Numerical Analysis Report 1

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Program Structure Description

Classes and Their Structures

Mathematical Theories Used

0.1 pp form splines

From Theorem 3.3, we have $m_i = s'(f; x_i)$, for $s \in \mathcal{S}_3^2$, for $i = 2, 3, \cdots, N-1$, there is:

$$\lambda_i m_{i-1} + 2m_i + \mu_i m_{i+1} = 3\mu_i f[x_i, x_{i+1}] + 3\lambda_i f[x_{i-1}, x_i] \tag{1}$$

In the formula,

$$\mu_i = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} = \frac{x_i - x_{i-1}}{x_{i+1} - x_i + x_i - x_{i-1}} = \frac{h_i}{h_{i+1} + h_i}, \quad \lambda_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} = \frac{h_{i+1}}{h_{i+1} + h_i}, \tag{2}$$

0.1.1 Cubic Splines

Cubic splines satisfy the boundary conditions

$$s'(f; a) = f'(a) \;\; and \;\; s'(f; b) = b \tag{3}$$

That is, the first-order derivatives of the curve at the two endpoints of the interval are given. Let $b_i = 3\mu_{i+1} f[x_{i+1}, x_{i+2}] + 3\lambda_{i+1} f[x_i, x_{i+1}]$, for $i = 1, \ldots, N-2$, then we have:

$$\begin{bmatrix} 2 & \mu_2 & & & & & \\ \lambda_3 & 2 & \mu_3 & & & & \\ & & \ddots & & & & \\ & & \lambda_i & 2 & \mu_i & & \\ & & & & \ddots & & \\ & & & & \lambda_{N-2} & 2 & \mu_{N-2} \\ & & & & & \lambda_{N-1} & 2 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \\ \vdots \\ m_i \\ \vdots \\ m_{N-2} \\ m_{N-1} \end{bmatrix} = \mathrm{b} = \begin{bmatrix} b_1 - \lambda_2 m_1 \\ b_2 \\ \vdots \\ b_{i-1} \\ \vdots \\ b_{N-3} \\ b_{N-2} - \mu_{N-1} m_N \end{bmatrix} \tag{4}$$

0.1.2 Natural Cubic Splines

Natural cubic splines are known for the second-order derivative values at the two endpoints:

$$s''(f;a) = 0 \quad s''(f;b) = 0 \tag{5}$$

From Theorem 3.3, we know

$$s_i(x) = f_i + (x - x_i)m_i + (x - x_i)^2 \frac{K_i - m_i}{x_{i+1} - x_i} + (x - x_i)^2(x - x_{i+1})\frac{m_i + m_{i+1} - 2K_i}{(x_{i+1} - x_i)^2} \tag{6}$$

Taking the second derivative of the above formula, we get

$$s_i''(x) = 2\frac{K_i - m_i}{x_{i+1} - x_i} + 2(x - x_{i+1})\frac{m_i + m_{i+1} - 2K_i}{(x_{i+1} - x_i)^2} + 4(x - x_i)\frac{m_i + m_{i+1} - 2K_i}{(x_{i+1} - x_i)^2} \tag{7}$$

Organizing, we get:

$$s_i''(x) = \frac{6x - 4x_i - 2x_{i+1} - 2h_i}{h_i^2}m_i + \frac{6x - 4x_i - 2x_{i+1}}{h_i^2}m_{i+1} + K_i\frac{-12x + 8x_i + 4x_{i+1} + 2h_i}{h_i^2} \tag{8}$$

The above formula can be transformed into:

$$s_i''(x) = \frac{6x - 2x_i - 4x_{i+1}}{h_i^2}m_i + \frac{6x - 4x_i - 2x_{i+1}}{h_i^2}m_{i+1} + \frac{-12x + 6x_i + 6x_{i+1}}{h_i^2}K_i \tag{9}$$

If the curve is on $[x_1, x_2]$, let $s_i''(x_1) = 0$, then we have:

$$s_1''(x_1) = -\frac{4}{h_1}m_0 - \frac{2}{h_1}m_1 + \frac{6}{h_1}\frac{f_2 - f_1}{h_1} \tag{10}$$

Then we have:

$$2m_0 + m_1 = 3f[x_1, x_2] - \frac{2}{h_1}s_1''(x_1) = 3f[x_1, x_2] \tag{11}$$

Similarly, we can get $m_{N-1} + 2m_N = 3f[x_{N-1}, x_N]$

So we have:

$$\begin{cases} 2m_1 + m_2 = 3f[x_1, x_2] = g_0 \\ m_{N-1} + 2m_N = 3f[x_{N-1}, x_N] = g_{N-1} \end{cases} \tag{12}$$

Thus, we get:

$$\begin{bmatrix} 2 & 1 & & & & & \\ \lambda_2 & 2 & \mu_2 & & & & \\ & & \ddots & & & & \\ & & \lambda_i & 2 & \mu_i & & \\ & & & & \ddots & & \\ & & & & \lambda_{N-1} & 2 & \mu_{N-1} \\ & & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ m_i \\ \vdots \\ m_{N-1} \\ mN \end{bmatrix} = \mathrm{b} = \begin{bmatrix} g_0 \\ b_1 \\ \vdots \\ b_{i-1} \\ \vdots \\ b_{N-2} \\ g_{N-1} \end{bmatrix} \tag{13}$$

Among them

0.1.3 Periodic Cubic Splines

Periodic cubic splines satisfy:

$$s(f;b) = s(f;a), s'(f;b) = s'(f;a), s''(f;b) = s''(f;a) \tag{14}$$

That is

$$s(x_1) = s(x_N) \quad s'(x_1) = s'(x_N) \quad s''(x_1) = s''(x_N) \tag{15}$$

From the condition of equal first derivatives, we get:

$$s_1'(x_1) = m_1 \tag{16}$$

$$s_{N-1}'(x_N) = m_N \tag{17}$$

From the condition of equal second derivatives, we get:

$$-\frac{4}{h_1}m_1 - \frac{2}{h_1}m_2 + \frac{6}{h_1}K_1 = \frac{2}{h_{N-1}}m_{N-1} + \frac{4}{h_{N-1}}m_N - \frac{6}{h_{N-1}^2}K_{N-1} \tag{18}$$

Further derivation gives:

$$\frac{1}{h_1}m_2 + \frac{1}{h_{N-1}}m_{N-1} + (\frac{1}{h_{N-1}} + \frac{1}{h_1})m_N = 3\frac{1}{h_1}K_1 + 3\frac{1}{h_{N-1}}K_{N-1} \tag{19}$$

Let $\lambda_N = \frac{x_2-x_1}{x_2-x_1+x_N-x_{N-1}}$ and $\mu_N = \frac{x_N-x_{N-1}}{x_2-x_1+x_N-x_{N-1}}$, then we have:

$$\begin{cases} m_1 = m_N \\ \mu_N m_2 + 2\lambda_N m_{N-1} + 2m_N = 3(\mu_N f[x_1,x_2] + \lambda_N f[x_{N-1},x_N]) \end{cases} \tag{20}$$

Thus, we have the system of equations:

$$\begin{bmatrix} 2 & \mu_2 & & & & & & \\ \lambda_3 & 2 & \mu_3 & & & & & \\ & & \ddots & & & & & \\ & & \lambda_i & 2 & \mu_i & & & \\ & & & & \ddots & & & \\ & & & & \lambda_{N-2} & 2 & \mu_{N-2} & \\ & & & & & \lambda_{N-1} & 2 & \mu_{N-1} \\ \mu_N & & & & & & 2\lambda_N & 2 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \\ \vdots \\ m_i \\ \vdots \\ m_{N-2} \\ m_{N-1} \\ m_N \end{bmatrix} = \mathrm{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_{N-2} \\ g_N \end{bmatrix} \tag{21}$$

0.2 B-splines

From Theorem 3.57, there exists a unique $S(x) \in \mathcal{S}_3^2$ that interpolates $f(x)$ at points $1, 2, \cdots, N$, and has $S'(1) = f'(1)$ and $S'(N) = f'(N)$. Therefore, the B-spline interpolation function is:

$$S(x) = \sum_{i=-1}^{N} a_i B_i^3(x) \tag{22}$$

The coefficients a_i are obtained by solving the linear system $Ma = b$.

$$f(t_i) = S(t_i) = a_{i-2}B_{i-2}^3(t_i) + a_{i-1}B_{i-1}^3(t_i) + a_i B_i^3(t_i) \tag{23}$$

Based on the above equation and introducing boundary conditions, the coefficients $a = [a_{-1}, a_0, a_1, \cdots, a_N]^T$ can be solved.

0.2.1 Clamped Cubic Splines

For clamped cubic splines, there are boundary conditions where the first derivatives at the endpoints are equal:

$$S'(t_1) = f'(t_1) \tag{24}$$

$$S'(t_N) = S'(t_N) \tag{25}$$

And $S'(x) = \sum_{i=-1}^{N} a_i B_i'^3(x)$. Thus, the linear system is:

$$M = \begin{bmatrix} B_{-1}'^3(t_1) & B_0'^3(t_1) & B_1'^3(t_1) & & \\ B_{-1}^3(t_1) & B_0^3(t_1) & B_1^3(t_1) & & \\ & \ddots & \ddots & \ddots & \\ & & B_{N-2}^3(t_N) & B_{N-1}^3(t_N) & B_N^3(t_N) \\ & & B_{N-2}'^3(t_N) & B_{N-1}'^3(t_N) & B_N'^3(t_N) \end{bmatrix} \tag{26}$$

$$b^T = [f'(t_1), f(t_1), \cdots, f(t_N), f'(t_N)] \tag{27}$$

0.2.2 Periodic Boundary Conditions

For periodic boundary conditions, we have:

$$S(t_1) = S(t_N) \tag{28}$$

$$S'(t_1) = S'(t_N) \tag{29}$$

$$S''(t_1) = S''(t_N) \tag{30}$$

Thus, we have:

$$M = \begin{bmatrix} -B_{-1}'^3(t_1) & -B_0'^3(t_1) & -B_1'^3(t_1) & 0 & \cdots & 0 & B_{N-2}'^3(t_N) & B_{N-1}'^3(t_N) & B_N'^3(t_N) \\ B_{-1}^3(t_1) & B_0^3(t_1) & B_1^3(t_1) & & & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & & & & 0 & B_{N-2}^3(t_N) & B_{N-1}^3(t_N) & B_N^3(t_N) \\ -B_{-1}''^3(t_1) & -B_0''^3(t_1) & -B_1''^3(t_1) & 0 & \cdots & 0 & B_{N-2}''^3(t_N) & B_{N-1}''^3(t_N) & B_N''^3(t_N) \end{bmatrix} \tag{31}$$

$$b^T = [0, f(t_1), \cdots, f(t_{N-1}), f(t_N), 0]^T \tag{32}$$

0.2.3 Natural Splines

The natural boundary conditions are:

$$S''(t_1) = S''(t_N) = 0 \tag{33}$$

Thus, by solving the linear system $Ma = b$, the coefficients of the B-splines are obtained:

$$M = \begin{bmatrix} B_{-1}''^3(t_1) & B_0''^3(t_1) & B_1''^3(t_1) & & \\ B_{-1}^3(t_1) & B_0^3(t_1) & B_1^3(t_1) & & \\ & \ddots & \ddots & \ddots & \\ & & B_{N-2}^3(t_N) & B_{N-1}^3(t_N) & B_N^3(t_N) \\ & & B_{N-2}''^3(t_N) & B_{N-1}''^3(t_N) & B_N''^3(t_N) \end{bmatrix} \tag{34}$$

$$b^T = [0, f(t_1), \cdots, f(t_N), 0]^T \tag{35}$$