ORF 505: Statistical Analysis of Financial Data

Tuesday Precept 1: Review Sesion of Chapter 1

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Before we get started, I can answer some quick questions about the martials or the homework problem set 1.

1 Background

1.1 Raw and Log Returns

- Raw Returns: (natural for continuous time finance)

$$RR_t = \frac{S_{t+\Delta t} - S_t}{S_t} = \frac{S_{t+\Delta t}}{S_t} - 1$$

- Log Returns: (natural for discrete time finance)

$$LR_t = \log \frac{S_{t+\Delta t}}{S_t}$$

- Relation:

$$LR_t = \log\left(1 + \frac{S_{t+\Delta t} - S_t}{S_t}\right) \sim \frac{S_{t+\Delta t} - S_t}{S_t} = RR_t$$

- Note: Both definitions lead to same results.
- Motivation: Stock prices are assumed to be log-normal in early days.

where PDF is the Probability Density Function and CDF is the Cumulative Distribution Function, and $\varphi_X(t)$ is the Characteristic Function

If you have any questions about the material of normal and log-normal distributions. Please refer to Section 1.1.1.3 and 1.1.1.4 of [1].

2 Chapter 1

2.1 Tail Comparisons

- Gaussian distribution: $X \sim N(\mu, \sigma^2)$, where μ is the mean or expectation of the distribution (and also its median and mode), while the parameter σ^2 is the variance.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Table 1: Comparison between Gaussian and Cauchy Distributions

Property	**Gaussian Distribution**	**Cauchy Distribution**
Support	$x \in (-\infty, \infty)$	$x \in (-\infty, \infty)$
PDF	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\frac{1}{\pi} \frac{\lambda}{\lambda^2+(x-m)^2}$
CDF	$\frac{1}{2}\left[1 + \text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$	$\frac{1}{\pi} \arctan\left(\frac{x-m}{\lambda}\right) + \frac{1}{2}$
Parameters	Mean (μ) Standard deviation (σ)	Location parameter (m) Scale parameter (λ)
Mean	μ (finite)	Undefined
Median	μ	m
Variance	σ^2 (finite)	Undefined
Skewness	0 (symmetric)	Undefined
Kurtosis	3 (excess kurtosis: 0)	Undefined

- Cauchy distribution: $X \sim C(m, \lambda)$, where m is the location parameter and μ is the scale parameter.

$$f_{m,\lambda}(x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + (x-m)^2}, \qquad x \in \mathbb{R}$$

- Key idea of camparison: The tails of a Gaussian distribution are thinner than Cauchy distribution. Comparison see 1

- Intuitative understanding: if $X \sim N(\mu, \sigma^2)$

$$\mathbb{P}\{-\sigma \leq X - \mu \leq \sigma\} = 0.683$$
$$\mathbb{P}\{-2\sigma \leq X - \mu \leq 2\sigma\} = 0.955$$
$$\mathbb{P}\{-3\sigma \leq X - \mu \leq 3\sigma\} = 0.997$$

More information refer to Section 1.1.1.4 and 1.1.1.8 of [1].

2.2 Quantiles and Q-Q Plots

- Definition: the p-quantile, or $100p$th percentile is the number q satisfying

$$F(q) = \mathbb{P}\{X \leq q\} = p$$

- Notation: $q = \pi_p = \pi_p(F)$

- Inverse: $q = \pi_p = F^{-1}(p)$ where F^{-1} is called the quantile function.

- Q-Q plots: the plot of the quantiles of one distribution against the same quantiles of another distribution. One typical example can be the theoretical quantiles v.s. sample quantiles, figure see 2.

More information see Section 1.1.3.1, 1.1.3.5, 1.1.3.6, and 1.1.3.7 of [1].

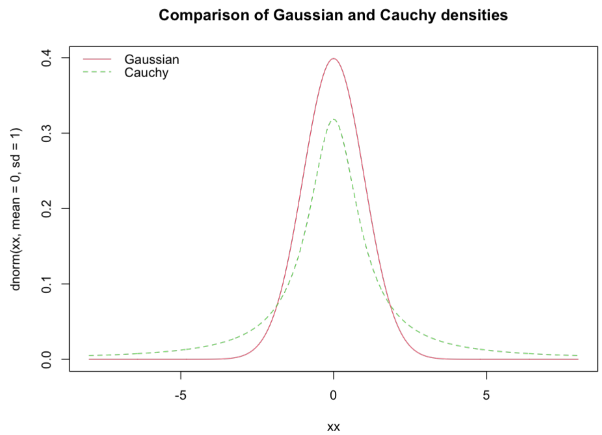

Figure 1: Tail comparison between Gaussian and Cauchy distribution

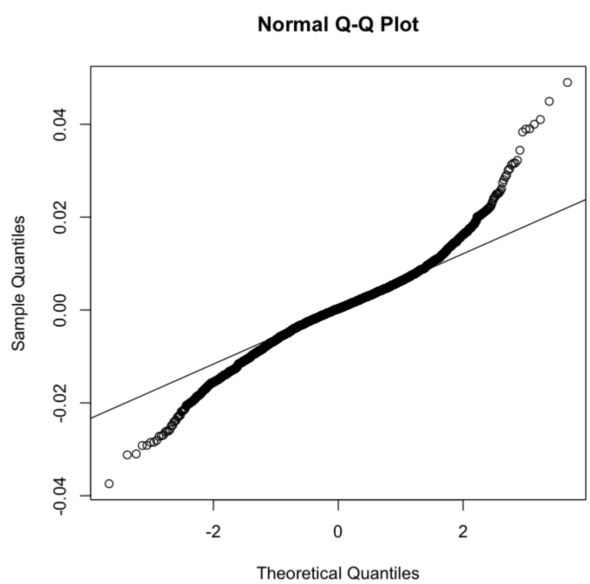

Figure 2: Q-Q Plot example

2.3 Value at Risk (VaR) as a Quantile

- P_t: the value of the portfolio at time t.
- RC_t: the capital needed at time $t + \Delta t$ with probability no greater than p, i.e.

  $$\mathrm{P}\{P_{t+\Delta t} + RC_t < 0\} = p$$

  .
- VaR: the Value at Risk at time t, i.e.

  $$VaR = P_t + RC_t$$

  it is the sum of the current endowment and the required capital.
- expression in unit of P_t:

  $$VaR_t = \widetilde{VaR}_t * P_t$$

  then the definition can be rewritten as

  $$\mathbb{P}\{P_{t+\Delta t} - P_t + VaR_t < 0\} = p \quad \sim \quad \mathbb{P}\{\frac{P_{t+\Delta t} - P_t}{P_t} + \widetilde{VaR}_t < 0\} = p$$

  since

  $$\frac{P_{t+h} - P_t}{P_t} \sim \log \frac{P_{t+h}}{P_t}$$

  , $-\widetilde{VaR}_t$ appears as the p-percentile (quantile) of the distribution of the log-return

References

[1] R. Carmona. *Statistical analysis of financial data in R*, volume 2. Springer, 2014.