ANALYTIC FUNCTIONS.

$f(z) = u(x,y) + i\,v(x,y)$ analytic on Ω

$$\Leftrightarrow \begin{cases} ① & u(x,y) \text{ and } v(x,y) \text{ differentiable on } \Omega \\ ② & \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \text{ and } \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\Leftrightarrow \frac{\partial f(z)}{\partial \bar{z}} = 0$$

power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$

$L = \overline{\lim_{k\to\infty}} |a_k|^{\frac{1}{k}}$ convergence $R = \frac{1}{L}$.

M – Test

$|a_k (z - z_0)^k| \leq M_k$, $\sum_{k=0}^{\infty} M_k$ converges $\Rightarrow \sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges.

Taylor expansion. $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$

where $a_k = \dfrac{f^{(k)}(z_0)}{k!}$

CAUCHY THEOREM & INTEGRAL FORMULA

Ω smooth, f continuous on $\bar{\Omega}$, f analytic on Ω.

$$\boxed{\int_{\partial\Omega} f\, dz = 0}$$

$f(z_0) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial\Omega} \dfrac{f(z)}{z - z_0}\, dz$ $\qquad \forall z_0 \in \Omega$.

$|f^{(n)}(z_0)| \leq \dfrac{n!\, M}{r^n}$ $\qquad |f(z)| \leq M$ on Ω.

$0 < r \leq \text{dist}(z_0, \partial\Omega)$.

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Liouville's Theorem A bounded entire function is constant.

Fundamental Theorem of Algebra Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Maximum - Modulus Theorem The maximum is always assumed on the boundary of the domain.

Minimum Modulus Theorem If is a non-constant analytic function in a region D, then no point $z \in D$ can be a relative minimum of f unless $f(z) = 0$.

POWER SERIES

Singularity

- removable singularity

  $\exists\, g$ analytic at z_0 and $f = g \ \forall z \in D \setminus \{z_0\}$.

  $\lim_{z \to z_0} (z - z_0) f(z) = 0$

- pole of order k

  for $z \neq z_0$, $f(z) = \frac{A(z)}{B(z)}$ $A(z)$, $B(z)$ analytic at z_0.

  $A(z_0) \neq 0$ and $B(z_0) = 0$.

  $\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0$ but $\lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0$

- essential singularity

  neither removable nor pole

Laurent expansion

$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k$ $R_1 < |z - z_0| < R_2$ analytic.

$$R_1 = \frac{1}{\lim_{k \to \infty} \sup |a_k|^{\frac{1}{k}}}$$

$$R_2 = \lim_{k \to \infty} \sup |a_{-k}|^{\frac{1}{k}}$$

$$a_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{k+1}} dz$$

RESIDUE

$\mathrm{Res}\,(f; z_0) = C_{-1}$ $\quad f(z) = \sum_{-\infty}^{+\infty} C_k (z - z_0)^k$

- f has a simple pole at z_0 $\quad f(z) = \dfrac{A(z)}{B(z)}$

$$C_{-1} = \lim_{z \to z_0} (z - z_0) f(z) = \frac{A(z_0)}{B'(z_0)}$$

- f has a pole of order k at z_0

$$C_{-1} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right]$$

- determine the residue directly from the Laurent expansion

Cauchy's Residue Theorem

$$\int_{\partial\Omega} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(f; z_k)$$

f analytic in $\Omega \setminus \{z_k\}$.

Argument Principle

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} = \mathbb{Z} - \mathbb{P} \to \text{numbers of poles inside}$$

$\mathbb{Z}$: numbers of zeroes inside

Rouché's Theorem

f and g analytic inside closed γ and

$|f(z)| > |g(z)| \quad \forall\, z \in \gamma$

Then $\quad \mathbb{Z}(f + g) = \mathbb{Z}(f)$

CONFORMAL MAPPING.

f analytic at z_0 and $f'(z_0) \neq 0$

$\Rightarrow$ f conformal and locally 1-1 at z_0

f is k-to-1 $\Leftrightarrow$ $f^{(k)}(z_0) \neq 0$ k is the least positive integer

$\Uparrow$

$f(z) = \alpha$ has k roots $\forall \alpha$.

Automorphism of the unit disc

$$g(z) = e^{i\theta}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) \qquad |\alpha| < 1.$$

Conformal mappings of upper half-plane $\rightarrow$ unit disc

$$h(z) = e^{i\theta}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) \qquad \operatorname{Im} \alpha > 0$$

Automorphism of the upper half-plane

$$h(z) = \frac{az+b}{cz+d}$$

Cross ratio

$$(z_1, z_2, z_3, z_4) = \left(\frac{z_4-z_2}{z_4-z_1}\right)\left(\frac{z_3-z_1}{z_3-z_2}\right)$$

Unique bilinear mapping sends z_1, z_2, z_3 into $\infty, 0, 1$

$$T(z) = \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)} = (z_1, z_2, z_3, z_4)$$

Unique bilinear transformation $w = f(z)$ maps z_1, z_2, z_3 into w_1, w_2, w_3

$$\frac{(w-w_2)(w_3-w_1)}{(w-w_1)(w_3-w_2)} = \frac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$$

$$(w_1, w_2, w_3, w) = (z_1, z_2, z_3, z).$$

$\int_{-\infty}^{+\infty} f(x)dx$

$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(f, z_k)$ z_k are singularities of f on $\{z : \mathrm{Im}\, z \geq 0\}$

$\lim_{z\to\infty} z f(z) = 0$

$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(\frac{P(z)}{Q(z)}, z_k)$ $\deg Q - \deg P \geq 2.$

$\int_{-\infty}^{+\infty} e^{i\alpha x} f(x)dx$

$\int_{-\infty}^{+\infty} e^{i\alpha x} f(x)\,dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(e^{i\alpha z} f(z), z_k).$

$\int_{-\infty}^{+\infty} f(x) \cos \alpha x\,dx = \mathrm{Re}[\,2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(e^{i\alpha z} f(z), z_k)]$

$\int_{-\infty}^{+\infty} f(x) \sin \alpha x\,dx = \mathrm{Im}[\,2\pi i \sum_{k=1}^{n} \mathrm{Res}\,(e^{i\alpha z} f(z), z_k)]$

$\int_0^{\infty} \frac{P(x)}{Q(x)}\,dx$

$\int_0^{\infty} \frac{P(x)}{Q(x)}\,dx = -\sum \mathrm{Res}\,(\frac{P(z)}{Q(z)} \log z, z_k)$

$\int_a^{\infty} \frac{P(x)}{Q(x)}\,dx = \int_0^{\infty} \frac{P(x)}{Q(x)}\,dx - \int_0^{a} \frac{P(x)}{Q(x)}\,dx$

$\int_0^{\infty} \frac{x^{\alpha-1}}{P(x)}\,dx = ?$ $0 < \alpha < 1.$

$[1 - e^{2\pi i(\alpha-1)}] \int_0^{\infty} \frac{x^{\alpha-1}}{P(x)}\,dx = 2\pi i \sum \mathrm{Res}\,(\frac{z^{\alpha-1}}{P(z)}, z_k)$

$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$ $z = e^{i\theta}$

$\parallel$

$\int_{|z|=1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right) \frac{dz}{iz}$

$\int_0^{2\pi} R(\sin n\theta, \cos n\theta)\,d\theta = \int_{|z|=1} R\left(\frac{1}{2}(z^n + \frac{1}{z^n}), \frac{1}{2i}(z^n - \frac{1}{z^n})\right)$