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SOCIAL LEARNING WITH TIME-VARYING WEIGHTS*

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Abstract This paper investigates a non-Bayesian social learning model, in which each individual updates her beliefs based on private signals as well as her neighbors' beliefs. The private signal is involved in the updating process through Bayes' rule, and the neighbors' beliefs are embodied in through a weighted average form, where the weights are time-varying. The authors prove that agents eventually have correct forecasts for upcoming signals, and all the beliefs of agents reach a consensus. In addition, if there exists no state that is observationally equivalent to the true state from the point of view of all agents, the authors show that the consensus belief of the whole group eventually reflects the true state.

Keywords Consensus, social learning, social networks, time-varying weights.

1 Introduction

It is well known that beliefs and opinions affect decisions in the daily life. Examples to support this statement are prevalent all around us. The kind of clothes we prefer to buy is shaped by our opinions. Similarly, in a voting situation, our ideological beliefs will determine which candidate we decide to support. Because of the importance of beliefs, the study of how beliefs are formed has gradually become a hot topic in past decades, and created a new research area — Social learning theory.

Social learning models can be classified into two categories: Bayesian and non-Bayesian. By Bayesian, we refer to those in which rational individuals use Bayes' rule to form the best mathematical estimate of the relevant unknowns given their priors and observed signals[1–4].

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Generally, the strategies of individuals must be taken into account in order to make the rational decision based on the observations. Therefore, Bayesian learning problems are mainly analyzed in the framework of game theory. The fully rational inference facing Bayesian agents, especially when they are situated in complex social networks, is quite challenging. The existing literature studies relatively simple scenarios, such as agents interacting sequentially, and the structures of networks are not concretely taken into account. Due to the complexity of analysis and computing of Bayesian inference, it is not realistic to expect individuals to adopt Bayesian learning in the real world. Moreover, a series of experiments suggests that people may process information in a less rational way[5, 6], which emphasizes the necessity of investigating the collective behavior of boundedly rational individuals in the non-Bayesian learning models. Most non-Bayesian learning approaches focus on simple updating rules, such as imitation and replication[7–9]. Traditional non-Bayesian learning can be considered as a consensus problem in cooperative control of multi-agent networks[10–13], where the mathematical models and the analysis methods are very similar. With abundant mathematical tools, such as algebraic graph theory and non-negative matrix theory, these models are convenient for analysis, and indeed, lots of meaningful results are obtained. However, one problem facing most non-Bayesian models is that they mainly focus on the agreement of beliefs of different agents, rather than acquiring the correct belief, and the compromised belief is only a mixture of the initial beliefs of all agents, which might not reflect the underlying truth.

To solve the problems facing both Bayesian and non-Bayesian models, Jadbabaie, et al.[14] recently developed a non-Bayesian social learning model, in which the boundedly rational agent updates her belief as a convex combination of the Bayesian posterior belief based on her private signal and the beliefs of her neighbors. In their model, signals are caused by the underlying true state of the social event concerned, and the posterior beliefs are obtained simply by Bayes' rule based on the private signals, which is quite different from the traditional Bayesian learning models, where the signals are obtained from other people's actions. Besides the personal Bayesian updating, individuals also learn by communicating with others in a naive way, i.e., simply adopting the weighted averages of neighbors' beliefs. Since complex deduction is avoided, this model sharply decreases the complexity of analysis and computing. The learning algorithm is proved to enable the individuals to successfully aggregate information and find the true state under mild assumptions. The major advantage of the model is that all agents are boundedly rational such that the belief updating rule is quite simple, as that in most non-Bayesian learning models, and meanwhile, all agents can learn the truth, as they do in most Bayesian learning models, where much complicated deduction is needed to achieve this goal.

In Jadbabaie's model, it is supposed that each individual assigns fixed weights on the beliefs of neighbors during the learning process, which means the trust levels among individuals are fixed. However, in the real world, the weights are affected by many factors. The literature in social sciences shows that the social structure may be shaped by certain principles, such as "birds of a feather fly together", i.e., individuals are inclined to trust others who have similar beliefs[15]. This makes the weights change along the process of social learning. Another scenario usually observed in the real world is that individuals may gradually become more self-confident, or conversely, more dependent on others, as the private signals accumulate. The changes on weights are usually captured by (or converted into) a function of time. Therefore, it is of practical importance to investigate social learning in networks with time-varying weights. In this paper, we formulate the evolution of beliefs over a period of time in the form of product of stochastic matrices plus a error term decreasing to zero. A major difficulty when analyzing the model with time-varying weights is to demonstrate the consistency of the group beliefs, or more mathematically, to testify the compressibility of time-varying stochastic matrices. As a treatment, we introduce the definition of coefficient of ergodicity, and turn the product of stochastic matrices into a scrambling matrix, the coefficient of ergodicity of which is strictly less than one. Therefore, by the compressibility of scrambling matrix, we prove that the convergence of the learning algorithm presented in Jadbabaie's model can be extended to a class of networks with time-varying weights. Simulations further demonstrate the validity of our model, and shed a light on the choice of the time-varying weights.

The remainder of this paper is organized as follows. In Section 2, we introduce basic notations, terminologies, and environment settings that will be used in the subsequent analysis. In Section 3, we investigate the social learning model with time-varying weights. Simulations are provided in Section 4. Section 5 contains concluding remarks.

2 Preliminaries

2.1 Social Networks

Consider a social network as a directed graph $G = (V, E)$, where $V = \{1, 2, \cdots, n\}$ is the node set and $E \subset V \times V$ is the edge set. We focus on the strongly connected networks, such that there is a directed path from any node to any other one. Each node in V represents an agent, and the edge from i to j, denoted by the order pair $(i, j) \in E$, captures the fact that agent i is a neighbor of agent j, and information flows from i to j. The set of neighbors of agent i is denoted by $N_i = \{j \in V : (j, i) \in E\}$.

2.2 States of Social Event and Beliefs of Individuals

Let θ denote a state of the social event concerned, and all the possible states compose a state set Θ, in which the true state is denoted by θ^*. The belief of agent i at time t is denoted by $\mu_{i,t}(\theta)$, which is the probability that she believes the state θ is the underlying true state. Thus, $\{\mu_{i,t}(\theta), \theta \in \Theta\}$ is a probability distribution over the state set Θ.

2.3 Outside Signals and Private Signal Structures

Conditional on the underlying true state, at each time period $t > 0$, a signal vector $s_t = (s_t^1, s_t^2, \cdots, s_t^n) \in S$ is generated according to the likelihood function $\ell(s_t|\theta^*)$, where s_t^i is the signal observed by agent i and S is the signal space. For each observed signal s and each possible state θ, agent i holds a corresponding private signal structure $\ell_i(s|\theta)$, representing the probability that she beliefs signal s appears if the true state is θ. We assume that the private signal structure of agent i about the true state θ^* is the i-th marginal of $\ell(\cdot|\theta^*)$, which means the agent has a perfect prior information about the true state. If there exists a state $\overline{\theta} \neq \theta^*$

satisfying that $\ell_i(s|\overline{\theta}) = \ell_i(s|\theta^*)$ for any signal s, we call $\overline{\theta}$ observationally equivalent to the true state. That is to say, state $\overline{\theta}$ and the underlying true state θ^* arouse exactly the same signals according to the same probability in agent i's eyes, and thus, she cannot tell these two state apart only by observing the signals. All the states that observationally equivalent to θ^* from the point of view of agent i compose a set $\overline{\Theta}_i = \{\theta \in \Theta : \ell_i(s|\theta) = \ell_i(s|\theta^*) \text{ for any signal } s\}$.

2.4 Definitions of Social Learning

"Social learning" may have different meanings in different contexts. Here we follow the definitions in [14], where two sorts of learning have been defined.

Definition 2.1 (see [14]) Let $m_{i,t}(\cdot) = \sum_{\theta\in\Theta} \ell_i(\cdot|\theta)\mu_{i,t}(\theta)$. Agent i can correctly forecast on a path $\{s_t\}_{t=1}^{\infty}$ if, along that path,

$$m_{i,t}(\cdot) \to \ell_i(\cdot|\theta^*) \quad \text{as} \quad t \to \infty.$$

Definition 2.2 (see [14]) Agent i asymptotically learns the true state θ^* on a path $\{s_t\}_{t=1}^{\infty}$ if, along that path,

$$\mu_{i,t}(\theta^*) \to 1 \quad \text{as} \quad t \to \infty.$$

3 Social Learning Model with Time-Varying Weights

3.1 Belief Updating Rule

Each agent updates her belief as a combination of two aspects: One is the Bayesian posterior belief based on the outside signal she observes, and the other is the weighted average of beliefs of her neighbors. The belief updating rule can be described as, for any $\theta \in \Theta$,

$$\mu_{i,t+1}(\theta) = a_{ii}(t)\mu_{i,t}(\theta)\frac{\ell_i(s_{t+1}^i|\theta)}{m_{i,t}(s_{t+1}^i)} + \sum_{j\in V, j\neq i} a_{ij}(t)\mu_{j,t}(\theta), \tag{1}$$

where $a_{ii}(t)$ is the self-reliance of agent i, and $a_{ij}(t)$ is the weight that agent i assigns to the belief of agent j ($j \neq i$) at time t. If agent j is a neighbor of agent i, then $a_{ij}(t) > 0$, and $a_{ij}(t) = 0$ otherwise. The belief updating rule (1) represents two learning methods in the real world: Deducting based on personal observations, and communicating with other individuals. If we let each agent's self-reliance be zero, the updating rule has the same form as that in traditional non-Bayesian learning models. It is shown in the literature that, under certain conditions, the beliefs of agents eventually converge to a common value, which is a mixture of the initial beliefs of the whole group. On the other hand, if the agents have their self-reliances to be one, the updating rule turns into the standard Bayesian learning in personal situation. By Savage's statement in [16], it is well known that any informed agent can learn the true state by herself without any neighbors' information.

The updating rule (1) can be rewritten as follows:

$$\mu_{i,t+1}(\theta) = \sum_{j\in V} a_{ij}(t)\mu_{j,t}(\theta) + a_{ii}(t)\mu_{i,t}(\theta)\left[\frac{\ell_i(s_{t+1}^i|\theta)}{m_{i,t}(s_{t+1}^i)} - 1\right]. \tag{2}$$

We will show later in this paper that, when agents correctly forecast the upcoming signals, the second term on the right hand side of (2) can be viewed as a disturbance, which decreases to zero almost surely as $t \to \infty$. From this point of view, the updating rule turns into a consensus protocol (see, e.g., [10–13] for details). We only need to analyze its convergence, and figure out whether the final belief is the correct one.

We can further write (2) in matrix form as

$$\mu_{t+1}(\theta) = A(t)\mu_t(\theta) + \mathrm{diag}\Bigg(a_{11}(t)\left[\frac{\ell_1(s^1_{t+1}|\theta)}{m_{1,t}(s^1_{t+1})} - 1\right], a_{22}(t)\left[\frac{\ell_2(s^2_{t+1}|\theta)}{m_{2,t}(s^2_{t+1})} - 1\right], \cdots, a_{nn}(t)\left[\frac{\ell_n(s^n_{t+1}|\theta)}{m_{n,t}(s^n_{t+1})} - 1\right]\Bigg)\mu_t(\theta), \tag{3}$$

where $\mu_t(\theta) = [\mu_{1,t}(\theta), \mu_{2,t}(\theta), \cdots, \mu_{n,t}(\theta)]^{\mathrm{T}}$, and diag of a vector is a diagonal matrix with the vector on its diagonal. The $n \times n$ matrix $A(t) = [a_{ij}(t)]$ captures the social interaction of the agents at time t, such as the neighboring relationships and the weights assigned to one another, and it must be a row-stochastic matrix, i.e., $\sum_{j=1}^n a_{ij}(t) = 1$ for any i to maintain a proper probability distribution.

Jadbabaie, et al.[14] have investigated the updating rule (1) in the fixed-weighted case, i.e., the matrix $A(t) \equiv A$, which is a constant matrix. In order to extend (1) to the networks with time-varying weights, we introduce a time-varying parameter $\eta(t) \in (0, 1]$, and define the time-varying weight matrix as

$$A(t) = \big(1 - \eta(t)\big)I + \eta(t)A, \tag{4}$$

where I is the identity matrix; $\eta(t)$ tunes the weights assigned to the neighbors' beliefs. Social learning with time-varying weights represents more realistic situations, where an individual may become more self-confident as the amount of observation increases, or more dependent on others after making a series of mistakes in forecast. In our model, a smaller $\eta(t)$ implies stronger self-reliance, and a larger $\eta(t)$ implies more dependent on others. In the case with $\eta(t) = 1$, our model (1) specializes to the fixed-weighted case investigated in [14].

3.2 Assumptions and Main Results

If $\eta(t)$ is too small, which means that the agents are stubborn and stick to their own beliefs, the updating process may be extremely long, or the agreement among all agents may not be reached in any reasonable amount of time. To rule this out, we make the following assumption.

Assumption 3.1 There exists a strictly positive constant $\sigma > 0$ such that $\eta(t) \geq \sigma$ for all $t \geq 0$.

Our work shows that, in networks with time-varying weights, repeated observation and sufficient communication can lead agents to the asymptotic learning. This result can be considered as a generalization of the result in [14], which focuses on the fixed-weighted case.

Theorem 3.2 *Consider the belief updating rule* (1) *with weights changing according to* (4)*. Suppose that:*

(a) *The social network is strongly connected;*

(b) *All agents have strictly positive self-reliances, i.e.,* $A(t)$ *has positive diagonal elements;*

(c) *There exists an agent with positive prior belief on the true state* θ^*. *Then correct forecast can be achieved for all agents. Furthermore, suppose that*

(d) *Assumption* 3.1 *holds. Then the beliefs of all agents converge to consensus almost surely. If, in addition,*

(e) *There exists no state* $\theta \neq \theta^*$ *that is observationally equivalent to* θ^* *from the point of view of all agents in the network. Then all agents can asymptotically learn the true state.*

Before giving the proof of Theorem 3.2, we need to introduce several related definitions which will be used later. For a stochastic matrix $P = [p_{ij}]$, define its coefficient of ergodicity as

$$\tau(P) = 1 - \min_{i,j} \sum_{s=1}^{n} \min\{p_{is}, p_{js}\}.$$

If $\tau(P) < 1$, then P is called a scrambling matrix. For any stochastic matrices P_1 and P_2, it follows that

$$\tau(P_1 P_2) \leq \tau(P_1)\tau(P_2). \tag{5}$$

Furthermore, the function $\tau(\cdot)$ also has the following property.

Lemma 3.3 (see [17]) *Let* $y = [y_1, y_2, \cdots, y_n]^{\mathrm{T}} \in R^n$ *be an arbitrary vector and* P *be a stochastic matrix. If* $z = Py$, $z = [z_1, z_2, \cdots, z_n]^{\mathrm{T}}$, *then*

$$\max_i z_i - \min_i z_i \leq \tau(P) \left(\max_i y_i - \min_i y_i \right). \tag{6}$$

Now, we give the proof of Theorem 3.2 in detail.

Proof of Theorem 3.2 We need to point out a feature of the weight matrix with the form of $A(t) = (1 - \eta(t))I + \eta(t)A$. Although $A(t)$ is time-varying, it has a fixed left eigenvector corresponding to the unit eigenvalue. In fact, since fixed weight matrix A corresponds to a strongly connected network, it must has a unique nonnegative left eigenvector $v^{\mathrm{T}} = [v_1, v_2, \cdots, v_n]$ satisfying $v^{\mathrm{T}} A = v^{\mathrm{T}}$[18]. Meanwhile, the vector v^{T} is also a left eigenvector of the time-varying matrix $A(t)$, since $v^{\mathrm{T}} A(t) = v^{\mathrm{T}}(1 - \eta(t))I + v^{\mathrm{T}}\eta(t)A = v^{\mathrm{T}}$. Similar to the proof in [14], using submartingale convergence theorem, we can prove that $\sum_{i=1}^{n} v_i \mu_{i,t}(\theta^*)$ and $\sum_{i=1}^{n} v_i \log \mu_{i,t}(\theta^*)$ converge almost surely as $t \to \infty$. Furthermore, in the same way as the proof of Proposition 1 in [14], we can deduce that $E^*[\frac{\ell_i(s_{t+1}^i|\theta^*)}{m_{i,t}(s_{t+1}^i)}|F_t] \to 1$ almost surely for all i, where $E^*[\cdot]$ denotes the expectation operator associated with true state θ^*, and F_t denotes the past history of signals up to the time period t.

Furthermore, we have

$$E^*\left[\frac{\ell_i(s^i_{t+1}|\theta^*)}{m_{i,t}(s^i_{t+1})}|F_t\right]-1=\sum_{s\in P^i_{t+1}}\ell_i(s|\theta^*)\frac{\ell_i(s|\theta^*)}{m_{i,t}(s)}-1$$
$$=\sum_{s\in P^i_{t+1}}\ell_i(s|\theta^*)\frac{\ell_i(s|\theta^*)}{m_{i,t}(s)}-\sum_{s\in P^i_{t+1}}2\ell_i(s|\theta^*)+\sum_{s\in P^i_{t+1}}m_{i,t}(s)$$
$$=\sum_{s\in P^i_{t+1}}\frac{[\ell_i(s|\theta^*)-m_{i,t}(s)]^2}{m_{i,t}(s)}\to 0,$$

where P^i_{t+1} is the space of all possible options of s^i_{t+1}. We can conclude that

$$m_{i,t}(s)\to\ell_i(s|\theta^*)\quad\text{almost surely,}$$

which means the forecast of all agents are eventually correct.

For the second statement, we first prove it in the set $\overline{\Theta}=\overline{\Theta}_1\cap\overline{\Theta}_2\cap\cdots\cap\overline{\Theta}_n$, i.e., the set of states that observationally equivalent to the true state from the point of view of all agents, then its complement set.

We can rewrite the updating rule (2) as

$$\mu_{i,t+1}(\theta)=\sum_{j=1}^{n}a_{ij}(t)\mu_{j,t}(\theta)+\omega_{i,t}(\theta),\tag{7}$$

where

$$\omega_{i,t}(\theta)=a_{ii}(t)\mu_{i,t}(\theta)\left[\frac{\ell_i(s^i_{t+1}|\theta)}{m_{i,t}(s^i_{t+1})}-1\right].$$

For any state $\theta\in\overline{\Theta}$, the first statement guarantees that $m_{i,t}(s^i_{t+1})\to\ell_i(s^i_{t+1}|\theta^*)=\ell_i(s^i_{t+1}|\theta)$ almost surely, which means, on almost all sample paths,

$$\omega_{i,t}(\theta)\to 0,\quad\text{as}\quad t\to\infty\quad\text{for any agent } i.$$

Equation (7) can be further written in matrix form as

$$\mu_{t+1}(\theta)=A(t)\mu_t(\theta)+\omega_t(\theta),\tag{8}$$

where $\omega_t(\theta)=[\omega_{1,t}(\theta),\omega_{2,t}(\theta),\cdots,\omega_{n,t}(\theta)]^{\mathrm{T}}$.

In our subsequent development, we use transition matrices to capture the evolution of beliefs over a period of time. In particular, we introduce the transition matrices $\Phi(t,s)$ for any t and s with $t\ge s$, as follows:

$$\Phi(t,s)=A(t-1)A(t-2)\cdots A(s),\quad \Phi(s,s)=I.$$

Using the transition matrices and the evolution of (8), the relation between $\mu_{t+1}(\theta)$ and $\mu_0(\theta)$ is given by

$$\mu_{t+1}(\theta)=\Phi(t+1,0)\mu_0(\theta)+\sum_{k=0}^{t}\Phi(t+1,k+1)\omega_k(\theta).$$

For a non-negative vector $z = [z_1, z_2, \cdots, z_n]^{\mathrm{T}}$, let $\Delta z = \max_i z_i - \min_i z_i$. By Lemma 3.3, we have

$$\Delta\mu_{t+1}(\theta) \leq \tau(\Phi(t+1,0))\Delta\mu_0(\theta) + \sum_{k=0}^{t} \tau(\Phi(t+1,k+1))\Delta\omega_k(\theta). \tag{9}$$

Since the social network is strongly connected with self-loops, $A(t)$ is an SIA (stochastic-indecomposable-aperiodic) matrix. As shown in [18], $\Phi(t+L,t) = \prod_{k=t}^{t+L-1} A(k)$ is a scrambling matrix, where $L = (n-1)/2$. Therefore, we have $\tau(\Phi(t+L,t)) < 1$. More specifically, based on Assumption 3.1, it is easy to see that any non-zero element of $A(t)$ has a unanimous lower bound α, which is the minimum element of A times σ, the lower bound of $\eta(t)$. Hence, for all $t \geq 0$,

$$\tau(\Phi(t+L,t)) \leq 1 - \alpha^L \triangleq \beta < 1.$$

For any $kL < h \leq (k+1)L$, $k \geq 0$, by Equation (5), we have

$$\begin{aligned}\tau(\Phi(t+h,t)) &\leq \tau(\Phi(t+h,t+kL))\tau(\Phi(t+kL,t+(k-1)L))\cdots\tau(\Phi(t+L,t))\\ &\leq \beta^k \leq \beta^{h/L-1} = \beta^{-1}(\beta^{1/L})^h.\end{aligned}$$

Let $C = \beta^{-1}$ and $\gamma = \beta^{1/L}$. Then $0 < \gamma < 1$ and $\tau(\Phi(t+h,t)) \leq C\gamma^h$, where C and γ are independent of t and h. From (9), we have

$$\Delta\mu_{t+1}(\theta) \leq C\gamma^{t+1}\Delta\mu_0(\theta) + \sum_{k=0}^{t} C\gamma^{t-k}\Delta\omega_k(\theta). \tag{10}$$

Obviously, $\lim_{t\to\infty} C\gamma^{t-k} = 0$ for any k. Since $\sum_{k=0}^{t} C\gamma^{t-k} < \frac{C}{1-\gamma}$ and $\Delta\omega_k(\theta) \to 0$ with probability 1 as $k \to \infty$, by Toeplitz Lemma, we have

$$\lim_{t\to\infty} \sum_{k=0}^{t} C\gamma^{t-k}\Delta\omega_k(\theta) = 0.$$

Thus, from (10) we have that

$$\lim_{t\to\infty} \Delta\mu_{t+1}(\theta) = 0.$$

which means that $\mu_{i,t}(\theta) - \mu_{j,t}(\theta) \to 0$ with probability 1 for all $i, j \in V$ and all $\theta \in \overline{\Theta}_1 \cap \overline{\Theta}_2 \cap \cdots \cap \overline{\Theta}_n$.

In order to complete the proof of this part, we need to show the existence of $\lim_{t\to\infty} \mu_{i,t}(\theta)$. Since $\mu_{i,t}(\theta) - \mu_{j,t}(\theta) \to 0$ as $t \to 0$, for any given $\varepsilon > 0$, there exists a large enough T such that for any $t \geq T$, we have $-\varepsilon < \mu_{i,t}(\theta) - \mu_{j,t}(\theta) < \varepsilon$ uniformly for all i and j. Thus, for any finite positive integer p,

$$\mu_{j,t}(\theta) - \varepsilon < \sum_{i=1}^{n} [\Phi(t+p,t)]_{ji}\mu_{i,t}(\theta) < \mu_{j,t}(\theta) + \varepsilon.$$

Note that the term $\sum_{i=1}^{n}[\Phi(t+p,t)]_{ji}\mu_{i,t}(\theta)$ can be make arbitrarily close to $\mu_{j,t+p}(\theta)$ for large enough t, implying that $-\varepsilon < \mu_{j,t+p}(\theta) - \mu_{j,t}(\theta) < \varepsilon$. Therefore, $\{\mu_{j,t}(\theta)\}_{t=1}^{\infty}$ is a Cauchy sequence for all j and hence, converges.

For the states in the complement set of $\overline{\Theta} = \overline{\Theta}_1 \cap \overline{\Theta}_2 \cap \cdots \cap \overline{\Theta}_n$, i.e., $\theta \notin \overline{\Theta}_i$ for some agent i, using the same analysis applied in Proposition 3 in [14], we know that $\mu_{i,t}(\theta) \to 0$ almost surely as $t \to 0$. Since

$$\mu_{i,t+1}(\theta) = a_{ii}(t)\mu_{i,t}(\theta)\frac{\ell_i(s_{t+1}^i)}{m_{i,t}(s_{t+1}^i)} + \sum_{j\in N_i} a_{ij}(t)\mu_{j,t}(\theta),$$

we know that $\mu_{i,t}(\theta) \to 0$ means $\sum_{j\in N_i} a_{ij}\mu_{j,t}(\theta) \to 0$, which implies that $\mu_{j,t}(\theta) \to 0$ almost surely for all $j \in N_i$. Since the network is strongly connected, by successive inference, all agents assign an asymptotic belief of zero to $\theta \notin \overline{\Theta}_i$. The same argument can be extended to other states not in the set $\overline{\Theta}$.

For the third statement, the condition (e) guarantees that there is only one state belonging to $\overline{\Theta}_1 \cap \overline{\Theta}_2 \cap \cdots \cap \overline{\Theta}_n$, i.e., the underlying true state θ^*. By the second statement we know that for any $i \in V$, $\mu_{i,t}(\theta) \to 0$ for any $\theta \neq \theta^*$ and $\mu_{i,t}(\theta^*) \to 1$, which means asymptotic learning is achieved. This completes the proof. ▮

Remark 3.4 Personal learning through Bayes' rule is extensively studied by Savage in the foundations of statistics[16]. A learning model with only the first term on the right hand side of (1) is studied there. It has been shown that, barring two banal exceptions, an agent becomes almost certain of the truth when the amount of her observation increases infinitely. One exception is that the initial belief of the true state is zero. This is very easy to understand. If the belief is zero, then, no matter what signal is observed, the posterior belief of the true state is still zero. The other exception occurs when there exists a state which arouses exactly the same signals as the true state does, i.e., observationally equivalent state exists. When an agent is situated in social networks, she might receive more information to learn the truth. As Theorem 3.2 states, the conditions for learning in social networks can be relaxed as: At least one agent has a nonzero initial belief assigned to the true state, and the intersection of all agents' observationally equivalent states is only the true state. We may view these relaxations as the advantage of social interactions.

Remark 3.5 A typical non-Bayesian learning model has the same form as the second term on the right hand side of the belief updating rule (1), which captures the relatively naive social learning approaches, such as imitation and replication. It is apparent that learning simply in naive ways cannot help agents discover the underlying true state, which is a main problem facing traditional non-Bayesian models. In our model, an "attraction of truth" is introduced to address the problem. Agents have access to signals aroused by the underlying true state, and through the Bayesian inference, observed signals turn into the "attraction of truth" that moves the beliefs in the correct direction.

4 Simulations

4.1 Verification of the Model

To intuitively verify that agents situated in networks with time-varying weights can learn the true state using the belief updating rule (1), we perform simulations in a group of 100 agents. Suppose that the set of states is $\Theta = \{\theta_1, \theta_2, \theta_3\}$, in which the true state $\theta^* = \theta_3$. Initial beliefs of any agent i, i.e., $\{\mu_{i,0}(\theta_1), \mu_{i,0}(\theta_2), \mu_{i,0}(\theta_3)\}$ are adopted randomly in interval $[0, 1]$, satisfying $\sum_{k=1}^{3} \mu_{i,0}(\theta_k) = 1$. We set the set of signals to be $\{H, T\}$, where signal H appears with possibility of 0.8 and T with 0.2. We classify the agents into two groups: The first group view states θ_1 and θ_3 as observationally equivalent states, meanwhile $\ell_i(H|\theta_2) = 0.5$ for all i in the first group; the second group view θ_2 and θ_3 as the observationally equivalent states, meanwhile $\ell_i(H|\theta_1) = 0.5$. Therefore, the intersection of all agents' observationally equivalent states is only the true state θ_3. To realize a strongly connected network with time-varying weights, we randomly choose $\eta(t)$ in the interval $(0, 1]$ at each time period such that $A(t) = (1 - \eta(t))I + \eta(t)A$ is time-varying, where the fixed weight matrix A is generated corresponding to a strongly connected graph.

The result is shown in Figure 1, where the upper, middle and lower sub-figure shows the evolution of beliefs on θ_1, θ_2, and θ_3, respectively. Each thin line in the sub-figure denotes an agent's belief. The figure is plotted on a semilog scale to illustrate both the dynamic aggregation in the incipient stage and the final convergence in the long run. Figure 1 shows that social interaction is dominant at the beginning, when the cumulation of outside signals is relatively scarce. The communication among local neighbors first leads the beliefs of the whole group towards the center of the initial beliefs, then move the beliefs towards the truth as the observed information increases. Since we set θ_3 to be the true state, the fact that beliefs on state θ_1 and θ_2 converging to 0 and beliefs on the true state θ_3 converging to 1 means that the asymptotic learning has been achieved, and the true state θ_3 is discovered.

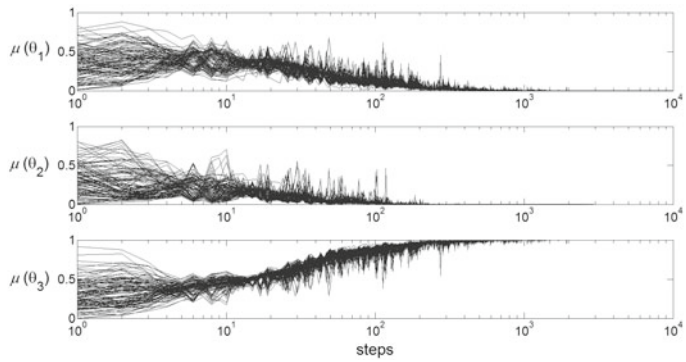

Figure 1 The belief evolution with time-varying weights

Remark 4.1 Since the traditional Bayesian social learning models are generally analyzed in the framework of game theory, where simulations are rarely used to verify the results, we only compare our model with traditional non-Bayesian models in this secion. Strictly speaking, our model also belongs to non-Bayesian models, since the agents in our model are not fully rational, and they do not perform the standard Bayes' rule to update beliefs. However, our model is essentially different from traditional non-Bayesian models, where agents update beliefs in a totally naive way. The feature that distinguishes our model from others is illustrated in Figure 1. Our model can be viewed as two parts: Learning through maintaining consensus with other agents and learning through Bayesian inference based on observed signals. At the beginning, the beliefs of all agents converge to the center of initial beliefs, showing the effect of social interaction, i.e., the second term in (1). The trend of consensus can also be seen in most non-Bayesian models under certain conditions. However, agents in our model do not only rely on information possessed by neighboring agents and are satisfied with the agreement on beliefs. Instead, they attempt to learn the correct belief and discover the underlying true state by continuously observing the outside signals.

4.2 Further Conjecture About $\eta(t)$

Setting a lower bound for non-zero weights is to maintain the minimum amount of communication for social interaction, which helps the whole group learn the true state collectively. As we have mentioned in Remark 3.4, the advantage of social learning compared with personal learning is that agents can learn the true state even in the situation with two obstacles: Belief of zero assigned to the true state and existence of observationally equivalent states. Once all agents obtain the correct belief, i.e., the true state is assigned the belief of one and other states are eliminated by assigned a belief of zero, both obstacles that prevent single agent from successful learning disappear. Even though observationally equivalent states still exist, it would not get any belief since the belief on it is zero, i.e., the possibility that the relevant state is the true state has already been eliminated. It would be clear that once the social learning is achieved, there is no need to maintain communication any more, which means the assumption of lower bound for time-varying weights is redundant, at least at the later stage of the social learning process.

The next simulation suggests that the assumption of lower bound is indeed a little too strict. Figure 2 shows that even if $\eta(t)$ is set to be $1/t$, which has no lower bound as t goes to infinity, the asymptotic learning can also be achieved, though the learning time is much longer than the previous simulation. Compared with Figure 1, it has three features: Fast concentrating at the initial stage, tending to be separated at the middle stage, and slowly converging to the correct belief. Since $\eta(t) = 1/t$ and $A(t) = (1-\eta(t))I + \eta(t)A$, agents have very low self-reliances at the beginning, and meanwhile, the levels of dependance on other agents are very high, which make the beliefs of the whole group concentrate fast through social interaction. Then the weights assigned to other agents decrease as time goes on, resulting in that the agents become more dependent on their own. The beliefs of the whole group are separated to certain extent, which, however, emerge into one cluster after long term running because of the social interaction.

Could $\eta(t)$ decrease without any constraint? Our simulation in Figure 3 shows that the answer is No. Comparing Figures 3 and 2, we may understand the importance of sufficient communication. It is shown in Figure 3 that, if we choose $\eta(t) = 1/t^2$, the group still concentrate at first then are separated. However, as weights on other agents decrease too fast, all agents become stubborn with their own beliefs, and there is not sufficient information exchange among the group, which makes social learning failed. We can see from Figure 3 that all agents almost do not change their beliefs after about 100 steps of learning and stick to their beliefs from then on. In fact, from the lowest sub-figure in Figure 3, we can find that no agent obtains the correct belief.

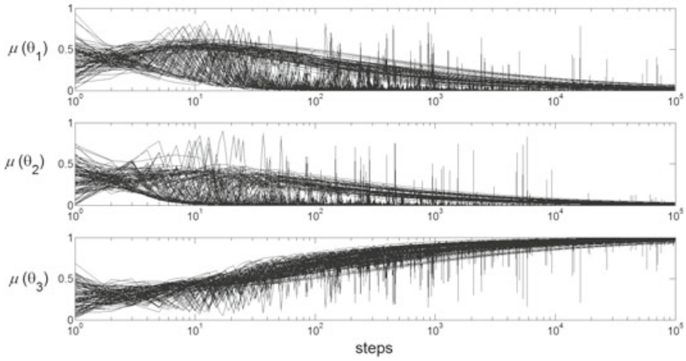

Figure 2 The belief evolution with $\eta(t) = 1/t$

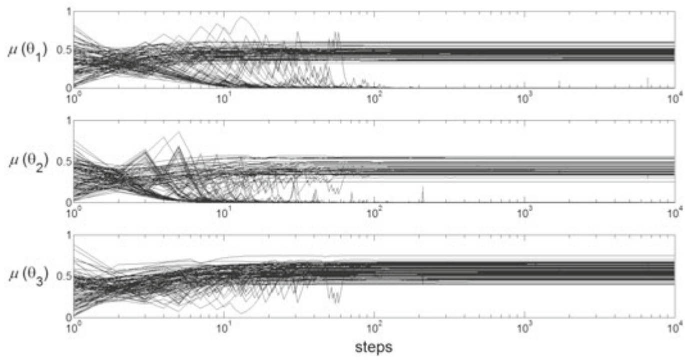

Figure 3 The belief evolution with $\eta(t) = 1/t^2$

To summarize, the simulations in this subsection suggest that $\eta(t)$ is not necessary to have a lower bound. Instead, it should have a constraint on decreasing rate to guarantee sufficient amount of information exchange among agents for asymptotic learning.

5 Conclusions

We have analysed a social learning model with time-varying weights. By introducing a variable $\eta(t)$ to adjust the weights, our model can represent more realistic situations. We have proved that asymptotic learning can be achieved in social networks with time-varying weights under similar conditions as that in the fixed-weighted scenario. We set a lower bound for the variable $\eta(t)$ for simplicity of theoretical analysis, but simulations show that this assumption is a little too strict. Based on simulations we make a further conjecture about the choice of $\eta(t)$. It remains a challenge to theoretically analyse the choice of the variable, and furthermore, to investigate the convergence of the belief updating rule in more general time-varying networks, which are important areas for our further works.

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