AMCS Written Preliminary Exam
Part I, August 27, 2018

1. A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}$$

   (a) Apply the Gram-Schmidt process to find two orthonormal vectors u_1 and u_2 which also span V.

   (b) Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.

   (c) Find the best possible (i.e., least squared error) solution to the linear system

$$Q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

2. Find the Singular Value Decomposition (SVD) of the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

   (a) Find the eigenvalues and unit length eigenvectors for A^TA and AA^T. (What is the rank of A?)

   (b) Calculate the three matricies U, D, and V^T in the SVD and observe that $A = UDV^T$. Please explain clearly how you obtain these matrices.

3. (a) For any two positive real numbers a and b, prove that

$$1 + \ln a - \ln b \leq \frac{a}{b}.$$

(b) Assume that $\{x_n\}$ is a sequence of nonzero complex numbers satisfying that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\left|\frac{x_{n+1}}{x_n}\right| = r < 1.$$

Show that the series $\sum_{n=1}^{\infty} x_n$ must be absolutely convergent.

4. Suppose $\{a_n\}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty}|a_n| < \infty$. Show that

$$\lim_{r\to 1^-}\sum_{n=1}^{\infty} a_n r^n = \sum_{n=1}^{\infty} a_n.$$

5. Find all analytic functions on the disk $\{z \in \mathbb{C} : |z| < 1\}$ which satisfy the functional equation

$$f(iz) = (f(z))^2.$$

6. Calculate the following integral

$$\int_{|z|=\rho} \frac{z+1}{z^2-2z}\,dz,$$

both when $0 < \rho < 1$ and when $1 < \rho < \infty$.

7. Suppose $\{x_n\}$ is a sequence of real numbers in $[0,1]$ which is randomly constructed according to the rule that given the values of x_{n-1}, x_n is uniformly distributed on the interval $[0, x_{n-1}]$. Compute the expected sum of the infinite series

$$\sum_{n=1}^{\infty} x_n.$$

You may assume that this sum is finite with probability 1.

Part I Aug 2018

1. (a)

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

let $\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

let $u_1 = \frac{\beta_1}{\|\beta_1\|} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

$$u_2 = \frac{\beta_2}{\|\beta_2\|} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

1. (b)

$Q = [u_1, u_2]$

Prove that $\exists\, k_1, k_2 \in \mathbb{R}$, s.t. $QQ^T X = k_1 u_1 + k_2 u_2 \quad \forall X \in \mathbb{R}^3$

let $Q = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \\ u_{13} & u_{23} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$QQ^T X = \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \\ u_{13} & u_{23} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \\ u_{13} & u_{23} \end{bmatrix} \begin{bmatrix} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 \\ u_{21}x_1 + u_{22}x_2 + u_{23}x_3 \end{bmatrix} = \begin{bmatrix} k_1 u_{11} + k_2 u_{21} \\ k_1 u_{12} + k_2 u_{22} \\ k_1 u_{13} + k_3 u_{23} \end{bmatrix} = k_1 u_1 + k_2 u_2$$

where $k_1 = u_{11}x_1 + u_{12}x_2 + u_{13}x_3$

$k_2 = u_{21}x_1 + u_{22}x_2 + u_{23}x_3$.

$$Q^T Q = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} [u_1, u_2] = \begin{bmatrix} u_1^T u_1 & u_1^T u_2 \\ u_2^T u_1 & u_2^T u_2 \end{bmatrix} = I \implies Q \text{ orthogonal.}$$

1. (c) $b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$$Q \begin{pmatrix} x \\ y \end{pmatrix} = b$$

normal equation: $Q^T Q \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = Q^T b$

since $Q^T Q = I$

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = Q^T b = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{3}} \\ \frac{5}{\sqrt{6}} \end{pmatrix}$$

2. (a)

rank $(A) = 2$

$$B = A^T A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$Bx = \lambda x$.

$$\det(B - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = (1-\lambda)(3-\lambda) = 0.$$

$\lambda_1 = 1$:

$$(B - \lambda_1 I) x_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad x_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\lambda_2 = 3$:

$$(B - \lambda_2 I) x_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$C = AA^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$Cx = \lambda x$

$$\det(C - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda) - (1-\lambda) - (1-\lambda)$$

$$= (1-\lambda)\big((1-\lambda)(2-\lambda) - 2\big) = (1-\lambda)(-3\lambda + \lambda^2) = (1-\lambda)\lambda(\lambda - 3) = 0.$$

$\lambda_1 = 0.$

$$(C - \lambda_1 I) X_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad X_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$\lambda_2 = 1.$

$$(C - \lambda_2 I) X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} X_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$\lambda_3 = 3.$

$$(C - \lambda_3 I) X_3 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

2. (b).

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

singular value : $1, \sqrt{3}$

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$$

3. (a)

$f(x) = x - \ln x - 1$

$f'(x) = 1 - \frac{1}{x}$.

$f'(1) = 0$.

$f(x) \geq f(1) = 0$.

let $x = \frac{a}{b}$.

$\frac{a}{b} - \ln\frac{a}{b} - 1 \geq 0$.

$\frac{a}{b} \geq \ln a - \ln b + 1$.

3. (b).

Let $y_n = \frac{x_{n+1}}{x_n}$ where $x_0 = 1$.

then. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |y_n| = r < 1$.

also $|x_n| = |y_{n-1}| \cdot \cdots \cdot |y_1| \cdot |x_1| = a_n |x_1|$. where $a_1 = 1$.

$\sum_{n=1}^{\infty} |x_n| = |x_1| \sum_{n=1}^{\infty} a_n$

$a_n = |y_{n-1}| \cdot \cdots \cdot |y_1| \leq \left(\frac{\sum_{k=1}^{n-1} |y_k|}{n-1}\right)^{n-1} = b_{n-1}$ $(b_0 = 1)$.

$\lim_{n\to\infty} \sqrt[n]{b_n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |y_k| = r < 1$.

$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ converges.

$\sum_{n=1}^{\infty} |x_n| = |x_1| \sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \sum_{n=1}^{\infty} x_n$ converges absolutely.

4.

Let $f_n(r) = \sum_{k=1}^{n} a_k r^k \quad 0<r<1, \quad r \in \mathbb{R}.$

then $f_n(r) = \sum_{k=1}^{n} a_k r^k < \sum_{k=1}^{n} |a_k| r^k < \sum_{k=1}^{\infty} |a_k| r^k < \sum_{k=1}^{\infty} |a_k| < \infty$

since $\sum_{k=1}^{n} |a_k|$ is increasing and $\sum_{k=1}^{\infty} |a_k| r^k < \infty$, $\sum_{k=1}^{\infty} |a_k| r^k$ converges.

$\Leftrightarrow \sum_{k=1}^{\infty} a_k r^k$ converges absolutely.

Then $\sum_{k=1}^{\infty} a_k r^k$ converges. Also $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ converges

$\forall \varepsilon > 0$:

$\exists N_1$, s.t. $\left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{\infty} a_n \right| < \frac{\varepsilon}{3} \quad \forall N \geqslant N_1$, since $\sum_{n=1}^{\infty} a_n$ converges.

$\exists N_2$ s.t. $\left| \sum_{n=1}^{N} a_n r^n - \sum_{n=1}^{\infty} a_n r^n \right| < \frac{\varepsilon}{3} \quad \forall N \geqslant N_2$, since $\sum_{n=1}^{\infty} a_n r^n$ converges

$0<r<1$.

since polynomials are continuous,

$\exists \delta > 0$, s.t. $\left| \sum_{n=1}^{N} a_n r^n - \sum_{n=1}^{N} a_n \cdot 1^n \right| < \frac{\varepsilon}{3} \quad \forall 1-\delta < r < 1.$

Then $\left| \sum_{n=1}^{\infty} a_n r^n - \sum_{n=1}^{\infty} a_n \right| \leqslant \left| \sum_{n=1}^{\infty} a_n r^n - \sum_{n=1}^{N} a_n r^n \right|$

$+ \left| \sum_{n=1}^{N} a_n r^n - \sum_{n=1}^{N} a_n \right| + \left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{\infty} a_n \right| = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

when $N \geqslant \max\{N_1, N_2\}$, $1-\delta < r < 1$.

5.

Let $z = x+iy$ and $f(z) = f(x,y) = u(x,y) + i\,v(x,y)$.

Since f analytic on $\{z \in \mathbb{C} : |z|<1\}$, $\dfrac{\partial u(x,y)}{\partial x} = \dfrac{\partial v(x,y)}{\partial y}$, $\dfrac{\partial u(x,y)}{\partial y} = -\dfrac{\partial v(x,y)}{\partial x}$

Then $f(iz) = f(-y+ix) = u(-y,x) + i\,v(-y,x)$

let $f(iz) = u(x_1,y_1) + i\,v(x_1,y_1)$, where $x_1 = -y$, $y_1 = x$

$$\frac{\partial u(x_1,y_1)}{\partial x_1} = \frac{\partial v(x_1,y_1)}{\partial y_1}$$

$$\frac{\partial u(x_1,y_1)}{\partial y_1} = -\frac{\partial v(x_1,y_1)}{\partial x_1}$$

let $f(iz) = u_1(x,y) + i\,V_1(x,y)$, where $u_1(x,y) = u(-y,x)$, $V_1(x,y) = v(-y,x)$

$$\frac{\partial u_1(x,y)}{\partial x} = \frac{\partial u(-y,x)}{\partial x} = \frac{\partial v_1(x,y)}{\partial y} = \frac{\partial v(-y,x)}{\partial y}$$

$$\frac{\partial u_1(x,y)}{\partial y} = \frac{\partial u(-y,x)}{\partial y} = -\frac{\partial v_1(x,y)}{\partial x} = \frac{\partial v(-y,x)}{\partial x}$$

$$\frac{\partial v_1(x,y)}{\partial x} = \frac{\partial v(-y,x)}{\partial y} \quad \text{and} \quad \frac{\partial v_1(x,y)}{\partial y} = -\frac{\partial v(-y,x)}{\partial x}$$

$$f(z)^2 = (u^2(x,y) - v^2(x,y)) + 2i\,u(x,y)\,v(x,y)$$
$$= u_2(x,y) + v_2(x,y).$$

$$\frac{\partial u_2(x,y)}{\partial x} = 2u(x,y)\frac{\partial u(x,y)}{\partial x} - 2v(x,y)\frac{\partial v(x,y)}{\partial x}$$

$f(iz) = (f(z))^2 \Rightarrow f(z) = (f(z))^8$

$f(z)\,(f(z)^7 - 1) = 0$

$f(z) = 0$ or $e^{\frac{2\pi k i}{7}}$ $\quad k = 0, \cdots, 14.$

6.

$0 < \rho < 1$:

Let $f(z) = \frac{z+1}{z-2}$, $f(0) = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{f(z)}{z} dz$ $f(0) = -\frac{1}{2}$

$I = 2\pi i f(0) = -\pi i$

$1 < \rho < 2$: the same

$\rho = 2$: does not exist.

$\rho > 2$: $g = \frac{z+1}{z^2-2z}$

$I = 2\pi i \left(\operatorname{Res}(g, 0) + \operatorname{Res}(g, 2) \right).$

$\operatorname{Res}(g, 0) = \frac{z+1}{2z-2}\Big|_{z=0} = -\frac{1}{2}$ and $\operatorname{Res}(g, 2) = \frac{z+1}{2z-2}\Big|_{z=2} = \frac{3}{2}.$

$\Rightarrow I = 2\pi i$

7.

$p_{X_n}(x_n) = \begin{cases} \frac{1}{x_{n-1}} , & 0 < x_n < x_{n-1} \\ 0 , & \text{o. w.} \end{cases}$

$E(X_n | X_{n-1}) = \int_0^{x_{n-1}} \frac{x_n}{x_{n-1}} dx_n = \frac{x_n^2}{2x_{n-1}}\Big|_0^{x_{n-1}} = \frac{x_{n-1}}{2}.$

$E(X_n) = E(X_n | X_{n-1}) E(X_{n-1}) = \frac{1}{2} E(X_{n-1}).$

$E(\sum_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} E(X_n) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} E(X_1) = E(X_1) \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = E(X_1).$

If we consider $x_1 = 1$, then $E(\sum_{n=1}^{\infty} x_n) = 1$.

AMCS Written Preliminary Exam
Part II, August 27, 2018

1. Let V be the vector space of C^∞ functions y on the interval $(-1, 1)$ which satisfy the ODE $y''' - 2y'' + y' = 0$. Show that the derivative operator $\frac{d}{dx}$ maps V to itself and find a basis of V such that the matrix of $\frac{d}{dx}$ in this basis is in Jordan canonical form.

2. Find the eigenvalues and eigenvectors of the following matrix A:

$$A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

What is the $\lim_{k\to\infty} A^k =$?

3. Suppose that $\vec{F}$ is the following vector field:

$$\vec{F}(x, y, z) = \left\langle \frac{1}{3}x^3 - 3xy^2, \frac{1}{3}y^3 - 3yz^2, \frac{1}{3}z^3 - 3zx^2 \right\rangle.$$

Compute the flux integral

$$\int_S \vec{F} \cdot \vec{n} \, dA,$$

where $\vec{n}$ is the outward normal to the surface

$$S := \{(x, y, z) \ : \ x^2 + y^2 + z^2 = 1\}.$$

4. Show that there exists a real-valued function f on the interval $(-1, 1)$ which is differentiable at every point with a derivative f' which is not continuous.

5. Compute the Laurent series in the annulus $1 < |z| < 2$ of the function

$$\frac{1}{z^8 - 17z^4 + 16}.$$

6. Let z_n be the real zero of the polynomial

$$1 - \frac{1}{2!}\left(\frac{z}{2}\right)^2 + \frac{1}{4!}\left(\frac{z}{2}\right)^4 - \cdots + \frac{(-1)^n}{(2n)!}\left(\frac{z}{2}\right)^{2n}$$

which is closest to π.

(a) Prove that this sequence of zeroes denoted $\{z_n\}$ converges as $z_n \to \pi$ as $n \to \infty$.

(b) Show that $R^n|z_n - \pi| \to 0$ as $n \to \infty$ for any positive real number R.

You may solve this for example by using the mean value theorem.

7. A fair coin is tossed repeatedly until it lands on tails for the k-th time, at which point the game ends. During the game, a sequence of repeated, consecutive heads is called a winning streak. Show that there is probability greater than $1 - e^{-1}$ that the longest winning streak is at least $\log_2 k$.

Part II Aug 2018

1. $y''' - 2y'' + y' = 0$

characteristic function: $r^3 - 2r^2 + r = 0$. $r(r-1)^2 = 0$

$r_1 = 0, \quad r_2 = r_3 = 1$

$\Rightarrow \quad y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 x e^{r_3 x}$

$y = C_1 + C_2 e^x + C_3 x e^x$ $\quad C_1, C_2, C_3$ constant

Then $V = \{ y = C_1 + C_2 e^x + C_3 x e^x, \; x \in (-1, 1) \mid C_1, C_2, C_3 \in \mathbb{C} \}$

$\forall \; C_1, C_2, C_3 \in \mathbb{C}$:

$\frac{d}{dx}(y) = \frac{d}{dx}(C_1 + C_2 e^x + C_3 x e^x) = C_2 e^x + C_3 e^x + C_3 x e^x = (C_2 + C_3) e^x + C_3 e^x$

let $D_1 = 0, \quad D_2 = C_2 + C_3, \quad D_3 = C_3$,

then $\frac{d}{dx}(y) = D_1 + D_2 e^x + D_3 x e^x \quad x \in (-1, 1) \quad D_1, D_2, D_3 \in \mathbb{C}$.

$\frac{d}{dx}(y) \in V$.

$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$ the matrix of $\frac{d}{dx}$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

basis : $\langle 1, e^x, x e^x \rangle$.

2.

$$\det(A-\lambda I) = \det\begin{pmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{pmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06$$

$$= 0.56 - 1.5\lambda + \lambda^2 - 0.06 = \lambda^2 - 1.5\lambda + 0.5$$

$$= (\lambda-1)(\lambda-0.5) = 0$$

eigenvalues:

$\lambda_1 = 0.5$, $\lambda_2 = 1$.

when $\lambda_1 = 0.5$

$$(A-\lambda_1 I)X_1 = \begin{pmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{pmatrix} X_1 = 0 \Rightarrow X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

when $\lambda_2 = 1$

$$(A-\lambda_2 I)X_2 = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{pmatrix} X_2 = 0 \Rightarrow X_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

X_1, X_2 are eigenvectors.

let $\beta_1 = \frac{X_1}{\|X_1\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\beta_2 = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix}$

$S = (X_1, X_2) = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ and $S^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

$$A = S\Lambda S^{-1} \Rightarrow A^k = S\Lambda^k S^{-1}$$

$$= \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} (\frac{1}{2})^k & 0 \\ 0 & 1^k \end{pmatrix}\frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} (\frac{1}{2})^k & 3 \\ -(\frac{1}{2})^k & 1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} (\frac{1}{2})^{k-1} + 3 & -3(\frac{1}{2})^k + 3 \\ -(\frac{1}{2})^{k-1} + 1 & 3(\frac{1}{2})^k + 1 \end{pmatrix}$$

$$\lim_{k\to\infty} A^k = \lim_{k\to\infty} \frac{1}{5}\begin{pmatrix} (\frac{1}{2})^{k-1} + 3 & -3(\frac{1}{2})^k + 3 \\ -(\frac{1}{2})^{k-1} + 1 & 3(\frac{1}{2})^k + 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

3.

Let $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$

$= \langle \frac{1}{3}x^3 - 3xy^2,\ \frac{1}{3}y^3 - 3yz^2,\ \frac{1}{3}z^3 - 3zx^2 \rangle$

$\int_S \vec{F}\cdot\vec{n}\,dA = \int_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$

$\overset{\text{div theorem}}{=} \int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$ where $V = \{(x,y,z),\ x^2+y^2+z^2 \le 1\}$

$= \int_V \left((x^2-3y^2) + (y^2-3z^2) + (z^2-3x^2)\right) dx\,dy\,dz$

$= \int_V -2(x^2+y^2+z^2)\,dx\,dy\,dz$

$= \int_0^1 dr \int_0^\pi d\varphi \int_0^{2\pi} -2(r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi)\,r^2\sin\varphi\,d\theta$

where $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$

$= -2\int_0^1 r^4\,dr \int_0^\pi \sin\varphi\,d\varphi \int_0^{2\pi} d\theta$

$= -2\left(\frac{1}{5}r^5\right)\Big|_0^1 (-\cos\varphi)\Big|_0^\pi\ \theta\Big|_0^{2\pi}$

$= -2\cdot\frac{1}{5}\cdot(1+1)\cdot 2\pi = -\frac{8}{5}\pi$

4.

Consider Riemann function:

$R(x) = \begin{cases} \frac{1}{p},\ x \in [0,1],\ x = \frac{q}{p},\ (p,q)=1. \\ 0,\ x \in (0,1) \setminus \mathbb{Q} \\ 1,\ x = 0 \end{cases}$

integrable.

Let $f'(x) = \begin{cases} R(x) & x \geqslant 0 \\ R(-x) & x < 0. \end{cases}$

f differentiable everywhere, but f' not continuous.

5.

$$\frac{1}{z^8-17z^4+16} = \frac{1}{(1-z^4)(16-z^4)} = \frac{1}{15}\,\frac{(16-z^4)-(1-z^4)}{(1-z^4)(16-z^4)}$$

$$= \frac{1}{15}\left(\frac{1}{1-z^4} - \frac{1}{16-z^4}\right)$$

$$= \frac{1}{15}\left(\frac{-\frac{1}{z^4}}{1-\frac{1}{z^4}} - \frac{\frac{1}{16}}{1-(\frac{z}{2})^4}\right)$$

$$= -\frac{1}{15z^4}\,\frac{1}{1-(\frac{1}{z})^4} - \frac{1}{240}\,\frac{1}{1-(\frac{z}{2})^4}$$

$$= -\frac{1}{15z^4}\left(1+\left(\frac{1}{z^4}\right)+\left(\frac{1}{z^4}\right)^2+\cdots\right)$$

$$-\frac{1}{240}\left(1+\left(\frac{z}{2}\right)^4+\left(\frac{z}{2}\right)^8+\cdots\right)$$

$$= -\frac{1}{15z^4}\sum_{k=0}^{\infty}\frac{1}{z^{4k}} - \frac{1}{240}\sum_{k=0}^{\infty}\frac{z^{4k}}{2^{4k}}$$

$$= -\frac{1}{15}\sum_{k=-\infty}^{-1} z^{4k} - \frac{1}{240}\sum_{k=0}^{+\infty}\frac{1}{2^{4k}}\,z^{4k}$$

6.(a)

Let $P_n(z) = 1 - \frac{1}{2!}\left(\frac{z}{2}\right)^2 + \frac{1}{4!}\left(\frac{z}{2}\right)^4 + \cdots + \frac{(-1)^n}{(2n)!}\left(\frac{z}{2}\right)^{2n}$ uniformly continuous on $\mathbb{C}$

$f(z) = \cos\frac{z}{2} = \sum_{k=0}^{+\infty} (-1)^k \frac{1}{(2k)!}\left(\frac{z}{2}\right)^{2k}$ uniformly continuous on $\mathbb{C}$.

then $\cos\frac{z}{2} = P_n(z) + (-1)^{n+1}\frac{\cos\frac{\theta z}{2}}{(n+1)!}\left(\frac{z}{2}\right)^{2(n+1)}$ $0<\theta<1$.

then $P_n(\pi) = \cos\frac{\pi}{2} - (-1)^{n+1}\frac{\cos\frac{\theta\pi}{2}}{(n+1)!}\left(\frac{\pi}{2}\right)^{2(n+1)}$

$P_n(\pi) = (-1)^n \frac{\cos\frac{\theta\pi}{2}}{(n+1)!}\left(\frac{\pi}{2}\right)^{2(n+1)}$

~~$\forall\ \delta>0,\ \exists N_1 \in \mathbb{N}$ s.t. $|P_n(\pi) - P_n(z_n)| = |P_n(\pi)| = \left|\frac{\cos\frac{\theta\pi}{2}}{(n+1)!}\left(\frac{\pi}{2}\right)^{2(n+1)}\right| < \delta\ \forall n > N_1$~~

And $f(z_n) = P_n(z_n) + (-1)^{n+1}\frac{\cos\frac{\theta z_n}{2}}{(n+1)!}\left(\frac{z_n}{2}\right)^{2(n+1)}$, $f(\pi) = 0$.

$f(z_n) = \cos\frac{z_n}{2} = (-1)^{n+1}\frac{\cos\frac{\theta z_n}{2}}{(n+1)!}\left(\frac{z_n}{2}\right)^{2(n+1)}$

since $f(z)$ continuous, $\forall\ \delta>0,\ \exists N \in \mathbb{N}^*$ s.t. $|f(z_n) - f(\pi)| < \delta\ \ \forall n > \mathbb{N}$

$\forall\ \varepsilon>0,\ \exists\ \delta>0$, s.t. $|z_n - \pi| < \varepsilon\ \ \forall\ |f(z_n) - f(\pi)| < \delta$

i.e. $\forall\ \varepsilon>0,\ \exists N \in \mathbb{N}^*$ s.t. $|z_n - \pi| < \varepsilon\ \ \forall n > N$.

6.(b). $R<1$, $R^n \to 0$ as $n\to\infty$, when $R \geq 1$

$\forall \varepsilon>0$, $\exists\ \lambda>0$ $(\lambda = \frac{\varepsilon}{R^n})$. s.t. $\exists N \in \mathbb{N}^*$ s.t. $|z_n - \pi| < \lambda\ \ \forall n > N$

then $|z_n - \pi| < \frac{\varepsilon}{R^n}$, $R^n|z_n - \pi| < \varepsilon$.

7.

~~Let W_k to be the waiting time for k^{th} tails.~~

~~$P(W_k = n) =$~~ $\binom{n-1}{k-1}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{n-k} = \binom{n-1}{k-1}\left(\frac{1}{2}\right)^n$

i.e. to show there is probability less than e^{-1} that all the winning streak is less than $\log_2 k$

consider all the sequences of consective heads, suppose they have length of a_i; and all the sequences of consective tails, suppose they have length of b_i.

Suppose there are n different sequences of consective heads in total ($n \leq k$)

$$P(\max\{a_i\} < [\log_2 k]) = P(a_i < [\log_2 k]\ \forall i)$$

$$= \sum_{\sum b_i = k} \prod_{i=1} \left(1 - \frac{1}{2^{a_i - 1}}\right) \frac{1}{2^{b_i}}$$

$$\leq 2^k \cdot \frac{1}{2^k} \prod_{i=1}^{n} \left(1 - \frac{1}{2^{[\log_2 k]}}\right)$$

$$\leq \left(1 - \frac{1}{k}\right)^{-k}$$

$$\leq e^{-1}.$$