Automatica 49 (2013) 610–618

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Consensus seeking over directed networks with limited information communication☆

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ARTICLE INFO

Article history:
Received 14 June 2011
Received in revised form
5 May 2012
Accepted 24 September 2012
Available online 13 December 2012

Keywords:
Consensus
Directed network
Distributed algorithm
Multi-agent systems
Uniform quantization

ABSTRACT

Existing works that deal with the problem of distributed average consensus with quantized information communication assume that the update matrices are doubly stochastic, which amounts to agents evolving on balanced directed networks with digital channels. This paper is concerned with the problem of seeking consensus via quantized information communication over a general unbalanced directed digital network. It is established that, by designing a protocol with a finite-level uniform quantization scheme, merely one bit quantized information transmitted along each connected digital channel suffices for achieving weighted average consensus with an exponential convergence rate, as long as the directed unbalanced network is strongly connected. An explicit characterization of the convergence rate of consensus is also given. By avoiding the double stochasticity assumption on the update matrix, the proposed quantized protocol is particularly suitable for the scenarios where no bidirectional and/or balanced information communication among agents is available.

1. Introduction

In recent years, distributed consensus algorithms have received increasing attention in various contexts such as parallel computation (Tsitsiklis, 1984), distributed optimization (Nedic, Ozdaglar, & Parrilo, 2010), and cooperative control of autonomous vehicles (Jadbabaie, Lin, & Morse, 2003; Lin, Broucke, & Francis, 2004; Olfati-Saber, 2006; Olfati-Saber, Fax, & Murray, 2007; Ren & Beard, 2008; Tanner, Jadbabaie, & Pappas, 2007). Consensus algorithms involve appropriate distributed control laws that use local neighboring information to cause the states of the agents to eventually reach a common value. Special interest has been devoted to average consensus, where the agents are required to agree on the exact average or the centroid of their initial values. Such average consensus provides an elegant distributed way for computing the average of a set of measurements across a network and is of particular interest in applications such as information fusion in sensor networks (Xiao & Boyd, 2004) and load balancing in processor networks (Kashyap, Basar, & Srikant, 2007).

Network topology and the information that can be transmitted across the network are important factors that have to be considered in the analysis and design of distributed consensus algorithms. Early consensus algorithms assume that agents have access to their neighbors' information flow without distortion, that is, without any restriction on the information communicated across the links. Such an assumption of perfect communication is equivalent to the communication channels among agents having unlimited capacity and the algorithms being carried out with an infinite precision (Elia & Mitter, 2001; Fu & Xie, 2005), which are not the case as digital communication channels are subject to bandwidth and energy constraints. Indeed, in a real world scenario, information transmitted among agents of the digital networks is usually quantized prior to being communicated and agents can only exchange their symbolic data with the neighboring agents. As a result, the analysis and design of distributed consensus algorithms with quantized communication has been an active research topic (Aysal, Coates, & Rabbat, 2008; Carli & Bullo, 2009; Carli, Bullo, & Zampieri, 2010; Carli, Fagnani, Frasca, & Zampieri, 2010; Carli, Fagnani, Speranzon, & Zampieri, 2008; Dimarogonas & Johansson, 2010; Frasca, Carli, Fagnani, & Zampieri, 2008; Huang, Dey, Nair, &

☆ Work supported in part by the State Major Basic Research Development Program of China (the 973 Program) under Grant No. 2010CB731400 and in part by the Natural Science Foundation of China under grant Nos. 60928008, 61073102, 61074125 and 61273105. The material in this paper was partially presented at the 9th IEEE International Conference on Control and Automation, December 19–21, 2011, Santiago, Chile. This paper was recommended for publication in revised form by Associate Editor Huijun Gao under the direction of Editor Ian R. Petersen.

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doi:10.1016/j.automatica.2012.11.041

Manton, 2010; Kar & Moura, 2010; Kashyap et al., 2007; Li, Fu, Xie, & Zhang, 2011; Yuan, Xu, Zhao, & Chu, 2010).

Recently, by respectively introducing a zoom in–zoom out factor and a scaling function in encoding/decoding of the uniform quantization scheme, both Carli et al. (2010) and Li et al. (2011) showed that average consensus is achievable with a finite number of quantization levels. In particular, the work of Li et al. (2011) provides a way to reduce the number of transmitting bits along each digital channel to merely one bit by judiciously designing the control parameters. Extensions of Li et al. (2011) are further discussed in the presence of communication delay (Liu, Li, & Xie, 2011) and when the network topology is dynamically-switching (Li & Xie, 2011). All these works are based on bidirectional information exchange, modeled by undirected graphs.

Indeed, a critical and standard assumption that underlies the available literature on quantized average consensus is that the weight update matrices associated with the networks are doubly stochastic. In practice, this translates into requiring information digraphs being balanced (Gharesifard & Cortes, 2010; Olfati-Saber & Murray, 2004). As argued in Benezit, Blondel, Thiran, Tsitsiklis, and Vetterli (2010), networks operating under adverse conditions may be prone to packet losses or node failure, which may render symmetric exchange protocols and bidirectional communication infeasible. Thus, the balanced network topology is very difficult to enforce in a distributed manner over directed networks using unidirectional communication, and double stochasticity implies more stringent restrictions on the communication protocols. Because of the potential asymmetry in pairwise information communication between different agents, the asymptotic value of consensus is not guaranteed to be the exact average of their initial states. Instead, agents may asymptotically agree on a weighted average of their initial states, if the corresponding information digraph is strongly connected (Ren & Beard, 2005; Touri & Nedic, 2011).

The objective of this paper is to extend the results of Li et al. (2011) for undirected networks to general case of directed networks, not necessarily symmetric and balanced, and to present conditions for achieving weighted average consensus. We assume that the digraph is strongly connected but not necessarily balanced, and thus the proposed protocol admits the update matrices that need not be doubly stochastic. Allowing unidirectional information transmission, rather than requiring bidirectional information exchange, increases robustness to communication failures, and potentially reduces the communication energy and the amount of information flow.

We will propose a quantized consensus algorithm based on dynamic encoding and decoding, under which, each agent (maybe nonreciprocally) sends merely one bit of quantized information to each of its neighbors and itself will cause the network to achieve weighted average consensus exponentially. An explicit expression of the convergence rate of consensus will also be given. In Li et al. (2011), average consensus can be achieved with an exponential convergence rate only when each pair of neighboring agents reciprocally sends one bit of quantization information to each other. Also in contrast to Li et al. (2011), where the eigenvalue spectrum of the Laplacian matrix and the matrix decomposition method are employed to derive stability conditions for the closed-loop system, in this paper, motivated by recent work of Touri and Nedic (2011) and by an intricate interpretation of the left eigenvector associated with the maximal eigenvalue of the weight update matrix of the digraph, an alternative convergence analysis method is developed based on the Lyapunov technique. The construction of the Lyapunov function is intimately related to topology properties of the directed network. Thus, the obtained results address the distributed consensus problem by simultaneously taking both network topology and quantized information communication into consideration.

In Wu (2005), Fiedler's notion of algebraic connectivity is generalized from undirected to directed graphs. Several properties of Fiedler's definition are shown to remain valid for directed graphs and some intrinsic properties of directed graphs are presented. The results of Wu (2005) imply that algebraic graph theory, especially the algebraic spectral theory of undirected graphs, cannot be trivially generalized to directed graphs. As a result, it is not obvious how the method of Li et al. (2011), which heavily utilizes the algebraic spectral theory, can be applied to directed graphs we consider in this paper.

The work of this paper is closely related to Cai and Ishii (2011), where the asynchronous quantized gossip consensus problem is considered under a setting in which the states of agents are integers and the networks are directed and randomized. The main objective of Cai and Ishii (2011) is to derive connectivity conditions on graphs that ensure weighted average consensus. On the other hand, in this paper, we explicitly design, under the assumption that the directed network is strongly connected, a suitable synchronous protocol with a finite-level uniform quantization scheme to guarantee weighted average consensus.

The remainder of this paper is organized as follows. In Section 2, we recall some basic concepts from graph theory. In Section 3, we formally state the problem to be studied and propose the distributed protocol with a finite-level uniform quantization scheme. In Section 4, we prove the weighted average preservation and establish the convergence results. Simulation results are given in Section 5. Section 6 draws the conclusions to the paper.

We will use standard notation. For a positive number a, the maximum integer less than or equal to a is denoted by $\lfloor a \rfloor$. The transposes of a vector v and a matrix M are denoted by v^T and M^T, respectively. For a vector or matrix A, its ∞-norm is denoted by $\|A\|_\infty$ and its Euclidean norm is denoted by $\|A\|_2$. For a vector v and a positive definite matrix D of appropriate dimensions, $\|v\|_D = \|v^T D v\|_2$. For a set $\mathcal{N}$, the number of the elements in $\mathcal{N}$ is denoted by $|\mathcal{N}|$.

2. Preliminary results

A square matrix $W = (w_{ij}) \in R^{N\times N}$ is said to be nonnegative if its entries $w_{ij} \geq 0$ for all i and j. W is said to be stochastic if it is nonnegative and satisfies $W\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = [1\ 1\ \cdots\ 1]^T \in R^N$. Moreover, W is said to be doubly stochastic if both W and W^T are stochastic.

The information flow communicated between agents of a fixed network can be modeled by a digraph $\mathbb{G} = (\mho, \mathscr{E}, W)$, where $\mho = \{1, 2, \ldots, N\}$ is the nonempty set of N nodes and each node denotes an agent, $\mathscr{E} = \{(i, j) : i, j \in \mho\}$ denotes the directed edge set. A directed edge $(i, j) \in \mathscr{E}$ means that agent j can obtain information from agent i, that is agent i is an in-neighbor of agent j, but not necessarily vice versa. The in-neighbor set of agent j is denoted as $\mathcal{N}_j = \{i \in \mho, i \neq j, (i, j) \in \mathscr{E}\}$. A directed path in a digraph is a sequence of edges in a digraph of the form of $(i_1, i_2), (i_2, i_3), \ldots$, where $i_s \in \mho$. A digraph $\mathbb{G}$ is strongly connected if for any distinct agents i and j, there exists a directed path that connects i and j. $W = (w_{ij}) \in R^{N\times N}$ is the stochastic weight adjacent matrix, with its elements w_{ij} encoding the relative confidence of each agent's information or relative reliability of different communication or sensing links, and $w_{ij} > 0 \iff (j, i) \in \mathscr{E}$, $w_{ij} = 0$ otherwise. We assume that each agent contains a self-loop, that is $w_{ii} > 0$ for $\forall i \in \mho$. Such an assumption is realistic as each agent has access to its own information. For $\forall i \neq j$, if $(i, j) \in \mathscr{E}$ implies $(j, i) \in \mathscr{E}$, the graph is said to be undirected. Otherwise, it is directed. The in-degree and out-degree of agent i are, respectively, defined as $\deg_{\text{in}}(i) = \sum_{j=1}^{N} w_{ij}$ and $\deg_{\text{out}}(i) = \sum_{j=1}^{N} w_{ji}$. A digraph is said to be balanced if the in-degree and the out-degree are equal at each of its nodes. Hence, a digraph is balanced if its weight adjacent matrix

W is doubly stochastic. It is noted that the weight matrix W associated with an undirected graph is symmetric and doubly stochastic.

Strong connectivity of the digraph $\mathbb{G}$ is equivalent to the irreducibility of the weight adjacent matrix W associated with it (Horn & Johnson, 1985), *i.e.*, there is no permutation matrix M such that $W = M \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} M^T$.

3. Consensus seeking with limited-information communication

Consider a directed network composing of N agents,

$$x_i(k+1) = x_i(k) + u_i(k), \quad k = 0, 1, \ldots, i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(k) \in R$ and $u_i(k) \in R$ are the state and control input of agent i. We make the following assumption on the underlying information digraph $\mathbb{G} = (\mho, \mathscr{E}, W)$.

Assumption 1. The unbalanced digraph $\mathbb{G}$ is strongly connected. There exists a constant $\rho > 0$ such that $w_{ii} = 1 - \sum_{j \in \mathcal{N}_i} w_{ij} \geq \rho$ for all $i \in \mho$, and $w_{ij} \in \{0\} \bigcup [\rho, 1]$ for all $i \neq j$.

If W is a stochastic matrix with positive diagonal elements such that the graph $\mathbb{G}$ is strongly connected, then W is primitive and 1 is a simple and maximal eigenvalue of W with algebraic multiplicity one. Furthermore, by the Perron–Frobenius Theorem (Horn & Johnson, 1985), there exists a unique normalized positive left eigenvector $\pi = [\pi_1 \ \pi_2 \ \cdots \ \pi_N]^T$ corresponding to eigenvalue 1 such that

$$\pi^T W = \pi^T, \qquad \pi^T \mathbf{1} = \sum_{i=1}^{N} \pi_i = 1, \tag{2}$$

and $\lim_{k\to\infty} W^k = \mathbf{1}\pi^T$. Here, by positive left eigenvector π we mean that $\pi_i > 0$ for $i = 1, 2, \ldots, N$. In addition, if the ith component in π is positive, then agent i has some influence on the consensus value and the influence is quantified by the corresponding component of π (Ren & Beard, 2005; Wieland, Kim, Scheu, & Allgower, 2008). Hence, the left eigenvector π reveals the fundamental topology properties of the directed network. For a balanced digraph, $\pi_1 = \pi_2 = \cdots = \pi_N = \frac{1}{N}$, which means that all agents have equal influence on the network.

In this paper, we aim to design a distributed protocol $u_i(k)$ with a finite-level uniform quantization scheme, such that the N agents as described by (1) achieve weighted average consensus, that is, for any initial values $x_i(0)$,

$$\lim_{k\to\infty} x(k) = \left(\sum_{i=1}^{N} \pi_i x_i(0) \right) \mathbf{1}, \tag{3}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_N(k)]^T$.

3.1. Finite-level uniform quantization scheme

In a digital communication network, only symbolic data can be communicated between agents through digital channels. Thus for each digital channel, the state of the sender is first encoded into symbolic data and then transmitted. After the data is received, the receiver uses a decoder to get an estimate of the sender's state. For any $i, j \in \mho$, if the edge weight $w_{ij} > 0$, *i.e.*, $j \in \mathcal{N}_i \cup \{i\}$, then agent j sends its information to agent i, and the encoder ϕ_j associated with j for the directed digital channel (j, i) is defined as Li et al. (2011),

$$\begin{cases} \xi_j(0) = 0, \\ \xi_j(k) = \xi_j(k-1) + g(k-1)\Delta_j(k), \\ \Delta_j(k) = q\left(\dfrac{x_j(k) - \xi_j(k-1)}{g(k-1)} \right), \qquad k = 1, 2, \ldots, \end{cases} \tag{4}$$

where $x_j(k)$ is the state of agent j, $\xi_j(k)$ is the internal state of encoder ϕ_j, the symbolic data $\Delta_j(k)$ is the output of the encoder ϕ_j and will be broadcast through the digital channels, and $q(\cdot)$ is a finite-level uniform quantizer, in which $g(k) = g_0\gamma^k$ is a scaling function with $g_0 > 0$ and $\gamma \in (0, 1)$ being design parameters. Here, the scaling function $g(k)$ is used to avoid the saturation of the finite-level quantizer.

When agent i receives the symbolic data $\Delta_j(k)$, the following decoder φ_{ji} of agent i associated with the directed digital channel (j, i) estimates $x_j(k)$,

$$\begin{cases} \hat{x}_{ji}(0) = 0, \\ \hat{x}_{ji}(k) = \hat{x}_{ji}(k-1) + g(k-1)\Delta_j(k), \quad k = 1, 2, \ldots, \end{cases} \tag{5}$$

where $\hat{x}_{ji}(k)$, the output of the decoder φ_{ji}, is an estimate of $x_j(k)$ obtained by agent i.

In (4), the function $q : R \to \Gamma$ represents a uniform symmetric quantizer with finite levels, which maps from R to a discrete set of quantized levels Γ and is defined as:

$$q(m) = \begin{cases} 0, & \text{if } -\dfrac{1}{2} < m < \dfrac{1}{2}, \\ i, & \text{if } \dfrac{2i-1}{2} \leq m < \dfrac{2i+1}{2}, \\ \quad i = 1, 2, \ldots, K-1, & \\ K, & \text{if } m \geq \dfrac{2K-1}{2}, \\ -q(-m), & \text{if } m < -\dfrac{1}{2}, \end{cases} \tag{6}$$

where K is a positive integer. Clearly, the set of quantization levels is $\Gamma = \{0, \pm 1, \pm 2, \ldots, \pm K\}$, and the number of quantization levels (data rate) is $2K + 1$. When $K = 1$, the quantizer is referred to as a one-bit uniform quantizer (Li et al., 2011).

Remark 1. Instead of quantizing $x_j(k)$ directly at each time instant, the uniform quantizer quantizes the "prediction error" $x_j(k) - \xi_j(k-1)$ so a fewer number of bits will be needed (Carli et al., 2010; Li et al., 2011). Meanwhile, (4) and (5) imply that $\hat{x}_{ji}(k) = \xi_j(k)$ for all $i \in \mho, j \in \mathcal{N}_i \cup \{i\}, k = 0, 1, 2, \ldots$, which means that agent i has $|\mathcal{N}_i| + 1$ decoders. In Li et al. (2011), agent i, in the absence of self-loop, has $|\mathcal{N}_i|$ decoders.

3.2. Quantized protocol with limited data rate

With the limited quantized information communication, we propose the following distributed protocol with a finite-level uniform quantization scheme for agents (1),

$$u_i(k) = \alpha \sum_{j \in \mathcal{N}_i} w_{ij} \left(\hat{x}_{ji}(k) - \hat{x}_{ii}(k) \right), \quad i = 1, 2, \ldots, N, \tag{7}$$

where w_{ij} are the entries of the stochastic weight adjacent matrix W, and α is a known control gain that satisfies $\alpha \in (0, 1]$. From (4) to (6), it can be seen that the quantized protocol (7) is characterized by the scaling function $g(k) = g_0\gamma^k$ and the finite quantization level parameter K. As will be seen later, reducing the value of α leads to fewer bits of quantized information that need to be transmitted.

Rewrite (7) as,

$$\begin{aligned} u_i(k) &= \alpha \sum_{j \in \mathcal{N}_i} w_{ij}(\hat{x}_{ji}(k) - \hat{x}_{ii}(k)) \\ &= \alpha \sum_{j \in \mathcal{N}_i} w_{ij}[\hat{x}_{ji}(k) - x_j(k) + x_j(k) \\ &\quad - x_i(k) + x_i(k) - \hat{x}_{ii}(k)] \\ &= \alpha \sum_{j \in \mathcal{N}_i} w_{ij}[x_j(k) - x_i(k)] - \alpha \sum_{j \in \mathcal{N}_i} w_{ij}[x_j(k) - \hat{x}_{ji}(k)] \\ &\quad + \alpha \sum_{j \in \mathcal{N}_i} w_{ij}[x_i(k) - \hat{x}_{ii}(k)]. \end{aligned} \tag{8}$$

It can be seen that protocol (7) consists of three terms. The first term, $\alpha\sum_{j\in\mathcal{N}_i} w_{ij}[x_j(k)-x_i(k)]$, is the control input in the absence of quantized communication and plays the main role. The second term, $-\alpha\sum_{j\in\mathcal{N}_i} w_{ij}[x_j(k)-\hat{x}_{ji}(k)]$, is the aggregated estimation errors due to the estimation of the states $x_j(k)$ by agent i. The third term, $\alpha\sum_{j\in\mathcal{N}_i} w_{ij}[x_i(k)-\hat{x}_{ii}(k)] = \alpha(1-w_{ii})[x_i(k)-\hat{x}_{ii}(k)]$, represents the estimation error of the state $x_i(k)$ by agent i through its self-loop.

Remark 2. For the case of an undirected network, where each agent has no self-loop and information is transmitted through each connected digital channel bidirectionally, the following Laplacian-based quantized protocol was proposed in Li et al. (2011),

$$\begin{aligned}
u_i(k) &= h\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{x}_{ji}(k)-\xi_i(k)) \\
&= h\sum_{j\in\mathcal{N}_i} a_{ij}[\hat{x}_{ji}(k)-x_j(k)+x_j(k) \\
&\quad -x_i(k)+x_i(k)-\hat{x}_{ij}(k)] \\
&= h\sum_{j\in\mathcal{N}_i} a_{ij}[x_j(k)-x_i(k)] - h\sum_{j\in\mathcal{N}_i} a_{ij}[x_j(k)-\hat{x}_{ji}(k)] \\
&\quad + h\sum_{j\in\mathcal{N}_i} a_{ji}[x_i(k)-\hat{x}_{ij}(k)],
\end{aligned} \tag{9}$$

where h is the control gain to be determined and $a_{ij}=a_{ji}$ are the entries of the Laplacian matrix $\mathscr{L}$ associated with the undirected graph $\mathbb{G}$. Correspondingly, protocol (9) also consists of three terms. The first two terms play the same roles as their counterparts in (8). The third term, $h\sum_{j\in\mathcal{N}_i} a_{ji}[x_i(k)-\hat{x}_{ij}(k)]$, represents the aggregated estimation errors due to the estimation of the state $x_i(k)$ by agent i's neighbors. This term, referred to as the error-compensation term (Li et al., 2011), plays an important role in the protocol and is mainly based on the critical assumption that $a_{ij}=a_{ji}$. By the symmetry of the whole network, this further implies that the quantized protocol (9) has a symmetric error-compensation mechanism that preserves the average invariance of the whole network, that is, $\frac{1}{N}\sum_{j=1}^{N} x_j(k+1)=\frac{1}{N}\sum_{j=1}^{N} x_j(k)$, $k=0,1,2,\ldots$. This mechanism was first considered in Xiao and Boyd (2004) and is the key to recent results about quantized average consensus (Carli et al., 2010, 2008; Li et al., 2011; Li & Xie, 2011; Liu et al., 2011). It is noted that $(i,j)\in\mathscr{E}$ does not imply $(j,i)\in\mathscr{E}$ for a directed network. Therefore, the last term in (8) clearly shows that our protocol (7) does not rely on the symmetric error-compensation mechanism as with the case of undirected networks.

Let $\xi(k)=[\xi_1(k)\ \xi_2(k)\ \cdots\ \xi_N(k)]^T$. Then, the estimation error $e(k)=x(k)-\xi(k)\in R^N$. Recalling the fact that $\xi_j(k)=\hat{x}_{ji}(k)$ for all $(j,i)\in\mathscr{E}$ and for all k, and applying the control input (7) to agents (1), we have the following dynamics in a compact form,

$$x(k+1)=Px(k)+\alpha(I-W)e(k), \tag{10}$$

where $P\triangleq(1-\alpha)I+\alpha W=(p_{ij})\in R^{N\times N}$ with $\alpha\in(0,1]$ is the update matrix and I is the identity matrix.

Remark 3. Since I and W are stochastic, the update matrix P is also stochastic and induces the same digraph as W, but with different edge weights. Thus, if the stochastic matrix W satisfies Assumption 1, so does the stochastic matrix P. In particular, we have $p_{ij}\geq\alpha\rho$ whenever $w_{ij}>0$. Meanwhile, it follows from (10) that, with the proposed protocol (7), we have

$$\pi^T x(k+1)=\pi^T Px(k)+\alpha\pi^T(I-W)e(k)=\pi^T x(k). \tag{11}$$

That is, the closed-loop system preserves the weighted average invariance of the network (Touri & Nedic, 2011), which, by Remark 2, includes average invariance as a special case.

Let the consensus error $\delta(k)=(I-\mathbf{1}\pi^T)x(k)$. Then,

$$\delta(k+1)=P\delta(k)+\alpha(I-W)e(k), \tag{12}$$

and

$$\begin{aligned}
e(k+1) &= \alpha(W-I)\delta(k)+((1+\alpha)I-\alpha W)e(k)-g(k) \\
&\quad\times Q\left(\frac{\alpha(W-I)\delta(k)+((1+\alpha)I-\alpha W)e(k)}{g(k)}\right),
\end{aligned} \tag{13}$$

for the consensus error and the estimation error respectively, where each element of the vector quantizer $Q([m_1\ m_2\ \cdots\ m_N]^T)=[q(m_1),\ q(m_2),\ldots,q(m_N)]^T$ is as defined in (6).

The problem of consensus seeking on strongly connected directed networks with finite-level quantized communication is then the following: select suitable quantizer parameters g_0, γ and K such that, under the proposed protocol (7), all agents will reach weighted average consensus, that is, (3) holds. This problem is solved if and only if the consensus error dynamics (12) is asymptotically stable at the origin $\delta=0$.

Remark 4. Protocol (9) indicates that a special case of Laplacian-based symmetric update matrix $P=I-h\mathscr{L}$ is adopted in Li et al. (2011). As explained earlier, unidirectional and unbalanced information communication being considered in this paper leads to an asymmetric update matrix. As a result, the tools of Li et al. (2011), which are specific to symmetric update matrices, are not applicable in our current situation. The only assumption we make in this paper is that the stochastic weight matrix W associated with the directed network has positive diagonal entries $w_{ii}>0$, $i\in\mho$. It is of interest to note that, in Li et al. (2011), if the condition $h\in\left(0,\frac{2}{\lambda_N(\mathscr{L})}\right)$ is met, the stochastic update matrix $P=I-h\mathscr{L}$ satisfies that $p_{ii}>0$, $i\in\mho$ (Olfati-Saber & Murray, 2004; Xiao & Boyd, 2004). Thus, our quantized protocol (7) requires a milder restriction on network topology. In view of Remarks 2 and 3, our work is distinct from the work in Li et al. (2011) in terms of both the network topology and the nature of the protocol itself.

We next establish a lemma, which is instrumental to establishing our main results in Section 4.

Consider the following Lyapunov function $V(x(k))$, which measures the weighted spread of the components of vector $x(k)$ with respect to the weighted average value $\pi^T x(k)$,

$$\begin{aligned}
V(x(k)) &= x^T(k)(I-\pi\mathbf{1}^T)D(I-\mathbf{1}\pi^T)x(k) \\
&= x^T(k)(D-\pi\pi^T)x(k) \\
&= \sum_{i=1}^{N}\pi_i\left(x_i(k)-\pi^T x(k)\right)^2 \\
&= \|\delta(k)\|_D^2,
\end{aligned} \tag{14}$$

where the diagonal matrix $D=\mathrm{diag}\{\pi_1,\pi_2,\ldots,\pi_N\}$. The second equality in (14) is obtained by using $\mathbf{1}^T D=\pi^T$, $D\mathbf{1}=\pi$ and $\pi^T\mathbf{1}=1$. With the novel concept of infinite flow, Ref. (Touri & Nedic, 2011) constructs the function $V(x(k))$ to study the consensus problem over *randomized* directed networks without quantized communication. Here, we establish the following lemma on the evolution of $V(x(k))$ for *deterministic* directed networks without quantized communication.

Lemma 1. *Consider the closed-loop system without quantized communication, that is*

$$x(k+1)=((1-\alpha)I+\alpha W)x(k)=Px(k). \tag{15}$$

Suppose that Assumption 1 holds. Then,

$$V(x(k+1))\leq\left(1-\frac{\eta}{2(N-1)}\right)V(x(k)),\quad \forall x(k)\in R^N, \tag{16}$$

where $1>\eta=\alpha\rho\pi_{\min}>0$ and $\pi_{\min}=\min_{1\leq i\leq N}\pi_i$.

This lemma characterizes the asymptotic convergence of consensus error $\delta(k)$ and is essentially built upon Lemma 5 of Nedic, Olshevsky, Ozdaglar, and Tsitsiklis (2009) and Theorem 5 of Touri and Nedic (2011). The undirected graph case was studied in Theorem 1 of Touri, Nedic, and Ram (2010). This lemma also provides an upper bound for the rate of convergence of consensus in the absence of quantized communication. However, this bound is conservative since it is based on a worst-case scenario.

Proof. In view of (14) and (15), we have

$$\begin{aligned} V(x(k+1)) &= x^T(k+1)\left(D-\pi\pi^T\right)x(k+1) \\ &= x^T(k)P^T\left(D-\pi\pi^T\right)Px(k) \\ &= -\sum_{i<j} H_{ij}\left(x_i(k)-x_j(k)\right)^2 + V(x(k)), \end{aligned} \tag{17}$$

where H_{ij} is the ijth-entry of the symmetric matrix $H = P^TDP$, that is, $H_{ij} = \sum_{l=1}^N \pi_l p_{li} p_{lj}$. The third equality in (17) results from relationship (15) and is similar to Theorem 5 of Touri and Nedic (2011).

Since the digraph $\mathbb{G}$ is strongly connected, the associated weight adjacent matrix P with positive diagonal entries $p_{ii} > 0$ is irreducible. By its irreducibility, the stochastic matrix P cannot be transformed into lower-triangular form by symmetric permutation, that is, there always exists an edge $(j, i) \in \mathscr{E}$ such that $p_{ij} \geq \alpha\rho > 0 (j > i)$. Since the effect of a symmetric permutation is simply to relabel nodes on the digraph, and in view of $H_{ij} = \sum_{l=1}^N \pi_l p_{li} p_{lj} \geq \pi_{\min}\left(p_{ii}p_{ij} + p_{jj}p_{ji}\right) \geq \pi_{\min} p_{ii} p_{ij}$, we have $H_{ij} > 0$ for $j > i$ and $(j, i) \in \mathscr{E}$. On the other hand, if $j < i$ and $(i, j) \in \mathscr{E}$, thus $p_{ji} \geq \alpha\rho > 0$, we similarly have $H_{ij} \geq \pi_{\min} p_{jj} p_{ji} > 0$. Then by the fact that H is symmetric, we can conclude that $H_{ij} > 0 (i \neq j)$ whenever $(j, i) \in \mathscr{E}$ or $(i, j) \in \mathscr{E}$. Let $\tau = \min_{(j,i)\in\mathscr{E}, i\neq j} H_{ij}$, then $\tau \geq \alpha^2\rho^2\pi_{\min} > 0$.

For all $d \in \mathscr{F} = \{1, 2, \dots, N-1\}$, let $C_d = \{(i, j) \in \mathscr{E}, (j, i) \in \mathscr{E} | i \leq d, d+1 \leq j\}$, and $F_{ij} = \{d \in \mathscr{F} | (i, j) \in C_d \text{ or } (j, i) \in C_d\}$, which consists of all "cuts" d such that the edge (i, j) or (j, i) communicates across these cuts. By the strong connectivity of the digraph $\mathbb{G}$, for any $d \in \mathscr{F}$, we either have $x_d(k) = x_{d+1}(k)$, or there exist $i \leq d$ and $j \geq d+1$ such that $(j, i) \in \mathscr{E}$ or $(i, j) \in \mathscr{E}$. Note the fact that $H_{ij} > 0 (i \neq j)$ whenever $(j, i) \in \mathscr{E}$ or $(i, j) \in \mathscr{E}$, which implies that all the conditions of Lemma 5 of Nedic et al. (2009) are satisfied, and hence, if $(i, j) \in C_d$ for any $d \in \mathscr{F}$, there holds $\sum_{(i,j)\in C_d} H_{ij} \geq \frac{\eta}{2}$.

Moreover, letting $x(k) \in R^N$ be such that $x_1(k) \leq x_2(k) \leq \cdots \leq x_N(k)$, then as in the proof of Lemma 8 of Nedic et al. (2009), we have

$$\left(x_i(k) - x_j(k)\right)^2 \geq \sum_{d\in F_{ij}} \left(x_{d+1}(k) - x_d(k)\right)^2. \tag{18}$$

Together with the fact that $\sum_{(i,j)\in C_d} H_{ij} \geq \frac{\eta}{2}$, we further obtain

$$\begin{aligned} \sum_{i<j} H_{ij}\left(x_i(k)-x_j(k)\right)^2 &\geq \sum_{i<j} H_{ij} \sum_{d\in F_{ij}} \left(x_{d+1}(k)-x_d(k)\right)^2 \\ &= \sum_{d\in\mathscr{F}} \sum_{(i,j)\in C_d} H_{ij}(x_{d+1}(k)-x_d(k))^2 \\ &\geq \frac{\eta}{2} \sum_{d\in\mathscr{F}} (x_{d+1}(k)-x_d(k))^2 \\ &= \frac{\eta}{2} \sum_{d=1}^{N-1} (x_{d+1}(k)-x_d(k))^2. \end{aligned} \tag{19}$$

Furthermore, in view of the fact that π satisfies $\sum_{i=1}^N \pi_i = 1$ and $\pi_i > 0$, there holds $x_N(k) \geq \pi^T x(k) \geq x_1(k)$, and hence

$$V(x(k)) = \sum_{i=1}^N \pi_i \left(x_i(k) - \pi^T x(k)\right)^2 \leq \left(x_N(k) - x_1(k)\right)^2. \tag{20}$$

Noting that $x_N(k) - x_1(k) = \sum_{d=1}^{N-1} (x_{d+1}(k) - x_d(k))$ and making use of the convexity of the squared-norm, we have

$$\begin{aligned} (x_N(k) - x_1(k))^2 &= (N-1)^2 \left(\frac{1}{N-1}\sum_{d=1}^{N-1}(x_{d+1}(k) - x_d(k))\right)^2 \\ &\leq (N-1)\sum_{d=1}^{N-1}\left(x_{d+1}(k) - x_d(k)\right)^2, \end{aligned} \tag{21}$$

which, along with (17), (19) and (20), lead to (16). □

4. Convergence analysis

Our main results on the proposed protocol with limited quantized information communication and the resulting closed-loop system (10), (12) and (13) will be presented in Theorem 1, which answers the following three questions: (A) How to select the quantizer parameters g_0, γ and K such that all agents achieve the weighted average consensus under the proposed protocol (7)? (B) How many bits are required for each agent to transmit along each directed digital channel to its neighboring agent and itself? and (C) What is the relationship between the convergence rate of consensus and the scaling function?

Assumption 2. The agents' initial states $x(0)$ satisfy $\|x(0)\|_\infty = \max_{1\leq i\leq N} |x_i(0)| \leq C_x$, and the initial consensus errors $\delta(0)$ satisfy $\|\delta(0)\|_\infty = \max_{1\leq i\leq N} |\delta_i(0)| \leq C_\delta$, for some known constants C_x and C_δ.

Theorem 1. *Let Assumptions 1 and 2 be satisfied. For the normalized positive left eigenvector π satisfying (2), let $\pi_{\max} = \max_{1\leq i\leq N}\pi_i$. For any $\gamma \in (\rho_\eta, 1)$ with $\rho_\eta = \left(1 - \frac{\eta}{2(N-1)}\right)^{\frac{1}{2}}$, let*

$$M_1(\alpha, \gamma) = \frac{2\sqrt{2N}\alpha^2\pi_{\max}}{\pi_{\min}\gamma\left(\gamma - \rho_\eta\right)} + \frac{1+2\alpha}{2\gamma}, \tag{22}$$

and

$$K_1(\alpha, \gamma) = \left\lfloor M_1(\alpha, \gamma) - \frac{1}{2} \right\rfloor + 1. \tag{23}$$

For any given $K \geq K_1(\alpha, \gamma)$, take the scaling function $g(k) = g_0\gamma^k$ with g_0 satisfying

$$g_0 \geq \max\left\{\frac{C_x}{K+\frac{1}{2}}, \frac{(2\alpha C_x + \gamma C_\delta)(\gamma - \rho_\eta)}{\alpha}\right\}. \tag{24}$$

Then, under the proposed protocol (7) with the $(2K+1)$-level uniform quantizer (6), the closed-loop system (10), (12) and (13) achieves weighted average consensus, that is

$$\lim_{k\to\infty} x_i(k) = \sum_{j=1}^N \pi_j x_j(0), \quad i = 1, 2, \dots, N. \tag{25}$$

Moreover,

$$\lim_{\alpha\to 0, \gamma\to 1} M_1(\alpha, \gamma) = \frac{1}{2}, \tag{26}$$

that is, the lowest bound of the number of transmitting bits required for each connected directed digital channel is 1*. Furthermore, let the convergence rate of consensus* r_{asym} *be defined as*

$$r_{\text{asym}} = \sup_{x(0)\neq \mathbf{1}\pi^T x(0)} \lim_{k\to\infty} \left(\frac{\|x(k) - \mathbf{1}\pi^T x(k)\|_D}{\|x(0) - \mathbf{1}\pi^T x(0)\|_D} \right)^k,$$

then,

$$r_{\text{asym}} \le \gamma. \tag{27}$$

Proof. Let

$$y(k) = \frac{\delta(k)}{g(k)}, \qquad z(k) = \frac{e(k)}{g(k)},$$

where $g(k) = g_0\gamma^k$, with g_0 being chosen as in (24). Then, we obtain from (12) that,

$$y(k+1) = \gamma^{-1}\left(Py(k) + \alpha(I-W)z(k)\right), \tag{28}$$

and

$$z(k+1) = \gamma^{-1}\beta(k), \tag{29}$$

where $\beta(k)$ is defined as $\beta(k) := \tilde{e}(k) - q(\tilde{e}(k))$ with $\tilde{e}(k) = \frac{x(k+1)-\xi(k)}{g(k)} = \alpha(W-I)y(k) + ((1+\alpha)I - \alpha W)z(k)$.

Now, we are in the position to prove that, when the $(2K+1)$-level uniform quantizer (6) with any $K > K_1(\alpha,\gamma)$, g_0 satisfying (24) and $\gamma \in (\rho_\eta, 1)$ is applied for each connected digital channel, then under the quantized protocol (7), $\tilde{e}(k)$ will never cause the quantizers to saturate, that is $\|z(k)\|_\infty \le \frac{1}{2\gamma}$, which, in view of (6), is equivalent to $\|\tilde{e}(k)\|_\infty \le K + \frac{1}{2}$ for all k. We will prove this fact by induction.

Indeed, when $k = 0$, since $\hat{x}(0) = 0$, $e(0) = x(0) - \hat{x}(0) = x(0)$. Then, by Assumption 2 and (24), we have

$$\|\tilde{e}(0)\|_\infty \le K + \frac{1}{2}. \tag{30}$$

Therefore, when $k = 0$, the uniform quantizer is not saturated. For any given nonnegative integer n, suppose that when $k = 1, 2, \ldots, n$, the quantizer associated with each connected digital channel is unsaturated, thus the quantization error $\beta(k)$ of $\tilde{e}(k)$ satisfies $\|\beta(k)\|_\infty \le \frac{1}{2}$, $k = 0, 1, 2, \ldots, n$. It then follows from (29) that

$$\|z(k)\|_\infty \le \frac{1}{2\gamma}, \quad k = 1, 2, \ldots, n+1. \tag{31}$$

We next show that $\|\tilde{e}(n+1)\|_\infty \le K + \frac{1}{2}$. Noting that P is also a stochastic matrix with positive diagonal entries, we obtain from (28) that

$$\begin{aligned}
y^T(n+1)Dy(n+1) &= \gamma^{-2}y^T(n)P^TDPy(n) \\
&\quad + \gamma^{-2}z^T(n)(\alpha(I-W))^TD(\alpha(I-W))z(n) \\
&\quad + 2\gamma^{-2}y^T(n)P^TD(\alpha(I-W))z(n) \\
&\overset{(a)}{\le} 2\gamma^{-2}\left[y^T(n)P^TDPy(n)\right. \\
&\quad \left. + z^T(n)(\alpha(I-W))^TD(\alpha(I-W))z(n)\right] \\
&\overset{(b)}{\le} 2\gamma^{-2}\left(1 - \frac{\eta}{2(N-1)}\right)y^T(n)Dy(n) \\
&\quad + 2\gamma^{-2}z^T(n)(\alpha(I-W))^TD(\alpha(I-W))z(n).
\end{aligned} \tag{32}$$

In the above derivation, to obtain inequality (a), we have used the inequality $2\mathbf{a}^T\mathbf{b} \le \mathbf{a}^T\mathbf{a} + \mathbf{b}^T\mathbf{b}$ for any column vectors $\mathbf{a}$ and $\mathbf{b}$ of the same dimension. Inequality (b) follows (16).

Recall that, for any S column vectors of the same dimension $\mathbf{a}_s$, $s = 1, 2, \ldots, S$, the following inequality holds $\sqrt{\sum_{s=1}^S \|\mathbf{a}_s\|_D^2} \le \sum_{s=1}^S \|\mathbf{a}_s\|_D$. Then, noting the definition of ρ_η, and expanding the right-hand side of (32) and regrouping the terms, we have

$$\begin{aligned}
&\|y(n+1)\|_D \\
&\le \sqrt{2}\left[\left(\frac{\rho_\eta}{\gamma}\right)^{n+1}\|y(0)\|_D + \left(\frac{\rho_\eta}{\gamma}\right)^n \gamma^{-1}\|\alpha(I-W)z(0)\|_D\right. \\
&\quad \left. + \sum_{s=0}^{n-1}\left(\frac{\rho_\eta}{\gamma}\right)^s \gamma^{-1}\|\alpha(I-W)z(n-s)\|_D\right].
\end{aligned} \tag{33}$$

Next we will separately estimate the three terms on the right-hand side of (33). For the first term, we note that $\pi_{\min}\|y\|_2 \le \|y\|_D \le \pi_{\max}\|y\|_2$ and $\|y\|_2 \le \sqrt{N}\|y\|_\infty$. It then follows that,

$$\left(\frac{\rho_\eta}{\gamma}\right)^{n+1}\|y(0)\|_D \le \frac{\sqrt{N}\pi_{\max}C_\delta}{g_0}\left(\frac{\rho_\eta}{\gamma}\right)^n. \tag{34}$$

Since W is stochastic, $\|\alpha(W-I)\|_2 \le \alpha(\|W\|_2 + \|I\|_2) \le 2\alpha$. Thus, the second term can be estimated as follows,

$$\gamma^{-1}\left(\frac{\rho_\eta}{\gamma}\right)^n\|(W-I)z(0)\|_D \le \frac{2\sqrt{N}\alpha C_x\pi_{\max}}{g_0\gamma}\left(\frac{\rho_\eta}{\gamma}\right)^n. \tag{35}$$

By using the fact that $\sum_{s=0}^{n-1}\left(\frac{\rho_\eta}{\gamma}\right)^s = \frac{1-\left(\frac{\rho_\eta}{\gamma}\right)^n}{1-\frac{\rho_\eta}{\gamma}}$, and in view of (31), we can estimate the third term of (33) as

$$\begin{aligned}
&\gamma^{-1}\sum_{s=0}^{n-1}\left(\frac{\rho_\eta}{\gamma}\right)^s\|\alpha(W-I)z(n-s)\|_D \\
&\le \sqrt{N}\pi_{\max}\gamma^{-1}\sum_{s=0}^{n-1}\left(\frac{\rho_\eta}{\gamma}\right)^s\|z(n-s)\|_\infty\|\alpha(W-I)\|_2 \\
&\le \sqrt{N}\pi_{\max}\gamma^{-1}\sum_{s=0}^{n-1}\left(\frac{\rho_\eta}{\gamma}\right)^s\frac{2\alpha}{2\gamma} \\
&\le \frac{\sqrt{N}\alpha\pi_{\max}}{\gamma(\gamma-\rho_\eta)}\left(1 - \left(\frac{\rho_\eta}{\gamma}\right)^n\right).
\end{aligned} \tag{36}$$

Using (34)–(36) in the right-hand side of (33), we have

$$\begin{aligned}
&\|y(n+1)\|_D \\
&\le \max\left\{\frac{\sqrt{2N}\alpha\pi_{\max}}{\gamma(\gamma-\rho_\eta)}, \frac{\sqrt{2N}\pi_{\max}(2\alpha C_x + \gamma C_\delta)}{g_0\gamma}\right\}.
\end{aligned} \tag{37}$$

With the above preparation and in view of (24) and (31), we have

$$\begin{aligned}
&\|\tilde{e}(n+1)\|_\infty \le 2\alpha\|y(n+1)\|_2 + (1+2\alpha)\|z(n+1)\|_\infty \\
&\le \frac{2\alpha}{\pi_{\min}}\|y(n+1)\|_D + (1+2\alpha)\|z(n+1)\|_\infty \\
&\le \frac{2\alpha}{\pi_{\min}}\max\left\{\frac{\sqrt{2N}\alpha\pi_{\max}}{\gamma(\gamma-\rho_\eta)}, \frac{\sqrt{2N}\pi_{\max}(2\alpha C_x + \gamma C_\delta)}{g_0\gamma}\right\} \\
&\quad + \frac{1+2\alpha}{2\gamma} \\
&= M_1(\alpha,\gamma) < \left\lfloor M_1(\alpha,\gamma) - \frac{1}{2}\right\rfloor + \frac{3}{2} \\
&= K_1(\alpha,\gamma) + \frac{1}{2} \le K + \frac{1}{2},
\end{aligned} \tag{38}$$

which indicates that the quantizer does not saturate at time $k = n+1$. Therefore, by (30), (31) and (38), we have shown by induction that the $(2K+1)$-level uniform quantizer (6) with any $K > K_1(\alpha,\gamma)$ would never saturate.

Meanwhile, it follows from (37) that

$$\lim_{k\to\infty} \|\delta(k)\|_\infty = \lim_{k\to\infty} g_0\gamma^k \|y(k)\|_\infty = 0,$$

which together with (11) leads to (3) or (25). Thus the weighted average consensus over directed networks with limited communication data rate is achieved.

Noting that

$$\lim_{\gamma\to 1} M_1(\alpha,\gamma) = \frac{2\sqrt{2N}\alpha^2\pi_{\max}}{\pi_{\min}(1-\rho_\eta)} + \frac{1+2\alpha}{2}, \tag{39}$$

and that $\lim_{\alpha\to 0}\left(\frac{2\sqrt{2N}\alpha^2\pi_{\max}}{\pi_{\min}(1-\rho_\eta)} + \frac{1+2\alpha}{2}\right) = \frac{1}{2}$, we obtain (26). Finally, following similar derivation as in Li et al. (2011), we derive that

$$\lim_{k\to\infty} \frac{\|\delta(k)\|_D}{\gamma^k} \le \frac{\sqrt{2N}\alpha\pi_{\max}g_0}{(\gamma-\rho_\eta)}, \tag{40}$$

from which and by the definition of r_{asym}, (27) follows. □

Remark 5. When the network is undirected and the associated Laplacian matrix $\mathscr{L}$ is symmetric, based on matrix decomposition and the eigenvalue spectrum of $\mathscr{L}$, a set of conditions for consensus were established in Li et al. (2011) as $h \in \left(0, \frac{2}{\lambda_N(\mathscr{L})}\right)$ and $\gamma \in (\rho_h, 1)$ with $\rho_h = \max_{2\le i\le N} |1 - h\lambda_i(\mathscr{L})|$, where $0 = \lambda_1(\mathscr{L}) < \lambda_2(\mathscr{L}) \le \cdots \le \lambda_N(\mathscr{L})$ are the eigenvalues of Laplacian matrix $\mathscr{L}$. Thus, the parameters g_0, γ and K are specified by the eigenvalue spectrum of Laplacian matrix $\mathscr{L}$. On the other hand, in this paper, where the network is directed, unbalanced and strongly connected, based on the property of the normalized positive left eigenvector π associated with the eigenvalue 1 of P, a condition for consensus is established as $\gamma \in (\rho_\eta, 1)$ with $\rho_\eta = \left(1 - \frac{\eta}{2(N-1)}\right)^{\frac{1}{2}}$, where $1 > \eta = \alpha\rho\pi_{\min} > 0$. Then the parameters g_0, γ and K are specified by the left eigenvector π and the edge weights p_{ij}. Furthermore, when the network is balanced, then, $D = \frac{1}{N}I$ and $1 > \eta = \frac{\alpha\rho}{N} > 0$, the convergence analysis carried out in this paper is still valid and leads to average consensus. This, along with Remarks 2–4, indicate that the consensus conditions and update schemes of Li et al. (2011) have been generalized to a more general case. Indeed, the final achieved consensus value given in (25) clearly demonstrates the dependence on the topology properties of the directed network.

Remark 6. It can be seen that (40) establishes the relationship between the steady consensus error of the closed-loop system and the scaling function: $\lim_{k\to\infty} \frac{\|\delta(k)\|_D}{g_0\gamma^k} \le \frac{\sqrt{2N}\alpha\pi_{\max}}{(\gamma-\rho_\eta)}$. Then, if $\lim_{k\to\infty} g_0\gamma^k = 0$, the steady consensus error $\lim_{k\to\infty}\|\delta(k)\|_D = 0$ and (40) gives the convergence rate of the closed-loop system $\|\delta(k)\|_D = O(g(k))$. Thus, Theorem 1 provides an estimate of the convergence rate of consensus in the context of directed digital networks that strongly connected, which extends the corresponding results for undirected networks (Li et al., 2011; Xiao & Boyd, 2004), where it has been clarified that the convergence rate of consensus can be specified by the Laplacian's eigenvalue spectrum both with and without the quantized communication. In view of $\gamma \in (\rho_\eta, 1)$ and Lemma 1, (27) indicates that the obtained convergence rate of consensus is determined by the network scale N, the edge weights p_{ij} and the left eigenvector π. Moreover, it is clear from (27) that the smaller the value of γ is, the faster the consensus error converges to zero. However, (22) indicates that more bits are required to be communicated along each digital channel as the value of γ decreases. In particular, when γ approaches ρ_η, the required number of bits goes to infinity. Therefore, in view of (26), a tradeoff exists between the convergence rate of consensus and the required number of quantization levels.

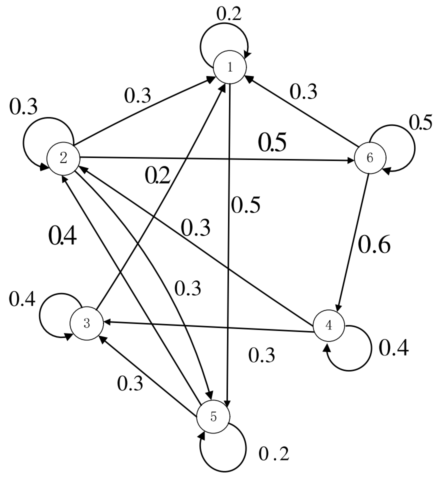

Fig. 1. The directed network topology.

Remark 7. Theorem 1 shows that as long as the directed unbalanced digital network is strongly connected with each agent containing a self-loop, no matter how large the network is, a suitable distributed protocol can always be designed to guarantee exponentially fast weighted average consensus, with each agent (maybe nonreciprocally) sending merely one bit of quantized information to each of its adjacent neighbors, together with one bit of quantized information to itself at each time instant. In Li et al. (2011), where each agent has no self-loop, average consensus can be guaranteed with an exponential convergence rate only when each pair of neighboring agents reciprocally sends one bit of quantized information to each other at every time instant.

Remark 8. The results of this work are mainly based on the assumption that the left eigenvector π associated with the eigenvalue 1 of the stochastic adjacent matrix W is known, with each component of π quantifying the influence for corresponding agent. The recent result of the distributed PageRank computation (Ishii & Tempo, 2010) may provide a hint to obtaining the value of this left eigenvector in a distributed manner, since PageRank is known as a crucial index which measures each web page's importance in the web graph in the search engine of Google. Note that many similarities exist between the algorithms for consensus and PageRank (Ishii & Tempo, 2010).

5. Simulation results

Consider a strongly connected and unbalanced information digraph $\mathbb{G}$ with 6 nodes and the weight update matrix

$$W = \begin{bmatrix} 0.2 & 0.3 & 0.2 & 0.0 & 0.0 & 0.3 \\ 0.0 & 0.3 & 0.0 & 0.3 & 0.4 & 0.0 \\ 0.0 & 0.0 & 0.4 & 0.3 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.4 & 0.0 & 0.6 \\ 0.5 & 0.3 & 0.0 & 0.0 & 0.2 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix},$$

which is stochastic but not doubly stochastic and admits unidirectional information communication (Fig. 1).

The unique normalized positive left eigenvector of W corresponding to eigenvalue 1 is

$$\pi = \begin{bmatrix} 0.0991 & 0.2922 & 0.0330 & 0.1626 & 0.1585 & 0.2546 \end{bmatrix}^T.$$

Let the initial value be randomly generated according to a uniform distribution over $[-36, 36]$ as $x(0) = [12.6456\ \ -5.2166\ \ 3.8145\ \ 10.7668\ \ 17.6725\ \ 20.1345]^T$. Then, the weighted average value is $\pi^T x(0) = 9.5320$.

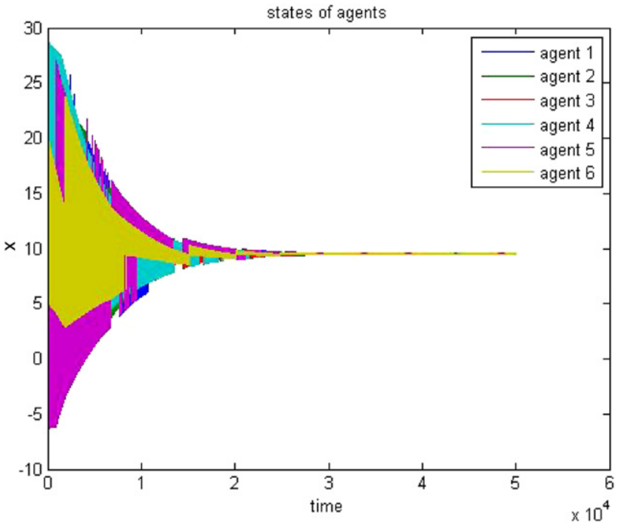

Fig. 2. The trajectories of the agents' states when $\alpha = 1$, $K = 1$ and $\gamma = 0.99981$.

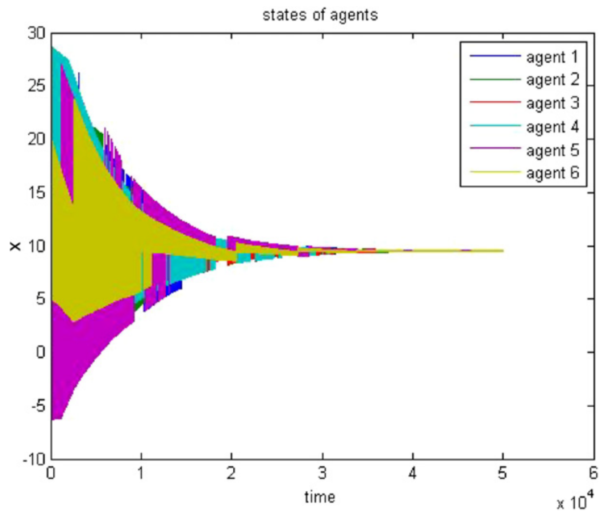

Fig. 3. The trajectories of the agents' states when $\alpha = 1$, $K = 1$ and $\gamma = 0.99986$.

We first select $\alpha = 1$ in (15), that is $P = W$. Assumption 1 holds with $\rho = 0.2$. According to Lemma 1, $\eta = \alpha\rho\pi_{\min} = 0.0066$. Then $\rho_\eta = \left(1 - \frac{\eta}{2(N-1)}\right)^{\frac{1}{2}} = 0.9997$ and $\gamma \in (0.9997, 1)$. We first choose $\gamma = 0.99981$. According to (23) and (24), we have $K_1 = 5575872$ and $g_0 \geq 0.01584$. In fact, since Lemma 1 is mainly based on a worst-case analysis, the obtained bit number K_1 by Theorem 1 is a conservative estimate. Meanwhile, (26) indicates that one-bit rate communication requires that α be sufficiently close to 0 and γ be sufficiently close to 1. In practice, fewer bits may suffice to achieve weighted average consensus. In what follows, we set $g_0 = 30$ and $K = 1$, that is, a one-bit uniform quantizer is applied to each connected digital channel. Shown in Fig. 2 is the evolution of the agents' states when $\gamma = 0.99981$. As expected, all these states converge exponentially to the weighted average value $\pi^T x(0) = 9.5320$. Moreover, as noted in Remark 6, there is a trade-off between the convergence rate of consensus and the required data rate. For comparison, Fig. 3 shows the evolution of the states when $\gamma = 0.99986$. The two figures clearly verify that the smaller the value of γ is, the faster the convergence is.

6. Conclusions

We have studied the consensus on general directed networks in the presence of finite bandwidth communication channels and in the absence of the double stochasticity assumption. We used directed graphs to represent information exchanges among multiple agents on digital networks, allowing unidirectional and unbalanced information exchange. With a simple requirement that the update matrix has positive diagonals, we derived conditions that guarantee all agents to exponentially approach weighted average consensus with limited communication data rate. Furthermore, the desired convergence rate of consensus can be achieved by properly selecting the number of the quantization levels. Numerical simulation showed the effectiveness of the theoretical results. It is hoped that the freedom our results allow for the network topology would open up the perspective of designing new efficient distributed quantized consensus algorithms for a broader class of directed networks. In particular, an extension of the results presented in this paper to more general directed networks with a directed spanning tree is among challenging problems that call for further investigation.

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