Orthogonality

orthogonal vectors

orthogonal subspaces.

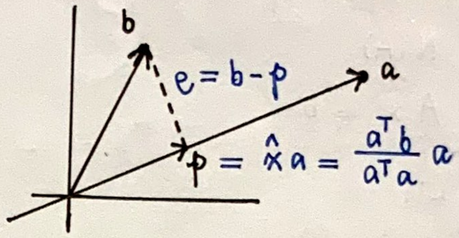
nullspace $\perp$ row space

$V^{\perp} = V$ perp

$\uparrow$

orthogonal complement of V.

projection onto a line. $p = \hat{x} a$



Least Square Problem. $\|Ax - b\| = E$. $\hat{x}$: least-square solution

$e = b - A\hat{x}$ normal equations $A^T A \hat{x} = A^T b$.

best estimation $\hat{x} = (A^T A)^{-1} A^T b$

projection $p = A\hat{x} = A(A^T A)^{-1} A^T b$

projection matrix $P = A(A^T A)^{-1} A^T$ $\begin{cases} P^2 = P \\ P^T = P \end{cases}$

Gram-Schmidt

$V_1, \cdots, V_n$.

$\beta_1 = V_1$

$\vdots$

$\beta_k = V_k - \sum_{i=1}^{k-1} \frac{(V_i, \beta_k)}{(V_i, V_i)} V_i$

$\vdots$

β_n.

EIGENVALUES & EIGENVECTOR

$Ax = \lambda x$ eigenvalues : $\lambda_1, \cdots, \lambda_n$

eigenvectors : $x_1, \cdots, x_n$.

$\boxed{\frac{du}{dt} = Au}$ solution : $u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$

matrix. initial condition : $c_1 x_1 + \cdots + c_n x_n = 0$.

diagonalization

$$S^{-1} A S = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad S = [x_1 \cdots x_n].$$

$S^{-1} A^k S = \Lambda^k$.

applications Fibonacci numbers $F_{k+2} = F_{k+1} + F_k$.

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Au_k$$

$$u_{k+1} = Au_k \Rightarrow u_k = A^k u_0$$

$A = S\Lambda S^{-1}$, then $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0$, write $S^{-1}u_0 = c$

$$u_k = S\Lambda^k c = [x_1 \cdots x_n] \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \lambda_1^k x_1 + \cdots + c_1 \lambda_n^k x_n.$$

$u_0 = c_1 x_1 + \cdots + c_n x_n$

Markov Matrices $Au_\infty = u_\infty$

$u_k = c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$

$$\begin{cases} \text{stable : all } |\lambda_i| < 1. \\ \text{neutrally stable : some } |\lambda_i| = 1, \text{ other } |\lambda_i| < 1. \\ \text{unstable : at least one } |\lambda_i| > 1. \end{cases}$$

$A^T = -A \Rightarrow e^{At}$ orthogonal matrix.

$\frac{d^2u}{dt^2} = Au$.

$u(t) = (c_1 e^{i\omega_1 t} + d_1 e^{-\omega_1 t}) x_1 + \cdots + (c_n e^{i\omega_n t} + d_n e^{-\omega_n t}) x_n$

$= (a_1 \cos\omega_1 t + b_1 \sin\omega_1 t) x_1 + \cdots + (a_n \cos\omega_n t + b_n \sin\omega_n t) x_n$

$u(0) = a_1 x_1 + \cdots + a_n x_n$.

$\omega_k = \sqrt{-\lambda_k}$

Differential Equations and e^{At}

differential equation: $\frac{du}{dt} = Au$

solution: $u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$

initial condition: $u(0) = c_1 x_1 + \cdots + c_n x_n = Sc.$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0).$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots = I + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1} t^2}{2!} + \frac{S\Lambda^3 S^{-1} t^3}{3!} + \cdots$$

$$= S\left(I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \frac{(\Lambda t)^3}{3!} + \cdots\right) S^{-1} = S e^{\Lambda t} S^{-1}$$

- stable : $e^{At} \to 0$ all $\mathrm{Re}\,\lambda_i < 0$.
- neutrally stable : all $\mathrm{Re}\,\lambda_i \leqslant 0$ and $\mathrm{Re}\,\lambda_1 = 0$
- unstable : e^{At} is bounded, some $\mathrm{Re}\,\lambda_i > 0$.

Complex Matrices

Every symmetric matrix (and Hermitian matrix) has real eigenvalues

$A^H = \bar{A}^T$ $\qquad (AB)^H = B^H A^H$

$(a_{ji} = \overline{a_{ij}})$

A real symmetric matrix can be factored into $A = Q\Lambda Q^T$ spectral theorem.

Hermitian matrix $\quad A = A^H = \bar{A}^T$

properties: ① $A = A^H \Rightarrow x^H A x \in \mathbb{R} \;\; \forall x \in \mathbb{C}^n$. pf: $(x^H A x)^H = x^H A x$

② $A = A^H \Rightarrow$ eigenvalues $\in \mathbb{R}$. pf: $x^H A x = \lambda x^H x \Rightarrow \lambda = \frac{x^H A x}{x^H x}.$

③ eigenvectors of a real symmetric matrix / a Hermitian matrix from different eigenvalues, are orthogonal.

Unitary matrix $\quad U^H U = I$, $UU^H = I$, and $U^H = U^{-1}$

properties: ① $(Ux)^T (Uy) = x^T U^T U y = x^T y$ and lengths are preserved by U

② Every eigenvalue of U has absolute value $|\lambda| = 1$.

③ Eigenvectors corresponding to different eigenvalues are orthogonal.

Similarity Transformation $B = M^{-1} A M$.

Spectral Theorem real $Q^{-1} A Q = \Lambda$

complex $U^{-1} A U = \Lambda$

Normal $NN^H = N^H N$

$T = U^{-1} N U$ is diagonal.

Jordan form

$$J = M^{-1} A M = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$

Jordan block $$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

POSITIVE DEFINITE MATRICES

positive definite (i) $x^T k x > 0 \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}$

(ii) $\lambda_i > 0 \quad \forall i$

(iii) upper left submatrices A_k have positive det $\forall k$.

(iv) pivots $d_k > 0 \quad \forall k$

The symmetric matrix A is positive definite $\Leftrightarrow$ $A = R^T R$, $\det(R) \neq 0$.

positive semidefinite (i') (ii') (iii') (iv') (v) $A = R^T R$

Singular Value Decomposition SVD.

$A = U \Sigma V^T$ → orthogonal.

U ↓ orthogonal, Σ → diagonal

properties ① positive definite matrix : $U \Sigma V^T = Q \Lambda Q^T$

symmetric matrix : negative eigenvalues in Λ → positive in Σ

complex matrix : U, V unitary

② first r columns of U : column space of A

last $m-r$ colums of U : left nullspace of A

first r columns of V : row space of A

last $n-r$ columns of V : null space of A.

③ $A = U \Sigma V^T$

$AV = U\Sigma$

④ eigenvectors of AA^T and A^TA go into columns of U and V.

$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = (U\Sigma V^T)(V\Sigma^T U^T) = U\Sigma\Sigma^T U^T$

$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = (V\Sigma^T U^T)(U\Sigma V^T) = V\Sigma^T\Sigma V^T$

QR decomposition

$A = QR$ where $R = \begin{bmatrix} \langle e_1, a_1\rangle & \langle e_1, a_2\rangle & \langle e_1, a_3\rangle \cdots \\ 0 & \langle e_2, a_2\rangle & \langle e_2, a_3\rangle \cdots \\ 0 & 0 & \langle e_3, a_3\rangle \cdots \end{bmatrix}$

$Q = [e_1 \cdots e_n]$ $\quad a_k = \sum_{j=1}^{k} \langle e_j, a_k\rangle e_j$

Rayleigh quotient

$R(x) = \frac{x^TAx}{x^Tx}$ ← minimize λ_1 / maximize λ_n

norm

$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$

condition number

$c = \|A\| \|A^{-1}\|$