AMCS Written Preliminary Exam

Part I, May 3, 2021

裴礼文 P375 / 4.4.2 1. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Prove that

$$\sup_{0 \le x \le 1} |f(x)| \le \sqrt{\int_0^1 |f'(x)|^2 dx}.$$

i.e. $|f(x)|^2 \leqslant \int_0^1 |f'(x)|^2 dx \quad \forall x \in [0,1]$

$\cancel{|f(x)|^2 = |f^2(x) - f^2(0)| = |\int_0^x (f^2(t))' dt|}$

$\cancel{= |\int_0^x 2f(t)f'(t) dt| \leqslant 2\int_0^x |f(t)f'(t)| dt \leqslant 2\int_0^1 |f(t)f'(t)| dt}$

$|f(x)|^2 = |f(x) - f(0)|^2 = |\int_0^x f'(t) dt|^2 \leqslant \int_0^x |f'(t)|^2 dt \leqslant \int_0^1 |f'(t)|^2 dt = \int_0^1 |f'(x)|^2 dx.$

2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions on $\mathbb{R}$ which converges uniformly to some limit function f. If $\lim_{n\to\infty} \lim_{x\to\infty} f_n(x)$ exists, prove that

$$\lim_{x\to\infty} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to\infty} f_n(x)$$

(and in particular, show that the limit on the left-hand side exists).

$\lim_{x\to\infty} (\lim_{n\to\infty} f_n(x)) = \lim_{x\to\infty} f(x)$

$\lim_{n\to\infty} (\lim_{x\to\infty} f_n(x)) = A$

$\forall \varepsilon_1 > 0, \exists N \in \mathbb{N}^*, \text{ s.t. } \forall n > N,$

$|\lim_{x\to\infty} f_n(x) - A| < \varepsilon_1$

$\forall \varepsilon_2 > 0, \exists X \in \mathbb{N}^*, \text{ s.t. } \forall x > X, \Rightarrow \lim_{x\to\infty} f(x)$ exists.

$|f_n(x) - f(x)| < \varepsilon_2$

3. Let

$$M := \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}.$$

Find a 2×2 matrix A and a 2×4 matrix B, both of which have orthonormal rows, and a diagonal 2×2 matrix D with positive diagonal entries such that $M = ADB$.

First solve for U, V, s.t. $M = U \Sigma V^T$:

$MM^T = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

$\det(MM^T - \lambda I) = \det\begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} = (5-\lambda)^2 - 9 = 0$

$\lambda_1 = 2, \quad \lambda_2 = 8.$

$\lambda_1 = 2: \quad (MM^T - \lambda_1 I) = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \Rightarrow X_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda_2 = 8: \quad (MM^T - \lambda_2 I) = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \Rightarrow X_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$

$M^T M = \begin{pmatrix} \frac{5}{2} & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{2} & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} \end{pmatrix}$

$\det(M^TM - \lambda I) = \det\begin{pmatrix} \frac{5}{2}-\lambda & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{2} & \frac{5}{2}-\lambda & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{2}-\lambda & \frac{5}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{5}{2}-\lambda \end{pmatrix}$

$\lambda_1 = 2, \ \lambda_2 = 8, \ \lambda_3 = \lambda_4 = 0.$

$\lambda_1 = 2: \quad (M^TM - \lambda_1 I) = \begin{pmatrix} \frac{1}{2} & \frac{5}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow Y_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

1

$\lambda_2 = 8: \quad Y_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad \lambda_3 = \lambda_4 = 0 \Rightarrow Y_3 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \quad Y_4 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$U = (X_1 \ X_2) \qquad V = (Y_1 \ Y_2 \ Y_3 \ Y_4) \qquad \Sigma = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{pmatrix}$

then $A = U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$

$B = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$

4. Suppose A and B are $n \times n$ complex matrices which commute.

a. Show that

$$(A+B)^j = \sum_{k=0}^{j} \binom{j}{k} A^k B^{j-k}$$ commute

for every integer $j \geq 1$.

b. Prove that

$$e^{A+B} = e^A e^B.$$

$$(A+B) + \frac{(A+B)^2}{2!} + \cdots \qquad \left(A + \frac{A}{2!} + \cdots\right)\left(B + \frac{B}{2!} + \cdots\right)$$

5. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}}$$

where n is a positive integer. Hint: The simplest approach is to evaluate a contour integral around a wedge-shaped region containing only a single pole.

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^{2n}}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^{2n}}\,dx = \lim_{R\to\infty} \left(\int_{C_R} \frac{1}{1+x^{2n}}\,dx - \int_{\Gamma_R} \frac{1}{1+x^{2n}}\,dx \right)$$

$$= 2\pi i \sum_k \operatorname{Res}\left(\frac{1}{1+x^{2n}}; z_k\right) = 2\pi i \sum_k \frac{1}{2n z_k^{2n-1}} = 2\pi i \sum_{k=0}^{2n-1} \frac{1}{2n\left(e^{\frac{2k+1}{2n}\pi i}\right)^{2n-1}}$$

$$= \frac{\pi i}{n} \sum_{k=0}^{2n-1} e^{\frac{2k+1-2n}{2n}\pi i}$$

6. For each integer $n \geq 1$, suppose $x_n \in (0, \infty)$ is randomly selected with pdf $\rho_k(x)$ given by

$$\rho_k(x) := \frac{k}{x^{k+1}} e^{-1/x^k}. \qquad \int \rho_k(x)\,dx = e^{-\frac{1}{x^k}} + C.$$

Assuming that the x_n are chosen independently,

a. What is

$$\mathbb{P}\left(\sup_{n\geq 1} x_n \leq \frac{1}{2}\right)?$$

b. What is

$$\mathbb{P}\left(\sup_{n\geq 1} x_n \leq 2\right)?$$

$$P\left(\sup_{n\geq 1} x_n \leq \tfrac{1}{2}\right) = P\left(x_n \leq \tfrac{1}{2}\ \forall n\right). = \left(P\left(x_n \leq \tfrac{1}{2}\right)\right)^n = \left(\int_0^{\frac{1}{2}} \rho_k(x)\,dx\right)^n = \left(e^{-\frac{1}{x^k}}\Big|_0^{\frac{1}{2}}\right)^n = \left(e^{-2^k}\right)^n = e^{-2^k n}$$

$$P\left(\sup_{n\geq 1} x_n \leq 2\right) = \left(e^{-\frac{1}{x^k}}\Big|_0^{2}\right)^n = e^{-\frac{n}{2^k}}$$

AMCS Written Preliminary Exam
Part II, May 3, 2021

7. Let

$$\Phi(x,y) := \left(\frac{x^3}{5} + \frac{\sin y}{5}, \frac{\cos x}{5} + \frac{y^3}{5}\right).$$

Show that there is a unique point $(x,y) \in [0,1]^2$ such that $\Phi(x,y) = (x,y)$.

$F\binom{x}{y} = \Phi\binom{x}{y} - \binom{x}{y} = \begin{pmatrix} \frac{x^3}{5} - x + \frac{\sin y}{5} \\ \frac{y^3}{5} - y + \frac{\cos x}{5} \end{pmatrix} = 0.$ then $\det DF\binom{x}{y} \neq 0$.

$F\binom{x}{y}$ is 1-1.

unique ✓

$DF\binom{x}{y} = (D_x F \quad D_y F) = \begin{pmatrix} \frac{3}{5}x^2 - 1 & \frac{1}{5}\cos y \\ -\frac{1}{5}\sin x & \frac{3}{5}y^2 - 1 \end{pmatrix}$

$\det DF\binom{x}{y} = 0$ when $(\frac{3}{5}x^2 - 1)(\frac{3}{5}y^2 - 1) + \frac{1}{25}\sin x \cos y = 0$

impossible because $(x,y) \in [0,1]^2$

8. Let $f(x)$ be a continuous function on $\mathbb{R}$ and let

$$f_n(x) := \int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|) f(x-y)\,dy$$

where n is any positive integer. Prove that, as $n \to \infty$, $f_n(x)$ converges to $f(x)$ uniformly on any bounded interval of $\mathbb{R}$.

$\int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|)\,dy = \int_{-\frac{1}{n}}^{0} (n + n^2 y)\,dy + \int_0^{\frac{1}{n}} (n - n^2 y)\,dy = (ny + \frac{1}{2}n^2y^2)\Big|_{-\frac{1}{n}}^{0} + (ny - \frac{1}{2}n^2y^2)\Big|_0^{\frac{1}{n}} = -(-1+\frac{1}{2}) + (1 - \frac{1}{2}) = 1$

$|f_n(x) - f(x)| = |\int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|) f(x-y)\,dy - f(x)| = |\int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|)(f(x-y) - f(x))\,dy| \leq \int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|)\,|f(x-y) - f(x)|\,dy$

9. Prove that the following is an inner product on the real vector space V polynomials of degree at most 2 and compute an example of a basis of V which is orthonormal with respect to this inner product.

$$\langle p, q\rangle := p(0)q(0) + \int_{-1}^{1} p(x)q(x)\,dx.$$

prof:

$\langle p, q\rangle = \langle q, p\rangle$

$\langle kp, q\rangle = k\langle p, q\rangle$

$\langle p+q, r\rangle = (p(0) + q(0))\,r(0) + \int_{-1}^{1} (p(x) + q(x))\,r(x)\,dx$

$= p(0)r(0) + \int_{-1}^{1} p(x)r(x)\,dx + q(0)r(0) + \int_{-1}^{1} q(x)r(x)\,dx$

$= \langle p, r\rangle + \langle q, r\rangle.$

$\langle p, p\rangle = p^2(0) + \int_{-1}^{1} p^2(x)\,dx \geq 0.$ if and only if $p(x) = 0$, $\langle p, p\rangle = 0$

example: $\langle 1, x, x^2\rangle$.

Frobenius norm

10. Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \leq 1.$$

Show that all eigenvalues of A lie in the closed unit disk $\{z \in \mathbb{C} \ : \ |z| \leq 1\}$ and that for any $r < 1$, there is an A of the sort described above which has at least one eigenvalue λ satisfying $|\lambda| \geq r$.

$\Sigma |\lambda|^2 = \mathrm{tr}(A^H A) = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 = \|A\|_F \leq 1.$

11. Find all zeros, poles, and essential singularities of the function

$$\frac{e^{\pi z} + 1}{e^{\pi/z} + 1}$$

in the complex plane. Compute the order of each zero or pole, and for each pole, compute the residue as well.

zeros: $e^{\pi z} + 1 = 0 \Rightarrow e^{\pi z} = e^{(2k+1)\pi i}$, $k \in \mathbb{Z}$

$\Rightarrow z = (2k+1)i$, $k \in \mathbb{Z}$. order 1. $\pm i$ removable.

poles: $e^{\frac{\pi}{z}} + 1 = 0 \Rightarrow z = \frac{1}{(2k+1)i}$, $k \in \mathbb{Z}$.

essential singularities: $z = 0$. $\lim_{z \to 0} \frac{e^{\pi z} + 1}{e^{\frac{\pi}{z}} + 1}$ doesn't exist.

$\mathrm{Res}\left(\frac{e^{\pi z}+1}{e^{\pi/z}+1}; z_k\right) = \frac{e^{\pi z}+1}{-\frac{\pi}{z^2} e^{\pi/z}}\Big|_{z=z_k} = -\frac{z^2(e^{\pi z}+1)}{\pi e^{\pi/z}}\Big|_{z=z_k} = -\frac{\left(\frac{1}{(2k+1)i}\right)^2 \left(e^{\frac{\pi}{(2k+1)i}}+1\right)}{\pi e^{(2k+1)\pi i}} = -\frac{e^{\frac{\pi}{(2k+1)i}}+1}{\pi(2k+1)^2}$

12. Let $C \subset \mathbb{R}^2$ be the circle $x^2 + y^2 = 4$. If u and v are two points chosen independently and uniformly at random on C, what is the probability that the line segment joining u and v will be entirely contained in the annulus A given by $3 \leq x^2 + y^2 \leq 4$?

~~$P(u \in \{3 \leq x^2+y^2 \leq 4\}) = \frac{1}{4}$~~

~~$P(u, v \in \{3 \leq x^2+y^2 \leq 4\}) = \frac{1}{16}$.~~

$\overset{\frown}{AB} < 60°$ $\quad \frac{1}{3}$.