58. Use the data of Exercise 7 to find a 95% prediction interval on the salary of a randomly selected individual with 15 years of formal education.

59. Use the data of Exercise 8 to find a 90% prediction interval on the energy consumption of a randomly selected household in which the yearly income is $50,000.

60. Use the data of Exercise 1 to

a. Find $X'X$, $(X'X)^{-1}$, and $X'\mathbf{y}$.
b. Estimate β_0 and β_1.
c. Estimate σ^2.
d. Estimate var b_0 and var b_1.
e. Estimate the average make-span required when six jobs are run.
f. Find a 95% confidence interval on the average make-span when six jobs are run.
g. Estimate the make-span required when a future group of six jobs is run.
h. Find a 95% prediction interval on the make-span required for a future run of six jobs.

Section 3.8

61. Use Corollary 2.3.4 to prove that the quadratic form

$$\frac{(\mathbf{b}-\boldsymbol{\beta})'X'X(\mathbf{b}-\boldsymbol{\beta})}{\sigma^2}$$

follows a chi-squared distribution with p degrees of freedom.

62. In the manufacture of commercial wood products, it is important to estimate the relationship between the density of a wood product and its stiffness. A new type of particleboard is being considered. Thirty of these boards are produced at densities ranging roughly from 8 to 26 pounds per cubic foot, and the stiffness is measured in pounds per square inch. The data below are obtained. (Based on data presented in "Investigation of Certain Mechanical Properties of a Wood-Foam Composite," Terrance E. Conners, M.S. thesis, Department of Forestry and Wildlife Management, University of Massachusetts, Amherst, Massachusetts, 1979.)

Density (x) lb/ft^3	Stiffness (y) lb/in^2	Density (x) lb/ft^3	Stiffness (y) lb/in^2
15.0	2,622	8.4	17,502
14.5	22,148	9.8	14,007
14.8	26,751	11.0	19,443
13.6	18,036	8.3	7,573
25.6	96,305	9.9	14,191
23.4	104,170	8.6	9,714
24.4	72,594	6.4	8,076
23.3	*49,512*	*7.0*	*5,304*
19.5	32,207	8.2	10,728
21.2	48,218	17.4	43,243
22.8	70,453	15.0	25,319
21.7	47,661	15.2	28,028
19.8	38,138	16.4	41,792
21.3	54,045	16.7	49,499
9.5	14,814	15.4	25,312

For these data, Show that

$$X'X = \begin{bmatrix} 30 & 464.1 \\ 464.1 & 8166.29 \end{bmatrix}, \quad (X'X)^{-1} = \begin{bmatrix} 0.2758892 & -0.0156791 \\ -0.0156791 & 0.001013517 \end{bmatrix},$$

$$X'\mathbf{y} = \begin{bmatrix} 1017405 \\ 19589339 \end{bmatrix}, \quad \text{and} \quad s^2 = 165242295.59$$

Use the simple linear regression model.

a. Estimate $\boldsymbol{\beta}$.
b. Find a point estimate for the mean stiffness reading when the density of the particleboard is 10 lb/ft^3. Find a 95% confidence interval on this mean reading.
c. Find a point estimate for the stiffness reading of an individual particleboard whose density is 10 lb/ft^3. Find a 95% prediction interval on the stiffness reading for such a board.
d. Find a 95% confidence interval on the slope of the regression line.
e. Find a 95% joint confidence region on the pair of parameters (β_0, β_1).

63. Consider the information given in Exercise 42. Use this to find a 95% joint confidence region on $(\beta_0, \beta_1, \beta_2)$.

Section 3.9

***64.** A theorem from the field of linear algebra states that "A necessary and sufficient condition for the symmetric matrix A to be positive definite is that there exist a nonsingular matrix P such that $A = PP'$" [17]. Use this result to prove that if V is symmetric and positive definite, then V^{-1} is also positive definite.

65. Verify that $\partial SS_{\mathrm{Res},V}/\partial\boldsymbol{\beta}^* = -2X'V^{-1}\mathbf{y} + 2X'V^{-1}X\boldsymbol{\beta}^*$ as claimed. (*Hint*: Remember that since V is a variance-covariance matrix it is symmetric.)

66. Verify that when $V = \sigma^2 I$, $\mathbf{b}^* = \mathbf{b}$.

67. Use the rules of variance and expectation given in Section 2.2 to show that $\mathbf{b}^*$ is an unbiased estimator for $\boldsymbol{\beta}$ and that var $\mathbf{b}^* = (X'V^{-1}X)^{-1}\sigma^2$.

68. Prove that the matrix $Z = RX$ defined in Section 3.9 is an $n \times p$ matrix of full rank and hence that $\mathbf{b} = (Z'Z)^{-1}Z'\mathbf{v}$ is the least squares estimator for $\boldsymbol{\beta}$ in the restructured model.

69. Verify that in weighted least squares the residual sum of squares can be expressed as

$$\sum_{i=1}^{n} \left(\frac{y_i - \hat{y}_i}{\sigma_i} \right)^2$$

as claimed.

(*) **70.** In solving the normal equations in generalized least squares it is found that

$$\mathbf{b}^* = (X'V^{-1}X)^{-1}X'V^{-1}\mathbf{y}$$

This exercise shows that $X'V^{-1}X$ is a nonsingular matrix as implied in the derivation of $\mathbf{b}^*$.

a. Argue that if V is an $n \times n$ positive definite symmetric matrix then V^{-1} is positive definite.

Section 5.4

21. Let $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \\ 0 \\ 4 \end{bmatrix}$$

a. Find $X'X$ and show that $r(X'X) = 3$.
b. Show that the system of normal equations is consistent.
c. Find the conditional inverse for $X'X$ based on the minor M where

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

d. Show that

$$(X'X)^c(X'X) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

e. Note that $\beta_0 = \mathbf{t}'\boldsymbol{\beta}$ where $\mathbf{t}' = [1 \quad 0 \quad 0 \quad 0]$. Show that β_0 is not estimable, thus showing that $\boldsymbol{\beta}$ is not estimable.
f. Write β_1 as a linear function of $\boldsymbol{\beta}$ and show that β_1 is *not* estimable.
g. Write β_3 as a linear function of $\boldsymbol{\beta}$ and show that β_3 is *not* estimable.
h. Consider the linear function $\beta_3 - \beta_1$. Is this function estimable?
i. Find two different solutions to the normal equations. Use each of them to estimate $\beta_3 - \beta_1$. Are these estimates identical as indicated in Theorem 5.4.3?

22. Consider the one-way classification model with fixed effects presented in Section 5.1. Assume that $k = 3$ and $n_1 = n_2 = n_3 = n$.

a. Show that

$$X'X = \begin{bmatrix} 3n & n & n & n \\ n & n & 0 & 0 \\ n & 0 & n & 0 \\ n & 0 & 0 & n \end{bmatrix}$$

b. Show that the conditional inverse for $X'X$ based on the minor M where

$$M = \begin{bmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{bmatrix}$$

Proof

To show that this system is consistent, it is sufficient to show that the conditions of Theorem 5.3.1 are satisfied. It is evident that $r(X'X) \leqslant r([X'X \mid X'\mathbf{y}])$. Now

$$\begin{aligned} r([X'X \mid X'\mathbf{y}]) &= r(X'[X \mid \mathbf{y}]) \\ &\leqslant r(X') \\ &= r(X'X) \end{aligned}$$

Thus it has been shown that

$$r(X'X) \leqslant r([X'X \mid X'\mathbf{y}]) \leqslant r(X'X)$$

This can only be true if

$$r([X'X \mid X'\mathbf{y}]) = r(X'X)$$

By Theorem 5.3.1 the system of normal equations is consistent.

As has been shown earlier, in the full rank model the system of normal equations has exactly one solution; in the less than full rank model, many solutions to the system exist. The immediate problem is to find a general method for solving the normal equations in the less than full rank model. To do so, the notion of a *conditional inverse* is needed.

Definition 5.3.1

Let A be an $n \times p$ matrix. A $p \times n$ matrix A^c such that

$$AA^cA = A$$

is called a conditional inverse for A.

Example 5.3.1 illustrates this definition and shows that conditional inverses are not necessarily unique.

Example 5.3.1 Consider the 3×3 matrix A of rank 2 given by

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 0 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

Let

$$A_1 = \begin{bmatrix} 0 & +1 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiplication will show that $AA_1A = A$. Now consider the matrix A_2 where

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It can also be verified by multiplication that $AA_2A = A$. Hence both A_1 and A_2 are conditional inverses for the matrix A. How such inverses are found will be demonstrated shortly.

Recall that for a matrix to have an ordinary inverse it must be $p \times p$ of rank p; it must be a full rank square matrix. This is not true of conditional inverses. Theorem 5.3.3 shows that every matrix has a conditional inverse.

Theorem 5.3.3 Let A be an $n \times p$ matrix. There exists a matrix A^c such that $AA^cA = A$. That is, A has a conditional inverse.

Proof

Let A be an $n \times p$ matrix of rank r. It is known that a series of elementary row and column operations can be performed on A to reduce it to the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r is an $r \times r$ identity matrix [11]. This means that nonsingular matrices P and Q exist such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = B$$

Multiplication will show that the matrix

$$B^c = \begin{bmatrix} I_r & U \\ V & W \end{bmatrix}$$

where U, V, and W are arbitrary is a conditional inverse for B. Since $B = PAQ$, $A = P^{-1}BQ^{-1}$. Now consider the matrix $A^c = QB^cP$. By substitution,

$$\begin{aligned} AA^cA &= P^{-1}BQ^{-1}QB^cPP^{-1}BQ^{-1} \\ &= P^{-1}BB^cBQ^{-1} \\ &= P^{-1}BQ^{-1} \\ &= A \end{aligned}$$

By definition, A^c is a conditional inverse for A. ■

It is easy to show that if A is nonsingular, then $A^c = A^{-1}$. The verification o
result is left as an exercise (see Exercise 17).

How does one go about finding a conditional inverse for a matrix? Of cours
method outlined in the existence proof could be used. However, this is no
easiest way to generate such an inverse. A computing algorithm was used to
the inverses of Example 5.3.1:

An Algorithm for Finding Conditional Inverses

Let A be $n \times p$ of rank r. To find a conditional inverse A^c,

1. Find *any* nonsingular $r \times r$ minor M.
2. Find M^{-1} and $(M^{-1})'$.
3. Replace M in A with $(M^{-1})'$.
4. Replace all other entries in A with zeros.
5. Transpose the resulting matrix.

You can verify that the matrix A of Example 5.3.1 was found by applying th
algorithm with

$$M_1 = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}$$

A_2 was based on minor M_2 where

$$M_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Conditional inverses are similar to traditional inverses in many ways. Some o
their more important properties are summarized below for easy reference. Proo
are straightforward and are left as exercises (Exercises 8–16).

Properties of Conditional Inverses

Let A be an $n \times p$ matrix of rank r where $n \geqslant p \geqslant r$. Then

1. A^cA and AA^c are idempotent.
2. $r(AA^c) = r(A^cA) = r$.
3. If A^c is a conditional inverse of A, then $(A^c)'$ is a conditional inverse of A'. That is, $(A^c)' = (A')^c$.
4. $A = A(A'A)^c(A'A)$ and $A' = (A'A)(A'A)^cA'$.
5. $A(A'A)^cA'$ is unique, symmetric, and idempotent. To say that this matri

is unique means that it is invariant to the choice of a conditional inverse. Furthermore, $r[A(A'A)^c A'] = r$.

6. $I - A(A'A)^c A'$ is unique, symmetric, and idempotent, and $r[I - A(A'A^c)A'] = n - r$.
7. If $n = r = p$, then $A^c = A^{-1}$. That is, in the full rank case the conditional inverse is the same as the traditional inverse.
8. $I - A^c A$ is idempotent.

CONDITIONAL INVERSES AND THE NORMAL EQUATIONS

Conditional inverses are particularly important because they can be used to solve consistent systems of the form

$$A\mathbf{x} = \mathbf{g}$$

Since the normal equations for any linear model are consistent and assume this form, conditional inverses provide a method for solving the normal equations in a less than full rank model. To see how a conditional inverse does this consider Theorem 5.3.4.

Theorem 5.3.4 Let $A\mathbf{x} = \mathbf{g}$ be consistent. Then $\mathbf{x} = A^c\mathbf{g}$ is a solution to the system where A^c is any conditional inverse for A.

Proof

Let $A\mathbf{x} = \mathbf{g}$ be consistent and let A^c denote any conditional inverse for A. By definition,

$$AA^cA\mathbf{x} = A\mathbf{x}$$

By assumption, $A\mathbf{x} = \mathbf{g}$. Substituting,

$$AA^c\mathbf{g} = \mathbf{g}$$

Let $\mathbf{x}_0 = A^c\mathbf{g}$. Then $A\mathbf{x}_0 = \mathbf{g}$, showing that $\mathbf{x}_0$ is a solution to the system. ■

Applying this theorem in the context of linear models, it is easy to see that

$$\mathbf{b} = (X'X)^c X'\mathbf{y}$$

is a solution to the system of normal equations $(X'X)\mathbf{b} = X'\mathbf{y}$. Any conditional inverse will generate a solution. However, in the less than full rank model the actual solution obtained depends on the choice of $(X'X)^c$; different conditional inverses result in different solutions.