Numerical Analysis homework week1

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I. Consider the bisection method starting with the initial interval [1.5, 3.5]. In the following questions "the interval" refers to the bisection interval whose width changes across different loops.

I-a What is the width of the interval at the nth step?

The initial interval is $[1.5, 3.5]$, and its width is:

$$b - a = 3.5 - 1.5 = 2$$

At each iteration, the width is halved. Therefore, at the nth step, the interval width is:

$$\mathrm{d} = \frac{2}{2^{n-1}}$$

I-b What is the supremum of the distance between the root r and the midpoint of the interval?

At the n-th step, the current interval width is $\frac{2}{2^{n-1}}$, and the midpoint is located at the center of this interval.

The root r can be at any position within the interval, and the farthest distance from the midpoint is half of the current interval width. If the current interval width is $\frac{2}{2^{n-1}}$, the maximum distance is:

$$d_n = \frac{1}{2} \times \frac{2}{2^n} = \frac{1}{2^n}$$

Therefore, at the n-th step, the supremum of the distance between the root r and the midpoint of the interval is $\frac{1}{2^n}$.

II. In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its relative error no greater than.Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$n \geq \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1.$$

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To ensure that the relative error of the root is no greater than ϵ using the bisection method with an initial interval $[a_0, b_0]$ where $a_0 > 0$, we need to choose the number of steps n appropriately.

The length of the interval after n steps is:

$$\frac{b_0 - a_0}{2^n}$$

We want this length to be at most $\epsilon \cdot a_0$, so：

$$\frac{b_0 - a_0}{2^n} \leq \epsilon \cdot a_0$$

Taking the logarithm of both sides:

$$\log\left(\frac{b_0 - a_0}{2^n}\right) \leq \log(\epsilon \cdot a_0)$$

$$\log(b_0 - a_0) - n\log 2 \leq \log\epsilon + \log a_0$$

$$-n\log 2 \leq \log\epsilon + \log a_0 - \log(b_0 - a_0)$$

$$n \geq \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2}$$

steps must be integers, the final number of steps n should be:

$$n \geq \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1$$

This ensures that the relative error of the root is no greater than ϵ.

III. Perform four iterations of Newton’ s method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

Calculate the function $p(x)$ and its derivative $p'(x)$, then use Newton’s method iteration formula:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$$

$$p(x) = 4x^3 - 2x^2 + 3$$

$$p'(x) = 12x^2 - 4x$$

First Iteration: $x_0 = -1$

$$p(-1) = 4(-1)^3 - 2(-1)^2 + 3 = -4 - 2 + 3 = -3$$

$$p'(-1) = 12(-1)^2 - 4(-1) = 12 + 4 = 16$$

$$x_1 = -1 - \frac{-3}{16} = -1 + \frac{3}{16} = -\frac{13}{16} \approx -0.8125$$

Second Iteration: $x_1 = -0.8125$

$$p(-0.8125) = 4(-0.8125)^3 - 2(-0.8125)^2 + 3$$

so:

$$p(-0.8125) = -2.14453125 - 1.3125 + 3 = -0.4658$$

$$x_2 = -0.8125 - \frac{-0.45703125}{11.125} = -0.8125 + 0.041 \approx -0.7708$$

Similarly, other values can be calculated iteratively. Below is the table summarizing the results:

Iteration	x_n	$p(x_n)$	$p'(x_n)$
0	-1.0000	-3.0000	16
1	-0.8125	-0.4658	11.1719
2	-0.7708	-0.0201	10.2129
3	-0.7688	-4.3708e-05	10.1686
4	-0.7688	-2.0741e-10	10.1685

IV. Consider a variation of Newton's method in which only the derivative at x_0 is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

Find C and s such that $e_{n+1} = Ce_n^s$, where en is the error of Newton's method at step n, s is a constant, and C may depend on x_n, the true solution α, and the derivative of the function f.

Let the error at step n be defined as $e_n = x_n - \alpha$, where α is the true root of $f(x) = 0$.
Using the modified Newton's method formula, we get:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

$$x_{n+1} - \alpha = x_n - \alpha - \frac{f'(\xi_n)(x_n - \alpha)}{f'(x_0)}$$

$$x_{n+1} - \alpha = (1 - \frac{f'(\xi_n)}{f'(x_0)})(x_n - \alpha)$$

$$e_{n+1} \approx (1 - \frac{f'(x_n)}{f'(x_0)})e_n$$

Thus:

$$s = 1, C = 1 - \frac{f'(\alpha)}{f'(x_0)}$$

V. Within $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

Within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $f(x)$ is continuous and monotonically increasing.

We need to analyze the convergence of the iteration $x_{n+1} = f(x_n)$. To check for convergence, we can find the fixed point of the function, which satisfies the equation:

$$x = \tan^{-1} x$$

Let $x^* = \tan^{-1} x^*$. By analyzing this equation, we find that the solution is $x^* = 0$, since $\tan^{-1}(0) = 0$.

Next, we analyze the convergence of the iteration. According to the fixed-point convergence theorem, if f is a contraction mapping near the fixed point, the iterative sequence will converge.

We compute $f'(x)$ to check its behavior near the fixed point $x^* = 0$:

$$f'(x) = \frac{1}{1 + x^2}$$

Within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $f'(x)$ is always positive and less than 1. Specifically, we have:

$$f'(0) = 1$$

To ensure that f is a contraction near the fixed point, we note that:

$$f'(x) < 1 \quad \text{for } x \neq 0$$

Since f is monotonically increasing and the slope near the fixed point is less than 1, we conclude that the iteration $x_{n+1} = \tan^{-1} x_n$ will converge to the fixed point $x^* = 0$.

The iteration $x_{n+1} = \tan^{-1} x_n$ converges within the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

VI. Let $p > 1$. What is the value of the following continued fraction?

$$x = \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}}$$

Prove that the sequence of values converges. (Hint: this can be interpreted as $x = \lim_{n\to\infty} x_n$, where $x_1 = \frac{1}{p}$, $x_2 = \frac{1}{p+\frac{1}{p}}$, $x_3 = \frac{1}{p+\frac{1}{p+\frac{1}{p}}}$, and so forth. Formulate x as a fixed point of some function.)

$$x_1 = \frac{1}{p}, \quad x_2 = \frac{1}{p + x_1}, \quad x_3 = \frac{1}{p + x_2}, \quad \ldots$$

express the general term x_n as:

$$x_n = \frac{1}{p + x_{n-1}}$$

Assuming the limit

$$x = \lim_{n\to\infty} x_n$$

exists, substituting x into the recurrence gives:

$$x = \frac{1}{p + x}$$

multiply both sides by $p + x$:

$$x(p + x) = 1$$

$$px + x^2 = 1$$

$$x^2 + px - 1 = 0$$

then:

$$x = \frac{-p \pm \sqrt{p^2 + 4}}{2}$$

Since $p > 1$:

$$x = \frac{-p + \sqrt{p^2 + 4}}{2}$$

Next, we need to prove that the sequence $\{x_n\}$ converges.

Boundedness:

To proof $x_n > 0$. By induction: - For $n = 1$, $x_1 = \frac{1}{p} > 0$ since $p > 1$. - Suppose $x_n > 0$. Then $x_{n+1} = \frac{1}{p+x_n} > 0$ because $p + x_n > p > 0$.

Therefore, by induction, $x_n > 0$ for all n.

Next, we show that $x_n < \frac{1}{p-1}$: - For $n = 1$:

$$x_1 = \frac{1}{p} < \frac{1}{p-1} \quad \text{(since } p > 1)$$

- Assume $x_n < \frac{1}{p-1}$. Then:

$$x_{n+1} = \frac{1}{p + x_n} > \frac{1}{p + \frac{1}{p-1}} = \frac{1}{\frac{p(p-1)+1}{p-1}} = \frac{p-1}{p^2 - p + 1}$$

because $p^2 - 2p + 1 > 0, \frac{1}{p+\frac{1}{p-1}} < \frac{1}{p-1}$. This is true for $p > 1$.

Thus, x_n is bounded above by $\frac{1}{p-1}$.

Monotonicity:

To prove that x_n is decreasing, we show $x_{n+1} < x_n$:

$$\frac{1}{p + x_n} < x_n \implies 1 < x_n(p + x_n) \implies 1 < px_n + x_n^2$$

This will hold true if we can show that $x_n^2 + px_n - 1 > 0$.

Given the roots of $x^2 + px - 1 = 0$ are $\frac{-p \pm \sqrt{p^2+4}}{2}$ and knowing that $x_n > 0$ for sufficiently large n, the quadratic $x^2 + px - 1$ is positive for $x > \frac{-p+\sqrt{p^2+4}}{2}$, which is valid for large n.

Thus, x_n is decreasing and bounded below by 0. Therefore, by the Monotone Convergence Theorem, the sequence $\{x_n\}$ converges.

In conclusion, the value of the continued fraction is:

$$x = \frac{-p + \sqrt{p^2 + 4}}{2}$$

VII. What happens in problem II if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in II. In this case, is the relative error still an appropriate measure?

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According to Problem II:

$$\frac{b_n - a_n}{2^n} \le \epsilon |r|$$

$$n \ge \frac{\log(b_0 - a_0) - \log \epsilon - \log a_0}{\log 2} - 1.$$

If the root is 0 or so close to 0, the relative error may become too large or undefined so, the relative error isn' t an appropriate measure,absolute error is better.

VIII. (*) Consider solving $f(x) = 0$ (where $f \in C^{k+1}$) by Newton's method with the starting point x_0 close to a root of multiplicity k. Note that α is a zero of multiplicity k of the function f if and only if $f^{(k)}(\alpha) \ne 0$ and $f^{(i)}(\alpha) = 0$ for all $i < k$.

VIII-a How can a multiple zero be detected by examining the behavior of the points $(x_n, f(x_n))$?

The standard Newton's iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As x_n approaches to a multiple zero, $f(x_n)$ may be always near to zero for several iterations and the iterates x_n remain very close together.

VIII-b Prove that if r is a zero of multiplicity k of the function f, then quadratic convergence in Newton' s iteration will be restored by making this modification:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In the case of a multiple root, can modify it to:

$$x_{n+1} = x_n - \frac{kf(x_n)}{f^{(k)}(x_n)}$$

Since r is a multiple root of k, we can represent $f(x)$ near r as:

$$f(x) = (x - r)^k g(x)$$

where $g(r) \ne 0$ (since $f^{(k)}(r) \ne 0$).

$$f'(x) = k(x - r)^{k-1} g(x) + (x - r)^k g'(x)$$

substitute x_n close to r:

$$f(x_n) = (x_n - r)^k g(x_n)$$

and

$$f'(x_n) \approx k(x_n - r)^{k-1} g(r)$$

when x_n is close to r, and $g(x_n) \approx g(r)$. So:

$$x_{n+1} = x_n - \frac{kf(x_n)}{f^{(k)}(x_n)} = x_n - \frac{k(x_n - r)^k g(x_n)}{g(r)}$$

Suppose x_n is close enough to r:

$$x_{n+1} - r = x_n - r - \frac{k(x_n - r)^k g(r)}{g(r)} = x_n - r - (x_n - r)^k$$

As n becomes large enough, $(x_n - r)^k$ will become very small. The quadratic term will yield:

$$|x_{n+1} - r| \approx C|x_n - r|^2$$

where C is a constant, thus confirming that x_n converges to r at a quadratic rate.