

# ECE 255, Diodes and Nonlinear Circuits

18 January 2018

In this lecture, the use of the  $pn$  junction as a diode will be discussed. The diode is one of the simplest semiconductor device, and finds applications in many modern electronic gadgets. The marked feature of a diode is that its  $I$ - $V$  relationship is nonlinear, making its analysis challenging. However, we will exploit as much of our knowledge in linear circuit analysis to analyze circuits with nonlinear diodes in them. It has been shown that as the voltage increases, the current cannot be linearly related to the applied voltage. This is unlike a resistor where one has  $V = IR$  where  $V$  is linearly related to  $I$ . For a diode made from a  $pn$  junction, this relation, as shown previously, is

$$I = I_S \left( e^{\frac{V}{V_T}} - 1 \right) \quad (0.1)$$

where  $V$  is the applied of biasing voltage and  $I$  is the diode current. This physical model is derived from semiconductor physics. It has the property that when  $V \gg V_T$ ,  $I$  tends to infinity. But when  $V \ll -V_T$ ,  $I \approx -I_S$  which is very small. Here,  $I_S$  is varyingly called the saturation current, the generation current, the leakage current, or the scale current; the last name follows from that this current scales as the cross-sectional area of the diode.

## 1 An Ideal Diode

The  $i$ - $v$  relation<sup>1</sup> of an ideal diode is such that the current becomes infinitely large when it is **on**, or in forward bias. Conversely, the current is zero when the diode is **off**, or in reverse bias. This idealization can be used to simplify the analysis of some simple nonlinear circuits. As shown in Figure 1, the use of such idealization turns a nonlinear circuit into set of linear circuits that can be analyze easily.

---

*Printed on March 14, 2018 at 10:21: W.C. Chew and S.K. Gupta.*

<sup>1</sup>Following the text book, the lower cases will be used for total current and total voltage in this lecture.

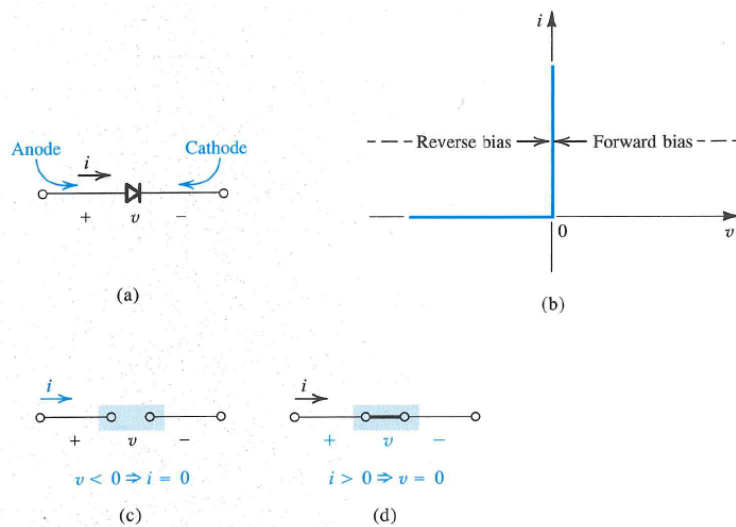


Figure 1: (a) The symbol of the diode. (b) The  $i$ - $v$  characteristic of a  $pn$  junction as an ideal diode. (c) When the diode is reverse biased, it is an open circuit or off. (b) When the diode is forward biased, it is a short circuit or on (Courtesy of Sedra and Smith).

Figure 2 shows the use of an ideal diode, where the on-off states are replaced with open and short circuits respectively. Then the circuit can be analyzed simply as such according to the state of the diode using linear circuit analysis. The behavior of the circuit can also be easily understood.

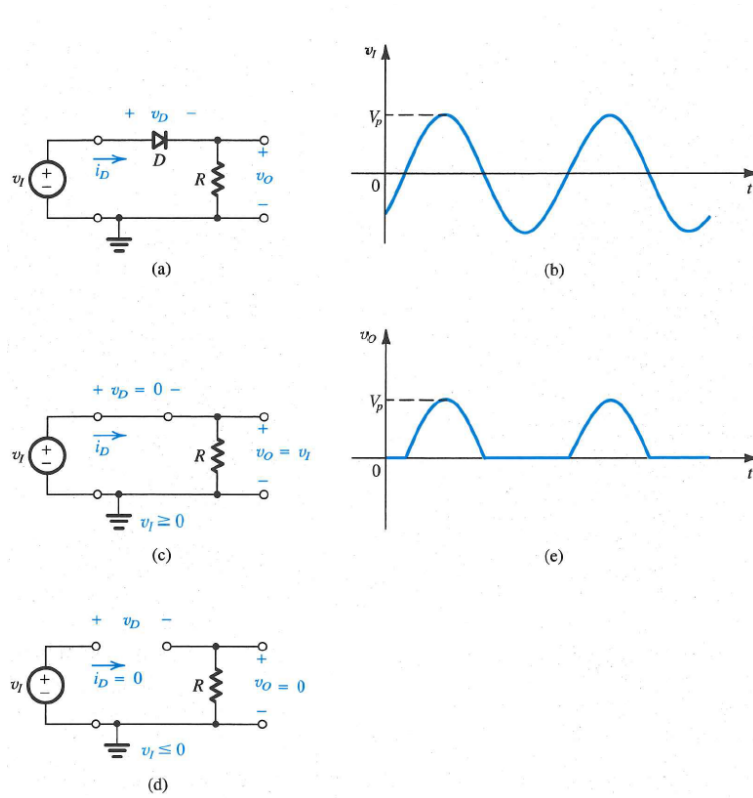


Figure 2: (a) The model of diode circuit. (b) The input voltage signal which is an AC signal. (c) The circuit model when the diode is on. (d) The circuit model when the diode is off. (e) The output voltage (Courtesy of Sedra and Smith).

The turning on of an ideal diode can be delayed by biasing it with the DC voltage source as shown in Figure 3. This also resembles the charging of a 12 V battery by an AC source whose peak voltage is 24 V.

### Example 1.<sup>2</sup>

For an ideal diode, it only has two states, the on or the off state. One will analyze the circuit by assuming that the diodes are either on or off, and analyze the voltage and current across the diodes accordingly using linear circuit analysis. In the on state, the voltage drop across the diode is zero, while in the off state, the current flow through the diode is zero. If the answer contradicts the assumption, it implies that the assumption is wrong, and it has to be revised.

For example in Figure 4(a), one assumes that both diodes are on. Then the

<sup>2</sup>Example 4.1 of textbook.

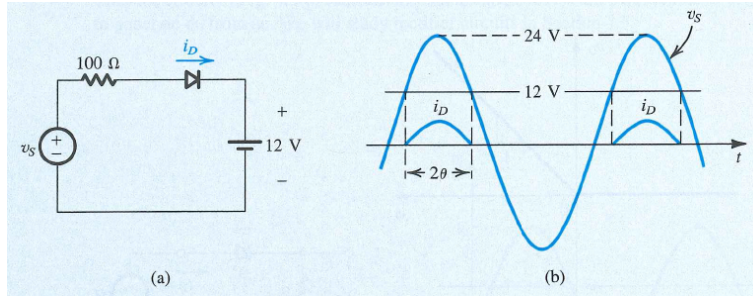


Figure 3: A DC voltage source can be used to delay the turning on of the ideal diode (Courtesy of Sedra and Smith).

voltage at node B is zero and

$$I_{D2} = \frac{10 - 0}{10} = 1 \text{ mA} \quad (1.1)$$

Writing KCL (Kirchhoff Current Law) at node B, then

$$I + 1 = \frac{0 - (-10)}{5} \quad (1.2)$$

giving  $I = 1 \text{ mA}$ . Hence, both diodes are on, and not contradicting the assumption.

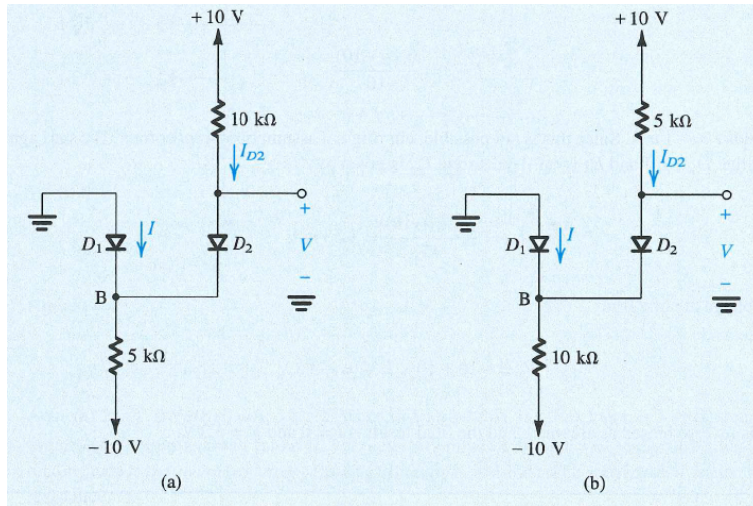


Figure 4: The circuit diagram for analyzing nonlinear circuits with ideal diodes for Example 1 (Courtesy of Sedra and Smith).

For the case in Figure 4(b), one first assumes that both diodes are on. Then

$V_B = 0$  and  $V = 0$ , and

$$I_{D2} = \frac{10 - 0}{5} = 2 \text{ mA} \quad (1.3)$$

Applying KCL at node B,

$$I + 2 = \frac{0 - (-10)}{10} \quad (1.4)$$

giving  $I = -1$  mA. This contradicts our assumption that the diode  $D_1$  is on.

To revise the wrong assumption, one assumes that  $D_1$  is off while  $D_2$  is on. Then the current through  $D_2$  is

$$I_{D2} = \frac{10 - (-10)}{15} = 1.33 \text{ mA} \quad (1.5)$$

The voltage at node B is obtained by applying KVL (Kichhoff voltage law) then

$$V_B = -10 + 10 \times 1.33 = +3.3 \text{ V} \quad (1.6)$$

Hence,  $V_B > 0$  implying that  $D_1$  is off or reverse biased giving rise to  $I = 0$  and  $V = 3.3$  V.

---

## 2 More on $I$ - $V$ Characteristic of Junction Diodes

The  $i$ - $v$  relation of a diode, as shown by using device physics, is given by

$$i = I_S \left( e^{\frac{v}{V_T}} - 1 \right) \quad (2.1)$$

where  $V_T = k_B T / q$  is about 25 mV at room temperature. When the voltage  $v$  is about 0.1 V, it is about 4 times  $V_T$ . And  $e^{v/V_T}$  is about  $e^4 \approx 54.6$  which is quite large. When  $v = 0.6$  V, a biasing voltage 6 times larger, then  $e^{4 \times 6} = e^{24} = 2.65 \times 10^{10}$  which is a very large number. A small increase in  $v$  can overcome the smallness of  $I_S$  which can be of the order of  $10^{-15}$ . Hence, the turn-on voltage of a diode is roughly 0.6-0.8 V. The above equation (2.1) can be inverted to give

$$v = V_T \left( \ln \frac{i}{I_S} + 1 \right) \quad (2.2)$$

When the bias voltage  $v$  is large, the  $i$ - $v$  relation (2.1) can be simplified and approximated with the **exponential model** as

$$i \approx I_S e^{\frac{v}{V_T}} \quad (2.3)$$

Since  $I_S$  is proportional to  $n_i^2$ , and that  $n_i \sim T^{3/2} e^{-E_g/(2k_B T)}$ , one deduces that

$$i \approx T^3 e^{-E_g/(k_B T)} e^{v/V_T} \sim T^3 e^{-(E_g/q)/V_T} e^{v/V_T} \quad (2.4)$$

For silicon,  $E_g/q \approx 1.1$  V. Defining  $V_g = E_g/q$ , the above becomes

$$i \approx C T^3 e^{-(V_g - v)/V_T} \quad (2.5)$$

Taking the natural log of the above, one arrives at

$$v = v_g + V_T \ln i - V_T \ln(CT^3) \quad (2.6)$$

Since  $v \approx 0.7$  V, then  $v < v_g$ , the above implies that  $\ln(CT^3) > \ln i$ , or that  $v$  becomes smaller as  $T$  increases.

**Revision of previous knowledge**—The formula for the saturation current is

$$I_S = Aqn_i^2 \left( \frac{D_p}{L_p N_D} + \frac{D_n}{L_n N_A} \right) \quad (2.7)$$

which is of the order of  $10^{-15}$  A. As have been learned earlier, the formula for  $n_i$  is

$$n_i = BT^{3/2} e^{-E_g/(2k_B T)} \quad (2.8)$$

where

$$B = 7.3 \times 10^{15} \text{K}^{-3/2} \text{cm}^{-3} \quad (2.9)$$

Hence, when the temperature  $T$  increases, there are more thermalized carriers  $n_i$  giving rise to larger  $I_S$ . It is generally assume that  $I_S$  doubles for every  $5^\circ$  C rise in temperature.

## 2.1 Temperature Dependence of the I-V Characteristic

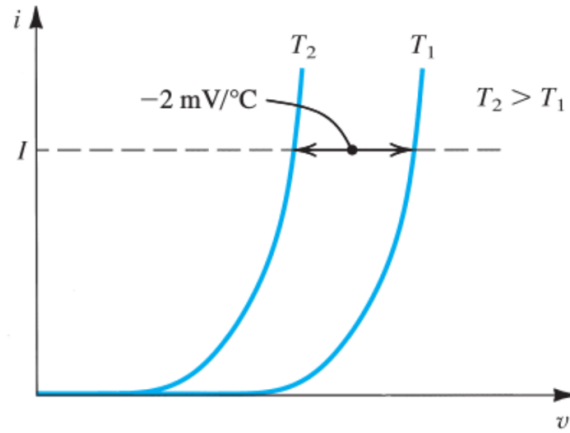


Figure 5: The temperature dependence of the  $i$ - $v$  characteristic of a diode (Courtesy of Sedra and Smith).

As shown in (??), for a fixed  $i$ , the voltage  $v$  is linearly proportional to  $V_T$  which is again linearly proportional to  $T$ . Hence  $v$  increases as  $T$  increases, giving

rise to the temperature dependence as shown in Figure 5. In the above,  $I_S$  is temperature dependent as well, but its temperature dependence is mitigated by the  $\ln$  function.

### 3 More on Diode Modeling

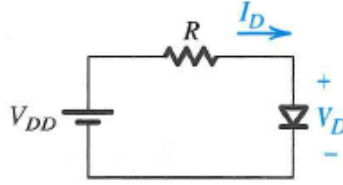


Figure 6: The case of a diode connected to a battery source via a load resistor (Courtesy of Sedra and Smith).

Assuming that the diode is connected to a resistor as shown in Figure 6, and one needs to find the diode current  $I_D$  and voltage  $V_D$ . Since the  $i$ - $v$  relation of a diode is nonlinear, there is no simple way to solve this problem. The solution has to be sought graphically or numerically.

#### 3.1 The Exponential Model

Let us assume that the bias voltage is large so that the  $I$ - $V$  relationship for a  $pn$  junction diode can be approximated with high accuracy with the exponential model to be

$$I_D = I_S e^{V_D/V_T} \quad (3.1)$$

The other equation for  $I$ - $V$  is governed by KVL for the voltage drop across the resistor  $R$ , or that

$$I_D = \frac{V_R}{R} = \frac{V_{DD} - V_D}{R} \quad (3.2)$$

The two unknowns to be sought are  $I_D$  and  $V_D$ , assuming the rest to be known constants. Here,  $I_D$  and  $V_D$  are common to both equations. The above constitute two simultaneous equations for two unknowns.

Albeit simple, these equations have no closed form. For instance, one can eliminate  $V_D$  by inverting the first equation, and substituting into the second equation, one gets

$$I_D = \frac{V_{DD} - V_T \ln(I_D/I_S)}{R} \quad (3.3)$$

The above equation has one unknown  $I_D$ , but it is a transcendental or nonlinear equation that cannot be solved in closed form.

### 3.1.1 Graphical Analysis

One way to solve nonlinear equation is via graphical method, as shown in Figure 7. One plot in the graph is for equation (3.1) representing the **diode characteristic** using the exponential model. The second plot, called the **load line**, is from equation (3.2) due to the resistive load of the circuit.

The solution is given by the point where the plots for the two equations meet. At this point, both equations share the same  $I_D$  and  $V_D$ , and is called the operating point or the  $Q$  point. This is a method of finding the solution of two simultaneous equations, one of which can be nonlinear. When the number of equations is large, this method is unwieldy, and one resorts to a numerical method of solving these equations.

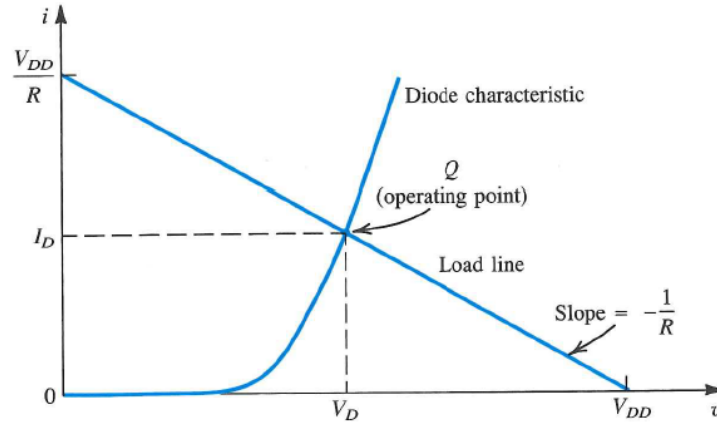


Figure 7: The graphical solution yields the solution of a transcendental equation quickly by visual inspection. (Courtesy of Sedra and Smith).



### 3.1.2 Iterative Analysis—Method of Successive Approximations

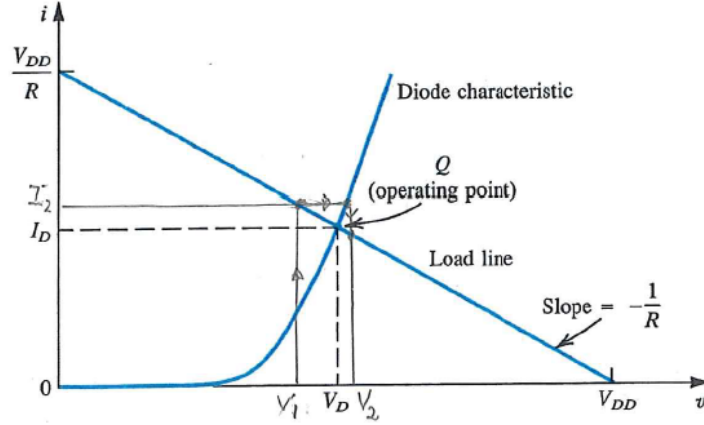


Figure 8: Iterative method such as the method of successive approximation can be converted into a computer program easily (Courtesy of Sedra and Smith).

The graphical method can be used by humans easily because of our gifted visual intelligence. However, it is difficult to program a computer to pick out the operating point or the  $Q$  point on a graph. For numerical or computer method, it is better to design an algorithm that can be converted to a program systematically: such is the spirit of iteration analysis or method. We will illustrate this with the **method of successive approximation** in the following example, which can be converted into a computer program easily.

---

#### Example 2.<sup>3</sup>

In this method, first, we guess a  $V_D$  which is not correct, unless we have clairvoyance. Say, one starts with  $V_D = 0.7$  V, and we can find out where on the load line the current should be if it were to satisfy (3.2). To this end, the correspond current through the resistor is given by

$$I_D = \frac{V_{DD} - V_D}{R} = \frac{5 - 0.7}{1} = 0.43 \text{ mA} \quad (3.4)$$

Next, given this new  $I_D$ , one needs to ascertain what  $V_D$  should be from the diode equation. One can invert equation (3.1) to obtain  $V_D$ , given  $I_D$ . However, we do not know  $I_S$ , but it can be found since it is given, in this example, that  $I_D = 1$  mA when  $V_D$  is 0.7 V. Alternatively, one can use the fact that

$$I_1 = I_S e^{V_1/V_T}, \quad I_2 = I_S e^{V_2/V_T} \quad (3.5)$$

---

<sup>3</sup>This is Example 4.2 of the textbook.

then

$$\frac{I_2}{I_1} = e^{(V_2 - V_1)/V_T} \quad (3.6)$$

Inverting the above gives

$$V_2 - V_1 = V_T \ln \frac{I_2}{I_1}, \quad V_2 - V_1 = 2.3V_T \log_{10} \frac{I_2}{I_1} \quad (3.7)$$

Using the above, and using that  $2.3V_T = 60$  mV, with  $V_1 = 0.7$  V,  $I_1 = 1$  mA, and  $I_2 = 4.3$  mA, gives  $V_2 = 0.738$  V. This process can be repeated until the solution converges. When convergence is reached, the solution changes little with iteration number.

---

The method of successive approximation is also shown in Figure 8. A word of caution is that this method does not always converge. Then other iterative or numerical methods have to be used, for instance, the secant method, or the Newton-Raphson method.

## 4 The Constant-Voltage-Drop Model

As can be seen previously, when nonlinear equations are involved, their solutions are often difficult. One method is to approximate the  $I$ - $V$  characteristics of a diode with piecewise linear approximation. When the diode is operating in the piecewise linear regime, simpler linear methods can be applied. The constant-voltage-drop model is such an attempt to simplify the analysis. The gist of this method is shown in Figure 9. When the diode is off, or the bias voltage is below 0.7 V, then it is replaced by an open circuit. When the diode is on, it is replaced by a short circuit with an internal battery with voltage of  $v_D$  as shown.

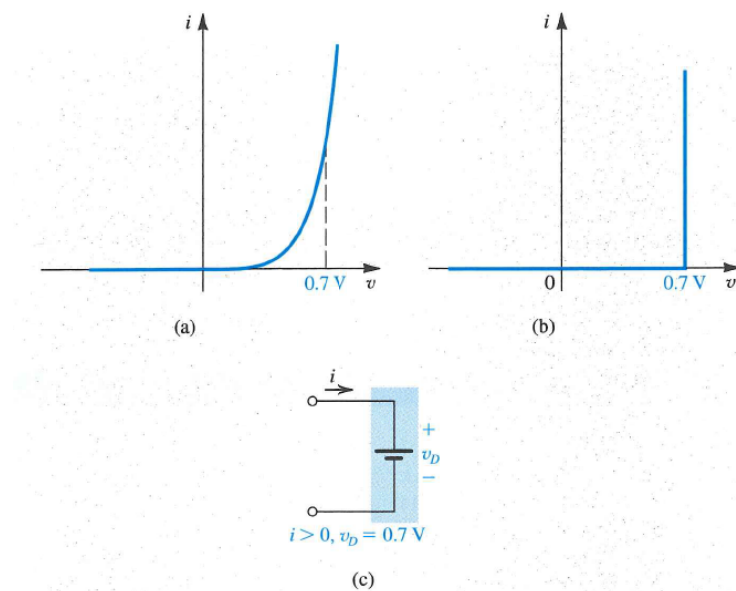


Figure 9: The constant-voltage-drop model can be use to simplify the analysis of diode circuits (Courtesy of Sedra and Smith).

## 5 Small-Signal Model

Another way of making a linear approximation to a nonlinear equation is to use the small signal model. The schematic for this model is shown in Figure 10.

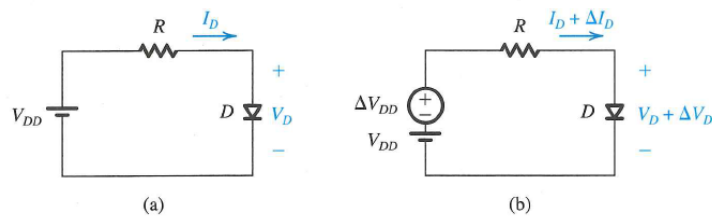


Figure 10: The circuit for a small signal model where a small voltage  $\Delta V_{DD}$  is superposed on top of a large voltage  $V_{DD}$  (Courtesy of Sedra and Smith).

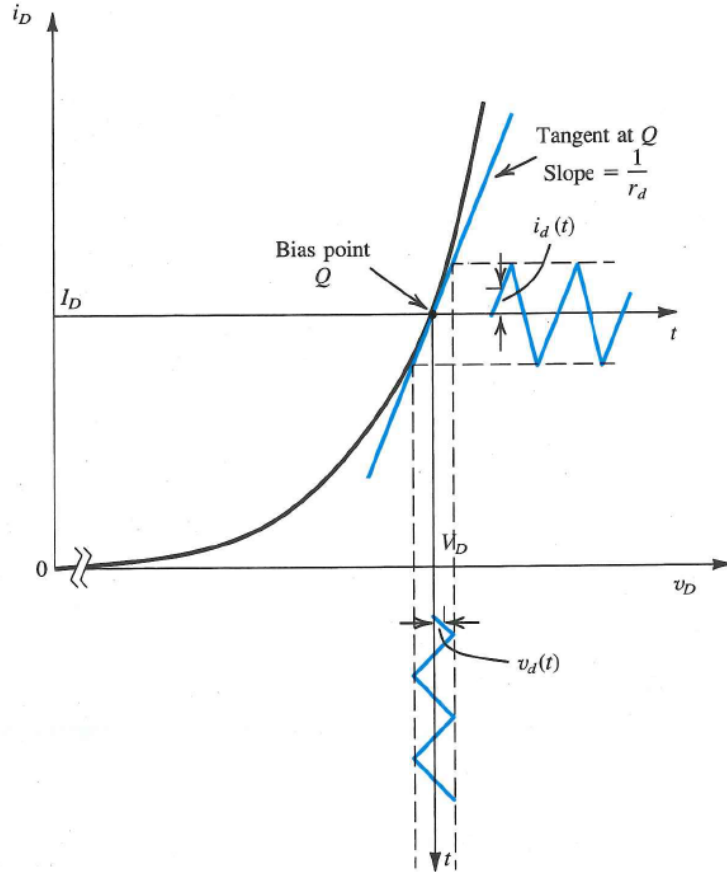


Figure 11: Graphical depiction of the small signal model where  $v_d(t)$  is assumed to be much smaller than  $V_D$  (Courtesy of Sedra and Smith).

### The Math Behind Small Signal Model

In this model, one assumes that the voltage across the diode

$$v_D(t) = V_D + v_d(t) \quad (5.1)$$

where  $v_d(t)$  is a small signal voltage compared to  $V_D$ , the quiescent DC voltage. The corresponding current through the diode is then

$$i_D(t) = I_S e^{v_D(t)/V_T} = I_S e^{(V_D + v_d(t))/V_T} \quad (5.2)$$

The above can be rewritten as

$$i_D(t) = I_S e^{V_D/V_T} e^{v_d(t)/V_T} = I_D e^{v_d(t)/V_T} \quad (5.3)$$

where  $I_D$  is time independent, and is

$$I_D = I_S e^{V_D/V_T} \quad (5.4)$$

Since  $v_d(t)/V_T \ll 1$  always, using  $e^x \approx 1+x$  when  $x$  is small, the above equation (5.3) can be approximated, namely,

$$i_D(t) \approx I_D \left( 1 + \frac{v_d(t)}{V_T} \right) = I_D + \frac{I_D}{V_T} v_d(t) \quad (5.5)$$

Writing

$$i_D(t) = I_D + i_d(t) \quad (5.6)$$

where  $i_d(t)$  is a small signal current, or that  $i_d(t) \ll I_D$ , then

$$i_d(t) = \frac{I_D}{V_T} v_d(t) \quad (5.7)$$

One can then define an **incremental resistance**, or **small-signal resistance**

$$r_d = \frac{V_T}{I_D} \quad (5.8)$$

The above approximation converts a nonlinear problem into a linear problem at the DC bias point, the **quiescent point**, or the  $Q$  point. The slope of the  $i$ - $v$  curve at this point is also inversely proportional to the incremental resistance, namely that

$$\left[ \frac{\partial i_D}{\partial v_D} \right]_{i_D=I_D} = \frac{1}{r_d} \quad (5.9)$$

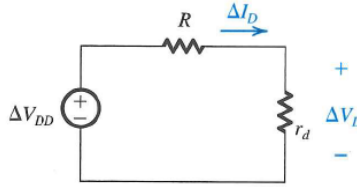


Figure 12: A nonlinear diode circuit can be replaced by a linear resistor circuit under the small signal approximation (Courtesy of Sedra and Smith).

The small-signal approximation is mathematically equivalent to Taylor series approximation. By so doing, a nonlinear function is replaced with a linear function around the  $Q$  point. This is also called the **linearization approximation**.

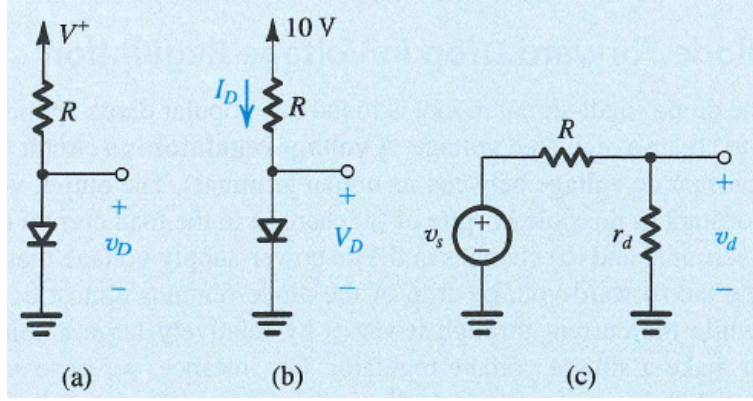


Figure 13: Figure for Example 4.5 of textbook illustrating the use of small-signal model to solve a nonlinear diode problem (Courtesy of Sedra and Smith).

**Example 3.<sup>4</sup>**

Consider a circuit shown in Figure 13 with  $R = 10 \text{ k}\Omega$ . The power source  $V^+$  has a DC value of 10 V, on top of which is superposed an AC signal with 1-V peak amplitude at 60 Hz. This models the imperfection of the power supply ripple. Find the DC voltage of the diode, and the sinusoidal signal across it, assuming that  $V_D$  of the diode is 0.7 V at then  $I_D = 1 \text{ mA}$ .

**Answer:**

Assuming DC signal only, and that  $V_D = 0.7 \text{ V}$ , as shown in Figure 13(b), then

$$I_D = \frac{10 - 0.7}{1} = 0.93 \text{ mA} \quad (5.10)$$

Since we are in the small-signal regime, then the incremental resistance is given by

$$r_d = \frac{V_T}{I_D} = \frac{25}{0.93} = 26.9 \Omega \quad (5.11)$$

The original circuit can now be replaced by the small signal model of Figure 13(c). Using the voltage divider rule, then

$$v_d(t) = v_s(t) \frac{r_d}{R + r_d} \quad (5.12)$$

The peak voltage is then

$$v_d(\text{peak}) = v_s(\text{peak}) \frac{r_d}{R + r_d} = 1 \frac{0.0269}{10 + 0.0269} = 2.68 \text{ mV} \quad (5.13)$$

This voltage is small compared to  $V_T$  affirming out small-signal assumption.

<sup>4</sup>Same as Example 4.5 of the textbook.