

HW1

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Question 1

(a)

$$\int_0^{\infty} x(1+x^2)^{-2} dx$$

Set $u = (1+x^2)$, then we can get: $\int_0^{\infty} x(1+x^2)^{-2} dx = \frac{1}{2} \int_1^{\infty} u^{-2} du = \frac{1}{2}$

In order to transform \int_0^{∞} to \int_0^1 , set $y = \frac{1}{1+x}$.

So $dx = -\frac{1}{y^2} dy$. Then we have $\int_1^0 -\frac{1}{y^2} \frac{(1/y-1)}{(1+(1/y-1)^2)^2} dy$.

```
# set seed
set.seed(1029)

# check theoretical value of the given integral using R:
integral_a = function(x) {
  x*(1+x^{2})^{-2}
}
integrate(integral_a, lower = 0, upper = Inf)$value
```

```
## [1] 0.5
```

```
# simulate using uniform distribution
s = runif(100000)
transform = (1/s^2)*(1/s-1)/(1+(1/s-1)^2)^2
mean(transform)
```

```
## [1] 0.4983382
```

(b)

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

Because we already know the probability density function of Normal distribution: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}$, we can let $\mu = 1$, $\sigma = 1/\sqrt{2}$. Then according to the property of probability density function, we have: $\int_{-\infty}^{\infty} e^{x^2} dx = \sqrt{\pi}$

To transform upper bound and lower bound, let $y = \frac{e^x}{1+e^x}$. Then we have $\int_0^1 e^{\frac{\log it^2(y)}{y(1-y)}} dy$

```
# set seed
set.seed(1029)

# check theoretical value of the given integral using R:
integral_b = function(x) {
  exp(-x^2)
}
integrate(integral_b, lower = -Inf, upper = Inf)$value
```

```
## [1] 1.772454
```

```
# simulate using uniform distribution
s = runif(100000)
transform = exp(-(log(s/(1-s)))^2)/(s*(1-s))
mean(transform)
```

```
## [1] 1.764829
```

(c)

$$\int_2^4 \ln x dx$$

$\int_2^4 \ln x dx = x \ln x \Big|_2^4 - \int_2^4 x d \ln x = 4 \ln 4 - 2 \ln 2 - 2 \approx 2.1588$. To transform the upper and lower bound, set $y = \frac{x-2}{2}$. Then we have $\int_0^1 2 \ln(2y+2) dy$.

```
# set seed
set.seed(1029)

# check theoretical value of the given integral using R:
integral_c = function(x) {
  log(x)
}
integrate(integral_c, lower = 2, upper = 4)$value
```

```
## [1] 2.158883
```

```
# simulate using uniform distribution
s = runif(1000000000)
transform = 2*log(2*s+2)
mean(transform)
```

```
## [1] 2.158887
```

(d)

$$\int_0^1 \int_0^1 e^{x+y} dx dy$$

The exact value of this integral is $\int_0^1 e^x dx \int_0^1 e^y dy = (e-1)^2 \approx 2.952492$.

```
# set seed
set.seed(1029)

# simulate using uniform distribution
x = runif(100000)
y = runif(100000)
transform = exp(x+y)
mean(transform)
```

```
## [1] 2.950352
```

Question 2

$$\text{Cov}(U, \sqrt{1-U^2}) = E(U^2\sqrt{1-U^2}) - E(U^2)E(\sqrt{1-U^2}).$$

$$E(U^2) = \int_0^1 u^2 du = u/3|_0^1 = \frac{1}{3}.$$

$$\text{Let } u = \sin\theta, du/d\theta = \cos\theta. \text{ Then } E(U^2\sqrt{1-U^2}) = \int_0^{\pi/2} \sin^2\theta \cos^2\theta d\theta = \int_0^{\pi/2} \frac{1}{4} \sin^2 2\theta d\theta = \frac{1}{8} \int_0^{\pi/2} 1 - \cos 4\theta d\theta = \frac{\pi}{16}$$

$$E(\sqrt{1-U^2}) = \int_0^1 \sqrt{1-u^2} du = \int_0^{\pi/2} \cos\theta d\sin\theta = \int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta =$$

$$\int_0^{\pi/2} (\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta) d\theta = \frac{\pi}{4}$$

$$\text{Cov}(U^2, \sqrt{1-U^2}) = \frac{\pi}{16} - \frac{1}{3} \times \frac{\pi}{4} \approx -0.06544985$$

```
# set seed
set.seed(1029)

# simulate
u = runif(1000000)
y = u^2
z = sqrt(1 - y)
cov(y, z)
```

```
## [1] -0.06543874
```

Question 3

```
# set seed
set.seed(1029)

# simulate
sim_n = 10000
for(i in 1:sim_n) {

}
```