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Quantized consensus over directed networks with switching topologies

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A B S T R A C T

This paper studies the quantized consensus problem for a group of agents over directed networks with switching topologies. We propose an effective distributed protocol with an adaptive finite-level uniform quantized strategy, under which consensus among agents is guaranteed with weaker communication conditions. In particular, we analytically prove that each agent sending 5-level quantized information to each of its neighbors, together with 3-level quantized information to itself at each time step, which suffices for attaining consensus with an exponential convergence rate as long as the duration of all link failures in the directed network is bounded. By dropping the typical common left eigenvector requirement for the existence of common quadratic Lyapunov function, we conduct the convergence analysis based on the notion of input-to-output stability. The proposed quantized protocol has favorable merits of requiring little communication overhead and increasing robustness to link unreliability, and it fits well into the digital network framework.

1. Introduction

Recently, the problem of distributed consensus in networked multi-agent systems has received significant attention due to its important applications. Roughly speaking, the purpose of the consensus problem is to design a distributed protocol in the presence of limited information communication and dynamically switching network topologies, such that a group of agents achieve some agreement between the states. If the final consensus value is the weighted linear combination of the initial states of all agents, then the weighted average consensus problem is solved. In particular, if the final agreed value is the exact average of the initial values of all agents, then average consensus is achieved, which is of particular interest in the applications of load balancing [1,2] and task assignment [3].

Early efforts on related distributed consensus problems focused primarily on the assumption that communication channels of networks have unlimited capacity. However, this assumption may not be true in practice, because digital channels are subject to bandwidth constraints and only finite number of bits of information can be transmitted along each channel. As such, information transmitted among agents has to be quantized prior to being sent. As a result, quantized consensus or consensus with quantized communication has attracted wide interest over the past few years [2,4–19]. Applications of quantized consensus can be found in [3,20], where a framework denoted as discrete consensus was proposed, in which quantized average consensus algorithms were performed via network gossiping. Serving as a generalization of quantized consensus, this framework is particularly suitable for applications in load balancing and task assignment, and thus sheds light on applications over directed networks.

While in the existing works about quantized consensus, it is commonly assumed that all weighted adjacency matrices (or Laplace matrices) associated to the directed networks having a common left eigenvector. With this basic assumption, state average or weighted average invariance of the networks is preserved, and thus the final consensus value can be specified [21,22]. More technically, the squared norm or weighted squared norm of the disagreement vector can be chosen as the common quadratic Lyapunov function to carry out the consensus convergence analysis [21,22]. For undirected or directed networks with fixed topologies, the above assumption is easily satisfied. While for general directed networks with dynamically switching topologies, the left eigenvectors are also time-dependent, except the case where the directed switching networks are always balanced (*i.e.* networks in which the in-degree and out-degree of each node are the same), or equivalently, the corresponding weighted adjacency matrices are double stochastic [21], otherwise the final achieved consensus value is time-varying and there does not exist a common quadratic

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Lyapunov function [23]. However, it is unrealistic to assume that the switching topologies remain balanced at each time step, particularly when networks operate under adverse environment [24]. Therefore, the distributed iterative algorithms proposed in [25] to construct doubly stochastic matrices for directed networks are not applicable for general directed networks with switching topologies. In order to enable wide range of applications, it is of great importance both theoretically and practically to study distributed quantized consensus protocols for general directed networks with switching topologies.

Recently, the authors of [13] proposed a novel distributed quantized protocol that demonstrates average consensus for jointed connected undirected switching networks when each pair of agent reciprocally exchanges 3-bit of quantized information at each time step. Furthermore, under the assumption that periodically connected directed switching networks are balanced, in [19] and its conference precursor [18], the authors established that average consensus can be exponential achieved by designing suitable uniform quantizers with quantization levels bigger than a certain finite constant. The primary objective of this paper is to extend the results derived in the well-done papers [13,18,19] to directed networks with switching topologies that are not necessarily balanced. However, the non-trivial challenging here is that the conservation property of state (weighted) average invariance no longer holds in this general scenario, which makes it difficult to characterize the final consensus value analytically and thus causes the analysis of consensus quite intricate. In a similar spirit of [13], we propose a quantized consensus protocol utilizing the adaptive finite-level dynamic uniform quantization strategy. Moreover, motivated by the recent input-to-output stability result [26], we show that agents reach consensus asymptotically under the proposed quantized protocol, and we also relate the convergence rate to the scaling function of the uniform quantizers. Particularly, we establish that as long as the duration of any link failure in the directed network is bounded, then by properly choosing the parameters of each quantizer such that at each time step, each agent (may be nonreciprocally) sends 3-bit quantized information to each of its intimate neighbors, together with 1-bit quantized information to itself, which causes agents to reach consensus exponentially.

As a comparison with [13], where *average consensus* is achievable by a symmetry error-compensation mechanism which heavily relies on bidirectional and symmetric information exchange [27]. Thus, certain overhead like message delivery acknowledgment and retransmission is required by the quantized average protocol. Compared to [18,19], where *average consensus* is still achievable for periodically connected directed switching networks that are balanced at each time step. However, not all directed networks admit balanced communication links between agents [24,28,29]. Additionally, the consensus convergence analyses of [13,19] are mainly conducted with the use of common quadratic Lyapunov function. Whereas in this paper, by removing the need for the symmetry error-compensation mechanism and the balancedness of network topologies, we show that *consensus* can be achieved under unidirectional and unbalanced connectivity conditions on communication topologies. In particular, the convergence to consensus for closed-loop multi-agent system in this paper is guaranteed by a recently developed input-to-output stability theorem [26].

In related work [29], under the assumption that the states of agents are integers and directed unbalanced networks are randomized, the authors designed distributed asynchronous gossip-based algorithms and derived connectivity conditions on directed unbalanced graphs that ensure (average) consensus. While in this paper, under the assumption that directed unbalanced networks are periodically strongly connected, we aim to explicitly design a suitable synchronous protocol based on adaptive gains that modify the quantization level of exchanged information to achieve consensus. The proposed protocol in this paper is motivated by applications such as those arising in mobile sensor networks, where the digital communication channels are of limited data rate and the communication links between agents are time-varying as agents enter or leave line-of-sight. Moreover, as agents usually have diverse transmission capabilities, the communication links between agents may be unidirectional and thus result in the unbalanced communication topology.

The remainder of this paper is organized as follows. In Section 2, we formally state the problem to be studied and give some assumptions. In Section 3, we establish the adaptive finite-level uniform quantization scheme and present the quantized protocol, then followed by one useful lemma. In Section 4, we show how to explicitly design the finite-level uniform quantizers that guaranteeing the convergence to consensus and establish the convergence results. Simulation results are given in Section 5. Section 6 is the conclusions.

We will use standard notation throughout the paper. For a given real number x, the minimum integer greater than or equal to x is denoted by $\lceil x \rceil$. The number of the elements in a given set Ξ is denoted by $|\Xi|$. A square matrix $W(k) = (w_{ij}(k))$ is said to be nonnegative if its entry $w_{ij}(k) \geq 0$ for all i and j. A square matrix $W(k) = (w_{ij}(k))$ is said to be stochastic, if $W(k)$ is nonnegative and the summation of each row of $W(k)$ is 1. A square matrix $W(k) = (w_{ij}(k))$ is said to be doubly stochastic, if $W(k)$ and $W(k)^{\mathrm{T}}$ are all stochastic, where $W(k)^{\mathrm{T}}$ is the transpose matrix of $W(k)$.

2. Problem formulation

We consider a multi-agent system with agent set $\mho = \{1, 2, \dots, N\} (N \geq 2)$, for which the dynamics of each agent is the discrete-time first-order integrator:

$$x_i(k+1) = x_i(k) + u_i(k), \quad k = 0, 1, 2, \dots, \; i \in \mho, \tag{1}$$

where $x_i(k) \in R$ represents the state of agent i and $u_i(k) \in R$ is the control input or the protocol of agent i.

The communication topology of the network of agents at time k is represented using a digraph (or directed graph) $\mathbb{G}(k) = (\mho, \mathcal{E}(k))$, where $\mathcal{E}(k) = \{(i, j) : i, j \in \mho\}$ denotes the edge set. With a slight abuse of terminology, we will use the terms *edge* and *channel* interchangeably. A directed edge $(i, j) \in \mathcal{E}(k)$ denotes agent j can obtain information from agent i at time k, and thus agent i is an in-neighbor of agent j, but not necessarily vice versa. If $(i, j) \in \mathcal{E}(k)$ implies $(j, i) \in \mathcal{E}(k)$ for $\forall i, j \in \mho$ and $i \neq j$, the network is called undirected. Otherwise, it is directed. The in-neighbor set of agent i at time k is denoted as $\mathcal{N}_i^+(k) = \{j \in \mho, (j, i) \in \mathcal{E}(k), i \neq j\}$, and similarly, the out-neighbor set of agent i at time k is denoted as $\mathcal{N}_i^-(k) = \{j \in \mho, (i, j) \in \mathcal{E}(k), i \neq j\}$. An undirected network just a special case of a directed network and it satisfies $\mathcal{N}_i^+(k) = \mathcal{N}_i^-(k) = \mathcal{N}_i(k)$. $W(k) = (w_{ij}(k)) \in R^{N \times N}$ is the stochastic adjacency matrix of the digraph $\mathbb{G}(k)$, with the edge weight $w_{ij}(k)$ encoding the relative confidence of each agent's information or relative reliability of different communications, and $w_{ij}(k) > 0 \iff (j, i) \in \mathcal{E}(k)$, $w_{ij}(k) = 0$ otherwise. A digraph $\mathbb{G}(k)$ is said to be balanced if $\sum_{j \in \mathcal{N}_i^+(k)} w_{ij}(k) = \sum_{j \in \mathcal{N}_i^-(k)} w_{ji}(k)$ for all i.

For the adjacency matrix $W(k)$ and the digraph $\mathbb{G}(k)$, we assume the following as [15,26,28,30].

Assumption 1 (*Weight Rule*)**.** For $W(k)$, there exists a constant $\rho > 0$, such that $w_{ii}(k) = 1 - \sum_{j \in \mathcal{N}_i^+(k)} w_{ij}(k) \geq \rho$ uniformly for all $k \geq 0$ and $i \in \mho$, and $w_{ij}(k) (i \neq j)$ satisfy $w_{ij}(k) \in \{0\} \bigcup [\rho, 1]$, $\forall k \geq 0$.

Assumption 2 (*Periodical Strong Connectivity*)**.** There exists a positive integer $B \geq 1$ such that for all time steps $k \geq 0$, the directed union graph $(\mho, \mathcal{E}(k) \cup \mathcal{E}(k+1) \cup \cdots \cup \mathcal{E}(k+B-1))$ is strongly connected.

Assumption 1 drops the double-stochasticity assumption on the adjacency matrix $W(k)$. While Assumption 2 implies the instantaneous digraph $\mathbb{G}(k)$ need not be connected at any time k, but for any k and any two agents $u, v \in \mho$, there exists a directed path connecting agent u and agent v with edges (j, i) in the set $(\mho, \mathcal{E}(k) \cup \mathcal{E}(k+1) \cup \cdots \cup \mathcal{E}(k+B-1))$. As such, proper information exchanging among agents is ensured.

The aim of this paper is to design a distributed protocol $u_i(k)$ with dynamic encoding-decoding uniform quantized scheme using limited quantized information, such that the N agents in (1) achieves consensus asymptotically in the sense of:

$$\lim_{k\to\infty} |x_j(k) - x_i(k)| = 0 \tag{2}$$

for $\forall i, j \in \mho$ and any initial values $x_i(0)$.

Remark 1. In existing works of quantized consensus [2,4–19], where the states of agents are required to agree on the exact average or weighted average of their initial values, here only the state differences between different agents are required to tend to zero, no matter what the final consensus value is. Moreover, this paper extends the works of [24,28,30] from the case of distributed consensus with perfect information communication to the case of distributed consensus with limited quantized information communication.

3. Protocol design using limited quantized information

In a digital communication network, only symbolic data can be communicated between agents through digital channels. For any $i \in \mho$, if the edge weight $w_{ij}(k) > 0$, *i.e.*, $j \in \mathcal{N}_i^+(k) \cup \{i\}$, or the directed digital channel (j, i) is activated at time k and agent j sent its information to agent i, then modified from [13], the dynamic encoder $\phi_{ji} : R \to \Gamma$ associated with j for the directed channel (j, i) is defined as

$$\begin{cases} \xi_{ji}(0) = 0, \\ \xi_{ji}(k) = \xi_{ji}(k-1) + g(k-1)I_{ji}(k)\Delta_{ji}(k), \\ \Delta_{ji}(k) = q_k^{ji}\left(\dfrac{x_j(k) - \xi_{ji}(k-1)}{g(k-1)}\right), \quad k = 1, 2, \ldots, \end{cases} \tag{3}$$

where $x_j(k)$ is the real-valued state of agent j, $\xi_{ji}(k)$ is the internal state of encoder ϕ_{ji} at time k; the symbolic data $\Delta_{ji}(k)$ is the output of the encoder ϕ_{ji} and will be sent to agent i along the directed channel (j, i). $q_k^{ji}(\cdot)$ is a finite-level uniform quantizer associated with the directed channel (j, i) at time k, in which $g(k)$ is a scaling function. By appropriately choosing the scaling function, the saturation of the finite-level quantizers can be avoided. $I_{ji}(k)$ is the index function for the directed channel (j, i) and is defined as

$$I_{ji}(k) = \begin{cases} 1, & \text{if } w_{ij}(k) > 0, \\ 0, & \text{if } w_{ij}(k) = 0. \end{cases} \tag{4}$$

The dynamic decoder $\varphi_{ji} : \Gamma \to R$ associated with agent i for the directed channel (j, i) is defined as

$$\begin{cases} \hat{x}_{ji}(0) = 0, \\ \hat{x}_{ji}(k) = \hat{x}_{ji}(k-1) + g(k-1)I_{ji}(k)\Delta_{ji}(k), \end{cases} \tag{5}$$

where $\hat{x}_{ji}(k)$ is the output of the decoder φ_{ji} at time k, which is the estimated state that agent i obtains for its neighbor j's state $x_j(k)$. In (3) and (5), $I_{ji}(k)$ acts as the updating gain of the encoder ϕ_{ji} associated with j and the decoder φ_{ji} associated with agent i for the directed channel (j, i), respectively.

The quantizer q_k^{ji} associated with the directed channel (j, i) is a uniform symmetric quantizer with finite levels, which takes the form [8,11,13]

$$q(m) = \begin{cases} 0, & \text{if } -\dfrac{1}{2} < m < \dfrac{1}{2}, \\ i, & \text{if } \dfrac{2i-1}{2} \leq m < \dfrac{2i+1}{2}, \\ K, & \text{if } m \geq \dfrac{2K-1}{2}, \\ -q(-m), & \text{if } m < 0, \end{cases} \tag{6}$$

and maps any $m \in R$ into the discrete set of quantized levels $\Gamma = \{0, \pm 1, \pm 2, \ldots, \pm K\}$. Thus, the needed channel quantization levels (data rate) is $2K + 1$ bits.

Since the network topology is dynamically switching, here we adopt the adaptive finite-level dynamic uniform quantization strategy proposed in [13], which implies that the encoder and the decoder can be adaptively tuned according to whether the associated directed channel is active or not at the last step. Thus, the number of quantization levels $2K_{ji}(k) + 1$ of the quantizer q_k^{ji} associated with the directed channel $(j, i)(j \in \mathcal{N}_i^+(k))$ also changes with time.

Remark 2. It follows from (3) and (5) that $\hat{x}_{ji}(k) = \xi_{ji}(k)$ hold for all $i \in \mho, j \in \mathcal{N}_i^+(k) \cup \{i\}, k = 0, 1, 2, \ldots$.

When limited quantized messages pass in the directed switching network, we propose the following protocol

$$u_i(k) = \alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)(\hat{x}_{ji}(k) - \xi_{ii}(k)),$$
$$k = 0, 1, 2, \ldots, \quad i = 1, \ldots, N, \tag{7}$$

where $w_{ij}(k)$ are the entries of the stochastic adjacency matrix $W(k)$, and $\alpha \in (0, 1]$ is a known constant control gain. As the case of directed networks with fixed topologies [17] and it will be seen later, reducing the value of α leads to smaller number of levels quantized information that need to be transmitted along each digital channel.

By the fact that $\hat{x}_{ii}(k) = \xi_{ii}(k)$, the protocol (7) can be further written as

$$\begin{aligned} u_i(k) = {} & \alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_j(k) - x_i(k)] \\ & - \alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_j(k) - \hat{x}_{ji}(k)] \\ & + \alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_i(k) - \hat{x}_{ii}(k)]. \end{aligned} \tag{8}$$

Therefore, the protocol (7) consists of three terms. The first term, $\alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_j(k) - x_i(k)]$, which is the control input in the absence of quantization communication and plays the main role. The second term, $-\alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_j(k) - \hat{x}_{ji}(k)]$ is the estimated errors for estimating the states $x_j(k)(j \in \mathcal{N}_i^+(k))$ by agent i. The third term, $\alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_i(k) - \hat{x}_{ii}(k)] = \alpha(1 - w_{ii}(k))[x_i(k) - \hat{x}_{ii}(k)]$ represents the estimated error of the state $x_i(k)$ by agent i through its self-loop.

Remark 3. For the case of undirected networks with switching topologies, wherein each agent has no self-loop, the following Laplace-based distributed quantized protocol was proposed in [13]:

$$u_i(k) = h \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)(\hat{x}_{ji}(k) - \xi_{ij}(k))$$
$$k = 0, 1, 2, \ldots, \quad i = 1, \ldots, N, \tag{9}$$

where h is the control gain to be designed, and $a_{ij}(k)$ is the entry on the ith row and jth of the Laplace matrix $L(k)$ associated to the undirected graph $\mathbb{G}(k)$, which denotes the edge weight of the undirected channel (j, i).

By the fact that $\hat{x}_{ji}(k) = \xi_{ji}(k)$, the protocol (9) can be written as

$$\begin{aligned} u_i(k) = & h \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)[x_j(k) - x_i(k)] \\ & - h \sum_{j\in\mathcal{N}_i(k)} a_{ij}(k)[x_j(k) - \hat{x}_{ji}(k)] \\ & + h \sum_{j\in\mathcal{N}_i(k)} a_{ji}(k)[x_i(k) - \hat{x}_{ij}(k)]. \end{aligned} \tag{10}$$

The protocol (10) also consists of three terms. The first and second terms play the same role as their counterparts in protocol (8). The third term, $h\sum_{j\in\mathcal{N}_i(k)} a_{ji}(k)[x_i(k) - \hat{x}_{ij}(k)]$ represents the estimated errors for the state $x_i(k)$ by agent i's neighbors. This term is obtained by the basic assumption that $a_{ij}(k) = a_{ji}(k)$, which is called an error-compensation term and plays an important role in the protocol [11,13]. Because of symmetry of the network, the last term implies that the quantized protocol (9) actually utilizes a symmetry error-compensation mechanism which preserves the state average invariance $\frac{1}{N}\sum_{j=1}^{N} x_j(k+1) = \frac{1}{N}\sum_{j=1}^{N} x_j(k)$ and is essential to reach quantized average consensus for undirected networks [7,8,11,14,13]. However, $(i, j) \in \mathcal{E}(k)$ does not imply $(j, i) \in \mathcal{E}(k)$ in directed networks. Thus compared to prior protocols of [7,8,11,14,13], the last term in (8) indicates that the protocol (7) drops this symmetry error-compensation mechanism.

Remark 4. For the case of periodically connected directed switching networks that are balanced, wherein each agent also has no self-loop, the following Laplace-based distributed quantized protocol was proposed in [18,19]:

$$u_i(k) = h\left(\sum_{j\in\mathcal{N}_i^+(k)} a_{ij}(k)\hat{x}_{ji}(k) - \sum_{j\in\mathcal{N}_i^-(k)} a_{ji}(k)\xi_{ji}(k)\right)$$
$$k = 0, 1, 2, \ldots, \quad i = 1, \ldots, N, \tag{11}$$

where h is also the control gain to be designed, and $a_{ij}(k)$ is the edge weight of the directed channel (j, i).

To implement the protocol (11), each agent needs to know the following information: both the in-neighbor and out-neighbor link weights, the output of its $|\mathcal{N}_i^+(k)|$ decoders and the output of its $|\mathcal{N}_i^-(k)|$ encoders. While to implement the protocol (7), each agent just needs to know the following information: the in-neighbor link weights, the output of its $|\mathcal{N}_i^+(k)|$ decoders and the output of its own encoder. Accordingly, both the main assumptions and the nature of the protocol proposed in this paper differ from those in [18,19].

Remark 5. In the fixed network topology case, where the quantizers q_K associated with the different digital channels are assumed identical, the output 0 of the quantizer need not be sent, and thus the quantizer q_K actually costs $\lceil \log_2(2K)\rceil$ bits, although the number of quantization levels of the quantizer q_K is $2K + 1$ [11,17]. To avoid confusion with the case of inactive channels, as [13], we let the output 0 also be sent by each agent to the neighbors explicitly, then if $i \neq j$, the number of bits of quantizer q_k^{ji} associated with the directed channel (j, i) is $\lceil \log_2(2K_{ji}(k) + 1)\rceil$. Note that (5) implies each agent can always obtain its own estimate state $\hat{x}_{ii}(k)$ through its self-loop. Then as the case of the fixed network topology, the output 0 of the quantizer need not be sent through each self-loop, and thus the lowest bound of the number of quantization bits required for each self-loop can be chosen as 1 [11,17].

Denote the estimated error $e_{ji}(k) = x_j(k) - \xi_{ji}(k)$ and the quantized error $\beta_{ji}(k) = \frac{x_j(k+1)-\xi_{ji}(k)}{g(k)} - \Delta_{ji}(k+1)$. Substituting the control input (7) into (1), together with (8) and using the fact that $w_{ii}(k) = 1 - \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k) > 0$, we obtain the closed-loop system of agent i as

$$x_i(k+1) = (1-\alpha)x_i(k) + \alpha\sum_{j=1}^{N} w_{ij}(k)x_j(k) + \Delta r_i(k), \tag{12}$$

where $\Delta r_i(k) = \alpha\sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)(e_{ii}(k) - e_{ji}(k))$.

Note that the directed network topology is dynamically switching and the conservation property of state (weighted) average invariance invalidates. Therefore, significant technical challenges arise for the convergence analysis for the closed-loop system (12).

Before proceeding further, we introduce some notation. Denote $x(k) = [x_1(k)\ \ x_2(k)\ \ \cdots\ \ x_N(k)]^{\mathrm{T}}$, $\Delta r(k) = [\Delta r_1(k)\ \ \Delta r_2(k)\ \ \cdots\ \ \Delta r_N(k)]^{\mathrm{T}}$. We define

$$M(k) = \max_{i\in\mho} x_i(k), \qquad m(k) = \min_{i\in\mho} x_i(k),$$
$$\Delta r_{\max}(k) = \max_{i\in\mho} \Delta r_i(k), \qquad \Delta r_{\min}(k) = \min_{i\in\mho} \Delta r_i(k),$$

and we further let

$$D(k) = M(k) - m(k), \qquad \Delta R(k) = \Delta r_{\max}(k) - \Delta r_{\min}(k).$$

Obviously, $D(k)$ characterizes the progress of the protocol towards consensus [1,28]. Denote $\max_{ij}|e_{ji}(k)| = \max_{i\in\mho, j\in\mathcal{N}_i^+(k)\cup\{i\}}|e_{ji}(k)|$, then $\max_{ij}|e_{ji}(k)| = \max_{ji}|e_{ij}(k)|$. Furthermore, by the definition of $\Delta r_i(k)$ and the stochasticity of $W(k)$, it follows that $\Delta r_{\max}(k) \leq 2\max_{ij}|e_{ji}(k)|$ and $\Delta r_{\min}(k) \geq -2\max_{ij}|e_{ji}(k)|$, and thus

$$\Delta R(k) \leq 4\max_{ij}|e_{ji}(k)|. \tag{13}$$

If one views $\Delta r_i(k)$ satisfying (13) as external disturbances, then under the protocol (7), the closed-loop system (12) just has the same form as the First-Order Dynamic Average Consensus protocol (the FODAC protocol for short) proposed by [26] (see Eq. (2) in [26], where $\alpha = 1$). Furthermore, if Assumptions 1 and 2 hold, all the conditions of Lemma 3.1 and Theorem 3.1 of [26] are satisfied. Thus, we have the following lemma, which is modified from Lemma 3.1 and Theorem 3.1 of [26] and is instrumental to derive our main results in Section 4.

Lemma 1. *Suppose that Assumptions 1 and 2 hold. Let $\eta = (\alpha\rho)^{\frac{1}{2}N(N+1)B-1}$ and $h(NB-1) = T_h$ for the integer $h \geq 1$. Furthermore, for all $k \geq 1$, let s be the largest integer such that $s(NB-1) = T_s \leq k < (s+1)(NB-1)$. Then for all $k \geq 1$,*

$$\begin{aligned} D(k) & \leq (1-\eta)D(T_s) + \sum_{\iota=T_s}^{k-1}\Delta R(\iota) \\ & \leq (1-\eta)^s D(0) + \Omega(k) \\ & = (1-\eta)^{s+1}\frac{1}{1-\eta}D(0) + \Omega(k) \\ & \overset{(a)}{\leq} (1-\eta)^{\frac{k}{NB-1}}\frac{1}{1-\eta}D(0) + \Omega(k) \\ & \leq (1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(0) + \Omega(k), \end{aligned} \tag{14}$$

where the inequality (a) *follows from the relations* $0 < 1-\eta < 1$ *and* $k < (s+1)(NB-1)$. $\Omega(k)$ *is*

$$\begin{aligned}\Omega(k) &= (1-\eta)^{s-1}\sum_{\iota=0}^{T_1-1}\Delta R(\iota)+\cdots \\ &\quad +(1-\eta)\sum_{\iota=T_{s-1}}^{T_s-1}\Delta R(\iota)+\sum_{\iota=T_s}^{k-1}\Delta R(\iota) \\ &\le (NB-2)\sup_{0\le t\le k-1}\Delta R(t)[(1-\eta)^{s-1}+\cdots+(1-\eta)+1] \\ &\le \frac{4(NB-2)}{\eta}\sup_{0\le t\le k-1}\max_{ij}|e_{ij}(t)|, \end{aligned} \tag{15}$$

where the last inequality of (15) *is followed from* (13). □

Remark 6. Lemma 1 establishes the relationship between the consensus error $D(k)$ and the estimated errors $e_{ji}(t)$. Based on the assumption with perfect information communication, [26] studied the problem of dynamic average consensus, where the agents were required to agree on the exact average of their time-varying reference input signals. For more about dynamic average consensus and the difference between dynamic average consensus and static average consensus, please refer to [26].

If the term $\sup_{0\le t\le k-1}\max_{ij}|e_{ji}(t)|$ in (15) is bounded, then it follows from (14) that the closed-loop system (12) is input-to-output stable and $D(k)$ is ultimately bounded [26]. Actually, by the definition of $e_{ji}(k)$, $\beta_{ji}(k)$ and the fact that $\hat{x}_{ji}(k)=\xi_{ji}(k)$, it follows from (3) that

$$e_{ji}(k+1) = g(k)\beta_{ji}(k). \tag{16}$$

Consequently, if the quantized error $\beta_{ji}(k)$ is bounded and $g(k)\to 0$ as $k\to\infty$, then the estimated error $e_{ji}(k+1)$ converges to zero asymptotically and thus $\sup_{0\le t\le k-1}\max_{ij}|e_{ji}(t)|$ is always bounded. According to (3) and (6), the key is that the uniform quantizers q_k^{ji} associated with the directed channels (j,i) are unsaturated. Thus, in the ensued section we will further prove that, if the scaling function $g(k)$ and the quantization level parameters $K_{ji}(k)$ are properly designed such that the uniform quantizers q_k^{ji} associated with the channels (j,i) are never saturated, then the system (1) exponentially achieves consensus in the sense of (2) with zero steady-state error under the protocol (7).

4. Convergence analysis

Assumption 3. For the agents' initial states $x_i(0), i\in\mho$, there holds $\|x(0)\|_\infty = \max_{1\le i\le N}|x_i(0)| \le C_x$ and $D(0)=M(0)-m(0)\le C_\delta$ for some known constants C_x and C_δ.

We are now ready to present our main results.

Theorem 1. *Suppose Assumptions 1–3 hold. Let* $\mu = \sup_{k\ge 1}\frac{g(k-1)}{g(k)}$. *For any* $(i,j)\in\mathcal{E}(k)$, *if* $\mu < \frac{1}{(1-\eta)^{\frac{1}{NB-1}}}$ *and* $K_{ij}(k)$ *satisfy*

$$K_{ij}(1) \ge \frac{C_x}{g(0)} - \frac{1}{2}, \tag{17}$$

$$K_{ij}(2) \ge \begin{cases} \dfrac{2\alpha C_\delta + (2\alpha+1)g(0)}{2g(1)} - \dfrac{1}{2}, \\ \quad w_{ji}(1) > 0, \\ \dfrac{\alpha C_\delta}{g(1)} + \dfrac{\left(K_{ij}(1)+\frac{1}{2}+\alpha\right)g(0)}{g(1)} - \dfrac{1}{2}, \\ \quad w_{ji}(1) = 0, \end{cases} \tag{18}$$

$$K_{ij}(k+1) \ge \begin{cases} K_{\alpha,\mu} + \dfrac{(2\alpha+1)\mu}{2} - \dfrac{1}{2}, \\ \quad w_{ji}(k) > 0, \\ K_{\alpha,\mu} + (K_{ij}(k)+\alpha)\mu + \dfrac{\mu-1}{2}, \\ \quad w_{ji}(k) = 0, \end{cases} \tag{19}$$

where

$$K_{\alpha,\mu} = \frac{\alpha\mu C_\delta}{g(0)(1-\eta)} + \frac{2\alpha(NB-2)\mu}{\eta}, \tag{20}$$

then, under the proposed protocol (7), *the closed-loop system* (12) *satisfies*

$$\lim_{k\to\infty}\sup\frac{\max_{ji}|x_j(k)-x_i(k)|}{g(k)} \le \frac{2(NB-2)\mu}{\eta}. \tag{21}$$

Furthermore, if $k\to\infty$ *and* $g(k)\to 0$, *the multi-agent system* (1) *achieves consensus asymptotically, that is*

$$\lim_{k\to\infty}|x_i(k)-x_j(k)| = 0, \quad \forall i,j\in\mho. \tag{22}$$

Remark 7. Expression (19) implies that the number of quantization levels of each quantizer adaptively tunes according to whether the associated directed channel is active or not at the last step. That is, for $\forall i\ne j$, if the directed channel (i,j) is activated at time k, then at the next step, the number of quantization levels $2K_{ij}(k+1)+1$ of the quantizer associated with this channel stays the same; otherwise, the number of quantization levels $2K_{ij}(k+1)+1$ of the quantizer will be increased for this directed channel. Thus, the impact of unreliable communication link on the directed channel (i,j) can be counteracted by increasing the information accuracy (*i.e.*, by increasing the number of quantization levels) for this directed channel [13]. While in [18,19], where the uniform quantizer designed for each directed channel need not be adaptively tuned, regardless whether the associated directed channel is active or not at the last step, *i.e.*, the quantizer is time-invarying. Then for the case of directed networks are balanced and periodically connected, a finite lower bound of the number of quantization levels between each pair of adjacent agents is obtained in [19] to ensure the exponential average consensus by properly choosing quantizer parameters. Additionally, the lower bound is proved to be merely 1-bit only for the case of the directed balanced network with fixed topology [19].

Proof. The proof of Theorem 1 is organized into two steps. Based on the input-to-output stability result (14), we firstly establish the relationship between the data $\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}$ that being quantized through the directed channel (i,j), the consensus error $\max_{ji}|x_j(k)-x_i(k)|$ and the quantized error $\max_{ji}|\beta_{ij}(k-1)|$. Secondly, by using mathematical induction, we show how to design the scaling function $g(k)$ and the quantization level parameters $K_{ij}(k)$ to achieve consensus.

Step 1: By the definitions of estimated error $e_{ji}(k)$, quantized error $\beta_{ji}(k)$ and the fact that $\hat{x}_{ji}(k)=\xi_{ji}(k)$, there holds

$$w_{ij}(k)e_{ji}(k) = w_{ij}(k)g(k-1)\beta_{ji}(k-1). \tag{23}$$

Thus, the closed-loop system (12) can be rewritten as

$$\begin{aligned} x_i(k+1) &= x_i(k) + \alpha\sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)[x_j(k)-x_i(k)] + \alpha g(k-1) \\ &\quad \times \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)(\beta_{ii}(k-1)-\beta_{ji}(k-1)). \end{aligned} \tag{24}$$

In view of (3) and (4), then for the directed channel (i,j), we have

$$\begin{cases}\xi_{ij}(k)=\xi_{ij}(k-1)+g(k-1)\Delta_{ij}(k),\ w_{ji}(k)>0\\ \xi_{ij}(k)=\xi_{ij}(k-1),\quad w_{ji}(k)=0.\end{cases} \tag{25}$$

If $w_{ji}(k)>0$, *i.e.*, the directed channel (i,j) is activated at time k or $(i,j)\in\mathcal{E}(k)$, then by the definition of quantized error $\beta_{ji}(k)$ and the fact that $\hat{x}_{ji}(k)=\xi_{ji}(k)$, together with the above equation and (16), we get

$$g(k-1)\beta_{ij}(k-1)=x_i(k)-\xi_{ij}(k-1)-g(k-1)\Delta_{ij}(k).$$

By the fact that $W(k)$ is stochastic and $\max_{ji}|\beta_{ij}(k)|=\max_{ij}|\beta_{ji}(k)|$, together with (24), we further get

$$\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right|\le\frac{\alpha\max_{ji}|x_j(k)-x_i(k)|}{g(k)}+\frac{(2\alpha+1)g(k-1)}{g(k)}\max_{ji}|\beta_{ij}(k-1)|, \tag{26}$$

when $w_{ji}(k)>0$, and

$$\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right|\le\left|\frac{x_i(k)-\xi_{ij}(k-1)}{g(k-1)}\right|\left|\frac{g(k-1)}{g(k)}\right|+\frac{\alpha\max_{ji}|x_j(k)-x_i(k)|}{g(k)}+\frac{2\alpha g(k-1)}{g(k)}\max_{ji}|\beta_{ij}(k-1)|, \tag{27}$$

when $w_{ji}(k)=0$ or $(i,j)\notin\mathcal{E}(k)$.

By (3) and Remark 2, the above two equations relate the data $\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}$ to the consensus error $\max_{ji}|x_j(k)-x_i(k)|$ and the quantized error $\max_{ji}|\beta_{ij}(k-1)|$ in the cases of $w_{ji}(k)>0$ and $w_{ji}(k)=0$.

On the other hand, it follows from (3) and (5) that $\xi_{ji}(0)=0$ and $\hat{x}_{ji}(0)=0$ for all $(i,j)\in\mathcal{E}(0)$. Then by (1) and (7), we know that $x(1)=x(0)$ and $D(1)=D(0)\le C_\delta$. Furthermore, by the fact that $\max_{ji}|x_j(k)-x_i(k)|\le M(k)-m(k)=D(k)$, along with the input-to-output stability result (14) and (23), we have

$$\frac{\max_{ji}|x_j(k)-x_i(k)|}{g(k)}\le\frac{(1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(1)}{g(k)}+\frac{4(NB-2)g(k-1)}{\eta g(k)}\sup_{0\le t\le k-1}\max_{ji}|\beta_{ij}(t-1)|. \tag{28}$$

Combining the above equation with (26) and (27), then if $w_{ji}(k)>0$, we have

$$\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right|\le\frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(1)}{g(k)}+\frac{4\alpha(NB-2)g(k-1)}{\eta g(k)}\sup_{0\le t\le k-1}\max_{ji}|\beta_{ij}(t-1)|+\frac{(2\alpha+1)g(k-1)}{g(k)}\max_{ji}|\beta_{ij}(k-1)|,\quad k=1,2,\ldots, \tag{29}$$

and if $w_{ji}(k)=0$, we obtain

$$\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right|\le\left|\frac{x_i(k)-\xi_{ij}(k-1)}{g(k-1)}\right|\left|\frac{g(k-1)}{g(k)}\right|+\frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(1)}{g(k)}+\frac{4\alpha(NB-2)g(k-1)}{\eta g(k)}\sup_{0\le t\le k-1}\max_{ji}|\beta_{ij}(t-1)|+\frac{2\alpha g(k-1)}{g(k)}\max_{ji}|\beta_{ij}(k-1)|,\quad k=1,2,3,\ldots. \tag{30}$$

Step 2: By Assumption 3 and (17), we have

$$\left|\frac{x_i(1)-\xi_{ij}(0)}{g(0)}\right|=\left|\frac{x_i(1)}{g(0)}\right|\le K_{ij}(1)+\frac{1}{2}, \tag{31}$$

which, along with Remark 2 and the uniform quantizer (6) implies

$$\max_{ji}|\beta_{ij}(0)|\le\frac{1}{2}, \tag{32}$$

hold for all $(i,j)\in\mathcal{E}(1)$. Therefore, all the uniform quantizers q_1^{ij} are not saturated when $k=1$.

Likewise, if $w_{ji}(1)>0$, then it follows from (26) and (18) that

$$\left|\frac{x_i(2)-\xi_{ij}(1)}{g(1)}\right|\le\frac{2\alpha C_\delta+(2\alpha+1)g(0)}{2g(1)}-\frac{1}{2}+\frac{1}{2}\le K_{ij}(2)+\frac{1}{2}. \tag{33}$$

Meanwhile, if $w_{ji}(1)=0$, (27) and (18) lead to

$$\left|\frac{x_i(2)-\xi_{ij}(1)}{g(1)}\right|\le\frac{\alpha C_\delta}{g(1)}+\frac{\left(K_{ij}(1)+\frac{1}{2}+\alpha\right)g(0)}{g(1)}-\frac{1}{2}+\frac{1}{2}\le K_{ij}(2)+\frac{1}{2}. \tag{34}$$

Therefore, the above two equations imply

$$\max_{ji}|\beta_{ij}(1)|\le\frac{1}{2}, \tag{35}$$

holds for all $(i,j)\in\mathcal{E}(2)$, which means all the uniform quantizers q_2^{ij} are also not saturated when $k=2$.

We are now in the place to prove that, if $\max_{ji}|\beta_{ij}(t)|\le\frac{1}{2}$, for all $t=0,1,2,\ldots,k-1$, $k=2,3,\ldots$, then there holds $\max_{ji}|\beta_{ij}(k)|\le\frac{1}{2}$, *i.e.*, all the quantizers q_{k+1}^{ij} at time $k+1$ also never saturate.

Suppose that $\sup_{0\le t\le k-1}\max_{ji}|\beta_{ij}(t)|\le\frac{1}{2}$. Then if $w_{ji}(k)>0$, from (29) we have

$$\begin{aligned}\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right|&\le\frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(1)}{g(k)}+\frac{2\alpha(NB-2)g(k-1)}{\eta g(k)}+\frac{(2\alpha+1)g(k-1)}{2g(k)}\\&\le\frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(1)}{g(0)}\prod_{j=1}^{k}\frac{g(j-1)}{g(j)}+\frac{2\alpha(NB-2)g(k-1)}{\eta g(k)}+\frac{(2\alpha+1)g(k-1)}{2g(k)}\\&\le\frac{\alpha\mu C_\delta}{g(0)(1-\eta)}+\frac{2\alpha(NB-2)\mu}{\eta}+\frac{(2\alpha+1)\mu}{2}-\frac{1}{2}+\frac{1}{2}.\end{aligned} \tag{36}$$

To obtain the last inequality of (36), we have used that $\mu=\sup_{k\ge1}\frac{g(k-1)}{g(k)}$ and $\mu<\frac{1}{(1-\eta)^{\frac{1}{NB-1}}}$. Similarly, if $w_{ji}(k)=0$, by $|\frac{x_i(k)-\xi_{ij}(k-1)}{g(k-1)}|\le K_{ij}(k)+\frac{1}{2}$ and (30) yield

$$\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right|\le\frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}}\frac{1}{1-\eta}D(1)}{g(k)}+\frac{2\alpha(NB-2)g(k-1)}{\eta g(k)}+\frac{\left(K_{ij}(k)+\frac{1}{2}+\alpha\right)g(k-1)}{g(k)}$$

$$\leq \frac{\alpha\mu C_\delta}{g(0)(1-\eta)} + \frac{2\alpha(NB-2)\mu}{\eta} + \left(K_{ij}(k) + \frac{1}{2} + \alpha\right)\mu - \frac{1}{2} + \frac{1}{2}, \quad k = 2, 3, \ldots, \tag{37}$$

which, together with (36) and (19) give $|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}| \leq K_{ij}(k+1)+\frac{1}{2}$. Thus, it follows that

$$\max_{ji} |\beta_{ij}(k)| \leq \frac{1}{2}, \tag{38}$$

holds for all $(i,j) \in \mathcal{E}(k+1)$, which indicates that all the uniform quantizers q_{k+1}^{ij} are also not saturated at time $k+1$. In summary, this together with (32), (35) and by induction, we conclude that

$$\sup_{k\geq 0} \max_{ji} |\beta_{ij}(k)| \leq \frac{1}{2}. \tag{39}$$

Using the fact that $\mu < \frac{1}{(1-\eta)^{\frac{1}{NB-1}}}$, we have

$$\frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}} \frac{1}{1-\eta} D(1)}{g(k)} = \frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}} D(1)}{g(0)(1-\eta)} \prod_{j=1}^{k} \frac{g(j-1)}{g(j)}$$
$$\leq \frac{\alpha(1-\eta)^{\frac{k-1}{NB-1}} D(1)}{g(0)(1-\eta)} \mu^k \to 0, \quad k \to \infty, \tag{40}$$

which, along with (28) leads to (21). Moreover, if $g(k) \to 0$ when $k \to \infty$, then by (21) we finally obtain (22). □

Remark 8. It can be seen that (21) reveals the relationship between the steady-error of the closed-loop system (12) and the scaling function $g(k)$. Particularly, if $\lim_{k\to\infty} g(k) = 0$, then the steady-error is zero, *i.e.* $\lim_{k\to\infty} \sup \max_{ji} |x_j(k) - x_i(k)| = 0$. Furthermore, the consensus convergence rate of the closed-loop system satisfies $\max_{ji} |x_j(k) - x_i(k)| = O(g(k))$. Hence, in order to achieve asymptotic consensus with zero steady-error, we can choose the scaling function as $g(k) = g_0\gamma^k$ with $g_0 > 0$ and $\gamma \in (0, 1)$. Then in this case, $\mu = \frac{1}{\gamma}$ and the closed-loop system (12) achieves asymptotic consensus with an exponential convergence rate γ.

Remark 9. Assumption 1 implies that each agent has access to its own information, then one may wonder why we do not adopt the following quantized protocol instead of (7):

$$u_i(k) = \alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)(\hat{x}_{ji}(k) - x_i(k)),$$
$$k = 0, 1, 2, \ldots, \quad i = 1, \ldots, N, \tag{41}$$

i.e., agent i ($i \in \mho$) uses its own exact state $x_i(k)$ instead of the internal state $\xi_{ii}(k)$ of encoder ϕ_{ii}. However, the quantized protocol (41) has obvious drawback.

Substituting the protocol (41) into (1) and by the definition of $e_{ji}(k)$, we obtain the following closed-loop system for agent i

$$x_i(k+1) = (1-\alpha)x_i(k) + \alpha \sum_{j=1}^{N} w_{ij}(k)x_j(k) - \tilde{e}_i(k), \tag{42}$$

where $\tilde{e}_i(k) = \alpha \sum_{j\in\mathcal{N}_i^+(k)} w_{ij}(k)e_{ji}(k)$. As will be shown by the simulation results in Section 5, the final consensus value of the closed-loop system (12) always lies in the convex hull of the initial values $x_i(0)$ ($i \in \mho$) even if it cannot be specified by Theorem 1. While for the closed-loop system (42), its final consensus value generally does not lie in the convex hull of the initial values $x_i(0)$ ($i \in \mho$), and even worse, the closed-loop system (42) may be divergent if $\tilde{e}_i(k)$ is a bounded white noise [27]. Thus, the protocol (7) is more suitable for practical implementations.

Note that the unreliable communication links, such as link failures or packet looses, and link recoveries can be modeled as the dynamically switching of network topologies: for the directed channel (i,j), the link failure at time k implies $w_{ji}(k) = 0$, otherwise, $w_{ji}(k) > 0$. Next, as a practical issue, we will investigate the robustness of the proposed protocol to unreliable links.

To this end, let $\mathcal{N}_i^+ = \bigcup_{k=1}^{\infty} \mathcal{N}_i^+(k)$ denote all the in-neighbors of agent i at all time steps. For each $j \in \mho$ and $i \in \mathcal{N}_j^+$, for the directed channel (i,j), we denote the first time step k such that $w_{ji}(k) > 0$ by $t_{ij}(1)$, $t_{ij}(t) = \min\{k : k > t_{ij}(t-1), w_{ji}(k-1) = 0, w_{ij}(k) > 0\}$, $t = 2, 3, \ldots$, and $s_{ij}(t) = \min\{k : k > t_{ij}(t), w_{ji}(k-1) > 0, w_{ji}(k) = 0\}$, $t = 1, 2, \ldots$. Then from the above definitions, $s_{ij}(t)$ denotes the start time for the tth failure of the directed channel (i,j) with $s_{ij}(0) = 0$, and $t_{ji}(t)$ denotes the start time for the tth recovery of the directed channel (i,j). Therefore, $t_{ij}(t+1) - s_{ij}(t)$ is the duration of the tth failure of the directed channel (i,j). For $t_{ij}(t)$ and $s_{ij}(t)$, we further assume [13]:

Assumption 4 (*Bounded Link Failure Duration*)**.** For all $j \in \mho$ and $i \in \mathcal{N}_j^+$, there holds that $\sup_{t\geq 0} |t_{ij}(t+1) - s_{ij}(t)| \leq T$, where T is a positive integer.

Assumption 4 means the duration of any link failure in the directed network is bounded and $B = T + 1$.

Theorem 2. *Let Assumptions 1–4 hold. For any $j \in \mho$, $i \in \mathcal{N}_j^+ \cup \{j\}$ and the integer $K \geq 1$, if there exists suitable $\mu \in (1, \frac{1}{(1-\eta)^{\frac{1}{NB-1}}})$ such that*

$$K_{\alpha,\mu} + \frac{\mu(2\alpha+1)}{2} \leq K + \frac{1}{2}, \tag{43}$$

and

$$\mu^T K + (K_{\alpha,\mu} + \mu\alpha)\frac{\mu^T - 1}{\mu - 1} + \frac{\mu^T - 1}{2} \leq K + 1. \tag{44}$$

Moreover, if for any scaling function $g(k)$ satisfies

$$g(0) \geq \frac{C_x}{K + \frac{1}{2}}, \tag{45}$$

$$g(1) \geq \max\left\{\frac{2\alpha C_\delta + (2\alpha+1)g(0)}{2K+1}, \frac{2\alpha C_\delta + (2K+1+2\alpha)g(0)}{2K+3}\right\}, \tag{46}$$

then, under the proposed protocol (7) with the quantization level parameters $K_{ij}(k)$ satisfying

$$K_{ij}(1) = K, \tag{47}$$

$$K_{ij}(k+1) = \begin{cases} K, & w_{ji}(k) > 0, \\ K+1, & w_{ji}(k) = 0, \end{cases} \tag{48}$$

the multi-agent system (1) achieves consensus asymptotically,

$$\lim_{k\to\infty} |x_i(k) - x_j(k)| = 0, \quad \forall i, j \in \mho.$$

Proof. The proof follows from a generalization of an argument in Theorem 2 of [13]. Firstly, by (20) and note that $\lim_{\alpha\to 0}(K_{\alpha,1} + \frac{2\alpha+1}{2}) = \frac{1}{2}$ and $\lim_{\alpha\to 0}(K_{\alpha,1} + \alpha) = 0$, *i.e.*, reducing the value of α leads to smaller number of bits of quantized information that need to be transmitted along each connected digital channel. Recall that the constant control gain $\alpha \in (0, 1]$, then for any integer $K \geq 1$, there always exists $\alpha^* \in (0, 1]$ such that

$$K_{\alpha^*,1} + \frac{2\alpha^*+1}{2} \leq K + \frac{1}{2}, \tag{49}$$

and

$$K + (K_{\alpha^*,1} + \alpha^*)T \leq K + 1. \tag{50}$$

Furthermore, note that

$$\lim_{\mu\to 1}\left(K_{\alpha^*,\mu} + \frac{\mu(2\alpha^*+1)}{2}\right) = K_{\alpha^*,1} + \frac{2\alpha^*+1}{2}, \tag{51}$$

and

$$\lim_{\mu\to 1}\left(\mu^T K + (K_{\alpha^*,\mu} + \mu\alpha^*)\frac{\mu^T-1}{\mu-1} + \frac{\mu^T-1}{2}\right) = K + (K_{\alpha^*,1} + \alpha^*)T, \tag{52}$$

then along with (49) and (50), it can be seen that there exist suitable $\alpha \in (0, 1]$ and $\mu \in (1, \frac{1}{(1-\eta)^{\frac{1}{NB-1}}})$ such that (43) and (44) hold. It is noteworthy that $\mu \in (1, \frac{1}{(1-\eta)^{\frac{1}{NB-1}}})$ implies that the scaling function $g(k)$ can be selected as $g(k) = g_0\gamma^k$ with $g_0 > 0$ and $\gamma \in (0, 1)$ (see Remark 8).

Then following steps are as those in the proof of Theorem 1. By (45) and (47), we have (17). Similarly, (46)–(48) lead to (18). Next, we need to prove that: if $\max_{ji}|\beta_{ij}(t)| \leq \frac{1}{2}$, for all $t = 0, 1, 2, \dots, k-1, k = 2, 3, \dots$, then $\max_{ji}|\beta_{ij}(k)| \leq \frac{1}{2}$, *i.e.*, all the quantizers q^{ij}_{k+1} at time $k+1$ also never saturate by choosing constant quantization parameter K. For the directed channel (i, j), if $t_{ij}(t) \leq k \leq s_{ij}(t) - 1, t = 1, 2, \cdots$, then by (47), (48), (43), (36) and Assumption 4, we have

$$\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right| \leq K_{\alpha,\mu} + \frac{(2\alpha+1)\mu}{2} \leq K + \frac{1}{2} \leq K_{ij}(k+1) + \frac{1}{2}. \tag{53}$$

On the other hand, if $s_{ij}(t) \leq k \leq t_{ij}(t+1) - 1, t = 0, 1, 2, \dots$, *i.e.*, at such time step k, the directed channel (i, j) is not activated, then by (44) and similar to (37) we have

$$\begin{aligned}\left|\frac{x_i(k+1)-\xi_{ij}(k)}{g(k)}\right| &\leq \left|\frac{x_i(k)-\xi_{ij}(k-1)}{g(k-1)}\right|\mu + K_{\alpha,\mu} + \alpha\mu \\ &\leq \mu^{k+1-s_{ij}(t)}\left|\frac{x_i(s_{ij}(t))-\xi_{ij}(s_{ij}(t)-1)}{g(s_{ij}(t)-1)}\right| \\ &\quad + (K_{\alpha,\mu}+\alpha\mu)\sum_{j=s_{ij}(t)}^{k}\mu^{k-j} \\ &\leq \mu^T\left(K+\frac{1}{2}\right) + (K_{\alpha,\mu}+\mu\alpha)\frac{\mu^T-1}{\mu-1} - \frac{1}{2} + \frac{1}{2} \\ &\leq K + 1 + \frac{1}{2} \leq K_{ij}(k+1) + \frac{1}{2},\end{aligned} \tag{54}$$

which, together with (53) indicate $\max_{ji}|\beta_{ij}(k)| \leq \frac{1}{2}$ for all $(i, j) \in \mathcal{E}(k+1)$. Therefore, we can conclude that under the protocol (7), the multi-agent system (1) achieves consensus with an exponential convergence rate γ. □

Remark 10. The above theorem implies K can be chosen as 1. Then by (47), (48) and Remark 4, it can be seen that by properly choosing the control gain and the scaling function, 5-level ($K = 2$ or 3-bit) quantizer q^{ji}_k is applied to each directed digital channel $(j, i) \in \mathcal{E}(k) (i \neq j)$, together with 3-level ($K = 1$ or 1-bit) quantizer q^{ii}_k is applied to each self-loop, which causes the closed-loop multi-agent system input-to-output stable and consensus be exponentially achieved. Additionally, it is worth pointing out that the final consensus value x^* of the closed-loop system (12) cannot be specified by Theorems 1 and 2. However, simulation results show that x^* always lies in the convex hull of agents' initial values $x_i(0)$ $(i \in \mho)$, *i.e.*, there always holds that $m(0) \leq x^* \leq M(0)$. This is in stark contrast to the existing works in the quantized consensus literature, where the final consensus values are usually dictated by the given communication topology and the initial states of agents.

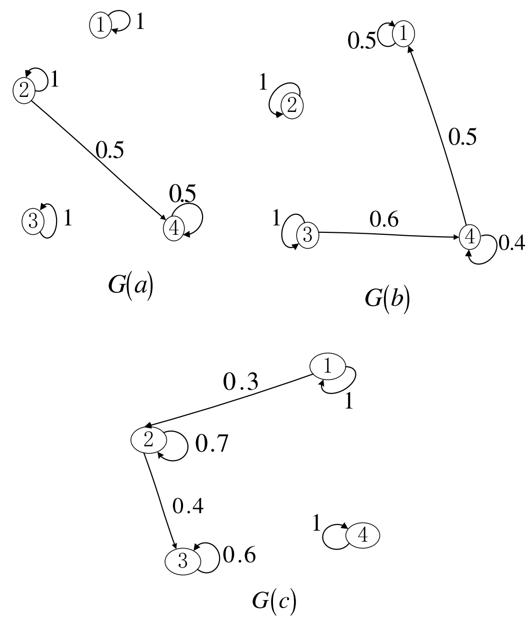

Fig. 1. A directed network with switching topology.

5. Simulation results

Consider a 4-agent directed network with periodically switching topologies shown in Fig. 1, with $\mathbb{G}(k) = \{\mho = \{1, 2, 3, 4\}, \mathcal{E}(3k) \cup \mathcal{E}(3k+1) \cup \mathcal{E}(3k+2)\} = \{\mho = \{1, 2, 3, 4\}, G(a) \cup G(b) \cup G(c)\}, k = 0, 1, 2, \dots$, which satisfies Assumption 2 and the period B is 3. The number associated with each connected directed channel is the edge weight $w_{ij}(k)$ of the stochastic adjacency matrix $W(k)$, which satisfies Assumption 1.

It follows from (47) and (48) that 1-bit rate communication requires α should be sufficiently close to 0 and γ should be sufficiently close to 1. Thus by Theorem 2, we choose the quantization level parameters as: $K_{ii}(k) = 1, k \geq 1, i \in \mho = \{1, 2, 3, 4\}$; meanwhile, when $i \neq j$, $K_{ij}(k) = 1$ if $w_{ji}(k) > 0$, or $K_{ij}(k) = 2$ if $w_{ji}(k) = 0$ for $k = 2, 3, \dots$ and $i, j \in \mho = \{1, 2, 3, 4\}$. The control gain α is chosen as 0.05, $g(0) = 10$, and the scaling function $g(k)$ is taken as $g_k = 10(0.99999)^k$, which means $\mu = \frac{1}{\gamma}$ with $\gamma = 0.99999$. The initial states of the agents are chosen randomly as $x(0) = (6.5143 \quad 1.9482 \quad 7.4341 \quad 2.7999)^T$. Then the evolution of the agents' states and $K_{12}(k)$ are shown in Figs. 2 and 3, respectively. It can be seen that, at each time step, if each agent just sends only 5-level quantized information to each of its neighbors, together with 3-level quantized information to itself, then consensus among agents can be achieved with an exponential convergence rate. Fig. 4 shows the evolution of the states when $\gamma = 0.99996$. Figs. 2 and 4 clearly verify that the smaller the value of γ is, the faster the convergence is.

Next we will compare the protocol (7) with the protocol (41). For this purpose, we select $\alpha = 0.05$ and $g_k = 10(0.99996)^k$,

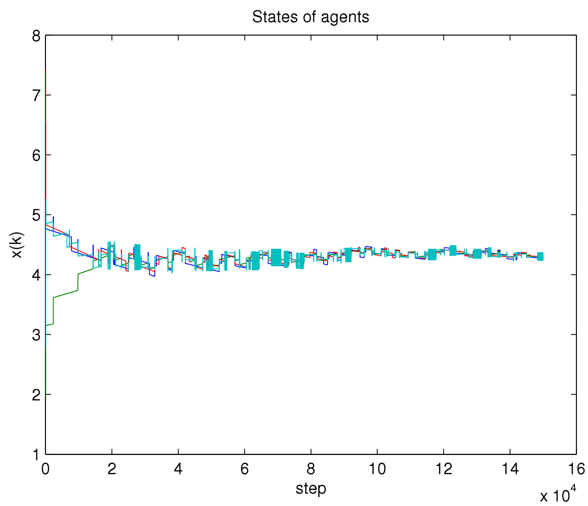

Fig. 2. The trajectories of agents' states when $\alpha = 0.05$ and $\gamma = 0.99999$.

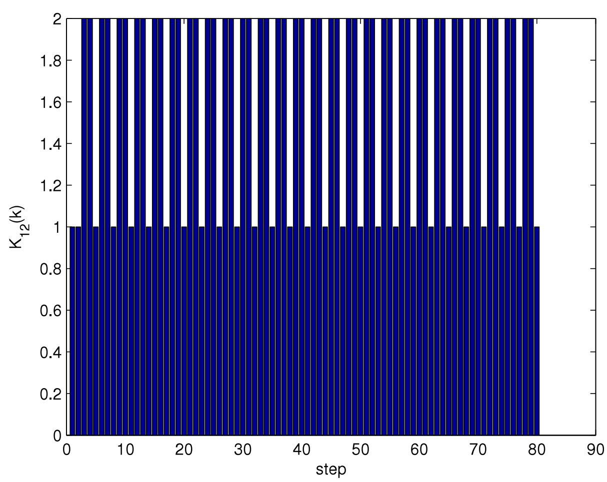

Fig. 3. The evolution of $K_{12}(k)$.

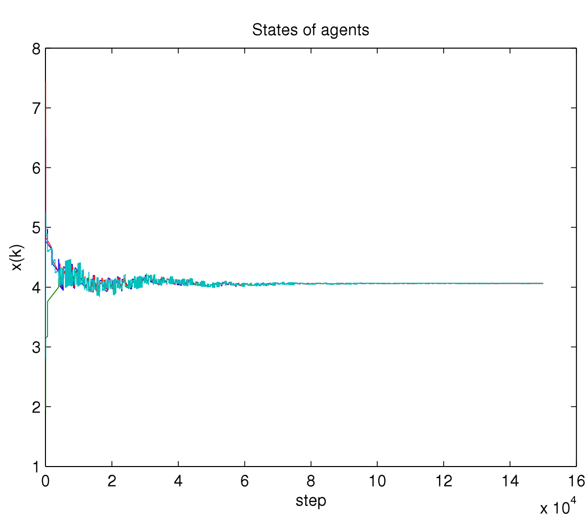

Fig. 4. The trajectories of agents' states when $\alpha = 0.05$ and $\gamma = 0.99996$.

$K_{ii}(k) = 1$ for $k \geq 1$ and $i \in \mho = \{1, 2, 3, 4\}$. When $i \neq j$, $K_{ij}(k) = 1$ if $w_{ji}(k) > 0$, or $K_{ij}(k) = 2$ if $w_{ji}(k) = 0$ for $k = 2, 3, \ldots$ and $i, j \in \mho = \{1, 2, 3, 4\}$. We let each of the protocols run 1000 times.

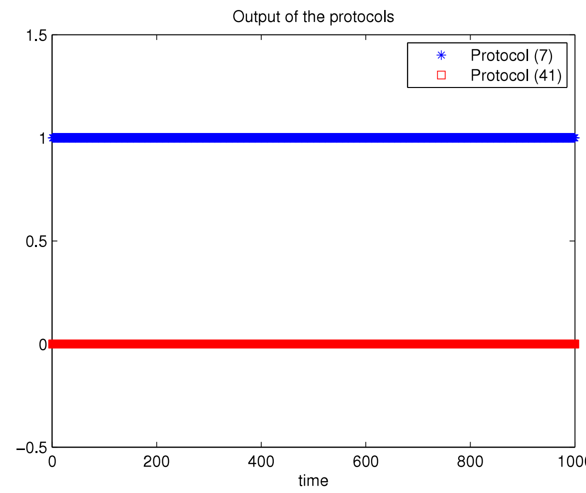

Fig. 5. Comparison protocol (7) with protocol (41).

For each time, the initial values $x_i(0), i = 1, 2, 3, 4$ are randomly generated according to a uniform distribution over $[-8, 8]$, and the two protocols are run 80 000 steps, respectively. Then at the end of each time, if the consensus value $x_f := \frac{1}{4}\sum_{j=1}^{4} x_j(80\,000)$ satisfies $\min_{i\in\mho} x_i(0) \leq x_f \leq \max_{i\in\mho} x_i(0)$, which means that the consensus value x_f is in the convex hull of the initial values $x_i(0)$ $(i = 1, 2, 3, 4)$, then the output of this time is 1, otherwise the output of this time is 0. The resulting evolution of the output of two protocols are shown in Fig. 5, which demonstrates that under the protocol (7), consensus is asymptotically achieved and the consensus value always lies in the convex hull of the initial values of agents. Therefore, we should choose protocol (7) rather than protocol (41) in practical applications.

6. Conclusions

In this paper, we have studied the problem of reaching consensus on general directed switching networks in the presence of limited quantized information communication and in the absence of double-stochasticity assumption. We have proposed a distributed protocol with an adaptive finite-level uniform quantized strategy. Under some mild connectivity assumptions that capturing the main features of multi-agent networks, we have derived conditions for consensus can be exponentially achievable with limited data rate. The proposed protocol features little communication overhead and suits well for directed switching digital networks. Interesting future work includes the extension of the obtained results to directed switching digital networks with time delays in the inter-agent communications. How to achieve consensus using as few as 1-bit quantized information communication in our setting is also very interesting.

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