AMCS Written Preliminary Exam
Part I, May 2, 2019

1. Calculate the Jordan form of the following matrix:

$$B = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

For this problem you may assume that the following equation holds:

$$\det(B - \lambda I) = (\lambda - 2)^2(\lambda - 1)$$

   (a) Find the algebraic and geometric multiplicities of both eigenvalues.

   (b) Find the Jordan form J of B.

   (c) Find an invertible matrix S so that $B = SJS^{-1}$. Be sure to clearly explain how you obtained the columns of this matrix.

2. Assume A is a symmetric $n \times n$ matrix with eigenvalues $1, 2, \ldots, n$ and consider $y : [0, \infty) \to \mathbb{R}^n$ which satisfies the ODE

$$\dot{y}(t) + Ay(t) = 0, \quad t > 0.$$

   (a) Compute the limit

$$\lim_{t\to\infty} e^t y(t).$$

   (b) Provided $y(0) \neq 0$, show $y(t) \neq 0$ for all $t > 0$ and compute

$$\lim_{t\to\infty} \frac{Ay(t) \cdot y(t)}{\|y(t)\|^2}.$$

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$f(tx) = tf(x)$$

for each $x \in \mathbb{R}^n$ and $t \geq 0$.

(a) Assume that f is differentiable away from 0, then verify that

$$\nabla f(x) \cdot x = f(x)$$

holds away from 0.

(b) Show that if f is differentiable at $x = 0$, then there is $a \in \mathbb{R}^n$ such that $f(x) = a \cdot x$ for all $x \in \mathbb{R}^n$.

4. Prove the summation by parts formula:
显然,

$$\sum_{n=1}^{N} a_n b_n = a_N B_N - a_1 B_0 - \sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n,$$

where $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$ are any finite sequences of complex numbers and $\{B_n\}_{n=0}^{N}$ is any sequence satisfying $B_n - B_{n-1} = b_n$ for any $n = 1, \ldots, N$.

5. Identify all poles, zeros, removable singularities, and essential singularities of the function

$$f(z) = \frac{z^4 + 1}{z^2 - z} \sin \frac{\pi z^2}{z^2 - 1}.$$

Be sure to consider the point at infinity.

6. Show that $\displaystyle\int_0^\infty \frac{x^\beta}{(x+1)^2} dx = \frac{\pi\beta}{\sin \pi\beta}$ assuming that $0 < \beta < 1$.

7. Suppose U, V are two independent random variables which are uniformly distributed in the interval $(0, 1)$. Find the joint distribution of the random variables X, Y defined via

$$X := \sqrt{-2 \ln U} \cos(2\pi V) \quad \text{and} \quad Y := \sqrt{-2 \ln U} \sin(2\pi V).$$

Part I May 2 2019

1. (a) algebraic multiplicities of 2 and 1 are 2 and 1.

when $\lambda_1 = 2$:

$$B - \lambda I = \begin{pmatrix} 4-\lambda & 2 & 1 \\ -2 & -\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ -2 & -2 & -1 \\ -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\langle 2, 2, 1 \rangle$, $\langle 1, 1, 1 \rangle$ independent

$\Rightarrow \dim(B - \lambda I) = 2. \Rightarrow$ geometric multiplicity $= 1$.

when $\lambda_2 = 1$:

$$B - \lambda I = \begin{pmatrix} 3 & 2 & 1 \\ -2 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

geometric multiplicity $= 1$.

(b). $J_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$J_2 = \lambda_2 = 1$

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c). $\lambda_1 = 2$, $x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1$, $x_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$B = SJS^{-1}$, $S = [x_1, x_2, x_3]$.

$BS = SJ \Rightarrow B[x_1, x_2, x_3] = [x_1, x_2, x_3]\, J$

$[Bx_1, Bx_2, Bx_3] = [\lambda_1 x_1, x_1 + \lambda_1 x_2, \lambda_2 x_3]$

then $Bx_2 = x_1 + \lambda_1 x_2$

$(B - \lambda_1 I)x_2 = x_1$

$$[B - \lambda_1 I \mid x_1] = \left(\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ -2 & -2 & -1 & -1 \\ -1 & -1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

then $x_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$S = [x_1, x_2, x_3] = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

2. (a)

$\dot{y}(t) + A\,y(t) = 0, \quad t > 0, \qquad$ where $A = S \Lambda S^T$, $\Lambda = \mathrm{diag}\{1, \cdots, n\}$

$\frac{dy}{dt} = -Ay \quad , \quad \lambda_1 = 1, \cdots, \lambda_n = n.$

$y(t) = c_1 e^{-\lambda_1 t} x_1 + \cdots + c_n e^{-\lambda_n t} x_n.$

$y(t) = c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n.$

$y(0) = c_1 x_1 + \cdots + c_n x_n. \quad , \quad c_1, \cdots, c_n$ are constants

where $x_1, \cdots, x_n$ are eigenvectors of A.

$S = [x_1, \cdots, x_n]$, $x_1, \cdots, x_n$ are orthogonal, $x_k^T x_k = 1$.

$$\lim_{t\to\infty} e^t y(t) = \lim_{t\to\infty} e^t (c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n)$$

$$= c_1 x_1 + \lim_{t\to\infty} (c_2 e^{-t} x_2 + \cdots + c_n e^{-(n-1)t} x_n)$$

$$= c_1 x_1.$$

2 (b).

$$A\,y(t) = S\Lambda S^T \left(c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n\right)$$

$$= (x_1, \cdots, x_n) \begin{pmatrix} 1 & & \\ & \ddots & \\ & & n \end{pmatrix} \begin{pmatrix} x_1^T \\ \vdots \\ x_n^T \end{pmatrix} \left(c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n\right).$$

$$= (x_1, \cdots, k x_k, \cdots, n x_n) \begin{pmatrix} c_1 e^{-t} x_1^T x_1 \\ \vdots \\ c_n e^{-nt} x_n^T x_n \end{pmatrix}$$

$$= (x_1, \cdots, n x_n) \begin{pmatrix} c_1 e^{-t} \\ \vdots \\ c_n e^{-nt} \end{pmatrix}$$

$$= c_1 e^{-t} x_1 + \cdots + k c_k e^{-kt} x_k + \cdots + n c_n e^{-nt} x_n.$$

$$A\,y(t) \cdot y(t) = \left(c_1 e^{-t} x_1 + \cdots + k c_k e^{-kt} x_k + \cdots + n c_n e^{-nt} x_n\right)^T \left(c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n\right)$$

$$= c_1^2 e^{-2t} x_1^T x_1 + \cdots + k c_k^2 e^{-2kt} x_k^T x_k + \cdots + n c_n^2 e^{-2nt} x_n^T x_n$$

$$= c_1^2 e^{-2t} + \cdots + k c_k^2 e^{-2kt} + \cdots + n c_n^2 e^{-2nt}$$

$$\|y(t)\|^2 = y(t)^T \cdot y(t) = \left(c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n\right)^T \left(c_1 e^{-t} x_1 + \cdots + c_n e^{-nt} x_n\right)$$

$$= c_1^2 e^{-2t} + \cdots + c_k^2 e^{-2kt} + \cdots + c_n^2 e^{-2nt}$$

$$\lim_{t\to\infty} \frac{A\,y(t) \cdot y(t)}{\|y(t)\|^2} = \lim_{t\to\infty} \frac{c_1^2 e^{-2t} + \cdots + k c_k^2 e^{-2kt} + \cdots + n c_n^2 e^{-2nt}}{c_1^2 e^{-2t} + \cdots + c_k^2 e^{-2kt} + \cdots + c_n^2 e^{-2nt}} = 1.$$

3. (a) (b).

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, then. $f(x) = f(x_1, \cdots, x_n)$.

$f(tx) = f(tx_1, \cdots, tx_n) = t f(x_1, \cdots, x_n) = t f(x)$. $\forall t > 0$. $\forall x \in \mathbb{R}^n$.

then $\frac{\partial f(tx_1, \cdots, tx_n)}{\partial x_k} = t \frac{\partial f(tx_1, \cdots, x_k, \cdots, tx_n)}{\partial x_k}$

and $\frac{\partial f(tx_1, \cdots, tx_n)}{\partial x_k} = \frac{\partial t f(x_1, \cdots, x_n)}{\partial x_k} = t \frac{\partial f(x_1, \cdots, x_n)}{\partial x_k}$

$\Rightarrow \frac{\partial f(tx_1, \cdots, x_k, \cdots, tx_n)}{\partial x_k} = \frac{\partial f(x_1, \cdots, x_k, \cdots, x_n)}{\partial x_k}$

Let $f_k(x_k) = f(0, \cdots, x_k, \cdots, 0)$, then $f(x_1, \cdots, x_n) = \sum_{k=1}^{n} f_k(x_k)$.

$\forall k \in \{1, \cdots, n\}$, $f_k(tx_k) = t f_k(x_k)$.

$$\frac{f_k(tx_k)}{tx_k} = \frac{t f_k(x_k)}{tx_k} = \frac{f_k(x_k)}{x_k} \quad \forall t > 0.$$

therefore, $\frac{f_k(x_k)}{x_k}$ is a constant.

let $\frac{f_k(x_k)}{x_k} = a_k$, then $f(x_1, \cdots, x_n) = \sum_{k=1}^{n} f_k(x_k) = a_k x_k$

$\nabla f(x) = (a_1, \cdots, a_n)^T$.

$\nabla f(x) \cdot x = (a_1, \cdots, a_n)^T \cdot (x_1, \cdots, x_n)^T = a_1 x_1 + \cdots + a_n x_n = f(x)$.

5.

$$f(z) = \frac{z^4+1}{z^2-z} \sin \frac{\pi z^2}{z^2-1}$$

(i) when $z_0^2 - 1 = 0$ (i.e. $z_0 = -1$ or 1), $\lim\limits_{z \to z_0} \sin \frac{\pi z_0^2}{z_0^2-1}$ does not exist

then $f(z)$ has essential singularities at $z_0 = -1$ or 1.

(ii) when $z_0^2 - z_0 = 0$, $z_0^2 - 1 \neq 0$ (i.e. $z_0 = 0$)

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to 0} z \cdot \frac{z^4+1}{z^2-z} \sin \frac{\pi z^2}{z^2-1} = \lim_{z \to 0} \frac{z^4+1}{z-1} \sin \frac{\pi z^2}{z^2-1} = 0.$$

not a pole.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 0} \frac{z^4+1}{z^2-z} \sin \frac{\pi z^2}{z^2-1} = \lim_{z \to 0} \frac{z^4+1}{z^2-z} \cdot \frac{\pi z^2}{z^2-1} = \lim_{z \to 0} \frac{\pi (z^4+1) z}{(z-1)^2 (z+1)} = 0.$$

$$\text{let } g(z) = \begin{cases} f(z), & z \neq 0 \\ 0, & z = 0. \end{cases}$$

$$g'(0) = \lim_{z \to 0} \frac{g(z) - g(0)}{z} = \lim_{z \to 0} \frac{\pi (z^4+1)}{(z-1)^2 (z+1)} = \pi$$ then $g(z)$ analytic.

then $f(z)$ has a removable singularity at $z_0 = 0$.

(iii) when $z_0^4 + 1 = 0$ or $\pi z_0^2 = 0$, but $z_0^2 - z_0 \neq 0$ and $z_0^2 - 1 \neq 0$.

(i.e. $z_0 = e^{\frac{\pi}{4} i}, e^{\frac{3\pi}{4} i}, e^{\frac{5\pi}{4} i}, e^{\frac{7}{4}\pi i}$).

$f(z) = 0$.

6.

Let $t = \frac{x}{1+x}$, then $x = \frac{t}{1-t}$, $dx = d\frac{t}{1-t} = \frac{(1-t)+t}{(1-t)^2} dt$

$$I = \int_0^\infty \frac{x^\beta}{(x+1)^2} dx = \int_0^1 \frac{\frac{t^\beta}{(1-t)^\beta}}{\left(\frac{1}{1-t}\right)^2} dt = \int_0^1 t^\beta (1-t)^{2-\beta} dt = B(1+\beta, 1-\beta).$$

$$= \frac{\Gamma(1+\beta)\Gamma(1-\beta)}{\Gamma(2)} = \frac{\beta \Gamma(\beta)\Gamma(1-\beta)}{\Gamma(2)} = \beta \frac{\pi}{\sin \pi \beta} = \frac{\pi \beta}{\sin \pi \beta}$$

7.

$$f_U(u) = \begin{cases} 1 & u \in (0,1) \\ 0 & u \notin (0,1) \end{cases}$$

$$f_V(v) = \begin{cases} 1 & v \in (0,1) \\ 0 & v \notin (0,1) \end{cases}.$$

U, V independent.

$$f_{X,Y}(x,y) = \begin{cases} 1 & (u,v) \in (0,1) \times (0,1) \\ 0 & (u,v) \notin (0,1) \times (0,1) \end{cases}$$

$$\begin{cases} x = \sqrt{-2\ln u}\,\cos(2\pi v) \\ y = \sqrt{-2\ln u}\,\sin(2\pi v) \end{cases}$$

$$x^2 + y^2 = -2\ln u \quad \Rightarrow \quad u = e^{-\frac{x^2+y^2}{2}}$$

$$\frac{y}{x} = \tan(2\pi v). \quad \Rightarrow \quad v = \frac{1}{2\pi}\arctan\frac{y}{x}.$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -x e^{-\frac{x^2+y^2}{2}} & -\frac{1}{2\pi}\frac{y}{x^2+y^2} \\ -y e^{-\frac{x^2+y^2}{2}} & \frac{1}{2\pi}\frac{x}{x^2+y^2} \end{vmatrix} = -\frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} & \left(e^{-\frac{x^2+y^2}{2}},\ \frac{1}{2\pi}\arctan\frac{y}{x}\right) \in (0,1) \times (0,1) \\ 0 & \text{o.w.} \end{cases}$$

AMCS Written Preliminary Exam
Part II, May 2, 2019

1. Let M be an $n \times n$ matrix with real entries and eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Set
$$\rho(M) := \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$
Show that $\rho(M) < 1$ if and only if $\lim_{k\to\infty} M^k x = 0 \in \mathbb{R}^n$ for every $x \in \mathbb{R}^n$.

2. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, set
$$\|x\|_1 := \sum_{i=1}^n |x_i| \quad \text{and} \quad \|x\|_2 := \left(\sum_{i=1}^n x_i^2\right)^{1/2}.$$
Find the largest constant c_1 and smallest constant c_2 such that
$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1$$
for each $x \in \mathbb{R}^n$.

3. Prove that
   (a) $\sum_{n=2}^\infty \frac{1}{(\log n)^3}$ diverges.
   (b) $\sum_{n=2}^\infty \frac{1}{(\log n)^{\log n}}$ converges.

4. Let D be the open unit disk in $\mathbb{R}^2$ and suppose that $u \in C(\overline{D}) \cap C^2(D)$ satisfies
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0$$
in D for some real number λ and that
$$u|_{\partial D} = 0.$$
Show that $\lambda \ge 0$.

5. (a) What kind of singularity does $\frac{1}{(e^z - 1)}$ have at $z = 0$?
   (b) Find the residue at $z = 0$.

(c) Find the first three non-zero terms of the Laurent series expansion about $z = 0$.

(d) Find the largest number R such that the Laurent series converges on $0 < |z| < R$.

6. Show that all five roots of $z^5 + 15z + 1 = 0$ lie inside the circular disk $|z| < 2$, but that only one of the roots lies inside the disk $|z| < 3/2$.

7. Suppose that $\{X_n\}$ is a sequence of independent discrete random variables such that

$$P(X_n = k) = \frac{e^{-n}n^k}{k!} \quad \text{for all } k = 0, 1, 2, \ldots.$$

Prove that the expected value of following sum

$$\sum_{i=1}^{n} X_i$$

is $\frac{n(n+1)}{2}$.

Part II May 2 2019

1. $\rho(M) < 1 \iff |\lambda_t| < 1 \quad \forall\, t \in \{1, \cdots, n\}$.

$M = S \Lambda S^{-1}$, where $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$.

$M^k = S \Lambda^k S^{-1}$, where $\Lambda^k = \mathrm{diag}\{\lambda_1^k, \cdots, \lambda_n^k\}$

$\Rightarrow \lim\limits_{k\to\infty} \Lambda^k = 0$

$\lim\limits_{k\to\infty} S\Lambda^k S^{-1} = 0$.

$\lim\limits_{k\to\infty} M^k x = \lim\limits_{k\to\infty} S\Lambda^k S^{-1} x = 0 \quad \forall\, x \in \mathbb{R}^n$.

2.

Let $y_i = |x_i|$, then $y_i \geq 0$ and

$\|x\|_1 = \sum_{i=1}^n y_i$ and $\|x\|_2 = \left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}$

Let $c = \|x\|_1$. consider $f(y_1, \cdots, y_n) = \|x\|_2$

$F(y_1, \cdots, y_n) = f(y_1 \cdots, y_n) + \lambda(\|x\|_1 - c)$

$$\begin{cases} \dfrac{\partial F}{\partial y_k} = \dfrac{y_k}{\left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}} - \lambda = 0. \quad k \in \{1, \cdots, n) \\ \dfrac{\partial F}{\partial \lambda} = \sum_{i=1}^n y_i - c = 0. \end{cases}$$

$\Rightarrow y_k = \frac{c}{n}$

When $y_k = \frac{c}{n}$, $k = 1, \cdots, n$, $\|x\|_2$ achieves its min value $\frac{c}{\sqrt{n}}$

Suppose $y_1 \leq \cdots \leq y_n$, for y_i and y_j, $i<j$, let

$y_i' = 0$, $y_j' = y_i + y_j < c$.

then $f(y_1, \cdots, y_i', \cdots, y_j', \cdots, y_n)^2 = \sum_{k=1}^n y_k^2 - y_i^2 - y_j^2 + (y_i + y_j)^2$

$= \sum_{k=1}^n y_k^2 + 2y_iy_j \geq f(y_1, \cdots, y_n)$.

then when, $y_k = 0$, $k = 1, \cdots, (n-1)$, $y_n = c$, $\|x\|_2$ achieves its max value c.

i.e. $\frac{c}{\sqrt{n}} \leq \|x\|_2 \leq c \qquad c_1 = \frac{1}{\sqrt{n}}, \quad c_2 = 1$.

3. [Lemma].

If $\exists\, \alpha > 0$, s.t. $n \geq n_0$, $\frac{\log \frac{1}{a_n}}{\log n} \geq 1+\alpha$ $(a_n > 0)$, then $\sum_{n=1}^{\infty} a_n$ converges

If $\exists\, \alpha > 0$, s.t. $n \geq n_0$, $\frac{\log \frac{1}{a_n}}{\log n} \leq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

proof of lemma:

$$\frac{\log \frac{1}{a_n}}{\log n} \geq 1+\alpha \Rightarrow \frac{1}{a_n} \geq n^{1+\alpha} \Rightarrow a_n \leq \frac{1}{n^{1+\alpha}} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \text{ converges}$$

then $\sum_{n=1}^{\infty} a_n$ converges.

$$\frac{\log \frac{1}{a_n}}{\log n} \leq 1 \Rightarrow \frac{1}{a_n} \leq n \Rightarrow a_n \geq \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

then $\sum_{n=1}^{\infty} a_n$ diverges

(a)

$$a_n = \frac{1}{(\log n)^3}$$

$$\frac{\log \frac{1}{a_n}}{\log n} = \frac{\log (\log n)^3}{\log n.} \leq 1. \quad \text{diverges.}$$

consider $f(n) = (\log n)^3 - n$ $\quad \forall\, n \in \mathbb{N}^* \setminus \{1\}$.

$$f'(n) = \frac{3(\log n)^2}{n} - 1.$$

let $g(n) = 3(\log n)^2 - n$ $\quad \forall\, n \in \mathbb{N}^* \setminus \{1\}$.

$$g'(n) = \frac{6 \log n}{n} - 1 \leq 0$$

$\Rightarrow g(n) \leq g(2) < 0. \Rightarrow f'(n) < 0 \Rightarrow f(n) \leq f(2) < 0.$

3(b).

$$a_n = \frac{1}{(\log n)^{\log n}}$$

$$\frac{\log \frac{1}{a_n}}{\log n} = \frac{\log (\log n)^{\log n}}{\log n} = \frac{(\log n) \cdot \log (\log n)}{\log n} = \log (\log n) \geq 1+\alpha$$

for some $n \geq n_0$. $(\alpha > 0)$. converges.

4.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0 \quad \text{and} \quad u|_{\partial D} = 0.$$

seperation of variables: $u(r, \theta) = A(r) B(\theta)$.

where $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\nabla^2 u = -\lambda u$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\lambda u$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r A'(r) B(\theta)\right) + \frac{1}{r^2} A(r) B''(\theta) = -\lambda A(r) B(\theta).$$

$$\frac{r(r A'(r))'}{A(r)} + \frac{B''(\theta)}{B(\theta)} = -\lambda r^2$$

$$\frac{r(r A'(r))'}{A(r)} = n^2 > 0. \quad n \text{ can be arbitrary}.$$

$$r(r A'(r))' - n^2 A(r) = 0.$$

then $$\frac{B''(\theta)}{B(\theta)} = -n^2 - \lambda r^2$$

$$B''(\theta) + (n^2 + \lambda r^2) B(\theta) = 0$$

$$B(\theta) = e^{\pm i \sqrt{n^2 + \lambda r^2}\,\theta}$$

4.

Let $u(x,y) = A(x)\,B(y)$.

$$\nabla^2 u = A''(x)\,B(y) + A(x)\,B''(y) = -\lambda\, A(x)\,B(y).$$

$$\frac{A''(x)}{A(x)} + \frac{B''(y)}{B(y)} = -\lambda$$

$$\frac{A''(x)}{A(x)} = -\lambda_1 \quad \text{and} \quad \frac{B''(y)}{B(y)} = -\lambda_2 \qquad \lambda = \lambda_1 + \lambda_2.$$

$$\begin{cases} A''(x) - \lambda_1 A(x) = 0. \\ B''(y) - \lambda_2 B(y) = 0 \end{cases}$$

$$\begin{cases} A(x) = C_1 e^{\sqrt{\lambda_1}x} + C_2 e^{-\sqrt{\lambda_1}x} \\ B(y) = D_1 e^{\sqrt{\lambda_2}y} + D_2 e^{-\sqrt{\lambda_2}y} \end{cases}$$

$$u|_{\partial D} = A(\cos\theta)\,B(\sin\theta) = (C_1 e^{\sqrt{\lambda_1}\cos\theta} + C_2 e^{-\sqrt{\lambda_1}\cos\theta})(D_1 e^{\sqrt{\lambda_2}\sin\theta} + D_2 e^{-\sqrt{\lambda_2}\sin\theta})$$

$$= 0 \qquad A(\cos\theta) = 0 \quad \text{or} \quad B(\sin\theta) = 0.$$

If any of λ_1, λ_2 < 0, then $A(x)\,B(y) \equiv 0$.

$\Rightarrow$ $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ then $\lambda = \lambda_1 + \lambda_2 \geq 0$.

5.(a)

$$\lim_{z\to 0} \frac{z}{e^z-1} = 1 \qquad \text{pole of order } 1.$$

5.(b)

$$\operatorname{Res}(f, 0) = \frac{1}{(e^z-1)'\big|_{z=0}} = \frac{1}{e^z\big|_{z=0}} = 1.$$

5.(c)

$$f(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots$$

$$f(z) = \frac{1}{e^z-1} \Rightarrow (e^z-1) f(z) = 1$$

$$\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1\right)\left(\frac{a_{-1}}{z} + a_0 + a_1 z + \cdots\right) = 1.$$

$$\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(\frac{a_{-1}}{z} + a_0 + a_1 z + \cdots\right) = 1.$$

$$a_{-1} + \left(\frac{a_{-1}}{2!} + a_0\right) z + \left(\frac{a_0}{2!} + a_1 + \frac{a_{-1}}{3!}\right) z^2 + \cdots = 1.$$

$$\Rightarrow a_{-1} = 1.$$

$$a_0 = -\frac{1}{2}$$

$$a_1 = \frac{1}{12}$$

5.(d)

$$\frac{a_{-1}}{2!} + a_0 = 0$$

$$\frac{a_{-1}}{3!} + \frac{a_0}{2!} + a_1 = 0$$

$$\frac{a_{-1}}{4!} + \frac{a_0}{3!} + \frac{a_1}{2!} + a_2 = 0$$

$$\cdots$$

R can be ∞

6.

$z^5 + 15z + 1 = 0$

(i) number of zeroes inside $|z| < 2$:

$f(z) = z^5$, $g(z) = 15z + 1$.

$|g(z)| \leq 15|z| + 1 = 31 < 32 = |z|^5 = |f(z)|$.

$f(z)$ has 5 zeroes inside $|z| < 2$

$\Rightarrow f(z) + g(z) = z^5 + 15z + 1$ has 5 zeroes inside $|z| < 2$.

(ii) number of zeroes inside $|z| < \frac{3}{2}$.

$f(z) = 15z$, $g(z) = z^5 + 1$.

$|g(z)| \leq |z|^5 + 1 = (\frac{3}{2})^5 + 1 = \frac{243}{32} + 1 < \frac{45}{2} = 15|z| = |f(z)|$

$f(z)$ has 1 zero inside $|z| < \frac{3}{2}$

$\Rightarrow f(z) + g(z) = z^5 + 15z + 1$ has 1 zero inside $|z| < \frac{3}{2}$.

7.

$$E(X_i) = \sum_{k=1}^{+\infty} k\, P(X_i = k).$$

$$= \sum_{k=1}^{+\infty} k \frac{e^{-i}\, i^{k}}{k!}$$

$$= \sum_{k=1}^{+\infty} e^{-i} \frac{i^{k-1}}{(k-1)!}\, i$$

$$= i\, e^{-i} \sum_{k=1}^{+\infty} \frac{i^{k-1}}{(k-1)!}$$

$$= i\, e^{-i} e^{i}$$

$$= i.$$

$$E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$