BSTT 565- Computational Statistics-Homework 1

September 14, 2020

Due: September 23, 2020

1) Use simulation to approximate the following integrals. Compare your estimate with the exact answer.

a)$\int_0^\infty x(1+x^2)^{-2}dx$.

b)$\int_{-\infty}^\infty e^{-x^2}dx$.

c)$\int_2^4 ln(x)dx$.

d)$\int_0^1\int_0^1 e^{x+y}dxdy$.

2) Let U be uniform on $(0,1)$. Use simulation to approximate $Cov(U^2, \sqrt{1-U^2})$. Compare it with the exact answer.

3) For uniform $(0,1)$ random variables $U_1, U_2, ...$ define $N = Min[n : \sum_{i=1}^n U_i > 1]$. That is, N is equal to the number of random numbers that must be summed to exceed 1. Estimate $E[N]$ by generating 100 values of N. Then, do the same thing by generating $10,000$ values of N.

4) For uniform $(0,1)$ random variables $U_1, U_2, ...$ define $N = Max[n : \prod_{i=1}^n U_i \geq e^{-3}]$. Estimate $E[N]$ by simulation.

5) A pair of dice are to be continually rolled until all the possible outcomes $2, 3, ..., 12$ have occurred at least once. Develop a simulation study to estimate the expected number of dice rolls that are needed.

6) The negative binomial probability mass function with parameters (r, p), where r is a positive integer and $0 < p < 1$, is given by

$p_j = \frac{(j-1)!}{(j-r)!(r-1)!} p^r (1-p)^{j-r}$ where $j = r, r+1, ...$

a) Find a relationship between p_{j+1} and p_j.

b) Give an algorithm for generating negative binomial random variables.

c) Compare empirical mean and variance with the theoretical ones for $p = 0.4$ and $r = 5$.

7) If Z is a standard normal random variable, show that $E[|Z|] = \sqrt{2/\pi} = 0.798$ by simulation and by brute force.

8) Give a method for generating a random variable having density function $f(x) = \frac{e^x}{e-1}$, $0 \le x \le 1$. Compare empirical mean and variance with the theoretical ones.

9) Use the inverse transform method to generate a random variable having the distribution function $F(x) = \frac{x^2+x}{2}$, $0 \le x \le 1$. Compare empirical mean and variance with the theoretical ones.

10) Let X be an exponential random variable with mean 1. Give an algorithm for simulating a random variable whose distribution is the conditional distribution of X given that $X < 0.5$. Estimate $E[X|X < 0.5]$ by simulation and compare what you have found with the exact answer.

11) Give an algorithm that generates a random variable having density

$f(x) = 30(x^2 - 2x^3 + x^4)$, $0 \le x \le 1$.

12) An urn contains n balls numbered from 1 through n. A random sample of n balls is selected from the urn, one at a time. A match occurs if ball numbered i is selected on the i^{th} draw. For a large n, find the probability of at least one match if the sampling is done with and without replacement.

Each problem is worth 10 points.