sample	Poi λ_1	λ_2	λ_3	......	λ_K
1	Y_{11} 0	Y_{12} 1	Y_{13} 0	......	Y_{1K} 0
2	Y_{21} 0	Y_{22} 0	Y_{23} 0	......	Y_{2K} 1
3	Y_{31} 0	Y_{32} 0	Y_{33} 1	......	Y_{3K} 0
⋮	⋮				⋮
n	Y_{n1} 1	Y_{n2} 0	Y_{n3} 0		Y_{nK} 0

$\Rightarrow$ Real dataset = mixture of Poisson family

Y_{12}
Y_{2K}
Y_{33}
⋮
Y_{n1}

↓ we don't observe j

but from the data we can estimate the probability from jth Poisson

n = sample size

K = # of Poisson distribution

i = obs of data

j = jth Poisson

---

① Define $\delta_{ij} = \begin{cases} 1, & \text{if } i\text{th observation comes from } j\text{th Poisson distribution} \\ 0, & \end{cases}$ $\ni y_i \sim f_j = e^{-\lambda_j} \cdot \frac{\lambda_j^{y_i}}{y_i!}$

$\parallel$ indicator variable

$i = 1, 2, \dots, n$

$j = 1, 2, \dots, K$

only one of δ_{ij} can be 1. others must be zero

② obviously, $\underset{\sim}{\delta_i} = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \dots, \delta_{iK})$

$\ni \underset{\sim}{\delta_i} = MN(\text{sample size} = 1, \underset{\sim}{p} = (p_1, p_2, p_3, \dots, p_K))$

$\sum_{j=1}^{K} p_j = p_1 + p_2 + \dots + p_K = 1$

p_j = the probability that the ith obs from jth Poisson

$\underset{\sim}{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_K)$

☆ Computation R code

$K = 2, 3, 4, 5, 6$

λ_j need large differences

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③ $y_i \mid \underset{\sim}{z}_i \sim \text{Poisson}(\lambda_j)$

likelihood for the observation i is

$$f(y_i, \underset{\sim}{z}_i) = f(y_i \mid \underset{\sim}{z}_i) \cdot f(\underset{\sim}{z}_i)$$

④ Assume observations y_i are independent ∋ random sampling
then the likelihood to describe the data is

$$f(\underset{\sim}{y}, \underset{\sim}{z}) = \prod_{i=1}^{n} [f(y_i \mid \underset{\sim}{z}_i) f(\underset{\sim}{z}_i)]$$

$$= \prod_{i=1}^{n} \Big[\prod_{j=1}^{K} \Big(e^{-\lambda_j} \frac{\lambda_j^{y_i}}{y_i!} \Big)^{z_{ij}} \Big] \times \prod_{j=1}^{K} p_j^{z_{ij}}$$

Ⓐ Ⓑ

$$= \prod_{i=1}^{n} \Big\{ \prod_{j=1}^{K} \Big[\Big(e^{-\lambda_j} \frac{\lambda_j^{y_i}}{y_i!} \Big) (p_j) \Big]^{z_{ij}} \Big\}$$

⑤ log likelihood

$$\ln f(\underset{\sim}{y}, \underset{\sim}{z}) = \sum_{i=1}^{n} \Big\{ \sum_{j=1}^{K} z_{ij} [\ln p_j - \lambda_j + y_i \ln \lambda_j - \ln y_i!] \Big\} \#$$

Example, the first obs from the dataset is Y_{12} [The truth: it is from $\text{POI}(\lambda_{K=2})$]

$$Y_{12} = \begin{matrix} i = 1 \\ K = 2 \end{matrix} \ni \text{Ⓐ} \Big(p_1^{z_{11}} \cdot p_2^{z_{12}} \cdots p_K^{z_{1K}} \Big) = p_1^0 \cdot p_2^1 \cdot p_3^0 \cdots p_K^0 = p_2$$

$\times$

$\Downarrow$

$$\begin{cases} z_{12} = 1 \\ \Downarrow \\ \text{other } z_{ij} = 0 \end{cases} \quad \text{Ⓑ} \Big(e^{-\lambda_1} \frac{\lambda_1^{y_1}}{y_1!} \Big)^{z_{11}} \cdot \Big(e^{-\lambda_2} \frac{\lambda_2^{y_2}}{y_2!} \Big)^{z_{12}} \cdots \Big(e^{-\lambda_K} \frac{\lambda_K^{y_K}}{y_K!} \Big)^{z_{1K}}$$

$$= 1 \cdot e^{-\lambda_2} \cdot \Big(\frac{\lambda_2^{y_2}}{y_2!} \Big) \cdots\cdots 1$$

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$y_1, y_2, \dots, y_n \sim f, \quad f(y) = p_1 f_1 + p_2 f_2 + \dots + p_K f_K$

$$f_j = e^{-\lambda_j} \frac{\lambda_j^{y_j}}{y_j!},$$

$$\delta_{ij} = \begin{cases} 1, & y_i \sim f_j \\ 0, & \end{cases}$$

$j=1 \Rightarrow \delta_{i1} = \begin{cases} 1 \\ 0 \end{cases} \Rightarrow y_i \sim f_1$

$$\Rightarrow E(\delta_{i1}) = 1 \cdot P(y_i \sim f_1) = \frac{f_1 \, p_1}{\sum_{j=1}^{K} f_j p_j}$$

$j=2$

$$E(\delta_{i2}) = \frac{f_2 \cdot p_2}{\sum_{j=1}^{K} f_j p_j}$$

(6) E-step

$$Q = E_{\delta_{ij}}[\ln f(\underset{\sim}{y}, \underset{\sim}{\delta})]$$

$$= \sum_{i=1}^{n} \Big[E(\delta_{i1}) (\ln p_1 - \lambda_1 + y_i \ln \lambda_1 - y_i!) + E(\delta_{i2}) (\ln p_2 - \lambda_2 + y_i \ln \lambda_2 - y_i!) + \vdots + E(\delta_{iK}) (\ln p_K - \lambda_K + y_i \ln \lambda_K - y_i!) \Big]$$

$$\propto \sum_{i=1}^{n} \left[\frac{f_j \cdot p_j}{\sum_{j=1}^{K} f_j p_j} \right]$$

$$\propto \sum_{i=1}^{n} \left[\frac{e^{-\lambda_j} \frac{\lambda_j^{y_i}}{y_i!} \cdot p_j}{\sum_{s=1}^{K} e^{-\lambda_s} \frac{\lambda_s^{y_s}}{y_s!} p_s} \right]$$

⑦ M-step

1) $$\frac{\partial \ln f(\underset{\sim}{y}, \underset{\sim}{\delta})}{\partial \lambda_j} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \left[\delta_{ij} \left(\frac{y_i}{\lambda_j} - 1 \right) \right] = 0.$$

$$\Rightarrow \sum_{i=1}^{n} \delta_{ij} \cdot \frac{y_i}{\lambda_j} = \sum_{i=1}^{n} \delta_{ij}$$

$$\Rightarrow \lambda_j = \frac{\sum_{i=1}^{n} (\delta_{ij})(y_i)}{\sum_{i=1}^{n} \delta_{ij}}$$

2) $$\frac{\partial \ln f(\underset{\sim}{y}, \underset{\sim}{\delta})}{\partial p_j} = 0 \quad , j=1,2,\cdots,K-1$$

↓ we don't need to estimate p_K because $p_K = 1 - \sum_{j=1}^{K-1} p_j$

$$\Rightarrow \sum_{i=1}^{n} \left[\delta_{ij} \left(\frac{1}{p_j} \right) - \frac{1-\delta_{i1}-\delta_{i2}-\cdots-\delta_{iK-1}}{1-p_1-p_2-\cdots-p_{K-1}} \right] = 0$$

$$\Rightarrow \frac{1}{p_j} \sum_{i=1}^{n} \delta_{ij} = \frac{1}{1-p_1-p_2-\cdots-p_{K-1}} \sum_{i=1}^{n} (1-\delta_{i1}-\delta_{i2}-\cdots-\delta_{iK-1})$$

$$\Rightarrow \frac{p_j}{p_K} = \frac{\sum_{i=1}^{n} \delta_{ij}}{\sum_{i=1}^{n} (1-\delta_{i1}-\delta_{i2}-\cdots-\delta_{iK-1})}, \quad j=1,2,\cdots,K-1$$

$$\Rightarrow \sum_{j=1}^{K-1} \frac{p_j}{p_K} = \frac{\sum_{j=1}^{K-1} \sum_{i=1}^{n} \delta_{ij}}{\sum_{i=1}^{n} \delta_{iK}} = \frac{n - \sum_{i=1}^{n} \delta_{iK}}{\sum_{i=1}^{n} \delta_{iK}}$$

$$\Rightarrow \frac{1-p_K}{p_K} = \frac{n}{\sum_{i=1}^{n} \delta_{iK}} - 1$$

$$\Rightarrow p_K = \frac{\sum_{i=1}^{n} \delta_{iK}}{n}$$

$$\Rightarrow p_j = \frac{\sum_{i=1}^{n} \delta_{ij}}{\sum_{i=1}^{n} (1-\delta_{i1}-\delta_{i2}-\cdots-\delta_{iK-1})} \cdot p_K = \frac{\sum_{i=1}^{n} \delta_{ij}}{n} \quad \#$$

$$\ln f(\underset{\sim}{y}, \underset{\sim}{\delta}) = \sum \delta_{ij} \ln p_j + \text{constant}$$

$$= \sum_{j=1}^{K-1} \delta_{ij} \ln p_j + \sum_{j=K}^{K} \delta_{ij} \ln p_j$$

$$= \left[\sum_{j=1}^{K-1} \delta_{ij} \ln p_j \right] + \left\{ \delta_{iK} \cdot \ln \left[1 - \sum_{j=1}^{K-1} p_j \right] \right\}$$

$$\sum_{j=1}^{K-1} \sum_{i=1}^{n} \delta_{ij} = \begin{matrix} \delta_{11} + \delta_{12} + \cdots + \delta_{1K-1} \\ \delta_{21} + \delta_{22} + \cdots + \delta_{2K-1} \\ + \\ \vdots \\ \delta_{n1} + \delta_{n2} + \cdots + \delta_{nK-1} \end{matrix}$$

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