Assigment 1

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Qishi Intermediate Machine Learning

Ex 1 in 2.4; Ex 5 in 3.7(ISL), Ex. 2.7 and 2.9(ESL)

Problem 1. For SLR(simple linear regression), prove the least square estimators(LSE) satisfy the following

a)
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$$

where $\bar{y} = \frac{1}{n}\sum_i y_i, \bar{x} = \frac{1}{n}\sum_i x_i, S_{xy} = \sum_i (x_i - \bar{x})(y_i - \bar{y}), S_{xx} = \sum_i (x_i - \bar{x})^2$

b) $\hat{\beta}_0$ and $\hat{\beta}_1$ are both unbiased estimators of β_0 and β_1, respectively.

c)
$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}, \mathrm{Var}(\hat{\beta}_0) = \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]$$

Problem 2.

a) Suppose $y_i \sim N(\mu_i, \sigma^2), i = 1, \ldots, n$, and μ_i is known. Prove the MLE of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (y_i - \mu_i)^2$$

b) Now we can explain why

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^n (y_i - \hat{y}_i)^2$$

can be an estimator of variance in SLR. Only difference is the degrees of freedom with a). Prove this estimator is unbiased.

Problem 3. For SLR, prove $R^2 = r^2$. We prove this step by step. Let $e_i = y_i - \hat{y}_i$ is the residual

a) Prove $\sum_i e_i = 0$

b) Prove $\sum_i x_i e_i = 0$

c) Prove $\sum_i \hat{y}_i e_i = 0$

d) Based on the above three lemma, we can prove the following

$$\sum_{i=1}^n (y_i - \bar{y})^2 = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

Then our total variation can be decomposed into two parts, variation due to regression and residual variation(can not be explained by model), i.e. $\mathrm{TSS} = \mathrm{SSR} + \mathrm{RSS}$.

e) Prove

$$\mathrm{SSR} = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 = \hat{\beta}_1^2 S_{xx}$$

f) Prove $R^2 = r^2$. Then we can say that linear regression gives an statistical explanation of the formula of r(correlation)

Problem 4. Prove $F_0 = t_0^2$ for SLR.