Linear Model Project

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Question 1

In this problem, Let $\mathbf{y}$ denote stiffness (lb/in^2) and $\mathbf{X}$ denote the design matrix. We want to investigate the relationship between stiffness and density(lb/ft^3). Since here we have 30 observations, and we only have one predictor and one outcome, the dimension of $\mathbf{y}$ is 30×1 and the dimension of $\mathbf{X}$ is 30×2. Then we have:

$$\mathbf{y} = \begin{bmatrix} 2622 \\ 22148 \\ 26751 \\ 18036 \\ 96305 \\ \vdots \\ 49499 \\ 25312 \end{bmatrix}_{30\times 1}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 15.0 \\ 1 & 14.5 \\ 1 & 14.8 \\ 1 & 13.6 \\ 1 & 25.6 \\ \vdots & \vdots \\ 1 & 16.7 \\ 1 & 15.4 \end{bmatrix}_{30\times 2}$$

Here the first column in the design matrix allows estimation of the y-intercept and the second column contains the values of our variable (density).
Then we construct a linear model:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

Here, we also have:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}_{2\times 1}$$

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{30} \end{bmatrix}_{30\times 1}$$

Here we also assume that

$$\boldsymbol{\epsilon} \sim N(0, \sigma^2)$$

Before running into questions, we firstly want to show

$$result1 : \mathbf{X}^{'}\mathbf{X} = \begin{bmatrix} 30 & 464.1 \\ 464.1 & 8166.29 \end{bmatrix}$$

$$result2 : (\mathbf{X}^{'}\mathbf{X})^{-1} = \begin{bmatrix} 0.2758892 & -0.0156791 \\ -0.0156791 & 0.001013517 \end{bmatrix}$$

$$result3 : \mathbf{X}^{'}\mathbf{y} = \begin{bmatrix} 1017405 \\ 19589339 \end{bmatrix}$$

$$result4 : s^2 = 165242295.59$$

Below is my code to show these four results.

```
# Import data
x = c(rep(1, 30), 15.0, 14.5, 14.8, 13.6, 25.6, 23.4, 24.4, 23.3, 19.5, 21.2, 22.8, 21.7, 19.8, 21.3, 9
      8.4, 9.8, 11.0, 8.3, 9.9, 8.6, 6.4, 7.0, 8.2, 17.4, 15.0, 15.2, 16.4, 16.7, 15.4)
y = c(2622, 22148, 26751, 18036, 96305, 104170, 72594, 49512, 32207, 48218, 70453, 47661, 38138, 54045,
      17502, 14007, 19443, 7573, 14191, 9714, 8076, 5304, 10728, 43243, 25319, 28028, 41792, 49499, 253

# Create design matrix
x.matrix = matrix(x, ncol = 2, nrow = 30, byrow = F)

# Result 1
result1 = t(x.matrix) %*% x.matrix
result1
```

```
##       [,1]    [,2]
## [1,]  30.0  464.10
## [2,] 464.1 8166.29
```

```
# Result 2
solve(result1)
```

```
##             [,1]         [,2]
## [1,]  0.27588920 -0.015679112
## [2,] -0.01567911  0.001013517
```

```
# Result 3
t(x.matrix) %*% y
```

```
##          [,1]
## [1,]  1017405
## [2,] 19589339
```

```
# Result 4
density = c(15.0, 14.5, 14.8, 13.6, 25.6, 23.4, 24.4, 23.3, 19.5, 21.2, 22.8, 21.7, 19.8, 21.3, 9.5,
            8.4, 9.8, 11.0, 8.3, 9.9, 8.6, 6.4, 7.0, 8.2, 17.4, 15.0, 15.2, 16.4, 16.7, 15.4)
data = data.frame(y, density)
model1 = lm(y~density, data = data)
summary(model1)$sigma^2
```

```
## [1] 165242296
```

All these four results are confirmed.

(a)

Estimate $\boldsymbol{\beta}$.

```r
# Fit a linear model
model1 = lm(y~density, data = data)
summary(model1)
```

```
##
## Call:
## lm(formula = y ~ density, data = data)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -29458  -8036   1905   4394  39313
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -26452.4     6751.9  -3.918 0.000524 ***
## density       3902.1      409.2   9.535 2.72e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12850 on 28 degrees of freedom
## Multiple R-squared:  0.7645, Adjusted R-squared:  0.7561
## F-statistic: 90.92 on 1 and 28 DF,  p-value: 2.724e-10
```

From the result, we could see that, the estimates of $\boldsymbol{\beta}$ are: $\beta_0 = -26452.4, \beta_1 = 3902.1$. The p-values for these two estimates are both less than 0.001, which indicates that they are significant different from 0. Since the number of parameters we are estimating is 2 (p = 2), and the total sample size is 30 (n = 30), the degree of freedom for the residual standard error is 30 - 2 = 28.

(b)

Find a point estimate for the mean stiffness reading when the density of the particleboard is 10 lb/ft^3. Find a 95% confidence interval on this mean reading. The 95% confidence interval for $E(y|x^*)$ is calculated using the following formula:

$$\hat{y} \pm t^{\star}_{n-2}\, s_y \sqrt{\frac{1}{n} + \frac{(x^{\star} - \bar{x})^2}{(n-1)s_x^2}}$$

where s_y is the standard deviation of the residuals, calculated as

$$s_y = \sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{n-2}}.$$

Figure 1: Confidence Interval Calculation

Here the x^* is the given value for the density particleboard.

```r
newdata = data.frame(
  density = 10
)
predict(model1, newdata = newdata, interval = "confidence")
```

```
##        fit      lwr     upr
## 1 12568.87 5925.235 19212.5
```

So the point estimate for the stiffness reading when density $= 10$ is 12568.87 lb/in^2. The 95% confidence interval of the point estimate is [5925.235, 19212.5]. This means that we are 95% confident that the true mean stiffness reading when the density is 10 falls into the interval [5925.235, 19212.5].

(c)

Find a point estimate for the stiffness reading of an individual particleboard whose density is 10 lb/ft^3. Find a 95% prediction interval on the stiffness reading for such a board.

```r
newdata = data.frame(
  density = 10
)
predict(model1, newdata = newdata, interval = "prediction")
```

```
##        fit      lwr      upr
## 1 12568.87 -14587.9 39725.63
```

The point estimate is the same as what has been shown in part (b). The 95% prediction interval for this individual is [-14587.9, 39725.63]. This means

(d)

Find a 95% confidence interval on the slope of the regression line. The 95% confidence interval for $E(y|x^*)$ is calculated using the following formula:

$$\hat{y} \pm t^{\star}_{n-2}\, s_y \sqrt{\frac{1}{n} + \frac{(x^{\star} - \bar{x})^2}{(n-1)s_x^2}}$$

Figure 2: Prediction Interval Calculation

Here the x^* is the given value for the density particleboard.

```r
# Calculate the confidence interval for the slope of the regression line
confint(model1, 'density', level = 0.95)
```

```
##            2.5 %   97.5 %
## density 3063.84 4740.413
```

The 95% confidence interval on the slope of the regression line is [3063.87, 4740.413]. Therefore, we are 95% confident that the new observation of stiffness reding when density $= 10$ falls into the interval [3063.87, 4740.413].

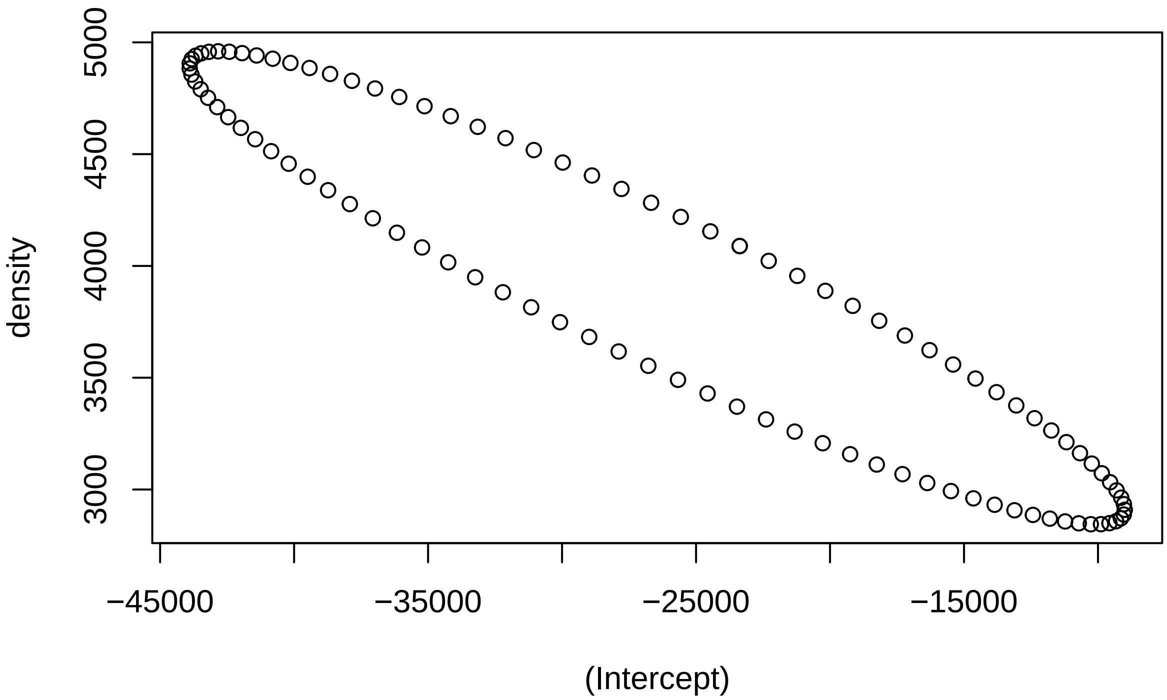
(e)

Find a 95% joint confidence region on the pair of parameters (β_0, β_1). Here the $H_0 : \boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$. Then the confidence for $\boldsymbol{\beta}$ is:

$$c_\alpha = \frac{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\boldsymbol{X'X})^{-1}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{\sigma^2} \leq F_{0.05,2,28}$$

. Solve this inequality, we will get the confidence region for β_0, β_1. It will be of an elliptical form.

```
# Calculate joint 95% confidence region
plot(ellipse(model1, level = 0.95, type = "l"))
```



The pairwise confidence region is plotted (using 100 scatter points). Because here we have 2 parameters to estimate, we will have a confidence rigion rather than just a interval. The shape of this region is an elliptical form. Here we plotted 100 points of this ellipse. Therefore, we are 95% confident that the joint pair of true (β_0, β_1) falls into this region.

Question 2

Let $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{6\times 4}$$

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \\ 0 \\ 4 \end{bmatrix}_{6\times 1}$$

(a)

Find $\mathbf{X}^{'}\mathbf{X}$ and show that $r(\mathbf{X}^{'}\mathbf{X}) = 3$

```
# Import data
x = matrix(c(1, 1, 0, 0,
             1, 1, 0, 0,
             1, 0, 1, 0,
             1, 0, 1, 0,
             1, 0, 0, 1,
             1, 0, 0, 1), nrow = 6, ncol = 4, byrow = T)

# Find X'X
t(x) %*% x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    6    2    2    2
## [2,]    2    2    0    0
## [3,]    2    0    2    0
## [4,]    2    0    0    2
```

Since we have 3 pairs of identical rows in the matrix, only 3 rows are linearly independent (They cannot be expressed by the linear combination of other rows). We can also verify this by R:

```
# Find the rank of X
library(Matrix)
```

```
##
## Attaching package: 'Matrix'

## The following objects are masked from 'package:tidyr':
##
##     expand, pack, unpack
```

```
rankMatrix(x)[[1]]
```

```
## [1] 3
```

So the rank of $\boldsymbol{X}$ is 3.

(b)

Show that the system of normal equations is consistent. The normal equation is: $\boldsymbol{X'X\beta} = \boldsymbol{X'y}$. To show this system is consistent, we need to show that $\boldsymbol{X'y}$ falls into the column space of $\boldsymbol{X'X}$. It is sufficient to show that $rank(\boldsymbol{X'X}|\boldsymbol{X'y}) \leq rank(\boldsymbol{X'X})$.

```
# Construct matrix
y = matrix(c(3, 1, 2, 2, 0, 4),
           nrow = 6, ncol = 1)
Xprime.y = t(x) %*% y
Xprime.x = t(x) %*% x

# Create matrix X'X|X'y
XprimeX.XprimeY = cbind(Xprime.x, Xprime.y)

# Compare the rank of X'X|X'y and X'X
rankMatrix(XprimeX.XprimeY)[[1]]
```

```
## [1] 3
```

```r
rankMatrix(Xprime.x)[[1]]
```

```
## [1] 3
```

From the results, we could see that $rank(\boldsymbol{X'X}|\boldsymbol{X'y}) = rank(\boldsymbol{X'X})$. Therefore, the system of normal equations is consistent since $\boldsymbol{X'y}$ falls into the column space of $\boldsymbol{X'X}$.

(c)

Find the conditional inverse of $\boldsymbol{X'X}$ based on the minor M where

$$\boldsymbol{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

```r
# Step 1, find (M^{-1})'
M = matrix(c(2, 0, 0,
             0, 2, 0,
             0, 0, 2), nrow = 3, ncol = 3, byrow = T)
M.new = t(solve(M)); M.new #(M^{-1})'
```

```
##      [,1] [,2] [,3]
## [1,]  0.5  0.0  0.0
## [2,]  0.0  0.5  0.0
## [3,]  0.0  0.0  0.5
```

```r
# Step 2, Replace M in X'X with (M^{-1})' and make all other entries in X'X zeros
Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    6    2    2    2
## [2,]    2    2    0    0
## [3,]    2    0    2    0
## [4,]    2    0    0    2
```

```r
Xprime.x.replaced = matrix(c(0, 0, 0, 0,
                             0, 0.5, 0, 0,
                             0, 0, 0.5, 0,
                             0, 0, 0, 0.5), ncol = 4, nrow = 4, byrow = T)

# Step 3, transpose the resulting matrix
XprimeX.inverse = t(Xprime.x.replaced)
XprimeX.inverse
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    0  0.0  0.0  0.0
## [2,]    0  0.5  0.0  0.0
## [3,]    0  0.0  0.5  0.0
## [4,]    0  0.0  0.0  0.5
```

```r
# Verify the result
Xprime.x %*% XprimeX.inverse %*% Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    6    2    2    2
## [2,]    2    2    0    0
## [3,]    2    0    2    0
## [4,]    2    0    0    2
```

Here the conditional inverse matrix of $\boldsymbol{X'X}$ based on M is

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

(d)

Show that

$$(\boldsymbol{X'X})^c(\boldsymbol{X'X}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

```
XprimeX.inverse %*% Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    0    0    0    0
## [2,]    1    1    0    0
## [3,]    1    0    1    0
## [4,]    1    0    0    1
```

(e)

Note that $\beta_0 = \boldsymbol{t'\beta}$ where

$$\boldsymbol{t'} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Show that β_0 is not estimable, thus showing that $\boldsymbol{\beta}$ is not estimable. By the definition of estimability, we want to check whether there is a solution the the system$\boldsymbol{(X'X)z = t}$. It is equivalent to see whether $r([\boldsymbol{X'X}|\boldsymbol{t}]) = r(\boldsymbol{X'X})$. We can simlify it, and it is also equivalent to verify whether $\boldsymbol{t'}(\boldsymbol{X'X})^c(\boldsymbol{X'X}) = \boldsymbol{t'}$. If this statement holds, we could conclude that $\boldsymbol{\beta}$ is not estimable.

```
t = matrix(c(1, 0, 0, 0), nrow = 4, ncol = 1)
t(t)%*% XprimeX.inverse %*% Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    0    0    0    0
```

Here we saw that this expression is not equal to $\boldsymbol{t'}$. Therefore β_0 is not estimable.

(f)

$\beta_1 = \boldsymbol{l'\beta}$ where

$$\boldsymbol{l'} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Similarly, we can use the same strategy to check whether β_1 is estimable or not.

```
l = matrix(c(0, 1, 0, 0), nrow = 4, ncol = 1)
t(l)%*% XprimeX.inverse %*% Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    1    0    0
```

Still, it is not equal to $\boldsymbol{l'}$. So β_1 is also not estimable.

(g)

$\beta_3 = \boldsymbol{h'\beta}$ where

$$\boldsymbol{h'} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly, we can use the same strategy to check whether β_3 is estimable or not.

```
h = matrix(c(0, 0, 0, 1), nrow = 4, ncol = 1)
t(h)%*% XprimeX.inverse %*% Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    0    0    1
```

Still, it is not equal to $\boldsymbol{h'}$. So β_3 is also not estimable.

(h)

Consider the linear function $\beta_3 - \beta_1$. Is this function estimable? $\beta_3 - \beta_1 = \boldsymbol{k'\beta}$ where

$$\boldsymbol{k'} = \begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}$$

We can also check the estimability using the same method.

```
k = matrix(c(0, -1, 0, 1), nrow = 4, ncol = 1)
t(k)%*% XprimeX.inverse %*% Xprime.x
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    0   -1    0    1
```

Since $\boldsymbol{k'}(\boldsymbol{X'X})^c(\boldsymbol{X'X}) = \boldsymbol{k'}$. We can conclude that $\beta_3 - \beta_1$ is estimable!!

(i)

Find two different solutions to the normal equations. Use each of them to estimate $\beta_3 - \beta_1$. Are these estimates identical as indicated in Theorem 5.4.3? Let $\boldsymbol{z_1}$ and $\boldsymbol{z_2}$ be two solutions for the normal equation:

$$\boldsymbol{(X'X)z = X'y}$$

Then we can easily find two solutions:

$$\boldsymbol{z_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{z_1} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

```
# Estimates using z1
z1 = matrix(c(1, 1, 1, 1), ncol = 1, nrow = 4)
t(k) %*% z1
```

```
##      [,1]
## [1,]    0
```

```
# Estimates using z2
z2 = matrix(c(0, 2, 2, 2), ncol = 1, nrow = 4)
t(k) %*% z2
```

```
##      [,1]
## [1,]    0
```

Therefore, the estimates are the same no matter what solution we used. This estimates are identical as indicated in Theorem 5.4.3.