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DISTRIBUTED QUANTIZED CONSENSUS FOR AGENTS ON DIRECTED NETWORKS*

LI Dequan · LIU Qipeng · WANG Xiaofan

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Abstract Communication bandwidth and network topology are two important factors that affect performance of distributed consensus in multi-agent systems. The available works about quantized average consensus assume that the adjacency matrices associated with the digraphs are doubly stochastic, which amounts to that the digital networks are balanced. However, this assumption may be unrealistic in practice. In this paper, without assuming double stochasticity, the authors revisit an existing quantized average consensus protocol with the logarithmic quantization scheme, and investigate the quantized consensus problem in general directed digital networks that are strongly connected but not necessarily balanced. The authors first derive an achievable upper bound of the quantization precision parameter to design suitable logarithmic quantizer, and this bound explicitly depends on network topology. Subsequently, by means of the matrix transformation and the Lyapunov techniques, the authors provide a testable condition under which the weighted average consensus can be achieved with the proposed quantized protocol.

Keywords Consensus protocol, digraph, logarithmic quantization, multi-agent systems.

1 Introduction

Recently, the consensus problem has received significant attention as a fundamental research topic in the coordinated control of networks of dynamic agents. Consensus means the agents

LI Dequan
Department of Automation, Shanghai Jiao Tong University, and Key Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240*, China; School of Science, Anhui University of Science and Technology, Huainan* 232001*, China.*
Email: leedqseu@gmail.com.
LIU Qipeng · WANG Xiaofan
Department of Automation, Shanghai Jiao Tong University, and Key Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240*, China.*
Email: sjtu_lqp@sjtu.edu.cn; xfwang@sjtu.edu.cn.
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achieving an agreement on their common state merely through local interaction. A more refined concept, average consensus, further requires agents to agree on the exact average or the centroid of their initial states. Due to the crucial role in distributed systems, the consensus problem, and the various closely related formulations have been intensively studied in the context of information fusion[1], load balancing[2], distributed and parallel computation[3,4], distributed coordination of mobile autonomous agents[5−9].

In the analysis and design of distributed consensus algorithms in the real world scenario, some communication constraints must be considered. In the most general setup, each agent can only communicate with its neighbors, and the communication topology is commonly modeled as a directed graph (also called digraph). Furthermore, since the digital communication channels are subject to limited data rate[10,11], information exchange within digital networks typically involves quantization. As a result, recent years have witnessed a growing number of works designing effective average consensus algorithms or protocols with quantized communication for multi-agent systems with single-integrator dynamics[12−20]. However, a critical and standard assumption for the foregoing literature is that the adjacency matrices corresponding to the networks are doubly stochastic, or equivalently, the corresponding information digraphs are always balanced[21,22]. The double stochasticity assumption is very restrictive for implementing the distributed algorithms in practice, since the balanced network topology is very difficult to be enforced in a distributed manner with unidirectional communication[23]. In such a case, due to the potential asymmetric information exchange between agents, average consensus is not guaranteed to be achievable. Instead, if the corresponding digraph is strongly connected, then weighted average consensus is achieved, i.e., agents agree on some value that is the weighted linear combination of their initial states[4,24].

For a general directed fixed network, by jointly taking quantized information communication and network topology into consideration, we recently proposed a quantized protocol with the finite-level uniform quantization scheme[25]. By constructing a generalized quadratic Lyapunov function, we proved that if the information digraph is strongly connected, then the weighted average consensus can be achieved. In this paper, we continue to focus on the weighted average consensus problem with quantized information communication, and present an extended analysis of a consensus protocol with the infinite-level logarithmic quantization scheme developed in [16], where the information graph is undirected (and thus trivially balanced). As in [25], we assume that the information digraph is strongly connected but not necessarily balanced, and thus the convergence analysis drops the double stochasticity assumption on the adjacency matrix, while this assumption underlies the main results of [16]. For the quantized consensus problem of the directed network, a non-trivial technical difficulty complicating the convergence analysis is that the asymmetric adjacency matrix can no longer be written as a clean diagonal matrix. Therefore, it is hard to use the method of [16], which heavily relies on the eigenvalue decomposition technique of symmetric matrices. To overcome this technical difficulty, in this paper, an alternative convergence analysis method is developed based on the matrix transformation and the Lyapunov techniques. Moreover, the stability condition of the closed-loop system is characterized by an easily testable linear matrix inequality (LMI).

Unidirectional and unbalanced information flow among agents, the chief characteristic of directed networks, increases robustness to communication failures and potentially reduces the amount of information flow required to ensure consensus. So, it is of practical importance to study quantized consensus problem in general directed digital networks that are possibly unbalanced. However, the unidirectional and unbalanced information flow among agents often leads to significant technical challenges when establishing convergence results of consensus algorithms, particularly in the presence of quantized information communication. The results of this paper and [25] provide two more examples in support of this assertion about the technical challenges for the case of quantized consensus in directed digital networks with fixed topologies.

The paper is organized as follows. In Section 2, we recall some basic concepts of graph theory. In Section 3, we formulate the problem of interest and propose the quantized consensus protocol. In Section 4, we derive consensus convergence results for directed networks. The results are illustrated by an example in Section 5, and finally, we present the concluding remarks.

Notation We will use standard notation. $\mathbf{1} = (1, 1, \cdots, 1)^{\mathrm{T}} \in R^N$ and $\mathbf{0} = (0, 0, \cdots, 0)^{\mathrm{T}} \in R^N$. $I \in R^{N\times N}$ is the identity matrix. For a complex number λ, its real part is denoted by $\mathrm{Re}(\lambda)$, its imaginary part is denoted by $\mathrm{Im}(\lambda)$, its complex conjugate is denoted by $\overline{\lambda}$, and its modulus is denoted by $|\lambda|$. The transposes of a vector $v \in R^N$ and a matrix $M \in R^{N\times N}$ are denoted by v^{T} and M^{T}. The inverse and the pseudoinverse of a matrix M are denoted by M^{-1} and $M^{\dagger}$. Furthermore, $\mathrm{span}\{v\}$ and $\mathrm{span}\{v\}^{\perp}$ denote the space and the orthogonal complementary space spanned by vector $v \in R^N$. For a set $\boldsymbol{R}$, $\mathrm{Co}(\boldsymbol{R})$ denotes the convex hull of $\boldsymbol{R}$.

2 Preliminaries

The information flow among nodes on a network can be described by a digraph $G = (\mathcal{V}, \mathcal{E}, W)$, where the set of nodes (or agents) $\mathcal{V} = \{1, 2, \cdots, N\}$, the set of directed edge $\mathcal{E}$ represents the interaction among agents, and the nonnegative weighted adjacency matrix $W = (w_{ij}) \in R^{N\times N}$. The directed edge (or channel) $(j, i) \in \mathcal{E}$ means agent i can receive information from agent j, and the adjacency weight w_{ij} associated with this edge is positive, i.e., $w_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$, otherwise $w_{ij} = 0$. Since any agent knows exactly the information of itself, we assume that each agent has a self-loop, i.e., $(i, i) \in \mathcal{E}$ and $w_{ii} > 0$ for all $i \in \mathcal{V}$. The neighbor set of agent i is denoted by $N_i = \{j \mid j \neq i, (j, i) \in \mathcal{E}, j \in \mathcal{V}\}$. For any $i \neq j$, if $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$, the graph is called undirected; otherwise the graph is called directed. A directed path in a digraph G is an ordered sequence of nodes $(i_1, i_2, \cdots, i_r)$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for each $j \in \{1, 2, \cdots, r-1\}$. A digraph is called strongly connected if any two distinct nodes of the graph can be connected via a directed path.

In a weighted digraph G, the in-degree and the out-degree of agent i are defined as $\deg_{\mathrm{in}}(i) = \sum_{j=1}^{N} w_{ij}$ and $\deg_{\mathrm{out}}(i) = \sum_{j=1}^{N} w_{ji}$. Furthermore, the digraph G is called balanced if $\deg_{\mathrm{in}}(i) = \deg_{\mathrm{out}}(i)$ for all $i \in \mathcal{V}$. A nonnegative matrix $W = (w_{ij}) \in R^{N\times N}$ is said to be (row) stochastic if it satisfies $\sum_{j=1}^{N} w_{ij} = 1$ for all $i \in \mathcal{V}$. Moreover, W is said to be doubly stochastic if it satisfies $\sum_{j=1}^{N} w_{ji} = \sum_{j=1}^{N} w_{ij} = 1$ for all $i \in \mathcal{V}$. Hence, if its weighted adjacency matrix W

is doubly stochastic, then the digraph G is balanced. In a special case where the weighted adjacency matrix W is symmetric and doubly stochastic, then G is undirected.

3 Seeking Consensus with Logarithmic Quantization Scheme

We consider a directed fixed network composed of N agents and each agent has the following discrete-time single-integrator dynamics

$$x_i(k+1) = x_i(k) + u_i(k), \quad k = 0, 1, \cdots; \ i = 1, 2, \cdots, N, \tag{1}$$

where $x_i(k)$ is the real-value state of agent i, and the initial value $x_i(0)$ is known. $u_i(k)$ is the control input of agent i. We make the following assumption on the digraph $G = (\mathcal{V}, \mathcal{E}, W)$.

Assumption 1 The unbalanced digraph G is strongly connected. The adjacency matrix W is stochastic and has positive diagonal entries, i.e., $w_{ii} = 1 - \sum_{j\in\mathcal{N}_i} w_{ij} > 0$ for all $i \in \mathcal{V}$.

3.1 Consensus Protocol Without Quantized Communication

For a group of agents with ideal communication, the standard distributed consensus protocol or control input is

$$u_i(k) = \sum_{j\in N_i} w_{ij}(x_j(k) - x_i(k)), \quad i = 1, 2, \cdots, N, \tag{2}$$

where w_{ij} are the entries of the weighted adjacency matrix W associated with the digraph G.

Combining (1) and (2), we obtain the closed-loop system

$$x_i(k+1) = x_i(k) + \sum_{j\in N_i} w_{ij}(x_j(k) - x_i(k)), \quad i = 1, 2, \cdots, N, \tag{3}$$

which can be written in a compact form

$$x(k+1) = Wx(k), \tag{4}$$

where $x(k) = (x_1(k), x_2(k), \cdots, x_N(k))^{\mathrm{T}}$ is the stacked state vector.

Under Assumption 1, the weighted adjacency matrix W is primitive, and thus 1 is a simple and maximal eigenvalue of W with algebraic multiplicity one [26]. Without loss of generality, we assume that all the eigenvalues of W are ordered as $1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_N|$. Furthermore, there exists a unique normalized positive left eigenvector $\pi = (\pi_1, \pi_2, \cdots, \pi_N)^{\mathrm{T}}$ corresponding to eigenvalue 1 such that

$$\pi^{\mathrm{T}} W = \pi^{\mathrm{T}}, \quad \pi^{\mathrm{T}}\mathbf{1} = \sum_{i=1}^{N} \pi_i = 1, \tag{5}$$

and $\lim_{k\to\infty} W^k = \mathbf{1}\pi^{\mathrm{T}}$ [26]. Here, the positive left eigenvector π means $\pi_i > 0$ for $i = 1, 2, \cdots, N$. Then the protocol (2) solves the weighted average consensus problem, namely

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} W^k x(0) = \left(\sum_{i=1}^{N} \pi_i x_i(0)\right)\mathbf{1}, \tag{6}$$

Figure 1 Encoder/decoder pair for a digital channel

If the edge weight $w_{ij} > 0$, i.e., agent j sends its information to agent i, then the encoder Φ_j associated with agent j is defined as [16]

$$\begin{cases} \xi_j(0) = 0, \\ \xi_j(k) = \xi_j(k-1) + \alpha_j(k), \\ \alpha_j(k) = \lg q_\delta(x_j(k) - \xi_j(k-1)), \quad k = 1, 2, \cdots , \end{cases} \tag{8}$$

where $\xi_j(k)$ is the internal state of the encoder $\varPhi_j$; the symbolic data $\alpha_j(k)$ is the output of the encoder $\varPhi_j$, which will be sent through the digital channels. The function $\lg q_\delta(\cdot)$ represents a

Figure 2 The logarithmic quantizer

After $\alpha_i(k)$ is received by agent i, the following decoder Ψ_{ji} of agent i associated with the directed digital channel (j,i) is used to estimate the real-value state $x_j(k)$,

$$\begin{cases} \widehat{x}_{ji}(0) = 0, \\ \widehat{x}_{ji}(k) = \widehat{x}_{ji}(k-1) + \alpha_j(k), \quad k = 1, 2, \cdots, \end{cases} \tag{11}$$

where $\widehat{x}_{ji}(k)$, the output of the decoder Ψ_{ji}, is the estimate of $x_j(k)$ obtained by agent i.

Remark 1 As each agent can obtain its own estimate through the self-loop, then by (8) and (11), we have $\widehat{x}_{ji}(k) = \xi_j(k)$ for all $i \in \mathcal{V}$, $j \in N_i \bigcup \{i\}$ and $k = 0, 1, \cdots$. Meanwhile, instead of quantizing $x_j(k)$ directly at each time instant, the logarithmic quantizer (9) actually quantizes the "prediction error" $x_j(k)-\xi_j(k-1)$.

Equation (8) indicates that the internal state $\xi_j(k)$ satisfies the following recursive relation

$$\xi_j(k+1)=\xi_j(k)+\lg q_\delta(x_j(k+1)-\xi_j(k)), \quad j=1,2,\cdots,N. \tag{12}$$

As shown in [11], the logarithmic quantizer (9) is easily bounded by a sector bound, namely

$$|\lg q_\delta(m)-m|\le\delta|m|, \quad m\in R. \tag{13}$$

Accordingly, we can rewrite (12) as

$$\xi_j(k+1)=\xi_j(k)+(1+\triangle_j(k))\left(x_j(k+1)-\xi_j(k)\right), \quad j=1,2,\cdots,N, \tag{14}$$

where the "uncertainty" $\triangle_j(k)\in[-\delta,\delta]$ for all $k\ge 0$.

Then in the presence of the logarithmic quantized information communication, we propose the consensus protocol of the form

$$u_i(k)=\sum_{j\in N_i}w_{ij}\left(\widehat{x}_{ji}(k)-\widehat{x}_{ii}(k)\right), \quad i=1,2,\cdots,N. \tag{15}$$

From (8)–(11), it can be seen that the quantized protocol (15) is characterized by the quantization precision parameter $\delta\in(0,1)$.

Applying the quantized protocol (15) to (1), we obtain

$$x_i(k+1)=x_i(k)+\sum_{j\in N_i}w_{ij}\left(\widehat{x}_{ji}(k)-\widehat{x}_{ii}(k)\right), \quad i=1,2,\cdots,N. \tag{16}$$

Note that $\widehat{x}_{ji}(k)=\xi_j(k)$ for all $i\in\mathcal{V}$, $j\in N_i\bigcup\{i\}$ and $w_{ii}=1-\sum_{j\in\mathcal{N}_i}w_{ij}>0$ for all $i\in\mathcal{V}$, then (16) is equivalent to

$$x_i(k+1)=x_i(k)-\xi_i(k)+\sum_{j=1}^{N}w_{ij}\xi_j(k), \quad i=1,2,\cdots,N, \tag{17}$$

which can be reformulated in a compact form

$$x(k+1)=x(k)+(W-I)\xi(k), \tag{18}$$

where $\xi(k)=(\xi_1(k),\xi_2(k),\cdots,\xi_N(k))^{\mathrm{T}}$. Denoting $\mathit{\Omega}(k)=\mathrm{diag}\,(\triangle_1(k),\triangle_1(k),\ \cdots\ \triangle_N(k))$, we obtain from (14) that

$$\xi(k+1)=\xi(k)+(I+\mathit{\Omega}(k))(x(k+1)-\xi(k)). \tag{19}$$

Let the estimation error of agent i be $e_i(k)=\xi_i(k)-x_i(k)$ and the stacked estimation error vector be $e(k)=\xi(k)-x(k)\in R^N$. Then by (17), we have

$$x_i(k+1) \quad = \quad \sum_{j=1}^{N}w_{ij}x_j(k)+\sum_{j=1}^{N}w_{ij}e_j(k)-e_i(k), \quad i=1,2,\cdots,N,$$

which can be further written in a compact form

$$x(k+1) = Wx(k) + (W - I)e(k). \tag{20}$$

By the definition of $e(k)$, together with (19) and (20), then after a short calculation, we get

$$e(k+1) = \Omega(k)[(W - I)x(k) + (W - 2I)e(k)]. \tag{21}$$

Furthermore, combining (20) and (21), we obtain the closed-loop system

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \Omega(k) \end{bmatrix} \begin{bmatrix} W & W - I \\ W - I & W - 2I \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}. \tag{22}$$

and the initial conditions are $x(0)$ and $e(0) = -x(0)$.

Remark 2 According to (5), then under protocol (15) it follows from (18) that

$$\pi^{\mathrm{T}} x(k+1) = \pi^{\mathrm{T}} x(k) + \pi^{\mathrm{T}} (W - I)\xi(k) = \pi^{\mathrm{T}} x(k), \tag{23}$$

which means the weighted average invariance of the whole network is preserved, namely

$$\sum_{i=1}^{N} \pi_i x_i(k+1) = \sum_{i=1}^{N} \pi_i x_i(k), \quad k = 0, 1, \cdots . \tag{24}$$

This contains the average invariance or "average along the way"

$$\frac{1}{N} \sum_{i=1}^{N} x_i(k+1) = \frac{1}{N} \sum_{i=1}^{N} x_i(k), \quad k = 0, 1, \cdots \tag{25}$$

as a special case, which is key to recent results about quantized average consensus.

Remark 3 In [16], where the network is undirected, average consensus can be achieved only by bidirectional and symmetric quantized information communication among agents, which implies that certain overhead like message delivery acknowledgment and retransmission is required by the protocol. In this paper, unidirectional and unbalanced quantized information flow among agents is allowed. Thus, the protocol (15) features little protocol overhead.

Remark 4 In [25], where the finite-level uniform quantization scheme is adopted, the resulting closed-loop system is a nonlinear system due to nonlinear quantization, and the convergence analysis is conducted by constructing a generalized quadratic Lyapunov function. The construction of the generalized quadratic Lyapunov function is intimately related to the topology property of the directed network. In this paper, thank to the sector bound condition (13), the resulting closed-loop system (22) is a linear parameter varying (LPV) system. As it will be seen later, the linearity allows us to employ certain matrix tools in convergence analysis. Thus, although both [25] and this paper do not require the balanced network topology, the convergence analysis methods used in [25] and this paper are different.

4 Convergence Analysis

For the proposed quantized protocol (15) and the resulting closed-loop system (22), a natural question may be asked: How to design the quantization precision parameter δ to achieve the weighted average consensus under the proposed quantized protocol (15)? This section is dedicated to answering this question, and the main contributions of this paper are shown in Lemma 2 and Theorem 1.

Let $z(k) = \left[x^{\mathrm{T}}(k)\ e^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ and define

$$F(k) = \begin{bmatrix} I & 0 \\ 0 & \varOmega(k) \end{bmatrix} \begin{bmatrix} W & W-I \\ W-I & W-2I \end{bmatrix} \in R^{2N\times 2N}, \tag{26}$$

then the closed-loop system (22) becomes

$$z(k+1) = F(k)z(k). \tag{27}$$

For simplicity of presentation, we borrow notations from [16], and let

$$\varXi = \left\{\mathrm{diag}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \in R^{N\times N} | \varepsilon_i \in \{-1, 1\}, i = 1, 2, \cdots, N\right\}.$$

Then one can observe that $\varXi$ contains 2^N elements, and hence $\varXi$ can be denoted by $\varXi = \{E_1, E_2, \cdots, E_{2^N}\}$. Without loss of generality, we assume $E_1 = I$. Furthermore, we define another set $\varXi_\delta = \{\delta E_1, \delta E_2, \cdots, \delta E_{2^N}\}$. Then we have that $\varOmega(k) \in \mathrm{Co}(\varXi_\delta)$ for all $k \geq 0$. With these definitions in hands, we introduce the following set of constant matrices

$$\boldsymbol{R} = \left\{ R_h = \begin{bmatrix} I & 0 \\ 0 & \delta E_h \end{bmatrix} \begin{bmatrix} W & W-I \\ W-I & W-2I, \end{bmatrix} \middle| \ h = 1, 2, \cdots, 2^N \right\},$$

and noting that $E_1 = I$, we have

$$R_1 = \begin{bmatrix} I & 0 \\ 0 & \delta I \end{bmatrix} \begin{bmatrix} W & W-I \\ W-I & W-2I \end{bmatrix}. \tag{28}$$

Then for the matrix sequence $\{F(k)\}_{k=0}^{+\infty}$, we have $F(k) \in \mathrm{Co}(\boldsymbol{R})$ for all $k \geq 0$. More specifically, there exist 2^N nonnegative real numbers $\mu_1(k), \mu_2(k), \cdots, \mu_{2^N}(k)$ satisfying

$$F(k) = \sum_{h=1}^{2^N} \mu_h(k) R_h \quad \text{and} \quad \sum_{h=1}^{2^N} \mu_h(k) = 1. \tag{29}$$

With these preparations, now we will give some lemmas that will be used later.

Lemma 1 *For $\upsilon = (\mathbf{1}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$ and $\varphi = (\pi^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$, we have $R_h\upsilon = \upsilon$ and $\varphi^{\mathrm{T}} R_h = \varphi^{\mathrm{T}}$ for all $h = 1, 2, \cdots, 2^N$, which means that υ and φ are the right and the left eigenvectors of R_h corresponding to eigenvalue 1, respectively.*

Proof By the fact that $W\mathbf{1} = \mathbf{1}$ and $\pi^{\mathrm{T}} W = \pi^{\mathrm{T}}$, then after some straightforward calculations, the lemma can be easily proved. ▮

Lemma 2 *Suppose that Assumption* 1 *holds. If the quantization precision parameter* δ *is chosen as*

$$0 < \delta \leq \min_{\lambda_i \neq 1} \frac{1 + \mathrm{Re}(\lambda_i)}{3 - \mathrm{Re}(\lambda_i)} = \overline{\delta}, \quad i = 2, 3, \cdots, N, \tag{30}$$

where λ_i *is the ith eigenvalue of* W, *then* 1 *is the only maximal eigenvalue of the matrix* R_1, *and all the other eigenvalues of* R_1 *are strictly within the unit circle.*

Proof The proof of this lemma is organized into four steps.

Step 1 To analyze the eigenvalues of the matrix R_1.

To compute the eigenvalues of R_1, we first calculate the characteristic polynomial of R_1

$$\det\,(sI - R_1) = \det \begin{bmatrix} sI - W & -(W - I) \\ -\delta(W - I) & sI - \delta(W - 2I) \end{bmatrix}. \tag{31}$$

Since each block in the above matrix can commute with each other, we have from [28] that

$$\det\,(sI - R_1) = \det\,\left[s^2 I - s(\delta(W - 2I) + W) - \delta I\right] = \prod_{i=1}^{N} \left[s^2 - s(\delta(\lambda_i - 2) + \lambda_i) - \delta\right]. \tag{32}$$

Recall that all the eigenvalues of W are ordered as $1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_N|$. Then the eigenvalues of R_1 are the solutions of the following equations

$$s^2 - s(1 - \delta) - \delta = 0, \tag{33}$$

and

$$s^2 - s(\delta(\lambda_i - 2) + \lambda_i) - \delta = 0, \quad i = 2, 3, \cdots, N. \tag{34}$$

Equation (33) has the trivial solutions 1 and $-\delta$.

Step 2 To discuss the solutions of (34).

Let $s_i^{(1)}$ and $s_i^{(2)}$ denote the two roots of (34). Then we have to analyze the conditions under which $|s_i^{(1)}| < 1$ and $|s_i^{(2)}| < 1$ for all $i = 2, 3, \cdots, N$. For this purpose, we next apply bilinear transformation $s = (1 + z)/(1 - z)$ to (34), and it yields

$$(1 + \delta)(1 - \lambda_i)z^2 + 2(1 + \delta)z + 1 + \lambda_i + \delta(\lambda_i - 3) = 0. \tag{35}$$

Since the bilinear transformation is a one-to-one map from the interior of the unit circle to the open left half plane (LHP), we know that (34) has all roots within the unit circle in z plane if and only if all the roots of (35) are in the open LHP. It is noted that the coefficients in (35) may be complex due to the directed unbalanced network topology, which implies that the analysis method used in [16, 17] for the undirected network cannot be applied to the case we are considering in this paper. It is known from [29] that the roots of the complex coefficients equation (35) are in the open LHP if and only if the roots of

$$\begin{aligned} &[(1 + \delta)(1 - \lambda_i)z^2 + 2(1 + \delta)z + 1 + \lambda_i + \delta(\lambda_i - 3)] \\ &\cdot[(1 + \delta)(1 - \overline{\lambda}_i)z^2 + 2(1 + \delta)z + 1 + \overline{\lambda}_i + \delta(\overline{\lambda}_i - 3)] = 0, \end{aligned} \tag{36}$$

are in the open LHP for all $i = 2, 3, \cdots, N$. After some tedious algebraic manipulations, (36) can be rewritten as

$$z^4 + p_1 z^3 + p_2 z^2 + p_3 z + p_4 = 0, \quad i = 2, 3, \cdots, N, \tag{37}$$

where

$$\begin{aligned} p_1 &= \frac{4(1-\mathrm{Re}(\lambda_i))}{(1-\lambda_i)(1-\overline{\lambda}_i)}, \quad p_2 = \frac{2[3-|\lambda_i|^2+\delta(4\mathrm{Re}(\lambda_i)-|\lambda_i|^2-1)]}{(1+\delta)(1-\lambda_i)(1-\overline{\lambda}_i)}, \\ p_3 &= \frac{4[1+\mathrm{Re}(\lambda_i)+\delta(\mathrm{Re}(\lambda_i)-3)]}{(1+\delta)(1-\lambda_i)(1-\overline{\lambda}_i)}, \\ p_4 &= \frac{(1+2\mathrm{Re}(\lambda_i)+|\lambda_i|^2)+2\delta(|\lambda_i|^2-2\mathrm{Re}(\lambda_i)-3)+\delta^2(9-6\mathrm{Re}(\lambda_i)+|\lambda_i|^2)}{(1+\delta)^2(1-\lambda_i)(1-\overline{\lambda}_i)}. \end{aligned} \tag{38}$$

Then following Theorem 2.4 in [29], all roots of (37) are in the open LHP if and only if

$$p_1 > 0, \quad p_2 > 0, \quad p_3 > 0, \quad p_4 > 0, \text{ and } p_1p_2p_3 > p_3^2 + p_1p_4. \tag{39}$$

Since for all $i = 2, 3, \cdots, N$,

$$(1-\lambda_i)(1-\overline{\lambda}_i) = (1-\lambda_i)\overline{(1-\lambda_i)} = 1 - 2\mathrm{Re}(\lambda_i) + |\lambda_i|^2 > 0,$$

then after some tedious algebraic manipulations for (39), we obtain the following five inequalities

$$\begin{aligned} &1-\mathrm{Re}(\lambda_i) > 0, && (40)\\ &3-|\lambda_i|^2+\delta(4\mathrm{Re}(\lambda_i)-|\lambda_i|^2-1) > 0, && (41)\\ &1+\mathrm{Re}(\lambda_i)+\delta(\mathrm{Re}(\lambda_i)-3) > 0, && (42)\\ &[1+\lambda_i+\delta(\lambda_i-3)]\overline{[1+\lambda_i+\delta(\lambda_i-3)]} > 0, && (43)\\ &8(1-\mathrm{Re}(\lambda_i))[3-|\lambda_i|^2+\delta(4\mathrm{Re}(\lambda_i)-|\lambda_i|^2-1)][1+\mathrm{Re}(\lambda_i)+\delta(\mathrm{Re}(\lambda_i)-3)] \\ &>(1-2\mathrm{Re}(\lambda_i)+|\lambda_i|^2)\{4[1+\mathrm{Re}(\lambda_i)+\delta(\mathrm{Re}(\lambda_i)-3)]^2 \\ &\quad + (1-\mathrm{Re}(\lambda_i))[1+\lambda_i+\delta(\lambda_i-3)]\overline{[1+\lambda_i+\delta(\lambda_i-3)]}\}. && (44) \end{aligned}$$

Step 3 To find conditions under which (40)–(44) hold.

Note that (40) holds because $\mathrm{Re}(\lambda_i) < 1$ $(i = 2, 3, \cdots, N)$. It is easy to verify that

$$[1+\lambda_i+\delta(\lambda_i-3)]\overline{[1+\lambda_i+\delta(\lambda_i-3)]} \geq [1+\mathrm{Re}(\lambda_i)+\delta(\mathrm{Re}(\lambda_i)-3)]^2, \tag{45}$$

then if (42) holds, (43) also holds. Furthermore, as the proof of [30] and after tedious algebraic manipulations, then by Vieta's Theorem, it follows that (44) always holds. Therefore, we next just need to further analyze (41) and (42).

Firstly, noting that $-1 < \mathrm{Re}(\lambda_i) < 1$, then from (42), we have

$$\delta < \frac{1+\mathrm{Re}(\lambda_i)}{3-\mathrm{Re}(\lambda_i)} = \delta_1(\mathrm{Re}(\lambda_i)), \quad i = 2, 3, \cdots, N. \tag{46}$$

Meanwhile, for any $y_1, y_2 \in (-1,1)$ and $y_1 > y_2$, we get

$$\frac{1+y_1}{3-y_1} - \frac{1+y_2}{3-y_2} = \frac{4(y_1-y_2)}{(3-y_1)(3-y_2)} > 0,$$

which means that $\delta_1(\mathrm{Re}(\lambda_i))$ is an increasing function on $\mathrm{Re}(\lambda_i)$ with $\delta(-1) = 0$ and $\delta(1) = 1$. Thus, we immediately have that $\delta_1(\mathrm{Re}(\lambda_i)) \in (0,1)$.

Secondly, since

$$3 - |\lambda_i|^2 + \delta(4\mathrm{Re}(\lambda_i) - |\lambda_i|^2 - 1) = 3 - \mathrm{Re}^2(\lambda_i) - (1+\lambda_i)\mathrm{Im}^2(\lambda_i) + \delta(4\mathrm{Re}(\lambda_i) - \mathrm{Re}^2(\lambda_i) - 1),$$

and $|\mathrm{Re}(\lambda_i)| \le |\lambda_i| < 1$ for $i = 2,3,\cdots,N$, we next will discuss the following two cases.

Case 1 When $2 - \sqrt{3} \le \mathrm{Re}(\lambda_i) < 1$, noting that $4\mathrm{Re}(\lambda_i) - |\mathrm{Re}(\lambda_i)|^2 - 1 \ge 0$ and $3 - \mathrm{Re}^2(\lambda_i) > 2 > (1+\delta)\mathrm{Im}^2(\lambda_i) \ge 0$, we know (41) always holds for $\delta > 0$. Together with (46), we have that if (30) is satisfied, then (41) and (42) hold.

Case 2 When $-1 < \mathrm{Re}(\lambda_i) < 2 - \sqrt{3}$, noting that $4\mathrm{Re}(\lambda_i) - |\lambda_i|^2 - 1 \le 4\mathrm{Re}(\lambda_i) - |\mathrm{Re}(\lambda_i)|^2 - 1 < 0$, then from (41), we obtain

$$\delta < \frac{3 - |\lambda_i|^2}{1 + |\lambda_i|^2 - 4\mathrm{Re}(\lambda_i)} = \delta_2(\mathrm{Re}(\lambda_i)), \quad i = 2,3,\cdots,N. \tag{47}$$

For $\delta_1(\mathrm{Re}(\lambda_i))$ and $\delta_2(\mathrm{Re}(\lambda_i))$, we will prove the fact: if $-1 < \mathrm{Re}(\lambda_i) < 2 - \sqrt{3}$, then $\delta_1(\mathrm{Re}(\lambda_i)) \le \delta_2(\mathrm{Re}(\lambda_i))$ for $i = 2,3,\cdots,N$. Indeed, after some straightforward calculations, we have

$$\delta_1(\mathrm{Re}(\lambda_i)) < \delta_2(\mathrm{Re}(\lambda_i)) \Leftrightarrow 2 \ge |\lambda_i|^2 - \mathrm{Re}^2(\lambda_i) = \mathrm{Im}^2(\lambda_i), \tag{48}$$

which always holds because $|\lambda_i| < 1$ $(i = 2,3,\cdots,N)$.

Hence, Cases 1 and 2 show that if (30) is satisfied, then (41) and (42) hold. In a word, Step 3 establishes that if δ is chosen as in(30), then (40)–(44) hold.

Step 4 From Step 2, we know the roots of the complex coefficients equation (35) (equivalently, (36)) are in the open LHP if and only if (40)–(44) hold. On the other hand, from Step 3, we know if (30) is satisfied, then (40)–(44) hold. Hence, if δ satisfies (30), the roots of (35) (equivalently, (36)) are in the open LHP. That is, the roots of (34) are within the unit circle. Then together with (33), we can conclude that if δ is chosen as in(30), except the unique maximal eigenvalue 1, all the other eigenvalues of the matrix R_1 are strictly within the unit circle. In other words, the matrix R_1 is neutral stable. ▮

Remark 5 Equation (30) explicitly indicates that the parameter δ of the infinite-level logarithmic quantizer (9) is only specified by the real part of the eigenvalues λ_i $(i = 2,3,\cdots,N)$ of the adjacency matrix W. While in [25], the parameters of the finite-level uniform quantizer are specified by the left eigenvector π and the positive entries w_{ij} of the adjacency matrix W.

Remark 6 If the network is undirected and the corresponding adjacency matrix W is symmetric, then the eigenvalues of W are real, and the coefficients of Equation (35) are also real. In this case, (30) becomes

$$0 < \delta \le \min_{\lambda_i \ne 1} \frac{1+\lambda_i}{3-\lambda_i} = \overline{\delta}, \quad i = 2,3,\cdots,N,$$

which is exactly the expression obtained in [16].

It follows from Lemma 1 that $R_h \in \boldsymbol{R}$ is not Hurwitz stable. Therefore, although (27) is a linear model, most existing convergence results for LPV systems cannot be directly applied. Next we will introduce the stability criterions in the common Lyapunov sense for discrete-time LPV systems.

Let $\{A(k)\}_{k=0}^{+\infty} \subset \mathrm{Co}\{A_1, A_2, \cdots, A_n\}$ denote the sequence of matrices taking values in the convex hull of given constant matrices $A_1, A_2, \cdots, A_n \in R^{N\times N}$. Then for the LPV system

$$x(k+1) = A(k)x(k), \tag{49}$$

the following lemma holds.

Lemma 3[17] *Assume that* 1 *is a simple eigenvalue with left eigenvector* $\pi \in R^N$ *and right eigenvector* $\mathbf{1}$ *for constant matrices* $A_1, A_2, \cdots, A_n \in R^{N\times N}$. *For any nonzero vector* $z \in \mathrm{span}\{\mathbf{1}\}^{\perp}$ *and any* $i, j \in \{1, 2, \cdots, n\}$, *if there exists a positive semidefinite matrix* $P \in R^{N\times N}$ *such that*

$$P\mathbf{1} = \mathbf{0}, \tag{50}$$

$$z^{\mathrm{T}}Pz > 0, \tag{51}$$

$$z^{\mathrm{T}}\left(\frac{1}{2}(A_i^{\mathrm{T}}PA_j + A_j^{\mathrm{T}}PA_i)\right)z < 0, \tag{52}$$

then, for any initial value $x(0)$ *and any matrix sequence* $\{A(k)\}_{k=0}^{+\infty} \subset \mathrm{Co}\{A_1, A_2, \cdots, A_n\}$, *the asymptotic solution of* (49) *is*

$$\lim_{k\to\infty} x(k) = a\mathbf{1}, \quad a = \pi^{\mathrm{T}}x(0).$$

With these preparations in hands, we now proceed to the main part of the convergence analysis for the closed-loop system (27) (equivalently, (22)).

Theorem 1 *Suppose that Assumption* 1 *holds. For the closed-loop system* (27), *if the quantization precision parameter* δ *is chosen as in* (30), *then for all initial value* $z(0) \in R^{2N}$ *and for any matrix sequence* $\{F(k)\}_{k=0}^{+\infty}$ *with* $F(k) \in \mathrm{Co}\{\boldsymbol{R}\}$, *we have*

$$\lim_{k\to\infty} z(k) = \begin{bmatrix} a\mathbf{1} \\ \mathbf{0} \end{bmatrix} \text{ with } a = \left(\sum_{i=1}^{N} \pi_i x_i(0)\right). \tag{53}$$

This means the weighted average consensus for agents on the directed unbalanced network with logarithmic quantized communication is achieved, namely

$$\lim_{k\to\infty} x(k) = \left(\sum_{i=1}^{N} \pi_i x_i(0)\right)\mathbf{1} \quad \text{and} \quad \lim_{k\to\infty} e(k) = \mathbf{0}.$$

Proof The proof of Theorem 1 is organized into five steps.

Step 1 To turn the consensus of a LPV system into the consensus of a linear invariant system.

By Lemma 1, we know $R_h \upsilon = \upsilon$ and $\phi^{\mathrm{T}} R_h = \phi^{\mathrm{T}}$ for all $h = 1, 2, \cdots, 2^N$. Then for δ as in (30), by Remark 2 and Lemma 3, it is clear that to get (53), we only need to prove that there exists a positive semidefinite matrix $L \in R^{2N \times 2N}$ satisfying

$$L(\mathbf{1}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{0}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}, \tag{54}$$

$$z^{\mathrm{T}} L z > 0, \tag{55}$$

$$z^{\mathrm{T}} \left(\frac{1}{2}(R_i^{\mathrm{T}} L R_j + R_j^{\mathrm{T}} L R_i) - L \right) z < 0, \quad \text{for all } R_i, R_j \in \boldsymbol{R}, \tag{56}$$

where $z \in \operatorname{span}\left((\mathbf{1}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}} \right)^{\perp}$ is an arbitrary nonzero vector.

For this purpose, we choose the following candidate positive semidefinite matrix L

$$L = \begin{bmatrix} P & 0 \\ 0 & \gamma I \end{bmatrix}, \tag{57}$$

where $P \in R^{N \times N}$ is a positive semidefinite matrix and $\gamma > 0$ is a constant. P and γ will be determined subsequently.

By the structure of L, it is easy to verify that $R_i^{\mathrm{T}} L R_j^{\mathrm{T}} = R_j^{\mathrm{T}} L R_i^{\mathrm{T}}$ for any pair of matrices $R_i, R_j \in \boldsymbol{R}$. Hence, if (56) holds, then it is equivalent to

$$z^{\mathrm{T}}(R_i^{\mathrm{T}} L R_j^{\mathrm{T}} - L) z < 0, \quad \text{for all } R_i, R_j \in \boldsymbol{R}. \tag{58}$$

Straightforward calculations lead us to

$$R_i^{\mathrm{T}} L R_j^{\mathrm{T}} - L = R_1^{\mathrm{T}} L R_1^{\mathrm{T}} - L - Q, \tag{59}$$

where

$$Q \ = \gamma\delta^2 \begin{bmatrix} (W-I)^{\mathrm{T}}(I - E_iE_j)(W-I) & (W-I)^{\mathrm{T}}(I - E_iE_j)(W - 2I) \\ (W-2I)^{\mathrm{T}}(I - E_iE_j)(W-I) & (W-2I)^{\mathrm{T}}(I - E_iE_j)(W - 2I) \end{bmatrix} \tag{60}$$

and $R_i^{\mathrm{T}} L R_j^{\mathrm{T}} - L$ is specified in the following

$$\begin{aligned}
& R_i^{\mathrm{T}} L R_j^{\mathrm{T}} - L \\
= & \begin{bmatrix} W^{\mathrm{T}} P W + \gamma\delta^2 (W-I)^{\mathrm{T}} E_iE_j (W-I) & W^{\mathrm{T}} P (W-I) + \gamma\delta^2 E_iE_j (W-I)^{\mathrm{T}} (W-2I) \\ (W-I)^{\mathrm{T}} P W + \gamma\delta^2 (W-2I)^{\mathrm{T}} E_iE_j (W-I) & (W-I)^{\mathrm{T}} P (W-I) + \gamma\delta^2 E_iE_j (W-2I)^{\mathrm{T}} (W-I) \end{bmatrix} \\
& - \begin{bmatrix} P & 0 \\ 0 & \gamma I \end{bmatrix} \\
= & \begin{bmatrix} W^{\mathrm{T}} P W - P + \gamma\delta^2 (W-I)^{\mathrm{T}} (W-I) & W^{\mathrm{T}} P (W-I) + \gamma\delta^2 (W-I)^{\mathrm{T}} (W-2I) \\ (W-I)^{\mathrm{T}} P W + \gamma\delta^2 (W-2I)^{\mathrm{T}} (W-I) & (W-I)^{\mathrm{T}} P (W-I) + \gamma\delta^2 (W-2I)^{\mathrm{T}} (W-I) - \gamma I \end{bmatrix} \\
& - \gamma\delta^2 \begin{bmatrix} (W-I)^{\mathrm{T}}(I - E_iE_j)(W-I) & (W-I)^{\mathrm{T}}(I - E_iE_j)(W-2I) \\ (W-2I)^{\mathrm{T}}(I - E_iE_j)(W-I) & (W-2I)^{\mathrm{T}}(I - E_iE_j)(W-2I) \end{bmatrix} \\
= & R_1^{\mathrm{T}} L R_1 - L - Q.
\end{aligned} \tag{61}$$

Clearly, $Q = Q^{\mathrm{T}} \geq 0$ and $(\mathbf{1}^{\mathrm{T}} \;\; \mathbf{0}^{\mathrm{T}})Q(\mathbf{1}^{\mathrm{T}} \;\; \mathbf{0}^{\mathrm{T}})^{\mathrm{T}} = 0$. Meanwhile, if (58) holds for $i = j = 1$, then (58) holds for all $R_i, R_j \in \boldsymbol{R}$. Therefore, if we can verify that, for any nonzero vector $z \in \operatorname{span}\big((\mathbf{1}^{\mathrm{T}} \;\; \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}\big)^{\perp}$, there exists a positive semidefinite matrix $L \in R^{2N\times 2N}$ satisfying

$$z^{\mathrm{T}}(R_1^{\mathrm{T}} L R_1 - L)z < 0, \tag{62}$$

which, along with (59), will lead to (56).

For this purpose, we next consider the following linear invariant system instead of the LPV system (27)

$$z(k+1) = \begin{bmatrix} I & 0 \\ 0 & \delta I \end{bmatrix} \begin{bmatrix} W & W - I \\ W - I & W - 2I \end{bmatrix} z(k). \tag{63}$$

That is, $F(0) = F(1) = \cdots = R_1$ for all $k \geq 0$ in (27), and R_1 is defined in (28). Thus, we now just need to verify that there exists a suitable candidate L satisfying (54), (55), and (62). In what follows, we will use the matrix transformation and the Lyapunov techniques to carry out the analysis.

Step 2 To turn the problem of consensus, i.e., the problem of convergence to the eigenspace, into the problem of Hurwitz stability at the origin.

For the convenience of performing a Lyapunov analysis of (63), we introduce the variable

$$\overline{z}(k) = \begin{bmatrix} \overline{x}(k) \\ \overline{e}(k) \end{bmatrix} = \begin{bmatrix} I - \mathbf{1}\pi^{\mathrm{T}} & 0 \\ 0 & I - \mathbf{1}\pi^{\mathrm{T}} \end{bmatrix} z(k).$$

Note that the ith $(1 \leq i \leq N)$ component of $\overline{z}(k)$ represents the distance of the state of agent i from the weighted average value at time k. Meanwhile, $(\pi^{\mathrm{T}} \;\; \pi^{\mathrm{T}})\overline{z}(k) = 0$ holds for all $k \geq 0$.

Since $(W - \alpha I)(I - \mathbf{1}\pi^{\mathrm{T}}) = (I - \mathbf{1}\pi^{\mathrm{T}})(W - \alpha I)$, where α is a constant, then from (63), we have

$$\begin{bmatrix} \overline{x}(k+1) \\ \overline{e}(k+1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \delta I \end{bmatrix} \begin{bmatrix} W & W - I \\ W - I & W - 2I \end{bmatrix} \begin{bmatrix} \overline{x}(k)) \\ \overline{e}(k) \end{bmatrix}. \tag{64}$$

It is easy to see that 0 is a simple eigenvalue of $(I - \mathbf{1}\pi^{\mathrm{T}})$ with left eigenvector π and right eigenvector $\mathbf{1}$, and that 1 is another eigenvalue of $(I - \mathbf{1}\pi^{\mathrm{T}})$ with algebraic multiplicity $N - 1$. Hence, it follows from (64) that $\overline{z}(k) = (\overline{x}(k)^{\mathrm{T}} \;\; \overline{e}(k)^{\mathrm{T}})^{\mathrm{T}} = \mathbf{0} \in R^{2N}$ if and only if $z_1(k) = z_2(k) = \cdots = z_N(k)$. In other words, the consensus problem, or the problem of convergence to the eigenspace of the system (63) is solved, if and only if the system (64) is Hurwitz stable at the origin.

Step 3 To turn the problem of Hurwitz stability at the origin of a linear invariant system into the problem of Hurwitz stability at the origin of its subsystem.

By (5), we know that 1 is a simple eigenvalue of the stochastic matrix W with left eigenvector π and right eigenvector $\mathbf{1}$. Therefore, there exists a nonsingular matrix $T = (\mathbf{1} \;\; M)$, $M \in$

$R^{N\times(N-1)}$, which transforms W to the Jordan canonical form of [26]

$$T^{-1}WT = \begin{bmatrix} 1 & 0_{1\times(N-1)} \\ 1_{(N-1)\times 1} & \Lambda \end{bmatrix} = \widetilde{W}, \tag{65}$$

where $T^{-1} = \begin{bmatrix} \mathbf{1}^{\dagger} \\ M^{\dagger} \end{bmatrix} = \begin{bmatrix} \pi^{\mathrm{T}} \\ M^{\dagger} \end{bmatrix}$ is such that $M^{\dagger}M = I_{N-1}$, $M^{\dagger}\mathbf{1} = 0_{(N-1)\times 1}$ and $\pi^{\mathrm{T}}M^{\dagger} = 0_{1\times(N-1)}$; Λ is an upper-triangular matrix with its diagonal blocks being the subunit eigenvalues of the stochastic matrix W. Defining the variable $\widetilde{z}(k) = (T^{-1}\otimes I_2)\overline{z}(k)$, where $\otimes$ denotes the Kronecker product operation of matrix, we can further formulate (64) as

$$\begin{bmatrix} \widetilde{x}(k+1) \\ \widetilde{e}(k+1) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \Lambda \end{bmatrix} & \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \Lambda - I_{N-1} \end{bmatrix} \\ \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \delta(\Lambda - I_{N-1}) \end{bmatrix} & \begin{bmatrix} -1 & 0_{1\times(N-1)} \\ 0_{(N-1)\times 1} & \delta(\Lambda - 2I_{N-1}) \end{bmatrix} \end{bmatrix} \begin{bmatrix} \widetilde{x}(k) \\ \widetilde{e}(k) \end{bmatrix}. \tag{66}$$

For the variable $\widetilde{z}_1(k) = (\widetilde{x}_1(k)\ \widetilde{e}_1(k))^{\mathrm{T}}$, by the definition of T^{-1} and the fact $\pi^{\mathrm{T}}(I - \mathbf{1}\pi^{\mathrm{T}}) = \mathbf{0}^{\mathrm{T}}$, it can be seen that $\widetilde{x}_1(k) = \pi^{\mathrm{T}}\overline{x}(k) = 0$, $\widetilde{e}_1(k) = -\pi^{\mathrm{T}}\overline{e}(k) = 0$, which means $\widetilde{z}_1(k) = (0\ 0)^{\mathrm{T}}$. Moreover, note that the transformation $\widetilde{z}(k) = (T^{-1}\otimes I_2)\overline{z}(k)$ is nonsingular, then $\overline{z}(k) = (\overline{x}(k)^{\mathrm{T}}\ \overline{e}(k)^{\mathrm{T}})^{\mathrm{T}} = \mathbf{0} \in R^{2N}$ is equivalent to that $\widetilde{z}_2(k) = \widetilde{z}_3(k) = \cdots = \widetilde{z}_N(k) = \mathbf{0} \in R^2$. Therefore, the consensus of agents can be achieved if and only if the variables $\widetilde{x}_2(k), \widetilde{x}_3(k), \cdots, \widetilde{x}_N(k)$ and $\widetilde{e}_2(k), \widetilde{e}_3(k), \cdots, \widetilde{e}_N(k)$ asymptotically converge to zero, or in other words, if and only if the following subsystem

$$\begin{bmatrix} \widetilde{x}_2(k+1) \\ \vdots \\ \widetilde{x}_N(k+1) \\ \widetilde{e}_2(k+1) \\ \vdots \\ \widetilde{e}_N(k+1) \end{bmatrix} = \begin{bmatrix} \Lambda & \Lambda - I_{(N-1)} \\ \delta(\Lambda - I_{(N-1)}) & \delta(\Lambda - 2I_{(N-1)}) \end{bmatrix} \begin{bmatrix} \widetilde{x}_2(k) \\ \vdots \\ \widetilde{x}_N(k) \\ \widetilde{e}_2(k) \\ \vdots \\ \widetilde{e}_N(k) \end{bmatrix} \tag{67}$$

is Hurwitz stable at the origin.

Step 4 To establish the corresponding Lyapunov stability condition.

According to [31, 32], if the following linear matrix inequality (LMI)

$$\begin{aligned} &\begin{bmatrix} \Lambda & \Lambda - I_{(N-1)} \\ \delta(\Lambda - I_{(N-1)}) & \delta(\Lambda - 2I_{(N-1)}) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & 0 \\ 0 & \gamma I_{(N-1)} \end{bmatrix} \begin{bmatrix} \Lambda & \Lambda - I_{(N-1)} \\ \delta(\Lambda - I_{(N-1)}) & \delta(\Lambda - 2I_{(N-1)}) \end{bmatrix} \\ &- \begin{bmatrix} P_1 & 0 \\ 0 & \gamma I_{(N-1)} \end{bmatrix} < 0, \end{aligned} \tag{68}$$

is satisfied, where P_1 is a $(N-1)\times(N-1)$ positive definite matrix and the constant $\gamma > 0$, then the system (67) is Hurwitz stable at the origin.

Based on the preceding analysis, it can be seen that the consensus problem of the closed-loop system (27) is now boiled down to determining positive definite matrix P_1 and $\gamma > 0$, such that the LMI (68) holds. Whether the LMI (68) holds can be easily tested by the LMI box in Matlab[32]. Note that Lemma 2 shows that if δ is chosen as in (30), then the matrix R_1 has a simple eigenvalue 1 and all its other eigenvalues are strictly inside the unit circle. This fact, together with the nonsingular matrix transformation (65), indicates that the subsystem (67) is always Hurwitz stable at the origin. Thus, there always exist a positive definite matrix P_1 and a constant $\gamma > 0$ satisfying the LMI (68).

Step 5 To determine the matrix L.

If the testable LMI (68) is satisfied, then the positive semidefinite matrix L defined in (57) can be constructed as

$$L = (T\otimes I_2)\begin{bmatrix}\begin{bmatrix}0 & 0_{1\times(N-1)}\\ 0_{(N-1)\times 1} & P_1\end{bmatrix} & 0_{N\times N}\\ 0_{N\times N} & \gamma I_N\end{bmatrix}(T^{-1}\otimes I_2), \tag{69}$$

correspondingly, the positive semidefinite matrix P in L is

$$P = T\begin{bmatrix}0 & 0_{1\times(N-1)}\\ 0_{(N-1)\times 1} & P_1\end{bmatrix}T^{-1} = (\mathbf{1}\ M)\begin{bmatrix}0 & 0_{1\times(N-1)}\\ 0_{(N-1)\times 1} & P_1\end{bmatrix}\begin{bmatrix}\pi^{\mathrm{T}}\\ M^{\dagger}\end{bmatrix}.$$

Hence, from (65) and by the definitions of T and T^{-1}, we can validate that $P\mathbf{1} = \mathbf{0}$, i.e., 0 is a simple eigenvalue of matrix P and the null space is span$\{\mathbf{1}\}$. Similarly, we can prove that $L(\mathbf{1}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{0}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$, i.e., L also has a simple eigenvalue 0 and the null space is span$\left((\mathbf{1}^{\mathrm{T}}\ \mathbf{0}^{\mathrm{T}})^{\mathrm{T}}\right)$. Therefore, the positive semidefinite matrix L obtained in (69) satisfies (54), (55) and (62). Then together with (59), we know that L also satisfies (54)–(56). Thus, we have (53). That is, the quantized protocol (15) solves the weighted average consensus problem. ▮

Remark 7 In [16], for the case of an undirected network, the candidate matrix L is chosen as

$$L = \begin{bmatrix}I-W & 0\\ 0 & \gamma I\end{bmatrix}.$$

Then based on the fact that the adjacency matrix W is symmetric and can be written as a clean diagonal matrix by matrix decomposition, an explicit expression of γ is derived in [16]. This explicit expression shows that γ is specified by the quantization precision parameter δ and the eigenvalues of W. It is worth pointing out that the double stochasticity of the adjacency matrix W would greatly simplify the analysis of consensus as in [16], but unfortunately, this feature disappears if W is stochastic but not doubly stochastic. As a result, for the case of a directed network, it is difficult to get an explicit expression of γ in this paper. Note that the computational effort required to solve the LMI (68) strictly depends on the network scale, which may bring about numerical computation issue, particularly when the network scale is very large. Thus, this is a drawback of the convergence analysis method presented in this paper.

Remark 8 If W is doubly stochastic and $\pi_1 = \pi_2 = \cdots = \pi_N = 1/N$, an immediate consequence of Theorem 1 is that the average consensus is achieved. Hence, Theorem 1 extends the results of [16] to a general case. Indeed, the final achieved consensus value shown in (53) clearly demonstrates the intimate dependence on the topology property of the directed network.

Remark 9 Similar to the proof of [16], we can further prove that the consensus can be achieved with an exponential convergence rate, and the detailed analysis is omitted here.

Remark 10 This paper and [25] focus on fixed and directed unbalanced networks that are strongly connected, where the adjacency matrices have time-independent left eigenvectors corresponding to the maximal eigenvalue 1. Therefore, the weighted average invariance is preserved. However, for general directed networks with switching topologies, the corresponding left eigenvectors are time-dependent and the weighted average of the states are not necessarily invariant at each time instant, except the case where the directed networks are always balanced. Note that undirected networks with switching topologies are balanced, and thus the average invariance is still preserved if the undirected switching networks are jointly connected[5]. Based on a novel adaptive finite-level uniform quantization scheme, the quantized average consensus problem for undirected networks with switching topologies is recently studied in [33]. For directed networks with switching topologies that are not always balanced, since the property of (weighted) average invariance is no longer preserved, the consensus convergence analyses carried out in [25,33] and this paper are no longer valid for this more general case, and new convergence analysis methods need to be developed. For the quantized consensus problem of this more general case, please refer to our recent work [34].

5 Simulation

In the simulation, we consider a strongly connected and unbalanced information digraph G with 6 nodes (see Figure 3), and the corresponding weighted adjacency matrix W is

$$W = \begin{bmatrix} 0.2 & 0.3 & 0.2 & 0 & 0 & 0.3 \\ 0 & 0.3 & 0 & 0.3 & 0.2 & 0.2 \\ 0 & 0 & 0.4 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0.5 & 0.3 & 0 & 0 & 0.2 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix},$$

which is stochastic but is not doubly stochastic, and thus unidirectional information flow among agents is admitted.

Figure 3 The directed network

The unique normalized positive left eigenvector of W corresponding to eigenvalue 1 is $\pi = (0.0536\ 0.3164\ 0.0179\ 0.1671\ 0.00858\ 0.3592)^{\mathrm{T}}$, and the other eigenvalues of W are $\lambda_2 = 0.5084,\ \lambda_{3,4} = 0.0783 \pm 0.3573j,\ \lambda_{5,6} = 0.1675 \pm 0.0759j$. Then the Jordan canonical form of W is

$$\widetilde{W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5084 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0286 & 0.1307 & 0 & 0 \\ 0 & 0 & -0.1307 & 0.0286 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0308 & 0.0140 \\ 0 & 0 & 0 & 0 & -0.0140 & 0.0308 \end{bmatrix}.$$

From (30), we know the quantization precision parameter δ satisfies $0 < \delta \leq \overline{\delta} = 0.3691$. In the simulation, we choose $\delta = 0.26$, and the initial states are randomly generated as $x(0) = (3.9221\ 5.5949\ 2.5574\ -3.6429\ 4.4913\ 5.3399)^{\mathrm{T}}$, then the weighted average value is $\pi^{\mathrm{T}}x(0) = 3.7209$. Furthermore, by using the LMI box in Matlab [32], the corresponding positive definite matrix P_1 and the constant $\gamma > 0$ satisfying LMI (68) are obtained as

$$P_1 = \begin{bmatrix} 0.9867 & 0 & 0 & 0 & 0 \\ 0 & 0.8381 & 0 & 0 & 0 \\ 0 & 0 & 0.8381 & 0 & 0 \\ 0 & 0 & 0 & 0.8240 & 0 \\ 0 & 0 & 0 & 0 & 0.8240 \end{bmatrix}$$

and $\gamma = 1.5858$.

Figure 4 The trajectories of the agents' states

Figure 5 The evolution of the square error

6 Conclusions

In this paper, in the absence of the double stochasticity assumption, we studied the problem of distributed quantized consensus in strongly connected directed networks with the logarithmic quantization scheme. We first derived an upper bound of the quantization precision parameter to design a suitable logarithmic quantizer. The resulting upper bound is explicitly specified by network topology. Then by employing the matrix transformation and the Lyapunov techniques, we provided a testable condition to guarantee the achievement of the weighted average consensus with the proposed quantized protocol. For future work, we will seek other less conservative ways to characterize the stability condition instead of using LMI. The extension of the obtained results to directed digital networks with communication delay is also interesting.

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