATM option decreases, so statement c. is correct. Theta is greater (in absolute value) for short-term ATM options, so statement d. is incorrect.

Example 14.11: Vega and Gamma

a. Long positions in options have positive gamma and vega. Gamma (or instability in delta) increases near maturity; vega decreases near maturity. So, to obtain positive gamma and negative vega, we need to buy short-maturity options and sell long-maturity options.

Example 14.12: Vega and Theta

a. Such a portfolio is short vega (volatility) and short theta (time). We need to implement a hedge that is delta-neutral and involves buying and selling options with different maturities. Long positions in short-dated options have high negative theta and low positive vega. Hedging can be achieved by selling short-term options and buying long-term options.

Example 14.13: FRM Exam 2005—Question 130

b. The linear VAR is derived from the worst move in the index value, which is $\alpha S\sigma\sqrt{T} = 1.645 \times 3{,}000(24\%/\sqrt{260})\sqrt{10} = 232.3$. Multiplying by the delta of 0.6 gives 139.

Example 14.14: FRM Exam 2009—Question 4-6

c. The delta must be around 0.5, which implies a linear VAR of $\$100{,}000 \times 10.4\% \times 0.5 = \$5{,}200$. The position is long an option and has positive gamma. As a result, the quadratic VAR must be lower than \$5,200.

PART

Five

Market Risk Management

CHAPTER 15

Advanced Risk Models: Univariate

We now turn to more advanced risk models. First, we consider univariate risk models. Multivariate models are presented in the next chapter.

This chapter covers improvements to traditional risk models. In practice, the implementation of risk models for large institutions involves many shortcuts, simplifications, and judgment calls. The role of the risk manager is to design a system that provides a reasonable approximation to the risk of the portfolio with acceptable speed and cost. The question is how to judge whether accuracy is reasonable.

This is why risk models must always be complemented by a backtesting procedure. This involves systematically comparing the risk forecast with the subsequent outcome. The framework for backtesting is presented in Section 15.1.

Next, we examine a method to improve the estimation of the tail quantile beyond the traditional historical-simulation and delta-normal methods, which can be defined as nonparametric and parametric, respectively. Section 15.2 turns to **extreme value theory** (EVT), which can be used to fit an analytical distribution to the left tail. This method, which can be described as nonparametric, gives more precise value at risk (VAR) estimates. In addition, the analytical function can be used to extrapolate VAR to other confidence levels.

Finally, Section 15.3 considers properties for risk measure. A risk measure that satisfies the selected properties is called coherent. It shows that, in some cases, VAR is not coherent. Expected shortfall, however, satisfies this property.

15.1 BACKTESTING

Any risk model should be checked for consistency with reality. **Backtesting** is a process to compare systematically the VAR forecasts with actual returns. This process is uniquely informative for risk managers. It should detect weaknesses in the models and point to areas for improvement.

Backtesting is also important because it is one of the reasons that bank regulators allow banks to use their internal risk measures to determine the amount of regulatory capital required to support their trading portfolios. Thus this section

FRM Exam Part 2 topic.

also covers the backtesting framework imposed by the Basel Committee in the Market Risk Amendment to the Basel I Capital Accord.[1]

Backtesting compares the daily VAR forecast with the realized profit and loss (P&L) the next day. If the actual loss is worse than the VAR, the event is recorded as an **exception.** The risk manager then counts the number of exceptions x over a window with T observations.

15.1.1 Measuring Exceptions

But first, we have to define the **trading outcome.** One definition is the profit or loss from the **actual portfolio** over the next day. This return, however, does not exactly correspond to the previous day's VAR. All VAR measures assume a *frozen* portfolio from the close of a trading day to the next, and ignore fee income. In practice, trading portfolios do change. Intraday trading will generally increase risk. Fee income is more stable and decreases risk. Although these effects may offset each other, the actual portfolio may have more or less volatility than predicted by VAR.

This is why it is recommended that **hypothetical portfolios** be constructed so as to match the VAR measure exactly. Their returns are obtained from fixed positions applied to the actual price changes on all securities, measured from close to close.

The Basel framework recommends using both hypothetical and actual trading outcomes in backtests. The two approaches are likely to provide complementary information on the quality of the risk management system. For instance, suppose the backtest fails using the actual but not the hypothetical portfolio. This indicates that the model is sound but that actual trading increases volatility. Conversely, if the backtest fails using the hypothetical model, the conclusion should be that the risk model is flawed.

15.1.2 Binomial Distribution

Consider a VAR measure over a daily horizon defined at the 99% level of confidence c. This implies a probability of $p = 1 - c = 1\%$ for daily exception. The window for backtesting is $T = 250$ days.

The number of exceptions is a random variable X, which is the result of T independent **Bernoulli trials,** where each trial results in an outcome of $y = 0$ or $y = 1$, pass or fail. As a result, X has a **binomial distribution,** with density

$$f(x) = \binom{T}{x} p^x (1-p)^{T-x}, \quad x = 0, 1, \ldots, n \tag{15.1}$$

where $\binom{T}{x}$ is the number of combinations of T things taken x at a time, or

$$\binom{T}{x} = \frac{T!}{x!(T-x)!} \tag{15.2}$$

[1] Basel Committee on Banking Supervision, *Supervisory Framework for the Use of "Backtesting" in Conjunction with the Internal Models Approach to Market Risk Capital Requirements* (Basel: Bank for International Settlements, 1996).

and the parameter p is the probability of an exception, and is between zero and one. The binomial variable has mean and variance $E[X] = pT$ and $V[X] = p(1 - p)T$.

For instance, we want to know what is the probability of observing $x = 0$ exceptions out of a sample of $T = 250$ observations when the true probability is 1%. We should expect to observe $p \times T = 2.5$ exceptions on average across many such samples. There will be, however, some samples with no exceptions at all simply due to luck. This probability is

$$f(X = 0) = \frac{T!}{x!(T - x)!} p^x (1 - p)^{T-x} = \frac{250!}{1 \times 250!} 0.01^0 0.99^{250} = 0.081$$

So, we would expect to observe 8.1% of samples with zero exceptions under the null hypothesis. We can repeat this calculation with different values for x. For example, the probability of observing eight exceptions is $f(X = 8) = \binom{250}{8}\, 0.01^8\,(0.99)^{242} = 0.02\%$. Because this probability is so low, this outcome should raise questions as to whether the true probability is 1%.

EXAMPLE 15.1: FRM EXAM 2003—QUESTION 11

Based on a 90% confidence level, how many exceptions in backtesting a VAR would be expected over a 250-day trading year?

a. 10
b. 15
c. 25
d. 50

EXAMPLE 15.2: FRM EXAM 2007—QUESTION 101

A large, international bank has a trading book whose size depends on the opportunities perceived by its traders. The market risk manager estimates the one-day VAR, at the 95% confidence level, to be USD 50 million. You are asked to evaluate how good a job the manager is doing in estimating the one-day VAR. Which of the following would be the most convincing evidence that the manager is doing a poor job, assuming that losses are identical and independently distributed (i.i.d.)?

a. Over the past 250 days, there are eight exceptions.
b. Over the past 250 days, the largest loss is USD 500 million.
c. Over the past 250 days, the mean loss is USD 60 million.
d. Over the past 250 days, there is no exception.

15.1.3 Normal Approximation

When T is large, we can use the central limit theorem and approximate the binomial distribution by the normal distribution

$$z = \frac{x - pT}{\sqrt{p(1-p)T}} \approx N(0, 1) \tag{15.3}$$

which provides a convenient shortcut. If the decision rule is defined at the two-tailed 95 percent test confidence level, then the cutoff value of $|z|$ is 1.96.

For instance, the z-value for observing eight exceptions is $z = (8 - 2.5)/1.573 = 3.50$, which is very high. So, it is unlikely that a well-calibrated model with a confidence level of 99% would produce eight exceptions.

15.1.4 Decision Rule for Backtests

On average, we would expect 1% of 250, or 2.5 instances of exceptions over the past year. Too many exceptions indicate that either the model is understating VAR or the trader is unlucky. How do we decide which explanation is more likely?

Such statistical testing framework must account for two types of errors:

- **Type 1 errors**, which describe the probability of rejecting a correct model, due to bad luck
- **Type 2 errors**, which describe the probability of not rejecting a model that is false

Ideally, one would want to create a decision rule that has low type 1 and type 2 error rates. In practice, one has to trade off one type of error against the other. Most statistical tests fix the type 1 error rate, say at 5%, and structure the test so as to minimize the type 2 error rate, or to maximize the test's power.[2]

Table 15.1 shows how the cumulative distribution can be used to compute a cutoff value for the number of exceptions as a function of the type 1 error rate. For example, the cumulative probability of observing five exceptions or more is 10.78%. This is one minus the cumulative probability of observing up to four observations, which is 0.8922 from the middle column. Using this cutoff point will penalize VAR models that are correct in about 11% of the cases. A higher cutoff point would lower this type 1 error rate. Say, for example, that the decision rule is changed to reject after 10 or more exceptions. This will reduce the type 1 error rate from 10.78% to 0.03%. So, we will almost never reject models that are correct.

However, this will make it more likely that we will miss VAR models that are misspecified. Suppose, for example, that the trader tries to willfully understate

[2] The **power of a test** is also one minus the type 2 error rate.

TABLE 15.1 Distribution for Number of Exceptions ($T = 250$, $p = 0.01$)

Number of Exceptions	Probability	Cumulative Probability	Type 1 Error Rate
0	0.0811	0.0811	100.00%
1	0.2047	0.2858	91.89%
2	0.2574	0.5432	71.42%
3	0.2150	0.7581	45.68%
4	0.1341	0.8922	24.19%
5	0.0666	0.9588	10.78%
6	0.0275	0.9863	04.12%
7	0.0097	0.9960	01.37%
8	0.0030	0.9989	00.40%
9	0.0008	0.9998	00.10%
10	0.0002	0.9999	00.03%

VAR, using a confidence level of 96% instead of 99%. This leads to an expected number of exceptions of $4\% \times 250 = 10$. As a result, the type 2 error rate is very high, around 50%. So, this cheating trader will not be easy to catch.

15.1.5 The Basel Rules for Backtests

The Basel Committee put in place a framework based on the daily backtesting of VAR. Having up to four exceptions is acceptable, which defines a green zone. If the number of exceptions is five or more, the bank falls into a yellow or red zone and incurs a progressive penalty, which is enforced with a higher capital charge. Roughly, the capital charge is expressed as a multiplier of the 10-day VAR at the 99% level of confidence. The normal multiplier k is 3. After an incursion into the yellow zone, the multiplicative factor, k, is increased from 3 to 4, or by a **plus factor** described in Table 15.2.

An incursion into the red zone generates an *automatic*, nondiscretionary penalty. This is because it would be extremely unlikely to observe 10 or more exceptions if the model was indeed correct.

TABLE 15.2 The Basel Penalty Zones

Zone	Number of Exceptions	Potential Increase in k
Green	0 to 4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	≥ 10	1.00

If the number of exceptions falls within the yellow zone, the supervisor has discretion to apply a penalty, depending on the causes for the exceptions. The Basel Committee uses these categories:

- *Basic integrity of the model:* The deviation occurred because the positions were incorrectly reported or because of an error in the program code. This is a very serious flaw. In this case, a penalty should apply and corrective action should be taken.
- *Deficient model accuracy:* The deviation occurred because the model does not measure risk with enough precision (e.g., does not have enough risk factors). This is a serious flaw, too. A penalty should apply and the model should be reviewed.
- *Intraday trading:* Positions changed during the day. Here, a penalty "should be considered." If the exception disappears with the hypothetical return, the problem is not in the bank's VAR model.
- *Bad luck:* Markets were particularly volatile or correlations changed. These exceptions "should be expected to occur at least some of the time" and may not suggest a deficiency of the model but simply bad luck.

15.1.6 Evaluation of Backtesting

Finally, we should note that exception tests focus only on the frequency of occurrences of exceptions. This is consistent with the idea that VAR is simply a quantile. The counting approach, however, totally ignores the size of losses. This is a general weakness of VAR-based risk measures, which can be remedied with conditional value at risk (CVAR), which is discussed in a later section.

Another issue with the traditional backtesting approach is that it ignores the time pattern of losses. Ideally, exceptions should occur uniformly over the period. In contrast, a pattern where exceptions tend to *bunch* over a short period indicates a weakness of the risk measure. In response, the risk model should take into account time variation in risk, using, for example, the generalized autoregressive conditional heteroskedastic (GARCH) model explained in Chapter 5. For example, even if a backtest produces only four exceptions over the past year, which passes the Basel requirements, the fact that these four exceptions occurred in the last month should cause concern, because it is more likely that the portfolio will suffer large losses over the coming days.

EXAMPLE 15.3: FRM EXAM 2002—QUESTION 20

Which of the following procedures is essential in validating the VAR estimates?

a. Stress-testing
b. Scenario analysis
c. Backtesting
d. Once approved by regulators, no further validation is required.

EXAMPLE 15.4: PENALTY ZONES

The Market Risk Amendment to the Basel Capital Accord defines the yellow zone as the following range of exceptions out of 250 observations:

a. 3 to 7
b. 5 to 9
c. 6 to 9
d. 6 to 10

EXAMPLE 15.5: FRM EXAM 2002—QUESTION 23

Backtesting routinely compares daily profits and losses with model-generated risk measures to gauge the quality and accuracy of their risk measurement systems. The 1996 Market Risk Amendment describes the backtesting framework that is to accompany the internal models capital requirement. This backtesting framework involves

I. The size of outliers
II. The use of risk measure calibrated to a one-day holding period
III. The size of outliers for a risk measure calibrated to a 10-day holding period
IV. Number of outliers

a. II and III
b. II only
c. I and II
d. II and IV

EXAMPLE 15.6: FRM EXAM 2009—QUESTION 5-6

Tycoon Bank announced that there were eight days in the previous year for which losses exceeded the daily 99% VAR. As a result, concerns emerged about the accuracy of the VAR implementation. Assuming that there are 250 days in the year, which of the following statements is/are correct?

I. Using a two-tailed 99% confidence level z-score test, the current VAR implementation understates the actual risk in the bank's portfolio.
II. Using a two-tailed 99% confidence level z-score test, the current VAR implementation overstates the actual risk in the bank's portfolio.
III. The bank's exception rates for VAR may be inaccurate if the bank's portfolio changes incorporate the returns from low-risk but highly profitable intraday market making activities.
IV. If these eight exceptions all happened in the previous month, the model should be reexamined for faulty assumptions and invalid parameters.

a. I and III
b. I, III, and IV
c. III only
d. I, II, and IV

15.2 EXTREME VALUE THEORY

As we have seen in Chapter 12, VAR measures can be computed using either of two approaches. The first is *nonparametric* and relies on a simulation of recent history of returns. The second is *parametric* because it imposes an analytical density function for returns, such as the normal, which is summarized in a standard deviation (the parameter), from which VAR is computed. The nonparametric method is more general but not very powerful, which is usually the case when few assumptions are made. As a result, the VAR estimates can be very imprecise, due to the effect of sampling variation, especially at high confidence levels.

Instead, this section turns to a third method, which can be described as *semi-parametric*. **Extreme value theory** (EVT) can be used to fit an analytical distribution, but just to the left tail. This leads to more precise VAR estimates. In addition, the analytical function can be used to extrapolate VAR to other confidence levels.

15.2.1 EVT Distribution

EVT can be viewed as an extension of the central limit theorem, which states that the average of independent random variables tends to the normal distribution, irrespective of the original distribution. This deals with the mean, or center, of the distribution.

For risk management purposes, the tails of the distribution are of interest. The EVT theorem says that the limit distribution for values x beyond a cutoff point u belongs to the following family

$$\begin{aligned} F(y) &= 1-(1+\xi y)^{-1/\xi}, \ \xi \neq 0 \\ F(y) &= 1-exp(-y), \ \xi = 0 \end{aligned} \tag{15.4}$$

where $y = (x-u)/\beta$. To simplify, we defined the loss x as a positive number so that y is also positive. The distribution is characterized by $\beta > 0$, a *scale* parameter, and by ξ, a *shape* parameter that determines the speed at which the tail disappears.

This approach is called **peaks over threshold** (POT), where u is the fixed threshold. Note that this is a limit theorem, which means that the EVT distribution is only asymptotically valid (i.e., as u grows large).

This distribution is called the **generalized Pareto** (GP) distribution because it subsumes other distributions as special cases. For instance, the normal distribution corresponds to $\xi = 0$, in which case the tails disappear at an exponential speed. Typical financial data have $\xi > 0$, which implies *fat tails*. This class of distribution includes the Gumbel, Fréchet, and Weibull families, as $\xi \to 0$, $\xi > 0$, and $\xi < 0$, respectively. Among these, the Fréchet distribution is most relevant for financial risk management because most financial risk factors have fatter tails than the normal. This family includes the Student's t distribution and the Pareto distribution.

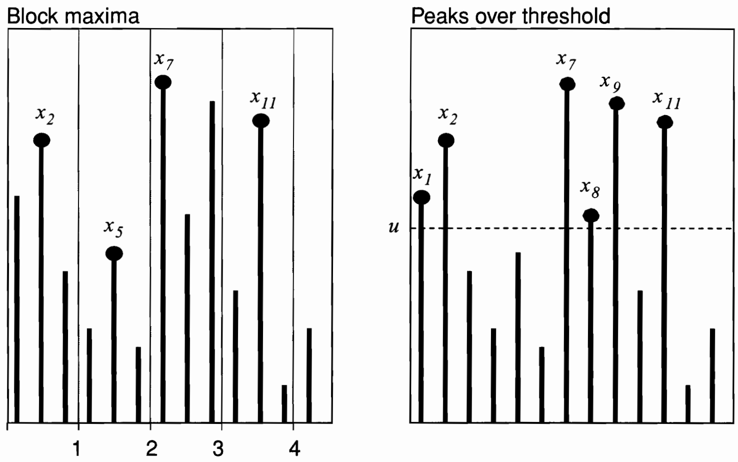

FIGURE 15.1 EVT Approaches

Another approach is the **block maxima**, where the sample is grouped into successive blocks, from which each maximum is identified. In this case, the limiting distribution of normalized maxima is the **generalized extreme value** (GEV) distribution.

The two approaches are compared in Figure 15.1. In the left panel, the selected observations are the maxima of groups of three. In the right panel, observations are selected whenever they are greater than u. The block maxima approach ignores extreme values x_9, x_1, and x_8, because they appear in a block with already one outlier.

In practice, the POT method is more widely used because it uses data more efficiently, even though it requires the choice of the threshold. It is better adapted to the risk measurement of tail losses because it focuses on the distribution of exceedances over a threshold.

Figure 15.2 illustrates the shape of the density function for U.S. stock market data using the GP distribution. The normal density falls off fairly quickly. With $\xi = 0.2$, the EVT density has a fatter tail than the normal density, implying a higher probability of experiencing large losses. This is an important observation for risk management purposes.

Note that the EVT density is only defined for the tail (i.e., when the loss x exceeds an arbitrary cutoff point, which is taken as 2 in this case). It says nothing about the rest of the distribution.

15.2.2 VAR and EVT

VAR, as well as CVAR, can be derived in closed-form solution from the analytical distribution in Equation (15.4). This requires estimation of the tail parameter ξ and of the dispersion parameter β.

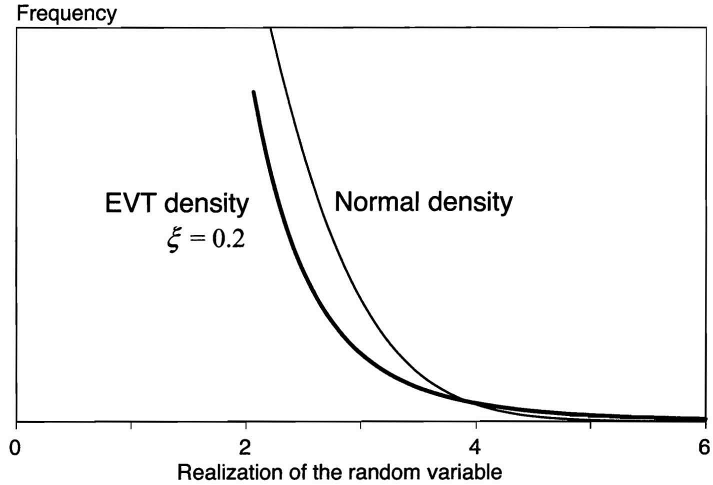

FIGURE 15.2 EVT and Normal Densities

This can be performed using a variety of statistical approaches. One method is **maximum likelihood**. First, we define a cutoff point u. This needs to be chosen so that there are a sufficient number of observations in the tail. However, the theory is most valid far into the tail. A good, ad hoc, choice is to choose u so as to include 5% of the data in the tail. For example, if we have $T = 1{,}000$ observations, we would consider only the 50 in the left tail. Second, we consider only losses beyond u and then maximize the likelihood of the observations over the two parameters ξ and β.

Another estimation method is the **method of moments**. This consists of fitting the parameters so that the GP moments equal the observed moments. This method is easier to implement but less efficient.

A third method, which is widely used, is **Hill's estimator**. The first step is to sort all observations from highest to lowest. The tail index is then estimated from

$$\widehat{\xi} = \frac{1}{k}\sum_{i=1}^{k} \ln X_i - \ln X_{k+1} \tag{15.5}$$

In other words, this is the average of the logarithm of observations from 1 to k minus the logarithm of the next observation. Unfortunately, there is no theory to help us choose k. In practice, one can plot ξ against k and choose the value in a flat area, where the estimator is not too sensitive to the choice of the cutoff point.

One issue with EVT is that it still relies on a small number of observations in the tail. Hence, estimates are sensitive to changes in the sample, albeit less so than with nonparametric VAR. More generally, EVT results also depend on the assumptions and estimation method. And in the end, it still relies on historical data, which may not give a complete picture of all financial risks.

EXAMPLE 15.7: FRM EXAM 2009—QUESTION 5-12

Extreme value theory (EVT) provides valuable insight about the tails of return distributions. Which of the following statements about EVT and its applications is *incorrect*?

a. The peaks over threshold (POT) approach requires the selection of a reasonable threshold, which then determines the number of observed exceedances; the threshold must be sufficiently high to apply the theory, but sufficiently low so that the number of observed exceedances is a reliable estimate.
b. EVT highlights that distributions justified by the central limit theorem (e.g., normal) can be used for extreme value estimation.
c. EVT estimates are subject to considerable model risk, and EVT results are often very sensitive to the precise assumptions made.
d. Because observed data in the tails of distribution is limited, EV estimates can be very sensitive to small sample effects and other biases.

EXAMPLE 15.8: FRM EXAM 2007—QUESTION 110

Which of the following statements regarding extreme value theory (EVT) is *incorrect*?

a. In contrast to conventional approaches for estimating VAR, EVT considers only the tail behavior of the distribution.
b. Conventional approaches for estimating VAR that assume that the distribution of returns follows a unique distribution for the entire range of values may fail to properly account for the fat tails of the distribution of returns.
c. EVT attempts to find the optimal point beyond which all values belong to the tail and then models the distribution of the tail separately.
d. By smoothing the tail of the distribution, EVT effectively ignores extreme events and losses that can generally be labeled outliers.

15.3 COHERENT RISK MEASURES

15.3.1 Desirable Properties for Risk Measures

The purpose of a risk measure is to summarize the entire distribution of dollar returns X by one number, $\rho(X)$. Artzner et al. (1999) list four desirable properties of risk measures for capital adequacy purposes:[3]

[3] See P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent Measures of Risk," *Mathematical Finance* 9 (1999): 203–228.

1. **Monotonicity:** if $X_1 \leq X_2$, $\rho(X_1) \geq \rho(X_2)$.
   In other words, if a portfolio has systematically lower values than another, that is, in each state of the world, it must have greater risk.
2. **Translation invariance:** $\rho(X + k) = \rho(X) - k$.
   In other words, adding cash k to a portfolio should reduce its risk by k. This reduces the lowest portfolio value. As with X, k is measured in dollars.
3. **Homogeneity:** $\rho(bX) = b\rho(X)$.
   In other words, increasing the size of a portfolio by a factor b should scale its risk measure by the same factor b. This property applies to the standard deviation.
4. **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.
   In other words, the risk of a portfolio must be less than, or at worst equal to, the sum of separate risks. If so, merging portfolios cannot increase risk.

The usefulness of these criteria is that they force us to think about ideal properties of risk measures and, more importantly, potential problems with simplified risk measures.

For instance, homogeneity seems reasonable in most cases. It is, however, questionable in the case of huge portfolios that could not be liquidated without substantial market impact. In this case, risk increases more than proportionately with the size of the portfolio. Thus, this property ignores liquidity risk. In practice, nearly all risk measures have this problem, which is very difficult to deal with.

Next, subadditivity implies that the risk of a portfolio must be less than the sum of risks for portfolio components. As we show in the next section, the quantile-based VAR measure fails to satisfy this property.

Assuming a normal distribution, however, the standard deviation–based VAR satisfies the subadditivity property. This is because the volatility of a portfolio is less than, or at worst equal to, the sum of volatilities: $\sigma(X_1 + X_2) \leq \sigma(X_1) + \sigma(X_2)$. More generally, subadditivity holds for **elliptical distributions**, for which contours of equal density are ellipsoids, such as the Student's t.

15.3.2 Example: VAR and Subadditivity

We now give an example where VAR fails to satisfy subadditivity. Consider a trader with an investment in a corporate bond with face value of \$100,000 and default probability of 0.5%. Over the next period, we can either have no default with a return of zero or default with a loss of \$100,000. The payoffs are thus −\$100,000 with probability of 0.5% and +\$0 with probability 99.5%. Since the probability of getting \$0 is greater than 99%, the VAR at the 99% confidence level is \$0, without taking the mean into account. This is consistent with the definition that VAR is the smallest loss such that the right-tail probability is at least 99%.

Now, consider a portfolio invested in three bonds (A, B, and C) with the same characteristics and independent payoffs. The VAR numbers add up to $\text{VAR}_S = \sum_i \text{VAR}_i = \$0$. We report the payoffs and probabilities in Table 15.3.

TABLE 15.3 Subadditivity and VAR: Example

State	Bonds	Probability	Payoff
No default		$0.995 \times 0.995 \times 0.995 = 0.9850749$	\$0
1 default	A, B, or C	$3 \times 0.005 \times 0.995 \times 0.995 = 0.0148504$	−\$100,000
2 defaults	AB, AC, or BC	$3 \times 0.005 \times 0.005 \times 0.995 = 0.0000746$	−\$200,000
3 defaults	A, B, and C	$0.005 \times 0.005 \times 0.005 = 0.0000001$	−\$300,000

Here, the probability of zero default is 0.9851, which is less than 99%. The portfolio VAR is therefore \$100,000, which is the lowest number such that the probability exceeds 99%. Thus the portfolio VAR is greater than the sum of individual VARs, which is zero. In this example, VAR is not subadditive. This is an undesirable property because it creates disincentives to aggregate the portfolio, since it appears to have higher risk.

Admittedly, this example is a bit contrived. Nevertheless, it illustrates the danger of focusing on VAR as a sole measure of risk.

15.3.3 Expected Shortfall

In contrast, **conditional VAR** (CVAR), also called **expected shortfall** or **expected tail loss**, does satisfy the subadditivity property. CVAR is the average of losses beyond VAR; CVAR $= E[-X \mid X < -\text{VAR}]$.

For each individual bond, there is only one observation in the tail, which leads to $\text{CVAR}_i = \$100{,}000$. The sum is $\text{CVAR}_S = \$300{,}000$. We now compute the CVAR for the portfolio. This is the probability-weighted average of losses worse than \$100,000, or $(0.0000746 \times \$200{,}000 + 0.0000001 \times \$300{,}000)/0.0000747 = \$200{,}167$. This is less than CVAR_S, hence showing that CVAR is a subadditive risk measure.

In addition, CVAR is better justified than VAR in terms of decision theory. Suppose an investor has to choose between two portfolios A and B with different distributions. Decision rules can be based on various definitions of stochastic dominance. One example is **first-order stochastic dominance** (FSD). This requires that the cumulative distribution function for A be systematically lower than that for B. So, B has a higher probability of a bad outcome. This is a very strict rule, however. Another example is **second-order stochastic dominance** (SSD). Portfolio A would dominate B if it has higher mean and lower risk. This is a more realistic rule than the first. Using CVAR as a risk measure is consistent with SSD, whereas VAR requires FSD, which is less realistic.

In practice, CVAR is rarely reported in the financial industry. It is more commonly used in the insurance industry, which has had traditionally greater focus on tail losses. In addition, statistical distributions for mortality rates and natural catastrophe events have long histories and are more amenable to expected shortfall analysis.

Even so, risk managers need to be aware of the shortcoming of summarizing an entire distribution with one number such as VAR. Traders might decide to create a portfolio with low VAR but very high CVAR, creating infrequent but very large losses. This is an issue with asymmetrical positions, such as short positions in options or undiversified portfolios exposed to credit risk. The next chapter gives an example of billions of losses suffered from senior tranches of collateralized debt obligations backed by subprime mortgages. Such senior tranches are similar to short positions in out-of-the-money options, which involve rare but catastrophic losses.

EXAMPLE 15.9: FRM EXAM 2008—QUESTION 2-25

A market risk manager uses historical information on 1,000 days of profit/loss information to calculate a daily VAR at the 99th percentile, or $8 million. Loss observations beyond the 99th percentile are then used to estimate the conditional VAR. If the losses beyond the VAR level, in millions, are $9, $10, $11, $13, $15, $18, $21, $24, and $32, then what is the CVAR?

a. $9 million
b. $32 million
c. $15 million
d. $17 million

EXAMPLE 15.10: FRM EXAM 2009—QUESTION 5-8

Greg Lawrence is a risk analyst at ES Bank. After estimating the 99%, one-day VAR of the bank's portfolio using historical simulation with 1,200 past days, he is concerned that the VAR measure is not providing enough information about tail losses. He decides to reexamine the simulation results. Sorting the simulated daily P&L from worst to best gives the following results:

Rank	1	2	3	4	5	6
P&L	−2,833	−2,333	−2,228	−2,084	−1,960	−1,751
Rank	7	8	9	10	11	12
P&L	−1,679	−1,558	−1,542	−1,484	−1,450	−1,428
Rank	13	14	15			
P&L	−1,368	−1,347	−1,319			

What is the 99%, one-day expected shortfall (ES) of the portfolio?

a. USD 433
b. USD 1,428
c. USD 1,861
d. USD 2,259

EXAMPLE 15.11: FRM EXAM 2009—QUESTION 5-14

Which of the following statements about expected shortfall (ES) is *incorrect*?

a. ES provides a consistent risk measure across different positions and takes account of correlations.
b. ES tells what to expect in bad states: It gives an idea of how bad the portfolio payoff can be expected to be if the portfolio has a bad outcome.
c. ES-based rule is consistent with expected utility maximization if risks are ranked by a second-order stochastic dominance rule.
d. Like VAR, ES does not always satisfy subadditivity (i.e., the risk of a portfolio must be less than or equal to the sum of the risks of its individual positions).

15.4 IMPORTANT FORMULAS

Expected number of exceptions in a sample of size T with VAR at confidence level $c = 1 - p$: $E[X] = p \times T$

Distribution of exceptions: $f(x) = \binom{T}{x} p^x (1-p)^{T-x}, \quad x = 0, 1, \ldots, n$

Basel rules for number n exceptions with $T = 252$, and $c = 99\%$:

Green zone: $0 \leq n \leq 4$

Yellow zone: $5 \leq n \leq 9$

Red zone: $10 \leq n$

Distribution of tails (EVT): $y = (x - u)/\beta \rightarrow$ generalized Pareto distribution

15.5 ANSWERS TO CHAPTER EXAMPLES

Example 15.1: FRM Exam 2003—Question 11

c. This is $p \times T = 10\% \times 250 = 25$.

Example 15.2: FRM Exam 2007—Question 101

d. We should expect $(1 - 95\%)250 = 12.5$ exceptions on average. Having eight exceptions is too few, but the difference could be due to luck. Having zero exceptions, however, would be very unusual, with a probability of $1 - (1 - 5\%)^{250}$, which is very low. This means that the risk manager is providing VAR estimates that are much too high. Otherwise, the largest or mean losses are not directly useful without more information on the distribution of profits.

Example 15.3: FRM Exam 2002—Question 20

c. VAR estimates need to be compared to actual P&L results to be validated, which is called backtesting.

Example 15.4: Penalty Zones

b. See Table 15.2.

Example 15.5: FRM Exam 2002—Question 23

d. The backtesting framework in the IMA only counts the number of times a daily exception occurs (i.e., a loss worse than VAR). So, this involves the number of outliers and the daily VAR measure.

Example 15.6: FRM Exam 2009—Question 5-6

b. The z-score gives $(8 - 2.5)/\sqrt{250 \times 0.01 \times 0.99} = 3.5$. This is too high (greater than 2), which leads to rejection of the null that the VAR model is well calibrated. Hence, VAR is too low and statement I. is correct. Statement II. is incorrect. However, this may be due to intraday trading, so III. is correct, too. Finally, if all eight exceptions occurred in the last month, there is bunching, and the model should be reexamined, so IV. is correct.

Example 15.7: FRM Exam 2009—Question 5-12

b. EVT estimates are subject to estimation risk, so statement c. and d. are correct. However, EVT does not apply the central limit theorem (CLT), which states that the average (as opposed to the tail) of i.i.d. random variables is normal.

Example 15.8: FRM Exam 2007—Question 110

d. EVT uses only information in the tail, so statement a. is correct. Conventional approaches such as delta-normal VAR assume a fixed probability density function (p.d.f.) for the entire distribution, which may understate the extent of fat tails, so statement b. is correct. The first step in EVT is to choose a cutoff point for the tail, and then to estimate the parameters of the tail distribution, so statement c. is correct. Finally, EVT does not ignore extreme events (as long as they are in the sample).

Example 15.9: FRM Exam 2008—Question 2-25

d. CVAR is the average of observations beyond VAR. This gives \$17 million. Answers a. and b. can be dismissed out of hand because they are too low and too high, respectively.

Example 15.10: FRM Exam 2009—Question 5-8

c. This looks like a computationally intensive question, but it can be answered using judgment. The 1% left tail for $T = 1{,}200$ is 12 observations, so VAR $= 1{,}428$. This rules out answers a. and b. The ES is then the average from observations 1 to 11. Using simple rank, the point in the middle is for observation 6, which is $-1{,}751$. The closest is 1,861, or answer c.

Example 15.11: FRM Exam 2009—Question 5-14

d. ES, like VAR, does provide a consistent measure of risk that takes diversification into account, so statement a. is correct. Unlike VAR, however, CVAR is a subadditive risk measure.

CHAPTER 16

Advanced Risk Models: Multivariate

We now turn to the measurement of risk across large portfolios. A risk system has three components: (1) a portfolio position system, (2) a risk factor modeling system, and (3) an aggregation system. The first component is described in Section 16.1. Portfolio positions must be collected and then processed through **mapping**, which consists of replacing each instrument by its exposures on selected risk factors. Mapping considerably simplifies the risk measurement process. It would not be feasible to model all instruments individually, because there are too many. The art of risk management consists of choosing a set of limited risk factors that will adequately cover the spectrum of risks for the portfolio at hand.

The second step is to describe the joint movements in the risk factors. Two approaches are possible. The first specifies an analytical distribution, for example, a normal joint distribution. More generally, the joint movements can be characterized by copulas, which are described in Section 16.2. In a second approach, the joint distribution could be simply taken from empirical observations, without making any additional assumptions.

The third step brings together positions and risk factors. Section 16.3 describes the three main value at risk (VAR) methods, which include the delta-normal method, the historical simulation (HS) method, and the Monte Carlo simulation method. The methods are illustrated with an example in Section 16.5.

A question of particular interest is the performance of VAR models during the recent credit crisis. This is discussed in Section 16.4, which also expands on general drawbacks of risk models.

16.1 RISK MAPPING

16.1.1 Risk Simplification

The fundamental idea behind modern risk measurement methods is to aggregate the portfolio risk at the highest level. In practice, it is often too complex to model

FRM Exam Part 2 topic.

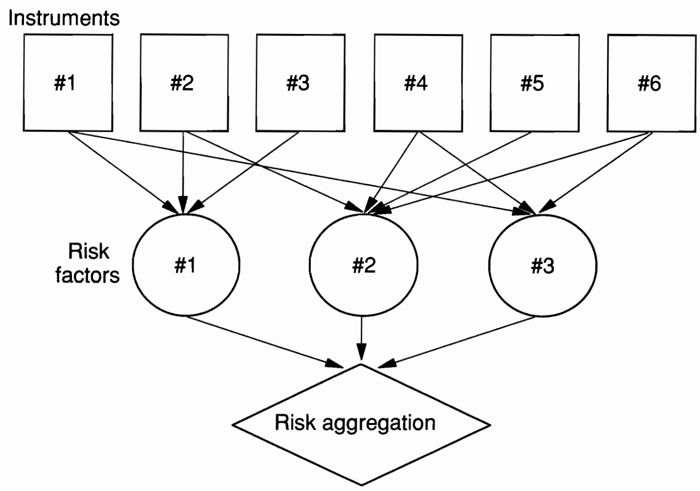

FIGURE 16.1 Mapping Approach

each position individually. Instead, some simplification is required. This is the role of **risk mapping**, which replaces exposures to individual positions by aggregate exposures to major risk factors.

Of course, this idea is not new. Consider, for example, a portfolio with many stock positions. William Sharpe devised a method to simplify the measurement of risk for these portfolios. His **diagonal model** decomposes individual stock return movements into a common index component and an idiosyncratic component. In large, well-diversified portfolios, the idiosyncratic component washes out, leaving the common component as the main driver of risk. This justifies mapping individual stocks to positions on the same index.

More generally, this methodology can be used in any market. Figure 16.1 illustrates the mapping process in a case with six instruments, say different forward contracts on the same currency but with different maturities. The risk manager judges that these positions can be replaced by exposures on three risk factors only. We give a fully worked-out example later in this chapter.

There are three steps in the process:

1. Replace each of the $N = 6$ positions with a $K = 3$ exposure on the risk factors. Define $x_{i,k}$ as the exposure of instrument i to risk factor k.
2. Aggregate the K exposures across the positions in the portfolio, $x_k = \sum_{i=1}^{N} x_{i,k}$. This generates $K = 3$ values for dollar exposures.
3. Derive the distribution of the portfolio return $R_{p,t+1}$ from the exposures and movements in risk factors, Δf, using one of the three VAR methods.

16.1.2 Mapping with Factor Models

The diagonal model is a simplified form of a factor model with one factor only, selected as a passive market index. This model starts with a statistical decomposition of the return on stock i into a marketwide return and a residual term, sometimes called idiosyncratic or specific. We decompose the return on stock i, R_i, into (1) a constant, (2) a component due to the market, R_M, and (3) a residual term:

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \tag{16.1}$$

where β_i is called systematic risk of stock i. Note that the residual is uncorrelated with R_M by assumption. The diagonal model adds the assumption that all specific risks are uncorrelated. Hence, any correlation across two stocks must come from the joint effect of the market.

The contribution of William Sharpe was to show that equilibrium in capital markets imposes restrictions on the α_i. For risk managers, however, the intercept is not the primary focus and is neglected in what follows. Instead, this model simplifies the risk measurement process.

Consider a portfolio that consists of simple, linear positions w_i on the various assets. We have

$$R_p = \sum_{i=1}^{N} w_i R_i \tag{16.2}$$

Using Equation (16.1), the portfolio return is also

$$R_p = \sum_{i=1}^{N} (w_i \beta_i R_M + w_i \epsilon_i) = \beta_p R_M + \sum_{i=1}^{N} (w_i \epsilon_i) \tag{16.3}$$

where the weighted average beta is

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i \tag{16.4}$$

The portfolio variance is

$$V[R_p] = \beta_p^2 V[R_M] + \sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i]\right) \tag{16.5}$$

since all the residual terms are uncorrelated. Suppose that, for simplicity, the portfolio is equally weighted, $w_i = w = 1/N$, and that the residual variances are all the same, $V[\epsilon_i] = V$. As the number of assets increases, the second term tends to

$$\sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i]\right) \rightarrow N \times [(1/N)^2 V] = (V/N)$$

which should vanish as N increases. In this situation, the only remaining risk is the general market risk, consisting of the beta squared times the variance of the market:

$$V[R_p] \rightarrow \beta_p^2 V[R_M]$$

So, this justifies ignoring specific risk in large, well-diversified portfolios. The mapping approach replaces a dollar amount of x_i in stock i by a dollar amount of $x_i\beta_i$ on the index:

$$x_i \text{ on stock } i \rightarrow (x_i\beta_i) \text{ on index} \tag{16.6}$$

More generally, this approach can be expanded to multiple factors. The appendix at the end of this chapter shows how this approach can be used to build a covariance matrix from general market factors. Each security is first mapped on the selected risk factors. Exposures are then added up across the entire portfolio, for which risk is aggregated at the top level.

This mapping approach is particularly useful when there is no history of returns for some positions. Instead, the positions can be mapped on selected risk factors. Consider, for example, a stock that just went through an **initial public offering** (IPO). This stock, has no history. Such stocks can be quite risky. A practical solution is to map this position on a stock index with similar characteristics (e.g., an index of small-cap, high-tech stocks).

16.1.3 Mapping Fixed-Income Portfolios

As another important example of portfolio simplification, we turn to the analysis of a risk-free bond portfolio. This portfolio will have different payments coming due at different points in time, ranging from the next day to 30 years from now. It would be impractical to model all these maturities individually. Instead, simplifications are used for the risk-free term structure.

One simplification is **maturity mapping**, which replaces the current value of each bond by a position on a risk factor with the same maturity. This ignores intervening cash flows, however. A better approach is **duration mapping**, which maps the bond on a zero-coupon risk factor with a maturity equal to the duration of the bond. A third approach, which is even more precise but more complex, is **cash flow** (CF) **mapping**, which maps the current value of each bond payment on a zero-coupon risk factor with maturity equal to the time to wait for each cash flow.

Consider now another example, which is a corporate bond portfolio. This increases the dimensionality of risk factors into both risk-free term structure factors and credit factors. Again, it would be impractical to try to model all securities individually. There may not be sufficient price history on each bond. In addition, the history may not be relevant if it does not account for the probability of default. More generally, the history may not represent the current credit rating nor the duration of this bond.

In what follows, we assume that the risk manager uses duration mapping and takes movements in yield curves rather than prices as risk factors. The risk manager judges that risk factors can be restricted to a set of J Treasury zero-coupon rates, z_j, and of K credit spreads, s_k, sorted by credit rating. Ideally, these should be sufficient to provide a good approximation of the risk of the portfolio.

We then model the movement in each corporate bond yield y_i by a movement in the Treasury factor z_j at the closest maturity and in the credit rating s_k class to which it belongs. Alternatively, we can interpolate. The remaining component is ϵ_i, which is assumed to be independent across i. We have $y_i = z_j + s_k + \epsilon_i$. This decomposition is illustrated in Figure 16.2 for a corporate bond rated BBB with a 20-year maturity.

The movement in the value of bond price i is

$$\Delta P_i = -\text{DVBP}_i \Delta y_i = -\text{DVBP}_i\, \Delta z_j - \text{DVBP}_i\, \Delta s_k - \text{DVBP}_i\, \Delta \epsilon_i \tag{16.7}$$

where DVBP is the total dollar value of a basis point for the associated risk factor, which is directly derived from modified duration times the bond value.

We now aggregate the exposures across N positions. We hold n_i units of each bond,

$$P = \sum_{i=1}^{N} n_i P_i \tag{16.8}$$

which gives a price change of

$$\Delta P = \sum_{i=1}^{N} n_i \Delta P_i = -\sum_{i=1}^{N} n_i \text{DVBP}_i \Delta y_i \tag{16.9}$$

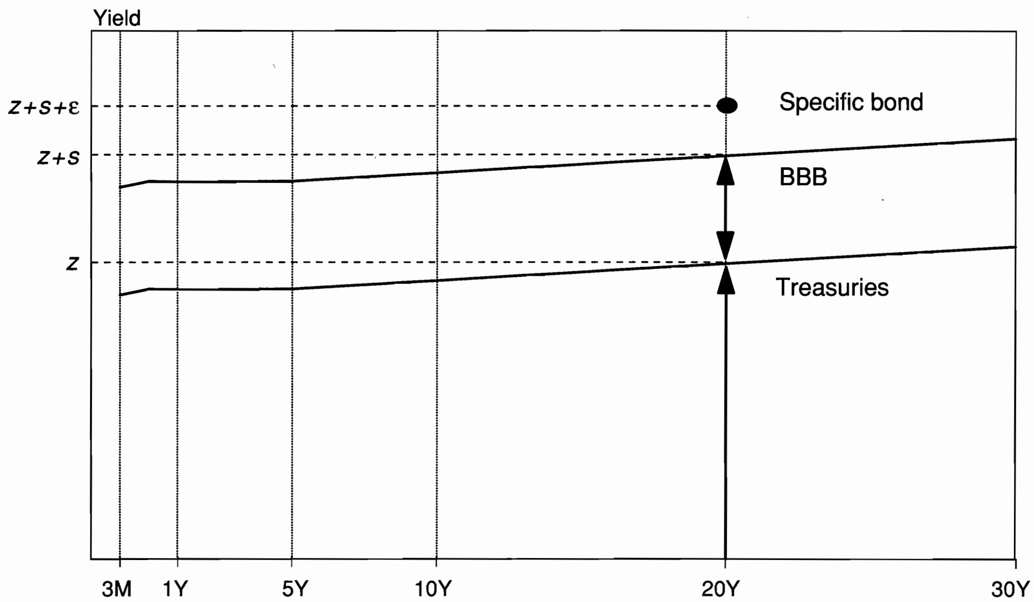

FIGURE 16.2 Yield Decomposition

Using the risk factor decomposition, the portfolio price movement is

$$\Delta P = -\sum_{j=1}^{J} \text{DVBP}_j^z \, \Delta z_j - \sum_{k=1}^{K} \text{DVBP}_k^s \, \Delta s_k - \sum_{i=1}^{N} n_i \text{DVBP}_i \, \Delta \epsilon_i \tag{16.10}$$

where DVBP_j^z results from the summation of $n_i\text{DVBP}_i$ for all bonds that are exposed to the jth maturity, and likewise for DVBP_k^s. As in Equation (16.5) for equity portfolios, the total variance can be decomposed into

$$V[\Delta P] = \text{General Risk} + \sum_{i=1}^{N} n_i^2 \text{DVBP}_i^2 \; V[\Delta \epsilon_i] \tag{16.11}$$

If the portfolio is well diversified, the general risk term should dominate. So, we could simply ignore the second term. Ignoring specific risk, a portfolio composed of thousands of securities can be characterized by its exposure to just a few government maturities and credit spreads. This is a considerable simplification.

The mapping approach replaces a dollar amount of x_i in bond i by dollar exposures on two risk factors:

$$x_i \text{ on bond } i \rightarrow (n_i\text{DVBP}_i) \text{ on yield}_j + (n_i\text{DVBP}_i) \text{ on spread}_k \tag{16.12}$$

16.1.4 Mapping: Choice of Risk Factors

The choice of risk factors should be driven by the nature of the portfolio. A diagonal risk model may be sufficient for portfolios of stocks that have many small positions well dispersed across sectors. For a portfolio with a small number of stocks concentrated in one sector, however, this approach will underestimate risk. Similarly, an equity market-neutral portfolio, which consists of long and short equity positions with essentially zero beta, will appear as having no risk, which is not the case.

A simple mapping approach on one interest rate risk factor may be perfectly adequate for a long-only portfolio. Essentially, all the risk exposures are summarized in one number, which is the dollar duration.

Consider next a trading portfolio where the portfolio manager is both long and short various bonds and has a net duration of zero. In this case, the duration model gives a total risk of zero, which is misleading. More factors are needed. Another example is a portfolio that has long and short positions in various options, across strike prices and maturities. In this case, simply considering the linear or even quadratic exposure to the underlying risk factor may not be sufficient. The risk manager should add movements in implied volatilities. As markets evolve toward more complex financial products, the risk manager has to make sure that the risk models do not lag behind and miss major risks.

KEY CONCEPT

The number of risk factors and the complexity of risk models depend on the depth of the trading strategies. In general, complex portfolios require more complex risk models.

EXAMPLE 16.1: FRM EXAM 2009—QUESTION 2-7

Which of these statements regarding risk factor mapping approaches is/are correct?

I. Under the cash flow (CF) mapping approach, only the risk associated with the average maturity of a fixed-income portfolio is mapped.
II. Cash flow mapping is the least precise method of risk mapping for a fixed-income portfolio.
III. Under the duration mapping approach, the risk of a bond is mapped to a zero-coupon bond of the same duration.
IV. Using more risk factors generally leads to better risk measurement but also requires more time to be devoted to the modeling process and risk computation.

a. I and II
b. I, III, and IV
c. III and IV
d. IV only

EXAMPLE 16.2: FRM EXAM 2002—QUESTION 44

The historical simulation (HS) approach is based on the empirical distributions and a large number of risk factors. The RiskMetrics approach assumes normal distributions and uses mapping on equity indices. The HS approach is more likely to provide an accurate estimate of VAR than the RiskMetrics approach for a portfolio that consists of

a. A small number of emerging market securities
b. A small number of broad market indices
c. A large number of emerging market securities
d. A large number of broad market indices

EXAMPLE 16.3: FRM EXAM 2007—QUESTION 11

A hedge fund manager has to choose a risk model for a large equity market-neutral portfolio, which has zero beta. Many of the stocks held are recent IPOs. Among the following alternatives, the best is

a. A single index model with no specific risk, estimated over the last year
b. A diagonal index model with idiosyncratic risk, estimated over the last year
c. A model that maps positions on industry and style factors
d. A full covariance matrix model using a very short window

16.2 JOINT DISTRIBUTIONS OF RISK FACTORS

The second component in risk systems is setting up the joint distribution of risk factors. One approach is nonparametric and consists of using recent observations. The other approach is parametric and requires specifying an analytical function for the joint distribution as well as its parameters. This section introduces the concept of the copula, which is central to the joint distribution of risk factors.

Copulas are used extensively for modeling financial instruments such as **collateralized debt obligations** (CDOs). CDOs are pools of N debt obligations, the value of each of which depends on the creditworthiness of the borrower. The risk of each credit taken one at a time can be described by a marginal distribution. Ultimately, however, what matters is the distribution of losses on the total portfolio. Hence, the financial engineering of CDOs requires modeling the joint movements in the individual credits. For portfolios of traded assets, the joint distributions can be assessed from historical data. Credit portfolios, however, do not have this luxury because the current credits in the portfolio do not have a history of defaults. Hence constructing the distribution of losses on a CDO must rely on a parametric approach, which requires the specification of a copula. Copulas are also used for enterprise-wide risk measurement, to aggregate market risk, credit risk, and operational risk.

16.2.1 Marginal Densities and Distributions

Marginal distributions consider each risk factor in isolation. This reduced dimensionality makes it easier to model the risk factor. In contrast, joint distributions are much more complex because of the higher dimensionality, which requires many more parameters to estimate. This creates serious difficulties, which in practice have caused major losses in financial markets, as we will see in a later section.

From Chapter 2, recall that a probability density function $f(u)$ describes the probability of observing a value around u. A normal density, for example, has the familiar bell-shaped curve. In contrast, a **distribution function** is the cumulative density $F(x) = \int_{-\infty}^{x} f(u)du$, and is denoted as an upper case. The value of this function is always between zero and one: $0 \leq F(x) \leq 1$.

Consider now a simple case with two risk factors. The question is how to link marginal densities, or distributions, across these risk factors.

16.2.2 Copulas

When the two variables are independent, the joint density is simply the product of the marginal densities. It is rarely the case, however, that financial variables are independent. Dependencies can be modeled by a function called the **copula**, which links, or attaches, marginal distributions into a joint distribution.

Formally, the copula is a function of the values of the marginal distributions $F(x)$ plus some parameters, θ, that are specific to this function (and not to the marginals). In the bivariate case, it has two arguments:

$$c_{12}[F_1(x_1), F_2(x_2); \theta] \tag{16.13}$$

The link between the joint and marginal distribution is made explicit by *Sklar's theorem*, which states that for any joint density there exists a copula that links the marginal densities:

$$f_{12}(x_1, x_2) = f_1(x_1) \times f_2(x_2) \times c_{12}[F_1(x_1), F_2(x_2); \theta] \tag{16.14}$$

With independence, the copula function is a constant always equal to one.

Thus the copula contains all the information on the nature of the dependence between the random variables but gives no information on the marginal distributions.

As an example, consider a normal multivariate density with N random variables. The joint normal density can be written as a function of the vector x, of the means μ, and of the covariance matrix Σ:

$$f^N(x_1, \ldots, x_N) = \frac{1}{(2\pi)^{N/2} \mid \Sigma \mid^{1/2}} \exp[-\frac{1}{2}(x-\mu)'\Sigma(x-\mu)] \tag{16.15}$$

Using the concept of copulas, this can be separated into N different marginal normal densities and a joint normal copula. With two variables,

$$f_{12}^N(x_1, x_2) = f_1^N(x_1) \times f_2^N(x_2) \times c_{12}^N[F_1(x_1), F_2(x_2); \rho] \tag{16.16}$$

Here, both f_1^N and f_2^N are normal marginals. They have parameters μ_1 and σ_1, and μ_2 and σ_2. In addition, c_{12}^N is the normal copula. Note that its sole parameter is the correlation coefficient ρ_{12}.

In particular, with standard normal variables, where $\mu_i = 0$ and $\sigma_i = 1$, the bivariate normal density reduces to

$$f_{12}^N(x_1, x_2) = \frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{-\frac{(x_1^2 + 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)}\right\} \tag{16.17}$$

When the correlation is zero, this gives

$$f_{12}^N(x_1, x_2) = \frac{1}{2\pi}\exp\left\{-\frac{(x_1^2 + x_2^2)}{2}\right\} = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x_1^2)}{2}\right\} \times \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x_2^2)}{2}\right\} \tag{16.18}$$

which is indeed the product of the marginal normal densities for x_1 and x_2, and of the copula, which is unity because the variables are independent.

Thus the copula can be separated from the marginals. In the case of a joint normal distribution, both the marginals and the copula are normal. This need not be the case, however. One could mix marginals from one family with a copula from another family.

The normal copula assumes a linear relationship between the risk factors, which is measured by the usual Pearson correlation coefficient. In general, however, the relationship does not need to be linear, which can be accommodated by other types of copulas.

The risk manager should choose the copula that provides the best fit to the data. This involves a choice of, first, an analytical function and, second, the best parameters using standard statistical tools such as maximum likelihood estimation.

An important feature of copulas is their **tail dependence**. This derives from the conditional probabilities of an extreme move in one variable given an extreme move in another variable. Formally, the upper and lower conditional probabilities are, for a given confidence level c,

$$P_U(c) = P[X_2 > F_{X_2}^{-1}(c) | X_1 > F_{X_1}^{-1}(c)] \tag{16.19}$$

$$P_L(c) = P[X_2 \leq F_{X_2}^{-1}(c) | X_1 \leq F_{X_1}^{-1}(c)] \tag{16.20}$$

Upper tail dependence is the limit when c goes to one; lower tail dependence when c goes to zero.

When the tail dependence parameters are zero, a copula is said to exhibit **tail independence**. This is the case for the normal copula. Consider for example a lower cutoff probability of 16%, which corresponds to $F_X^{-1} = -1$. Next, compute the conditional probability that $X_2 \leq -1$ given that $X_1 \leq -1$. Assuming a correlation of 0.50, this is $P_L(16\%) = P[X_2 \leq -1 | X_1 \leq -1] = 42\%$. Next, change the cutoff point from -1 to -1.645. The conditional probability then goes down from 42% to $P_L(5\%) = 22\%$. Continue with a lower cutoff point of -2.326. The conditional probability continues to decrease to $P_L(1\%) = 9\%$. Eventually, this converges to