AMCS Written Preliminary Exam
Part I, August 28, 2017

1. Prove that

$$\text{(1)} \quad \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$f(x) = \log x$ Taylor extension at $x = 1$.

2. Suppose that $f(z)$ is an analytic function in $D_R = \{z : |z| < R\}$ such that $|f(z)| < M$ for $z \in D_R$, and $f(0) = 1$. Find a number $0 < \rho < R$, such that $f(z) \neq 0$ for any z with $|z| \leq \rho$.

If not, $\forall \varepsilon > 0$, $\exists z_0$ s.t. $f(z_0) = 0$, $|z_0| < \varepsilon$ then $\lim_{z \to 0} f(z) = 0$ contradiction

3. There is an orthognal transformation O of $\mathbb{R}^3$ that transforms the quadratic form

$$\text{(2)} \quad q(x, y, z) = 2xy + 2xz + 2yz$$

$q(x, y, z) = x^T A x$

$A = S \Lambda S^{-1}$

to the quadratic form

$$\text{(3)} \quad Q(X, Y, Z) = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2,$$

here (X, Y, Z) are the transformed variables. What are the values of $\lambda_1, \lambda_2, \lambda_3$.

4. Let $\{a_1, \ldots, a_n\}$ and c be positive numbers with $c > n$. Use Lagrange multipliers to show that the minimum value of

$F(x) = \sum_{j=1}^{n} \frac{a_j}{x_j} - \lambda\left(\sum_{j=1}^{n} \frac{1}{1-x_j} - c\right)$

$$\text{(4)} \quad f(x) = \sum_{j=1}^{n} \frac{a_j}{x_j},$$

$\frac{\partial F}{\partial x_i} = -\frac{a_i}{x_i^2} + \frac{\lambda}{(1-x_i)^2} = 0$

$\Rightarrow x_i = \frac{\sqrt{a_i}}{\sqrt{a_i} + \sqrt{\lambda}}$

on the set

$$\text{(5)} \quad S = \left\{ 0 \leq x_j \leq 1 : \sum_{j=1}^{n} \frac{1}{1 - x_j} = c \right\},$$

$\lambda = \left(\frac{\sum \sqrt{a_j}}{c-n}\right)^2$

$f(x) = \sum a_j + \sqrt{\lambda} \sum \sqrt{a_j}$

is

$$\text{(6)} \quad \sum_{j=1}^{n} a_j + \frac{1}{c - n} \left(\sum_{j=1}^{n} a_j^{\frac{1}{2}} \right)^2.$$

5. Suppose that $g(z)$ is an entire function that never vanishes. What are all the possible values of the integrals

0.

on C, $z \neq 0$, $g(z) \neq 0$.

$$\text{(7)} \quad \int_C \frac{1}{z g(z)} dz,$$

$\rightarrow$ analytic

where C is any smooth curve that does not pass through 0 and goes from $z = 1$ to $z = z_0$.

large deviation theory

$\lim_{n \to \infty} \frac{1}{n} \log P(S_n \geq na) = -I(a)$

$I(x) = \begin{cases} \log 2 + x \log x + (1-x) \log(1-x) & 0 \leq x \leq 1 \\ \infty & \text{o.w.} \end{cases}$

$I(\frac{1}{2}) = 0$

6. In a set of 4000 independent fair coin flips, what is the probability of getting 3000 or more HEADS? Please answer to within a factor of 10. The following common logarithms are accurate to roughly one part in 4000: $\log 2 = 0.301$, $\log 3 = 0.477$.

De Moivre–Laplace limit theorem ~~bin(n, p)~~ $\rightarrow$ ~~$P(\lambda)$. $n = 4000$, $p = \frac{1}{2}$~~

$\lim_{n \to \infty} P\left(a \leq \frac{S_n - np}{\sqrt{npq}} \leq b\right) \rightarrow \Phi(b) - \Phi(a)$

~~$\lambda = np = 2000$ $f(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$~~

~~P(3000 or more HEADS) $= 1 - P$(less than 3000 HEADS) $= 1 - \sum_{k=0}^{2999} \frac{\lambda^k}{k!} e^{-\lambda}$~~

where X_i i.i.d. $S_n = \sum_{i=1}^{n} X_i$

$q = 1 - p$

7. Let A be a non-singular square matrix ($\det A \neq 0$). Show that there is a polynomial, $p(\lambda) = c_k\lambda^k + \cdots + c_1\lambda + c_0$ such that

$$A^{-1} = c_0 \,\mathrm{Id} + c_1 A + \cdots + c_k A^k. \qquad (8)$$

$\Leftrightarrow$ $Id = c_0 A + c_1 A^2 + \cdots + c_k A^{k+1}$.

char poly: $\det(A - \lambda I) = f(\lambda) = 0$ and $f(A) = 0$.

6.

$n = 4000$, $p = \frac{1}{2}$

$S_n = \sum_{i=1}^{4000} X_i$

$3000 \leq S_n \leq 4000$

$\Leftrightarrow 1000 \leq S_n - 2000 \leq 2000$

$\sqrt{1000} \leq \dfrac{S_n - 2000}{\sqrt{4000 \cdot \frac{1}{2} \cdot \frac{1}{2}}} \leq 2\sqrt{1000}$

$P(3000 \leq S_n \leq 4000) = P\left(\sqrt{1000} \leq \dfrac{S_n - 2000}{\sqrt{4000 \cdot \frac{1}{2} \cdot \frac{1}{2}}} \leq 2\sqrt{1000}\right)$

$= \Phi(2\sqrt{1000}) - \Phi(\sqrt{1000})$.

AMCS Written Preliminary Exam
Part II, August 28, 2017

1. Find a 1-to-1 conformal map from the sector $S_{\frac{\pi}{4}} = \{z : 0 < \arg z < \frac{\pi}{4}\}$ onto the interior of the unit disk $D_1 = \{z : |z| < 1\}$.

2. Let $< f_n >$ be a sequence of functions analytic in the unit disk, D_1, and let f be a continuous function also defined in D_1. Show that if

$$\lim_{n\to\infty} \iint_{D_1} |f_n(x,y) - f(x,y)|\,dx\,dy = 0, \tag{1}$$

then the function f is also analytic in D_1.

3. Suppose that A is an $n \times n$ matrix that commutes with all $n \times n$ diagonal matrices. What can we say about A; you must prove your answer.

Λ diag
$(\Lambda A)_{ij} = \lambda_i a_{ij} = (A\Lambda)_{ij} = \lambda_j a_{ij}$

4. (a) What is a Hermitian inner product? $\langle x, y\rangle = x^H y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$
   (b) What is unitary matrix? $UU^H = U^H U = I$
   (c) What is a Hermitian matrix? $A = \bar{A}^T$
   (d) Suppose that A is an Hermitian matrix. Prove that $(A - i\,\mathrm{Id})$ is invertible and the matrix $Z = (A + i\,\mathrm{Id})(A - i\,\mathrm{Id})^{-1}$ is a unitary matrix.

is a part
so invertible

$I^H I = ((A - iI)^{-1})^H (A + iI)^H$
$(A + iI)(A - iI)^{-1} = I$

5. Show that there is no non-constant polynomial $P(u, v)$ in two variables such that

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$$P(x, \cos x) = 0 \tag{2}$$

holds for all $x \in \mathbb{R}$.

6. Consider the equation $ye^y = x$. Show that this uniquely defines a function $y(x)$ on the interval $[0, \infty)$.
   (a) Sketch the graph of $y(x)$.
   (b) Find a formula for $\frac{dy}{dx}$ as a function of y.
   (c) Give a method to calculate the value of y such that $ye^y = 1$, to arbitrary accuracy.

(a) $F(x, y) = ye^y - x = 0$
$F_x' = -1, \quad F_y' = (y+1)e^y$

(b) $\frac{dy}{dx} = -\frac{F_x'}{F_y'} = \frac{1}{(y+1)e^y}$
$y(0) = 0$

(c)

$P(x, \cos x) = 0$ suppose p is the one with lowest degree
let $x = \frac{\pi}{2} + 2k\pi$, then $\cos(\frac{\pi}{2} + 2k\pi) = 0$
$P(x, \cos x) = P(\frac{\pi}{2} + 2k\pi, 0) = 0 \quad \forall k.$
then $P(x, \cos x)$ has infinite roots
$\Rightarrow P(x, 0) = 0 \quad \forall x$
$P(x, y) = Q(x) + y R(x, y).$
$P(x, 0) = Q(x) = 0 \quad \forall x$
then $P(x, y) = y R(x, y)$
let $y = \cos x$, $P(x, y) = \cos x \, Q(x, y)$
$\Rightarrow Q(x, y) = 0.$
but $\deg(Q) < \deg(P)$ contradiction!

? $U_{j-1} < U_j > U_{j+1} < U_{j+2} > U_{j+3}$

7. Fix a large number N and for $1 \le j \le N$ let $\{U_j\}$ be IID random variables uniform on the interval $[0, 1]$. Let X be the number of these that are local maxima, that is, the number of j such that $U_{j-1} < U_j > U_{j+1}$. The indices are considered modulo N, e.g., a maximum occurs at 1 if $U_N < U_1 > U_2$.
   (a) Compute the expected value $\mu := E(X)$.
   (b) Compute the second moment $M := E(X^2)$.
   (c) Compute the variance $V := \mathrm{Var}(X)$.
   (d) For what real number q can you prove a nontrivial upper bound on

$$\mathrm{Prob}(|X - \mu| > cN^q)$$

that does not depend on N? It will of course depend on c, and nontrivial means it has to be less than 1 for at least a range of values of c. You are not asked to get the best bound, just any bound uniform in N.

(a)

$Y_j = \begin{cases} 1, & \text{local max} \\ 0, & \text{o.w.} \end{cases} \Rightarrow X = \Sigma Y_j$ $N(N-1)$

$E(X) = E(\Sigma Y_j) = \Sigma E(Y_j) = \frac{N}{3}$

(b)

$E(X^2) = E(\Sigma Y_j)^2 = \Sigma E(Y_j^2) + 2 \sum_{i<j} E(Y_i Y_j)$

when $j = i+1$: $E(Y_i Y_j) = 0$. $2N$

when $j = i+2$: $E(Y_i Y_j) = \frac{2 \cdot 3! + 4}{5!} = \frac{2}{15}$ $2N$

when $j \geq i+3$: $E(Y_i Y_j) = E(Y_i) E(Y_j) = \frac{1}{9}$ $N(N-5)$.

$= \frac{N}{3} + \frac{4}{15} N + \frac{1}{9} N(N-5) = \frac{1}{9} N^2 + \left(\frac{1}{3} + \frac{4}{15} - \frac{5}{9}\right) N = \frac{1}{9} N^2 + \frac{1}{45} N$

(c)

$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{45} N$

(d)

$\mathrm{Prob}(|X - \mu| > cN^q) \leq \frac{\mathrm{Var}(X)}{(cN^q)^2} = \frac{\frac{1}{10} N}{c^2 N^{2q}}$

$q = \frac{1}{2}$

$Y_1 \quad Y_2 \quad Y_3$

2 1 3

5 4

4 5

3 1 2

Part II Aug 2017

1.

$f_1: S_{\#} \to S_1 = \{z: 0 < \arg z < \frac{\pi}{2}\}$

$z \mapsto z^2$

$f_2: S_1 \to S_2 = \{z: 0 < \arg(z-\frac{1}{2}) < \frac{\pi}{2}\}$

$z \mapsto z - \frac{1}{2}$

$f_3: S_2 \to S_3 = \{z: |z| < 1, \ \mathrm{Im}\, z < 0\}$

$f_3^{-1}(z) = \frac{1}{1+z} \Rightarrow f_3(z) = \frac{1-z}{z}$

$f_4: S_3 \to D_1$

$z \to z^2$

then $f: S_{\#} \to D_1$

$f = f_4 \circ f_3 \circ f_2 \circ f_1$

$f(z) = f_4 \circ f_3 \circ f_2 (z^2) = f_4 \circ f_3 (z^2 - 1) = f_4\left(\frac{1-(z^2-1)}{z^2-1}\right) = \left(\frac{2-z^2}{z^2-1}\right)^2$

2.

Let $z = x+iy$, $g_n(z) = f_n(x,y)$ $\qquad g(z) = f(x,y)$

$g_n(z) = u_n(x,y) + i v_n(x,y)$. $\qquad g(z) = u(x,y) + iv(x,y)$.

$D_1 = \{z: |z| < 1\}$.

$\lim_{n\to\infty} \iint_{D_1} |f_n(x,y) - f(x,y)|\, dx dy = 0$

$= \lim_{n\to\infty} \iint_{D_1} |g_n(z) - g(z)|\, dz$

$= \lim_{n\to\infty} \iint_{D_1} \sqrt{|u_n(x,y) - u(x,y)|^2 + |v_n(x,y) - v(x,y)|^2}\, dx\, dy$

$\geq \lim_{n\to\infty} \iint_{D_1} \frac{|u_n(x,y) - u(x,y)| + |v_n(x,y) - v(x,y)|}{2}\, dx\, dy$

$= \lim_{n\to\infty} \iint_{D_1} \frac{|u_n(x,y) - u(x,y)|}{2}\, dx\, dy + \lim_{n\to\infty} \iint_{D_1} \frac{|v_n(x,y) - v(x,y)|}{2}\, dx\, dy$

$\Rightarrow$ $u_n(x,y) \to u(x,y)$

$v_n(x,y) \to v(x,y)$.

$\Rightarrow$ analytic.