ORF 505: Statistical Analysis of Financial Data

Tuesday Precept 8: Markov Processes

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1 Markov Chains

- **Stochastic process background**
  Let $\{X_n\}_{n\geq 0}$ be a general stochastic process in discrete time $T = \mathbb{N}_0$ and with values in a finite state space $S = \{s_1, \ldots, s_N\}$. The process $\{X_n\}_{n\geq 0}$ is then completely determined. Consider the following probabilities:

  $$\begin{aligned}&\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n)\\ &\qquad\qquad = \mathbb{P}(X_0 = x_0) \cdot \prod_{k=1}^{n} \mathbb{P}(X_k = x_k | X_0 = x_0, \ldots, X_{k-1} = x_{k-1})\end{aligned}$$

  The Markov property simplified these conditional transitions by assuming that *ast independent of the future given the present*, then

  $$\mathbb{P}(X_k = x_k | X_0 = x_0, \ldots, X_{k-1} = x_{k-1}) = \mathbb{P}(X_k = x_k | X_{k-1} = x_{k-1}), \quad k \geq 1.$$

  By putting everything together,

  $$\mathbb{P}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \mathbb{P}(X_0 = x_0) \cdot \prod_{k=1}^{n} \mathbb{P}(X_k = x_k | X_{k-1} = x_{k-1})$$

- **Markov chain (MC)**
  A random process $\{X_n\}_{n\geq 0}$ where each X_n takes values in a finite set D is called a Markov chain (MC) if:

  $$\begin{aligned}\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0) &= \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)\\ &\qquad \forall n \geq 0, x_1, \ldots, x_{n+1} \in D\end{aligned}$$

- **Transition probability matrix**
  To quantify the Markov process, we still need to know $\mathbb{P}(X_k = x_k | X_{k-1} = x_{k-1})$ for all pairs $x_{k-1}, x_k \in S$ and $k \geq 1$. A further simplification can be granted by assuming each of these probabilities only depends on the two states, i.e., there exists N^2 numbers p_{ij} indexed by $i, j \in S$ such that for all $k \geq 1$,

  $$p_{ij} = \mathbb{P}(X_k = j | X_{k-1} = i), \quad k \geq 1$$

  We denote by $P = (p_{ij})_{i,j\in D}$ the transition probability matrix. Here, we have:

- $0 \le p_{ij} \le 1$
- $\sum_{j \in D} p_{ij} = \sum_{j \in D} \mathbb{P}(X_{n+1} = j | X_n = i) = 1$ (probabilities of going to any other states starting from the point i sum to 1)

- **Time-homogeneousity**
A Markov Chain (MC) $\{X_n\}_{n \ge 0}$ is said to be time-homogeneous if

$$\mathbb{P}(X_{n+1} = j | X_n = i) = P_{ij}, \quad \forall n, i, j \in D$$

In other words, this time-homogeneous finite Markov chain $\{X_n\}_{n \ge 0}$ is entirely determined by the initial distribution μ of X_0, and the transition matrix $P = (p_{ij})_{i,j \in S}$.

- **Ergodic and stationary probability**
A Markov chain $\{X_n\}$ is said to be ergodic if the limit $\pi(j) = \lim_{n \to \infty} \mathbb{P}_i(X_n = j)$ exists for every state j and does not depend on the initial state i. Then $\pi = (\pi(j))_{j \in D}$ is called the stationary probability of the Markov chain.

2 Example Problem

Let $\{X_n\}$ be a Markov chain in $D = \{a, b, c, d\}$ with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0.3 & 0 & 0.2 & 0.5 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0.7 & 0.3 \end{pmatrix}.$$

(a) Classify the states. Is the chain ergodic? If so, compute its stationary distribution.

[Solution] Since all states communicate with each other, the state space $D = \{a, b, c, d\}$ is one class. The Markov chain is irreducible and recurrent. In addition, the chain is aperiodic (one can check it by observing that the chain can return to state a in one step). As the chain is ergodic, the stationary distribution is given by the limiting probabilities.

We find the limiting distribution π_j, $j \in D$, by finding the normalized solution to the system $\pi = \pi P$:

$$(\pi_a, \pi_b, \pi_c, \pi_d) = (\pi_a, \pi_b, \pi_c, \pi_d) \begin{pmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0.3 & 0 & 0.2 & 0.5 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0.7 & 0.3 \end{pmatrix},$$

or in the form of linear equations:

$$\begin{aligned} \pi_a &= 0.2\pi_a + 0.3\pi_b, \\ \pi_b &= 0.3\pi_a + 0.4\pi_c, \\ \pi_c &= 0.4\pi_a + 0.2\pi_b + 0.7\pi_d, \\ \pi_d &= 0.1\pi_a + 0.5\pi_b + 0.6\pi_c + 0.3\pi_d. \end{aligned}$$

If we set $\pi_a = 1$, then

$$\pi_b = \frac{8}{3}, \quad \pi_c = \frac{71}{12}, \quad \pi_d = \frac{299}{42}.$$

The normalized solution (with $\sum_{j \in D} \pi_j = 1$) is

$$\pi = \frac{1}{1403}(84, 224, 497, 598).$$

(b) Let T be the first time that the Markov chain hits the state c or d. Why is T finite? Compute $\mathbb{P}_b\{X_T = c\}$ and $\mathbb{P}_b\{X_T = d\}$.

[Solution]

Since states c and d are recurrent, T is finite. To compute the desired probabilities, we will do a first step analysis. For $i = a, b$:

$$\begin{aligned} \mathbb{P}_i\{X_T = c\} &= \sum_{j \in D} \mathbb{P}(X_T = c | X_1 = j, X_0 = i) p_{ij} \\ &= \sum_{j \in D} \mathbb{P}(X_T = c | X_1 = j) p_{ij} \\ &= p_{ia}\mathbb{P}_a\{X_T = c\} + p_{ib}\mathbb{P}_b\{X_T = c\} + p_{ic}. \end{aligned}$$

where the second equality is due to the Markov property of X. Clearly $\mathbb{P}_c\{X_T = c\} = 1$ and $\mathbb{P}_d\{X_T = c\} = 0$. Substituting $i = a, b$ into (1) yields the system

$$\mathbb{P}_a\{X_T = c\} = 0.2 \cdot \mathbb{P}_a\{X_T = c\} + 0.3 \cdot \mathbb{P}_b\{X_T = c\} + 0.4,$$

$$\mathbb{P}_b\{X_T = c\} = 0.3 \cdot \mathbb{P}_a\{X_T = c\} + 0.2 \cdot \mathbb{P}_b\{X_T = c\}.$$

References