Chapter 5

Simulation

5.1 Simulation for BGW and BGWI processes

Consider a BGWI process $\mathfrak{z}$ with regeneration and immigration distributions given by random variables ξ and η with generating functions $f(s) := \mathbb{E}s^{\xi}$ and $g(s) := \mathbb{E}s^{\eta}$. We work under Assumption **(SL)** and so the generating functions f and g are given by

$$f(s) = s + c(1 \quad s)^{1+} \ \ell(1 \quad s) \ \text{ and } \ g(s) = 1 \quad d(1 \quad s) \ \hbar(1 \quad s),$$

where c, d are positive and $\quad \in (0,1]$; $\hbar$ and ℓ are slowly varying at 0 and $\hbar(x) \sim \ell(x)$ when $x \to 0$. For a given $\;$, we denote ξ corresponding to f by $\xi^{c,\ell}$ and similarly we denote η by $\eta^{d,\hbar}$.

For any $p \in [0,1]$, let $Be(1/p)$ denote a Bernoulli random variable with $\mathbb{P}(Be(1/p) = 1) = p$.

Claim 5.1. *Consider a function ℓ_1 on $[0,1]$ given by $\ell_1(x) = \ell(x)\Big(1 + \frac{x\ell'(x)}{(1+ \ )\ell(x)}\Big)$ for $x \in [0,1]$. It holds that $\ell_1(x) \sim \ell(x)$ as $x \to 0$ and so ℓ_1 is slowly varying at 0. Then the law of $\xi^{c,\ell}$ is*

$$\xi^{c,\ell} \overset{w}{=} Be\,(m+1)\,(m+1) \ \textit{given} \ \eta^{c(1+ \ ),\ell_1} = m.$$

Remark 5.2. To simulate ξ it is enough to simulate $m = \eta^{c(1+ \ ),\ell_1}$ and flip a coin with the probability $\frac{1}{m+1}$ of landing H. If the coin lands H, then $\xi \leftarrow m+1$, otherwise $\xi \leftarrow 0$.

Proof. Observe that

$$\begin{aligned} f'(1 \quad x) &= \ 1 \quad x + cx^{1+} \ \ell(x)\big)' = \ 1 + c(1+ \ )x \ \ell(x) + cx^{1+} \ \ell'(x) \\ &= \ 1 + c(1+ \ )x \ \ell(x)\Big(1 + \frac{x\ell'(x)}{(1+ \ )\ell(x)}\Big). \end{aligned}$$

Put $\ell_1(x) := \ell(x)\Big(1 + \frac{x\ell'(x)}{(1+\quad)\ell(x)}\Big)$, then $\frac{\ell_1(x)}{\ell(x)} \to 1$ as $x \to 0+$. Consider the following

$$f(s) = c + \int_0^s 1 \quad c(1+\quad)(1 \quad r) \;\ell_1(1 \quad r)dr = c + \int_0^s \mathbb{E}r^{\eta^{c\;1+\quad),\ell_1}}dr$$
$$= c + \mathbb{E}\frac{r^{\eta^{c\;1+\quad),\ell_1}+1}}{\eta^{c(1+\quad),\ell_1}+1} = c + \sum_{k\geq 1}\frac{1}{k}\mathbb{P}(\eta^{c(1+\quad),\ell_1} = k \quad 1)r^k$$

Thus $\mathbb{P}\;\;\xi^{c,\ell} = 0\big) = c$ and $\mathbb{P}(\xi^{c,\ell} = k) = \frac{1}{k}\mathbb{P}(\eta^{c(1+\quad),\ell_1} = k \quad 1)$ for $k \geq 1$. On the other hand, we derive, for $k \geq 1$

$$\mathbb{P}(Be\;\;\eta^{c(1+\quad),\ell_1}+1\big)\;(\eta^{c(1+\quad),\ell_1}+1) = k)$$
$$= \mathbb{P}(Be\;\;\eta^{c(1+\quad),\ell_1}+1\big)\mid \eta^{c(1+\quad),\ell_1}+1 = k)\mathbb{P}(\eta^{c(1+\quad),\ell_1}+1 = k)$$
$$= \frac{1}{k}\mathbb{P}(\eta^{c(1+\quad),\ell_1} = k \quad 1).$$

Claim 5.3. *Let $S^{d,\ell}$ be a random variable such that for $\lambda \geq 0$*

$$\mathbb{E}\exp(\quad\lambda S^{d,\ell}) = 1 \quad d\lambda\;\;\ell(\lambda).$$

Then $\eta^{d,\ell} \overset{w}{=} Pois(S^{d,\ell})$.

Proof. Indeed,
$$\mathbb{E}s^{\eta^{d,\ell}} = \mathbb{E}s^{Pois(S^{d,\ell})} = \mathbb{E}\exp(\quad S^{d,\ell}(1 \quad s)).$$

Corollary 5.4. *Consider an -stable random variable S such that, for some $d > 0$ and $\lambda \geq 0$,*
$$\mathbb{E}\exp(\quad\lambda S) = \exp(\quad d\lambda\;\;).$$

Then $\eta^{d,\ell} \overset{w}{=} Pois(S)$ with

$$\ell(1 \quad s) := \frac{e^{\quad d(1\quad s)} \quad 1}{d(1 \quad s)},\ \textit{for } s \in [0,1].$$

Note that with this choice of ℓ it holds that $\ell(1 \quad s) \to 1$ when $s \to 1$, hence it is slowly varying. Moreover, by Faà di Bruno's formula

$$g(s) = e^{\quad d}\sum_{n=0}^{\infty}\frac{1}{n!}B_n(\quad,\quad(1 \qquad),\ldots,\quad\ldots(n \quad 1 \qquad))s^n,$$

for Bell's complete polynomials B_n.

Claim 5.5. *The probability mass function of $\eta^{d,1}$, is $\mathbb{P}\;\;\eta^{d,1} = 0\big) = 1 \quad d$ and, for $k \geq 1$,*

$$\mathbb{P}\;\;\eta^{d,1} = n\big) = d(\quad 1)^{n+1}\binom{}{n} = d\frac{(1 \qquad)\ldots(n \quad 1 \qquad)}{n!}.$$

This implies that the generating function of $\eta^{d,1}$ *admits representation*

$$g(s) = (1 \quad d) + d \cdot s\left(\quad + (1 \qquad) \cdot s\left(\frac{}{2} + \frac{2}{2} \cdot s\left(\frac{}{3} + ...\right)\right)\right).$$

Remark 5.6. The following sampling algorithm for $\eta^{d,1}$ follows from the representation of the generating function. With probability $1 \quad d$ set $\eta^{d,1} \leftarrow 0$. Otherwise consider a sequence of coin tosses $(X_k)_{k\geq 1}$ where, for $k \geq 1$, let $\mathbb{P}(X_k = H) = \frac{}{k}$. Denote κ the index of the first H in this sequence. Set $\eta^{d,1} \leftarrow \kappa$. Overall

$$\eta^{d,1} \leftarrow Be(1/d) \cdot \kappa.$$

Note that the second Borel-Cantelli lemma implies that κ is finite. At the same time $\mathbb{E}\kappa = \infty$ in accordance with Lemma 2.1.

Remark 5.7. The assumptions $\ell \equiv 1$ and $\hbar \equiv 1$ imply the following bounds on c and d:

$$0 < d < 1 \quad \text{and} \quad 0 < c(1 + \quad) < 1.$$

The bound on d follows from Claim 5.5. Whereas the bound on c is a consequence of Claim 5.1, as we can see $\xi^{d,1} \stackrel{w}{=} Be(M+1)(M+1)$, with $M \stackrel{w}{=} Be(1/c(1 + \quad))\kappa$.

5.2 Simulation of CB and CBI processes

In this section, we show how to generate a random variable with distribution of $Z^z(t)$ for any $z \geq 0$ and $t \geq 0$. Recall that by (1.8), it has the law of the sum of independent random variables

$$Z^z(t) \stackrel{w}{=} Z^0(t) + \tilde{Z}^z(t)$$

with Laplace transforms given by

$$\mathbb{E}\exp(\quad \lambda Z^0(t)) = (1 + \quad c\lambda \ \ t) \ \ ^{\frac{d}{c}} \quad \text{and} \quad \mathbb{E}\exp(\quad \lambda \tilde{Z}^z(t)) = \exp\left(\frac{\lambda x_0}{(1 + \quad c\lambda \ \ t)^{1/}}\right).$$

De nition 5.8. For and β positive, a *Linnik*$(\ ,\beta)$ distribution has the Laplace transform given by $\lambda \mapsto 1/(1 + \lambda \ \ )^{\beta}$, for $\lambda \geq 0$.

The following sampling algorithm is described in [Dev90].

Theorem 5.9. *Linnik*$(\ ,\beta)$ *distribution can be sampled as a product* *Linnik*$(\ ,\beta) \sim (\Theta_\beta)^{1/} \ \Sigma$, *where where* Θ_β *is a Gamma random variable with Laplace transform* $\lambda \mapsto 1/(1+\lambda)^{\beta}$ *and* Σ *is a positive stable random variable with Laplace transform* $\lambda \mapsto e^{\ \ \lambda}$.

We recognise that the distribution of $Z^0(t)$ is an appropriately scaled *Linnik*$(\ ,\delta)$

distribution and we have

$$Z^0(t) \overset{w}{=} (\ ct)^{1/} \ Linnik(\ , \delta). \tag{5.1}$$

As for $\tilde{Z}^z(t)$, observe that it can be represented as a sum of a random number of random variables. In particular, let $k := (\ ct)^{1/}$ and $\eta \sim Pois(z/k)$ and let τ be an independent random variable distributed as

$$\mathbb{E}e^{\ \lambda\tau} = 1 \quad \left(\frac{\lambda}{1+\lambda}\right)^{1/}, \text{ for } \lambda \geq 0. \tag{5.2}$$

then for a sequence of independent random variables $(\tau_i)_{i\in\mathbb{N}}$ with the common distribution $\tau_1 \sim \tau$, we have

$$\tilde{Z}^z(t) \overset{w}{=} k \sum_{i=1}^{\eta} \tau_i. \tag{5.3}$$

Indeed, the Laplace transform of the right hand side is given by

$$\mathbb{E}\exp\left(\ \lambda k \sum_{i=1}^{\eta} \tau_i\right) = \exp\left(\ \frac{z}{k} \ 1 \quad \mathbb{E}e^{\ \lambda k\tau}\right)\Big) = \exp\left(\ \frac{z}{k}\left(\frac{(k\lambda)}{1+(k\lambda)}\right)^{1/} \ \right)$$
$$= \exp\left(\frac{\lambda z}{(1+ \ ct\lambda \ )^{1/}}\right) = \mathbb{E}\exp(\ \lambda\tilde{Z}^z(t)).$$

We will now be concerned with the sampling of τ.

5.2.1 Size-biasing and integrated tail transforms

De nition 5.10. Let ξ be a non-negative random variable with finite mean. We call ξ' a *size-biased transform* of ξ if $\mathbb{E}g(\xi') = \mathbb{E}\left[\xi g(\xi)\right]/\mathbb{E}\xi$ for any bounded continuous g.

De nition 5.11. Let ξ be a non-negative random variable with finite mean. We call ξ° an *integrated tail transform* of ξ if $\mathbb{P}(\xi^\circ \in dx) = \mathbb{P}(\xi > x)/\mathbb{E}\xi$.

Claim 5.12 (Properties of ξ°)**.**

1. *ξ° is a continuous non-negative random variable with $\mathbb{E}\xi^\circ = 1$.*

2. *For a continuously di erentiable function θ such that $\mathbb{E}|\theta(\xi)| < \infty$*

$$\mathbb{E}\theta(\xi) = \theta(0) + \mathbb{E}\xi \cdot \mathbb{E}\theta'(\xi^\circ).$$

3. *Let $\xi_1, \ldots, \xi_n$ be independent random variables with the common distribution $\xi_1 \sim \xi$. Then for a n-times continuously di erentiable function θ such that $\mathbb{E}|\theta(\xi_1 + \ldots +$*

$\xi_n)| < \infty$,

$$\mathbb{E}\theta(\xi_1 + \ldots + \xi_n) = \sum_{i=0}^{n} \binom{n}{i} (\mathbb{E}\xi)^i \, \mathbb{E}\theta^{(i)}(\xi_1^\circ + \ldots + \xi_i^\circ).$$

4. Let $\eta \sim Pois(\lambda)$, then for an infinitely di erentiable function θ

$$\mathbb{E}\left[\theta\left(\sum_{i=1}^{\eta} \xi_i\right)\right] = e^{\lambda}\, \mathbb{E}\left[(\mathbb{E}\xi)^{\eta} \cdot \theta^{(\eta)}\left(\sum_{i=1}^{\eta} \xi_i^\circ\right)\right].$$

Proof of 2. By the integration by parts, see Lemma 5.13, we get

$$\mathbb{E}\theta(\xi) = \theta(0) + \int_0^\infty \theta'(x)\mathbb{P}(\xi > x)dx.$$

Proof of 3. Consider the case $n = 2$. Applying Property 2 twice we get

$$\begin{aligned}\mathbb{E}\theta(\xi_1 + \xi_2) &= \mathbb{E}\theta(\xi_1) + \mathbb{E}\xi_2 \, \mathbb{E}\theta'(\xi_1 + \xi_2^\circ) \\ &= \theta(0) + 2\mathbb{E}\xi \, \mathbb{E}\theta(\xi_1^\circ) + (\mathbb{E}\xi_2)^2 \, \mathbb{E}\theta''(\xi_1^\circ + \xi_2^\circ).\end{aligned}$$

The general case, follows by induction.

Proof of 4. Recall that $\mathbb{P}(\eta = n) = e^{\ \lambda}\lambda^n/n!$. Then by Property 3

$$\begin{aligned}\mathbb{E}\left[\theta\left(\sum_{i=1}^{\eta} \xi_i\right)\right] &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \binom{n}{i} (\mathbb{E}\xi)^i \, \mathbb{E}\theta^{(i)}\left(\sum_{j=1}^{i} \xi_j^\circ\right) e^{\ \lambda}\frac{\lambda^n}{n!} \\ &= \sum_{i=0}^{\infty} (\mathbb{E}\xi)^i \, \mathbb{E}\theta^{(i)}\left(\sum_{j=1}^{i} \xi_j^\circ\right) e^{\ \lambda}\frac{\lambda^i}{i!}\sum_{n=i}^{\infty}\frac{\lambda^{n\ i}}{(n\ \ i)!} \\ &= e^{\lambda}\sum_{i=0}^{\infty} (\mathbb{E}\xi)^i \, \mathbb{E}\theta^{(i)}\left(\sum_{j=1}^{i} \xi_j^\circ\right) \mathbb{P}(\eta = i).\end{aligned}$$

Lemma 5.13 (Integration by parts)**.** *Let X be a non-negative random variable. Then*

$$\mathbb{E}\,X = \int_0^\infty \mathbb{P}(X > y)dy.$$

Moreover, for a once continuously di erentiable function g such that $\mathbb{E}\,|g(X)| < \infty$, it holds that

$$\mathbb{E}[g(X)\quad g(0)] = \int_0^\infty g'(x)\,\mathbb{P}(X > x)dx.$$

Proof. Note that for $x \geq 0$ it holds that $x = \int_0^x dy = \int_0^\infty 1_{x>y}dy$. Then by Fubini's theorem

$$\mathbb{E}\,X = \mathbb{E}\int_0^\infty 1_{\{X>y\}}dy = \int_0^\infty \mathbb{P}(X > y)dy.$$

For a once continuously differentiable function g consider an approximation g_n constructed as a linear interpolation of g at points $(k/n)_{k \in \mathbb{N}_0}$. Then for any $K > 0$ it holds that

$$\sup_{x<K} [|g_n(x) \quad g(x)| \vee |g_n'(x) \quad g'(x)|] \to 0, \text{ as } n \to \infty.$$

For $K > 0$, consider

$$\begin{aligned} \mathbb{E}\, g_n(X \wedge K) &= \sum_k \mathbb{E}\left[g_n(X \wedge K); X \in [k/n, (k+1)/n)\right] \\ &= g_n(0) + \int_0^K g_n'(x)\mathbb{P}(X > x)dx. \end{aligned}$$

Letting $n \to \infty$ and then $K \to \infty$ we get the result.

Sampling of the size-biased and integrated tail transforms

Let $S(n) := \sum_{i=1}^n \xi_i$, where $(\xi_i)_{i\in\mathbb{N}}$ are independent copies of ξ. Consider a renewal process $(N(t))_{t\geq 0}$, which is defined as $N(t) := \inf\{k \geq 0 : S(k) > t\}$. Consider the undershoot U, the overshoot V, and the current lifetime $W := U + V$ processes, given by:

$$G(t) := t \quad S(N(t) \quad 1), \;\; D(t) := S(N(t)) \quad t, \;\; \text{and} \;\; L(t) := S(N(t)) \quad S(N(t) \quad 1).$$

The following theorem can be found in, for example, [Iks16, Proposition 6.2.7].

Theorem 5.14. *Assume that the distribution of ξ is non-lattice. As $t \to \infty$, the marginal distributions of $G(t)$ and $D(t)$ converge to ξ°; and the marginal distribution of $W(t)$ converges to ξ'. Moreover, for $U \sim Unif[0,1]$, it holds that $\xi' U \overset{w}{=} \xi^\circ$.*

The next claim shows how to sample ξ, given a sampler for ξ'.

Claim 5.15. *Assume that $\mathbb{E}\xi = 1$. To produce a sample of ξ, sample $y \leftarrow \xi'$ and flip a coin which lands H with probability $1/y$. If the coin landed T then repeat the procedure, otherwise return y.*

Proof. Note that

$$\mathbb{E}\left[g(\xi'); Be(\xi') = 1\right] = \mathbb{E}\left[\frac{1}{\xi}\,\xi\, g(\xi)\right] = \mathbb{E}g(\xi).$$

Application to τ

By the integration by parts of the Laplace transform we get

$$\frac{1}{(1+\lambda\ )^{1/}} = \frac{1 \quad \mathbb{E}e^{\ \lambda\tau}}{\lambda} = \int_0^\infty \mathbb{P}(\tau > x)e^{\ \lambda x}dx. \tag{5.4}$$

We immediately recognise the left-hand side as the Laplace transform of a random variable ζ with *Linnik*(,1/) distribution, i.e., its density f_ζ is such that

$$\int_0^\infty e^{\ \lambda x} f_\zeta(x)dx = \left(\frac{1}{1+\lambda}\right)^{1/}.$$

By the uniqueness of the Laplace transforms $f_\zeta(x) = \mathbb{P}(\tau > x)$. This implies that $\zeta \sim \tau^\circ$ and $\mathbb{E}\tau = 1$. The next claim is an immediate corollary of Property 4 and representation (5.3).

Claim 5.16. *Let $k = (\ ct)^{1/}$ and $z > 0$. Then for an infinitely di erentiable function θ and $\eta \sim Pois(z/k)$, we have*

$$\mathbb{E}\,\theta(\tilde{Z}^z(t)) = e^{z/k}\,\mathbb{E}\left[k^\eta \cdot \theta^{(\eta)}\left(k\sum_{i=1}^{\eta}\zeta_i\right)\right].$$

where $(\zeta_i)_{i\in\mathbb{N}}$ are independent random variables distributed as Linnik(,1/).

Consider (5.4) with $\lambda \leftarrow \lambda/y$ for $y > 0$. After the change of variables we get

$$\int_0^\infty \mathbb{P}(\tau > xy)\,y\,e^{\ \lambda x}dx = \frac{1}{(1+(\lambda/y)\ )^{1/}}.$$

Take the derivative on both sides and evaluate it at $y = 1$. We obtain

$$\int_0^\infty x f_\tau(x)e^{\ \lambda x}dx = \int_0^\infty f_\zeta(x)e^{\ \lambda x}dx \qquad \frac{\lambda}{(1+\lambda\ )^{1+1/}} = \frac{1}{(1+\lambda\ )^{1+1/}}.$$

Hence the sized-biased transform of τ has the Laplace transform given as above and we recognise it to be *Linnik*(,1 + 1/) distribution. This, together with (5.3) and Claim 5.15, yields the following claim.

Claim 5.17. *Let $k = (\ ct)^{1/}$ and $z > 0$. Consider a sequence $(y_i, h_i)_i$ of independent identically distributed random variables with y_1 having the Linnik(,1+1/) distribution; and $h_1 \sim Be(y_1)$. Let $H(n) = \sum_{i=1}^n h_i$ and $H^{\ 1}(n) = \inf\{k \geq 0 : H(k) = n\}$. Then, upon letting $\eta \sim Pois(z/k)$, we have*

$$\tilde{Z}^z(t) \overset{w}{=} k \sum_{i=1}^{H^{\ 1}(\eta)} h_i y_i.$$

5.2.2 Sampling based on the Pólya property

Definition 5.18. A function $\varphi : \mathbb{C} \to \mathbb{C}$ is said to have the *Pólya's property* if $\varphi|_{\mathbb{R}}$ is real and even; φ is 1 at $z = 0$; it goes to 0 when $z \to \infty$; and φ is convex on $(0, \infty)$.

The following theorem can be found in, for example, [Fel71, Example XV.3.b].

Theorem 5.19. *A function with the Pólya's property is a characteristic function of some random variable.*

There exists a way to sample from a distribution given only by its characteristic function if it has the Pólya's property. Let us first introduce the Fejer-de la Vallee Poussin (FVP) distribution.

Definition 5.20. A random variable Y has the *Fejer-de la Vallee Poussin (FVP)* distribution if its characteristic function is given by

$$\mathbb{E}e^{izY} = (1 - |z|) \wedge 0, \text{ for } z \in \mathbb{C}.$$

Its density is given by

$$\mathbb{P}(Y \in dx) = \frac{1}{2\pi} \left(\frac{\sin(x/2)}{x/2} \right)^2, \text{ for } x \in \mathbb{R}.$$

The following two theorems are found in [Dev84].

Theorem 5.21. *Let X have a characteristic function φ which has the Pólya's property. Then $X \sim Y/Z$, where Y and Z are independent and Y has FVP distribution; and Z has the distribution function given by*

$$\mathbb{P}(Z > s) = \varphi(s) - s\varphi'(s), \text{ for } s > 0, \text{ and } \mathbb{P}(Z = 0) = 0.$$

Moreover, if the above expression is absolutely continuous, then Z has density given by

$$\mathbb{P}(Z \in ds) = s\varphi''(s).$$

Theorem 5.22. *To produce a sample of Y, sample (U, V) uniformly in $[-1, 1]^2$. If $U > 0$ and $|U| < V^2 \sin^2(1/V)$, then return $2V$. If $U < 0$ and $|U| < \sin^2(V)$, then return $2V$. Otherwise, repeat the procedure. The average number of iterations to produce a result is $4/\pi$.*

Application to τ

Denote the Laplace transform of τ by φ_τ, i.e., as is given in (5.2),

$$\varphi_\tau(\lambda) = 1 \quad \left(\frac{|\lambda|}{1+|\lambda|}\right)^{1/} \quad, \text{ for } \lambda \geq 0.$$

Consider the symmetrised version of τ,

$$\tau^s := h\tau_1 \quad (1 \quad h)\tau_2,$$

where h, τ_1 and τ_2 are independent and $h \sim Be(2)$, and τ_1 has the same distribution as τ_2 and $\tau_1 \sim \tau$. Then the characteristic function of τ^s is given by

$$\mathbb{E}e^{iz\tau^s} = \frac{1}{2}\varphi_\tau(\quad iz) + \frac{1}{2}\varphi_\tau(iz) = 1 \quad \left(\frac{|z|}{1+|z|}\right)^{1/} \quad =: \varphi_{\tau^s}(z), \text{ for } z \in \mathbb{C}.$$

The function φ_{τ^s} has the Pólya's property and so it can be sampled based on Theorem 5.21. Moreover, since $|\tau^s| \stackrel{w}{=} \tau$, this is also an algorithm for sampling τ. This, together with (5.3), yields the following claim.

Claim 5.23. *Let $k = (\quad ct)^{1/}$ and $z > 0$. Consider a sequence $(Y_i, Z_i)_{i\in\mathbb{N}}$ of independent identically distributed random variables with Y_1 having the FVP distribution; and Z_1 having the distribution function given by*

$$\mathbb{P}(Z_1 \leq z) = \frac{z^{\quad +1}}{(1+z^{\quad})^{1+1/}}, \text{ for } z > 0, \text{ and } \mathbb{P}(Z = 0) = 0, \tag{5.5}$$

and its density is given by $\mathbb{P}(Z_1 \in dz) = \frac{(1+ \quad)z}{(1+z^{\quad})^{2+1/}}$, for $z > 0$. Then, upon letting $\eta \sim Pois(z/k)$, we have

$$\tilde{Z}^z(t) \stackrel{w}{=} k \sum_{i=1}^{\eta} |Y_i| / Z_i.$$