BASIC TOPOLOGY

Matric Spaces $\forall$ two points $p, q \in X$.

(a) $d(p,q) > 0$ if $p \neq q$ $d(p,p) = 0$

(b) $d(p,q) = d(q,p)$

(c) $d(p,q) \leq d(p,r) + d(r,q)$ $\forall r \in X$

neighborhood ; limit point ; isolated point ;

closed ; interior ; open ; complement ; perfect ; bounded ; dense

closed + every point of E is a limited point.

E is dense in X if every point of X is a limit point of E.

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open sets $\bigcup G_\alpha$ open

closed sets $\bigcap F_\alpha$ closed

Compact Sets

A subset K of a matric space X is said to be compact if every open cover of K contains a finite subcover.

Weierstress

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$

Perpect sets

Let P be an nonempty perfect set in R^k. Then P is uncountable.

Connected Sets

Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.

A set $E \subset X$ is said to be connected if E is not a union of two nonempty separated sets.

ODE

1^{st} order differential equations $\frac{dy}{dt} = f(t,y)$.

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t).$$

$$\frac{d\mu(t)}{dt} = \mu(t)p(t) \Rightarrow \mu(t) = \exp\int p(t)dt$$

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t) \Rightarrow y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t}\mu(s)g(s)ds + C\right)$$

2^{nd} order differential equations $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$

(linear) $ay'' + by' + cy = 0$

characteristic equation: $ar^2 + br + c = 0$ r_1, r_2.

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Homogeneous linear systems with 1^{st} order differential equations

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

$\lambda_1, \cdots, \lambda_n$ eigenvalues

$\xi_1, \cdots, \xi_n$ eigenvectors

$$\vec{x} = C_1 e^{\lambda_1 t}\xi_1 + \cdots + C_n e^{\lambda_n t}\xi_n$$

INTEGRATION

$(\sin x)' = \cos x$ $\quad (\cos x)' = -\sin x$ $\quad (\tan x)' = \sec^2 x$ $\quad (\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x$ $\quad (\csc x)' = -\csc x \cot x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ $\quad (\arctan x)' = \frac{1}{1+x^2}$ $\quad (\text{arccot}\, x)' = -\frac{1}{1+x^2}$

$(a^x)' = a^x \ln a$ $\quad (\log_a x)' = \frac{1}{x \ln a}$

$(x\ln x - x)' = \ln x$

$I = \int R(\sin x, \cos x)\,dx$ $\quad t = \tan\frac{x}{2} \Leftrightarrow x = 2\arctan t$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$I = \int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}\right)dx$

$t^m = \frac{ax+b}{cx+d}$

Improper Integrals — infinite intervals

$$\int_0^{+\infty} e^{-ax}\cos bx\,dx = \frac{-a\cos bx + b\sin bx}{a^2+b^2}e^{-ax}\Big|_0^{+\infty} = \frac{a}{a^2+b^2}$$

$$\int_0^{+\infty} e^{-ax}\sin bx\,dx = \frac{-a\sin bx - b\cos bx}{a^2+b^2}e^{-ax}\Big|_0^{+\infty} = \frac{b}{a^2+b^2}$$

$\int_0^{+\infty} f\left(ax+\frac{b}{x}\right)dx = \frac{1}{a}\int_0^{+\infty} f\left(\sqrt{x^2+4ab}\right)dx$ $\quad$ let $t = ax - \frac{b}{x}$

$\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)} = l$ $\quad$ (i) $0 < l < +\infty$ $\quad \int_a^{+\infty} f(x)dx$ & $\int_a g(x)dx$ converge / diverge simultaneously

(ii) $l = 0$ $\quad \int_0^{+\infty} g(x)dx$ converges $\Rightarrow \int_0^{+\infty} f(x)dx$ converges

(iii) $l = +\infty$ $\quad \int_0^{+\infty} g(x)dx$ diverges $\Rightarrow \int_0^{+\infty} f(x)dx$ diverges

$\lim\limits_{x\to+\infty} x^p f(x) = \lambda$ $\quad$ (i) $p > 1$, $0 \le \lambda < +\infty$ $\quad \int_a^{+\infty} f(x)dx$ converges

(ii) $p \le 1$, $0 < \lambda \le +\infty$ $\quad \int_a^{+\infty} f(x)dx$ diverges

discontinuous integrands

discontinuous at $x = a$.

$\lim\limits_{x\to a^+}(x-a)^p f(x) = \lambda$ $\quad$ (i) $0 < p < 1$, $0 \le \lambda < +\infty$ $\quad \int_a^b f(x)dx$ converges

(ii) $p \ge 1$, $0 < \lambda \le +\infty$ $\quad \int_a^b f(x)dx$ diverges

$$\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx = (f(0)-\alpha)\ln\frac{b}{a}$$

$f(x)$ continuous on $[0, +\infty)$. $\alpha = \lim\limits_{x\to+\infty} f(x)$.

$$I = -\int_0^{+\infty}\left(\int_a^b f'(xy)\,dy\right)dx = -\int_a^b \frac{1}{y}\left(\int_0^{+\infty} y\, f'(xy)\,dx\right)dy = -\ln y\Big|_a^b f(xy)\Big|_a^b$$

SERIES $S=\sum_{n=1}^{\infty} a_n$

Ratio Test $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ $\begin{cases} L<1, & S \text{ converges absolutely} \\ L>1, & S \text{ diverges.} \end{cases}$

Root Test $L=\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}$ $\begin{cases} L<1, & S \text{ converges absolutely} \\ L>1, & S \text{ diverges} \end{cases}$

Raabe's Test $\underline{\lim}_{n\to\infty}\, n\left(\frac{a_n}{a_{n+1}}-1\right)>1$, S converges

$\overline{\lim}_{n\to\infty}\, n\left(\frac{a_n}{a_{n+1}}-1\right)<1$, S diverges

Integral Test for Convergence $\sum_{n=N}^{\infty} f(n)$ converges $\Leftrightarrow \int_N^{+\infty} f(x)dx$ finite.

Log Test $\exists \alpha>0$, s.t. $n\geq N$, $\frac{\ln\frac{1}{a_n}}{\ln n}\geq 1+\alpha \Rightarrow$ S converges

If $n\geq N$, $\frac{\ln\frac{1}{a_n}}{\ln n}\leq 1 \Rightarrow$ S diverges

2^n Test $\{a_n\}$ decreasing, $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Alternating Series Test $\{a_n\}$ decreasing, $\lim_{n\to\infty} a_n=0 \Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges

Dirichlet's Test $\left.\begin{matrix}\sum_{n=1}^{\infty} a_n \text{ is bounded} \\ \{b_n\} \text{ decreasing and } \lim_{n\to\infty} b_n=0\end{matrix}\right\}\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges

Abel's Test $\left.\begin{matrix}\sum_{n=1}^{\infty} a_n \text{ converges} \\ \{b_n\} \text{ is bounded and monotone}\end{matrix}\right\}\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges

$\prod_{n=1}^{+\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n=1$

$\underbrace{\prod_{n=1}^{+\infty}(1+a_n)}_{T_n}$ converges $\Leftrightarrow \underbrace{\sum_{n=1}^{+\infty}\ln(1+a_n)}_{S_n}$ converges $T_n=e^{S_n}$

$\prod_{n=1}^{+\infty}(1+a_n)$ converges $\Leftrightarrow \sum_{n=1}^{+\infty} a_n$ converges

Series of functions

Weierstrass M-Test $\left.\begin{matrix}\sum_{n=1}^{\infty} M_n \text{ converges} \\ |u_n(x)|\leq M_n \ \forall x.\end{matrix}\right\}\Rightarrow \sum_{n=1}^{\infty} u_n(x)$ converges uniformly

Uniform convergence $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$, s.t. $n>N$, $|S_n(x)-S(x)|<\varepsilon \ \forall x$.

NOT uniform convergent: $\exists \varepsilon_0>0$. s.t. $\forall N\in\mathbb{N}$, $\exists n'>N$ and $x'\in I$

s.t. $|f_{n'}(x')-f(x')|>\varepsilon_0$

Power series

$$e^x = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots \qquad -\infty<x<+\infty$$

$$\sin x = x-\frac{x^3}{3!}+\cdots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\cdots \qquad -\infty<x<+\infty$$

$$\cos x = 1-\frac{x^2}{2!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots \qquad -\infty<x<+\infty$$

$$(1+x)^m = 1+mx+\frac{m(m-1)}{2!}x^2+\cdots+\frac{m(m-1)\cdots(m-n+1)}{n!}x^n+\cdots \qquad -1<x<1$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n} \qquad -1<x<1.$$

convergence radius: $\sum_{n=1}^{+\infty} a_n x^n$

$$\rho = \overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \qquad R=\frac{1}{\rho}$$

Fourier Series on $(-l,l)$

$$f(x) \sim \frac{a_0}{2}+\sum_{n=1}^{+\infty}\left(a_n\cos\frac{n\pi}{l}x+b_n\sin\frac{n\pi}{l}x\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Bessel's inequality $\sum_{n=1}^{\infty}|\langle x,e_k\rangle|^2 \leq \|x\|^2$

$$\frac{a_0^2}{2}+\sum_{n=1}^{+\infty}(a_n^2+b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx.$$

Parseval's theorem

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

含参变量积分

$f(x,y)$ 在 $D=[a,b]\times[c,d]$ 连续 $\Rightarrow$ ① $I(x)=\int_c^d f(x,y)dy$ 存在且连续.

② $\int_a^b dx\int_c^d f(x,y)dy = \int_c^d dy\int_a^b f(x,y)dx$

③ 若 $f'_x(x,y)$ 也在 $D=[a,b]\times[c,d]$ 连续 $\Rightarrow$ $I'(x)=\int_c^d f'_x(x,y)dy$

④ 若 $p(x), q(x) \in [c,d]$, $x\in[a,b]$ 可微 $\Rightarrow$ $I'(x)=\int_{c(x)}^{d(x)} f'_x(x,y)dy + f(x,d(x))d'(x) - f(x,c(x))c'(x)$

反常积分类似

$$\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}\,dx = f(0)\ln\frac{b}{a} \qquad \int_a^{+\infty} \frac{f(x)}{x}\,dx \text{ exists}$$

$$\frac{e^{-ax}-e^{-bx}}{x} = \int_a^b e^{-xy}\,dy$$

$$\frac{x^b-x^a}{\ln x} = \int_a^b x^y\,dy$$

$$\frac{\arctan x}{x} = \int_0^1 \frac{dy}{1+x^2y^2}$$

$$I(a)=\int_0^{\frac{\pi}{2}} \ln(a^2\sin^2 x + b^2\cos^2 x)\,dx$$

$$I'(a) = \int_0^{\frac{\pi}{2}} \frac{2a\sin^2 x}{a^2\sin^2 x + b^2\cos^2 x}\,dx$$

$$I(a) = \int I'(a)\,da + C.$$

$$I(a) = \int_0^{\pi} \ln(1-2a\cos x + a^2)\,dx$$

$$I'(a) = \int_0^{\pi} \frac{-2\cos x + 2a}{1-2a\cos x + a^2}\,dx$$

$$I(a) = \int I'(a)\,da + C.$$

$$I = \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

let $x = ut$, $I = u\int_0^{+\infty} e^{-u^2t^2}\,dt$

$$\int_0^{+\infty} I e^{-u^2}\,du = I^2 = \int_0^{+\infty} e^{-u^2}du \int_0^{+\infty} ue^{-u^2t^2}\,dt$$

$$= \int_0^{+\infty} dt \int_0^{+\infty} e^{-(1+t^2)u^2}u\,du = \frac{1}{2}\int_0^{+\infty} \frac{dt}{1+t^2} = \frac{\pi}{4}$$

Gamma function. $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}\,dx \quad (s>0).$

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(s) = 2\int_0^{+\infty} x^{2s-1}e^{-x^2}\,dx \qquad (\text{let } x = t^2)$$

$$\Gamma''(s) > 0 \quad \text{and} \quad \ln\Gamma''(s) > 0$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s} \qquad 0<s<1.$$

Beta function $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad (p,q>0)$

$$B(p,q) = B(q,p) = \frac{p-1}{p+q-1}B(p-1,q)$$

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \cos^{2p-1}\theta\,\sin^{2q-1}\theta\,d\theta \qquad (x=\cos^2\theta).$$

$$B(p,q) = \int_0^{+\infty} \frac{x^{q-1}}{(1+x)^{p+q}}\,dx \qquad x = \frac{y}{1+y}$$

$$B(p,q) = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx \qquad y=\frac{1}{t} \text{ for } \int_1^{+\infty} \frac{x^{q-1}}{(1+x)^{p+q}}\,dx$$

MULTIVARIABLE CALCULUS

Inverse function theorem

$W = B(x_0; \varepsilon) \subset \mathbb{R}^m$, $\vec{f}: W \to \mathbb{R}^m$ continuously differentiable, $\vec{y}_0 = \vec{f}(\vec{x}_0)$

If $L = [D\vec{f}(\vec{x}_0)]$ invertible, then $\vec{f}$ is invertible in some neighborhood of $\vec{y}_0$

Implicit function theorem

$W = B(x_0; \varepsilon) \subset \mathbb{R}^n$, $\vec{F}: W \to \mathbb{R}^{n-k}$ differentiable with $\vec{F}(\vec{c}) = \vec{0}$ and $D\vec{F}(\vec{c})$ onto

If

$$L = \begin{bmatrix} [D_{y_1}\vec{F}(\vec{c}) & D_{y_2}\vec{F}(\vec{c}) & \cdots & D_{y_{n-k}}\vec{F}(\vec{c})] & [D_{x_1}\vec{F}(\vec{c}) & \cdots & D_{x_k}\vec{F}(\vec{c})] \\ & [0] & & & & I_k & \end{bmatrix}$$ invertible,

there exists a unique $\vec{g}(\vec{x})$, $\vec{F}\begin{pmatrix} \vec{g}(\vec{x}) \\ \vec{x} \end{pmatrix} = 0 \quad \forall \vec{x}$ $\qquad \vec{c} = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}$

$$D\vec{g}(\vec{b}) = -[D_{y_1}\vec{F}(\vec{c}) \cdots D_{y_{n-k}}\vec{F}(\vec{c})]^{-1} [D_{x_1}\vec{F}(\vec{c}) \cdots D_{x_k}\vec{F}(\vec{c})]$$

Tangent space to a manifold given by equation

$\vec{F}(\vec{z}) = \vec{0}$, $D\vec{F}(\vec{z}_0)$ is onto for some $\vec{z}_0 \in M$

Then $T_{\vec{z}_0} M = \ker [D\vec{F}(\vec{z}_0)]$, equation: $[D\vec{F}(\vec{z}_0)] \begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{bmatrix} = 0$

i.e. $D_1\vec{F}(\vec{z}_0)(z_1 - z_{0,1}) + \cdots D_n\vec{F}(\vec{z}_0)(z_n - z_{0,n}) = \vec{0}$

Quadratic forms and extrema

$f: U \subset \mathbb{R}^n \to \mathbb{R}$, $f \in C^2(a)$, a critical point, then

If $H(a)$ pos def, f has local min at a

If $H(a)$ neg def, f has local max at a

Vector fields in Euclidean space

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right) = \text{grad } f.$$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

divergence $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

curl $\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

conservative field $\int_C \vec{F} \cdot d\vec{r}$ independent of path

line integral $x = x(t), \quad y = y(t), \quad z = z(t), \qquad a \le t \le b$

$$\int_C f(x,y,z)\,dr = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

where $\frac{dr}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$$= \int_a^b P(x(t), y(t), z(t))\,x'(t)\,dt + \int_a^b Q(x(t), y(t), z(t))\,y'(t)\,dt + \int_a^b R(x(t), y(t), z(t))\,z'(t)\,dt$$

surface integral $x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \qquad (u,v) \in D$

$$\iint_S f(x,y,z)\,ds = \iint_D f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\,du\,dv$$

area of S: $z = f(x,y) \qquad S = \iint_D \sqrt{1 + f_x'^2(x,y) + f_y'^2(x,y)}\,dx\,dy$

$$A = \frac{\partial(y,z)}{\partial(u,v)} \qquad B = \frac{\partial(z,x)}{\partial(u,v)} \qquad C = \frac{\partial(x,y)}{\partial(u,v)}$$

$$n = (\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{A}{\sqrt{A^2+B^2+C^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}}\right)$$

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \vec{n}\,ds = \iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

$$= \iint_D (PA + QB + RC)\,du\,dv$$

Green's theorem $P(x,y)$, $Q(x,y)$ continuously differentiable on $D \subset \mathbb{R}^2$

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Divergence theorem $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ continuously differentiable on $D \subset \mathbb{R}^3$

(Gauss's theorem)

$$\iint_{\partial D} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$\vec{F} \cdot \vec{n}\,dS$ $\quad$ $\operatorname{div} \vec{F}$

Stoke's theorem $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ continuously differentiable on S

S: a surface on $\mathbb{R}^3$

$\vec{F} \cdot d\vec{r}$

$$\int_{\partial S} P\,dx + Q\,dy + R\,dz$$

$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$\operatorname{curl} \vec{F} \cdot d\vec{S}$

Improper integral.

Dirichlet's Test. $\exists M > 0$, s.t. $\forall X > a$, $\left|\int_a^X f(x)dx\right| \leq M$. $\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow \int_a^{+\infty} f(x)g(x)dx$ converges

$g(x)$ monotone, $\lim\limits_{x\to+\infty} g(x) = 0$

Abel's Test $\int_a^{+\infty} f(x)\,dx$ converges $\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow \int_a^{+\infty} f(x)g(x)dx$ converges

$g(x)$ monotone & bounded