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SOCIAL LEARNING IN NETWORKS WITH TIME-VARYING TOPOLOGIES

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ABSTRACT

Recently, Jadbabaie *et al*. presented a social learning model, where agents update beliefs by combining Bayesian posterior beliefs based on personal observations and weighted averages of the beliefs of neighbors. For a network with a fixed topology, they provided sufficient conditions for all of the agents in the network to learn the true state almost surely. In this paper, we extend the model to networks with time-varying topologies. Under certain assumptions on weights and connectivity, we prove that agents eventually have correct forecasts for upcoming signals and all the beliefs of agents reach a consensus. In addition, if there is no state that is observationally equivalent to the true state from the point of view of all agents, we show that the consensus belief of agents eventually reflects the true state.

Key Words: Social networks, social learning, consensus.

I. INTRODUCTION

It is well known that beliefs affect decisions. For example, our beliefs or opinions shape what kind of clothes we choose to buy. In a voting situation, our ideological beliefs will determine which candidate we decide to support. Because of the importance of beliefs, the study of how beliefs are formed has gradually become a hot topic over recent decades, and created a new research area—social learning theory.

A social learning model generally contains two aspects: a model of a belief updating rule and a model of the social structure. In the respect of a belief updating rule, the existing models can be classified into two categories: Bayesian and non-Bayesian. Bayesian models are those in which fully rational individuals use Bayes' rule to form the best mathematical estimate of the relevant unknowns [1]. Each individual updates his or her belief by performing complex conjecture on others' actions, in which case the strategies of others must be taken into account. Therefore, Bayesian learning models are mainly analyzed in the game theory framework [2–4]. In contrast to Bayesian models, non-Bayesian models focus on boundedly rational individuals who update their beliefs simply by linearly combining the beliefs of their neighbors [5–7]. Roughly, the mathematical models of traditional non-Bayesian updating rules are very similar to the consensus problem in the coordination control field [8–12]. Regarding social structure, it may be as simple as a chain graph representing the situation where agents communicate with one another sequentially and only once in the learning process or as complex as a time-varying belief-dependent graph where the beliefs of agents and the structure of the social network evolve collectively. Most of the Bayesian updating rules are investigated in simple networks because of the complexity of the fully rational inference itself, while most of the non-Bayesian models incorporate the analysis of the effect of social structure on learning performance.

In this paper, we examine a non-Bayesian belief updating rule presented by Jadbabaie *et al*. in [13]. In the model, each individual updates his or her belief by combining Bayesian posterior belief based on a private signal and the weighted average of the beliefs of neighbors. In fact, this updating rule represents two basic learning approaches in the real world: observing and deducing by oneself and communicating with other individuals. Although Bayesian inference is involved in this model, the observed signals are caused directly by the underlying true state of the world rather than being obtained from other individuals, which is an important feature that distinguishes this model from traditional Bayesian models. As for the social character of the model, each individual still interacts with others, but on the level of beliefs rather than actions. More specifically, an individual involves the weighted average of beliefs of his or her neighbors as part of the new belief. Since complex deduction is avoided, this model sharply decreases the complexity of analysis and computing. In [13], the belief updating rule is investigated in fixed networks and is proven to enable the individuals to

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successfully aggregate information and discover the underlying true state under relatively relaxed conditions, compared with learning in the personal situation [14]. The major advantage of the model is that all agents are boundedly rational such that the belief updating rule is quite simple, as in most of the non-Bayesian social learning models, and all of the agents can learn the truth under certain conditions, as they do in most of the Bayesian social learning models, where much more complicated deduction is needed to achieve this goal.

The assumption of fixed and strongly connected networks simplifies the analysis of Jadbabaie's model, but it is apparently quite restrictive. The underlying networks of interactions in the real world, however, are more likely to be time-varying. For example, neighboring relationships between individuals may be established or removed over time. Sometimes, even if the communications are maintained, the levels of trust are changed as individuals receive a growing amount of information. Motivated by this consideration, we extend Jadbabaie's model to networks with time-varying topologies in this paper. With assumptions on the structure of networks, such as the joint connectivity and the positive weight threshold, we show that all agents eventually have correct forecasts for upcoming signals. Then, the learning problem can be turned into a consensus problem with decreasing noise. It is proven that all of the beliefs of agents reach a consensus. Furthermore, without a state that is observationally equivalent to the true state from the point of view of all agents, the consensus belief eventually reflects the true state.

This paper is organized as follows. In Section II, the social learning model in networks with time-varying topologies is described. In Section III, assumptions and main results are presented. All proofs are provided in Section IV. Section V includes simulations to further verify our theoretical results. Finally, Section VI contains concluding remarks.

II. MODEL DESCRIPTION

2.1 Network structure

Consider a social network with time-varying topologies as a set of directed graphs: $\{G_t = (V, E_t), t \geq 0\}$, where $V = \{1, 2, \cdots, n\}$ is the vertex set and $E_t \subset V \times V$ is the edge set at time t. Each vertex in V represents an agent, and an edge connecting i to j, denoted by the ordered pair $(i, j) \in E_t$, captures the fact that information flows from agent i to agent j. In this, agent i is called a neighbor of agent j. The set of neighbors of j at time t is denoted by $N_j(t) = \{i \in V : (i, j) \in E_t\}$.

2.2 Possible states and beliefs of individuals

Let θ denote a state of the world, representing the quality of a new product, the underlying reason of a phenomenon, and so on. All of the possible states compose a finite set of states $\Theta = \{\theta_1, \theta_2, \cdots, \theta_m\}$, in which the true state is denoted by θ^*. Agent i's belief on state θ at time t is denoted by $\mu_{i,t}(\theta)$, which is the probability that he or she believes state θ is true. Note that $\{\mu_{i,t}(\theta), \theta \in \Theta\}$ is a probability distribution over the state set Θ.

2.3 Signal structure

Conditional on the true state, at each time period $t > 0$, signal vector $s_t = (s_t^1, \cdots, s_t^n) \in S$ is generated according to the likelihood function $\ell(s_t \mid \theta^*)$, where signal s_t^i is observed by agent i at period t and S denotes the signal space. For each observed signal s_t^i, agent i has a corresponding personal signal structure for each θ, which is denoted by $\ell_i(s_t^i|\theta)$, representing agent i's subjective probability of s_t^i arising if the true state is θ from agent i's viewpoint. These personal signal structures are decided by the agents themselves, and may be totally different from the truth. Nevertheless, we need to impose the following essential assumption on signal structures about the true state: the signal structure about the true state, *i.e.*, $\ell_i(s_t^i|\theta^*)$, is the ith marginal of $\ell(s_t|\theta^*)$, which means agents have exact knowledge of what will happen if the true state is θ^*. Similar to learning in the real world, observations do not necessarily contain valuable information that facilitates agents in finding the true state. For example, there might exist one false state among all possible states that causes exactly the same signals as the true state does in one's eyes. In this case, one certainly cannot discover the true state only by observing the outside signals. In [13], the concept of observational equivalent is used to denote the identification problem. The elements in the set $\bar{\Theta}_i = \{\theta \in \Theta : \ell_i(\cdot|\theta) = \ell_i(\cdot|\theta^*)$ are observationally equivalent to the true state θ^* from the point of view of agent i.

2.4 Belief updating rule

The belief updating rule adopted by any agent i can be described as, for all $\theta \in \Theta$,

$$\mu_{i,t+1}(\theta) = a_{ii}(t)\mu_{i,t}(\theta)\frac{\ell_i(s_{t+1}^i|\theta)}{m_{i,t}(s_{t+1}^i)} + \sum_{j \in N_i(t)} a_{ij}(t)\mu_{j,t}(\theta) \quad (1)$$

where $m_{i,t}(s_{t+1}^i) = \sum_{\theta \in \Theta} \ell_i(s_{t+1}^i \mid \theta)\mu_{i,t}(\theta)$, which is known as agent i's one-step-ahead forecast. Similarly, we define the k-step-ahead forecast of agent i at time t as

$$m_{i,t}(s_{t+1}^i, \cdots, s_{t+k}^i) = \sum_{\theta \in \Theta}\left(\prod_{r=1}^{k} \ell_i(s_{t+r}^i|\theta)\right)\mu_{i,t}(\theta),$$

which will be used in later analysis. Belief updating rule (1) can be interpreted as that the individual updates his or her belief as a convex combination of the Bayesian posterior belief based on a private signal and beliefs of neighbors. From the aspect of cooperative control of multi-agent networks, the first term on the right hand side of (1) can be interpreted as an outside input. The observed signal becomes the "attraction of truth" in a Bayesian manner that moves the beliefs in the desired direction. Meanwhile, all agents consider the beliefs of neighbors and seek a consensus.

To facilitate the analysis, we rewrite (1) in matrix form as

$$\mu_{t+1}(\theta) = A(t)\mu_t(\theta) + diag\left(a_{11}(t)\left[\frac{\ell_1(s^1_{t+1}|\theta)}{m_{1,t}(s^1_{t+1})} - 1\right], \cdots, a_{nn}(t)\left[\frac{\ell_n(s^n_{t+1}|\theta)}{m_{n,t}(s^n_{t+1})} - 1\right]\right)\mu_t(\theta) \tag{2}$$

where $\mu_t(\theta) = [\mu_{1,t}(\theta), \cdots, \mu_{n,t}(\theta)]^T$; $A(t)$ is called interaction matrix and has its entry on ith row and jth column as $a_{ij}(t)$; $diag$ of a vector is a diagonal matrix with the vector on its diagonal.

2.5 Definitions of social learning

"Social learning" may have different meanings in different contexts. Here, we adopt the definitions in [13], where two types of learning have been defined.

Definition 1 [13]**.** The k-step-ahead forecasts of agent i are eventually correct on a path $\{s_t\}_{t=1}^{\infty}$ if, along that path,

$$m_{i,t}(s^i_{t+1}, \cdots, s^i_{t+k}) \to \prod_{r=1}^{k} \ell_i(s^i_{t+r}|\theta^*) \quad \text{as } t \to \infty.$$

Moreover, if agent i's k-step-ahead forecasts are eventually correct for any natural number k on some path, we say his or her beliefs weakly merge to the truth.

Definition 2 [13]**.** Agent i asymptotically learns the underlying true state θ^* on a path $\{s_t\}_{t=1}^{\infty}$ if, along that path,

$$\mu_{i,t}(\theta^*) \to 1 \quad \text{as } t \to \infty.$$

The two notions of "social learning" are distinct and might not occur simultaneously. Nevertheless, a general phenomenon is that learning the true state will suffice for weak merging to the truth, which is also the case here. For more detailed discussion and examples about these two sorts of learning, please refer to [15].

2.6 Learning in social networks with fixed topologies

For a network with fixed topology, *i.e.*, $E_t = E$ and $A(t) = A$, the main results in [13] can be summarized as follows.

Theorem 1 [13]**.** Suppose that

(a) The network is strongly connected;
(b) For all i and j, $a_{ii} > 0$, and $a_{ij} > 0$ if agent i has access to the belief held by agent j; otherwise, $a_{ij} = 0$;
(c) The interaction matrix A is row stochastic, *i.e.*, $\sum_{j=1}^{n} a_{ij} = 1$ for all i.

If there exists at least one agent with positive initial belief on the true state θ^*, then,

1. the beliefs of all agents weakly merge to the truth almost surely;

In addition, suppose that there is no state $\theta \neq \theta^*$ that is observationally equivalent to θ^* from the point of view of all agents in the network. Then,

2. all agents learn the true state almost surely.

III. MAIN RESULT

3.1 Assumptions

Now, we consider the social learning model (1) with time-varying topologies. In this case, the network is not required to be strongly connected at every time slot, but a certain level of connectivity is still required. We make the following joint connectivity assumption:

Assumption 1 (Joint Connectivity): The directed graph (V, E_∞) is strongly connected, where $E_\infty = \{(i, j)$ | There exists an integer $B \geq 1$ such that edge (i, j) appears at least once every B consecutive time slots$\}$.

As a natural generalization of Condition (b) in Theorem 1, we assume that there exists a positive weight threshold, *i.e.*:

Assumption 2 (Weight Threshold): There exists a scalar $0 < \eta < 1$ such that, for all i and j,

1. $a_{ii}(t) \geq \eta$ for all $t \geq 0$;
2. if agent j is a neighbor of agent i at time t, then $a_{ij}(t) \geq \eta$.

Investigating the belief updating rule (1) in general time-varying networks is beyond us. Instead, our focus is on the networks satisfying the following assumption.

Assumption 3 (Double Stochasticity): The interaction matrix is doubly stochastic, *i.e.*, for any i and j,

$$\sum_{i=1}^{n} a_{ij}(t) = \sum_{j=1}^{n} a_{ij}(t) = 1.$$

In Assumption 3, the row stochasticity condition (c) in Theorem 1 is replaced by the double stochasticity. We make this assumption for a technical reason: to guarantee the existence of a common left eigenvector of $A(t)$ corresponding to the unit eigenvalue. We conjecture, and the simulations will also show, that our results still hold even in its absence.

3.2 Main result

With these assumptions in hand, we now present the main result of our work on social learning in time-varying networks and will cover the proof of it in the next section.

Theorem 2. Consider the social learning model (1). Suppose that Assumptions 1–3 hold and there exists at least one agent with positive initial belief on the true state θ^*. Then,

1. the beliefs of all agents weakly merge to the truth almost surely;
2. the beliefs of all agents reach a consensus almost surely.

In addition, suppose that there is no state $\theta \neq \theta^*$ that is observationally equivalent to θ^* from the point of view of all agents in the network. Then,

3. all agents learn the true state almost surely.

IV. THEORETICAL ANALYSIS

4.1 Proof of weak merging to the truth

In fact, the proof is along the line of that of Proposition 2 in [13], where fixed networks are considered. Therefore, we omit details here and just provide simple analysis of the proof.

Technically, the proof of weak merging to the truth in fixed networks depends on three conditions: (i) the interaction matrix A has a fixed left eigenvector with all positive entries corresponding to the unit eigenvalue; (ii) all agents have strictly positive self-reliances, *i.e.*, $a_{ii} > 0$ for all i; (iii) there exists an agent i such that $\mu_{i,0}(\theta^*) > 0$, and, with the connectivity of the network, after long enough time, all of the agents assign a strictly positive probability to the true state. As long as these three conditions are satisfied, the method of proof in [13] can be extended to our time-varying situation. Now, let us check which conditions are satisfied in our model. First, under the double stochasticity assumption, *i.e.*, $\sum_{i=1}^{n} a_{ij}(t) = \sum_{j=1}^{n} a_{ij}(t) = 1$, the interaction matrix $A(t)$ always has a fixed left eigenvector of all ones corresponding to the unit eigenvalue, even though $A(t)$ is time-varying. Second, Assumption 2 guarantees that all agents have strictly positive self-reliances bounded below by η. Finally, we also assume there exists an agent with positive initial belief assigned to the true state, and, based on the joint connectivity in Assumption 1 and the weight threshold in Assumption 2, all agents have positive beliefs on the true state after at most $(n-1)B$ time slots. To summarize, all of the conditions required to perform the analysis in [13] to prove the weak merging to the truth are satisfied. Although some parts of the proof need to be modified slightly to fit our time-varying settings, one can repeat the proof in [13] with the remarks we have given above. For more details and discussion, please refer to [13].

4.2 Proof of reaching a consensus on beliefs

We prove this in two parts. First, we focus on the set of states that are observationally equivalent to the true state from the point of view of all agents, *i.e.*, $\theta \in \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$. In the second part, we focus on the complement set.

Part 1 (For $\theta \in \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$). We pointed out in the preceding section that the beliefs of all agents weakly merge to the truth almost surely, *i.e.*, $m_{i,t}(s_{t+1}^i, \cdots, s_{t+k}^i) \to \prod_{r=1}^{k} \ell_i(s_{t+r}^i|\theta^*)$ stands for any agent i and any natural k as time goes to infinity. This implies that the one-step-forecast ($k = 1$) is also correct for any agent i, *i.e.*,

$$m_{i,t}(s_{t+1}^i) \to \ell_i(s_{t+1}^i|\theta^*) \quad \text{as } t \to \infty.$$

The above statement is correct for all $\theta \in \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$, since $\ell_i(\cdot|\theta) = \ell_i(\cdot|\theta^*)$. Then, according to the belief updating rule (2), we have

$$\mu_{t+1}(\theta) = A(t)\mu_t(\theta) + e_t(\theta) \tag{3}$$

where $e_t(\theta) = [e_{1,t}(\theta), \cdots, e_{n,t}(\theta)]^T$ and $e_{i,t}(\theta) = a_{ii}(t)\mu_{i,t}(\theta)\left[\frac{\ell_i(s_{t+1}^i|\theta)}{m_{i,t}(s_{t+1}^i)} - 1\right]$, which converges to zero almost surely as time goes to infinity.

Now, we have turned the social learning problem into a consensus problem with decreasing noise. We introduce the following transition matrices to describe the evolution of the beliefs:

$$\Phi(t, s) = A(t)A(t-1) \cdots A(s) \text{ for all } t, s \text{ with } t \geq s$$

where $\Phi(t, t) = A(t)$ for all t. Recent work has established explicit convergence rate results for these transition matrices [16–18].

Let Assumptions 1–3 hold. Then, according to [16], for all i, j and all t, s with $t \geq s$, we have:

$$\left|[\Phi(t, s)]_{ij} - \frac{1}{n}\right| \leq \alpha\beta^{t-s},$$

where $[\Phi(t, s)]_{ij}$ is the entry of $\Phi(t, s)$ on ith row and jth column, $\alpha = 2\frac{1+\eta^{-(n-1)B}}{1-\eta^{(n-1)B}}$ and $\beta = [1-\eta^{(n-1)B}]^{\frac{1}{(n-1)B}} < 1$.

Based on (3), the relation between $\mu_{i,t+1}(\theta)$ and $\mu_{1,0}(\theta), \cdots, \mu_{n,0}(\theta)$ is given by:

$$\mu_{i,t+1}(\theta) = \sum_{j=1}^{n} [\Phi(t,0)]_{ij}\mu_{j,0}(\theta) + e_{i,t}(\theta) + \sum_{r=1}^{t}\left(\sum_{j=1}^{n}[\Phi(t,r)]_{ij}e_{j,r-1}(\theta)\right) \tag{4}$$

Then, we define an auxiliary sequence $\{z_t(\theta),\ t \geq 0\}$, where $z_t(\theta)$ is given by:

$$z_t(\theta) = \frac{1}{n}\sum_{i=1}^{n}\mu_{i,t}(\theta) \text{ for all } t.$$

Combining these two equations implies:

$$z_t(\theta) = \frac{1}{n}\sum_{j=1}^{n}\mu_{j,0}(\theta) + \frac{1}{n}\sum_{r=1}^{t-1}\left(\sum_{j=1}^{n}e_{j,r-1}(\theta)\right) + \frac{1}{n}\sum_{j=1}^{n}e_{j,t-1}(\theta). \tag{5}$$

Using the relations in (4) and (5), we obtain for any i:

$$\begin{aligned}
|\mu_{i,t}(\theta) - z_t(\theta)| &= \Bigg|\sum_{j=1}^{n}\left([\Phi(t-1,0)]_{ij} - \frac{1}{n}\right)\mu_{j,0}(\theta) \\
&\quad + \sum_{r=1}^{t-1}\sum_{j=1}^{n}\left([\Phi(t-1,r)]_{ij} - \frac{1}{n}\right)e_{j,r-1}(\theta) \\
&\quad + \left(e_{i,t-1}(\theta) - \frac{1}{n}\sum_{j=1}^{n}e_{j,t-1}(\theta)\right)\Bigg| \\
&\leq \sum_{j=1}^{n}\left|[\Phi(t-1,0)]_{ij} - \frac{1}{n}\right||\mu_{j,0}(\theta)| \\
&\quad + \sum_{r=1}^{t-1}\sum_{j=1}^{n}\left|[\Phi(t-1,r)]_{ij} - \frac{1}{n}\right||e_{j,r-1}(\theta)| \\
&\quad + |e_{i,t-1}(\theta)| + \frac{1}{n}\sum_{j=1}^{n}|e_{j,t-1}(\theta)| \\
&\leq \alpha\beta^{t-1}\sum_{j=1}^{n}|\mu_{j,0}(\theta)| \\
&\quad + \sum_{r=1}^{t-1}\alpha\beta^{t-r-1}\sum_{j=1}^{n}|e_{j,r-1}(\theta)| \\
&\quad + |e_{i,t-1}(\theta)| + \frac{1}{n}\sum_{j=1}^{n}|e_{j,t-1}(\theta)|
\end{aligned}$$

Since β is less than one and $e_{i,t}(\theta)$ converges to zero almost surely for all i as time goes on, we have:

$$\lim_{t\to\infty}\alpha\beta^{t-1}\sum_{j=1}^{n}|\mu_{j,0}(\theta)| = 0$$

$$\lim_{t\to\infty}\left(|e_{i,t-1}(\theta)| + \frac{1}{n}\sum_{j=1}^{n}|e_{j,t-1}(\theta)|\right) = 0$$

Furthermore, $\lim_{t\to\infty}\alpha\beta^{t-r-1} = 0$ for any r. Since $\sum_{r=1}^{t-1}\alpha\beta^{t-r-1} < \dfrac{\alpha}{1-\beta}$ and $\sum_{j=1}^{n}|e_{j,r-1}(\theta)| \to 0$ almost surely as $r \to \infty$, by Toeplitz lemma *, we have

$$\lim_{t\to\infty}\sum_{r=1}^{t-1}\alpha\beta^{t-r-1}\sum_{j=1}^{n}|e_{j,r-1}(\theta)| = 0.$$

Thus,

$$\lim_{t\to\infty}|\mu_{i,t}(\theta) - z_t(\theta)| = 0$$

which implies $\mu_{i,t}(\theta) - \mu_{j,t}(\theta) \to 0$ almost surely for all i, j, and all $\theta \in \bar{\Theta}_i \cap \cdots \cap \bar{\Theta}_n$.

In order to complete the proof of Part 1, all we need to show is the existence of $\lim_{t\to\infty}\mu_{i,t}(\theta)$. It is shown in [13] that $\sum_{i=1}^{n} v_i\mu_{i,t}(\theta)$ converges almost surely, where $[v_1, \cdots, v_n]$ is the left eigenvector corresponding to the unit eigenvalue of interaction matrix A. The claim is also true in our time-varying scenario, *i.e.*, $\sum_{i=1}^{n}\mu_{i,t}(\theta)$ converges almost surely, since $[v_1, \cdots, v_n] = [1, 1, \cdots, 1]$ is an eigenvector of $A(t)$. Therefore, $\lim_{t\to\infty}\mu_{i,t}(\theta)$ exists for any i and any $\theta \in \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$, and the value does not depend on i.

Part 2 (For $\theta \notin \bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$). We now consider the case where $\theta \notin \bar{\Theta}_i$ for some agent i. Before processing the next steps of the proof, we state a lemma from [13], which shows that there should exist a long enough signal sequence that is more probable under θ^* than any other state that is not observationally equivalent to θ^*.

Lemma 4 in [13]. For any agent i and any time t, there exists a positive integer $\hat{k}_i$, a sequence of signals $\left(\hat{s}^i_{t+1}, \cdots, \hat{s}^i_{t+\hat{k}_i}\right)$, and a constant $\delta_i \in (0, 1)$ such that

$$\prod_{r=1}^{\hat{k}_i}\frac{\ell_i\left(\hat{s}^i_{t+r}|\theta\right)}{\ell_i\left(\hat{s}^i_{t+r}|\theta^*\right)} \leq \delta_i, \quad \forall\theta \notin \bar{\Theta}_i \tag{6}$$

Recall $m_{i,t}(s^i_{t+1}, \cdots, s^i_{t+k}) \to \prod_{r=1}^{k}\ell_i(s^i_{t+r}|\theta^*)$ almost surely for any natural number k. In particular, the claim is true for the sequence of signals $\left(\hat{s}^i_{t+1}, \cdots, \hat{s}^i_{t+\hat{k}_i}\right)$ satisfying (6):

* Let $\{a_{ni}\}_{n,i\geq 1}$ be a double infinite array, $\sup_n \sum_{i=1}^{n}|a_{ni}| < +\infty$, $\lim_{n\to\infty}\sum_{i=1}^{\infty}a_{ni} = a$, $|a| < \infty$. Assume that for any i, $a_{ni} \to 0$. Let b_n be a convergent sequence with limit b. Then $\sum_{i=1}^{\infty}a_{ni}b_i \to ab$ as $n \to \infty$.

$$1-\frac{m_{i,t}\left(s_{t+1}^i,\cdots,s_{t+\hat{k}_i}^i\right)}{\prod_{r=1}^{\hat{k}_i}\ell_i(s_{t+r}^i|\theta^*)}$$
$$=1-\sum_{\theta\in\Theta}\mu_{i,t}(\theta)\prod_{r=1}^{\hat{k}_i}\frac{\ell_i\left(\hat{s}_{t+r}^i|\theta\right)}{\ell_i\left(\hat{s}_{t+r}^i|\theta^*\right)}$$
$$=1-\sum_{\theta\in\bar{\Theta}_i}\mu_{i,t}(\theta)-\sum_{\theta\notin\bar{\Theta}_i}\mu_{i,t}(\theta)\prod_{r=1}^{\hat{k}_i}\frac{\ell_i\left(\hat{s}_{t+r}^i|\theta\right)}{\ell_i\left(\hat{s}_{t+r}^i|\theta^*\right)}$$
$$=\sum_{\theta\notin\bar{\Theta}_i}\mu_{i,t}(\theta)\left(1-\prod_{r=1}^{\hat{k}_i}\frac{\ell_i\left(\hat{s}_{t+r}^i|\theta\right)}{\ell_i\left(\hat{s}_{t+r}^i|\theta^*\right)}\right)\to 0\ a.s.$$

Since

$$1-\prod_{r=1}^{\hat{k}_i}\frac{\ell_i\left(\hat{s}_{t+r}^i|\theta\right)}{\ell_i\left(\hat{s}_{t+r}^i|\theta^*\right)}\geq 1-\delta_i>0\quad\forall\theta\notin\bar{\Theta}_i,$$

it must be the case that $\mu_{i,t}(\theta)\to 0$ as $t\to 0$ for any $\theta\notin\bar{\Theta}_i$.

Now, consider the belief updating rule of agent i given by (1), evaluated at some state $\theta\notin\bar{\Theta}_i$:

$$\mu_{i,t+1}(\theta)=a_{ii}(t)\mu_{i,t}(\theta)\frac{\ell_i(s_{t+1}^i|\theta)}{m_{i,t}(s_{t+1}^i)}+\sum_{j\in N_i(t)}a_{ij}(t)\mu_{j,t}(\theta)$$

We have already shown that $\mu_{i,t}(\theta)\to 0$ almost surely. Now, we prove by contradiction that any other agent j in the network also has $\mu_{j,t}(\theta)\to 0$ almost surely for the same θ, even though $\theta\in\bar{\Theta}_j$.

Suppose that there exists an agent j in the network satisfying the following statement with a positive probability: there exists a real number $\sigma>0$ satisfying that $\mu_{j,T}(\theta)>\sigma$ for an arbitrary large time T. Then, based on (1), any agent, k for instance, influenced by agent j has belief $\mu_{k,T+1}(\theta)>\eta\sigma$ at time $T+1$, because of the lower bound of positive weight in Assumption 2. Considering the joint connectivity property in Assumption 1, we know that the belief of agent j affects the belief of agent i in at most $(n-1)B$ time slots. Without loss of generality, suppose agent i is influenced directly or indirectly by agent j at time $T+s$, where $1\leq s\leq(n-1)B$. Then, agent i's belief $\mu_{i,T+s}(\theta)>\eta^s\sigma\geq\eta^{(n-1)B}\sigma$, where $\eta^{(n-1)B}\sigma$ is a positive constant. Note that $T+s$ can be arbitrarily large, which means $\mu_{i,t}(\theta)$ cannot converge to zero as time goes to infinity. This is incompatible with the fact that $\mu_{i,t}(\theta)\to 0$ almost surely. Thus, it proves the claim that any other agent j in the network must have $\mu_{j,t}(\theta)\to 0$ almost surely. Consequently, all agents assign an asymptotic belief of zero to any $\theta\notin\bar{\Theta}_1\cap\cdots\cap\bar{\Theta}_n$.

Combining the results of Part 1 and Part 2, we have shown that the beliefs of all agents converge to consensus almost surely. Moreover, the consensus belief on $\theta\notin\bar{\Theta}_1\cap\cdots\cap\bar{\Theta}_n$ is zero.

4.3 Proof of learning the true state

Recall that all agents assign an asymptotic belief of zero to $\theta\notin\bar{\Theta}_1\cap\cdots\cap\bar{\Theta}_n$. Here, we assume that there is only the true state $\theta^*\in\bar{\Theta}_1\cap\cdots\cap\bar{\Theta}_n$. This implies $\mu_{j,t}(\theta)\to 0$ for all $\theta\neq\theta^*$; therefore, $\mu_{j,t}(\theta^*)\to 1$, which means all agents learn the true state almost surely.

V. SIMULATIONS

Note that simulations of Bayesian inference involve complicated choices of parameters. Nevertheless, different choices would generally not affect simulations to an extent that changes their qualitative results. Since agents in our model conduct Bayesian inference (only the first term on the right hand side of (1)) in the same way as one standard Bayesian agent does in a personal situation, we can apply the statement in [14] to our case to support our argument. First, for the number of possible states, three (one true and two false) and countably infinite, have no essential difference. Considering the true state θ^* and any one of false states $\bar{\theta}$, we always have $\mu_{i,t}(\theta^*)/\mu_{i,t}(\bar{\theta})\to\infty$ as the amount of observation increases if the two states are not observationally equivalent, which means all of the beliefs of false states asymptotically converge to zero. Second, the choices of the signal structures do not affect the beliefs of true state converging to one, as long as no state is observationally equivalent to the true state from the point of view of all agents. Given the above considerations, we only need to keep the simulation environment as simple as possible.

We consider a simple network of three agents. The topologies of the network at different times are shown in Fig. 1. The correspondingly time-varying interaction matrices are

$$A(2t)=\begin{bmatrix}1&0&0\\0&1&0\\0.5&0&0.5\end{bmatrix},\ A(2t+1)=\begin{bmatrix}0.5&0.5&0\\0&0.5&0.5\\0&0&1\end{bmatrix}$$

for all $t\geq 0$. From Fig. 1, we know that the underlying network at each time is not strongly connected, but the union

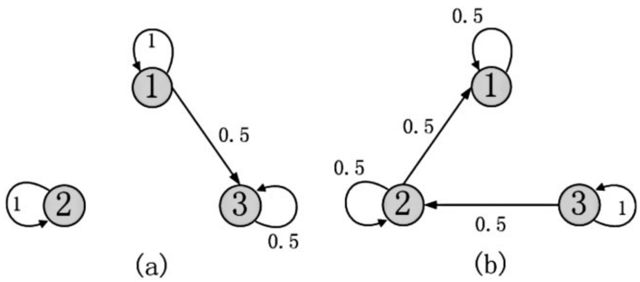

Fig. 1. (a) The topology of networks at even number of steps; (b) The topology of networks at odd number of steps.

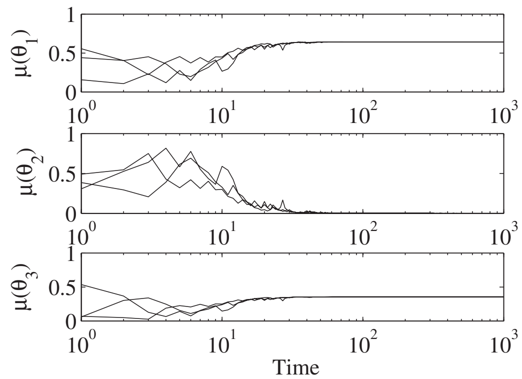

Fig. 2. Dynamic of beliefs with other states observationally equivalent to the true state.

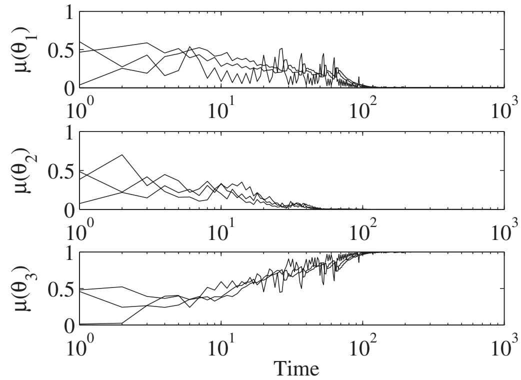

Fig. 3. Dynamic of beliefs with no other state observationally equivalent to the true state.

of networks at every two consecutive time slots is strongly connected.

Suppose that the set of states is $\Theta = \{\theta_1, \theta_2, \theta_3\}$, in which the true state $\theta^* = \theta_3$. The set of signals generated by the true state is $S = \{H, T\}$, and we assume that H appears with possibility of 20% and T with 80%. Initial beliefs of any agent i, *i.e.*, $\{\mu_{i,0}(\theta_1), \mu_{i,0}(\theta_2), \mu_{i,0}(\theta_3)\}$ are adopted randomly in interval [0, 1], satisfying $\sum_{k=1}^{3} \mu_{i,0}(\theta_k) = 1$. We perform two simulations, representing two different situations classified by whether the true state is the only state in $\bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$.

5.1 With multiple states in $\bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$

In the first simulation, we suppose that all three agents view states θ_1 and θ_3 as observationally equivalent states, while $\ell_i(H|\theta_2) = 0.5$ for i = 1, 2, 3. The corresponding belief evolution is shown in Fig. 2.

The figure is plotted on a semilog scale to illustrate both the detailed dynamic aggregation in the incipient stage and the convergence in the long run. It is shown that beliefs of all agents on θ_2 converge to zero because θ_2 is not equivalent observationally to the true state. The beliefs on the true state θ_3 and its observationally equivalent state θ_1 converge to nonzero constants, which implies that an agreement among agents is obtained, but agents cannot recognize the true state.

5.2 With only the true state in $\bar{\Theta}_1 \cap \cdots \cap \bar{\Theta}_n$

In the second simulation, we set that agent 1 and agent 2 view θ_1 and θ_3 as observationally equivalent states, while $\ell_i(H|\theta_2) = 0.5$ for i = 1, 2. From the point of view of agent 3, states θ_2 and θ_3 are observationally equivalent, while $\ell_3(H|\theta_1) = 0.8$. Therefore, the intersection of all agents' observationally equivalent states is only θ_3, the true state. Fig. 3 shows that beliefs on θ_1 and θ_2 converge to zero, and beliefs on θ_3 converge to one, which means the asymptotic learning has been achieved.

VI. CONCLUSIONS

In this paper, we extend a non-Bayesian social learning model from networks with fixed topologies to networks with time-varying topologies. With assumptions on the structure of networks, such as the existence of weight threshold and the joint connectivity of networks in finite time intervals, we have proven that agents eventually have correct forecasts for upcoming signals. Similar to most of the non-Bayesian models, consensus in beliefs of all agents can be reached in our model. Moreover, the consensus belief eventually reflects the true state if there is no state observationally equivalent to the true state.

It has been pointed out that we make the assumption of double stochasticity for technical reasons, and we conjecture that all our results still hold in its absence. The simulations support our argument to a certain extent, where the interaction matrices are not doubly stochastic. Examining the belief updating rule in more general networks is our feature work.

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