Supplementary Numerical Examples

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1 Advection-Diffusion-Reaction equations

Let $\Omega \subset \mathbb{R}^2$ be a polygonal convex bounded domain, with its boundary denoted by $\partial\Omega$. We shall consider the following advection-diffusion-reaction equation:

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \gamma u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

Here, u is the unknown quantity to be solved, $f \in L^2(\Omega)$ is a given source term, $\varepsilon > 0$ is the diffusive constant, $\mathbf{b} \in (L^\infty(\Omega))^2$ is the convection field, satisfying div $\mathbf{b} = 0$, γ is the the dissipative reaction constant.

The weak form of this problem is to find $u \in H_0^1(\Omega)$ such that

$$A(u,v) = (f,v) \qquad \forall v \in H_0^1(\Omega),$$

where

$$A(u,v) := \varepsilon(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + \gamma(u,v) \tag{2}$$

Here $(\cdot,\cdot)$ denotes L^2-inner product. The standard Galerkin finite element method is introduced in the following.First, let $\{\mathcal{T}_h\}_{0<h\leq 1}$ denote the shape-regular triangulation of $\bar{\Omega}$ into triangles. A generic triangle element is denoted as T.

Define the finite element space $U_h \subset H_0^1(\Omega)$, a standard finite dimensional space of continuous piecewise polynomials with respect to $\mathcal{T}_h$. We choose

$$U_h := \{v \in H_0^1(\Omega) : v|_T \in P_1(T), \forall T \in \mathcal{T}_h\} \tag{3}$$

where $P_1(T)$ denotes the space of linear polynomials over T. The Gelerkin method is stated as follow:

Find $u_h \in U_h$, such that $A(u_h, v) = (f, v)$, $\forall v \in U_h$.

It is well known that the standard Galerkin approximation, in the convection dominated case, results in a wildly oscillating solution in the presence of shape layers. So, a stabilization of this method is necessary.

I. SUPG Method

The SUPG method is one of the most popular stabilized finite element methods. Basically, this method adds diffusion in the direction of the streamlines to the Galerkin finite element method. Here we consider the problem(1) with $\gamma = 0$, then it reads as follows: Find $u_h \in U_h$ such that

$$A(u_h, v) + \sum_{T \in \mathcal{T}_h} (R_h(u_h), \delta_T \mathbf{b} \cdot \nabla v)_T = (f, v) \qquad \forall v \in U_h \tag{4}$$

$R_h(u_h) = -\varepsilon \Delta_h u_h + \mathbf{b} \cdot \nabla_h u_h - f$ is the residual of the strong form of the equation, the index h at the differential operators denotes their restriction to a mesh cell T, δ_T are the stablization parameters. Obviously, the SUPG method is a consistent, residual-based stabilization. The parameters are used as follow:

$$\delta_T(\mathbf{x}) = \frac{\bar{h}_T}{2p|\mathbf{b(x)}|} \xi(Pe_T(\mathbf{x})), \;\; Pe_T(\mathbf{x}) = \frac{|\mathbf{b(x)}|\bar{h}_T}{2p\varepsilon}, \;\; \xi(\alpha) = \coth\alpha - \frac{1}{\alpha} \tag{5}$$

where $\bar{h}_T$ is an approximation of the length of the mesh cell T in the direction of the convection, Pe_T is the local mesh cell Péclet number, and p is the degree of the local finite element space.

II. E.Burman Method

The method adds a term penalizing the gradient jumps across element boundaries of the type

$$J(u_h, v) = \sum_{F \in \mathcal{F}_h^{int}} \frac{1}{2} \int_F \beta h_F^2 [\nabla u_h] \cdot [\nabla v] ds = \sum_{F \in \mathcal{F}_h^{int}} \frac{1}{2} \int_F \beta h_F^2 [\nabla u_h \cdot \mathbf{n}][\nabla v \cdot \mathbf{n}] ds \tag{6}$$

Here, $\mathcal{F}_h^{int}$ is the interior boundary, h_F is the size of F, $[q]$ denotes the jump of q across $\mathcal{F}_h^{int}$, for $\mathcal{F}_h^{int} \cap \partial\Omega = \emptyset$, $[q] = 0$ on $\mathcal{F}_h^{int} \cap \partial\Omega \neq \emptyset$, n is the outward pointing unit normal to F, and β is a constant. We also introduce the local mesh size

$$h_T := \max_T h_F,$$

and we will assume that $h_T / h_F < C$ where C is a fixed constant. The finite element method then reads, find $u_h \in U_h$ such that

$$A(u_h, v) + J(u_h, v) = (f, v) \quad \forall v \in U_h \tag{7}$$

III. NEW Method-I

we shall propose a new method for the problem(1). This method employs the same finite element space as before (3). Different from E.Burman Method, the new method will consist of an additional stabilization term, with some ad hoc stabilization parameters.

$$B(u_h, v) = \sum_{F \in \mathcal{F}_h^{int}} \beta \tau_{int,F} \int_F [\nabla u \cdot \mathbf{n}][\nabla v \cdot \mathbf{n}] + \sum_{F \in \mathcal{F}_h^{\partial}} \alpha \tau_{\partial,F} \int_F (-\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + \gamma u)(-\varepsilon \Delta v + \mathbf{b} \cdot \nabla v + \gamma v)$$

$$L(v) = \sum_{F\in\mathcal{F}_h^\partial} \alpha\tau_{\partial,F} \int_F f(-\varepsilon\Delta v + \mathbf{b}\cdot\nabla v + \gamma v)$$

where

$$\tau_{int,F} := \frac{h_F^3\|\mathbf{b}\|_{0,\infty}^2}{\gamma h_T^2 + \|\mathbf{b}\|_{0,\infty}h_F + \varepsilon}$$

$$\tau_{\partial,F} := \frac{h_F^3}{\gamma h_T^2 + \|\mathbf{b}\|_{0,\infty}h_F + \varepsilon}$$

$\mathcal{F}_h^{int}$ is the interior boundary, $\mathcal{F}_h^\partial$ is the boundary on $\partial\Omega$. h_T is diameter of the triangle T, h_F is the size of F. β and α are constant. Then the finite element method reads, find $u_h \in U_h$ such that

$$A(u_h, v) + B(u_h, v) = (f, v) + L(v) \quad \forall v \in U_h \tag{8}$$

IV. NEW Method-II

Another new B(u,v) with shockcapturing:

$$B(u_h, v) = \sum_{F\in\mathcal{F}_h^{int}} \beta\tau_{int,F} \int_F [\nabla u\cdot\mathbf{n}][\nabla v\cdot\mathbf{n}] + \sum_{F\in\mathcal{F}_h^\partial} \alpha\tau_{\partial,F} \int_F (-\varepsilon\Delta u+\mathbf{b}\cdot\nabla u+\gamma u)(-\varepsilon\Delta v+\mathbf{b}\cdot\nabla v+\gamma v)$$

$$+ \sum_{F\in\mathcal{F}_h^\partial} \alpha\tau_{\partial,F} \int_F (\mathbf{b}^\perp\cdot\nabla u)(\mathbf{b}^\perp\cdot\nabla v)$$

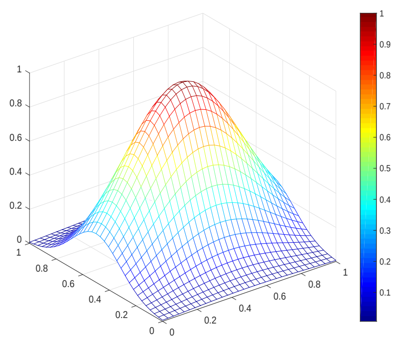
$$L(v) = \sum_{F\in\mathcal{F}_h^\partial} \alpha\tau_{\partial,F} \int_F f(-\varepsilon\Delta v + \mathbf{b}\cdot\nabla v + \gamma v)$$

2 Numerical examples of Burman's paper

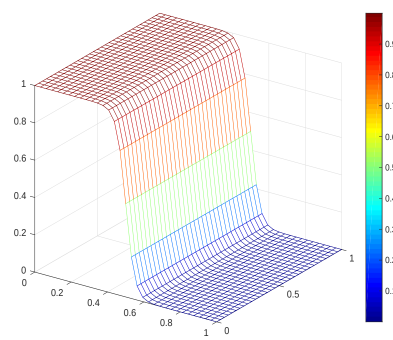
2.1 Convection-diffusion-reaction

Considering the equation (1),choosing $\gamma = 1, \mathbf{b} = (1, 0), \varepsilon = 10^{-5}$, corresponding to the convection dominated case. We let $\Omega = [0, 1] \times [0, 1]$ and use two different source terms f in order to get the following exact solutions, see Fig 1

- Test case 1: $u = exp(-\frac{(x-0.5)^2}{0.2} - \frac{3(y-0.5)^2}{0.2})$
- Test case 2: $u = \frac{1}{2}(1 - tanh(\frac{x-0.5}{0.05}))$



(a) Test case 1

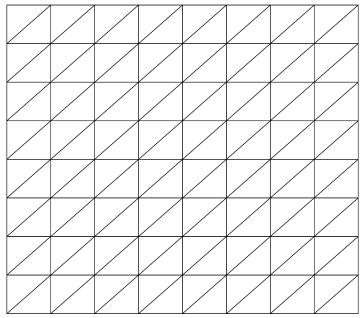


(b) Test case 2

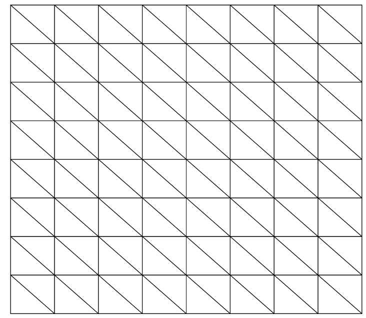
Fig 1: The two exact solutions.

There are four different types of meshes have been used, illustrated in Fig 2, all are based on square elements. In the first case(denoted mesh-1) the square elements are cut into two triangles approximately along the direction of the convection, in the second case (mesh-2) they are almost cut perpendicular to the direction of the convection, the third case(mesh-3) cut into two triangles with diagonal chosen randomly, in the last case(mesh-4) they are cut into four triangles.

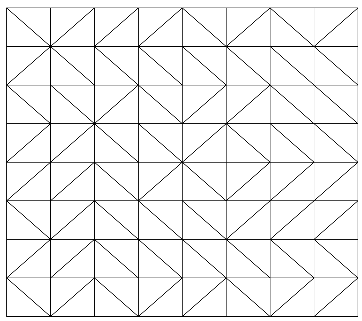
We have computered the solution using four methods as above: supg, bm, new-I and new-II under four meshes as follow. The stabilization parameter for the edge stabilization was chosen to $\alpha = 0.55, \beta = 0.0125$. The solutions were computered on four consecutive meshes , the standard mesh size is taken as $h^* = 1/32, 1/64, 1/128, 1/256$. We present the errors in the L_2 norm, the H^1 norm and the L_∞ norm for the four methods applied to the two test cases in table1-24.



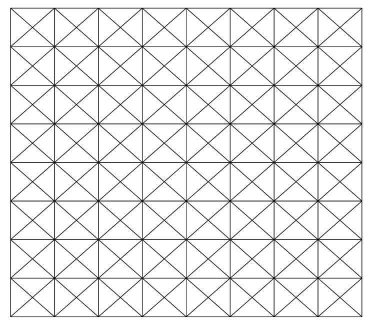
(c) mesh-1



(d) mesh-2



(e) mesh-3



(f) mesh-4

Fig 2: The four different meshes.

Table 1: L_2 Convergence results for test case 1 on mesh 1 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	5.5688e-04	1.3492e-04	3.3332e-05	8.3004e-06	2.0227e+00
BM	5.6524e-04	1.3803e-04	3.4075e-05	8.4620e-06	2.0206e+00
NEW-I	6.0242e-04	1.4370e-04	3.5065e-05	8.6527e-06	2.0405e+00
NEW-II	1.4629e-03	5.5104e-04	2.1619e-04	8.7064e-05	1.3569e+00

Table 2: H^1 Convergence results for test case 1 on mesh 1 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.1921e-01	5.9525e-02	2.9750e-02	1.4873e-02	1.0009e+00
BM	1.1958e-01	5.9646e-02	2.9786e-02	1.4883e-02	1.0021e+00
NEW-I	1.2046e-01	5.9872e-02	2.9841e-02	1.4896e-02	1.0052e+00
NEW-II	1.4262e-01	7.9742e-02	4.6901e-02	2.8879e-02	7.6804e-01

Table 3: L_∞ Convergence results for test case 1 on mesh 1 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	2.2905e-03	5.9068e-04	1.4910e-04	3.7268e-05	1.9805e+00
BM	2.4173e-03	6.0626e-04	1.5161e-04	3.8116e-05	1.9956e+00
NEW-I	2.3069e-03	5.5090e-04	1.3488e-04	3.3365e-05	2.0372e+00
NEW-II	9.2205e-03	4.6825e-03	2.3882e-03	1.2102e-03	9.7655e-01

Table 4: L_2 Convergence results for test case 1 on mesh 2 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	5.5689e-04	1.3492e-04	3.3333e-05	8.3004e-06	2.0227e+00
BM	5.6525e-04	1.3803e-04	3.4076e-05	8.4620e-06	2.0206e+00
NEW-I	6.0243e-04	1.4370e-04	3.5065e-05	8.6527e-06	2.0405e+00
NEW-II	1.4629e-03	5.5104e-04	2.1619e-04	8.7064e-05	1.3569e+00

Table 5: H^1 Convergence results for test case 1 on mesh 2 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.1921e-01	5.9525e-02	2.9750e-02	1.4873e-02	1.0009e+00
BM	1.1958e-01	5.9646e-02	2.9786e-02	1.4883e-02	1.0021e+00
NEW-I	1.2046e-01	5.9872e-02	2.9841e-02	1.4896e-02	1.0052e+00
NEW-II	1.4262e-01	7.9742e-02	4.6901e-02	2.8879e-02	7.6804e-01

Table 6: L_∞ Convergence results for test case 1 on mesh 2 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	2.2481e-03	5.7926e-04	1.4612e-04	3.6510e-05	1.9814e+00
BM	2.3761e-03	5.9453e-04	1.4860e-04	3.7358e-05	1.9970e+00
NEW-I	2.3517e-03	5.6069e-04	1.3688e-04	3.3852e-05	2.0394e+00
NEW-II	9.2205e-03	4.6825e-03	2.3882e-03	1.2102e-03	9.7655e-01

Table 7: L_2 Convergence results for test case 1 on mesh 3 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	6.4486e-04	1.6099e-04	3.9572e-05	9.8164e-06	2.0125e+00
BM	8.1790e-04	2.0919e-04	4.9835e-05	1.2398e-05	2.0146e+00
NEW-I	7.9634e-04	1.9266e-04	4.8828e-05	1.2141e-05	2.0118e+00
NEW-II	1.5161e-03	5.6549e-04	2.1733e-04	8.6898e-05	1.3750e+00

Table 8: H^1 Convergence results for test case 1 on mesh 3 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.2120e-01	6.0257e-02	3.0351e-02	1.5148e-02	1.0001e+00
BM	1.2737e-01	6.4898e-02	3.2187e-02	1.6071e-02	9.9550e-01
NEW-I	1.2821e-01	6.4114e-02	3.2257e-02	1.6063e-02	9.9892e-01
NEW-II	1.4926e-01	8.3266e-02	4.8220e-02	2.9431e-02	7.8081e-01

Table 9: L_∞ Convergence results for test case 1 on mesh 3 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	3.6878e-03	8.7126e-04	2.2737e-04	6.3326e-05	1.9546e+00
BM	3.3724e-03	1.0115e-03	2.6268e-04	8.5591e-05	1.7667e+00
NEW-I	3.7223e-03	8.6234e-04	2.6711e-04	7.7370e-05	1.8628e+00
NEW-II	9.5313e-03	4.7765e-03	2.4970e-03	1.2423e-03	9.7990e-01

Table 10: L_2 Convergence results for test case 1 on mesh 4 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	4.4337e-04	1.0962e-04	2.7282e-05	6.8045e-06	2.0086e+00
BM	3.3007e-04	8.2035e-05	2.0480e-05	5.1156e-06	2.0039e+00
NEW-I	4.4949e-04	1.0944e-04	2.6992e-05	6.6911e-06	2.0233e+00
NEW-II	8.1165e-04	3.1716e-04	1.3091e-04	5.4858e-05	1.2957e+00

Table 11: H^1 Convergence results for test case 1 on mesh 4 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	9.8601e-02	4.9281e-02	2.4612e-02	1.2274e-02	1.0020e+00
BM	8.5125e-02	4.2507e-02	2.1238e-02	1.0611e-02	1.0014e+00
NEW-I	8.5327e-02	4.2562e-02	2.1252e-02	1.0615e-02	1.0023e+00
NEW-II	1.0226e-01	5.7161e-02	3.4032e-02	2.1372e-02	7.5283e-01

Table 12: L_∞ Convergence results for test case 1 on mesh 4 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.7884e-03	4.5000e-04	1.1257e-04	2.7972e-05	1.9995e+00
BM	1.5821e-03	3.9520e-04	9.8453e-05	2.4529e-05	2.0037e+00
NEW-I	2.1461e-03	5.3182e-04	1.3188e-04	3.2749e-05	2.0114e+00
NEW-II	4.9475e-03	2.5220e-03	1.4012e-03	7.7252e-04	8.9302e-01

Table 13: L_2 Convergence results for test case 2 on mesh 1 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	2.5954e-03	5.0337e-04	1.1145e-04	2.6767e-05	2.1998e+00
BM	2.4847e-03	4.4566e-04	1.0727e-04	2.6499e-05	2.1837e+00
NEW-I	2.3997e-03	4.4566e-04	1.0727e-04	2.6499e-05	2.1669e+00
NEW-II	2.4276e-03	4.5293e-04	1.0894e-04	2.6806e-05	2.1669e+00

Table 14: H^1 Convergence results for test case 2 on mesh 1 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	4.2517e-01	2.0998e-01	1.0445e-01	5.2132e-02	1.0093e+00
BM	4.5742e-01	2.1044e-01	1.0446e-01	5.2130e-02	1.0444e+00
NEW-I	4.5694e-01	2.1044e-01	1.0446e-01	5.2130e-02	1.0439e+00
NEW-II	4.5586e-01	2.1037e-01	1.0444e-01	5.2126e-02	1.0428e+00

Table 15: L_∞ Convergence results for test case 2 on mesh 1 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.7377e-02	4.3445e-03	1.1151e-03	2.8067e-04	1.9841e+00
BM	2.3865e-02	4.3447e-03	1.1183e-03	2.8209e-04	2.1342e+00
NEW-I	2.3150e-02	4.3447e-03	1.1183e-03	2.8209e-04	2.1196e+00
NEW-II	2.0195e-02	4.4050e-03	1.1426e-03	2.8963e-04	2.0412e+00

Table 16: L_2 Convergence results for test case 2 on mesh 2 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	2.5954e-03	5.0337e-04	1.1145e-04	2.6767e-05	2.1998e+00
BM	2.4847e-03	4.4566e-04	1.0727e-04	2.6499e-05	2.1837e+00
NEW-I	2.3997e-03	4.4566e-04	1.0727e-04	2.6499e-05	2.1669e+00
NEW-II	2.4276e-03	4.5293e-04	1.0894e-04	2.6806e-05	2.1669e+00

Table 17: H^1 Convergence results for test case 2 on mesh 2 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	4.2517e-01	2.0998e-01	1.0445e-01	5.2132e-02	1.0093e+00
BM	4.5742e-01	2.1044e-01	1.0446e-01	5.2130e-02	1.0444e+00
NEW-I	4.5694e-01	2.1044e-01	1.0446e-01	5.2130e-02	1.0439e+00
NEW-II	4.5586e-01	2.1037e-01	1.0444e-01	5.2126e-02	1.0428e+00

Table 18: L_∞ Convergence results for test case 2 on mesh 2 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.7377e-02	4.3445e-03	1.1151e-03	2.8067e-04	1.9841e+00
BM	2.2011e-02	4.3447e-03	1.1183e-03	2.8209e-04	2.0953e+00
NEW-I	2.1360e-02	4.3447e-03	1.1183e-03	2.8209e-04	2.0809e+00
NEW-II	2.0870e-02	4.4050e-03	1.1426e-03	2.8963e-04	2.0570e+00

Table 19: L_2 Convergence results for test case 2 on mesh 3 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	3.6596e-03	8.2463e-04	1.9486e-04	4.6823e-05	2.0961e+00
BM	2.4567e-03	4.7681e-04	1.1286e-04	2.7858e-05	2.1541e+00
NEW-I	2.4619e-03	4.7620e-04	1.1242e-04	2.7845e-05	2.1554e+00
NEW-II	2.4756e-03	4.7960e-04	1.1447e-04	v2.8114e-05	2.1535e+00

Table 20: H^1 Convergence results for test case 2 on mesh 3 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	4.8474e-01	2.5007e-01	1.2427e-01	6.1014e-02	9.9666e-01
BM	4.6313e-01	2.1411e-01	1.0588e-01	5.2759e-02	1.0447e+00
NEW-I	4.7058e-01	2.1427e-01	1.0545e-01	5.2722e-02	1.0527e+00
NEW-II	4.6380e-01	2.1390e-01	1.0596e-01	5.2754e-02	1.0454e+00

Table 21: L_∞ Convergence results for test case 2 on mesh 3 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	2.6971e-02	7.6877e-03	2.0966e-03	5.4068e-04	1.8802e+00
BM	2.5864e-02	4.5532e-03	1.1099e-03	3.0910e-04	2.1289e+00
NEW-I	2.7737e-02	4.3703e-03	1.1097e-03	2.8370e-04	2.2038e+00
NEW-II	2.8278e-02	4.4200e-03	1.1604e-03	2.8799e-04	2.2058e+00

Table 22: L_2 Convergence results for test case 2 on mesh 4 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.5380e-03	3.7491e-04	9.2793e-05	2.3076e-05	2.0195e+00
BM	1.4964e-03	3.4309e-04	8.2835e-05	2.0487e-05	2.0636e+00
NEW-I	1.4964e-03	3.4309e-04	8.2835e-05	2.0487e-05	2.0636e+00
NEW-II	1.6829e-03	3.6901e-04	8.7342e-05	2.1315e-05	2.1010e+00

Table 23: H^1 Convergence results for test case 2 on mesh 4 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	4.2039e-01	2.0871e-01	1.0391e-01	5.1745e-02	1.0074e+00
BM	3.9517e-01	1.8035e-01	8.6941e-02	4.2988e-02	1.0668e+00
NEW-I	3.9517e-01	1.8035e-01	8.6941e-02	4.2988e-02	1.0668e+00
NEW-II	4.0436e-01	1.8208e-01	8.7311e-02	4.3064e-02	1.0770e+00

Table 24: L_∞ Convergence results for test case 2 on mesh 4 with four methods

Method	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
SUPG	1.4966e-02	4.1575e-03	1.0681e-03	2.6857e-04	1.9334e+00
BM	1.5114e-02	4.2055e-03	1.0842e-03	2.7281e-04	1.9306e+00
NEW-I	1.5114e-02	4.2055e-03	1.0842e-03	2.7281e-04	1.9306e+00
NEW-II	1.6810e-02	4.4669e-03	1.1491e-03	3.4211e-04	1.8729e+00

2.2 Outflow layers

In this section we will show qualitatively the loss of stability in outflow layers. We propose a classical testcase with a convection-diffusion problem, choose domian $\Omega := [0,1] \times [0,1]$, the convection field $\mathbf{b} = 1$, $f = 0$, $\gamma = 0, \varepsilon = 10^{-6}$. The geometry, the boundary conditions and the orientation of $\mathbf{b}$ are resumed in Fig 3.

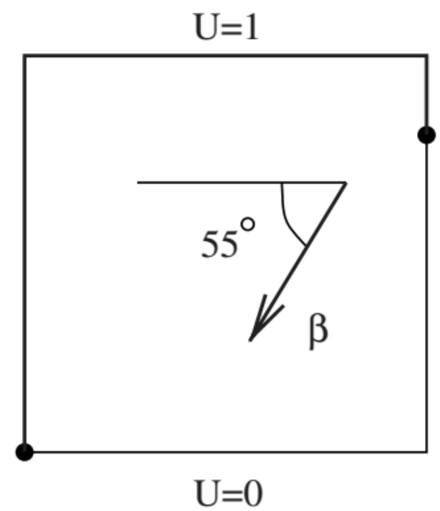


Fig 3: Boundary conditions and flow orientation, for outflow layer test case: U=1 along thick edge and U=0 along thin edge.

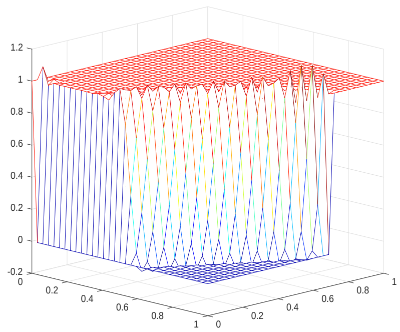
The results of the four methods mentioned above with the four types of meshes, where $h^* = 1/32, 1/64$.

The parameters in SUPG method: $\bar{h}_T$ is an approximation of the length of the mesh cell T in the direction of the convection, so $\bar{h}_T = h^*/sin(55°)$ in mesh-1; $\bar{h}_T = \sqrt{1/2}h^*/sin(80°)$ in mesh-2; $\bar{h}_T = h^*/sin(55°)$ or $\bar{h}_T = \sqrt{1/2}h^*/sin(80°)$ in mesh-3; $\bar{h}_T = 1/2h^*/sin(55°)$ or $\bar{h}_T = \sqrt{1/2}h^*/sin(80°)$ in mesh-4.

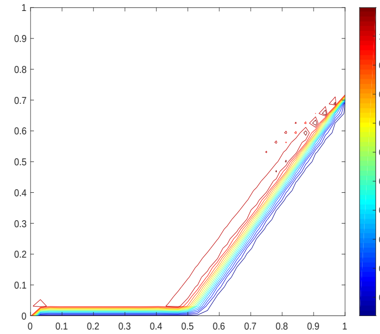
The parameters in E.Burman method: h_F is the size of $F \in \mathcal{F}_h^{int}$, that is $h_F = h^*$ or $h_F = \sqrt{2}h^*$ in mesh-1,mesh-2,mesh-3; $h_F = h^*$ or $h_F = \sqrt{1/2}h^*$ in mesh-4; $\beta = 0.0125$.

The parameters in New method: h_F is the same as in E.Burman method, and $h_T = \sqrt{2}h$ in mesh-1,mesh-2,mesh-3, $h_T = h^*$ in mesh-4. $\beta = 0.0125$, $\alpha = 0.55$ in New-method-I, $\alpha = 0.37$ in New-method-II .

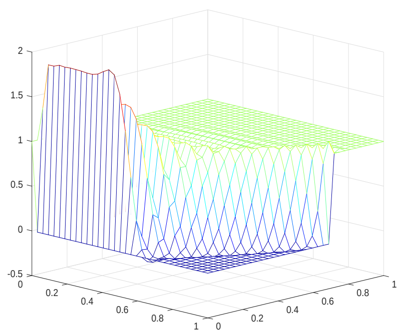
2.2.1 Results with mesh-1



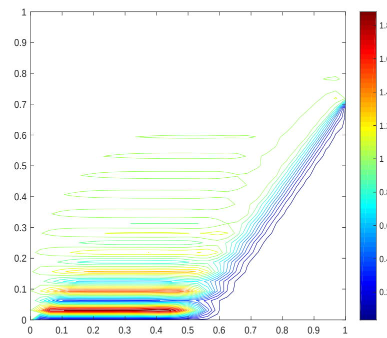
(a) SUPG Method with mesh-1.



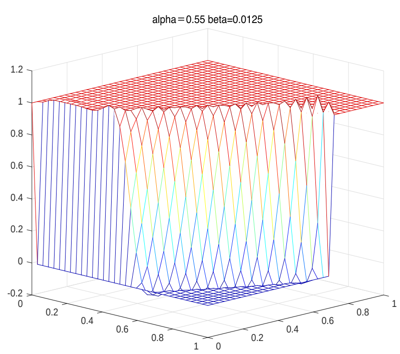
(b) SUPG Method with mesh-1.



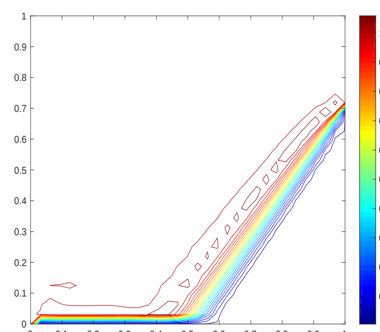
(c) E.Burman Method with mesh-1.



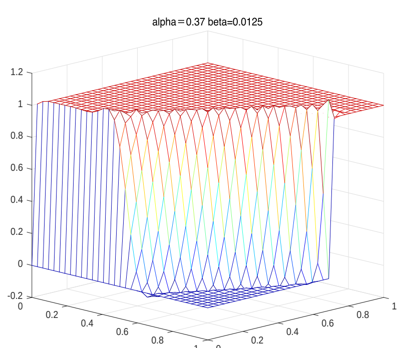
(d) E.Burman Method with mesh-1.



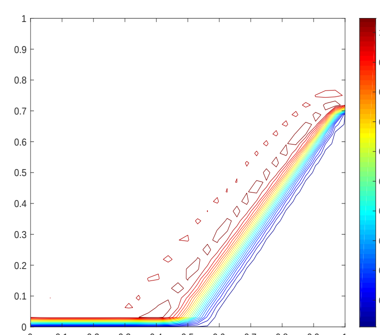
(e) New Method-I with mesh-1.



(f) New Method-I with mesh-1.

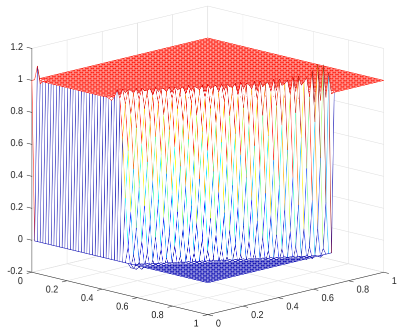


(g) New Method-II with mesh-1.

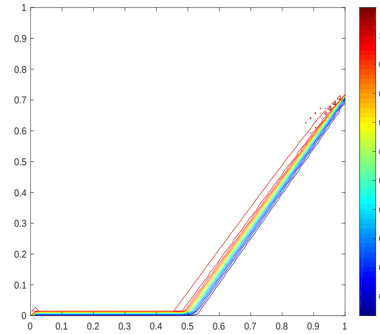


(h) New Method-II with mesh-1.

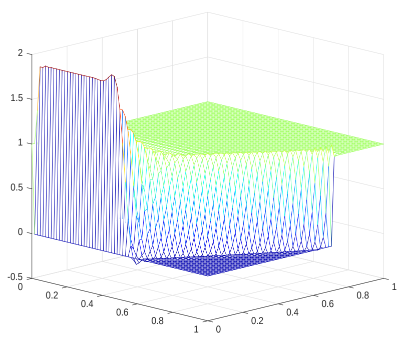
The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b}$ = $(-cos(55°), -sin(55°))$, γ=0, f=0, h=1/32.



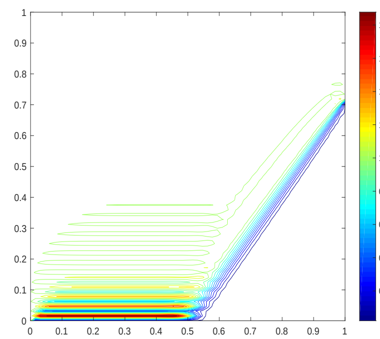
(a) SUPG Method with mesh-1.



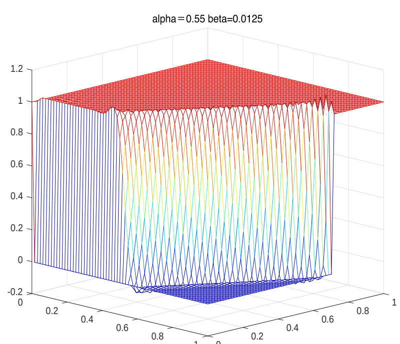
(b) SUPG Method with mesh-1.



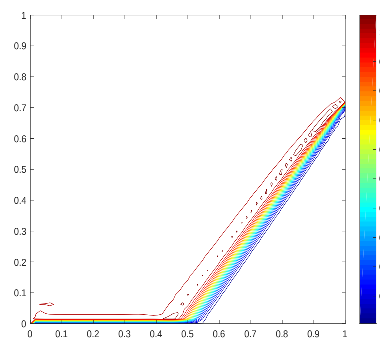
(c) E.Burman Method with mesh-1.



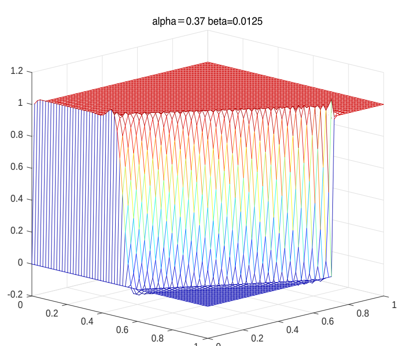
(d) E.Burman Method with mesh-1.



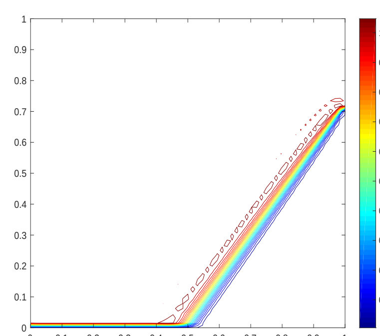
(e) New Method-I with mesh-1.



(f) New Method-I with mesh-1.



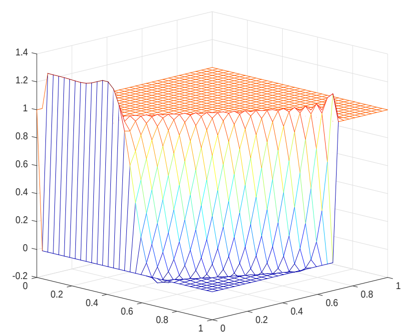
(g) New Method-II with mesh-1.



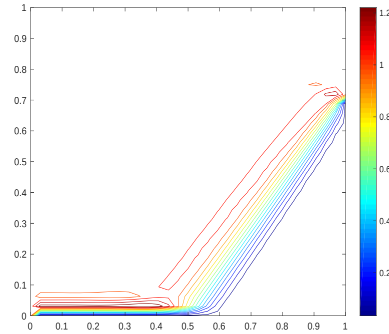
(h) New Method-II with mesh-1.

The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/64.

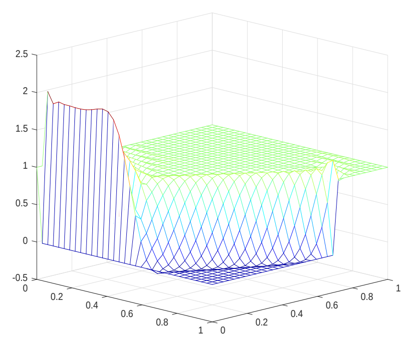
2.2.2 Results with mesh-2



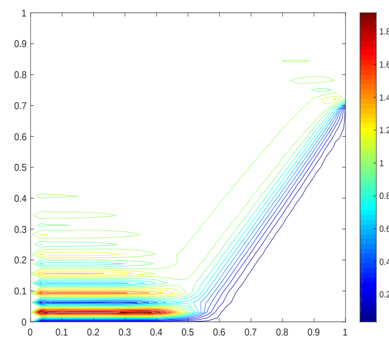
(a) SUPG Method with mesh-2.



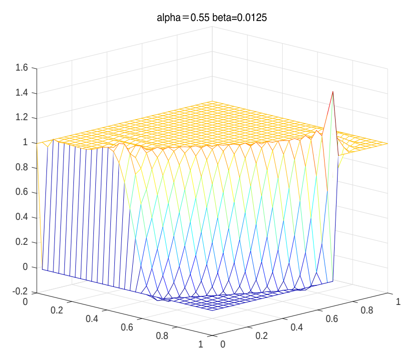
(b) SUPG Method with mesh-2.



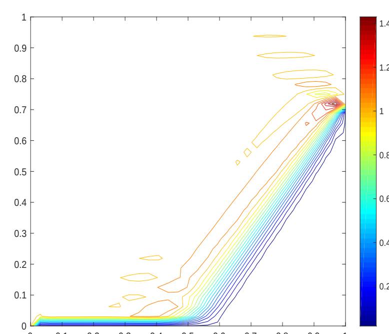
(c) E.Burman Method with mesh-2.



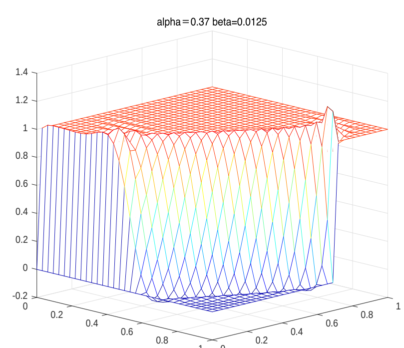
(d) E.Burman Method with mesh-2.



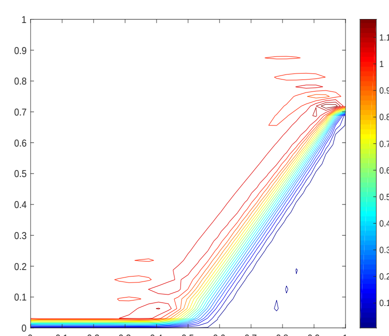
(e) New Method-I with mesh-2.



(f) New Method-I with mesh-2.

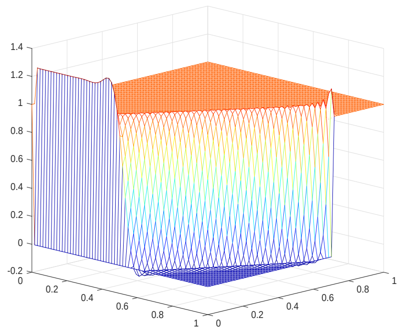


(g) New Method-II with mesh-2.

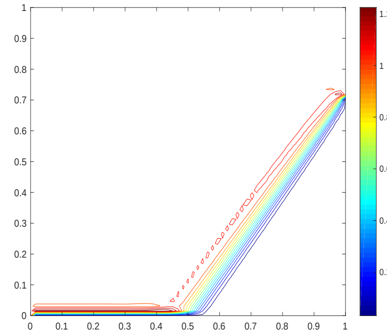


(h) New Method-II with mesh-2.

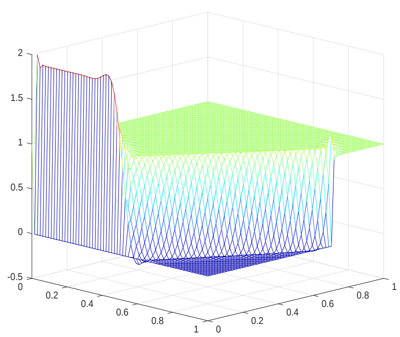
The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/32.



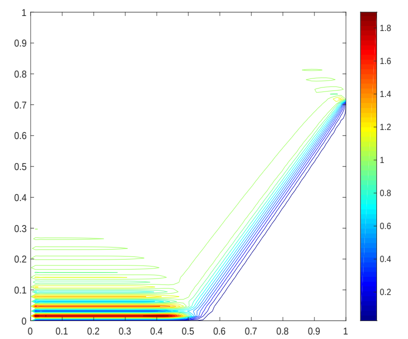
(a) SUPG Method with mesh-2.



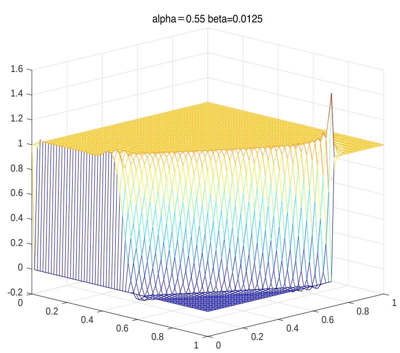
(b) SUPG Method with mesh-2.



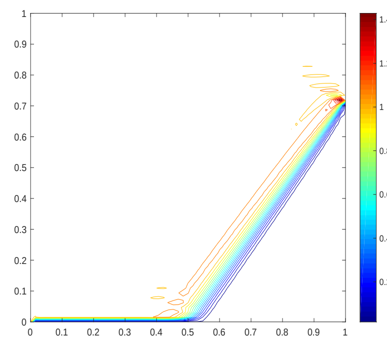
(c) E.Burman Method with mesh-2.



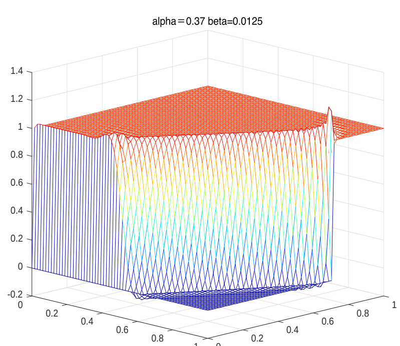
(d) E.Burman Method with mesh-2.



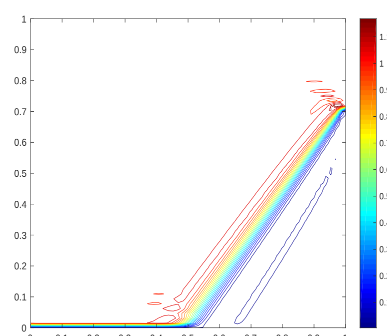
(e) New Method-I with mesh-2.



(f) New Method-I with mesh-2.



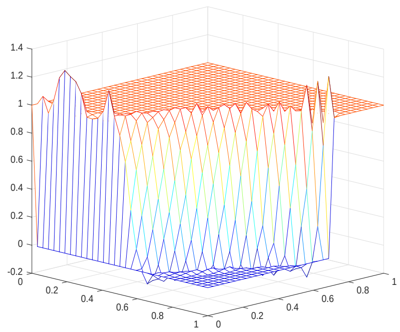
(g) New Method-II with mesh-2.



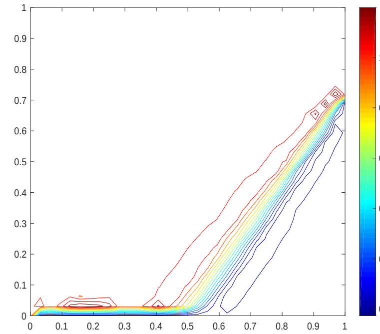
(h) New Method-II with mesh-2.

The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/64.

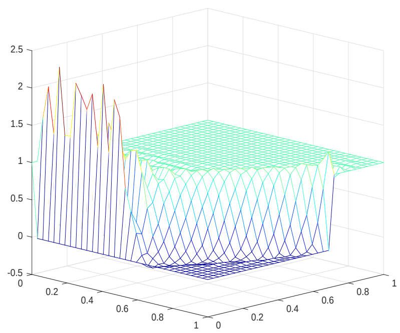
2.2.3 Results with mesh-3



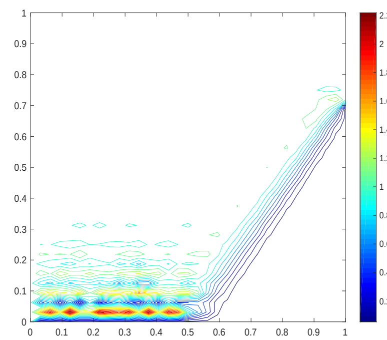
(a) SUPG Method with mesh-3.



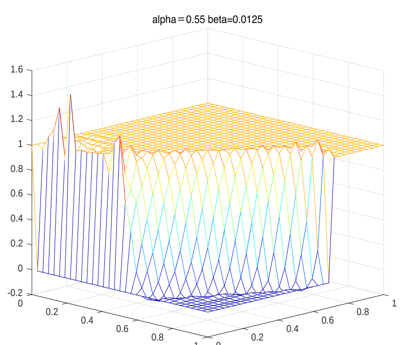
(b) SUPG Method with mesh-3.



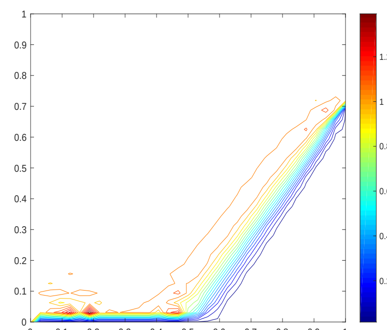
(c) E.Burman Method with mesh-3.



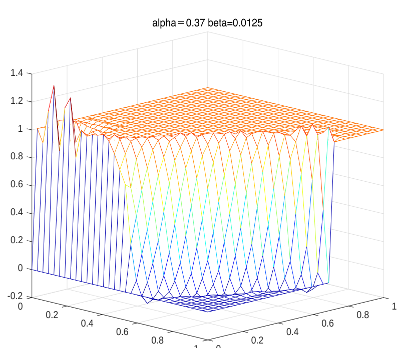
(d) E.Burman Method with mesh-3.



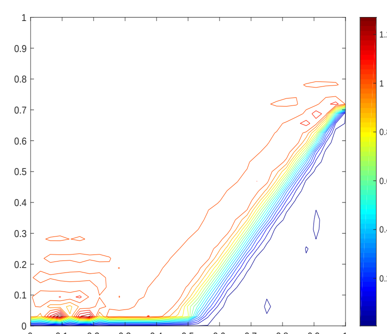
(e) New Method-I with mesh-3.



(f) New Method-I with mesh-3.

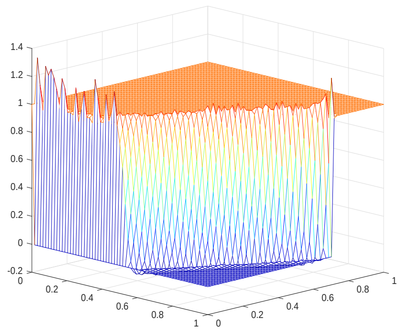


(g) New Method-II with mesh-3.

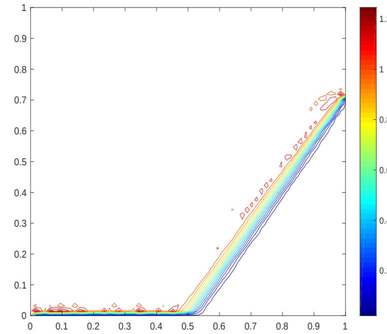


(h) New Method-II with mesh-3.

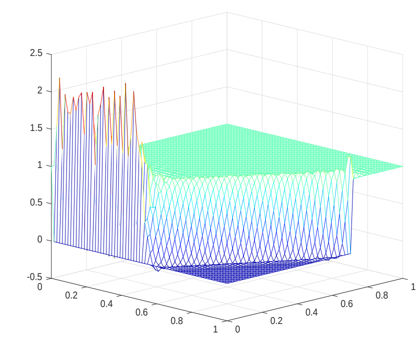
The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/32.



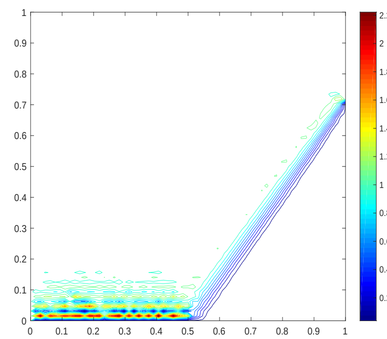
(a) SUPG Method with mesh-3.



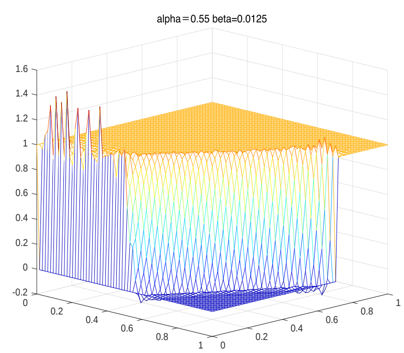
(b) SUPG Method with mesh-3.



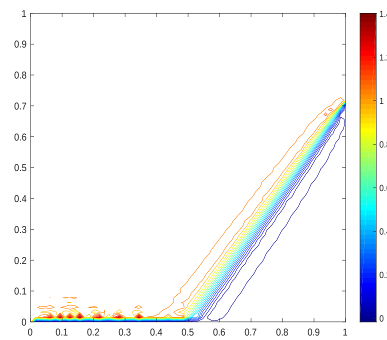
(c) E.Burman Method with mesh-3.



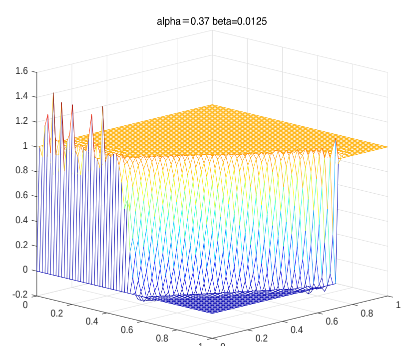
(d) E.Burman Method with mesh-3.



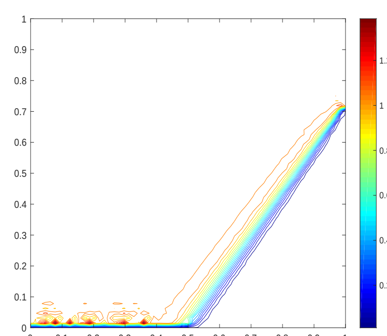
(e) New Method-I with mesh-3.



(f) New Method-I with mesh-3.



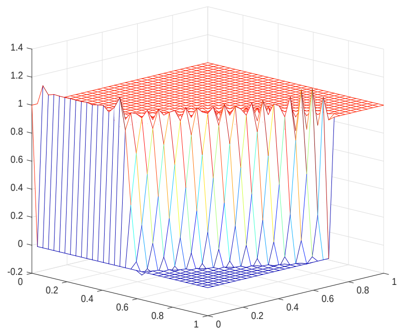
(g) New Method-II with mesh-3.



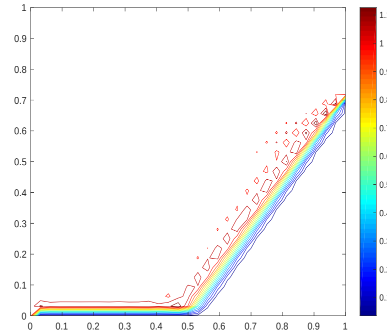
(h) New Method-II with mesh-3.

The elevation and contour of the finite element solution, with ε=10^{-5},$\mathbf{b}$ = $(-cos(55°), -sin(55°))$, γ=0, f=0, h=1/64.

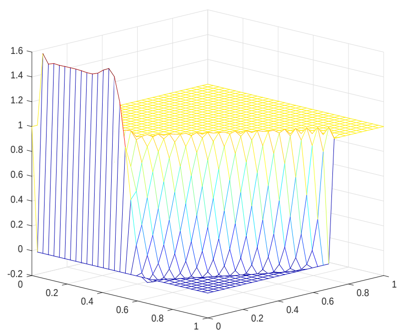
2.2.4 Results with mesh-4



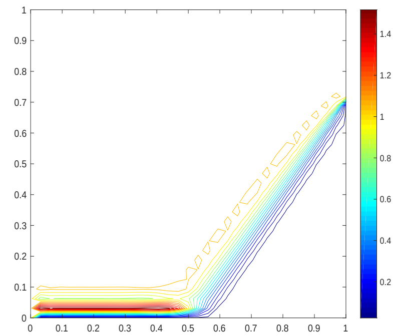
(a) SUPG Method with mesh-4.



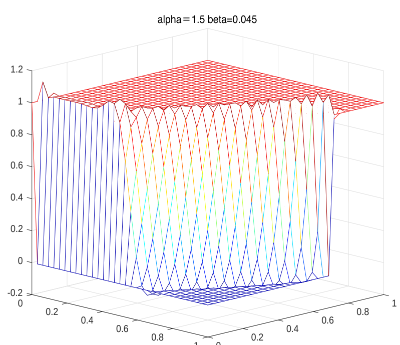
(b) SUPG Method with mesh-4.



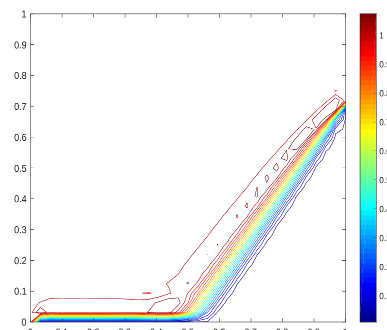
(c) E.Burman Method with mesh-4.



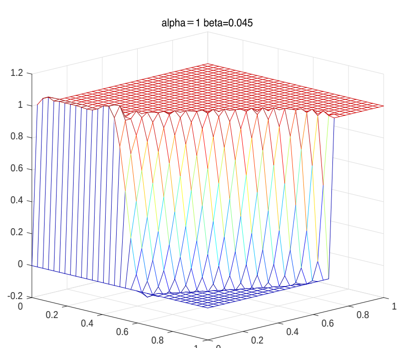
(d) E.Burman Method with mesh-4.



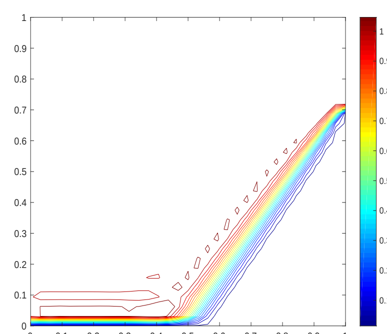
(e) New Method-I with mesh-4.



(f) New Method-I with mesh-4.

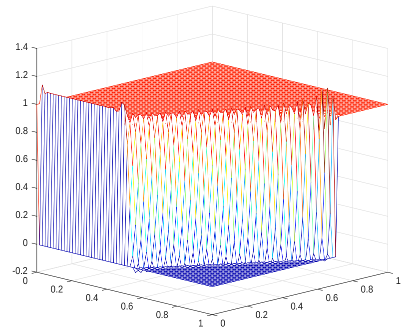


(g) New Method-II with mesh-4.

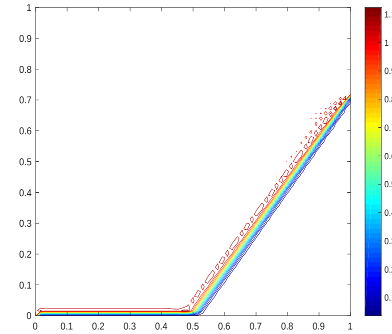


(h) New Method-II with mesh-4.

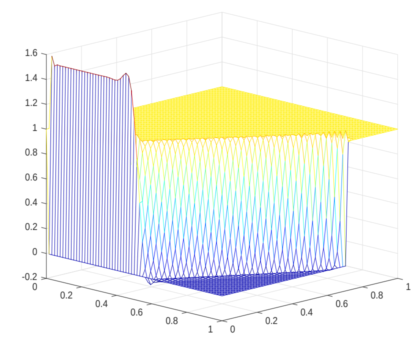
The elevation and contour of the finite element solution, with ε=10^{-5},$\mathbf{b}$ = $(-cos(55°), -sin(55°))$, γ=0, f=0, h=1/32.



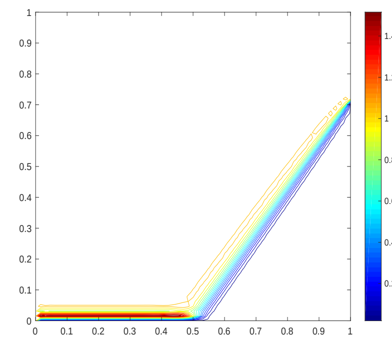
(a) SUPG Method with mesh-4.



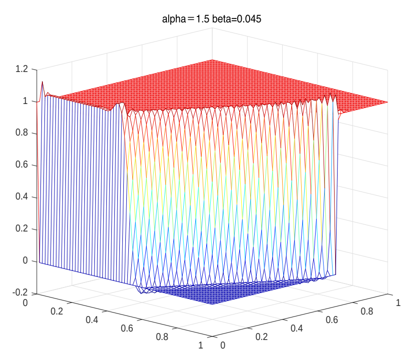
(b) SUPG Method with mesh-4.



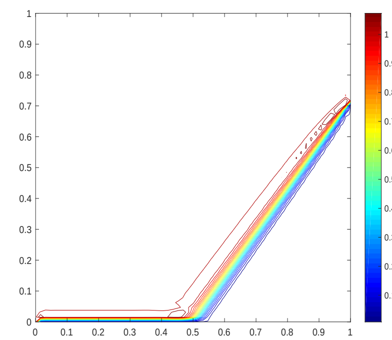
(c) E.Burman Method with mesh-4.



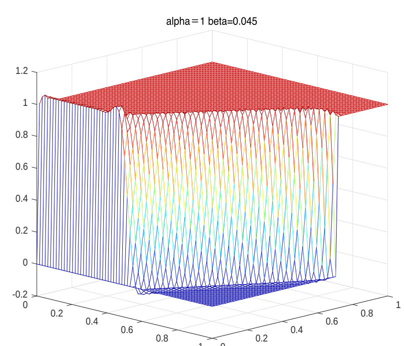
(d) E.Burman Method with mesh-4.



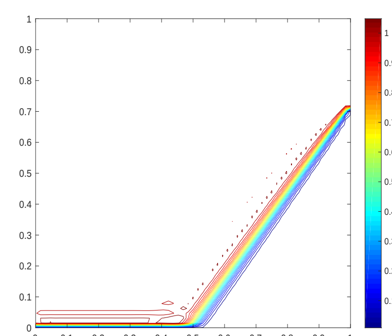
(e) New Method-I with mesh-4.



(f) New Method-I with mesh-4.



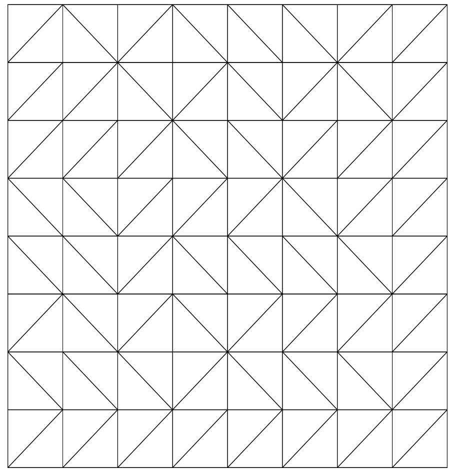
(g) New Method-II with mesh-4.



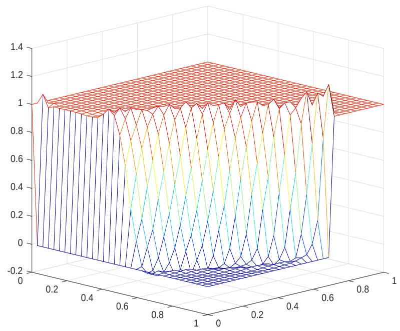
(h) New Method-II with mesh-4.

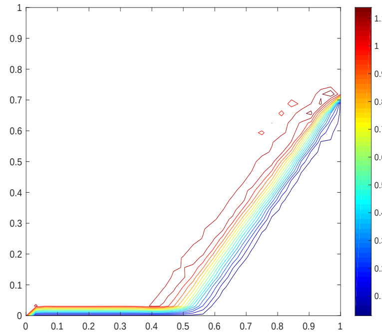
The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/64.

2.2.5 Numerical experiments with modify mesh-3

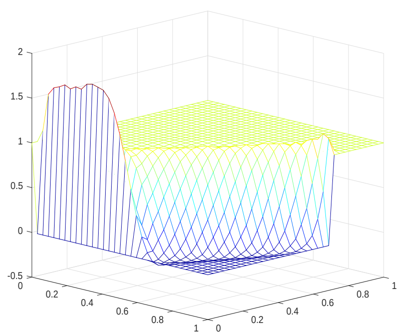


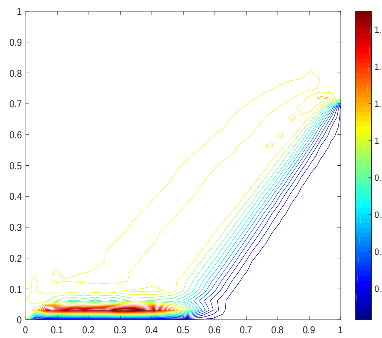
modify mesh-3: We fix the split direction of the bottom boundary, to the upper right corner.



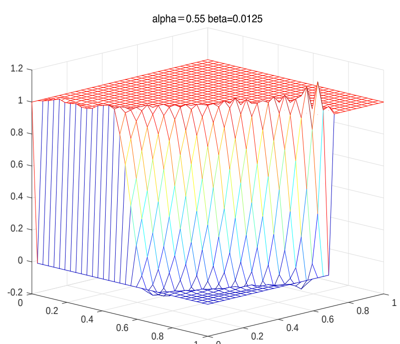


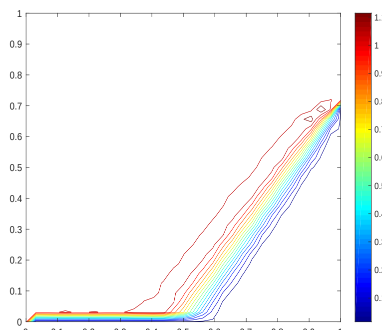
(a) SUPG Method with modify mesh-3. (b) SUPG Method with modify mesh-3.



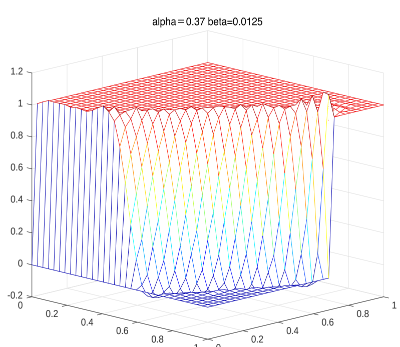


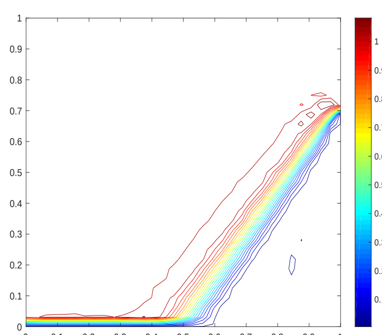
(c) E.Burman Method with modify mesh-3. (d) E.Burman Method with modify mesh-3.





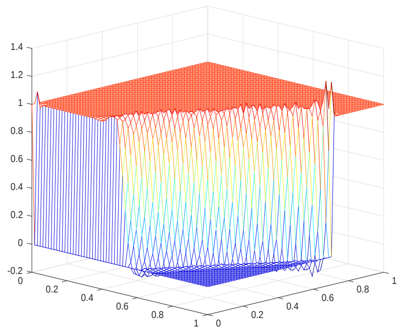
(e) New Method-I with modify mesh-3. (f) New Method-I with modify mesh-3.

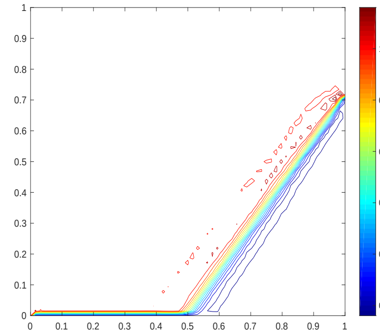




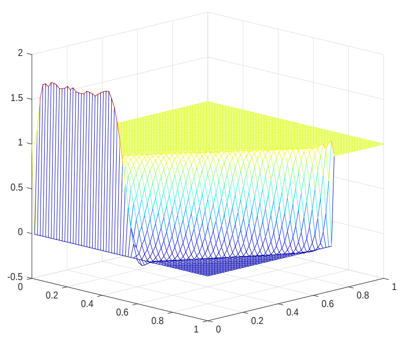
(g) Another Method-I modify mesh-3. (h) Another Method-I modify mesh-3.

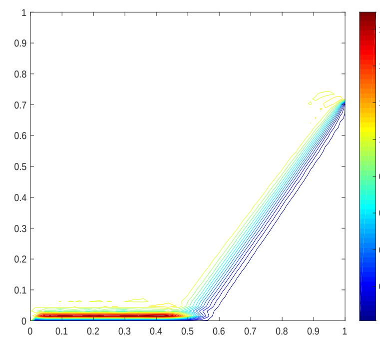
The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/32 .



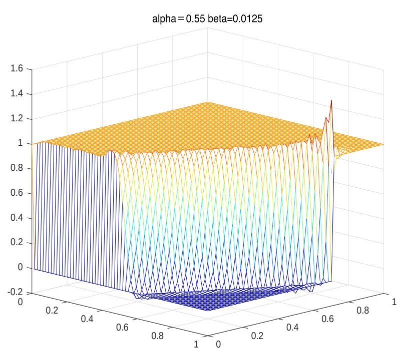


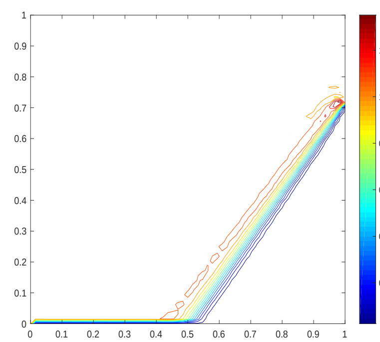
(a) SUPG Method with modify mesh-3. (b) SUPG Method with modify mesh-3.



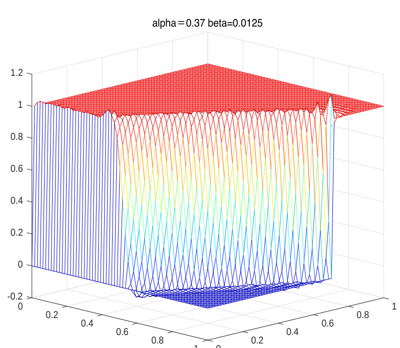


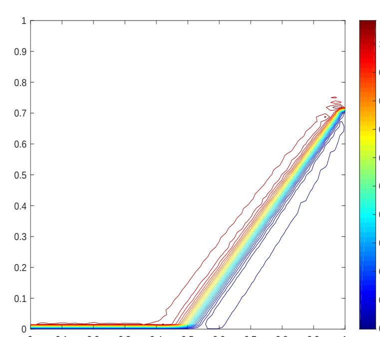
(c) E.Burman Method with modify mesh-3. (d) E.Burman Method with modify mesh-3.





(e) New Method-I with modify mesh-3. (f) New Method-I with modify mesh-3.





(g) Another Method-I modify mesh-3. (h) Another Method-I modify mesh-3.

The elevation and contour of the finite element solution, with ε=10^{-5}, $\mathbf{b} = (-cos(55°), -sin(55°))$, γ=0, f=0, h=1/64 .

3 Numerical examples of Qiu's paper

3.1 example 1: Exact Solution is Known

In this example, the exact solution of problem(1) is taken as follows:

$$u(x,y) = \left(\frac{x^2}{2a_1} + \frac{\varepsilon x}{a_1^2} + \left(\frac{1}{2a_1} + \frac{\varepsilon}{a_1^2}\right)\frac{e^{\frac{-a_1}{\varepsilon}} - e^{\frac{-a_1}{\varepsilon}(1-x)}}{1 - e^{\frac{-a_1}{\varepsilon}}}\right) \times \left(\frac{y^2}{2a_2} + \frac{\varepsilon y}{a_2^2} + \left(\frac{1}{2a_2} + \frac{\varepsilon}{a_2^2}\right)\frac{e^{\frac{-a_2}{\varepsilon}} - e^{\frac{-a_2}{\varepsilon}(1-y)}}{1 - e^{\frac{-a_2}{\varepsilon}}}\right) \tag{2}$$

Choose $\Omega = [0,1] \times [0,1]$, the convecttion field $\mathbf{b} = (b1, b2)^T = (1/2, \sqrt{3}/2)^T$. It can be verified that $u|_{\partial\Omega} = 0$. Defined the errors e_{L^2} in L^2 norm and e_{H^1} in H^1 norm as follows: $e_{L^2} = \|u - u_H\|_0$ and $e_{H^1} = \|u - u_h\|_1$. Take $\gamma = 10^{m_1}, -1 \leq m_1 \leq 2$, and $\varepsilon = 10^{-m_2}, 1 \leq m_2 \leq 2$, where m_1, m_2 are two integers. The result of L^2 norm and H^1 norm are listd in Table 1-4. $h_F = h_{\partial K}$ with mesh-1, we choose $\alpha = 0.55$. From the fomula $log_2(\|u - u_{2h^*}\|_m / \|u - u_{h^*}\|_m)$, where $m = 0, 1$, we can obtain an order, and then computer the average of these orders. In all the table, the order is the average.

Table 25: SUPG: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	5.0626e-04	1.2715e-04	3.1823e-05	7.9578e-06	1.9971e+00
10^{-1}	5.0463e-04	1.2674e-04	3.1720e-05	7.9320e-06	1.9971e+00
10^{0}	4.8785e-04	1.2250e-04	3.0658e-05	7.6663e-06	1.9972e+00
10^{1}	3.6863e-04	9.2237e-05	2.3062e-05	5.7656e-06	1.9995e+00
10^{2}	2.9015e-04	7.1198e-05	1.7704e-05	4.4196e-06	2.0123e+00

Table 26: SUPG: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.5926e-02	2.3049e-02	1.1535e-02	5.7689e-03	9.9764e-01
10^{-1}	4.5925e-02	2.3049e-02	1.1535e-02	5.7689e-03	9.9764e-01
10^{0}	4.5922e-02	2.3048e-02	1.1535e-02	5.7689e-03	9.9761e-01
10^{1}	4.5988e-02	2.3057e-02	1.1536e-02	5.7691e-03	9.9828e-01
10^{2}	4.6745e-02	2.3158e-02	1.1549e-02	5.7707e-03	1.0060e+00

Table 27: SUPG: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.1891e-02	3.7829e-03	1.0221e-03	2.6092e-04	1.8367e+00
10^{-1}	1.1892e-02	3.7825e-03	1.0220e-03	2.6088e-04	1.8368e+00
10^{0}	1.1882e-02	3.7736e-03	1.0192e-03	2.6013e-04	1.8378e+00
10^{1}	1.1089e-02	3.4749e-03	9.3508e-04	2.3846e-04	1.8464e+00
10^{2}	9.3292e-03	2.6304e-03	6.5862e-04	1.6337e-04	1.9452e+00

Table 28: SUPG: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.1617e+00	6.9036e-01	3.6477e-01	1.8511e-01	8.8326e-01
10^{-1}	1.1617e+00	6.9034e-01	3.6477e-01	1.8511e-01	8.8325e-01
10^{0}	1.1613e+00	6.9017e-01	3.6474e-01	1.8510e-01	8.8311e-01
10^{1}	1.1607e+00	6.8924e-01	3.6454e-01	1.8508e-01	8.8292e-01
10^{2}	1.2287e+00	7.1564e-01	3.6965e-01	1.8580e-01	9.0843e-01

Table 29: SUPG: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	5.3326e-04	1.3363e-04	3.3429e-05	8.3589e-06	1.9985e+00
10^{-1}	5.3098e-04	1.3306e-04	3.3286e-05	8.3229e-06	1.9985e+00
10^{0}	5.0830e-04	1.2736e-04	3.1859e-05	7.9661e-06	1.9985e+00
10^{1}	3.7343e-04	9.3265e-05	2.3312e-05	5.8281e-06	2.0006e+00
10^{2}	2.9366e-04	7.2017e-05	1.7909e-05	4.4716e-06	2.0124e+00

Table 30: SUPG: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6031e-02	2.3062e-02	1.1537e-02	5.7691e-03	9.9873e-01
10^{-1}	4.6031e-02	2.3062e-02	1.1537e-02	5.7691e-03	9.9872e-01
10^{0}	4.6027e-02	2.3062e-02	1.1537e-02	5.7691e-03	9.9868e-01
10^{1}	4.6092e-02	2.3070e-02	1.1538e-02	5.7693e-03	9.9935e-01
10^{2}	4.6838e-02	2.3170e-02	1.1551e-02	5.7709e-03	1.0069e+00

Table 31: SUPG: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2156e-02	3.8096e-03	1.0246e-03	2.6131e-04	1.8466e+00
10^{-1}	1.2158e-02	3.8094e-03	1.0245e-03	2.6128e-04	1.8467e+00
10^{0}	1.2148e-02	3.8010e-03	1.0218e-03	2.6059e-04	1.8476e+00
10^{1}	1.1288e-02	3.4907e-03	9.3557e-04	2.3843e-04	1.8550e+00
10^{2}	9.3553e-03	2.6265e-03	6.5758e-04	1.6326e-04	1.9468e+00

Table 32: SUPG: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.1713e+00	6.9368e-01	3.6534e-01	1.8519e-01	8.8702e-01
10^{-1}	1.1713e+00	6.9366e-01	3.6534e-01	1.8519e-01	8.8700e-01
10^{0}	1.1707e+00	6.9347e-01	3.6530e-01	1.8518e-01	8.8677e-01
10^{1}	1.1688e+00	6.9239e-01	3.6509e-01	1.8515e-01	8.8606e-01
10^{2}	1.2350e+00	7.1849e-01	3.7019e-01	1.8588e-01	9.1069e-01

Table 33: SUPG: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	5.2665e-04	1.3584e-04	3.3959e-05	8.5194e-06	1.9833e+00
10^{-1}	5.3336e-04	1.3496e-04	3.3844e-05	8.4845e-06	1.9914e+00
10^{0}	5.2693e-04	1.3086e-04	3.2683e-05	8.2206e-06	2.0007e+00
10^{1}	4.0832e-04	1.0007e-04	2.5006e-05	6.2494e-06	2.0099e+00
10^{2}	3.0451e-04	7.7686e-05	1.9607e-05	4.8959e-06	1.9863e+00

Table 34: SUPG: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6969e-02	2.3444e-02	1.1820e-02	5.9113e-03	9.9672e-01
10^{-1}	4.7536e-02	2.3663e-02	1.1812e-02	5.9096e-03	1.0026e+00
10^{0}	4.7283e-02	2.3830e-02	1.1797e-02	5.9313e-03	9.9831e-01
10^{1}	4.6878e-02	2.3573e-02	1.1786e-02	5.9171e-03	9.9532e-01
10^{2}	4.9085e-02	2.3810e-02	1.1781e-02	5.9068e-03	1.0183e+00

Table 35: SUPG: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2516e-02	3.9328e-03	1.0632e-03	2.7186e-04	1.8416e+00
10^{-1}	1.2491e-02	3.9304e-03	1.0651e-03	2.7199e-04	1.8404e+00
10^{0}	1.2246e-02	3.8204e-03	1.0603e-03	2.7213e-04	1.8306e+00
10^{1}	1.1850e-02	3.5607e-03	9.7769e-04	2.4912e-04	1.8573e+00
10^{2}	1.0008e-02	2.7264e-03	6.8798e-04	1.7497e-04	1.9460e+00

Table 36: SUPG: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.1349e+00	7.0553e-01	3.6785e-01	1.8966e-01	8.6035e-01
10^{-1}	1.1820e+00	6.9554e-01	3.7110e-01	1.9060e-01	8.7752e-01
10^{0}	1.1863e+00	7.1675e-01	3.7154e-01	1.8909e-01	8.8310e-01
10^{1}	1.2046e+00	7.0033e-01	3.7115e-01	1.9124e-01	8.8503e-01
10^{2}	1.2439e+00	7.5592e-01	3.8036e-01	1.8983e-01	9.0403e-01

Table 37: SUPG: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	2.3247e-04	5.8259e-05	1.4573e-05	3.6439e-06	1.9985e+00
10^{-1}	2.3208e-04	5.8160e-05	1.4549e-05	3.6377e-06	1.9985e+00
10^{0}	2.2804e-04	5.7145e-05	1.4295e-05	3.5742e-06	1.9985e+00
10^{1}	1.9810e-04	4.9601e-05	1.2405e-05	3.1015e-06	1.9991e+00
10^{2}	1.6112e-04	3.9978e-05	9.9714e-06	2.4913e-06	2.0050e+00

Table 38: SUPG: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.2504e-02	1.6285e-02	8.1465e-03	4.0738e-03	9.9873e-01
10^{-1}	3.2504e-02	1.6285e-02	8.1465e-03	4.0738e-03	9.9873e-01
10^{0}	3.2506e-02	1.6285e-02	8.1465e-03	4.0738e-03	9.9875e-01
10^{1}	3.2532e-02	1.6288e-02	8.1470e-03	4.0738e-03	9.9913e-01
10^{2}	3.2759e-02	1.6318e-02	8.1507e-03	4.0743e-03	1.0024e+00

Table 39: SUPG: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	6.7456e-03	2.0077e-03	5.3054e-04	1.3463e-04	1.8823e+00
10^{-1}	6.7442e-03	2.0073e-03	5.3043e-04	1.3460e-04	1.8823e+00
10^{0}	6.7306e-03	2.0032e-03	5.2934e-04	1.3433e-04	1.8823e+00
10^{1}	6.4868e-03	1.9359e-03	5.1182e-04	1.2989e-04	1.8807e+00
10^{2}	5.4486e-03	1.6529e-03	4.3418e-04	1.0965e-04	1.8783e+00

Table 40: SUPG: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	9.5713e-01	5.5628e-01	2.9186e-01	1.4784e-01	8.9822e-01
10^{-1}	9.5712e-01	5.5628e-01	2.9186e-01	1.4784e-01	8.9822e-01
10^{0}	9.5711e-01	5.5632e-01	2.9187e-01	1.4784e-01	8.9821e-01
10^{1}	9.6009e-01	5.5728e-01	2.9203e-01	1.4786e-01	8.9964e-01
10^{2}	1.0165e+00	5.7015e-01	2.9410e-01	1.4814e-01	9.2616e-01

Table 41: BM: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.7856e-04	9.4337e-05	2.3551e-05	5.8851e-06	2.0024e+00
10^{-1}	3.7742e-04	9.4080e-05	2.3489e-05	5.8695e-06	2.0023e+00
10^{0}	3.6811e-04	9.1933e-05	2.2963e-05	5.7385e-06	2.0011e+00
10^{1}	3.2623e-04	8.1582e-05	2.0390e-05	5.0967e-06	2.0001e+00
10^{2}	2.5967e-04	6.3964e-05	1.5923e-05	3.9762e-06	2.0097e+00

Table 42: BM: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.5989e-02	2.3057e-02	1.1536e-02	5.7691e-03	9.9829e-01
10^{-1}	4.5988e-02	2.3057e-02	1.1536e-02	5.7691e-03	9.9829e-01
10^{0}	4.5987e-02	2.3056e-02	1.1536e-02	5.7691e-03	9.9827e-01
10^{1}	4.6013e-02	2.3060e-02	1.1537e-02	5.7691e-03	9.9854e-01
10^{2}	4.6230e-02	2.3089e-02	1.1540e-02	5.7696e-03	1.0007e+00

Table 43: BM: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	9.4491e-03	2.7260e-03	7.0438e-04	1.7629e-04	1.9147e+00
10^{-1}	9.4354e-03	2.7194e-03	7.0334e-04	1.7617e-04	1.9143e+00
10^{0}	9.3732e-03	2.6863e-03	6.9705e-04	1.7530e-04	1.9135e+00
10^{1}	9.1641e-03	2.6223e-03	6.8052e-04	1.7164e-04	1.9129e+00
10^{2}	8.9147e-03	2.5208e-03	6.3996e-04	1.6001e-04	1.9333e+00

Table 44: BM: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2365e+00	7.0611e-01	3.6721e-01	1.8544e-01	9.1242e-01
10^{-1}	1.2389e+00	7.0640e-01	3.6722e-01	1.8544e-01	9.1334e-01
10^{0}	1.2488e+00	7.0771e-01	3.6731e-01	1.8545e-01	9.1716e-01
10^{1}	1.2463e+00	7.0804e-01	3.6735e-01	1.8545e-01	9.1618e-01
10^{2}	1.2308e+00	7.1106e-01	3.6826e-01	1.8558e-01	9.0981e-01

Table 45: BM: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.5939e-04	8.9541e-05	2.2357e-05	5.5873e-06	2.0024e+00
10^{-1}	3.5872e-04	8.9400e-05	2.2323e-05	5.5788e-06	2.0022e+00
10^{0}	3.5276e-04	8.8076e-05	2.2002e-05	5.4991e-06	2.0011e+00
10^{1}	3.2013e-04	8.0015e-05	2.0001e-05	5.0000e-06	2.0002e+00
10^{2}	2.6067e-04	6.4170e-05	1.5978e-05	3.9907e-06	2.0098e+00

Table 46: BM: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6063e-02	2.3066e-02	1.1537e-02	5.7692e-03	9.9905e-01
10^{-1}	4.6063e-02	2.3066e-02	1.1537e-02	5.7692e-03	9.9906e-01
10^{0}	4.6068e-02	2.3067e-02	1.1537e-02	5.7692e-03	9.9911e-01
10^{1}	4.6109e-02	2.3072e-02	1.1538e-02	5.7693e-03	9.9953e-01
10^{2}	4.6323e-02	2.3100e-02	1.1542e-02	5.7698e-03	1.0017e+00

Table 47: BM: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	9.2846e-03	2.6774e-03	6.9341e-04	1.7387e-04	1.9129e+00
10^{-1}	9.2748e-03	2.6715e-03	6.9246e-04	1.7377e-04	1.9127e+00
10^{0}	9.2349e-03	2.6425e-03	6.8680e-04	1.7301e-04	1.9127e+00
10^{1}	9.0897e-03	2.5904e-03	6.7281e-04	1.6993e-04	1.9137e+00
10^{2}	8.9121e-03	2.5066e-03	6.3660e-04	1.5939e-04	1.9350e+00

Table 48: BM: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2430e+00	7.0840e-01	3.6761e-01	1.8550e-01	9.1479e-01
10^{-1}	1.2455e+00	7.0870e-01	3.6762e-01	1.8550e-01	9.1575e-01
10^{0}	1.2562e+00	7.1010e-01	3.6772e-01	1.8550e-01	9.1985e-01
10^{1}	1.2563e+00	7.1078e-01	3.6780e-01	1.8551e-01	9.1989e-01
10^{2}	1.2426e+00	7.1412e-01	3.6875e-01	1.8565e-01	9.1422e-01

Table 49: BM: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.8742e-04	9.6620e-05	2.4026e-05	6.0494e-06	2.0003e+00
10^{-1}	3.9438e-04	9.5925e-05	2.4118e-05	6.0232e-06	2.0110e+00
10^{0}	3.6692e-04	9.5175e-05	2.3581e-05	5.9804e-06	1.9797e+00
10^{1}	3.5089e-04	8.6837e-05	2.1867e-05	5.5247e-06	1.9963e+00
10^{2}	2.7750e-04	7.5432e-05	1.8531e-05	4.6125e-06	1.9703e+00

Table 50: BM: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.5862e-02	2.3621e-02	1.1788e-02	5.9229e-03	9.8431e-01
10^{-1}	4.6673e-02	2.3395e-02	1.1752e-02	5.9086e-03	9.9390e-01
10^{0}	4.7417e-02	2.3350e-02	1.1752e-02	5.9311e-03	9.9968e-01
10^{1}	4.6187e-02	2.3607e-02	1.1837e-02	5.9150e-03	9.8835e-01
10^{2}	4.7578e-02	2.3633e-02	1.1745e-02	5.9066e-03	1.0033e+00

Table 51: BM: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	9.9791e-03	2.7300e-03	7.3847e-04	1.8299e-04	1.9230e+00
10^{-1}	9.2924e-03	2.9471e-03	7.3006e-04	1.8467e-04	1.8844e+00
10^{0}	9.1471e-03	2.8056e-03	7.2122e-04	1.8256e-04	1.8823e+00
10^{1}	8.9121e-03	2.8014e-03	7.1245e-04	1.8129e-04	1.8731e+00
10^{2}	8.7837e-03	2.6520e-03	6.7988e-04	1.7136e-04	1.8932e+00

Table 52: BM: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2190e+00	7.2174e-01	3.7167e-01	1.8978e-01	8.9442e-01
10^{-1}	1.2616e+00	7.2304e-01	3.7769e-01	1.9027e-01	9.0972e-01
10^{0}	1.2682e+00	7.1173e-01	3.7668e-01	1.9026e-01	9.1225e-01
10^{1}	1.2379e+00	7.2893e-01	3.7373e-01	1.8917e-01	9.0336e-01
10^{2}	1.2610e+00	7.2090e-01	3.7155e-01	1.9079e-01	9.0819e-01

Table 53: BM: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	2.0510e-04	5.1218e-05	1.2797e-05	3.1987e-06	2.0009e+00
10^{-1}	2.0480e-04	5.1152e-05	1.2781e-05	3.1947e-06	2.0008e+00
10^{0}	2.0237e-04	5.0603e-05	1.2647e-05	3.1613e-06	2.0001e+00
10^{1}	1.9045e-04	4.7675e-05	1.1921e-05	2.9803e-06	1.9993e+00
10^{2}	1.6240e-04	4.0392e-05	1.0079e-05	2.5184e-06	2.0036e+00

Table 54: BM: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.2527e-02	1.6288e-02	8.1469e-03	4.0738e-03	9.9906e-01
10^{-1}	3.2527e-02	1.6288e-02	8.1469e-03	4.0738e-03	9.9905e-01
10^{0}	3.2527e-02	1.6288e-02	8.1469e-03	4.0738e-03	9.9906e-01
10^{1}	3.2537e-02	1.6289e-02	8.1470e-03	4.0738e-03	9.9921e-01
10^{2}	3.2630e-02	1.6301e-02	8.1486e-03	4.0740e-03	1.0006e+00

Table 55: BM: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	6.3653e-03	1.8474e-03	4.7870e-04	1.2009e-04	1.9093e+00
10^{-1}	6.3192e-03	1.8406e-03	4.7796e-04	1.2002e-04	1.9061e+00
10^{0}	6.1253e-03	1.8082e-03	4.7357e-04	1.1948e-04	1.8933e+00
10^{1}	5.8675e-03	1.7567e-03	4.6283e-04	1.1724e-04	1.8817e+00
10^{2}	5.5010e-03	1.6802e-03	4.4100e-04	1.1099e-04	1.8771e+00

Table 56: BM: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	9.9352e-01	5.6415e-01	2.9310e-01	1.4801e-01	9.1561e-01
10^{-1}	9.9434e-01	5.6427e-01	2.9311e-01	1.4801e-01	9.1601e-01
10^{0}	9.9841e-01	5.6487e-01	2.9316e-01	1.4801e-01	9.1796e-01
10^{1}	9.9832e-01	5.6509e-01	2.9318e-01	1.4801e-01	9.1793e-01
10^{2}	1.0020e+00	5.6642e-01	2.9353e-01	1.4807e-01	9.1948e-01

Table 57: NEW-I: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	7.7041e-04	1.5564e-04	3.1886e-05	6.9481e-06	2.2643e+00
10^{-1}	7.4793e-04	1.5343e-04	3.1689e-05	6.9254e-06	2.2516e+00
10^{0}	6.0297e-04	1.3658e-04	2.9973e-05	6.7182e-06	2.1626e+00
10^{1}	3.6401e-04	9.2195e-05	2.2840e-05	5.5749e-06	2.0096e+00
10^{2}	2.6100e-04	6.4508e-05	1.6096e-05	4.0231e-06	2.0065e+00

Table 58: NEW-I: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.7202e-02	2.3192e-02	1.1548e-02	5.7699e-03	1.0107e+00
10^{-1}	4.7081e-02	2.3184e-02	1.1547e-02	5.7699e-03	1.0095e+00
10^{0}	4.6433e-02	2.3130e-02	1.1544e-02	5.7697e-03	1.0029e+00
10^{1}	4.5985e-02	2.3060e-02	1.1537e-02	5.7693e-03	9.9824e-01
10^{2}	4.6221e-02	2.3087e-02	1.1540e-02	5.7695e-03	1.0007e+00

Table 59: NEW-I: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.6797e-02	6.5191e-03	1.9275e-03	4.4482e-04	1.7463e+00
10^{-1}	1.5651e-02	6.0806e-03	1.8204e-03	4.2850e-04	1.7302e+00
10^{0}	1.0825e-02	4.0239e-03	1.2566e-03	3.2838e-04	1.6809e+00
10^{1}	9.0553e-03	2.6987e-03	7.4356e-04	1.9709e-04	1.8406e+00
10^{2}	8.9135e-03	2.5246e-03	6.4404e-04	1.6202e-04	1.9272e+00

Table 60: NEW-I: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2656e+00	7.9377e-01	4.1118e-01	1.9526e-01	8.9880e-01
10^{-1}	1.2362e+00	7.7301e-01	4.0420e-01	1.9418e-01	8.9015e-01
10^{0}	1.1632e+00	7.0146e-01	3.7531e-01	1.8871e-01	8.7462e-01
10^{1}	1.2233e+00	6.9960e-01	3.6563e-01	1.8527e-01	9.0769e-01
10^{2}	1.2299e+00	7.1041e-01	3.6803e-01	1.8553e-01	9.0960e-01

Table 61: NEW-I: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	6.4098e-04	1.3341e-04	2.8422e-05	6.3791e-06	2.2169e+00
10^{-1}	6.3820e-04	1.3300e-04	2.8355e-05	6.3666e-06	2.2158e+00
10^{0}	6.1356e-04	1.2930e-04	2.7744e-05	6.2516e-06	2.2056e+00
10^{1}	4.8914e-04	1.0979e-04	2.4344e-05	5.5811e-06	2.1512e+00
10^{2}	2.9302e-04	7.4763e-05	1.7992e-05	4.2900e-06	2.0313e+00

Table 62: NEW-I: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6579e-02	2.3121e-02	1.1542e-02	5.7695e-03	1.0044e+00
10^{-1}	4.6576e-02	2.3121e-02	1.1542e-02	5.7695e-03	1.0043e+00
10^{0}	4.6540e-02	2.3119e-02	1.1542e-02	5.7695e-03	1.0040e+00
10^{1}	4.6345e-02	2.3109e-02	1.1542e-02	5.7696e-03	1.0020e+00
10^{2}	4.6202e-02	2.3094e-02	1.1542e-02	5.7698e-03	1.0005e+00

Table 63: NEW-I: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.4378e-02	5.3254e-03	1.5180e-03	3.4786e-04	1.7897e+00
10^{-1}	1.4346e-02	5.3182e-03	1.5168e-03	3.4771e-04	1.7889e+00
10^{0}	1.4042e-02	5.2477e-03	1.5053e-03	3.4626e-04	1.7806e+00
10^{1}	1.1894e-02	4.6738e-03	1.4053e-03	3.3310e-04	1.7194e+00
10^{2}	8.9883e-03	2.9136e-03	9.5829e-04	2.6238e-04	1.6994e+00

Table 64: NEW-I: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2122e+00	7.4344e-01	3.8729e-01	1.8967e-01	8.9201e-01
10^{-1}	1.2115e+00	7.4320e-01	3.8725e-01	1.8966e-01	8.9178e-01
10^{0}	1.2058e+00	7.4088e-01	3.8683e-01	1.8962e-01	8.8961e-01
10^{1}	1.1751e+00	7.2284e-01	3.8319e-01	1.8928e-01	8.7804e-01
10^{2}	1.2127e+00	6.9490e-01	3.6885e-01	1.8733e-01	8.9817e-01

Table 65: NEW-I: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	6.5188e-04	1.3694e-04	2.9579e-05	6.7688e-06	2.1965e+00
10^{-1}	6.5736e-04	1.3656e-04	2.9647e-05	6.7408e-06	2.2025e+00
10^{0}	6.1036e-04	1.3329e-04	2.8933e-05	6.6763e-06	2.1715e+00
10^{1}	5.1391e-04	1.1530e-04	2.6013e-05	6.0781e-06	2.1339e+00
10^{2}	3.0551e-04	8.5128e-05	2.0384e-05	4.8859e-06	1.9888e+00

Table 66: NEW-I: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6240e-02	2.3618e-02	1.1789e-02	5.9226e-03	9.8828e-01
10^{-1}	4.7051e-02	2.3390e-02	1.1751e-02	5.9079e-03	9.9783e-01
10^{0}	4.7671e-02	2.3351e-02	1.1750e-02	5.9306e-03	1.0023e+00
10^{1}	4.6269e-02	2.3617e-02	1.1834e-02	5.9145e-03	9.8924e-01
10^{2}	4.7393e-02	2.3611e-02	1.1737e-02	5.9058e-03	1.0015e+00

Table 67: NEW-I: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.4108e-02	5.2936e-03	1.5299e-03	3.5315e-04	1.7733e+00
10^{-1}	1.3496e-02	5.5025e-03	1.5304e-03 3.5385e-04	1.7511e+00	
10^{0}	1.3411e-02	5.2939e-03	1.4987e-03	3.5104e-04	1.7519e+00
10^{1}	1.1410e-02	4.7762e-03	1.4250e-03	3.3925e-04	1.6906e+00
10^{2}	8.7348e-03	3.0053e-03	9.9057e-04	2.7018e-04	1.6716e+00

Table 68: NEW-I: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.1679e+00	7.4839e-01	3.8980e-01	1.9270e-01	8.6647e-01
10^{-1}	1.1921e+00	7.5297e-01	3.9310e-01	1.9267e-01	8.7645e-01
10^{0}	1.1918e+00	7.5330e-01	3.9178e-01	1.9297e-01	8.7556e-01
10^{1}	1.1543e+00	7.2591e-01	3.8688e-01	1.9124e-01	8.6454e-01
10^{2}	1.2223e+00	6.9254e-01	3.7005e-01	1.9091e-01	8.9288e-01

Table 69: NEW-I: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.9328e-04	9.4157e-05	1.8429e-05	3.8985e-06	2.3278e+00
10^{-1}	4.9151e-04	9.3955e-05	1.8405e-05	3.8945e-06	2.3265e+00
10^{0}	4.7544e-04	9.2060e-05	1.8169e-05	3.8562e-06	2.3153e+00
10^{1}	3.8488e-04	8.0158e-05	1.6478e-05	3.5727e-06	2.2504e+00
10^{2}	2.1247e-04	5.3864e-05	1.2407e-05	2.8459e-06	2.0741e+00

Table 70: NEW-I: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.4497e-02	1.6486e-02	8.1625e-03	4.0749e-03	1.0272e+00
10^{-1}	3.4490e-02	1.6485e-02	8.1625e-03	4.0749e-03	1.0271e+00
10^{0}	3.4429e-02	1.6482e-02	8.1624e-03	4.0749e-03	1.0263e+00
10^{1}	3.3994e-02	1.6461e-02	8.1615e-03	4.0749e-03	1.0201e+00
10^{2}	3.2846e-02	1.6378e-02	8.1583e-03	4.0748e-03	1.0036e+00

Table 71: NEW-I: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.3510e-02	5.0083e-03	1.4254e-03	3.1556e-04	1.8067e+00
10^{-1}	1.3488e-02	5.0025e-03	1.4243e-03	3.1541e-04	1.8061e+00
10^{0}	1.3278e-02	4.9455e-03	1.4135e-03	3.1392e-04	1.8008e+00
10^{1}	1.1630e-02	4.4702e-03	1.3205e-03	3.0080e-04	1.7576e+00
10^{2}	6.8714e-03	2.6572e-03	8.9021e-04	2.3322e-04	1.6269e+00

Table 72: NEW-I: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.3234e+00	7.7735e-01	3.7026e-01	1.6279e-01	1.0077e+00
10^{-1}	1.3225e+00	7.7700e-01	3.7019e-01	1.6278e-01	1.0074e+00
10^{0}	1.3132e+00	7.7358e-01	3.6952e-01	1.6272e-01	1.0042e+00
10^{1}	1.2340e+00	7.4314e-01	3.6336e-01	1.6212e-01	9.7605e-01
10^{2}	9.8661e-01	6.0858e-01	3.2812e-01	1.5811e-01	8.8052e-01

Table 73: NEW-II: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	8.0509e-04	1.6081e-04	3.2559e-05	7.0305e-06	2.2798e+00
10^{-1}	7.8048e-04	1.5839e-04	3.2346e-05	7.0068e-06	2.2665e+00
10^{0}	6.2229e-04	1.4003e-04	3.0496e-05	6.7897e-06	2.1727e+00
10^{1}	3.6623e-04	9.2756e-05	2.2963e-05	5.5984e-06	2.0105e+00
10^{2}	2.6102e-04	6.4517e-05	1.6098e-05	4.0238e-06	2.0065e+00

Table 74: NEW-II: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.7237e-02	2.3194e-02	1.1548e-02	5.7699e-03	1.0111e+00
10^{-1}	4.7113e-02	2.3185e-02	1.1547e-02	5.7699e-03	1.0098e+00
10^{0}	4.6447e-02	2.3131e-02	1.1544e-02	5.7697e-03	1.0030e+00
10^{1}	4.5986e-02	2.3060e-02	1.1537e-02	5.7693e-03	9.9824e-01
10^{2}	4.6221e-02	2.3087e-02	1.1540e-02	5.7695e-03	1.0007e+00

Table 75: NEW-II: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.9292e-02	7.4687e-03	2.1673e-03	4.8835e-04	1.7680e+00
10^{-1}	1.7970e-02	6.9547e-03	2.0423e-03	4.6953e-04	1.7528e+00
10^{0}	1.1968e-02	4.4717e-03	1.3778e-03	3.5375e-04	1.6934e+00
10^{1}	9.1073e-03	2.7345e-03	7.5695e-04	2.0087e-04	1.8343e+00
10^{2}	8.9143e-03	2.5255e-03	6.4452e-04	1.6219e-04	1.9268e+00

Table 76: NEW-II: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-1

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.3242e+00	8.2660e-01	4.2043e-01	1.9667e-01	9.1708e-01
10^{-1}	1.2873e+00	8.0167e-01	4.1234e-01	1.9546e-01	9.0648e-01
10^{0}	1.1766e+00	7.1080e-01	3.7831e-01	1.8929e-01	8.7867e-01
10^{1}	1.2220e+00	6.9961e-01	3.6573e-01	1.8530e-01	9.0709e-01
10^{2}	1.2298e+00	7.1038e-01	3.6803e-01	1.8553e-01	9.0956e-01

Table 77: NEW-II: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	6.6316e-04	1.3658e-04	2.8830e-05	6.4295e-06	2.2295e+00
10^{-1}	6.5994e-04	1.3611e-04	2.8756e-05	6.4163e-06	2.2281e+00
10^{0}	6.3172e-04	1.3194e-04	2.8088e-05	6.2948e-06	2.2163e+00
10^{1}	4.9580e-04	1.1086e-04	2.4494e-05	5.6009e-06	2.1560e+00
10^{2}	2.9338e-04	7.4858e-05	1.8010e-05	4.2927e-06	2.0316e+00

Table 78: NEW-II: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6590e-02	2.3121e-02	1.1542e-02	5.7695e-03	1.0045e+00
10^{-1}	4.6585e-02	2.3121e-02	1.1542e-02	5.7695e-03	1.0044e+00
10^{0}	4.6547e-02	2.3119e-02	1.1542e-02	5.7695e-03	1.0041e+00
10^{1}	4.6344e-02	2.3109e-02	1.1542e-02	5.7696e-03	1.0019e+00
10^{2}	4.6201e-02	2.3094e-02	1.1542e-02	5.7698e-03	1.0004e+00

Table 79: NEW-II: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.6448e-02	6.0566e-03	1.6874e-03	3.7646e-04	1.8164e+00
10^{-1}	1.6404e-02	6.0460e-03	1.6856e-03	3.7622e-04	1.8155e+00
10^{0}	1.5987e-02	5.9440e-03	1.6679e-03	3.7389e-04	1.8060e+00
10^{1}	1.3062e-02	5.1603e-03	1.5273e-03	3.5481e-04	1.7341e+00
10^{2}	9.0390e-03	2.9893e-03	9.9368e-04	2.7084e-04	1.6869e+00

Table 80: NEW-II: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-2

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2531e+00	7.6462e-01	3.9245e-01	1.9035e-01	9.0626e-01
10^{-1}	1.2522e+00	7.6429e-01	3.9240e-01	1.9035e-01	9.0591e-01
10^{0}	1.2434e+00	7.6106e-01	3.9184e-01	1.9030e-01	9.0264e-01
10^{1}	1.1919e+00	7.3650e-01	3.8720e-01	1.8988e-01	8.8338e-01
10^{2}	1.2102e+00	6.9527e-01	3.6987e-01	1.8763e-01	8.9641e-01

Table 81: NEW-II: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	6.7398e-04	1.4017e-04	2.9988e-05	6.8197e-06	2.2089e+00
10^{-1}	6.7874e-04	1.3972e-04	3.0071e-05	6.7905e-06	2.2144e+00
10^{0}	6.3197e-04	1.3611e-04	2.9293e-05	6.7237e-06	2.1848e+00
10^{1}	5.2199e-04	1.1666e-04	2.6192e-05	6.1020e-06	2.1395e+00
10^{2}	3.0623e-04	8.5297e-05	2.0410e-05	4.8895e-06	1.9896e+00

Table 82: NEW-II: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	4.6237e-02	2.3619e-02	1.1788e-02	5.9226e-03	9.8824e-01
10^{-1}	4.7074e-02	2.3394e-02	1.1750e-02	5.9079e-03	9.9808e-01
10^{0}	4.7685e-02	2.3355e-02	1.1750e-02	5.9306e-03	1.0024e+00
10^{1}	4.6254e-02	2.3617e-02	1.1834e-02	5.9145e-03	9.8908e-01
10^{2}	4.7401e-02	2.3609e-02	1.1737e-02	5.9058e-03	1.0016e+00

Table 83: NEW-II: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.5906e-02	6.0281e-03	1.6992e-03	3.8175e-04	1.7936e+00
10^{-1}	1.5413e-02	6.2435e-03	1.7012e-03	3.8216e-04	1.7780e+00
10^{0}	1.5169e-02	6.0092e-03	1.6566e-03	3.7890e-04	1.7744e+00
10^{1}	1.2481e-02	5.2607e-03	1.5455e-03	3.6117e-04	1.7036e+00
10^{2}	8.7646e-03	3.0800e-03	1.0251e-03	2.7878e-04	1.6582e+00

Table 84: NEW-II: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-3

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.2117e+00	7.7257e-01	3.9510e-01	1.9330e-01	8.8269e-01
10^{-1}	1.2292e+00	7.7685e-01	3.9829e-01	1.9334e-01	8.8950e-01
10^{0}	1.2257e+00	7.7301e-01	3.9655e-01	1.9366e-01	8.8736e-01
10^{1}	1.1641e+00	7.4286e-01	3.9079e-01	1.9196e-01	8.6678e-01
10^{2}	1.2186e+00	6.9349e-01	3.7144e-01	1.9123e-01	8.9062e-01

Table 85: NEW-II: when $\varepsilon = 0.1$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	5.1521e-04	9.7179e-05	1.8787e-05	3.9379e-06	2.3439e+00
10^{-1}	5.1304e-04	9.6929e-05	1.8758e-05	3.9337e-06	2.3423e+00
10^{0}	4.9364e-04	9.4629e-05	1.8483e-05	3.8924e-06	2.3289e+00
10^{1}	3.9194e-04	8.1261e-05	1.6628e-05	3.5917e-06	2.2566e+00
10^{2}	2.1293e-04	5.3972e-05	1.2425e-05	2.8486e-06	2.0747e+00

Table 86: NEW-II: when $\varepsilon = 0.1$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	3.4516e-02	1.6486e-02	8.1624e-03	4.0749e-03	1.0275e+00
10^{-1}	3.4509e-02	1.6485e-02	8.1624e-03	4.0749e-03	1.0274e+00
10^{0}	3.4443e-02	1.6482e-02	8.1623e-03	4.0749e-03	1.0265e+00
10^{1}	3.3993e-02	1.6460e-02	8.1614e-03	4.0749e-03	1.0201e+00
10^{2}	3.2844e-02	1.6377e-02	8.1582e-03	4.0748e-03	1.0036e+00

Table 87: NEW-II: when $\varepsilon = 0.01$, errors in L^2 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.4984e-02	5.6135e-03	1.5831e-03	3.4400e-04	1.8149e+00
10^{-1}	1.4956e-02	5.6054e-03	1.5815e-03	3.4376e-04	1.8144e+00
10^{0}	1.4692e-02	5.5268e-03	1.5654e-03	3.4142e-04	1.8091e+00
10^{1}	1.2664e-02	4.9056e-03	1.4376e-03	3.2253e-04	1.7651e+00
10^{2}	7.0710e-03	2.7677e-03	9.3058e-04	2.4233e-04	1.6223e+00

Table 88: NEW-II: when $\varepsilon = 0.01$, errors in H^1 norm under different values of γ in mesh-4

γ	$h^* = 1/32$	$h^* = 1/64$	$h^* = 1/128$	$h^* = 1/256$	Order
10^{-2}	1.3845e+00	8.1148e-01	3.8014e-01	1.6430e-01	1.0250e+00
10^{-1}	1.3835e+00	8.1107e-01	3.8006e-01	1.6429e-01	1.0247e+00
10^{0}	1.3732e+00	8.0707e-01	3.7926e-01	1.6422e-01	1.0213e+00
10^{1}	1.2846e+00	7.7184e-01	3.7206e-01	1.6352e-01	9.9128e-01
10^{2}	9.9321e-01	6.1751e-01	3.3244e-01	1.5907e-01	8.8082e-01

3.2 example 2: Boundary and Inner Layers Present

Choose $\Omega = [0,1] \times [0,1]$, the convection field $\mathbf{b} = (b1, b2)^T = (1/2, \sqrt{3}/2)^T$, $f = 0$, $\gamma = 0$, $h^* = 1/32$, and $h_F = h_{\partial K}$ with mesh-1. The domain with boundary data indicated is given in Fig 4.The following gives the elevation, the contour, and the cut of the finite element solution for $\varepsilon = 10^{-2}, \varepsilon = 10^{-6}, \gamma = 0, h^* = 1/32$.

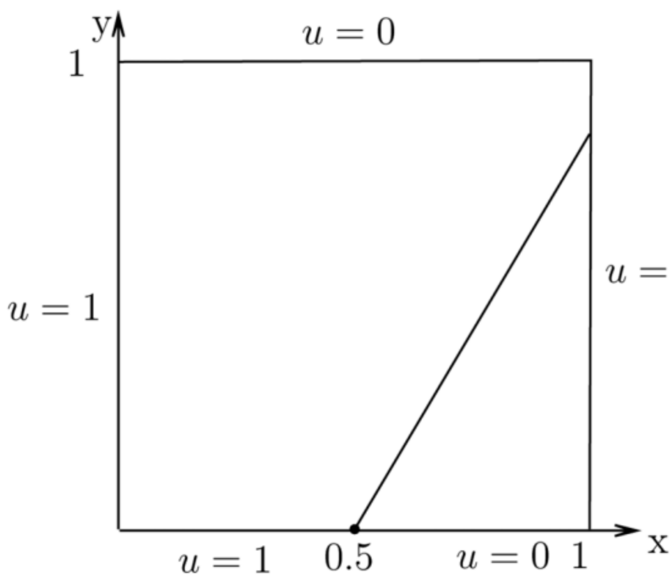
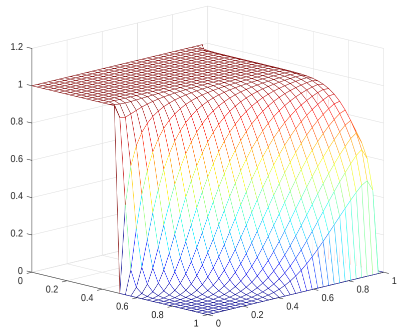
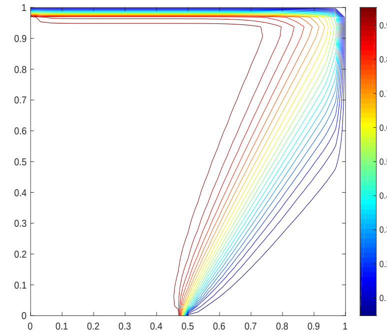


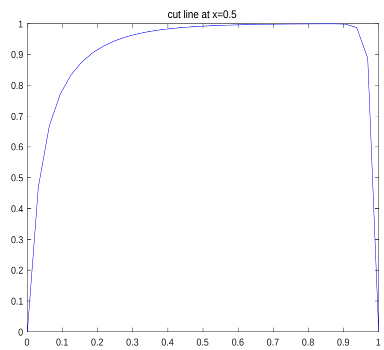
Fig 4: Domain and boundary data



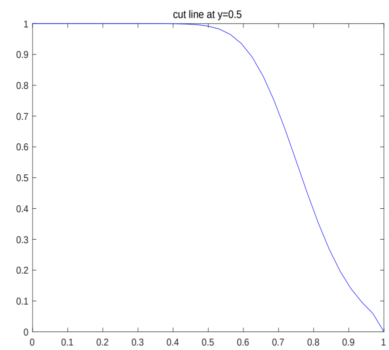
(a) Elevation



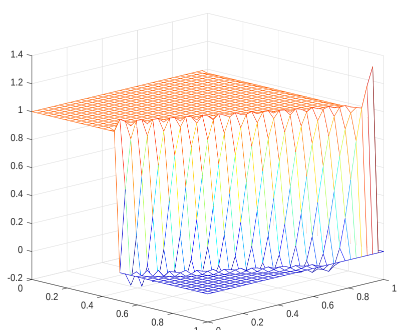
(b) Contour



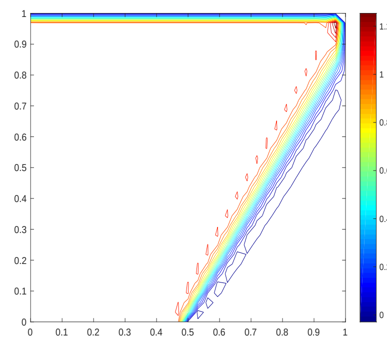
(c) cut line at $x = 0.5$



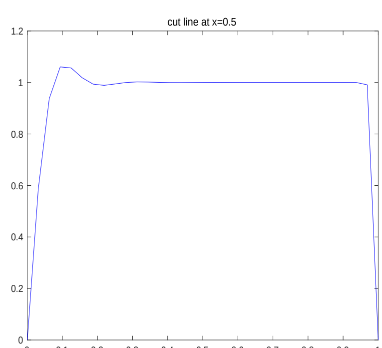
(d) cut line at $y = 0.5$



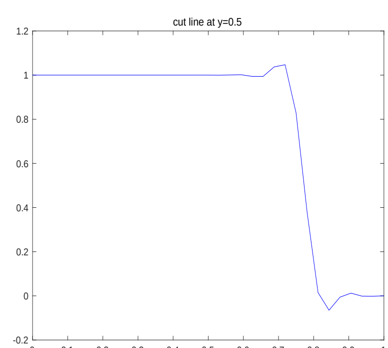
(e) Elevation



(f) Contour

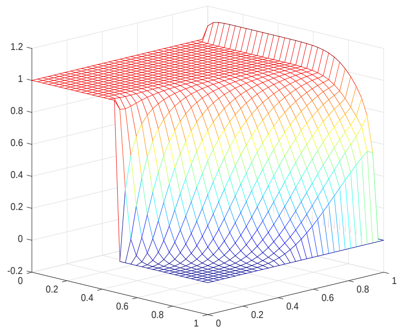


(g) cut line at $x = 0.5$

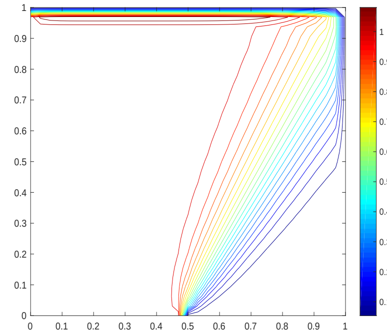


(h) cut line at $y = 0.5$

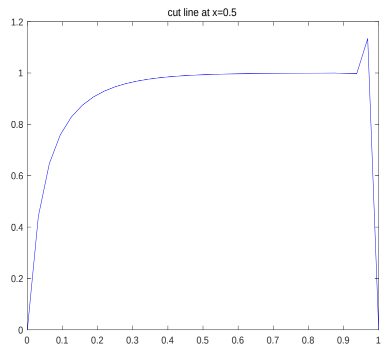
SUPG : The elevation,contour,and vertical and horizontal cuts of the finite element solution, with $\varepsilon = 10^{-2}, 10^{-6}$, $\mathbf{b} = (1/2, \sqrt{3}/2), \gamma = 0, h^* = 1/32, h_F = h_{\partial K}$ with mesh-1.



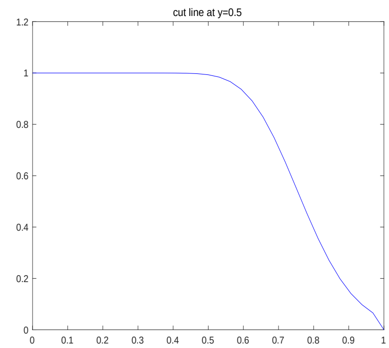
(a) Elevation



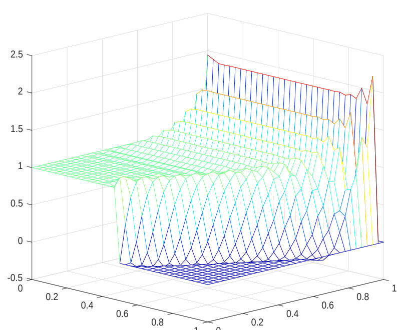
(b) Contour



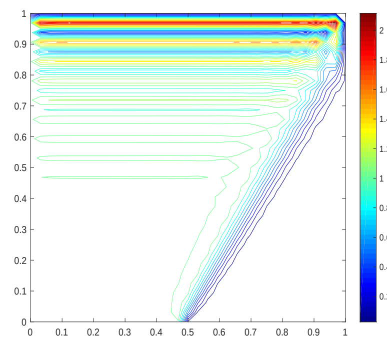
(c) cut line at $x = 0.5$



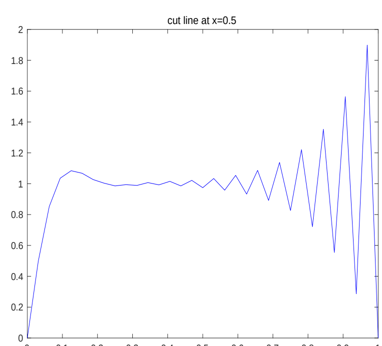
(d) cut line at $y = 0.5$



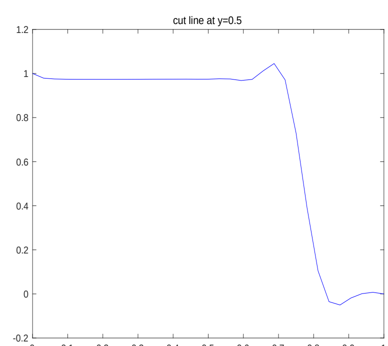
(e) Elevation



(f) Contour

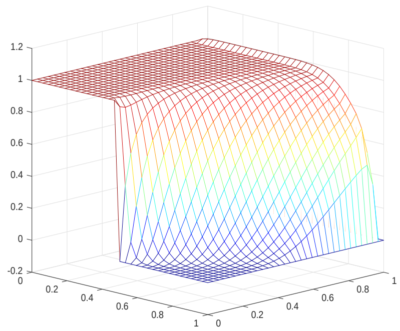


(g) cut line at $x = 0.5$

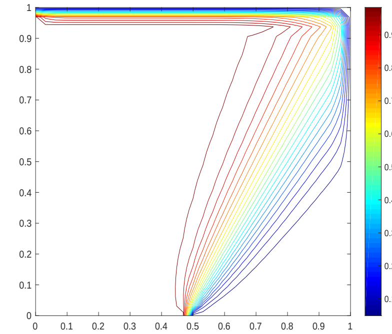


(h) cut line at $y = 0.5$

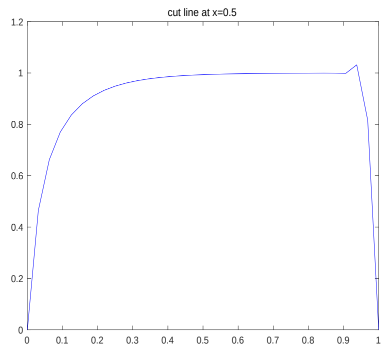
BM : The elevation,contour,and vertical and horizontal cuts of the finite element solution, with $\varepsilon = 10^{-2}, 10^{-6}$, $\mathbf{b} = (1/2, \sqrt{3}/2), \gamma = 0, h^* = 1/32, h_F = h_{\partial K}$ with mesh-1.



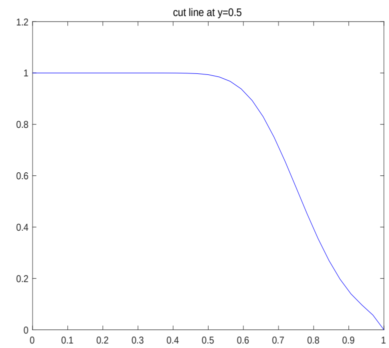
(a) Elevation



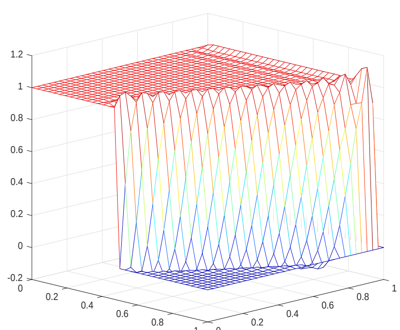
(b) Contour



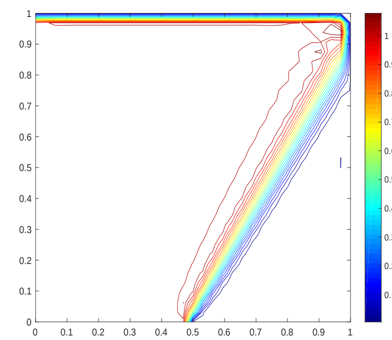
(c) cut line at $x = 0.5$



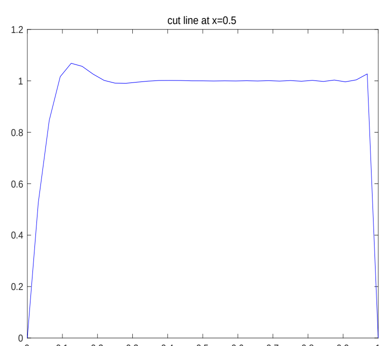
(d) cut line at $y = 0.5$



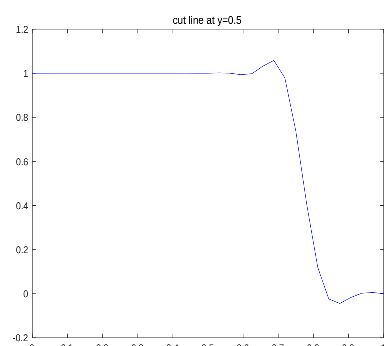
(e) Elevation



(f) Contour

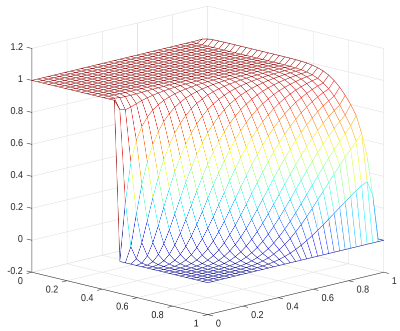


(g) cut line at $x = 0.5$

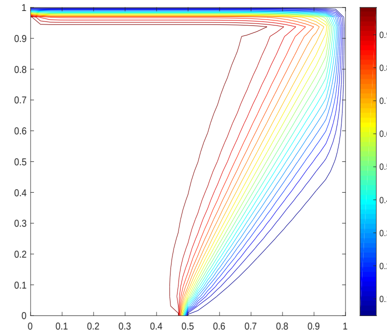


(h) cut line at $y = 0.5$

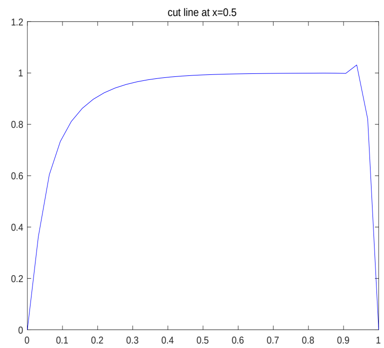
NEW-I : The elevation,contour,and vertical and horizontal cuts of the finite element solution, with $\varepsilon = 10^{-2}, 10^{-6}$, $\mathbf{b} = (1/2, \sqrt{3}/2), \gamma = 0, h^* = 1/32, h_F = h_{\partial K}$ with mesh-1.



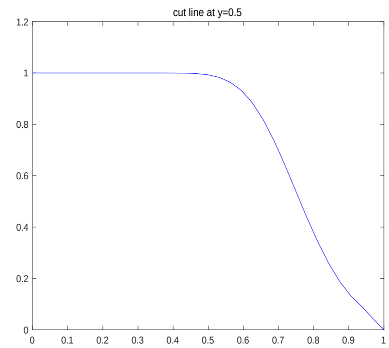
(a) Elevation



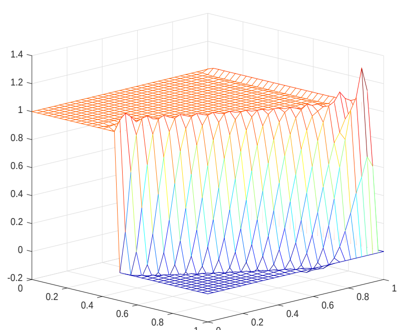
(b) Contour



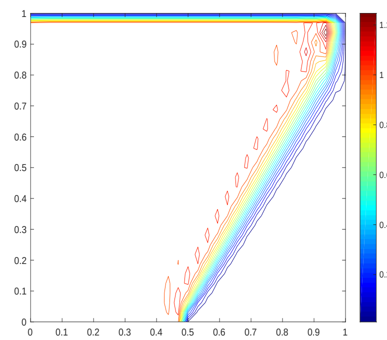
(c) cut line at $x = 0.5$



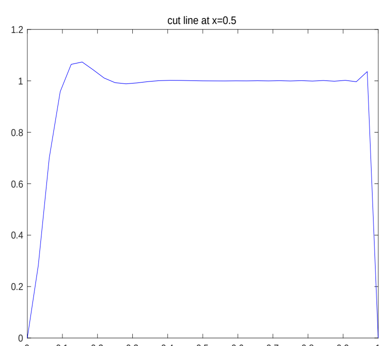
(d) cut line at $y = 0.5$



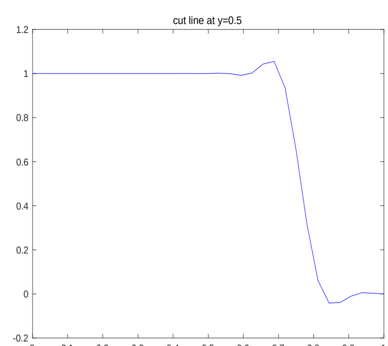
(e) Elevation



(f) Contour



(g) cut line at $x = 0.5$



(h) cut line at $y = 0.5$

NEW-II : The elevation,contour,and vertical and horizontal cuts of the finite element solution, with $\varepsilon = 10^{-2}, 10^{-6}$, $\mathbf{b} = (1/2, \sqrt{3}/2), \gamma = 0, h^* = 1/32, h_F = h_{\partial K}$ with mesh-1.

3.3 example 3: Front Problem

In this example, we consider a time-dependent problem containing transport when the time is evolving. Let $\Omega = (0,1) \times (0,1)$,we consider the following equation:

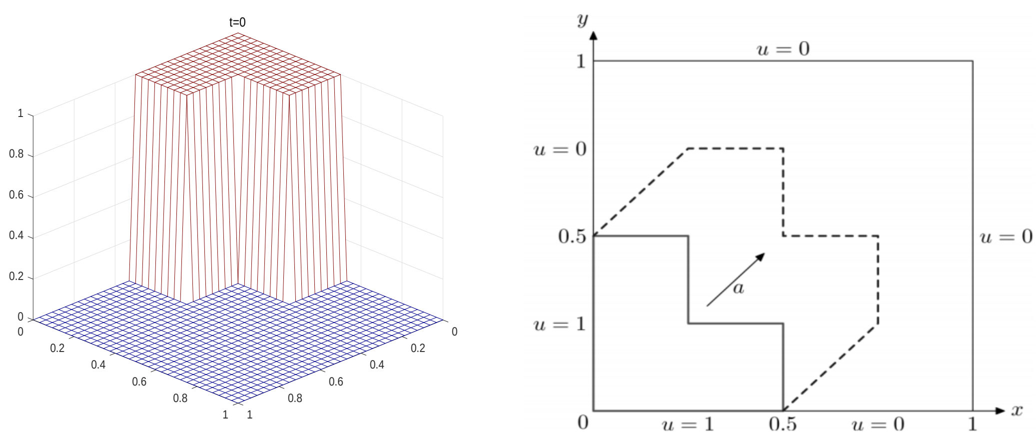
$$u_t - \varepsilon \Delta u + \mathbf{b} \cdot \nabla u + \gamma u = f \qquad in\ (0,T) \times \Omega$$

We use the first-order implicit Euler method to discretize the time terms, time step selected as δt, then:

$$\frac{u^{n+1} - u^n}{\delta t} - \varepsilon \Delta u^{n+1} + \mathbf{b} \cdot \nabla u^{n+1} + \gamma u^{n+1} = f^{n+1}$$

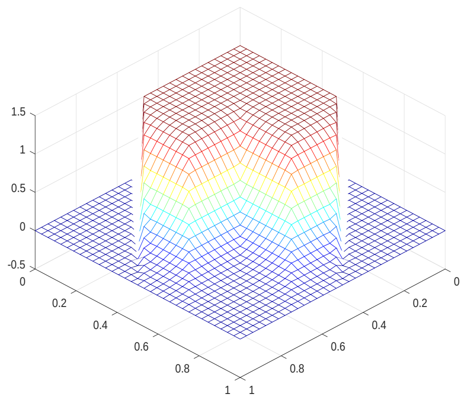
$$-\varepsilon \Delta u^{n+1} + \mathbf{b} \cdot \nabla u^{n+1} + (\gamma + \frac{1}{\delta t}) u^{n+1} = f^{n+1} + \frac{u^n}{\delta t}$$

$f = 0, \varepsilon = 10^{-6}, \gamma = 0, \mathbf{b} = (\sqrt{2}/2, \sqrt{2}/2)^T$. The initial and boundary conditions are described in Fig 5. For a given $h^* = 1/32$ while $h_F = h_{\partial K}$ with mesh-1, various time steps $\delta t = h^*, h^*/4, h^*/8$, are chosen to produce the stabilized finite element approximations.

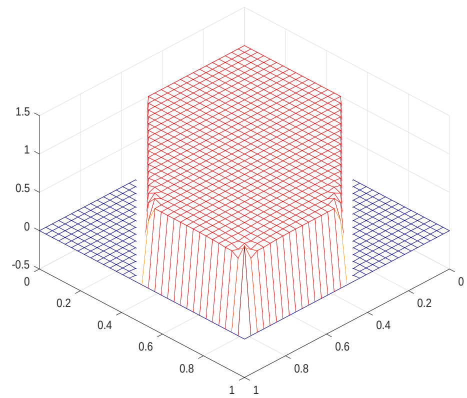


(i) (j)

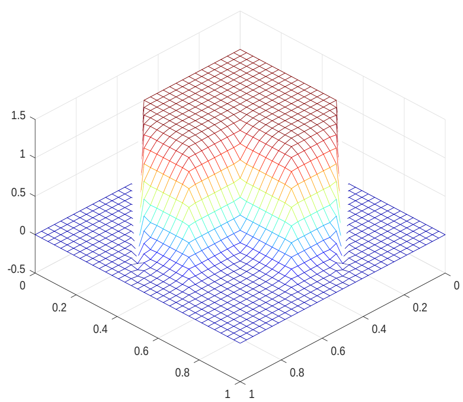
Fig 5: Initial and boundary conditions of L-shape



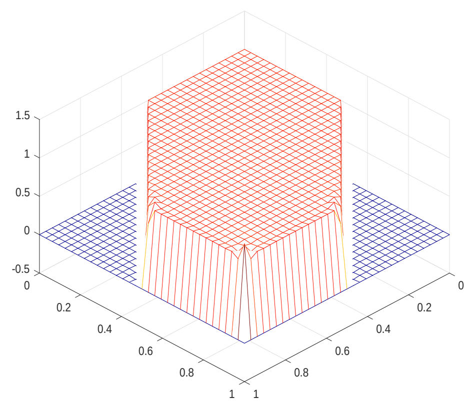
(a) $\delta t = h^*$, t=0.25



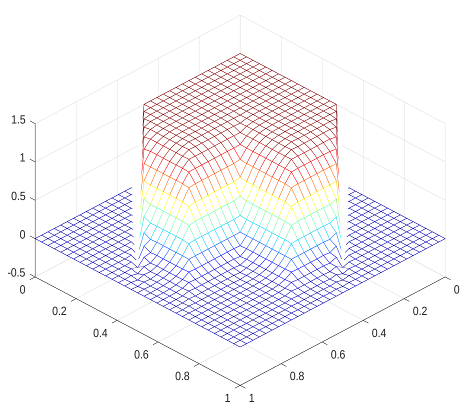
(b) $\delta t = h^*$, t=1.5



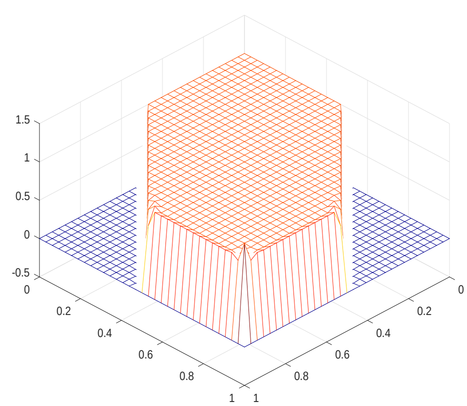
(c) $\delta t = h^*/4$, t=0.25



(d) $\delta t = h^*/4$, t=1.5

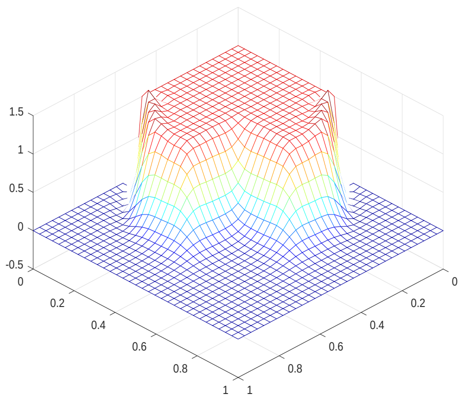


(e) $\delta t = h^*/8$, t=0.25

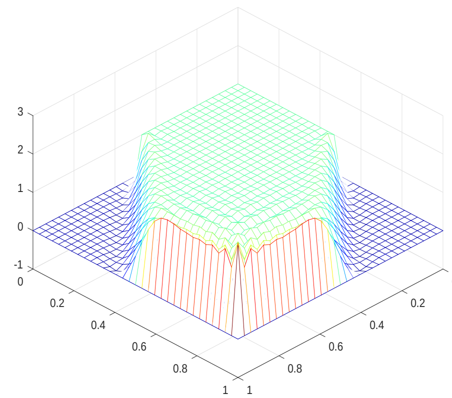


(f) $\delta t = h^*/8$, t=1.5

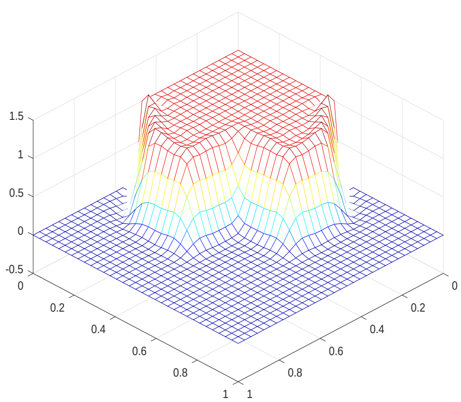
SUPG method



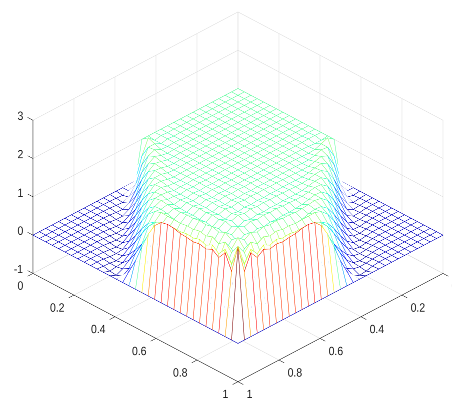
(a) $\delta t = h^*$, t=0.25



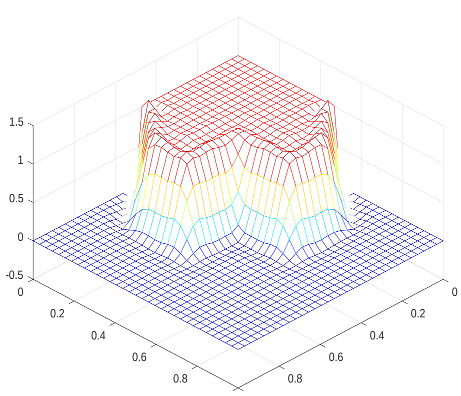
(b) $\delta t = h^*$, t=1.5



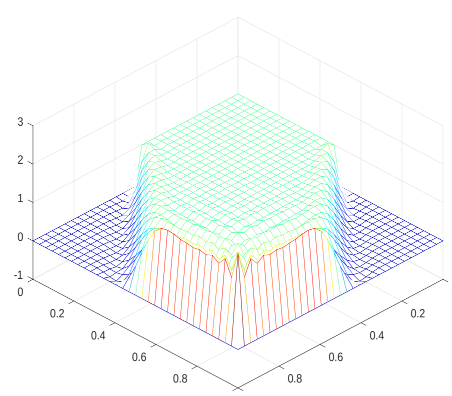
(c) $\delta t = h^*/4$, t=0.25



(d) $\delta t = h^*/4$, t=1.5

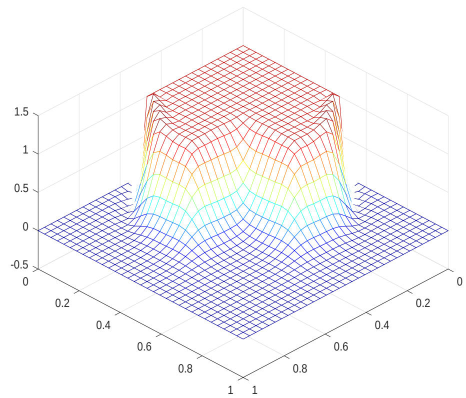


(e) $\delta t = h^*/8$, t=0.25

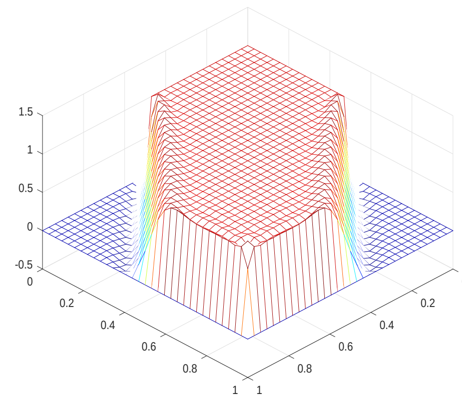


(f) $\delta t = h^*/8$, t=1.5

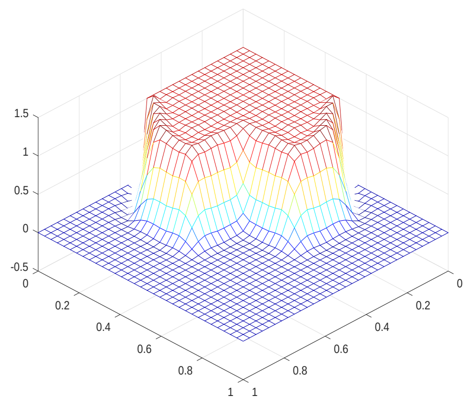
BM method



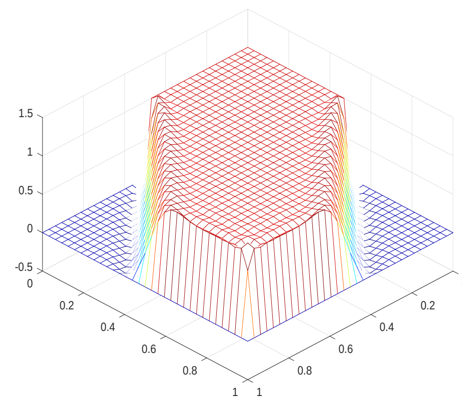
(a) $\delta t = h^*$, t=0.25



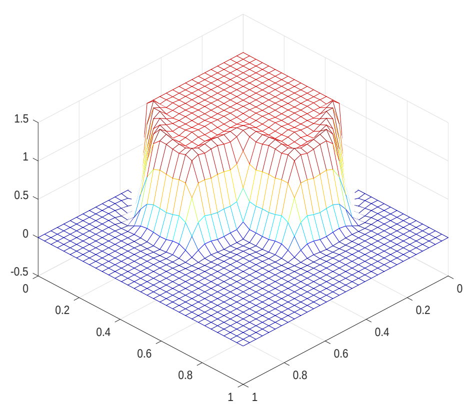
(b) $\delta t = h^*$, t=1.5



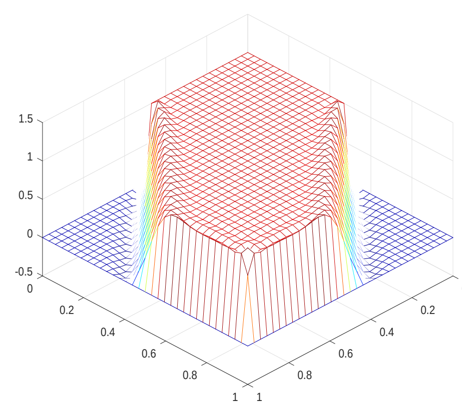
(c) $\delta t = h^*/4$, t=0.25



(d) $\delta t = h^*/4$, t=1.5

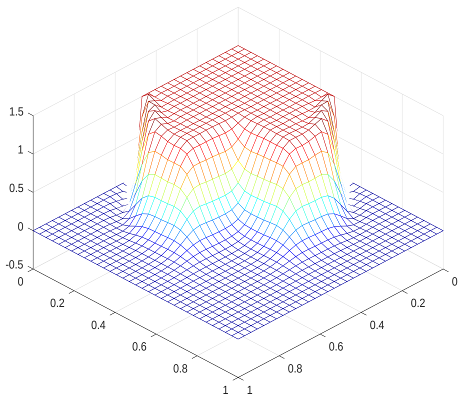


(e) $\delta t = h^*/8$, t=0.25

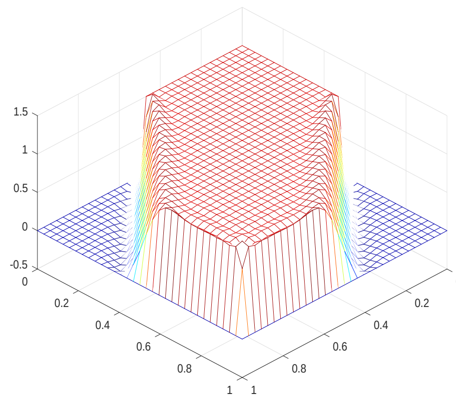


(f) $\delta t = h^*/8$, t=1.5

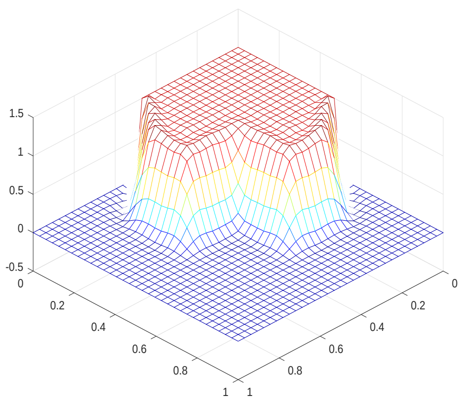
NEW-I method



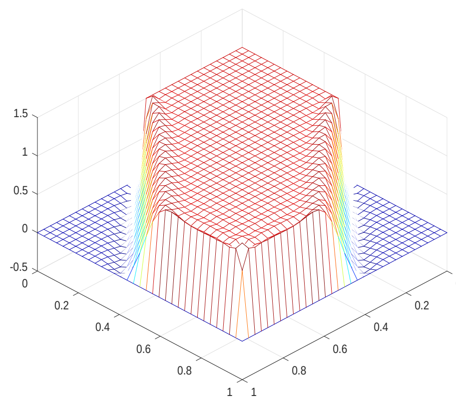
(a) $\delta t = h^*$, t=0.25



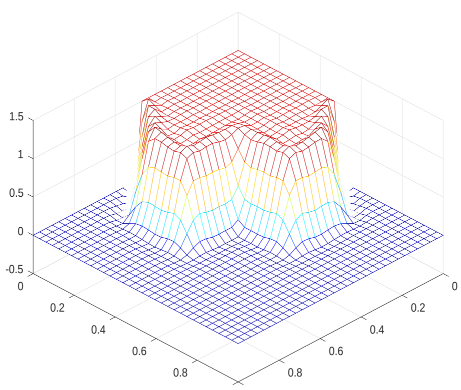
(b) $\delta t = h^*$, t=1.5



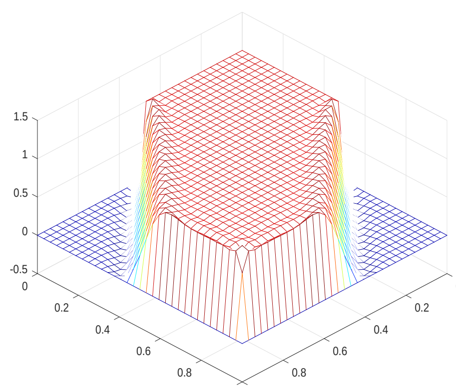
(c) $\delta t = h^*/4$, t=0.25



(d) $\delta t = h^*/4$, t=1.5



(e) $\delta t = h^*/8$, t=0.25



(f) $\delta t = h^*/8$, t=1.5

NEW-II method

3.4 example 4: Hemker Problem

In this example, we consider the Hemker problem. We use a unstructured mesh which is a nonuniform mesh and make a comparison to SUPG method. The Hemker problem is given as follows. Let $\Omega = \{[-3,9]\times[-3,3]\}/\{(x,y) : x^2+y^2 < 1\}$, the coefficients are $\mathbf{b} = (1,0)^T, \gamma = 0, f = 0$, and the boundary conditions are given by

$$u(x,y) = \begin{cases} 0, & x = -3, \\ 1, & x^2+y^2 = 1, \\ \varepsilon \nabla u \cdot n = 0, & else. \end{cases}$$

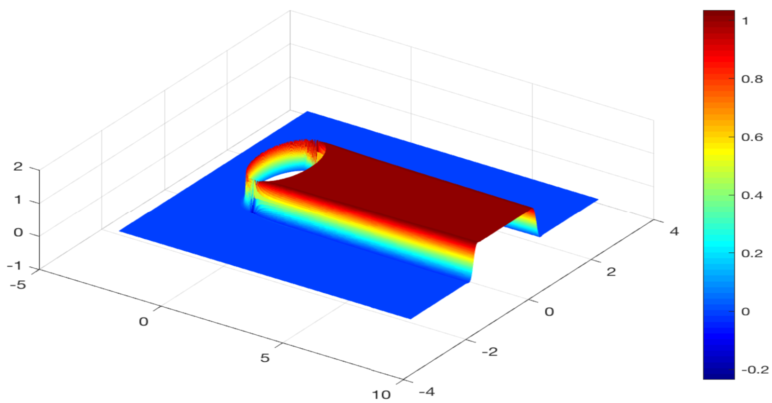
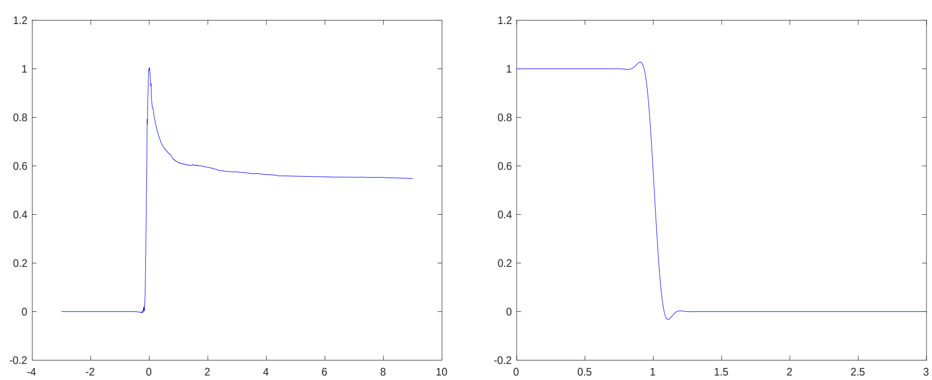


Figure 1



(a) y=1 (b) x=4

SUPG: Herker problem, cut line on level 4 (dof 326880) at y=1 and x=4