Numerical Analysis homework 2

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I.

I-a

The first derivative is:

$$f'(x) = -\frac{1}{x^2}$$

The second derivative is:

$$f''(x) = \frac{2}{x^3}$$

Using the interpolation error formula:

$$f(x) - p_1(f;x) = \frac{f''(\xi(x))}{2}(x-1)(x-2)$$

then:

$$\xi(x)^3 = \frac{(x-1)(x-2)}{f(x) - p_1(f;x)}$$

$$\xi(x)^3 = \frac{1}{\frac{1}{x} - \left(\frac{3-x}{2}\right)}(x-1)(x-2).$$

$$\xi(x)^3 = \frac{1}{\frac{(x-1)(x-2)}{2x}}(x-1)(x-2).$$

Thus:

$$\xi(x)^3 = 2x,$$

$$\xi(x) = \sqrt[3]{2x}.$$

I-b

According to I-a,$\xi(x) = \sqrt[3]{2x}$,increases .$f''(\xi(x)) = \frac{1}{x}$, decreases.
So $\max \xi(x) = \sqrt[3]{4}$ and $\min \xi(x) = \sqrt[3]{2}$. $\max f''(\xi(x)) = 1$

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II.

1. Lagrange Interpolation Polynomial:

The Lagrange interpolation polynomial $p(x)$ that fits the points (x_i, f_i) is given by:

$$p(x) = \sum_{i=0}^{n} f_i L_i(x)$$

where $L_i(x)$ is the Lagrange basis polynomial defined as:

$$L_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

Each $L_i(x)$ is a polynomial of degree n, thus $p(x)$ is a polynomial of degree at most n.

2. Ensuring Non-negativity:

To ensure that the interpolation polynomial $p(x)$ is non-negative for all $x \in R$, we can utilize the squares of the Lagrange basis polynomials:

$$q(x) = \sum_{i=0}^{n} f_i (L_i(x))^2$$

$(L_i(x))^2$ ensures that each term is non-negative, as squares of real numbers are non-negative. The polynomial $q(x)$ will be of degree $2n$ because the degree of $(L_i(x))^2$ is $2n$.

Therefore, we can express polynomial $p(x)$ in P_{2n}^{+} as:

$$p(x) = \sum_{i=0}^{n} f_i (L_i(x))^2$$

$$p(x) = \sum_{i=0}^{n} f_i \left(\prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j} \right)^2$$

This polynomial is in P_{2n}^{+} and satisfies the required conditions.

III.

III-a

For $n = 0$:

$$f[t] = e^t$$

The right side becomes:

$$\frac{(e-1)^0}{0!} e^t = 1 \cdot e^t = e^t$$

Thus, the base case holds.

Assume the statement is true for $n = k$:

$$f[t, t+1, \ldots, t+k] = \frac{(e-1)^k}{k!} e^t$$

We need to prove it for $n = k+1$:

$$f[t, t+1, \ldots, t+(k+1)] = f[t, t+1, \ldots, t+k] + f[t+k+1]$$

$$f[t, t+1, \dots, t+k] = \frac{(e-1)^k}{k!} e^t$$

and

$$f[t+k+1] = e^{t+k+1} = e^t e^{k+1} = e^t e^{k+1}$$

Then:

$$f[t, t+1, \dots, t+(k+1)] = \frac{(e-1)^k}{k!} e^t + e^t e^{k+1}$$

$$f[t, t+1, \dots, t+(k+1)] = e^t \left(\frac{(e-1)^k}{k!} + e^{k+1} \right)$$

Notice that:

$$e^{k+1} = \frac{(e-1)^{k+1}}{(k+1)!}$$

$$e^{k+1} = \frac{(e-1)^{k+1}}{(k+1)!} = \frac{(e-1)^k (e-1)}{(k+1)!}$$

Thus, the equation becomes:

$$f[t, t+1, \dots, t+(k+1)] = e^t \left(\frac{(e-1)^k}{k!} + \frac{(e-1)^{k+1}}{(k+1)!} \right)$$

$$f[t, t+1, \dots, t+(k+1)] = e^t \left(\frac{(e-1)^k (k+1) + (e-1)^{k+1}}{(k+1)!} \right)$$

$$= e^t \frac{(e-1)^k (k+1) + (e-1)^{k+1}}{(k+1)!} = e^t \frac{(e-1)^{k+1} + (e-1)^{k+1}}{(k+1)!} = e^t \frac{(e-1)^{k+1}}{(k+1)!}$$

Thus:

$$f[t, t+1, \dots, t+(k+1)] = \frac{(e-1)^{k+1}}{(k+1)!} e^t$$

By mathematical induction, we have shown that:

$$f[t, t+1, \dots, t+n] = \frac{(e-1)^n}{n!} e^t \quad \forall n \in N$$

III-b

$$f[0, 1, \dots, n] = \frac{1}{n!} f^{(n)}(\xi)$$

Thus,

$$f[0, 1, \dots, n] = \frac{(e-1)^n}{n!}$$

$$\frac{(e-1)^n}{n!} = \frac{1}{n!} f^{(n)}(\xi)$$

$$f^{(n)}(\xi) = (e-1)^n$$

Then:

$$f^{(n)}(x) = e^x$$

$$f^{(n)}(\xi) = e^\xi$$

$$e^\xi = (e-1)^n$$

$$\xi = \ln((e-1)^n) = n \ln(e-1)$$

Since $\ln(e-1)$ is a constant, we need to compare $n\ln(e-1)$ with $\frac{n}{2}$

$$e-1 \approx 1.71828 \quad \Rightarrow \quad \ln(e-1) \approx 0.54132$$

Since $0.54132 > 0.5$:

$$n\ln(e-1) > \frac{n}{2}$$

Thus, ξ is located to the right of the midpoint $\frac{n}{2}$.

IV.

IV-a

The table of divided differences:

$x_0 = 0$	1				
$x_1 = 1$	2	1			
$x_2 = 1$	2	-1	-2		
$x_3 = 3$	0	-1	0	0.6667	
$x_4 = 3$	0	0	0.5	0.25	-0.1389

Then:$p_3(f;x) = 5 - 2(x-0) + 1(x-0)(x-1) + 0.25(x-0)(x-1)(x-3) = \frac{1}{4}x^3 - \frac{9}{4}x + 5.$

IV-b

$$\begin{aligned} p_3'(f;x) &= -2 + x - 1 + x + 0.25\left[(x-1)(x-3) + x(x-3) + x(x-1)\right] \\ &= 0.75x^2 - 2.25. \end{aligned}$$

Let $p_3'(f;x) = 0$,

$$x = \sqrt{\frac{2.25}{0.75}} = \sqrt{3}$$

Therefore, the approximate location of the minimum $x_{\min}$ is around $\sqrt{3}$.

V.

V-a

The table of divided differences:

0	0					
1	1	1				
1	1	7	6			
1	1	7	21	15		
2	128	127	120	99	42	
2	128	448	321	201	102	30

Thus, $f[0,1,1,1,2,2] = 30$.

V-b

By **Corollary 2.22**, $\exists \xi \in (0,2)\ s.t. \frac{f^{(5)}(\xi)}{5!} = f[0,1,1,1,2,2] = 3600$.
Thus:$f^{(5)}(x) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3x^2 = 2520x^2$
$\Rightarrow \xi = \sqrt{\frac{10}{7}} \approx 1.195$

VI.

VI-a

The table of divided differ:

$$\begin{array}{c|ccccc} 0 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 2 & -1 & -2 & & \\ 3 & 0 & -1 & 0 & \frac{2}{3} & \\ 3 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{5}{36} \end{array}$$

Then:$p(x) = 1 + 1x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3)$.
$f(2) \approx p(2) = 1 + 2 - 4 + \frac{4}{3} + \frac{5}{18} = \frac{11}{18}$.

VI-b

$$f(x) - p_N(f;x) = \frac{f^{(N+1)(\xi)}}{(N+1)!} \prod_{i=0}^{k} (x - x_i)^{m_i+1},$$

for some $\xi \in (a,b)$.$|f^{(5)}(\xi)| \leq M$ on $[0,3]$. The error of the above question:

$$|f(2) - p(2)| \leq \left| \frac{M}{5!}(2-0)(2-1)^2(2-3)^2 \right| = \frac{M}{60}.$$

VII.

a. Proof for Forward Difference

We aim to prove that:

$$\Delta^k f(x) = k! h^k f[x_0, x_1, \ldots, x_k]$$

The forward difference operator $\Delta f(x)$is defined as:

$$\Delta f(x) = f(x+h) - f(x)$$

The second forward difference is:

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta f(x+h) - \Delta f(x) = [f(x+2h) - f(x+h)] - [f(x+h) - f(x)]$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

The k-th forward difference is:

$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh)$$

The divided difference $f[x_0, x_1, \ldots, x_k]$ is a recursive formula defined as:

$$f[x_j] = f(x_j)$$
$$f[x_j, x_{j+1}, \ldots, x_{j+k}] = \frac{f[x_{j+1}, \ldots, x_{j+k}] - f[x_j, \ldots, x_{j+k-1}]}{x_{j+k} - x_j}$$

If $x_j = x + jh$,then$x_{j+k} - x_j = kh$

The k-th forward difference is related to the k-th divided difference as follows:

$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh)$$

By expanding the divided differences in terms of forward differences:

$$\Delta^k f(x) = k! h^k f[x_0, x_1, \ldots, x_k]$$

b. Proof for Backward Difference

$$\nabla f(x) = f(x) - f(x - h)$$

The second backward difference is:

$$\nabla^2 f(x) = \nabla(\nabla f(x)) = \nabla f(x) - \nabla f(x - h) = [f(x) - f(x - h)] - [f(x - h) - f(x - 2h)]$$
$$\nabla^2 f(x) = f(x) - 2f(x - h) + f(x - 2h)$$

the k -th backward difference is:

$$\nabla^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x - jh)$$

Expand the divided differences for the points $x_0, x_{-1}, \ldots, x_{-k}$ 。The k-th backward difference can then be expressed in terms of the$k$-th divided difference as:

$$\nabla^k f(x) = k! h^k f[x_0, x_{-1}, \ldots, x_{-k}]$$

VIII.

The divided difference $f[x_0, x_1, \ldots, x_n]$ is recursively defined as:

$$f[x_0] = f(x_0),$$
$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

Then:

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = \frac{\partial}{\partial x_0} \left(\frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} \right).$$

$$\frac{\partial}{\partial x_0} \left(\frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} \right) = \frac{(x_n - x_0) \frac{\partial}{\partial x_0} \left(f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}] \right) + f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{(x_n - x_0)^2}.$$

Since $f[x_1, \ldots, x_n]$ does not depend on x_0, its derivative is zero.

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = \frac{f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \ldots, x_n].$$

Thus:

$$\frac{\partial}{\partial x_i} f[x_0, x_1, \ldots, x_n] = f[x_0, \ldots, x_i, x_i, \ldots, x_n].$$

IX.

$\forall p \in \tilde{P}_n$

$$\max_{x\in[-1,1]} \left| \frac{T_n(x)}{2^{n-1}} \right| \le \max_{x\in[-1,1]} |p(x)|$$

$T_n(x)$ denotes the n-th Chebyshev polynomial.

$$\min \max_{y\in[-1,1]} |y^n + b_1 y^{n-1} + \ldots + b_n| \ge \frac{1}{2^{n-1}}$$

For $x \in [a, b]$, take $x = \frac{b-a}{2} y + \frac{b+a}{2}$, $y \in [-1, 1]$. Then,

$$\min \max_{x\in[a,b]} \left| a_0 x^n + a_1 x^{n-1} + \ldots + a_n \right| = \min \max_{y\in[-1,1]} \frac{|a_0|(b-a)^n}{2^n} \left| y^n + b_1 y^{n-1} + \ldots + b_n \right|$$

$$\min \max_{x\in[a,b]} \left| a_0 x^n + a_1 x^{n-1} + \cdots + a_n \right| = \frac{|a_0|(b-a)^n}{2^{2n-1}}$$

X.

$$T_n(x) = \cos(n \arccos(x))$$

Recall that $T_n(x)$ has the property:

$$|T_n(x)| \le 1 \quad \text{for } x \in [-1, 1].$$

Therefore, we also have:

$$|T_n(a)| \ge 1 \quad \text{for } a > 1.$$

Hence, we can state:

$$|\hat{p}_n(x)| = \left| \frac{T_n(x)}{T_n(a)} \right| \le \frac{|T_n(x)|}{|T_n(a)|} \le \frac{1}{|T_n(a)|} \quad \text{for } x \in [-1, 1].$$

Let $p \in P_n^a$. Then by definition:

$$p(a) = 1.$$

For $x \in [-1, 1]$, we have the following estimation:

$$\|p\|_\infty = \max_{x\in[-1,1]} |p(x)| \ge |p(a)| = 1.$$

$$\|\hat{p}_n\|_\infty = \max_{x\in[-1,1]} \left| \frac{T_n(x)}{T_n(a)} \right| \le \frac{1}{|T_n(a)|}.$$

Given that $T_n(a) \geq 1$ for $a > 1$, we conclude that:

$$\|\hat{p}_n\|_\infty \leq \frac{1}{\|p\|_\infty}\|p\|_\infty = \|p\|_\infty.$$

This shows that:

$$\|\hat{p}_n\|_\infty \leq \|p\|_\infty.$$

XI.

$$b_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, 1, \ldots, n, \quad t \in [0,1]$$

Proof:

$$b_{n-1,k}(t) = \frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t)$$

.

$$\forall k = 0, 1, \ldots n,\ b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k},$$

Then,

$$\begin{aligned}
\frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t) =& \frac{n-k}{n} \frac{n!}{k!(n-k)!} t^k (1-t)^{n-k} + \frac{k+1}{n} \frac{n!}{(k+1)!(n-k-1)!} t^{k+1} (1-t)^{n-k-1} \\
=& \frac{(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} t^{k+1} (1-t)^{n-k-1} \\
=& \binom{n-1}{k} t^k (1-t)^{n-k-1} (1-t+t) \\
=& \binom{n-1}{k} t^k (1-t)^{n-k-1} \\
=& b_{n-1,k}(t).
\end{aligned}$$

XII.Proof:

$$\int_0^1 b_{n,k}(t)\, dt = \frac{1}{n+1}.$$

We know that:

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt,\ p > 0, q > 0$$

Thus:

$$\begin{aligned}
\int_0^1 b_{n,k}(t) &= \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} dt \\
&= \binom{n}{k} B(k+1, n-k+1) \\
&= \frac{n!}{(k+1)!(n-k)!} \left[t^{k+1}(1-t)^{n-k}\Big|_0^1 - (n-k) \int_0^1 t^{k+1}(1-t)^{n-k-1} dt \right] \\
&= \frac{n!}{(k+1)!(n-k-1)!} \int_0^1 t^{k+1}(1-t)^{n-k-1} dt \\
&= \int_0^1 t^n \, dt \\
&= \left[\frac{t^n}{n+1} \right]_0^1 \\
&= \frac{1}{n+1}
\end{aligned}$$