Name ______________________

AMCS Written Preliminary Exam
Part II, April 27, 2018

1. Suppose that $\sum_{n=0}^{\infty} a_n(z-1)^n$ is the power series expansion of the function

$$f(z) = \frac{1}{\cos z}$$

diverges. [prove by contradiction]

about the point $z = 1$. Does the series

$$\sum_{n=0}^{\infty} |a_n| \qquad (1)$$

if converge: when $z=\frac{\pi}{2}$. $\sum_{n=0}^{\infty} a_n(\frac{\pi}{2}-1)^n < \sum_{n=1}^{\infty}|a_n|$. converges.

but $f(z)=\frac{1}{\cos z} \to \infty$ at $z=\frac{\pi}{2}$.

converge or diverge? You must justify your answer.

$|e^{x+iy}| = |e^x\cos y + i e^x\sin y| = e^x \Rightarrow |f(z)| \le |e^z|$

2. Suppose that $f(z)$ is an entire function such that $|f(z)| \le e^x$, for all $z = x + iy$. What can be said about the function $f(z)$? you must prove your answer.

3. A real matrix A is skew symmetric if $A^t = -A$.

   (a) Show that a $(2n+1)\times(2n+1)$ skew symmetric matrix has a non-trivial null-space. $\det(A^T) = \det(-A) \Leftrightarrow \det(A) = -\det(A) \Rightarrow \det(A) = 0$.

   (b) Show that if λ is an eigenvalue of A, then so is $-\lambda$. $\det(A-\lambda I) = \det(A^T-\lambda I) = \det(-A-\lambda I) = -\det(A+\lambda I) = 0$

   spectrum of A: the set of eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.

   (c) Show that the spectrum of A is purely imaginary, i.e., consists of numbers of the form $\{i\lambda_j\}$ where the $\lambda_j \in \mathbb{R}$.

   ~~$\lambda x^T x = x^T\lambda x = x^T A x = x^T(-A^T)x = -(Ax)^T x = -(\lambda x)^T x = -\lambda x^T x$~~

   $\lambda x^H x = x^H \lambda x = x^H A x = x^H(-A^H)x = -(Ax)^H x = -(\lambda x)^H x = -\bar{\lambda} x^H x$.

   (d) Show that $\det A \ge 0$. $\det(A) = 0$.

4. Suppose that A is a real, upper triangular matrix, with strictly positive diagonal entries. Prove that there is a real, upper triangular matrix, with strictly positive diagonal entries, B, such that $B^2 = A$. Hint: Use induction.

   $A = S^{-1}\Lambda^2 S$, $B = S^{-1}\Lambda S$ also upper triangular

5. Evaluate the following limits and justify your answers:

   (a)
$$\lim_{n\to\infty}\int_0^1 ne^{-nx}(\cos x)^2 dx$$

$= \lim_{n\to\infty}\int_0^1 -d(e^{-nx})(\cos x)^2$

$= \lim_{n\to\infty}\left(-(\cos x)^2 e^{-nx}\Big|_0^1 + \int_0^1 e^{-nx}\, d(\cos x)^2\right)$

$= \lim_{n\to\infty}\left(-e^{-n}\cos^2 1 + 1 + \int_0^1 e^{-nx}\, 2\cos x(-\sin x)dx\right)$

$= 1$ since $|2\cos x\sin x| = |\sin 2x| \le 1$.

~~$Ei(x) = \int_x^\infty \frac{e^{-u}}{u}du$. $u = xt$~~

~~$= \int_1^\infty \frac{e^{-xt}}{xt}x\,dt = \int_1^\infty \frac{e^{-xt}}{t}dt$~~

~~$= \int_1^\infty -e^{-}$~~

   (b)
$$\lim_{t\to\infty}\left[te^t\int_t^\infty \frac{e^{-s}}{s}ds\right].$$

does not exist. let $I = te^t\int_t^\infty \frac{e^{-s}}{s}ds$.

~~$\frac{dI}{dt} = te^t(-\frac{e^{-t}}{t}) + (e^t + te^t)\int_t^\infty \frac{e^{-s}}{s}ds$~~

~~$= -1 + (e^t + te^t)\int_t^\infty \frac{e^{-s}}{s}ds$~~

~~$= -1 + \frac{1+t}{t}I$.~~

~~$\frac{dI}{dt} - \frac{1+t}{t}I = -1$~~

~~$M(t) = \exp\int -\frac{1+t}{t}dt = \frac{1}{te^t}$~~

~~$I(t) = te^t\int_{t_0}^t -\frac{1}{se^s}ds + C$.~~

~~$I(t)$ const.~~

$\lim_{t\to\infty} te^t\int_t^\infty \frac{e^{-s}}{s}ds$

$= \lim_{t\to\infty} te^t\left[-\left(\frac{e^{-s}}{s}\Big|_t^\infty - \int_t^\infty e^{-s}d(\frac{1}{s})\right)\right]$

$= \lim_{t\to\infty} te^t\cdot\frac{e^{-t}}{t} - \lim_{t\to\infty} te^t\int_t^\infty \frac{e^{-s}}{s^2}ds$

$= 1 - S. = 0$.

$|S| \le \left|te^t\cdot\frac{1}{t^2}\int_t^\infty e^{-s}ds\right| \to 0$.

6. Find a conformal map from the half disk

$$D_1^+ = \{z : |z| < 1 \text{ and } 0 < \operatorname{Im} z\}$$

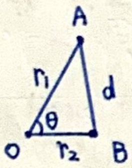
to the unit disk, $D_1 = \{z : |z| < 1\}$. $f(z) = z^2$

7. Two random points A and B are selected independently, and uniformly from the disk $\{(x, y) : x^2 + y^2 < 1\}$. A third random point C is selected uniformly from the larger disk $\{(x, y) : x^2 + y^2 < 4\}$, independently of A and B. What is the probability that the angle $\angle ACB$ is obtuse? Hint: First consider the answer for a fixed choice of A, B.

For fixed A & B, $\angle ACB$ is obtuse if C lies inside the circle whose diameter is AB. (radius r).

[C inside the larger disk]. $(r = \frac{d}{2})$

$P(\angle ACB \text{ is obtuse} \mid |AB| = d) = \frac{\pi r^2}{\pi \cdot 4} = \frac{r^2}{4} = \frac{d^2}{16} \quad (0 < r < 1)$.



$r_1 \sim U(0,1)$

$r_2 \sim U(0,1)$

$\theta \sim U(0,\pi)$. $\quad t = \cos\theta$, $\theta = \arccos t$

$p(\theta) = p(\arccos t) = \frac{1}{2\pi} \arccos t$

$d^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta$