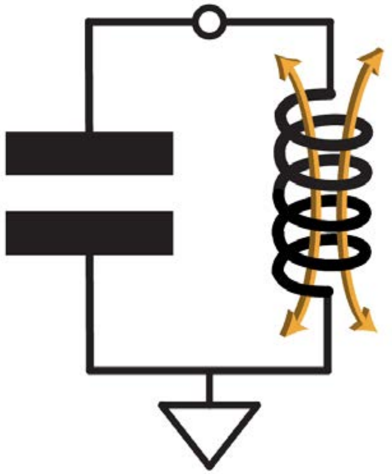
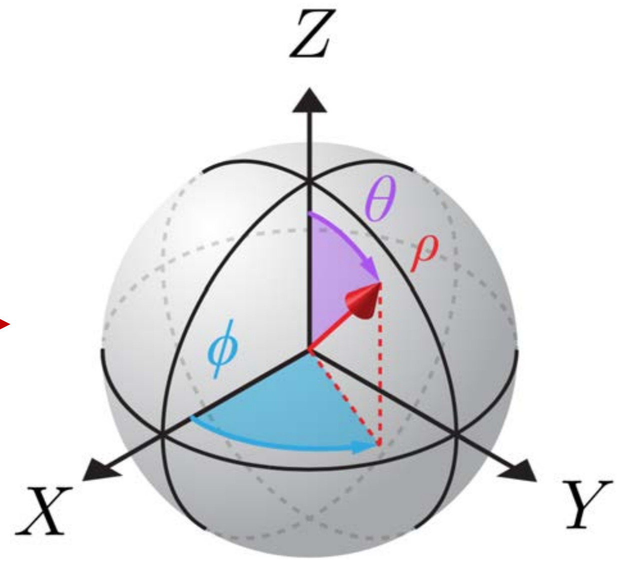
# Superconducting Qubits I:

## Making Your First Qubit From an Oscillator

### Introduction to Circuit Quantum Electrodynamics (cQED)







## Zlatko K. Minev

IBM Quantum
IBM T.J. Watson Research Center, Yorktown Heights, NY

@zlatko_minev    zlatko-minev.com

# What is a real qubit?

# How can you design, control, and measure a real qubit?

# Why?

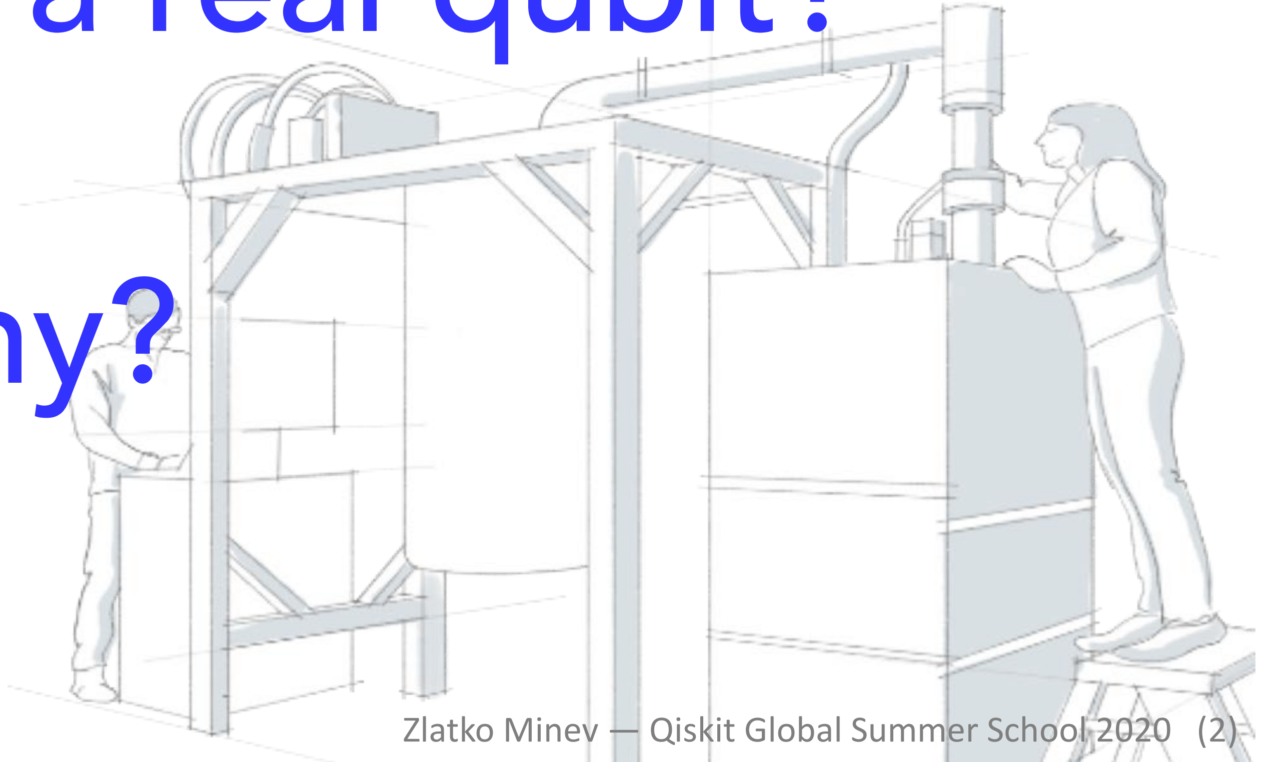
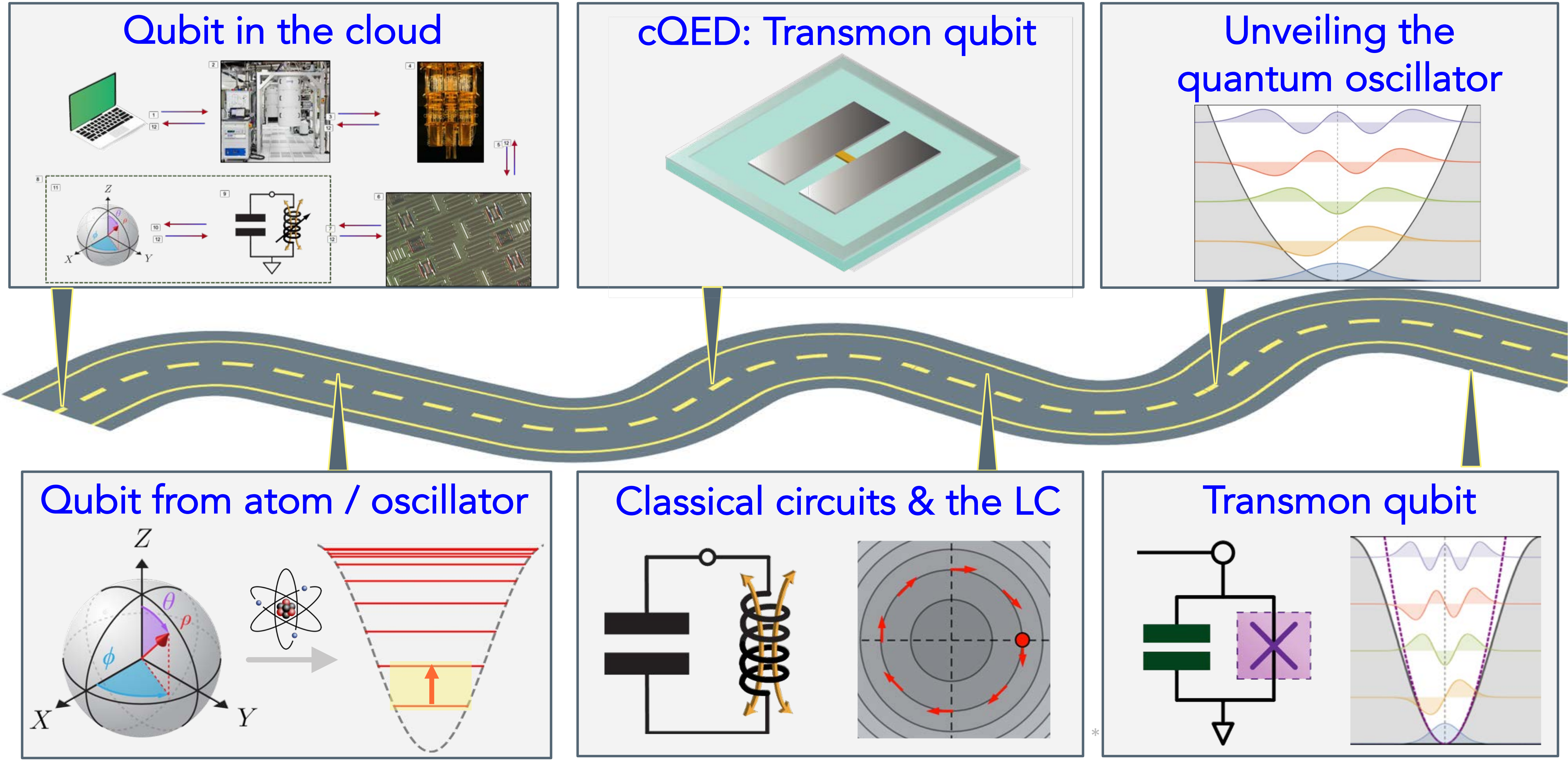


Illustration: IBM Qiskit Textbook

# On the road ahead



# This Lecture

Introductory and skill reaffirming

Don't need to know much going in, but we will go far

Advanced material

Examples: simplest, most practical examples

Step by step

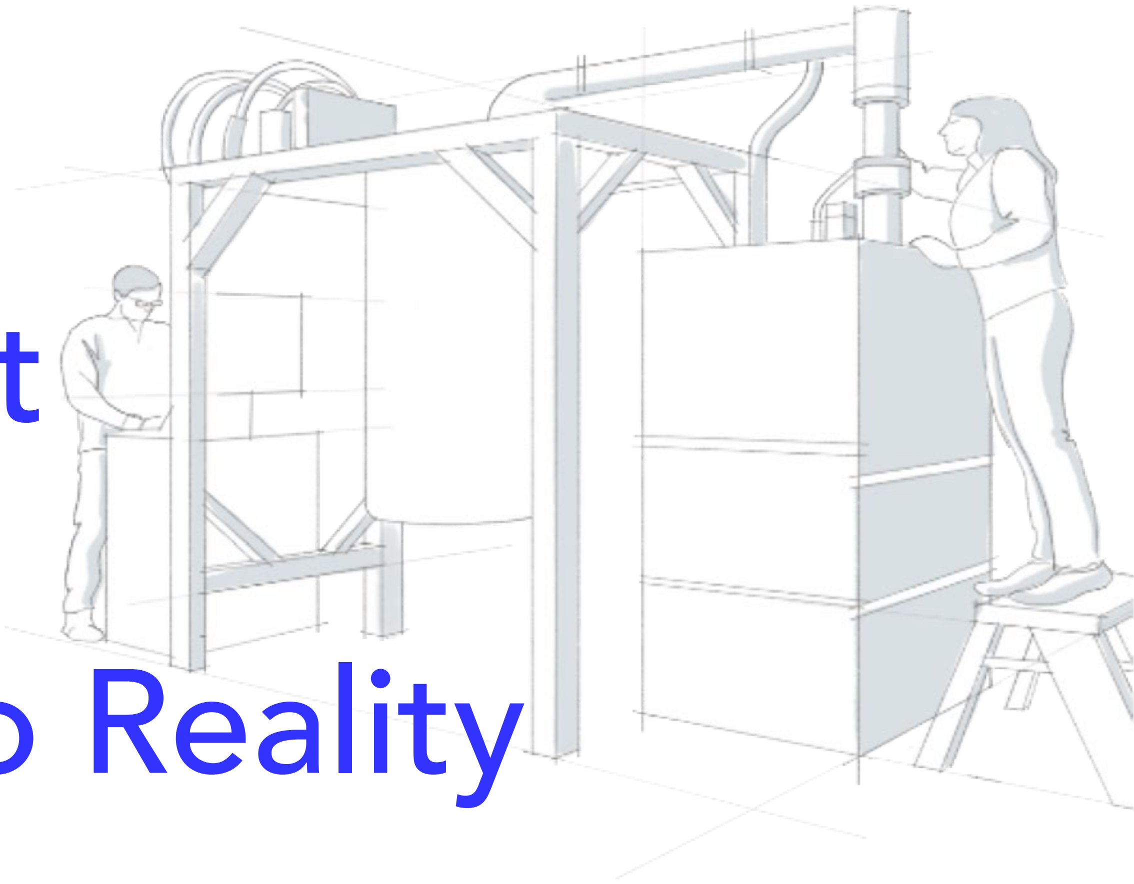
Ask questions!

Tightly integrated lab work by Dr. Nick Bronn and Co.!

Icons here by FreePik. Dangerous bend symbol based on Donald Knuth.

# Avoid firehose of information


Thanks to Fred Moxley for gif reference.



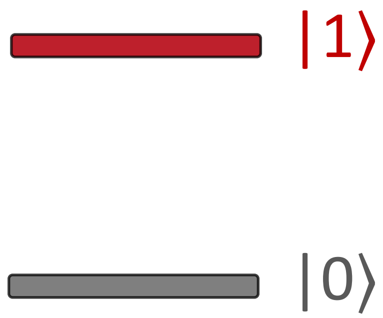
# Qubit

# From Idea to Reality

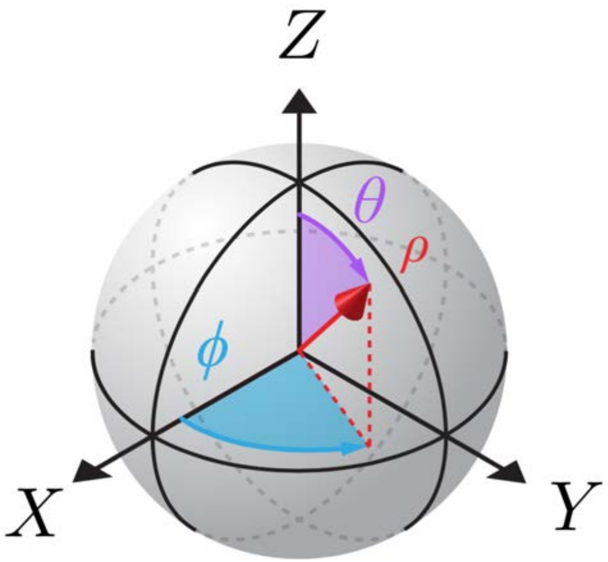
## THE BIG PICTURE
before calculations

# Qubit: idea

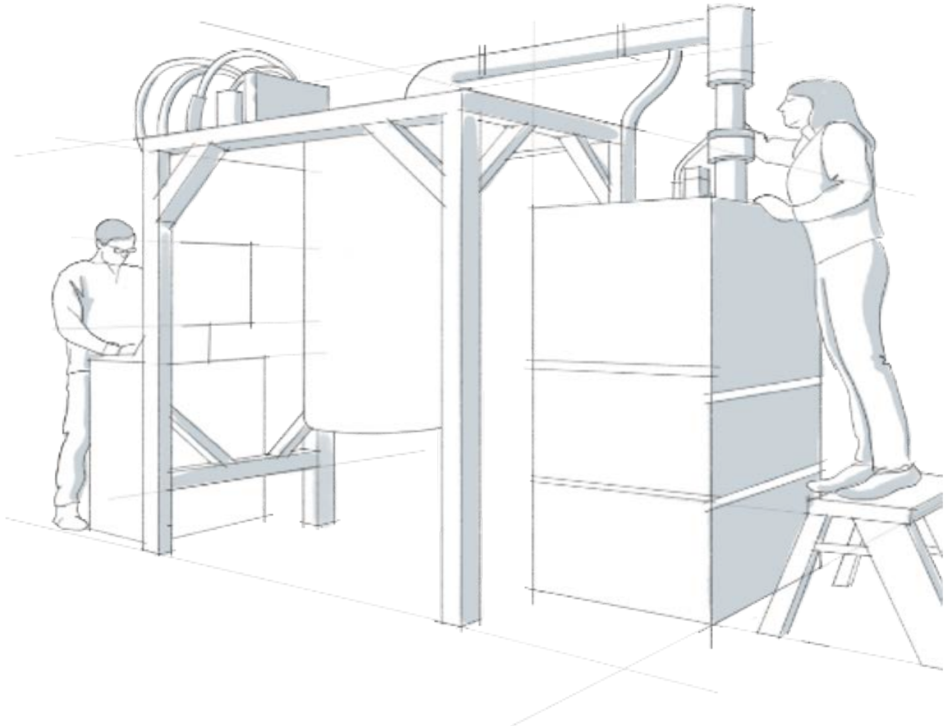
## Energy levels




## Hilbert space




## Quantum cloud?



* Images: Minev, arXiv:1902.10355; Illustration on right: IBM Qiskit Textbook;

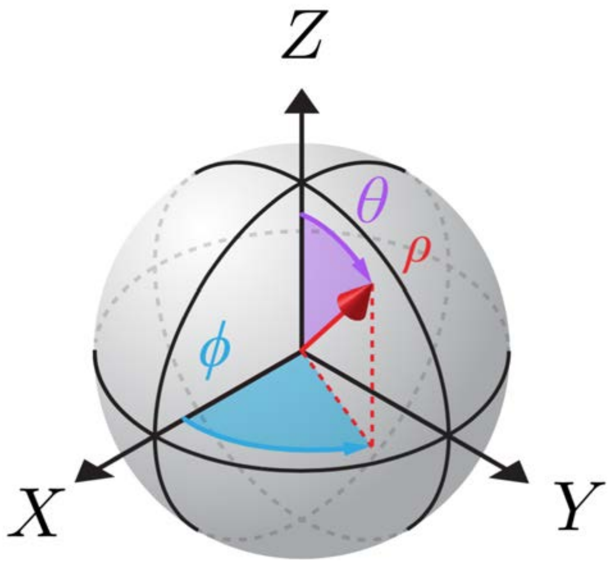
# Qubit: idea and reality

## Energy levels

$|1\rangle$

$|0\rangle$

## Hilbert space

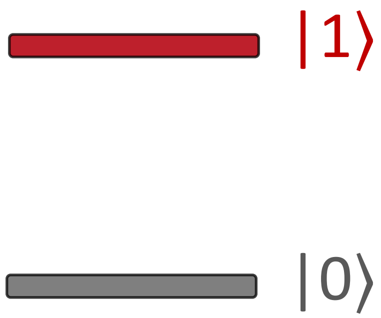




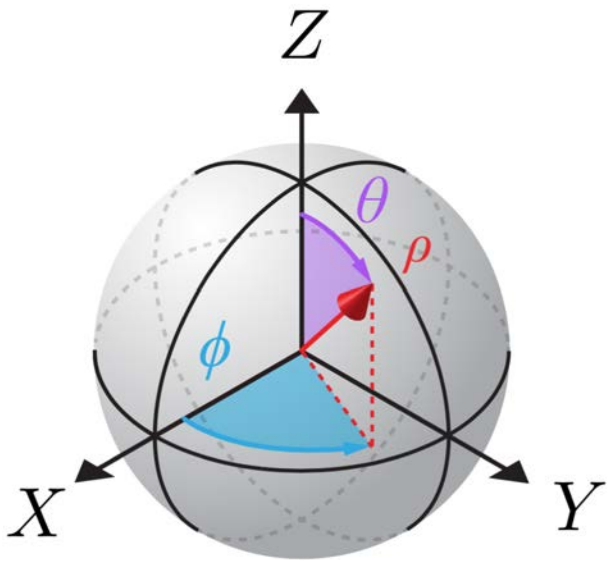
* Images: Minev, arXiv:1902.10355; Illustration on right: IBM, Qiskit Textbook

# Qubit: idea and reality

## Energy levels

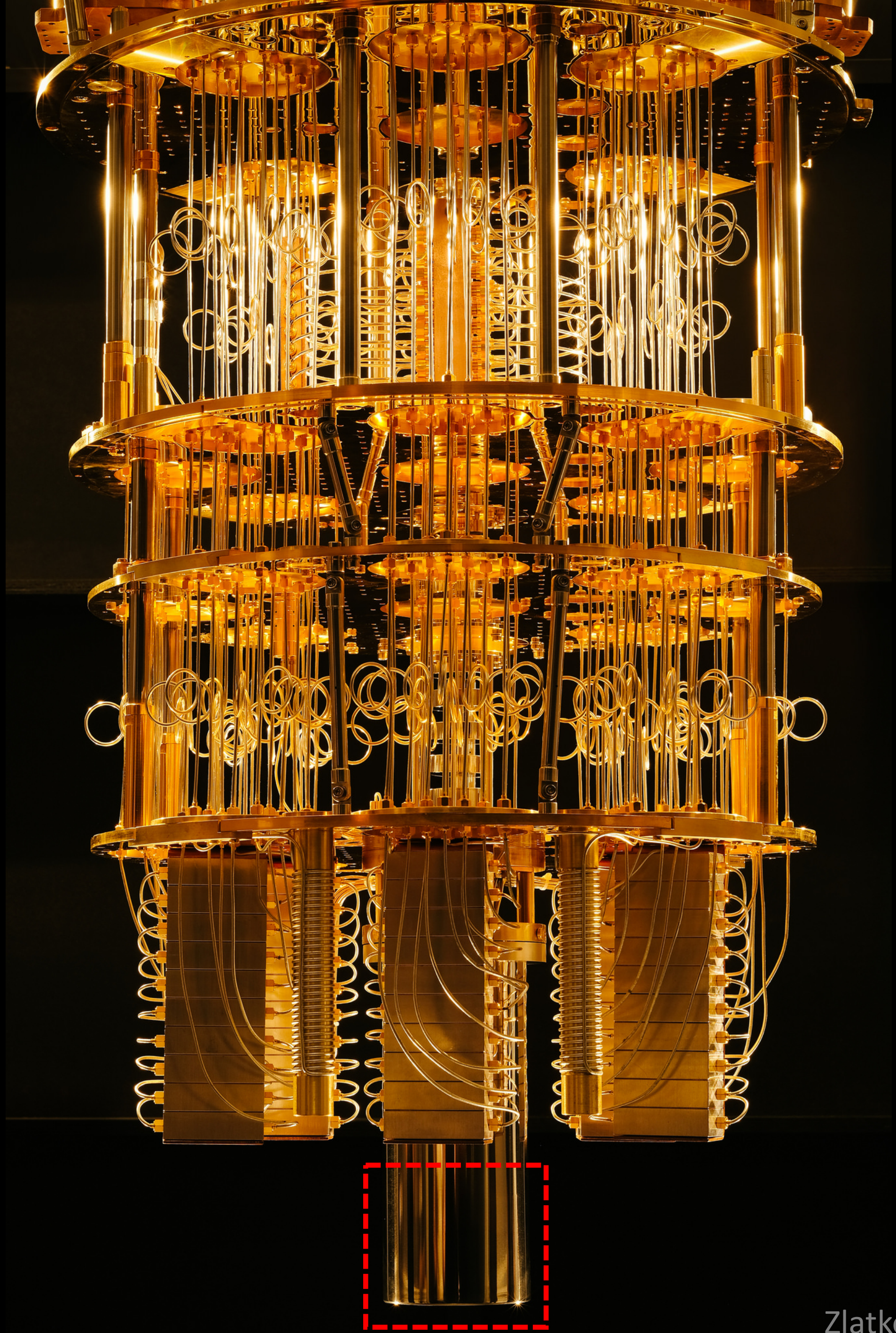



## Hilbert space

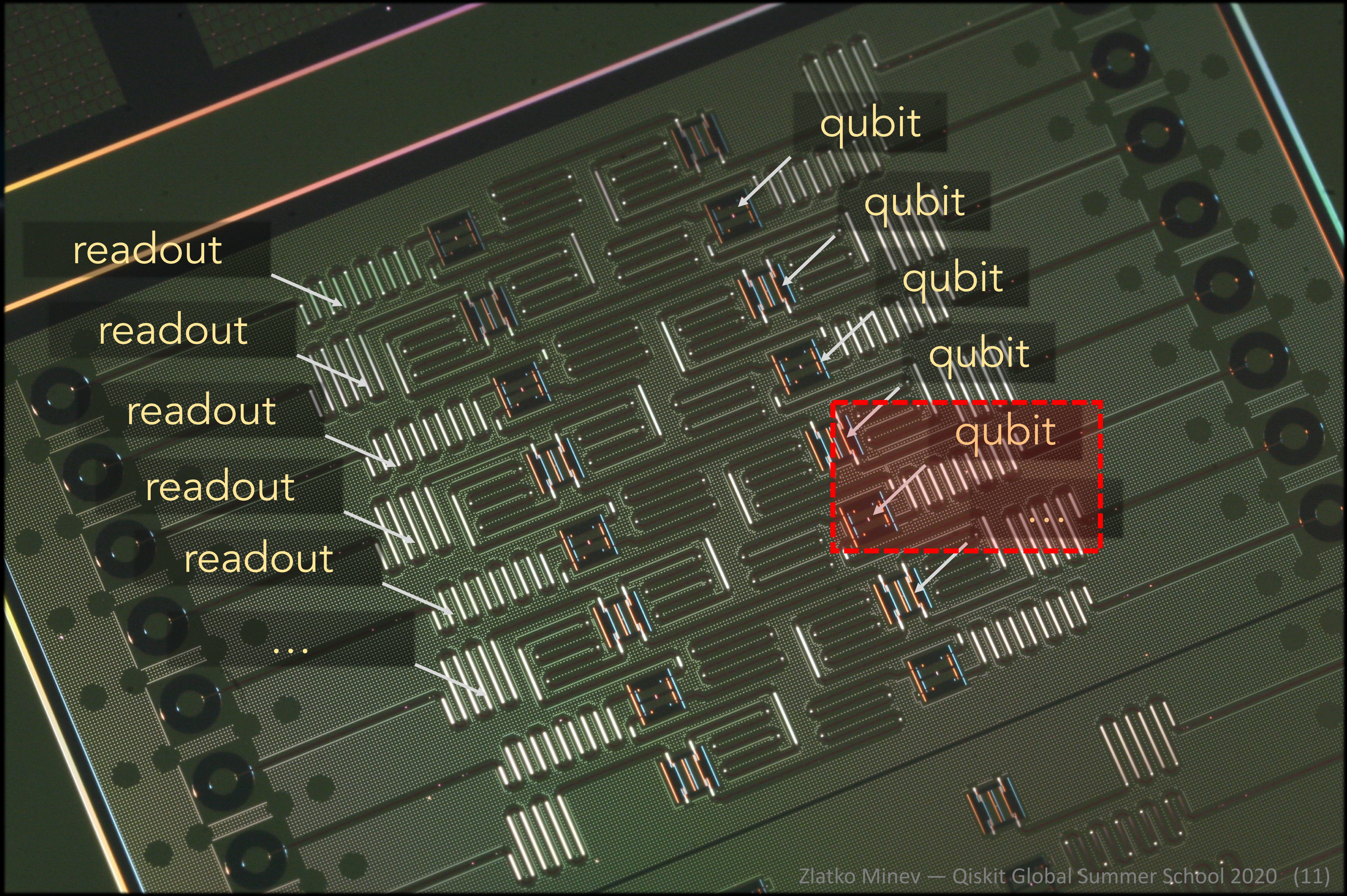


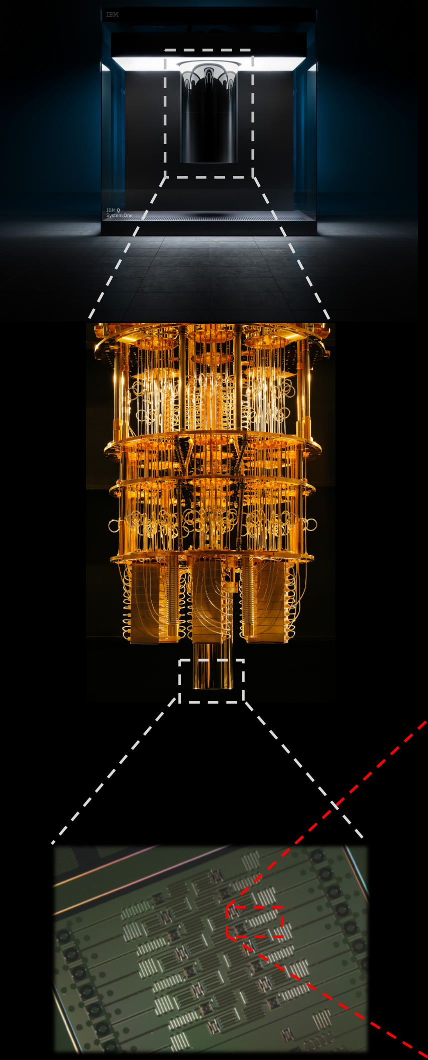


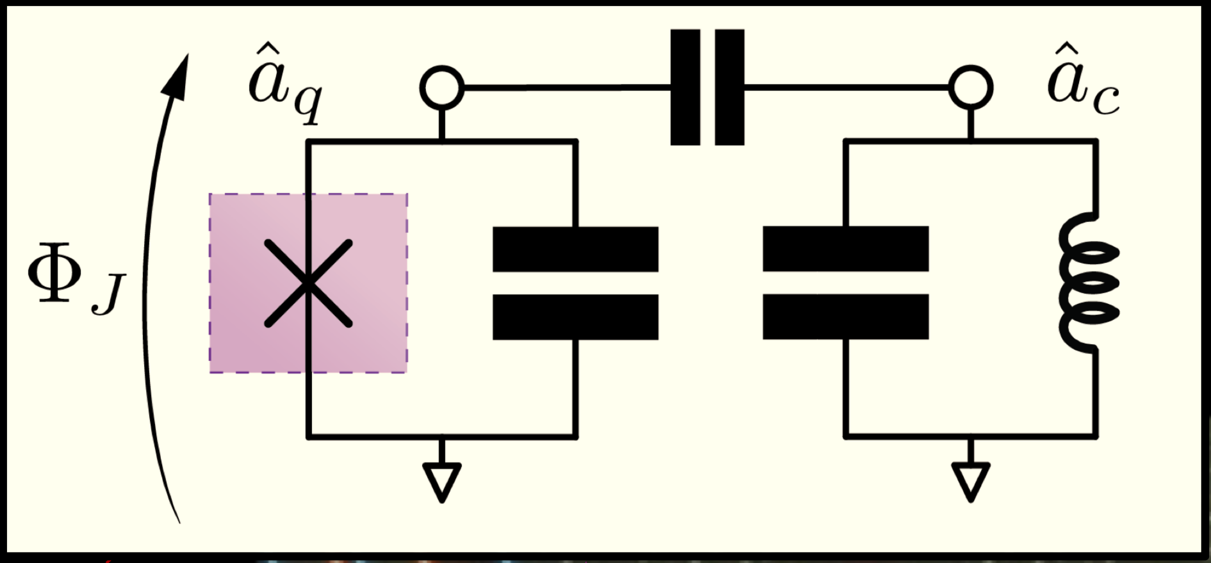
[*] Images: Minev, arXiv:1902.10355; Illustration on right: IBM, Qiskit Textbook

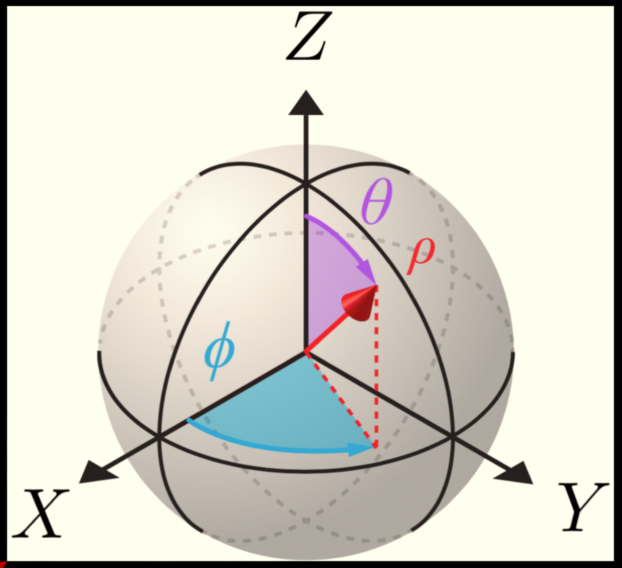


Operation at
15 mK (-273.13 °C)

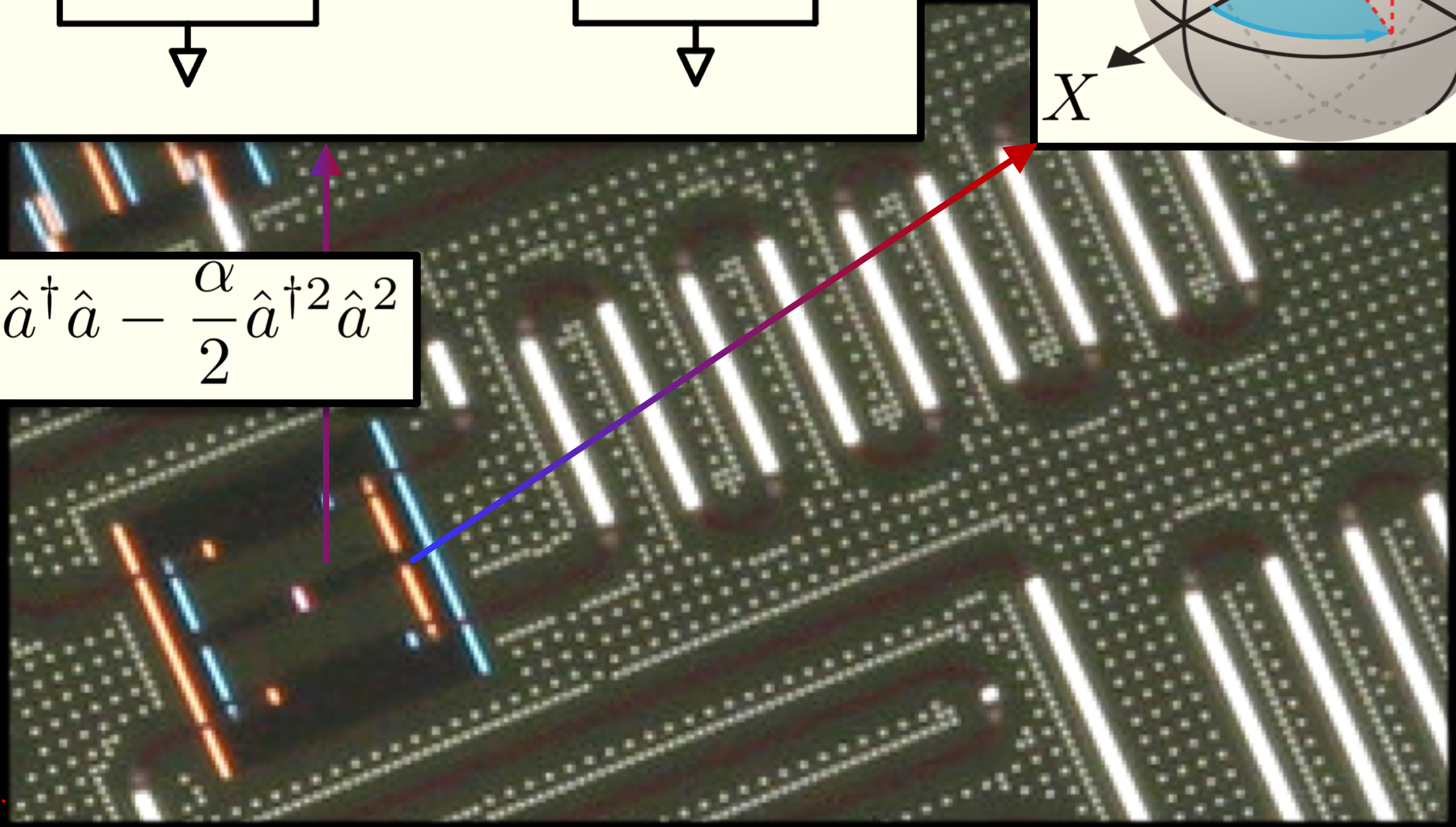

qubit
qubit
qubit
qubit
qubit
...
readout
readout
readout
readout
readout
...



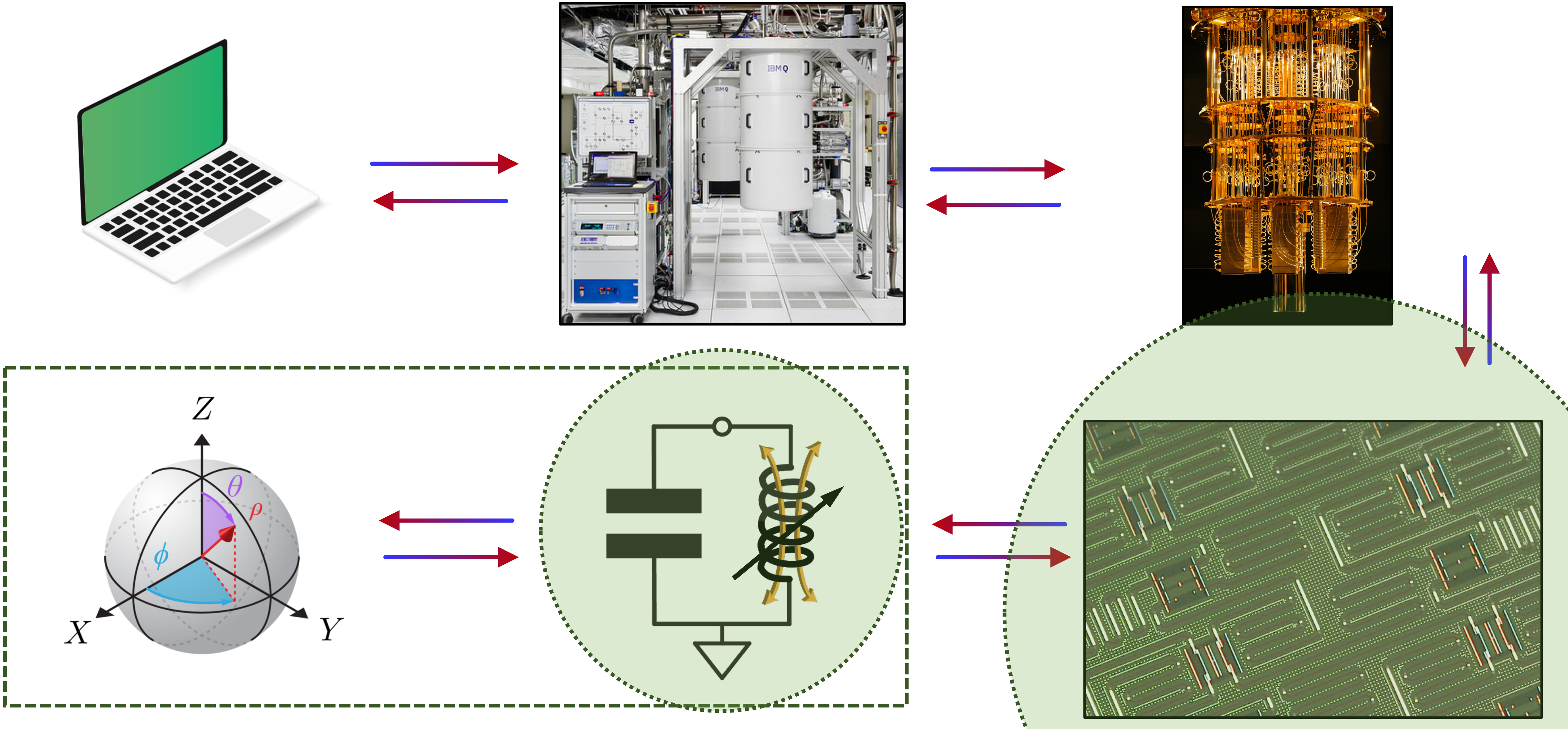



$$\hat{H} \approx \omega_0 \hat{a}^\dagger \hat{a} - \frac{\alpha}{2} \hat{a}^{\dagger 2} \hat{a}^2$$



# cQED qubit in the cloud: Summary of flow

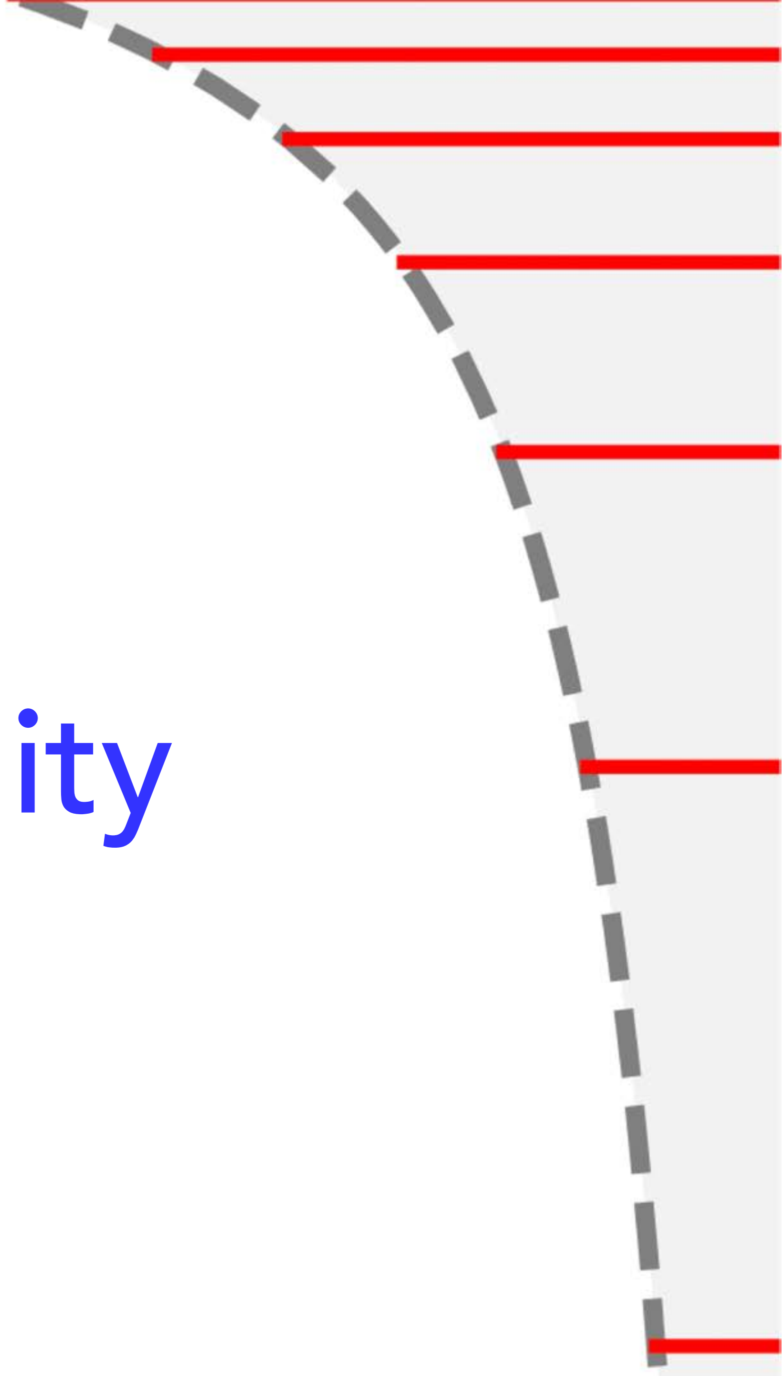



* Laptop image: rawpixel.com; Photos: IBM

# Qubit

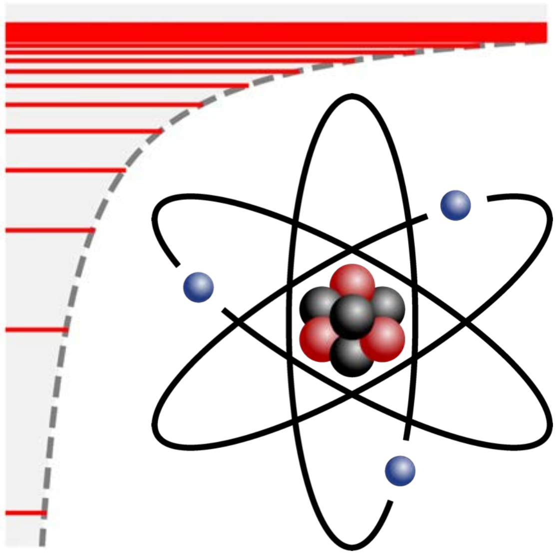
# From Idea to Reality

## Concepts

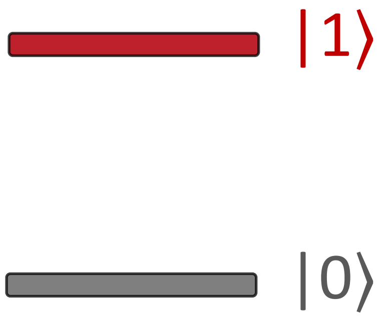


# From qubit representation to reality

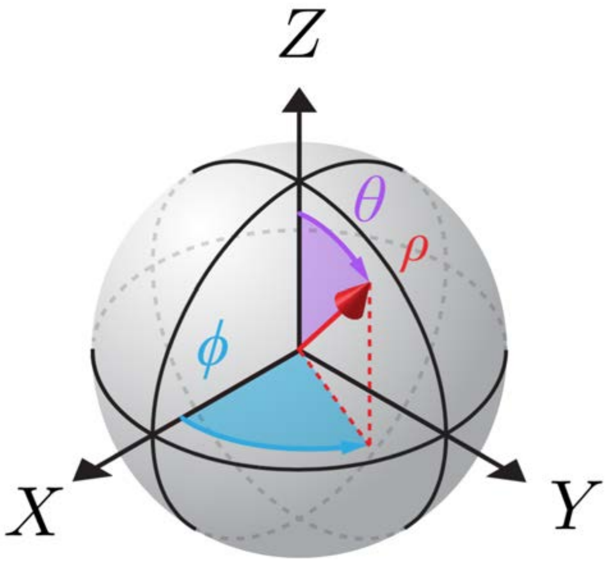
## Realization



## Energy levels




## Hilbert space*

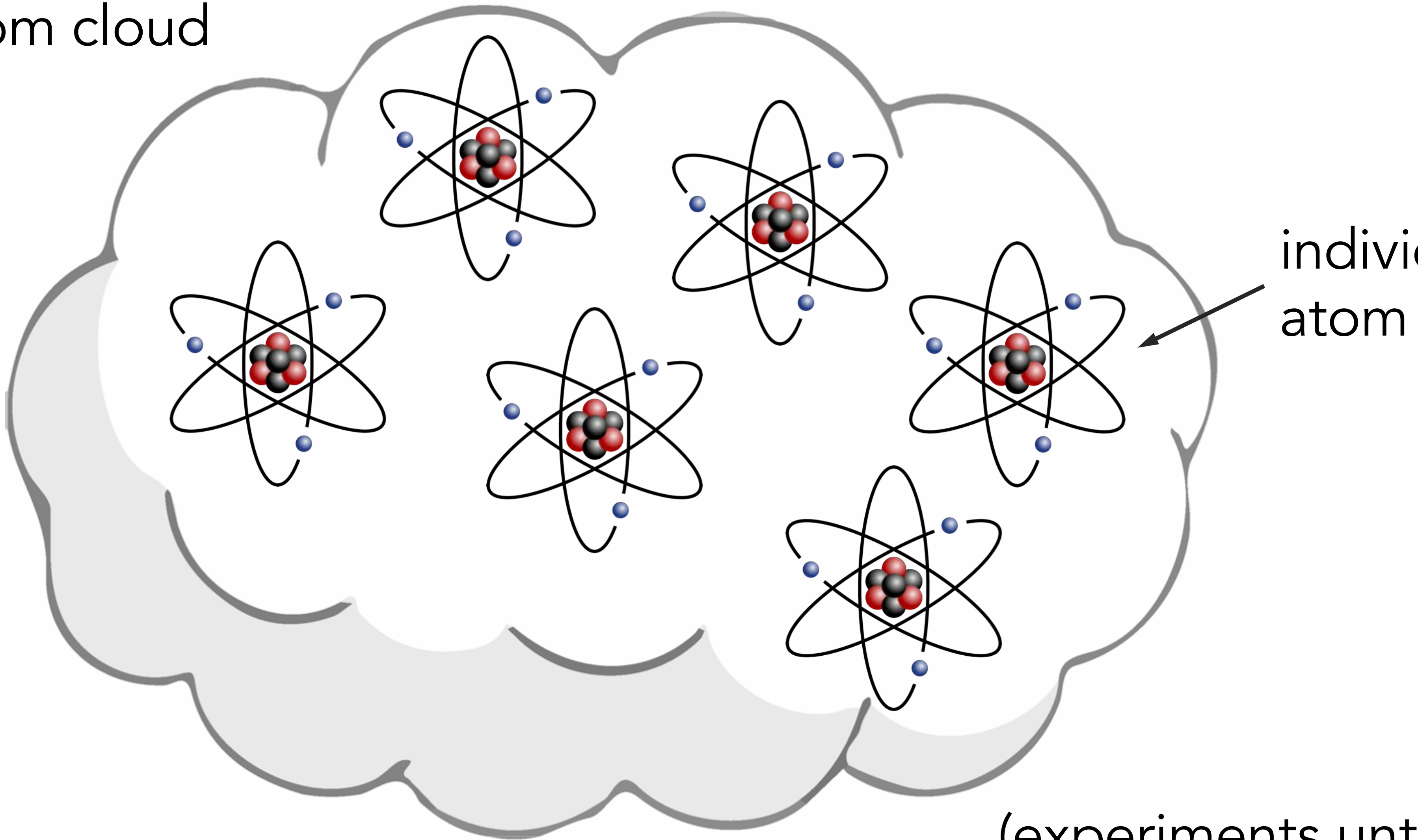



* Bloch sphere is a mere geometrical representation of SO(3), but the density matrix $\boldsymbol{\rho}$ is in SU(2), a double cover of SO(3).
* A density matrix operator lives not in the Hilbert space H but in the Liouville space H ⊗ H.

Images: Minev, arXiv:1902.10355; atom art: Indoleces.

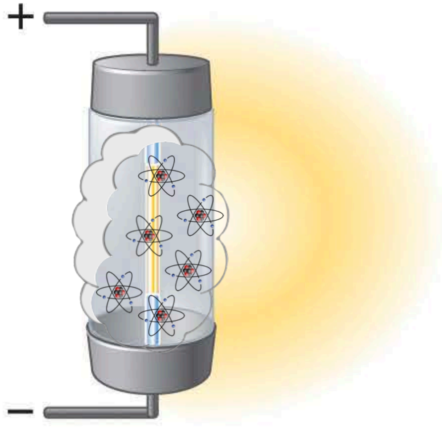
# Origins of quantum

e.g., atom cloud




(experiments until c. 1980)

# Atomic emission of light




Tube containing excited $H_2$ gas

Image: Averill and Eldredge, Principles of General Chemistry

# Atomic emission of light

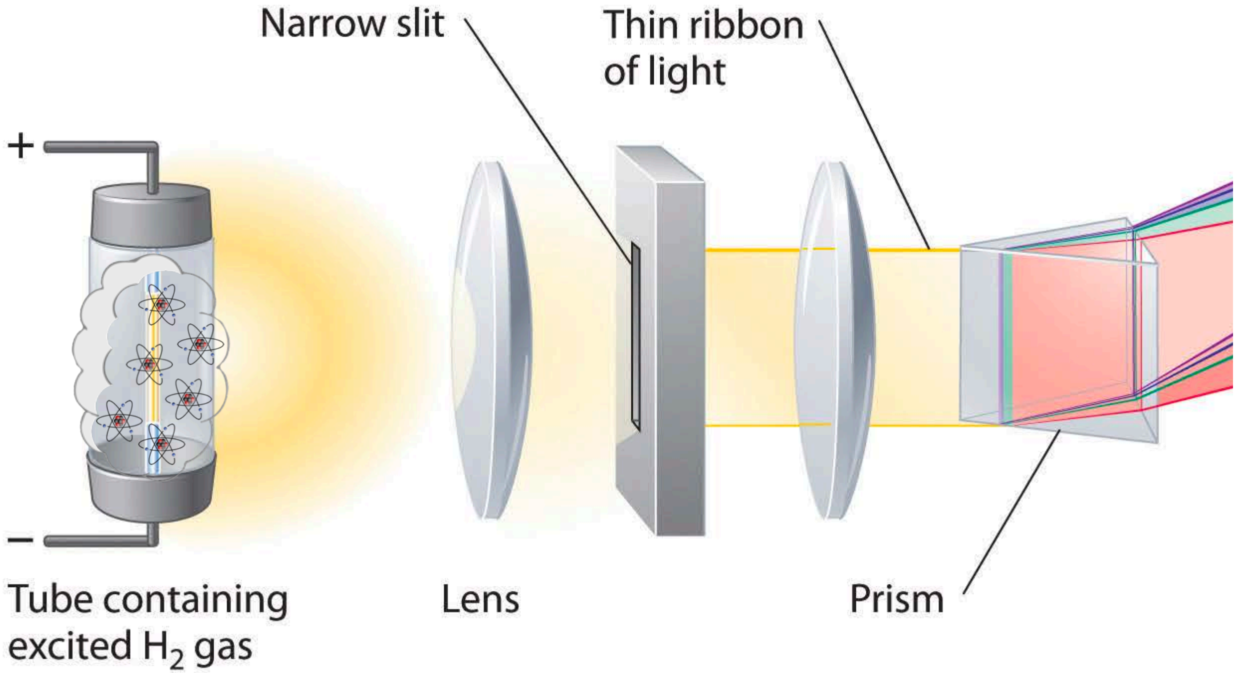



Image: Averill and Eldredge, Principles of General Chemistry

# Atomic emission of light

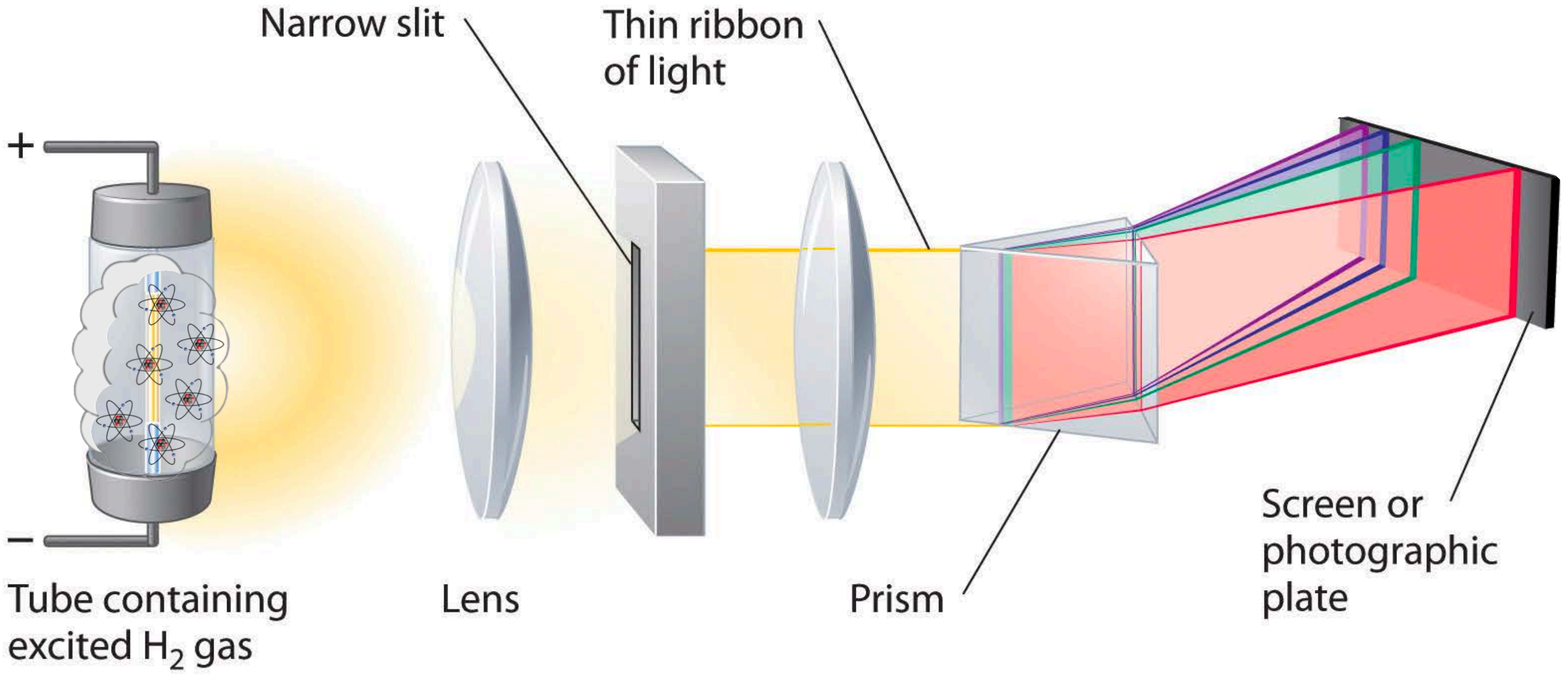



Image: Averill and Eldredge, Principles of General Chemistry

# Atomic emission of light

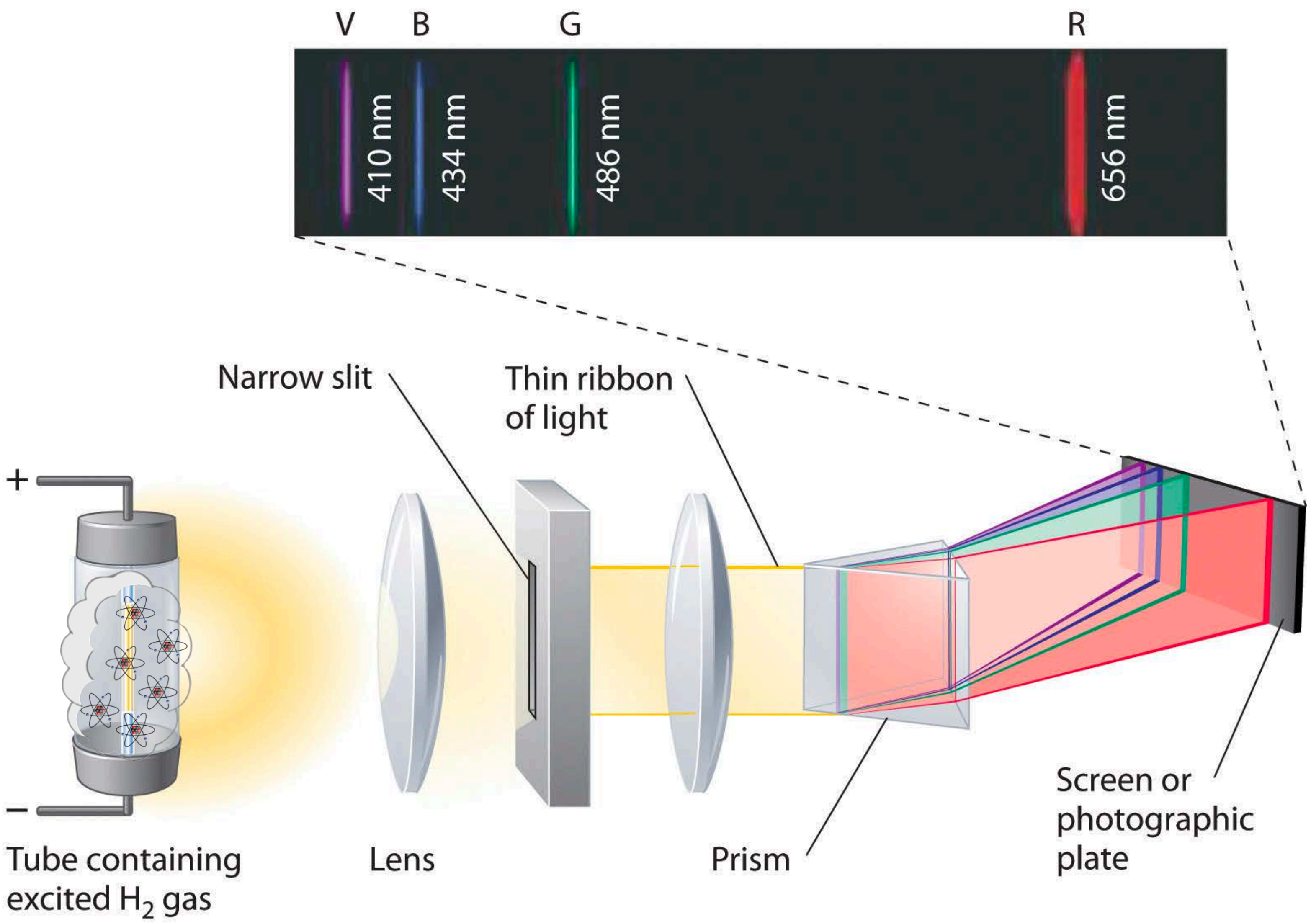



Image: Averill and Eldredge, Principles of General Chemistry

# Quantized levels

## Atomic emission spectra

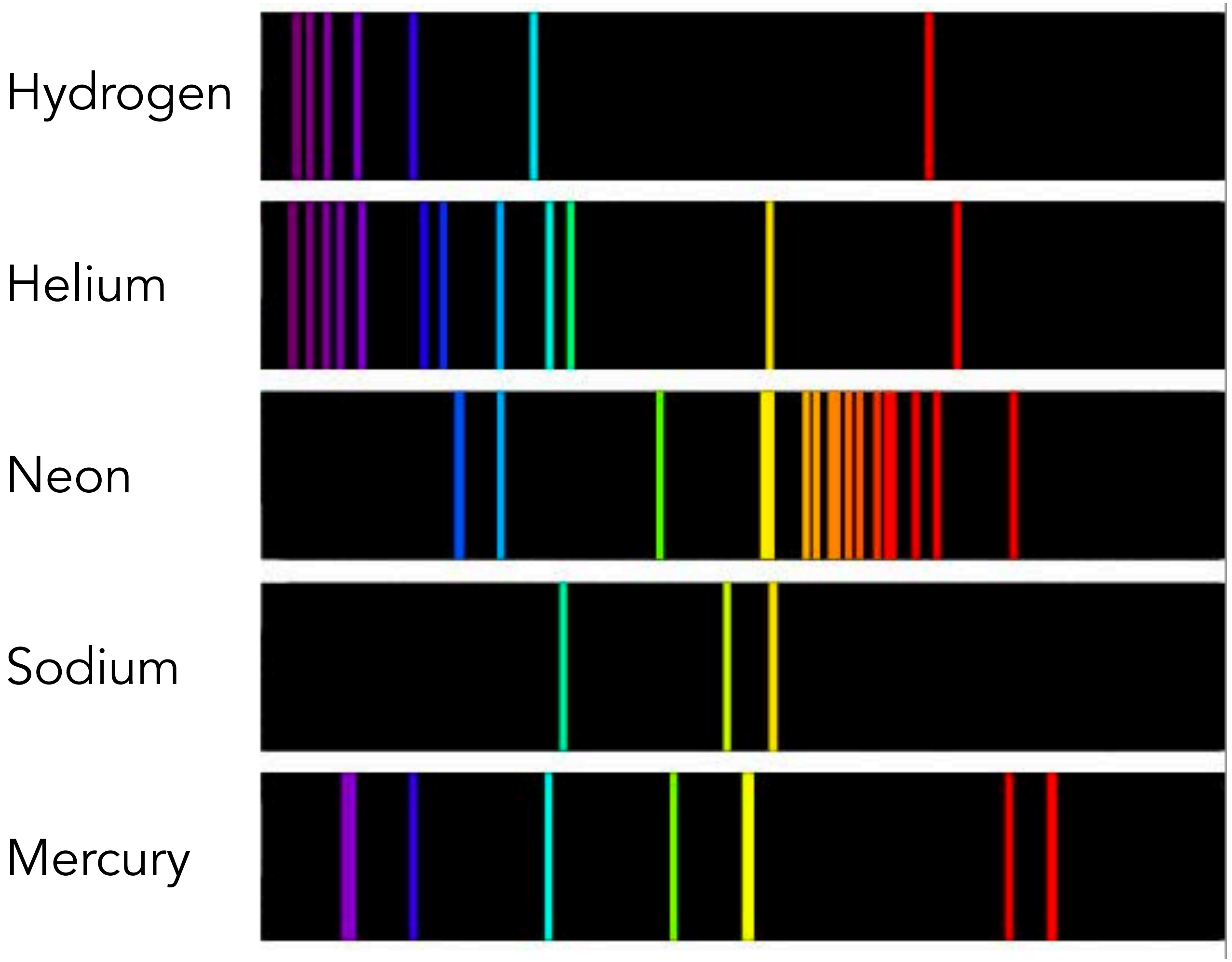
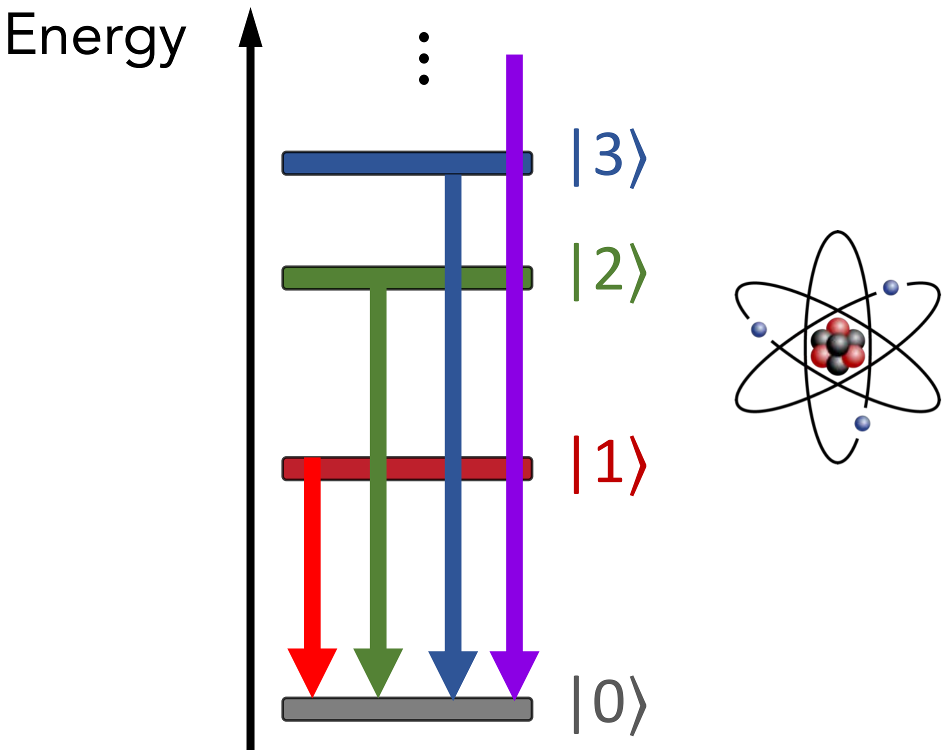



*Image credit: NMSU, N. Vogt*

## Atoms are quantum: discrete intrinsic energy levels*




Quantization of energy

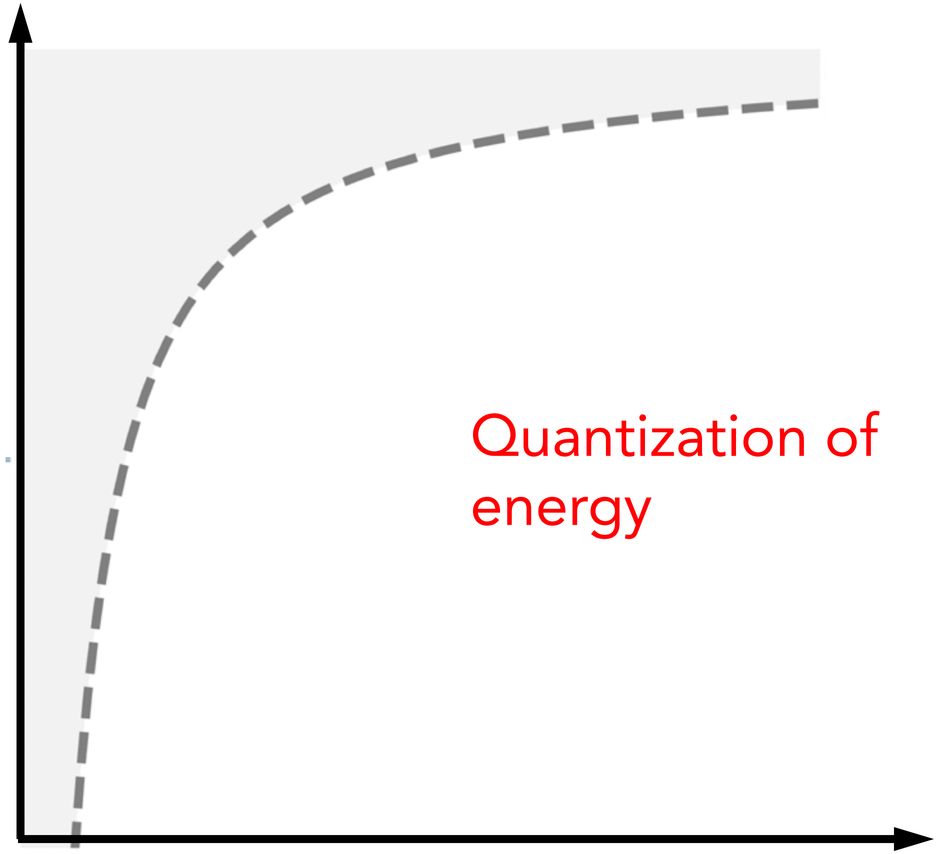
* The notion of an energy level was proposed by Bohr in 1913.

# The light of atoms


Image: Averill and Eldredge, Principles of General Chemistry

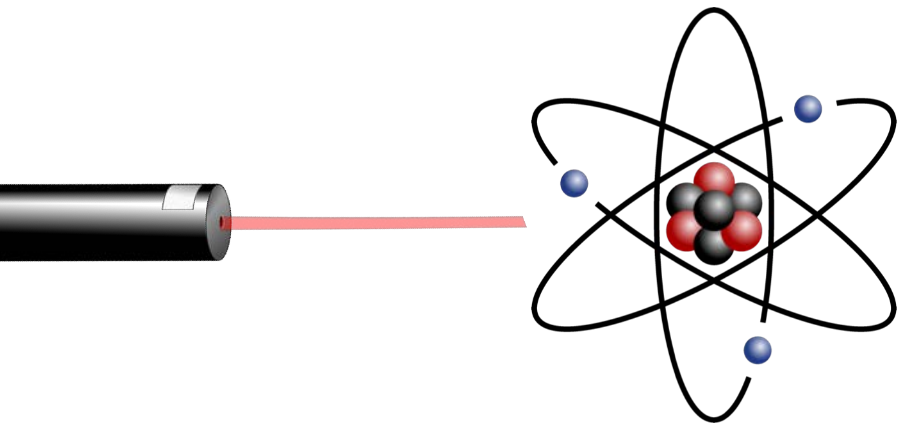
# Atomic energy levels and transitions

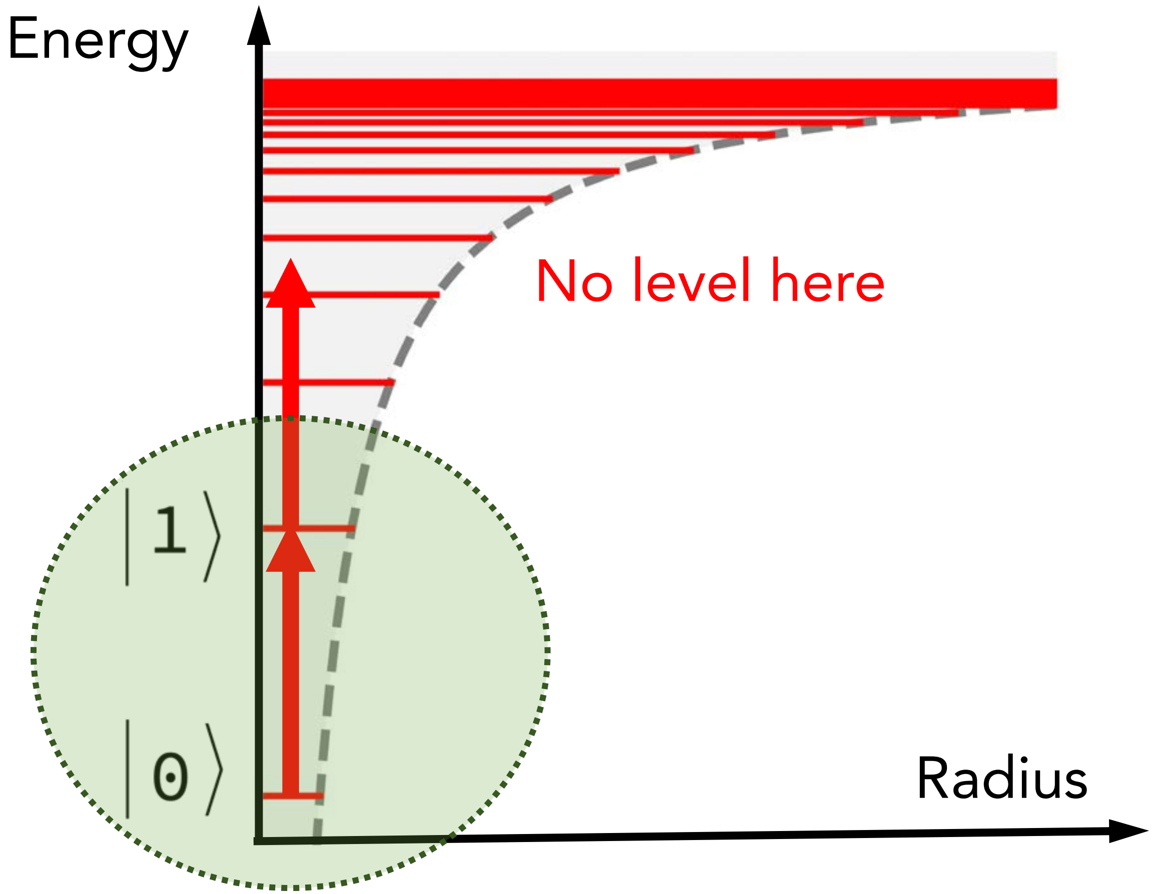
Electron potential-energy landscape



Quantization of energy

Radial distance from nucleus






# Qubit from Atom

Anharmonic degree of freedom and spin

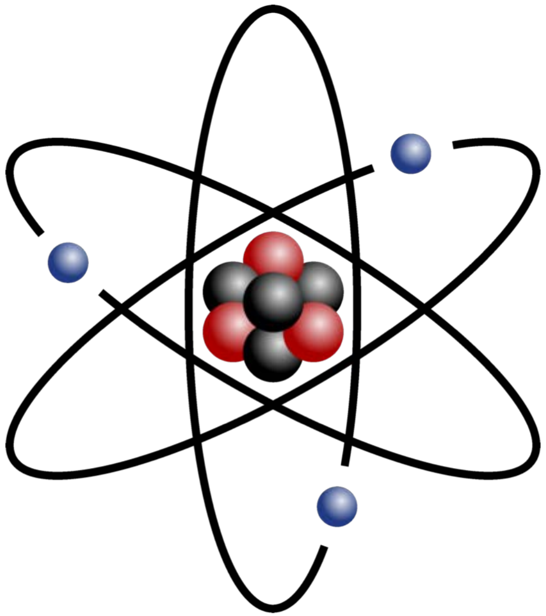
Isolated from environment and thermal bath

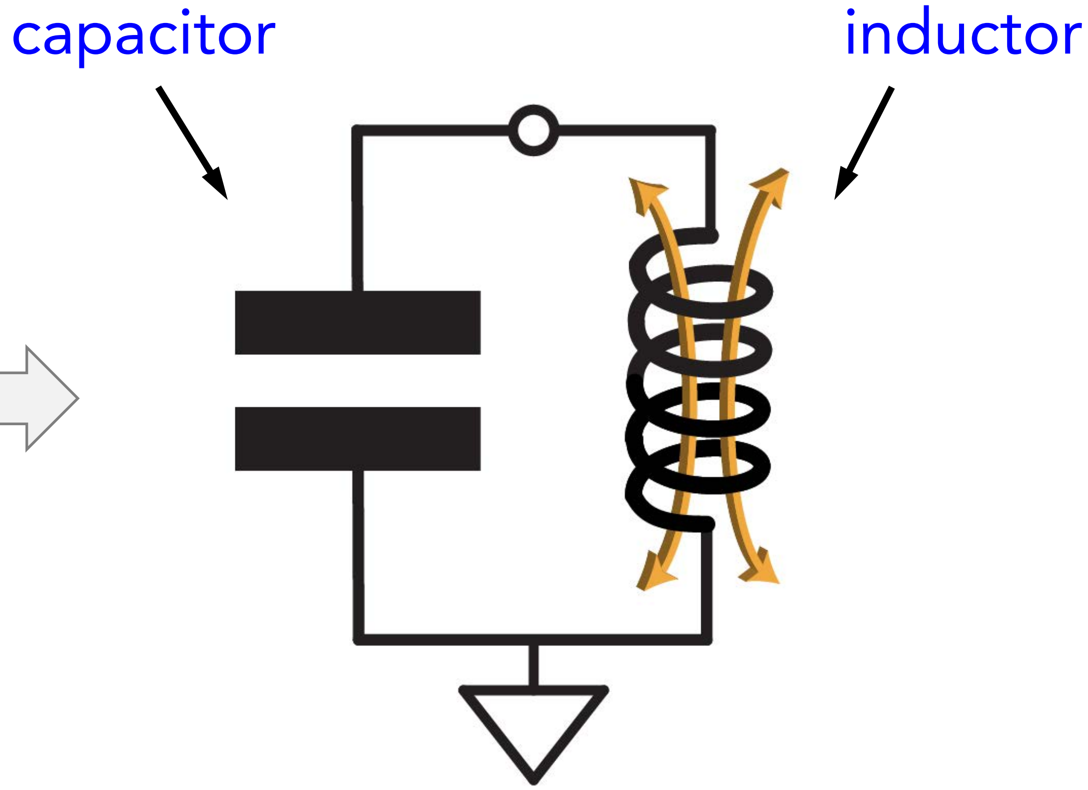
Low-loss

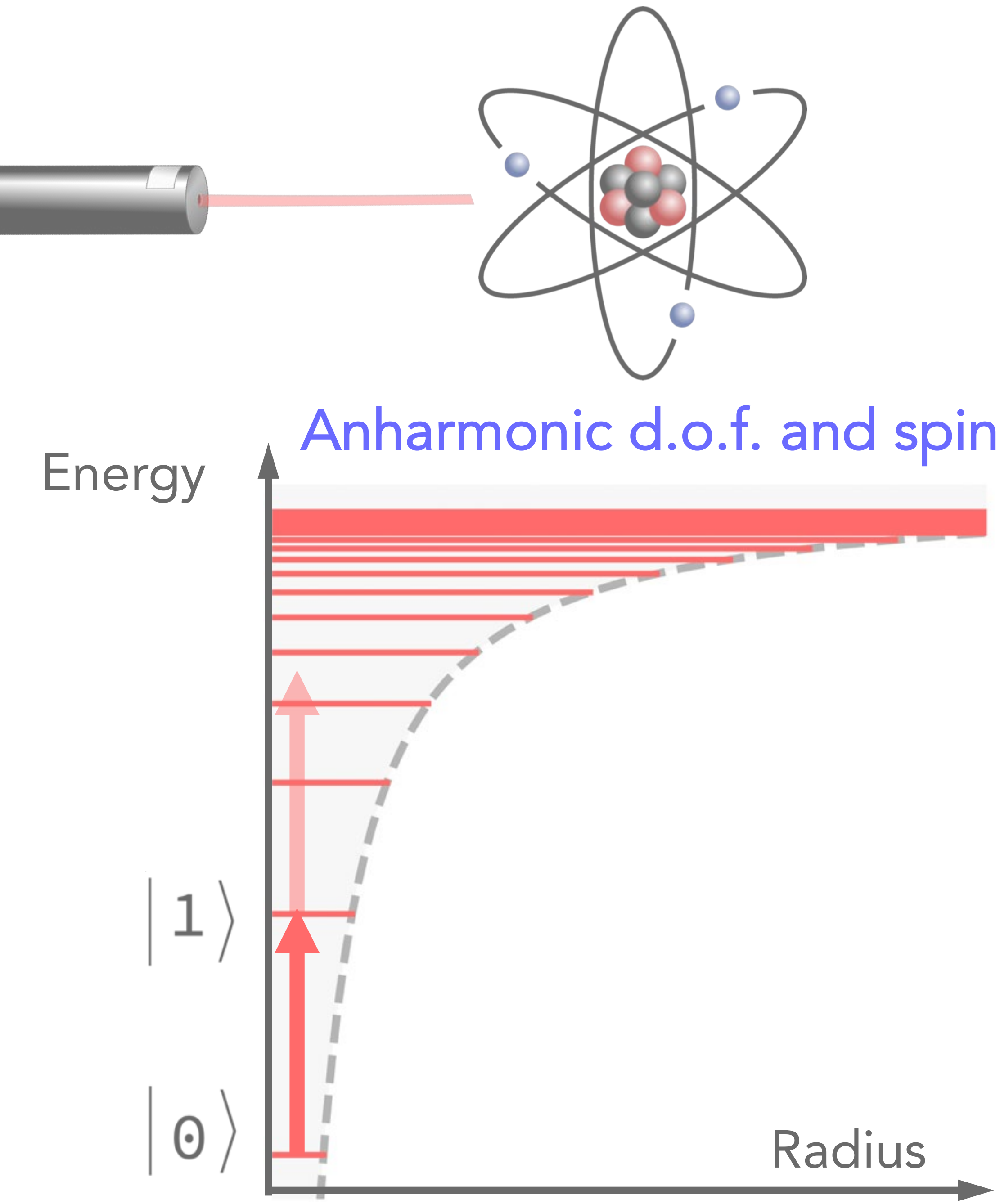
Level diagram allows for qubit-specific control and readout

There are always more than two levels

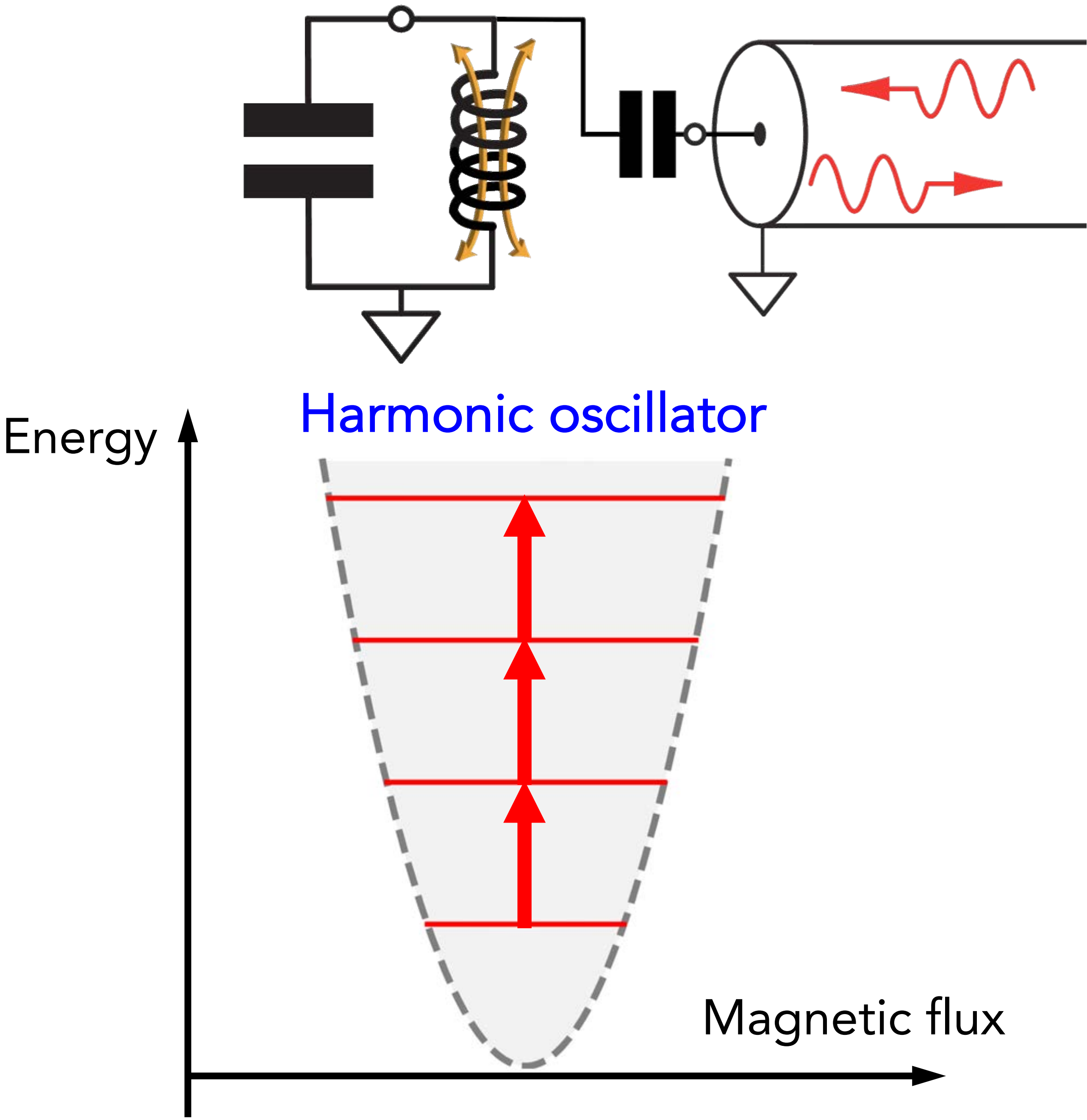
# Artificial atoms



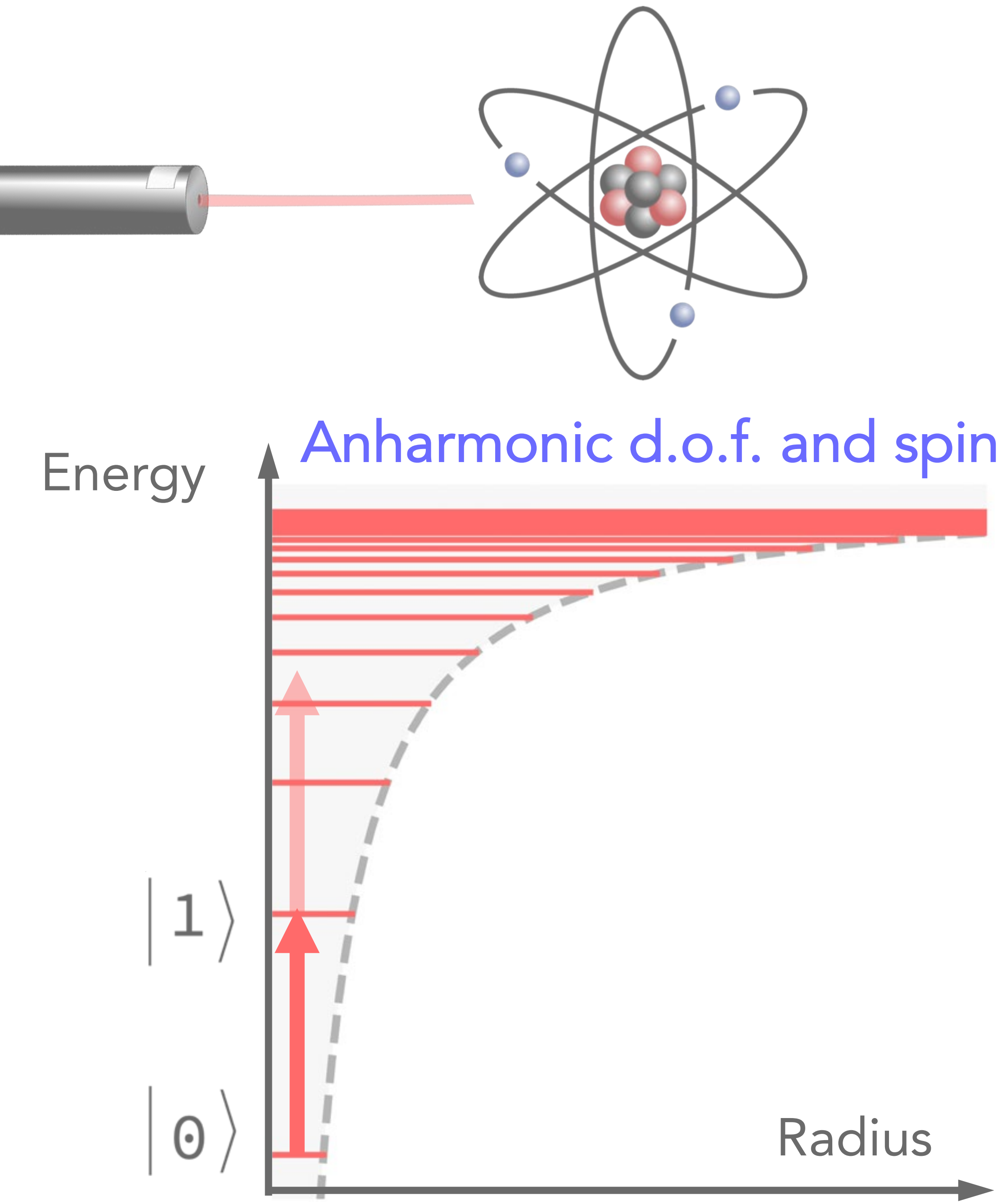


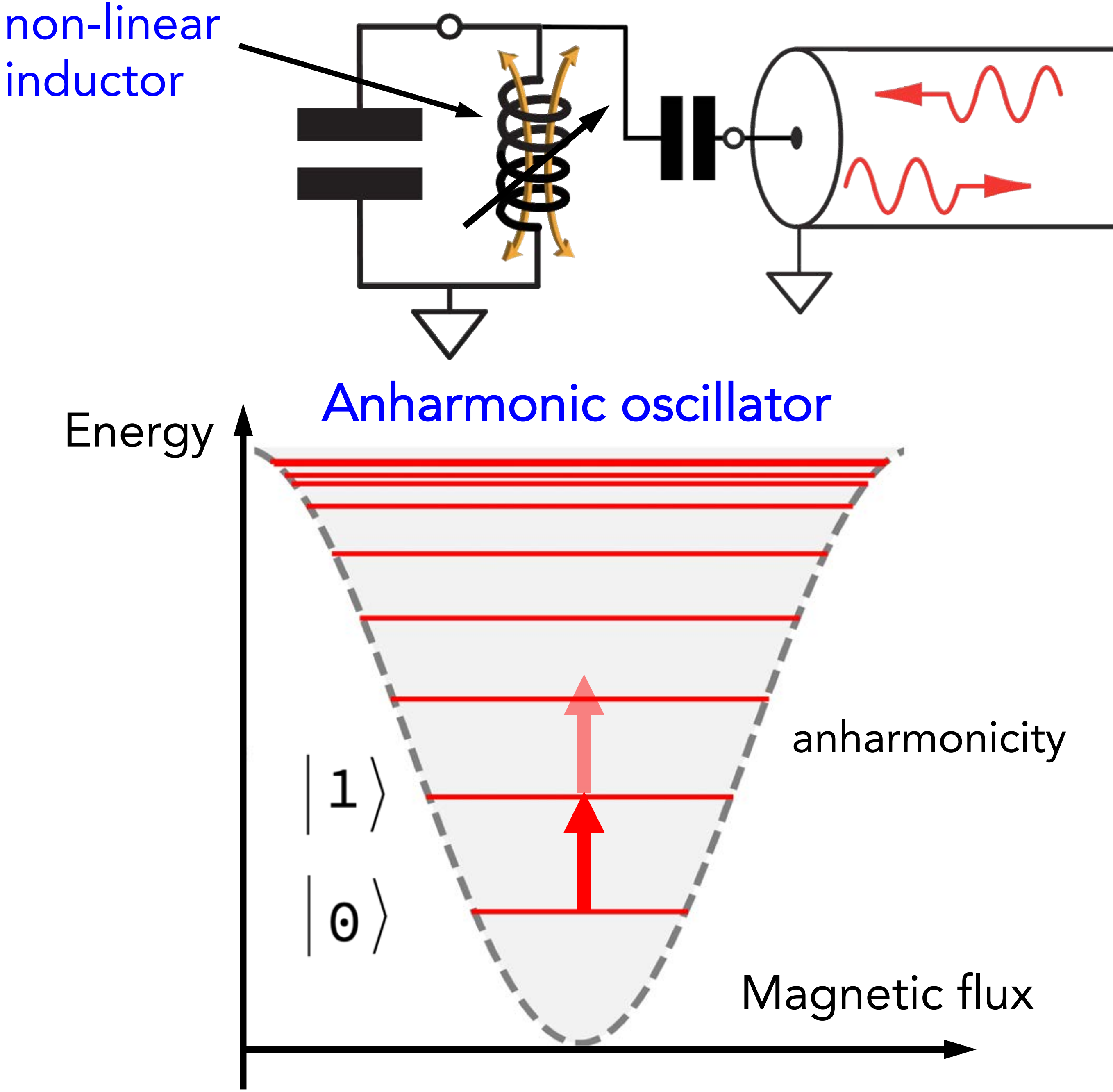


Anharmonic d.o.f. and spin
Energy
$|1\rangle$
$|0\rangle$
Radius

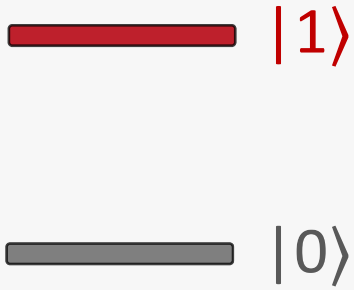
Harmonic oscillator
Energy
Magnetic flux


Anharmonic d.o.f. and spin
Energy
$|1\rangle$
$|0\rangle$
Radius

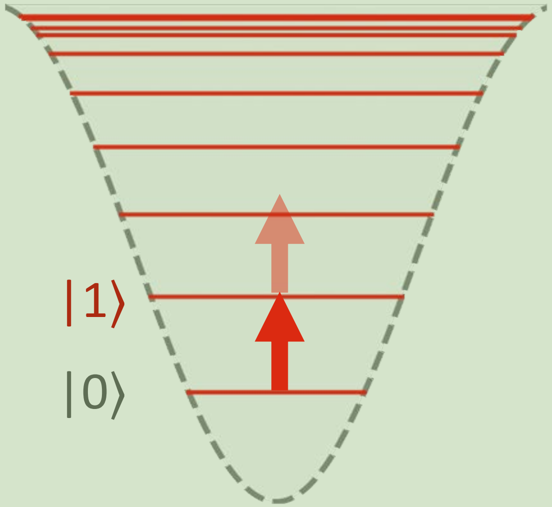

non-linear
inductor
Anharmonic oscillator
Energy
anharmonicity
$|1\rangle$
$|0\rangle$
Magnetic flux

# Big-picture connections

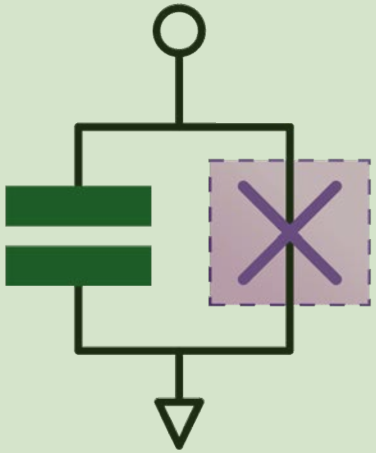
Idealization of *qubit*



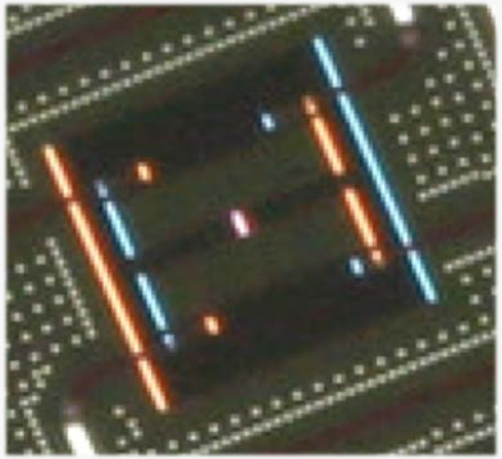

Anharmonic *oscillator*




Physical circuit model



Physical layout



Idealization → Physical reality

# Circuit Quantum Electrodynamics (cQED)

Macro
overview

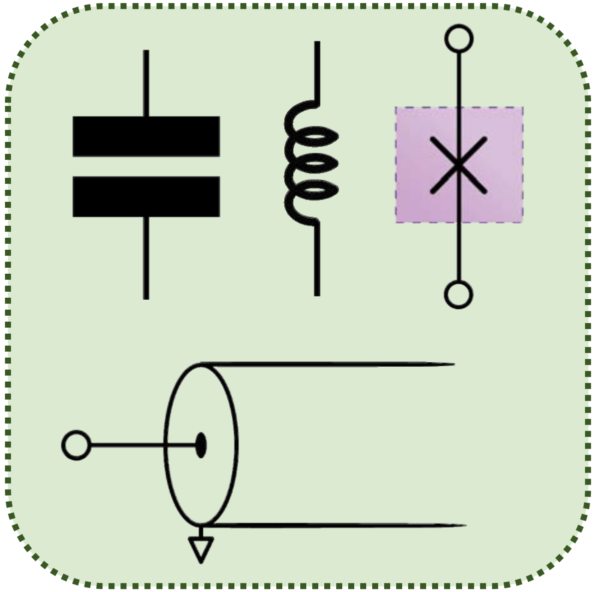
There are two kinds of physicists:

Those who believe all of physics is *spins*.

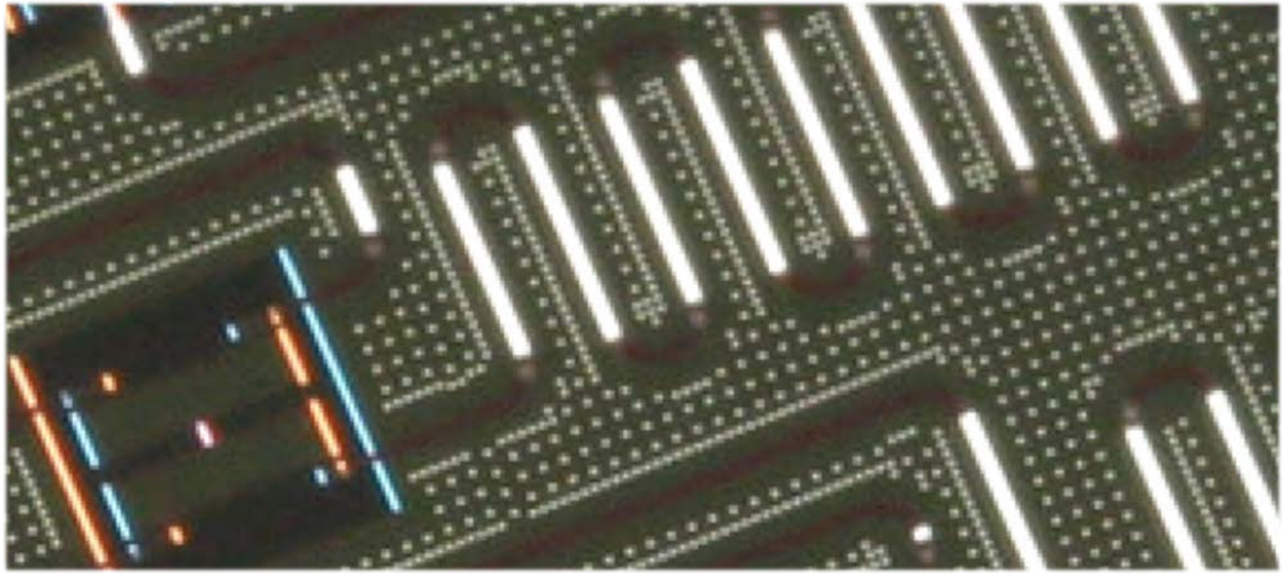
Those who believe all of physics is *oscillators*.

# cQED Ingredients

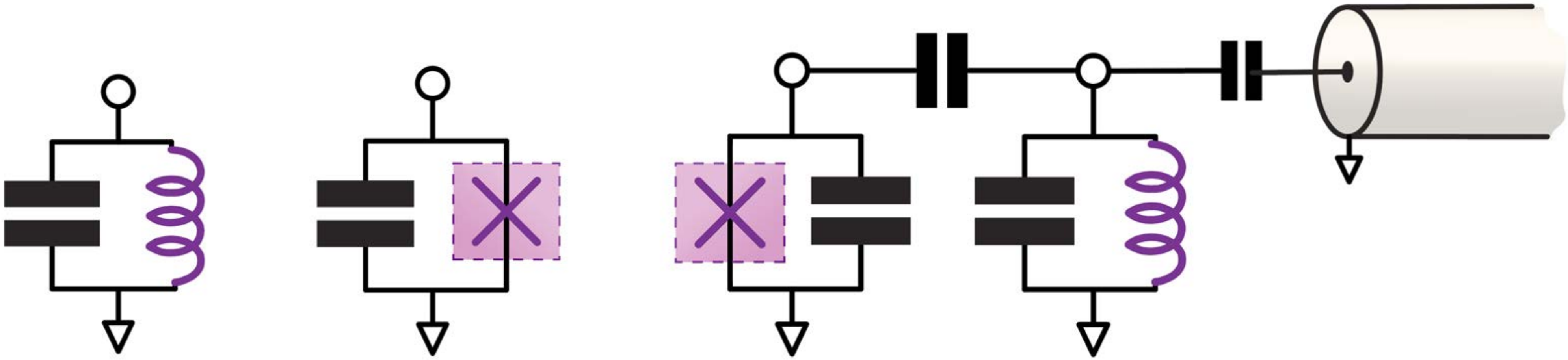
## Circuit elements



## Microwave oscillators

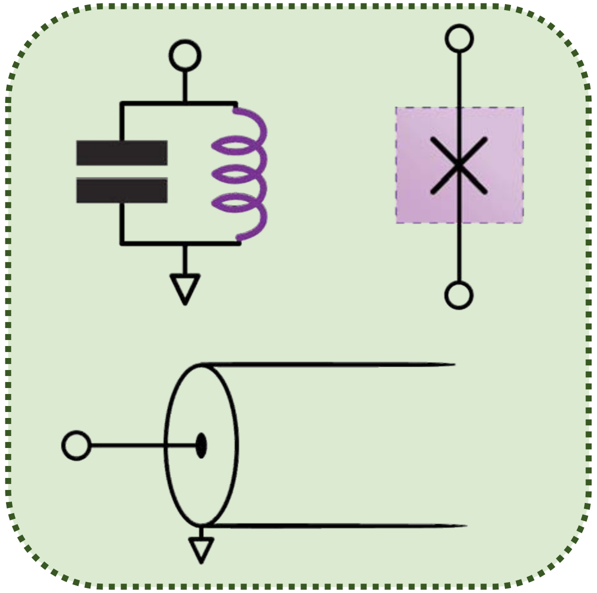


Combine



…

# cQED Ingredients



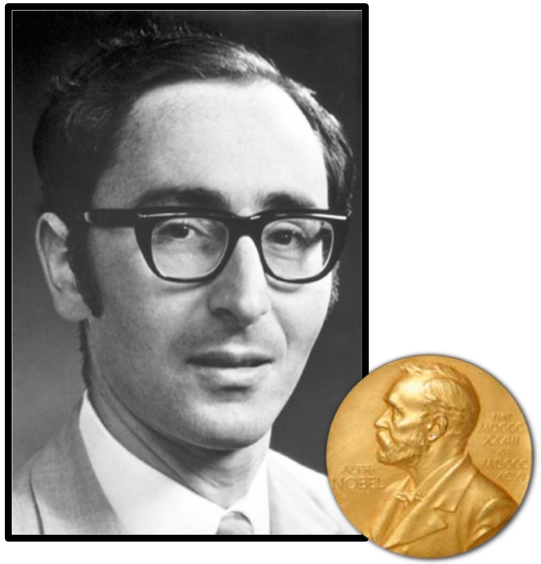

Small dissipation

Isolation from environment

Low temperature

Nonlinearity

Large vacuum fluctuations



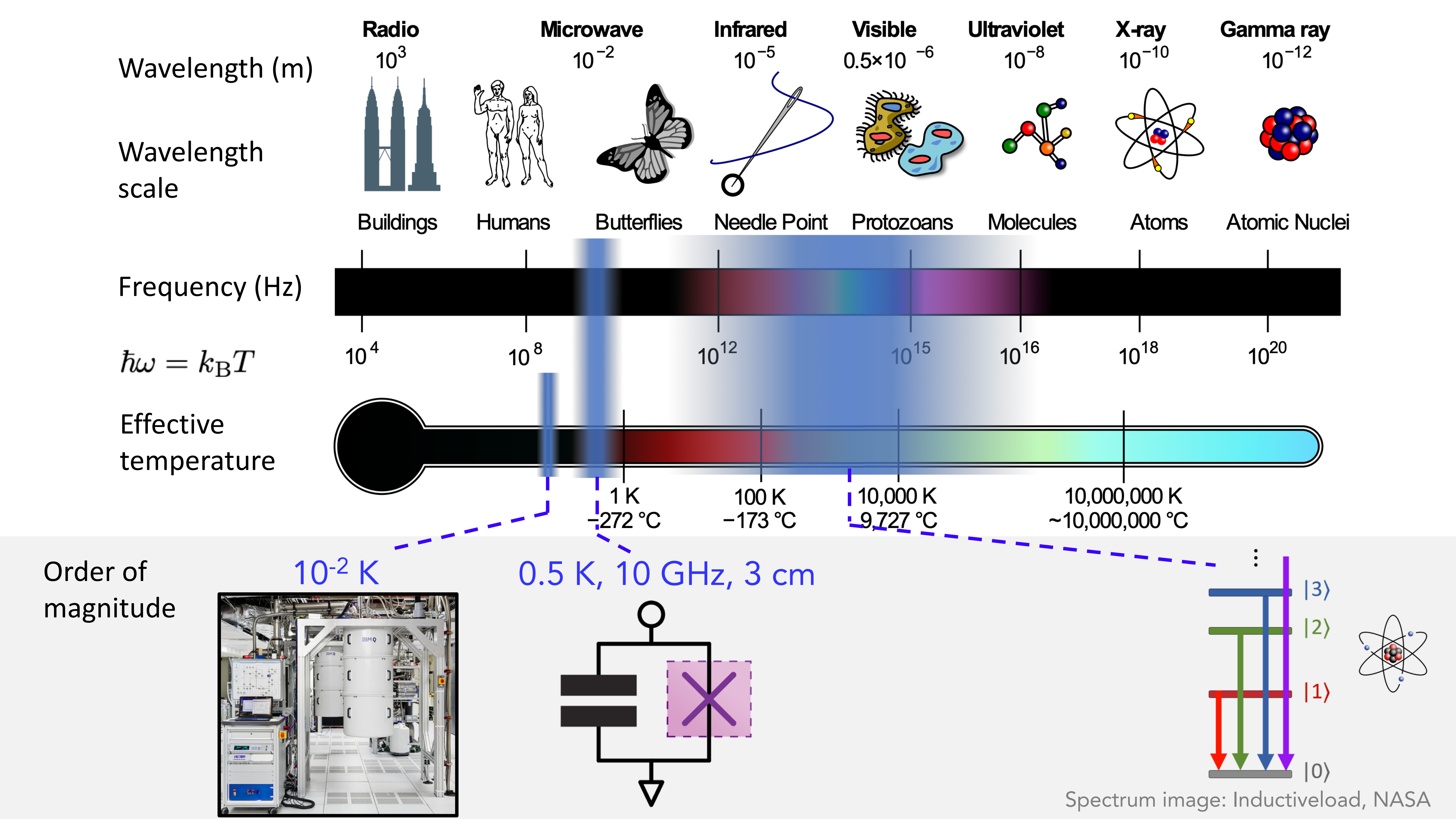
**Superconductivity**

Nominally zero intrinsic dissipation and heat

Nominal temperature far below energy level splitting

Non-linear, robust Josephson tunnel junction effect

* Low energy dynamics. Microwave image: vectorpocket; Nobel photograph: Jonathunder; Josephson photograph: Nobel Foundation archive.



Wavelength (m)
Radio
$10^3$
Microwave
$10^{-2}$
Infrared
$10^{-5}$
Visible
$0.5\times10^{-6}$
Ultraviolet
$10^{-8}$
X-ray
$10^{-10}$
Gamma ray
$10^{-12}$
Wavelength scale
Buildings
Humans
Butterflies
Needle Point
Protozoans
Molecules
Atoms
Atomic Nuclei
Frequency (Hz)
$10^4$
$10^8$
$10^{12}$
$10^{15}$
$10^{16}$
$10^{18}$
$10^{20}$
$\hbar\omega = k_B T$
Effective temperature
1 K
−272 °C
100 K
−173 °C
10,000 K
9,727 °C
10,000,000 K
~10,000,000 °C
Order of magnitude
$10^{-2}$ K
0.5 K, 10 GHz, 3 cm
IBM Q
|3⟩
|2⟩
|1⟩
|0⟩
Spectrum image: Inductiveload, NASA

# A few introductory reviews

And many more… check online or ask us for specific topic


Qiskit Textbook (2020; more chapters coming)

Blais, A., Grimsmo, A. L., Girvin, S. M., & Wallraff, A. (2020)
*Circuit Quantum Electrodynamics (arXiv:2005.12667)*

Kjaergaard, M., Schwartz, … Oliver, W. D. *(2020)*
*Superconducting Qubits: Current State of Play*
*Annual Reviews of Condensed Matter Physics 11, 369-395*

Krantz, P., Kjaergaard, M., Yan, F., … & Oliver, W. D. (2019)
A quantum engineer's guide to superconducting qubits
*Applied Physics Reviews*, *6*(2), 021318

Corcoles, A. D., Kandala, A., … Gambetta, J. M. (2019)
Challenges and Opportunities of Near-Term Quantum Computing Systems. *Proceedings of the IEEE*, 1–15.

Wendin, G. (2017)
Quantum information processing with superconducting circuits. *Reports on Progress in Physics*, *80*(10), 106001

Gambetta, J. M., Chow, J. M., & Steffen, M. (2017)
Building logical qubits in a superconducting quantum computing system. *Npj Quantum Information*, *3*(1), 2

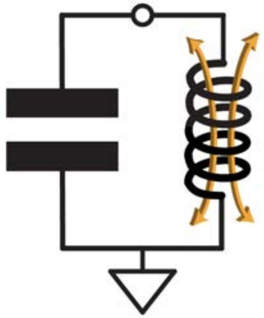
Girvin, S. M. (2011) Circuit QED: superconducting qubits coupled to microwave photons. *Quantum machines: measurement and control of engineered quantum systems*, *113*, 2.

Clerk, A. A., Girvin, S. M., Marquardt, F., & Schoelkopf, R. J. (2010)
Introduction to quantum noise, measurement, and amplification
*Reviews of Modern Physics*, *82*(2), 1155–1208

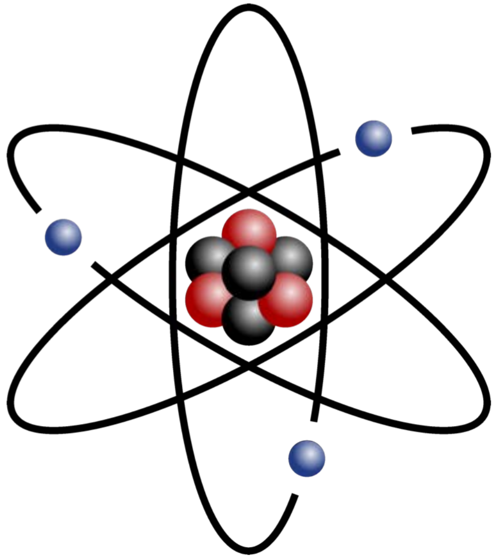
Clarke, J., & Wilhelm, F. K. (2008)
Superconducting quantum bits. *Nature*, *453*(7198), 1031–1042

Devoret, M. H. (1997)
Quantum Fluctuations in Electrical Circuits.
In *Fluctuations Quantiques/Quantum Fluctuations* (p. 351)

…



# Circuit Quantum Electrodynamics

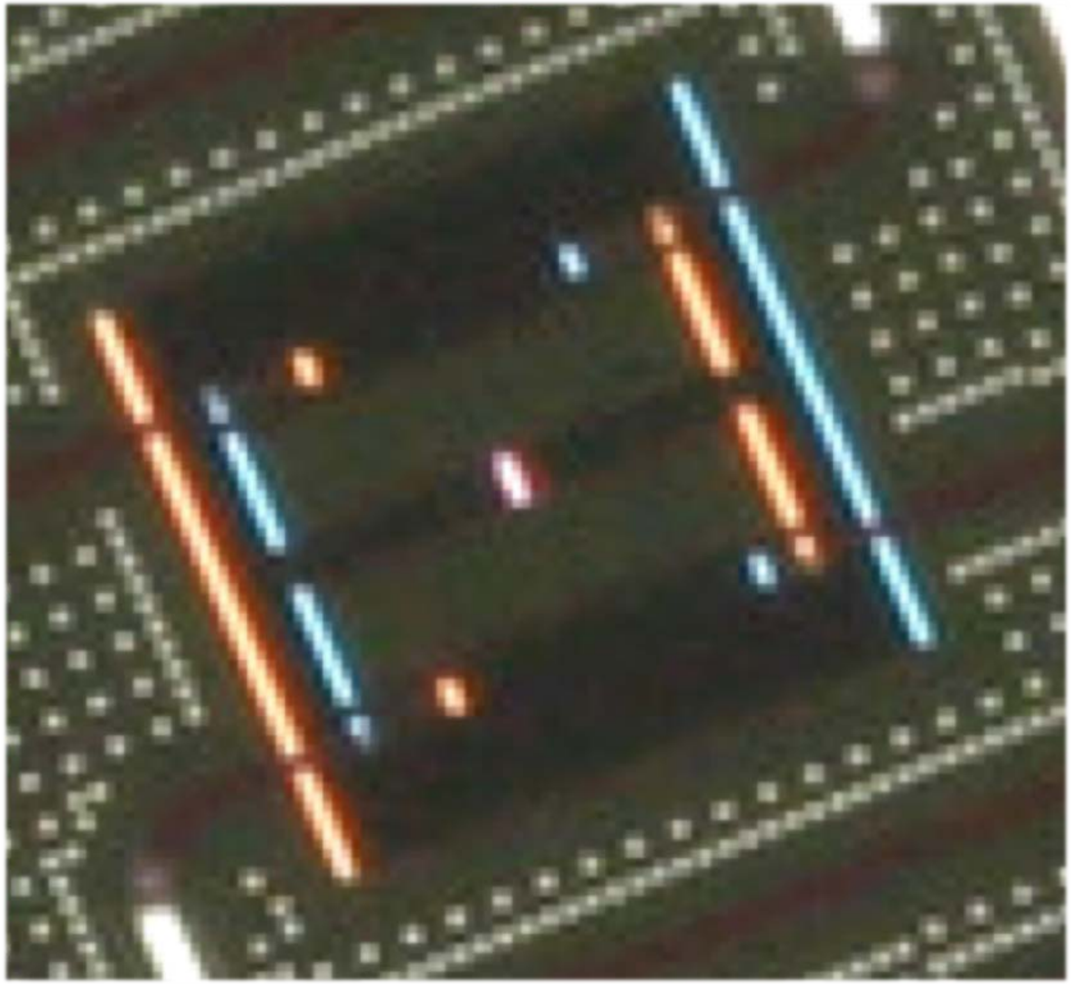


To do **quantum**,

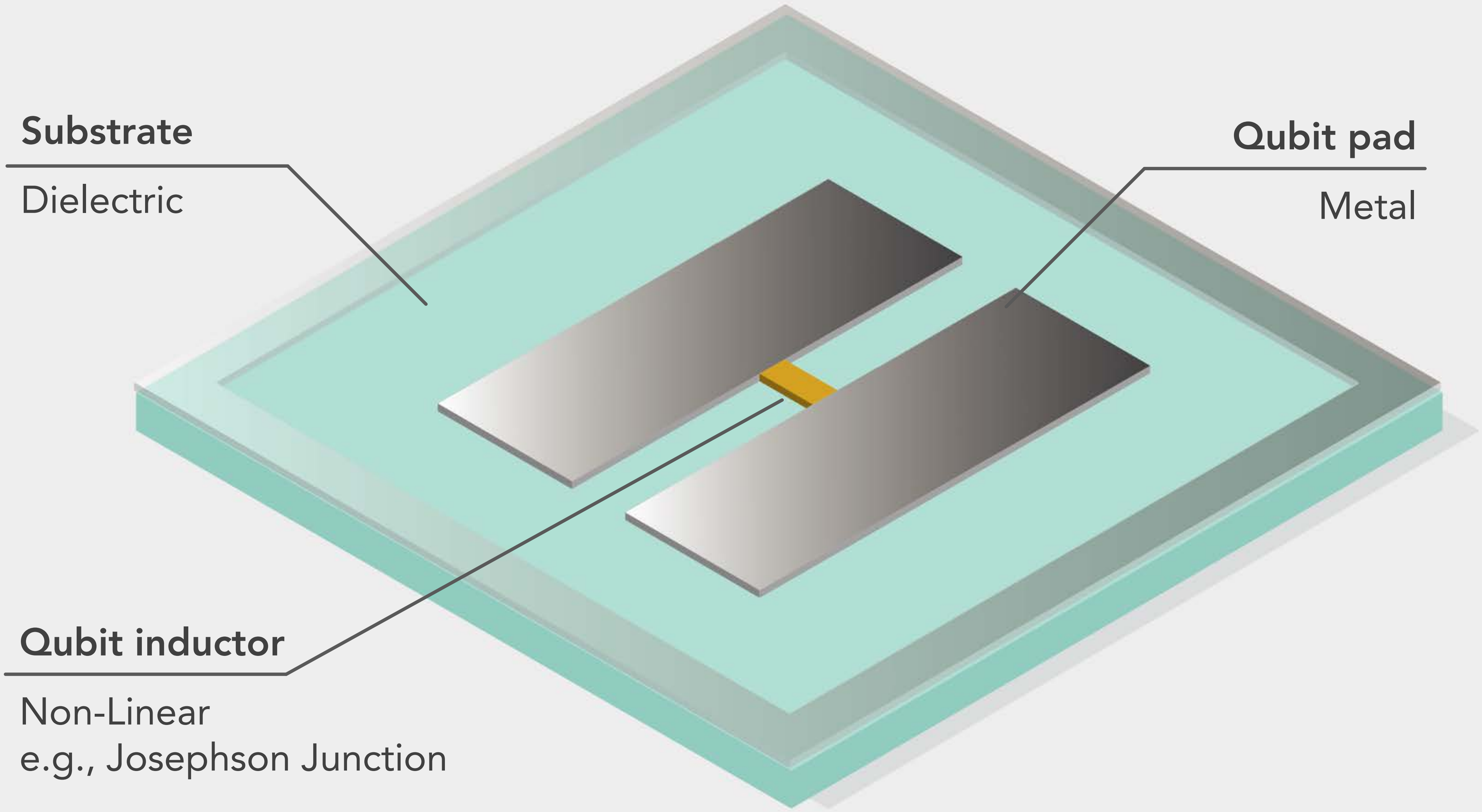
start with **classical**

Classical oscillator

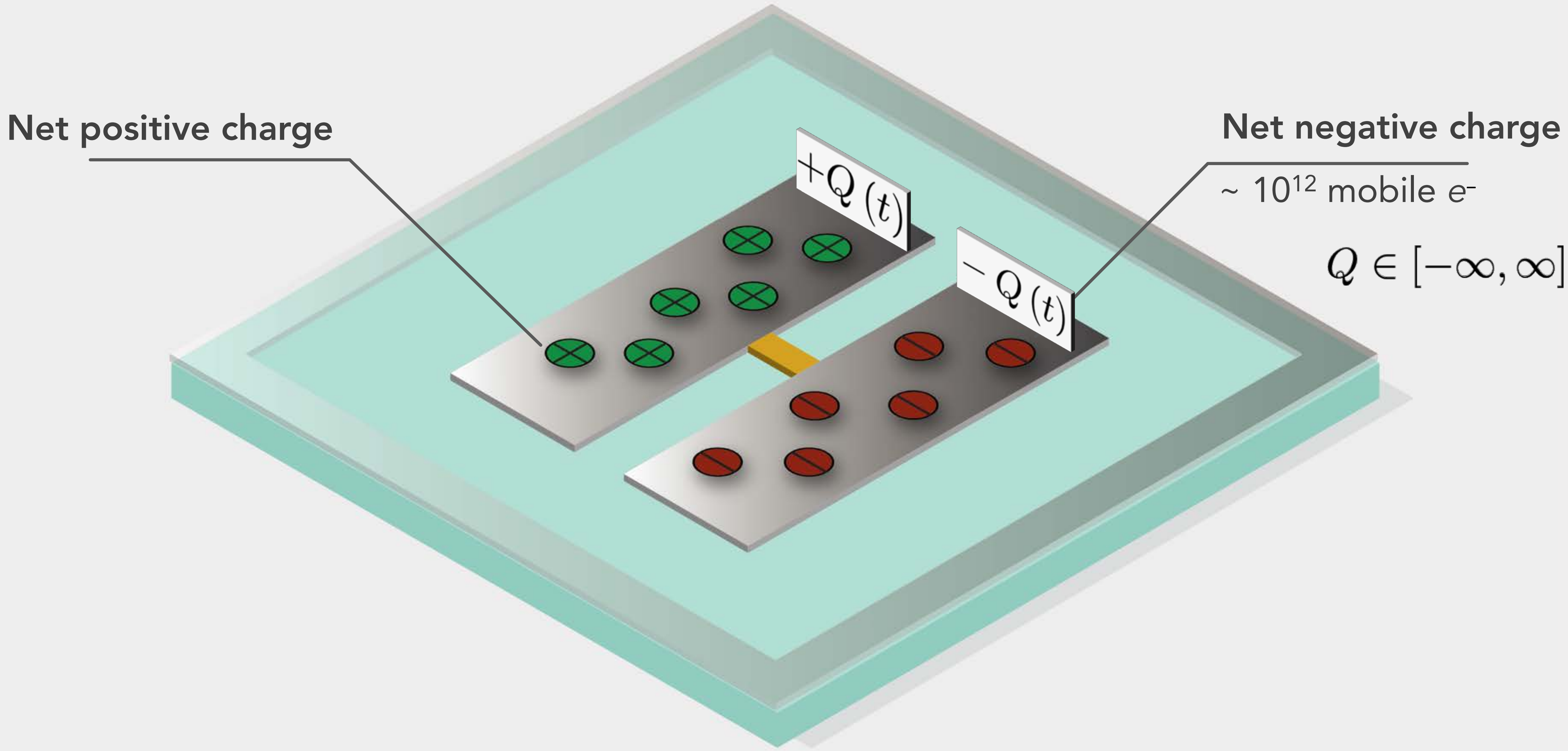
# Transmon qubit



# Transmon qubit

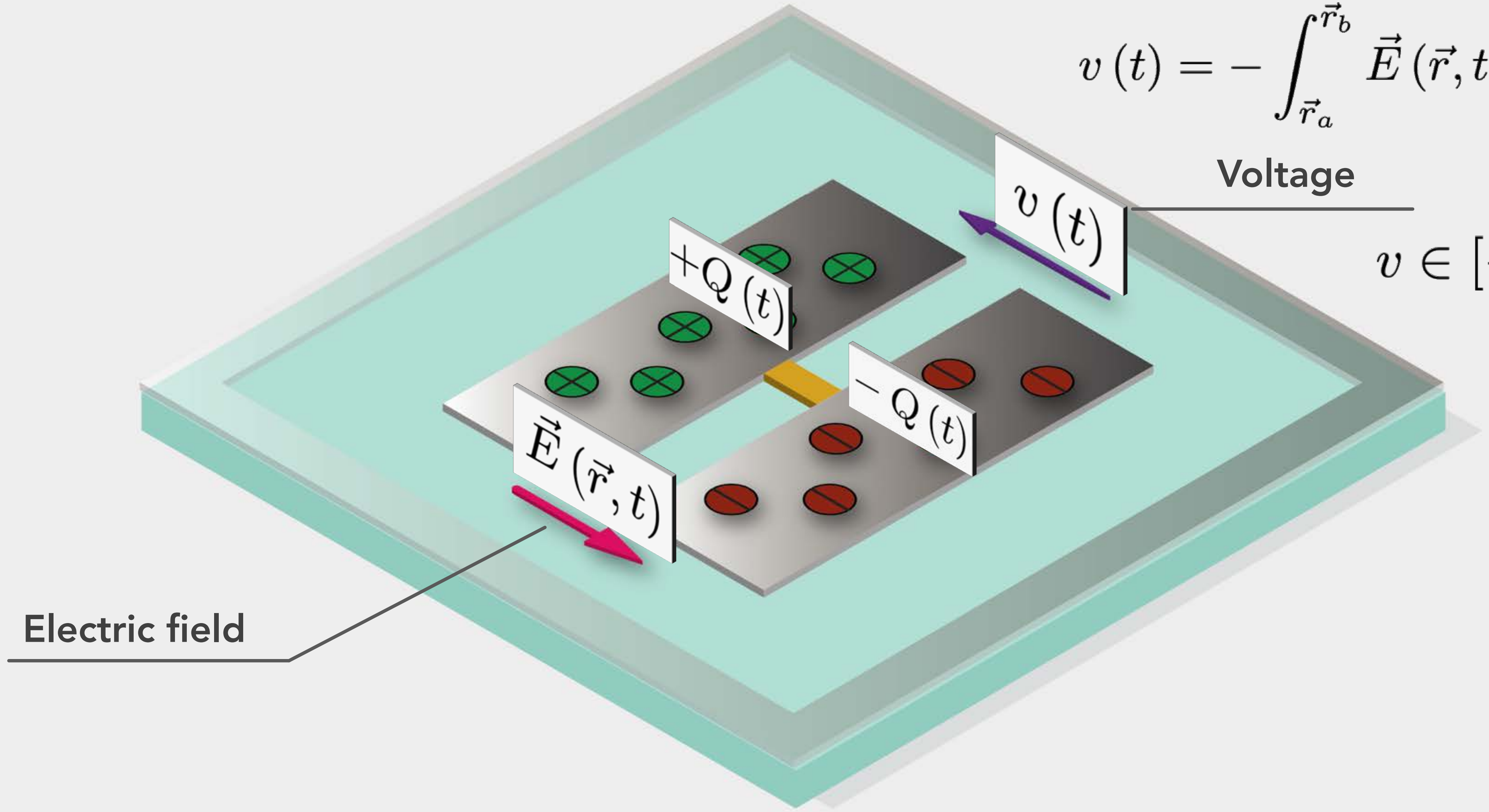


# Transmon qubit: charge



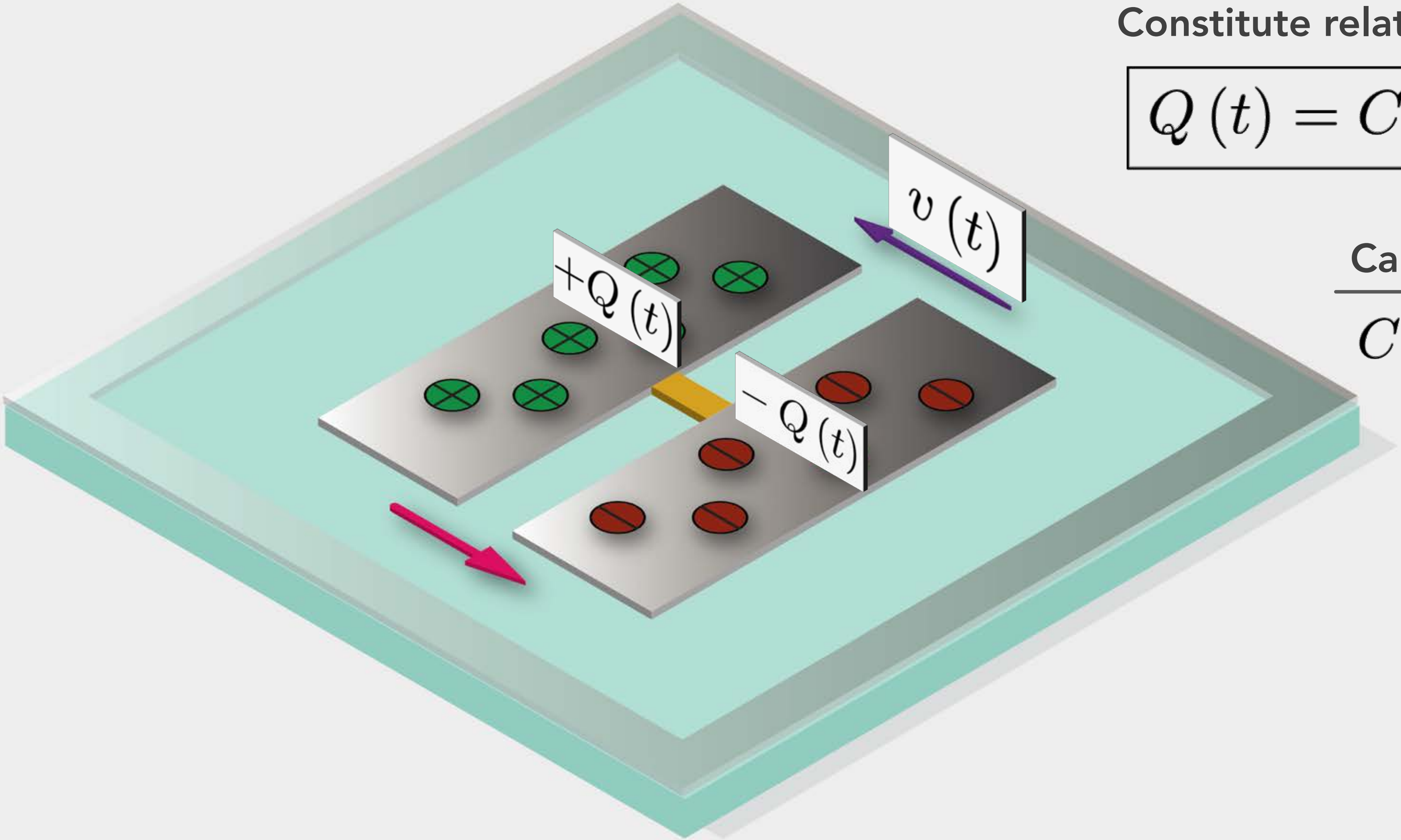
# Electric field and voltage

$$v(t) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E}(\vec{r}, t) \cdot \mathrm{d}\vec{l}(\vec{r})$$




$$v \in [-\infty, \infty]$$

# Charge and capacitance




**Constitute relationship**

$$\boxed{Q(t) = Cv(t)}$$

**Capacitance**

$$C \in (0, \infty)$$

For a good discussion, see "*The Feynman Lectures on Physics Vol. II Ch. 22: AC Circuits.*" Caltech.

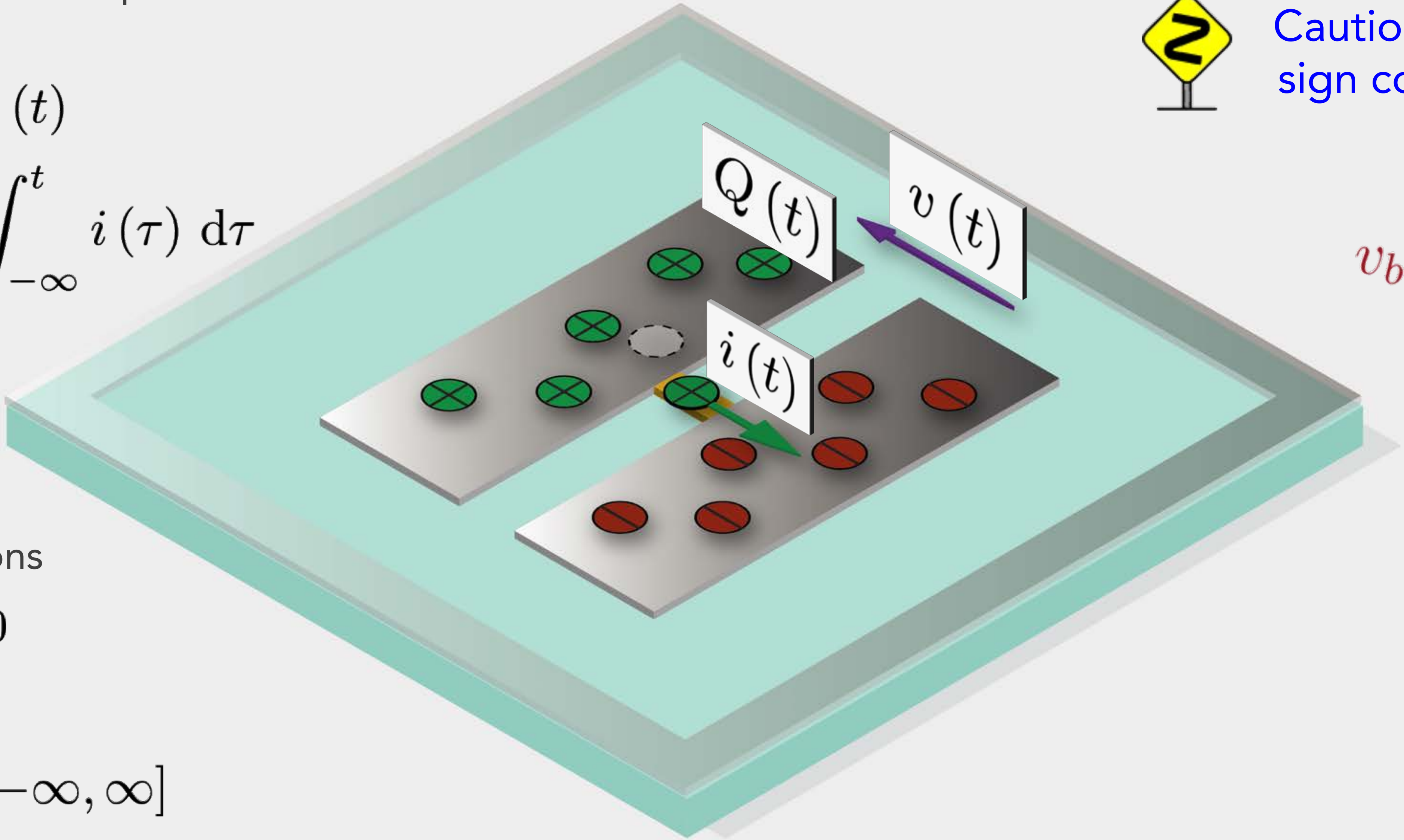
# Charge and current

**Conservation of charge**

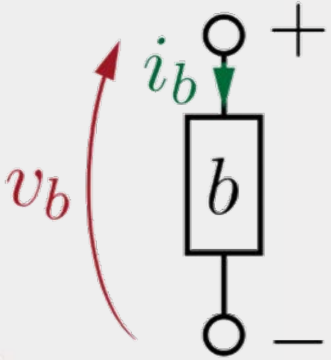
Universal relationship

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t) = i(t)$$

$$Q(t) = \int_{-\infty}^{t} i(\tau)\,\mathrm{d}\tau$$




Caution: Passive sign convention




Initial conditions

$$Q(-\infty) = 0$$

$$i \in [-\infty, \infty]$$

# Magnetic flux and inductance

**Faraday's law of induction**

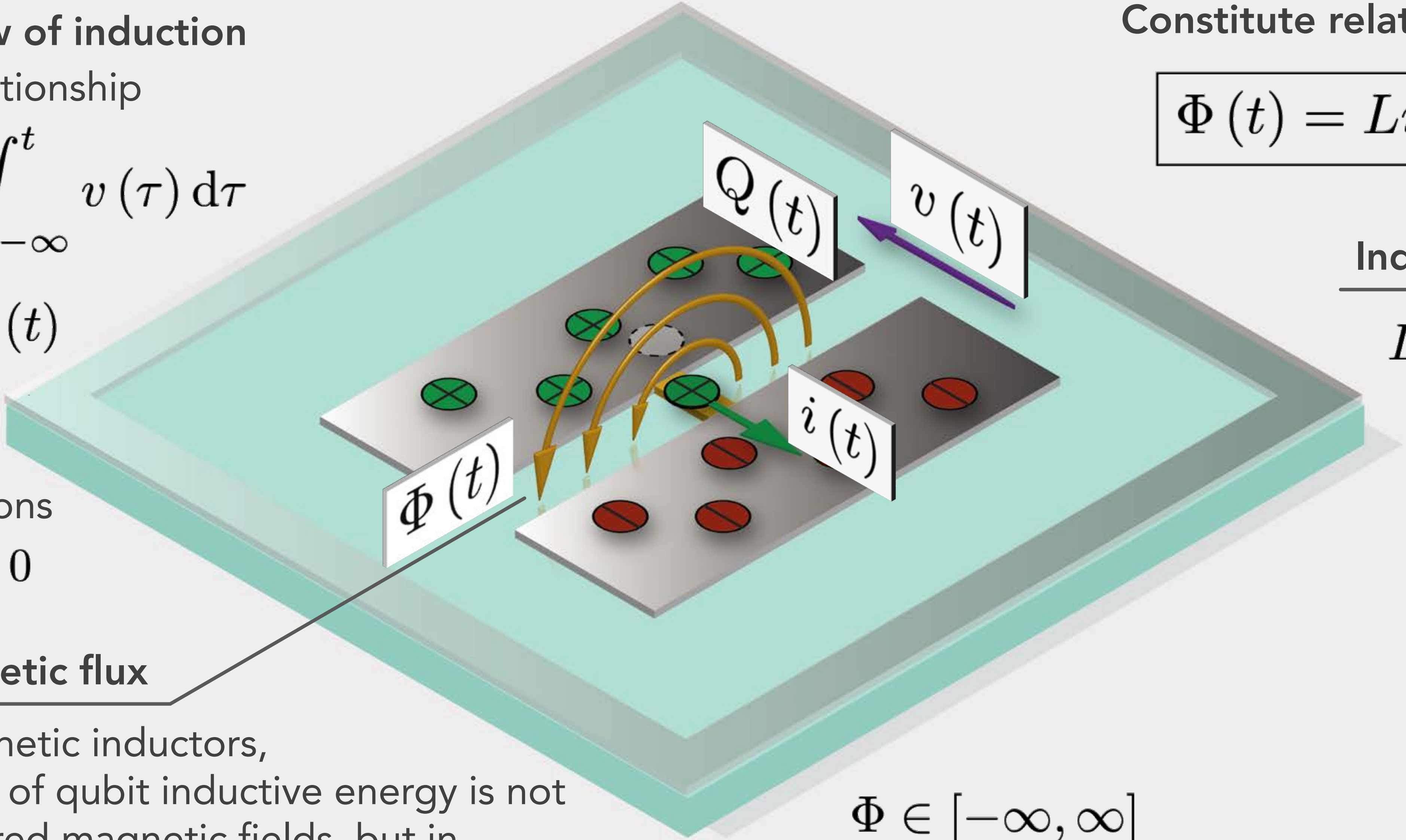
Universal relationship

$$\Phi\left(t\right) = \int_{-\infty}^{t} v\left(\tau\right) \mathrm{d}\tau$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(t\right) = v\left(t\right)$$

Initial conditions

$$\Phi\left(-\infty\right) = 0$$




**Magnetic flux**

For kinetic inductors,
~98% of qubit inductive energy is not in stored magnetic fields, but in kinetic inductance

$$\Phi \in \left[-\infty, \infty\right]$$

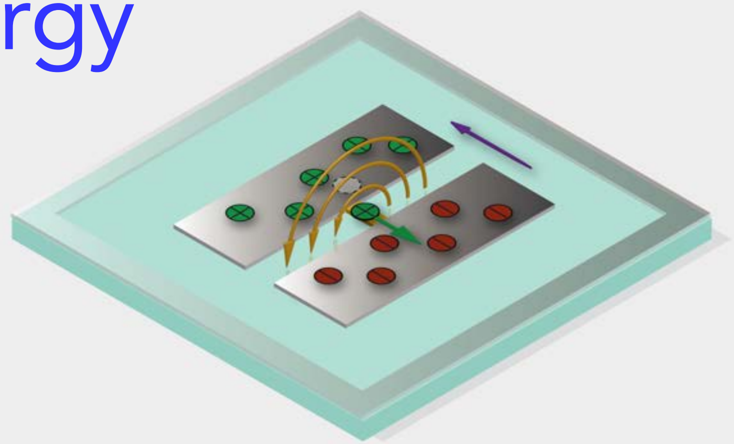
**Constitute relationship**

$$\boxed{\Phi\left(t\right) = Li\left(t\right)}$$

**Inductance**

$$L \in \left(0, \infty\right)$$

# Power and energy



## Universal

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = p(t) \equiv v(t)\, i(t)$$

Energy stored *in* (delivered *to*) component

Instantaneous power flowing *to* component

$$\mathcal{E}(t) = \int_{-\infty}^{t} p(\tau)\, \mathrm{d}\tau$$

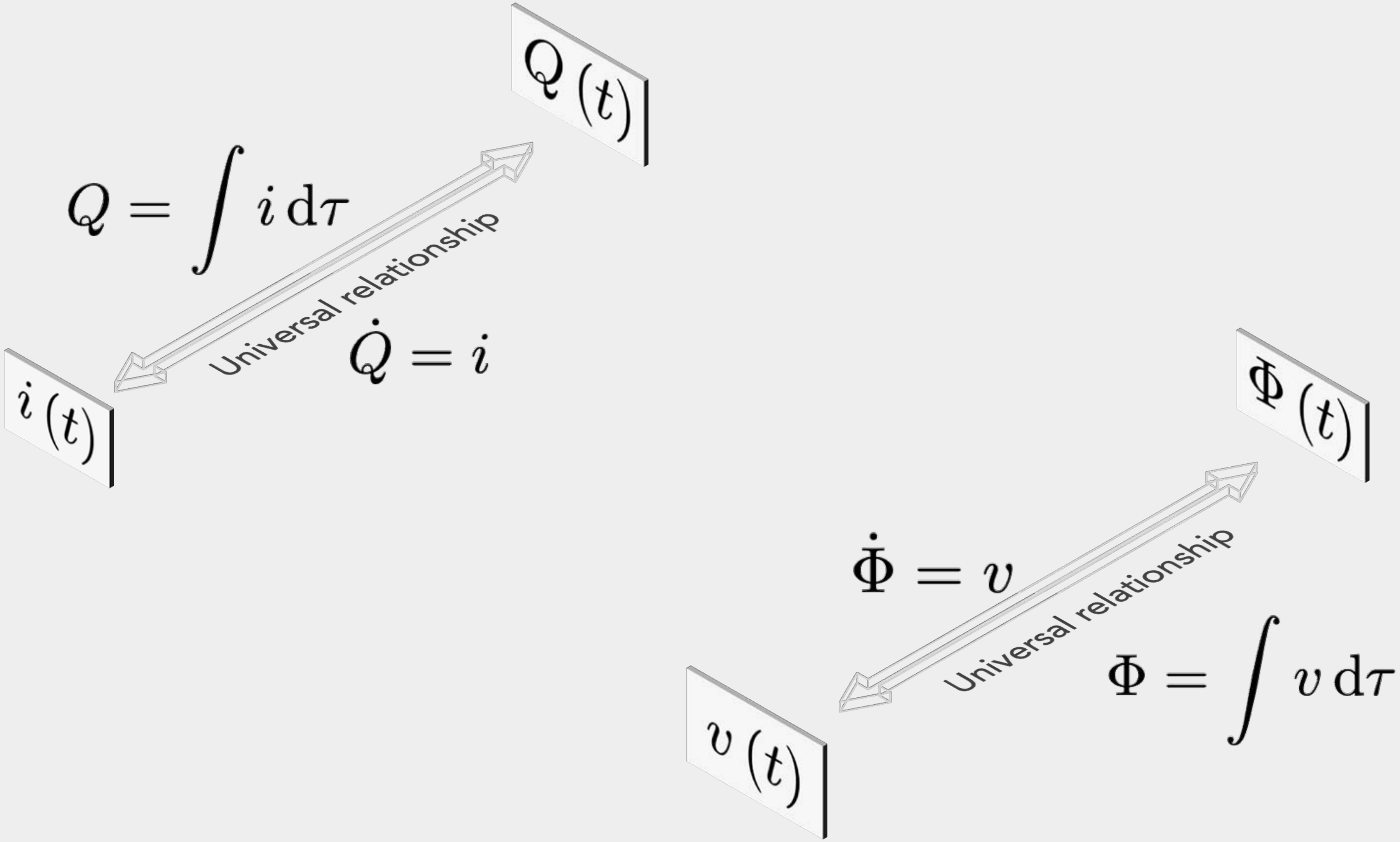
$$\mathcal{E}_{\mathrm{cap}}\left(\dot{\Phi}\right) = \frac{1}{2}C\dot{\Phi}^2$$

$$\mathcal{E}_{\mathrm{ind}}\left(\Phi\right) = \frac{\Phi^2}{2L}$$

$$\mathcal{E}_{\mathrm{cap}}\left(Q\right) = \frac{Q^2}{2C}$$

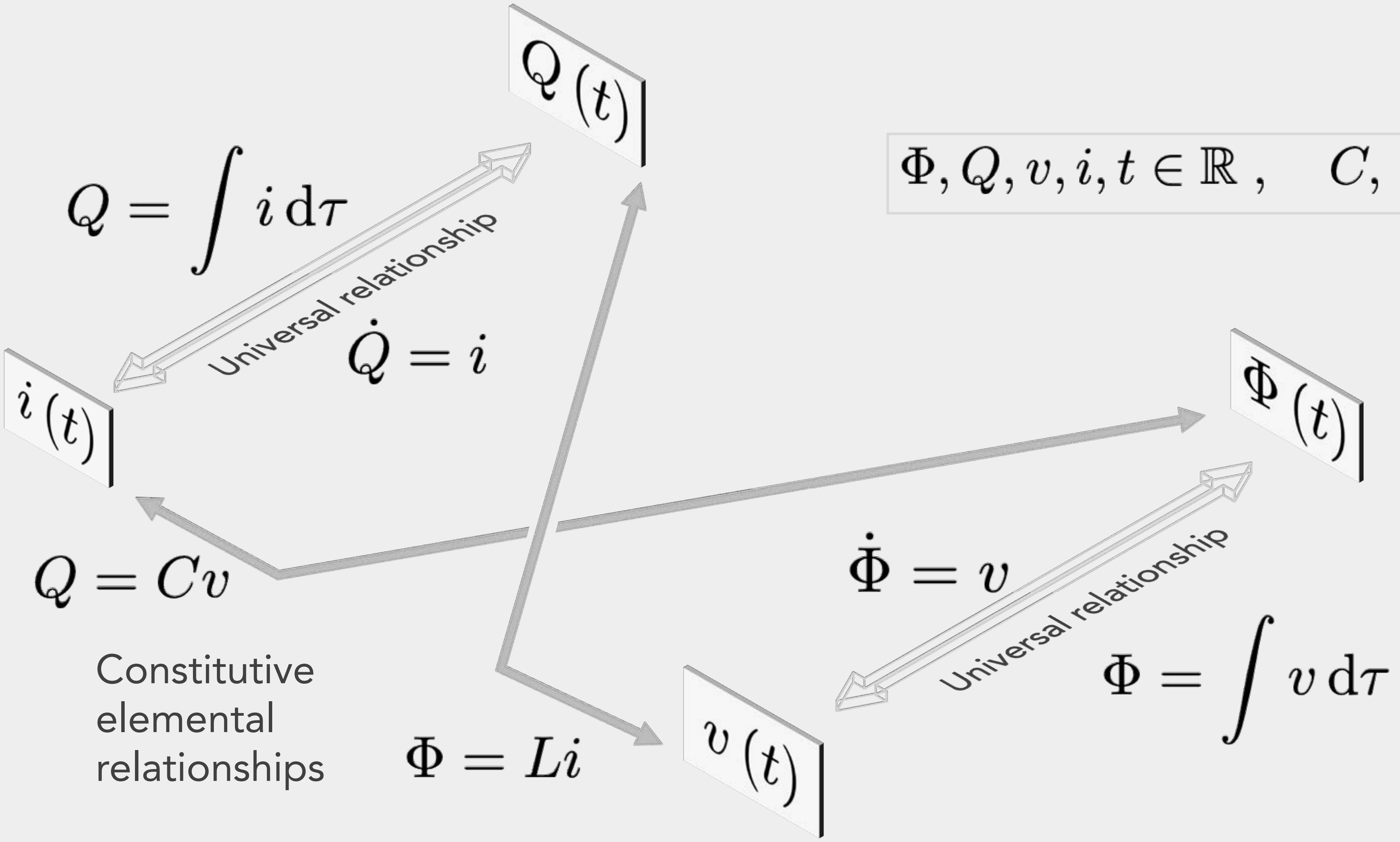
$$\mathcal{E}_{\mathrm{ind}}\left(\dot{Q}\right) = \frac{1}{2}L\dot{Q}^2$$

# Four fundamental manifestations of electricity

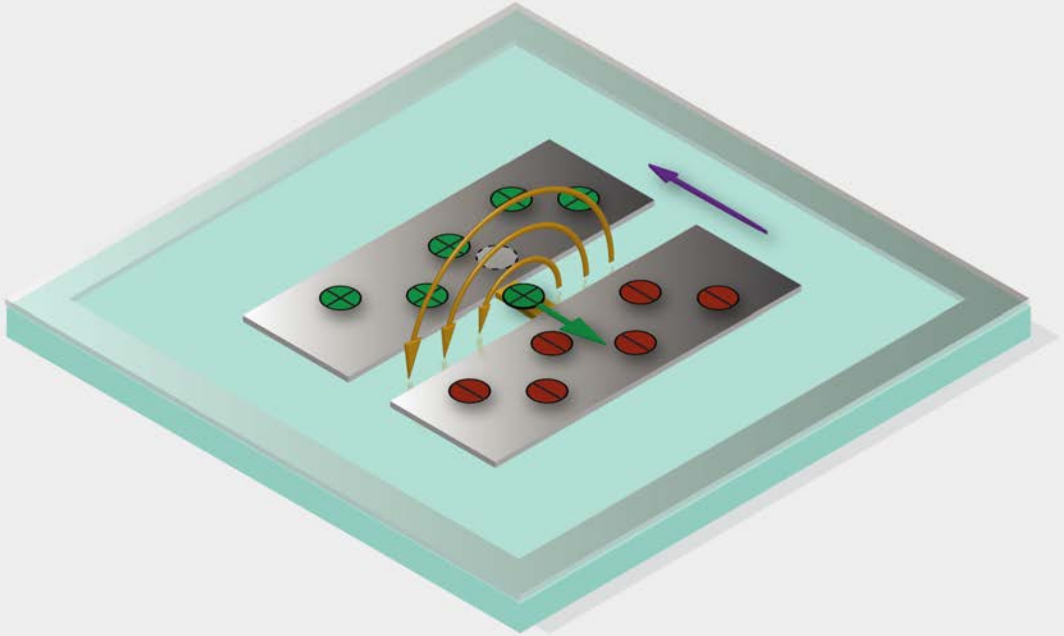


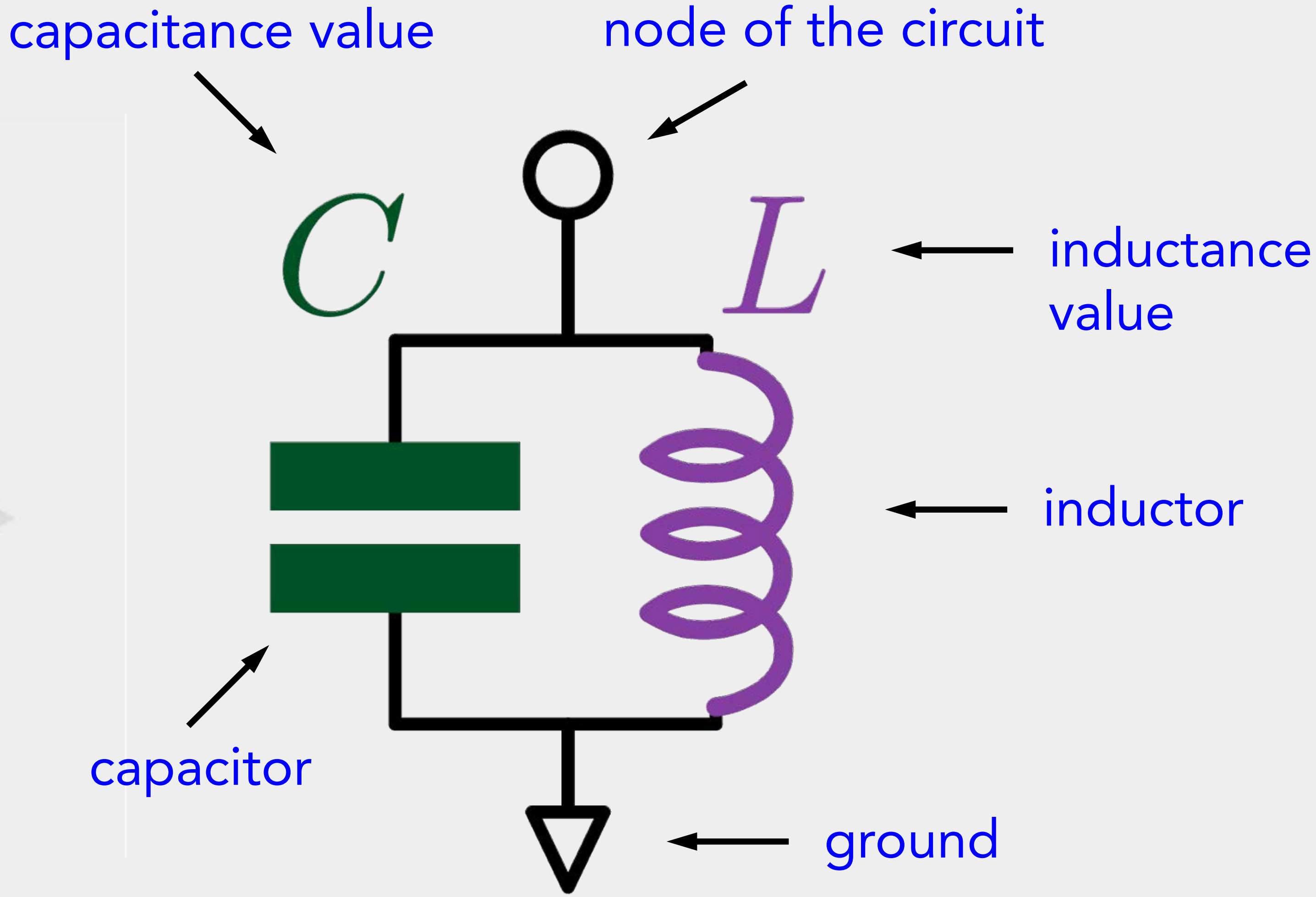
# Four fundamental manifestations of electricity

$$\Phi, Q, v, i, t \in \mathbb{R}\,, \quad C,\, L \in \mathbb{R}_{>0}$$



# Electromagnetic oscillator

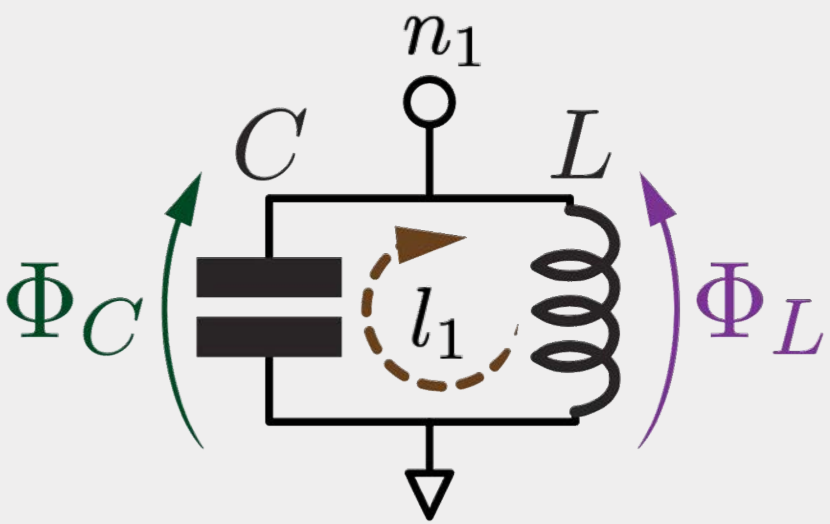




# Kirchhoff's network laws*

Conservation of charge
Kirchhoff's current law

$$\sum_{b \in \mathrm{node}} \pm \dot{Q}_b(t) = 0$$




Faraday's law of induction
Kirchhoff's voltage law

$$\sum_{b \in \mathrm{loop}} \pm \dot{\Phi}_b(t) = 0$$

$$n_1: \quad \dot{Q}_C + \dot{Q}_L = 0$$

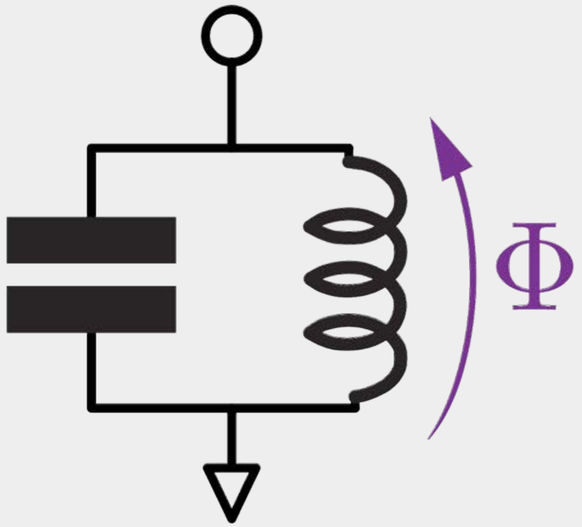
$$C\ddot{\Phi} + L^{-1}\Phi = 0$$

$$l_1: \quad \dot{\Phi}_C - \dot{\Phi}_L = 0$$

$$\Phi_C = \Phi_L$$

As we will see later, for the Lagrangian description in flux basis, KVL acts as a set of holonomic constraints and KCL as the equations of motion

# Oscillator analogy




$$C\ddot{\Phi} + L^{-1}\Phi = 0$$

$$\ddot{\Phi} = -\omega_0^2 \Phi \,, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Resonance frequency

$\dot{\Phi}(t)$ Velocity

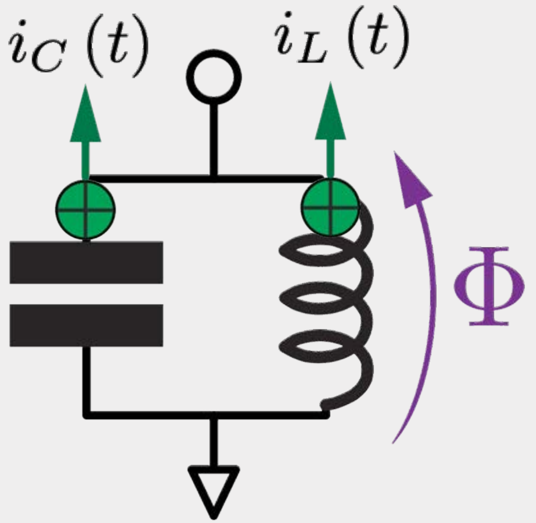
$\Phi(t)$ Deviation from equilibrium

Equilibrium position

$x = 0 \,, \quad \Phi = 0$

Spring: Svjo

# Lagrangian and Hamiltonian*




$$\underbrace{C\ddot{\Phi}}_{i_C(t)} + \underbrace{L^{-1}\Phi}_{i_L(t)} = 0$$

$$\mathcal{L}\left(\Phi, \dot{\Phi}\right) = \mathcal{E}_{\text{cap}}\left(\dot{\Phi}\right) - \mathcal{E}_{\text{ind}}\left(\Phi\right)$$

Kinetic energy – Potential energy

$$= \frac{1}{2}C\dot{\Phi}^2 - \frac{\Phi^2}{2L}$$

$$Q = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = C\dot{\Phi}$$

Canonically conjugate variable (momentum; charge)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{\partial \mathcal{L}}{\partial \Phi}$$

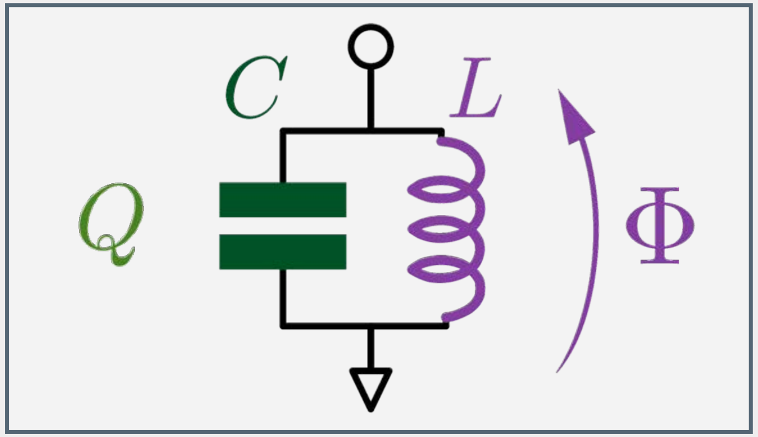
Euler-Lagrange equations, $F = ma$

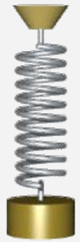
$$\mathcal{H}\left(\Phi, Q\right) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

Kinetic energy + Potential energy

* Temporarily going to assume some minimal knowledge of classical mechanics.
Since we eliminated the KVL constraints, this is a Lagrangian of the second kind.

# The LC classical harmonic oscillator

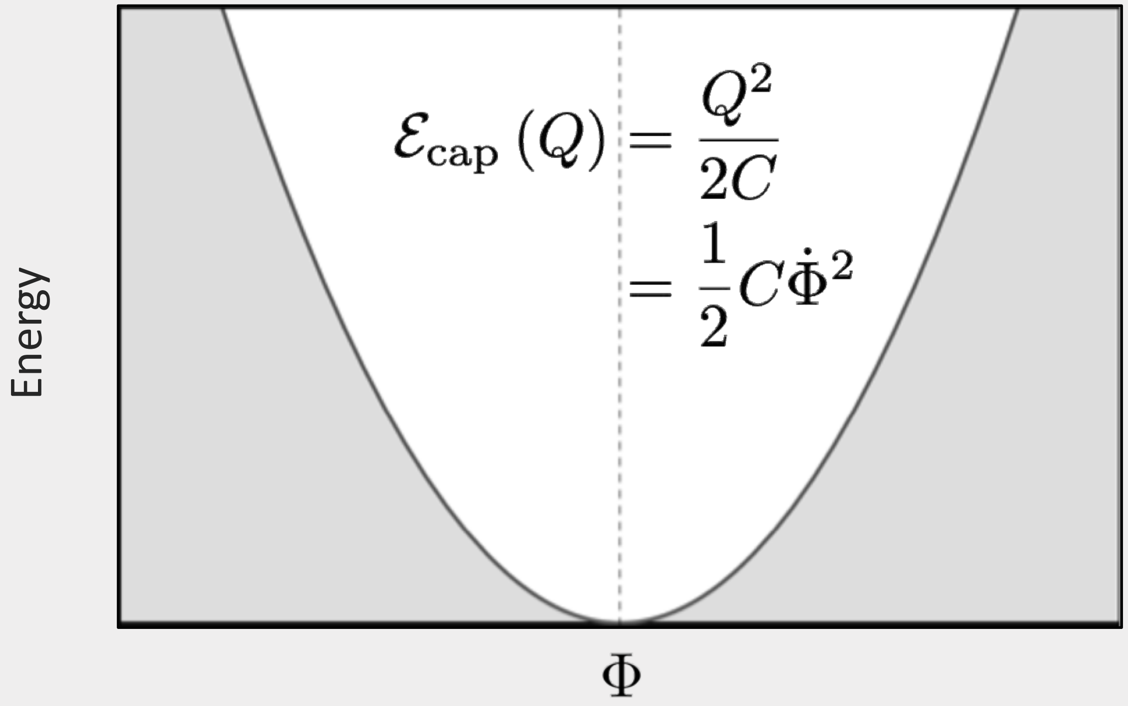


Position: $\Phi \mapsto x$ Momentum: $Q \mapsto p$

Mass: $C \mapsto m$ Spring constant: $L^{-1} \mapsto k$

$$\mathcal{E}_{\mathrm{ind}}\left(\Phi\right) = \frac{\Phi^2}{2L}$$




$$\mathcal{E}_{\mathrm{cap}}\left(Q\right) = \frac{Q^2}{2C} = \frac{1}{2}C\dot{\Phi}^2$$

$$\mathcal{H}\left(\Phi, Q\right) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$\omega_0^2 = \frac{1}{LC} \qquad Z_0 = \frac{L}{C}$$

Graph: Minev; Spring-mass: Svjo

"It is by *logic* that we prove,
but by *intuition* that we discover."

Henri Poincaré

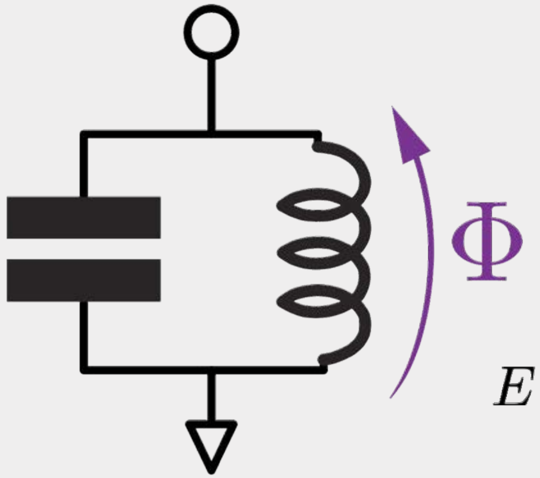
Photo by Eugène Pirou

# Hamiltonian dynamics and phase space

$$\mathcal{H}(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

$$E = \hbar\omega_0\left(n + \frac{1}{2}\right) = \frac{1}{2}\left(Q^2 + \Phi^2\right)$$

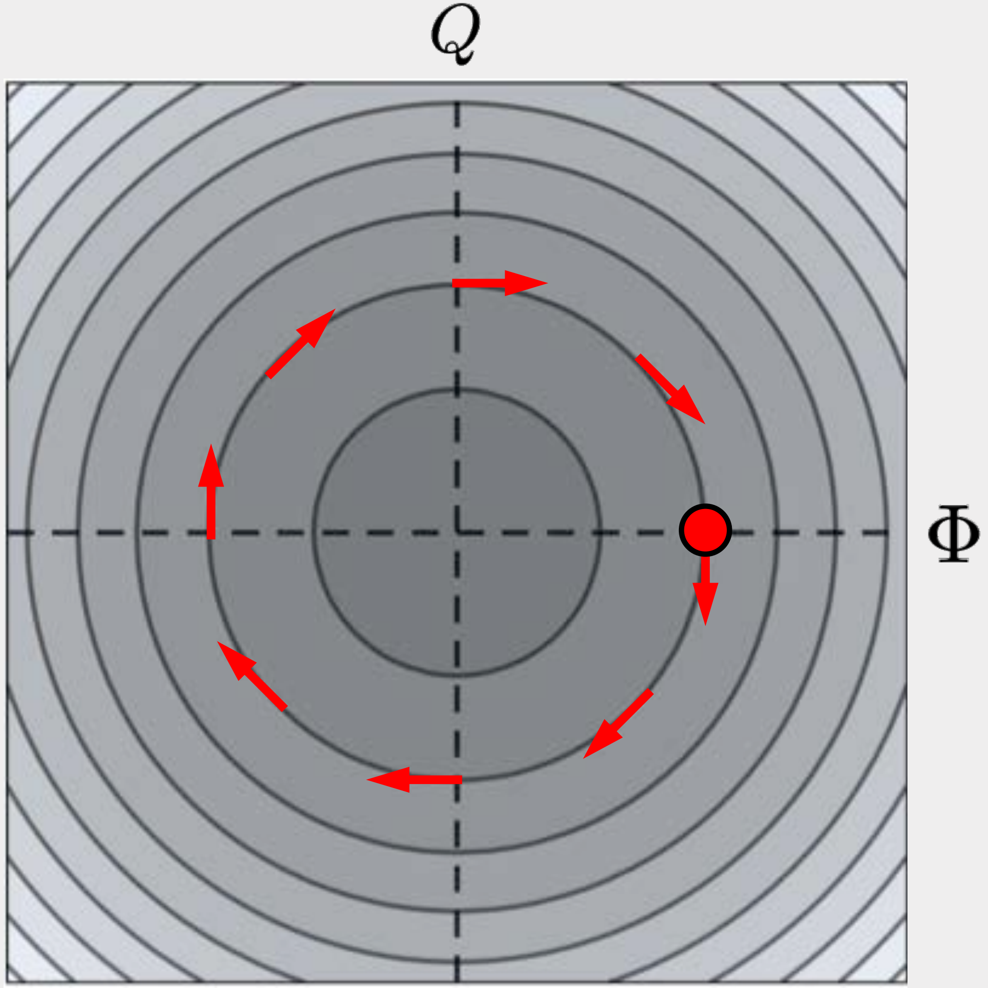
with $L = C = 1$




$\vdots$

$$E = \frac{3\hbar\omega_0}{2}$$

$$E = \frac{\hbar\omega_0}{2}$$




$$\dot{\Phi} = +\frac{\partial \mathcal{H}}{\partial Q} = +Q$$

$$\dot{Q} = -\frac{\partial \mathcal{H}}{\partial \Phi} = -\Phi$$

# Complex action-angle variable

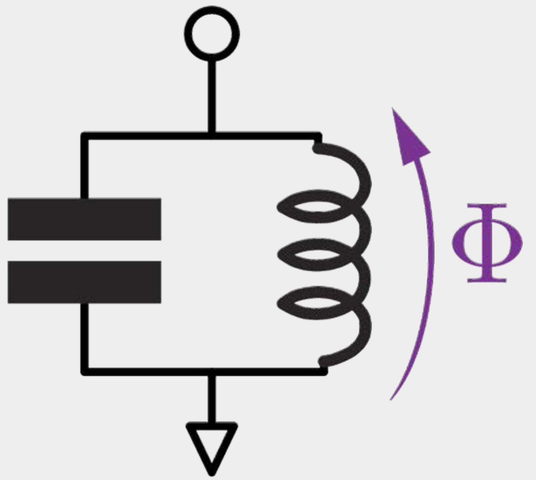
$$\mathcal{H}(\Phi, Q) = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

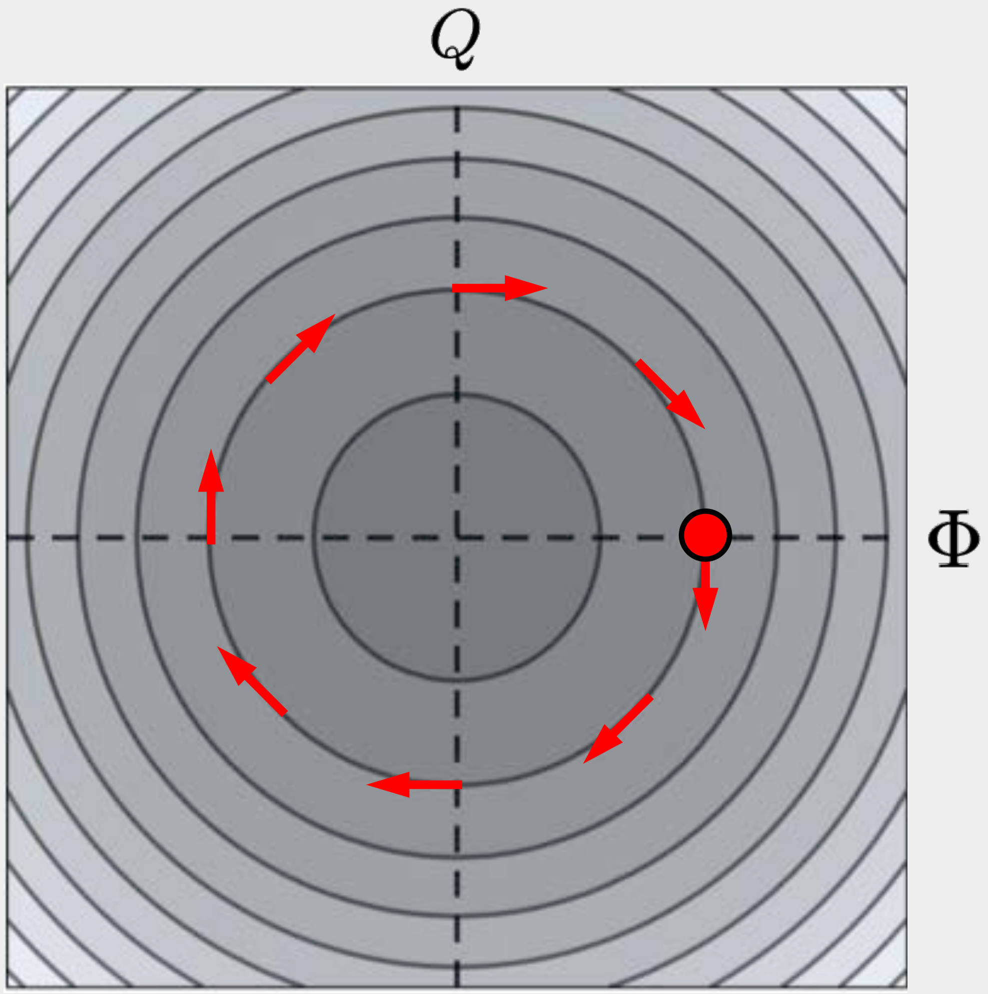
$$\mathcal{H}(\alpha, \alpha^*) = \frac{1}{2}\hbar\omega_0\left(\alpha^*\alpha + \alpha\alpha^*\right)$$

$$E = \hbar\omega_0\left(n + \frac{1}{2}\right)$$

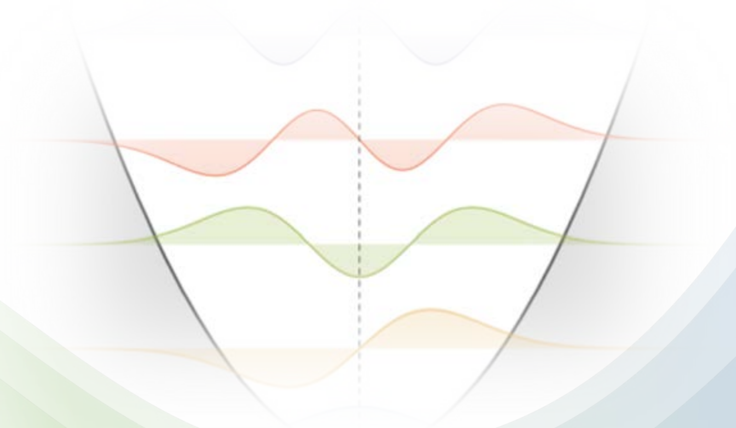
$$\alpha(t) = \sqrt{\frac{1}{2\hbar Z}}\left[\Phi(t) + iZQ(t)\right]$$

Classical analog of the bosonic ladder operator




$$\alpha(t) = \alpha(0)\, e^{-i\omega_0 t}$$

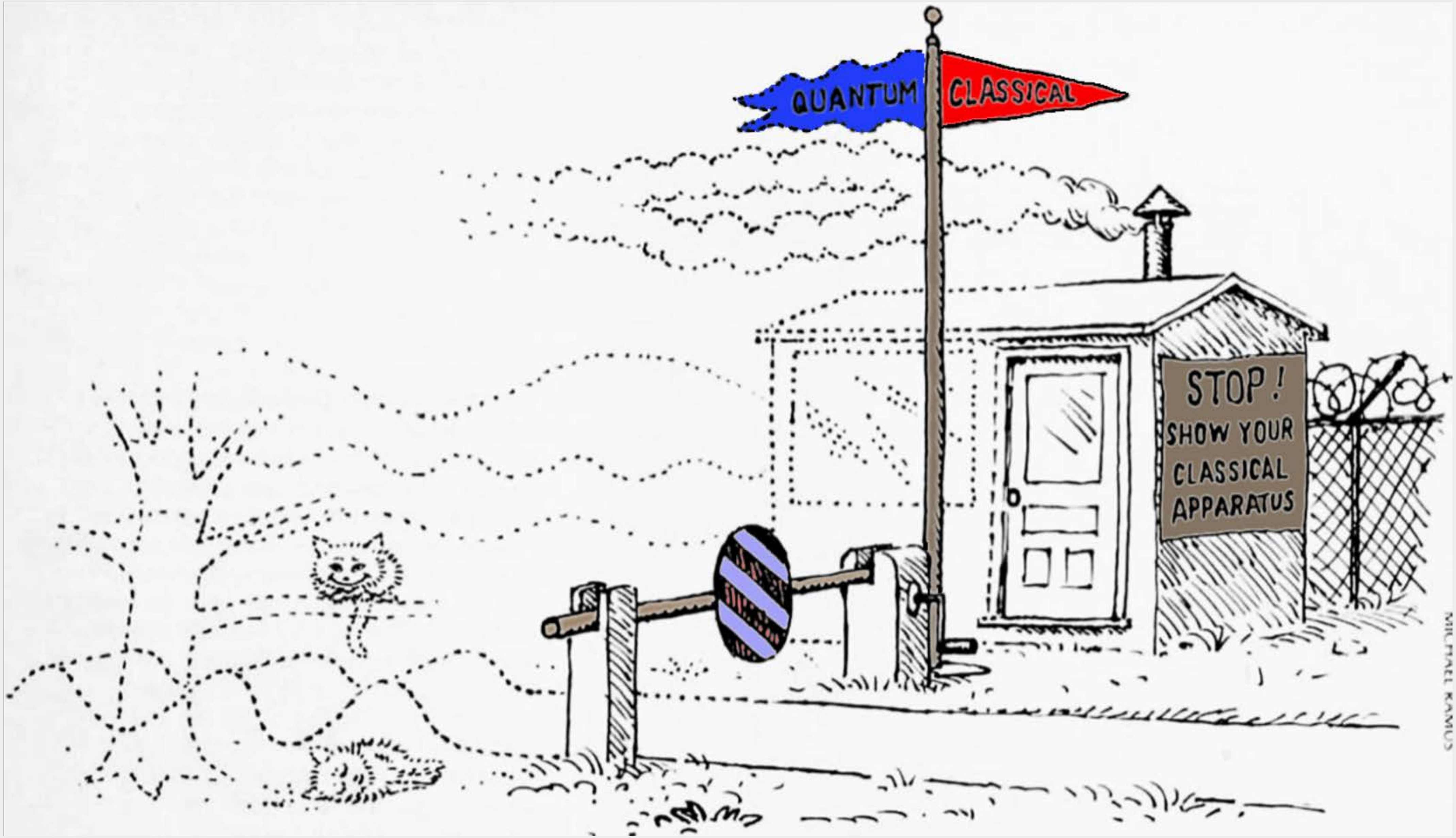


# Unveiling the quantum

## Quantum harmonic oscillator

# Unveiling the quantum

Dangerous bend
ahead on quantization




Drawing: Zurek, Physics Today (1991)

# Dirac's canonical quantization

Source: Cambridge University, Cavendish Laboratory / Wikimedia Commons

**Quantum (commutator)**

$$-\frac{i}{\hbar}\left[\hat{x},\hat{p}\right]=\hat{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}=-\frac{i}{\hbar}\left[\hat{O},\hat{H}\right]$$

where $\hat{O}\left(\hat{x},\hat{p};t\right),\hat{H}\left(\hat{x},\hat{p};t\right)$

These look a lot alike!

**Classical (Poisson bracket)**

$$\{x,p\}_P=1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}O=\{O,\mathcal{H}\}_P$$

where $O\left(x,p;t\right),\mathcal{H}\left(x,p;t\right)$

Procedure: Supplant the Poisson brackets by commutators


$$\{x,p\}\longmapsto\frac{1}{i\hbar}\left[\hat{x},\hat{p}\right]$$

$$\{O,O'\}\longmapsto\frac{1}{i\hbar}\left[\hat{O},\hat{O}'\right]$$

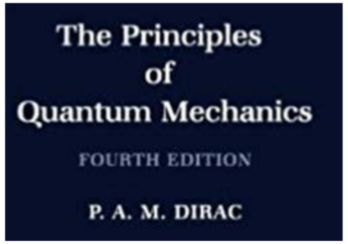
"There is, however, a fairly general method of obtaining quantum conditions, applicable to a very large class of dynamical systems. This is the method of classical analogy"
P.A.M. Dirac

# Dirac's canonical quantization: Quick exposure

Supplant classical Poisson bracket

and all quantum algebra follows…

$$\{O, O'\} \longmapsto \frac{1}{i\hbar}[\hat{O}, \hat{O}']$$




Derivation: Dirac derives the quantum form of the Poisson bracket — the commutator — from merely assuming that

1. Classical Poisson bracket rules hold (by analogy, the new theory must be consistent with the old!)
2. The dynamical variables do not commute; i.e., $xp \neq px$
3. The Poisson bracket has a single result and single unique meaning

Problems:

- Operator ordering ambiguities.
  Consider A and B are polynomials in $x$ and $p$; *e.g.*, $x^2 \times p$ or $p \times x^2$
- Curvilinear coordinate systems (potentially transmon if $\cos(\Phi/\varphi_0)$ considered wrapped)
- Quantum gravity …

If I knew what I was doing, it wouldn't be called research.

---

Albert Einstein

See Hawken *et al.* (2010)

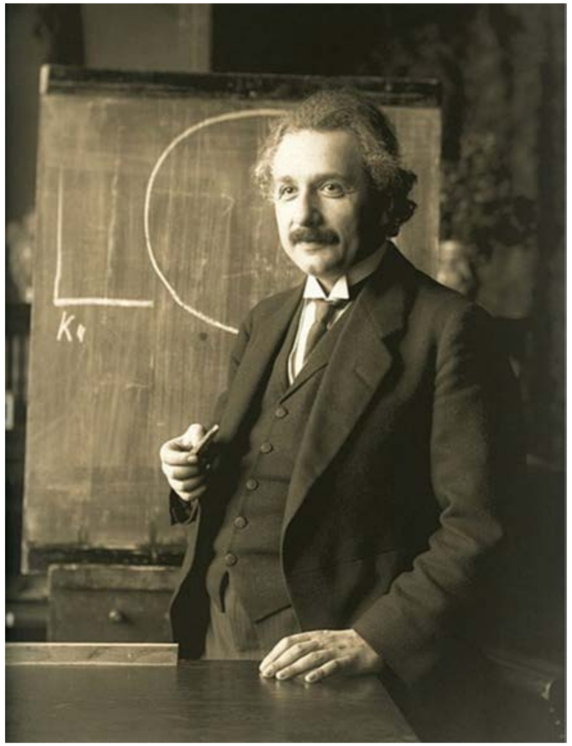
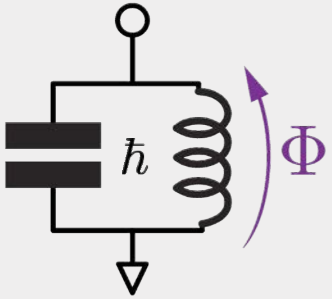



Photo: F. Schmutzer




# The classical and quantum oscillator

Classical | Quantum

$$\Phi(t) \mapsto \hat{\Phi}$$

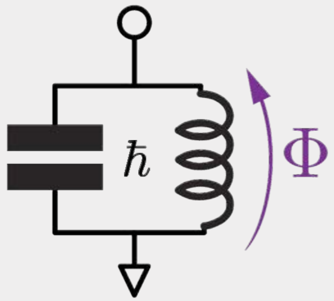
$$Q(t) \mapsto \hat{Q}$$

$$\mathcal{H}(\Phi, Q) \mapsto \hat{H}\left(\hat{\Phi}, \hat{Q}\right)$$

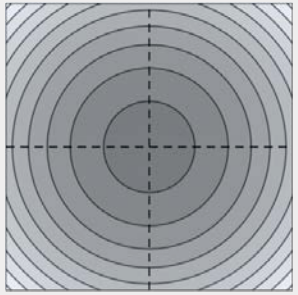
$$\{\Phi, Q\} = 1 \mapsto \left[\hat{\Phi}, \hat{Q}\right] = i\hbar\hat{1}$$

$$\{\alpha, \alpha^*\} = 1/(i\hbar) \mapsto \left[\hat{a}, \hat{a}^\dagger\right] = \hat{1}$$

$$\{A, B\} = \frac{\partial A}{\partial \Phi}\frac{\partial B}{\partial Q} - \frac{\partial B}{\partial \Phi}\frac{\partial A}{\partial Q} \qquad \left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$




# The classical and quantum oscillator



| | Classical | Quantum |
|---|---|---|
| **Hamiltonian** | $\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} = \frac{1}{2}\hbar\omega_0\left(\alpha^*\alpha + \alpha\alpha^*\right)$ | $\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$ |
| **Phase space** | $\alpha(t) = \sqrt{\frac{1}{2\hbar Z}}\left[\Phi(t) + iZQ(t)\right]$<br>$\alpha(t) = \alpha(0)\, e^{-i\omega_0 t}$<br>$\Phi(t) = \sqrt{\frac{\hbar Z}{2}}\left(\alpha^*(t) + \alpha(t)\right)$<br>$Q(t) = i\sqrt{\frac{\hbar}{2Z}}\left(\alpha^*(t) - \alpha(t)\right)$ | $\hat{a} = \sqrt{\frac{1}{2\hbar Z}}\left(\hat{\Phi} + iZ\hat{Q}\right)$<br>$\hat{a}(t) = \hat{a}(0)\, e^{-i\omega_0 t}$ (Heisenberg picture)<br>$\hat{\Phi} = \Phi_{\mathrm{ZPF}}\left(\hat{a}^\dagger + \hat{a}\right)$<br>$\hat{Q} = iQ_{\mathrm{ZPF}}\left(\hat{a}^\dagger - \hat{a}\right)$ |

# Ladder operators and matrix representation

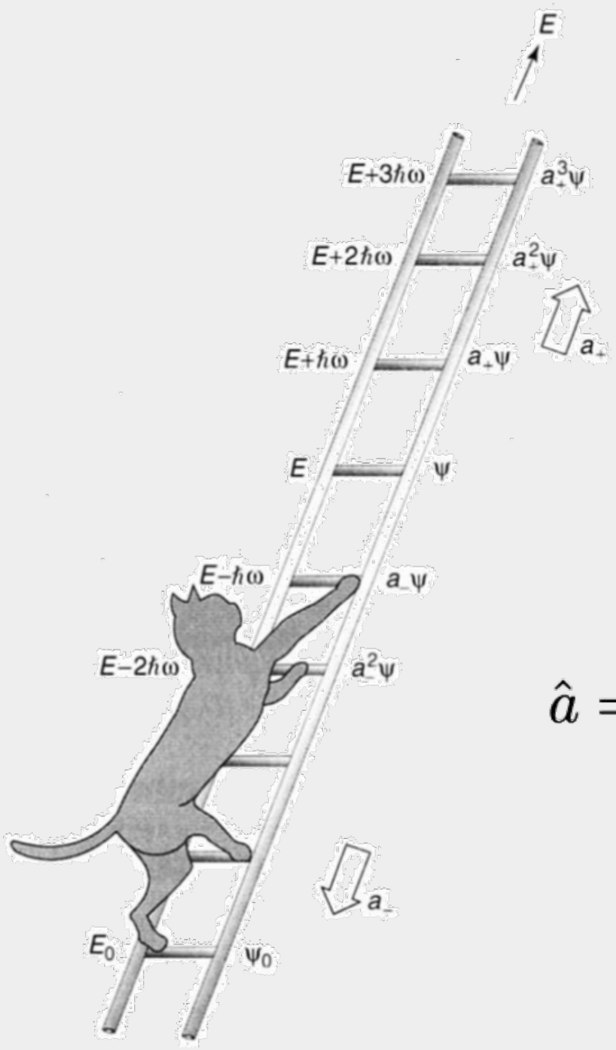



Image: Griffiths, D.J.

**annihilation**

$$\hat{a}\,|0\rangle = 0$$

$$\hat{a}\,|1\rangle = \sqrt{1}\,|0\rangle$$

**creation**

$$\hat{a}^\dagger\,|0\rangle = \sqrt{1}\,|1\rangle$$

$$\hat{a}^\dagger\,|1\rangle = \sqrt{2}\,|2\rangle$$

**general hopping**

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & \ddots \end{pmatrix} \quad \hat{a}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & \ddots \end{pmatrix} \quad \hat{a}^\dagger\hat{a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

# Hand-written notes

## Hamiltonian and energy

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger\right)$$

## Mean, variance, and RMS fluctuations

$$\left\langle 0 \middle| \hat{\Phi}^2 \middle| 0 \right\rangle = \Phi_{\mathrm{ZPF}}^2$$

$$\left\langle 0 \middle| \hat{Q}^2 \middle| 0 \right\rangle = Q_{\mathrm{ZPF}}^2$$

# Calculations of the energy

For notational simplicity, I will drop the hats on the operators temporarily and the 0 from $\omega_0$.

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$$

$$\hat{a}^\dagger a|n\rangle = n|n\rangle$$ ← Fock

$$\hat{a}^\dagger(a|n\rangle)$$

$$\hat{a}^\dagger\sqrt{n}\,|n-1\rangle$$ using $a^\dagger|n'\rangle = \sqrt{n'+1}\,|n'+1\rangle$

$$\downarrow$$

$$\sqrt{n+1-1}\,\sqrt{n}\,|n-1+1\rangle$$

$$\downarrow$$

$$= n|n\rangle$$ $a^\dagger a$ = Photon number $n$

$$[a, a^\dagger] = 1$$

$$aa^\dagger - a^\dagger a = 1$$

$$aa^\dagger = 1 + a^\dagger a$$

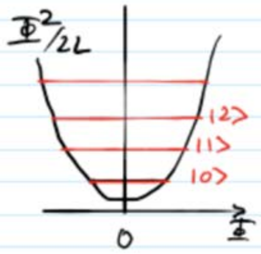
$$\hat{H} = \frac{\hbar\omega}{2}\left(a^\dagger a + aa^\dagger\right)$$

$$= \frac{\hbar\omega}{2}\left(a^\dagger a + a^\dagger a + 1\right)$$

$$= \frac{\hbar\omega}{2}\left(2a^\dagger a + 1\right)$$

$$\hat{H} = \hbar\omega_0\left(a^\dagger a + 1/2\right)$$

$n$ zero-point energy



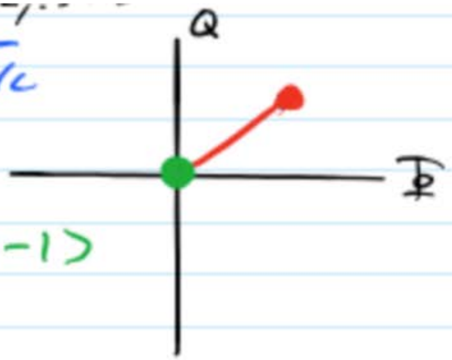

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$= \hbar\omega_0(n + 1/2)$$ $n = 0, 1, 2, \ldots$

# Expectation value of magnetic flux and charge

some # $= \sqrt{\frac{\hbar Z}{2}}$ $\quad Z = \sqrt{L/C}$

$$\langle \hat\Phi \rangle = \langle n| \Phi_{ZPF} (\hat a + a^\dagger) |n\rangle$$




$$\langle n| a |n\rangle = \sqrt{n}\, \langle n | n-1\rangle = 0$$

$$\langle n| a^\dagger |n\rangle = \sqrt{n+1}\, \langle n | n+1\rangle = 0$$

$$\langle \hat\Phi \rangle = \Phi_{ZPF} \left( \langle \hat a \rangle + \langle a^\dagger \rangle \right) = 0$$

$$\langle \hat Q \rangle = \langle n| -i Q_{ZPF} ( a^\dagger - a ) |n\rangle = -i Q_{ZPF} \left( \langle a^\dagger \rangle - \langle a \rangle \right) = 0$$

$$Q_{ZPF} = \sqrt{\frac{\hbar}{2Z}}$$

# Fluctuations of flux

$$\mathrm{Var}(\hat{\Phi}) = \langle \hat{\Phi}^2 \rangle - \langle \hat{\Phi} \rangle^2$$

$$= \langle \hat{\Phi}^2 \rangle$$

$$= \langle n | (a^\dagger + a)^2 | n \rangle \times \Phi_{ZPF}^2$$

$$= \langle n | a^{\dagger 2} + a^2 + a^\dagger a + a a^\dagger | n \rangle \times \cdots$$

$$= \cancel{\langle n | a^{\dagger 2} | n \rangle} \xrightarrow{} 0 + \cancel{\langle a^2 \rangle} + \langle a^\dagger a + a a^\dagger \rangle \times \Phi_{ZPF}^2$$

$c \langle n | n+2 \rangle$

$$= \Phi_{ZPF}^2 \langle a^\dagger a + a a^\dagger \rangle$$

$$= \Phi_{ZPF}^2 \langle a^\dagger a + a^\dagger a + 1 \rangle$$

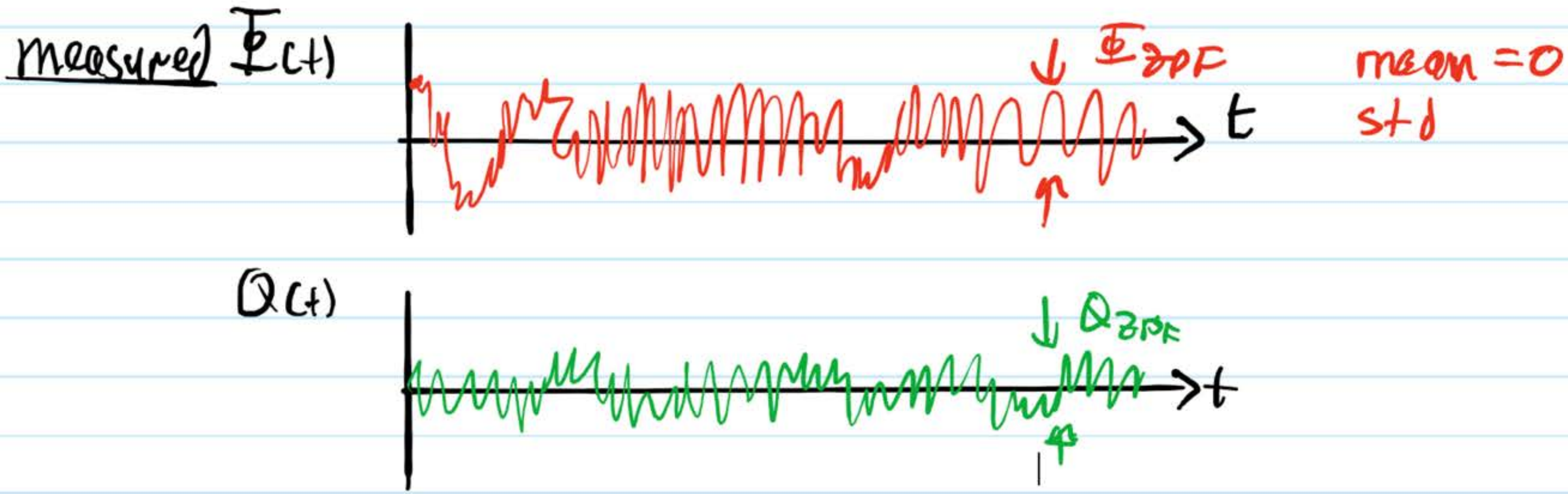
# Fluctuations of flux and charge

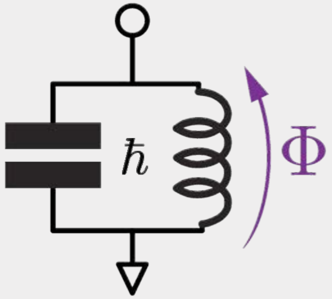
$$\mathrm{Var}(\hat{\Phi}) = \langle \hat{\Phi}^2 \rangle = (2n+1)\,\Phi_{ZPF}^2$$

For $|n\rangle = 0$ : $\langle \hat{\Phi}^2 \rangle = \Phi_{ZPF}^2$

i.e., RMS fluctuation of $\hat{\Phi}$: in ground $|0\rangle$

$$\sigma_{\hat{\Phi}} = \Phi_{ZPF}$$

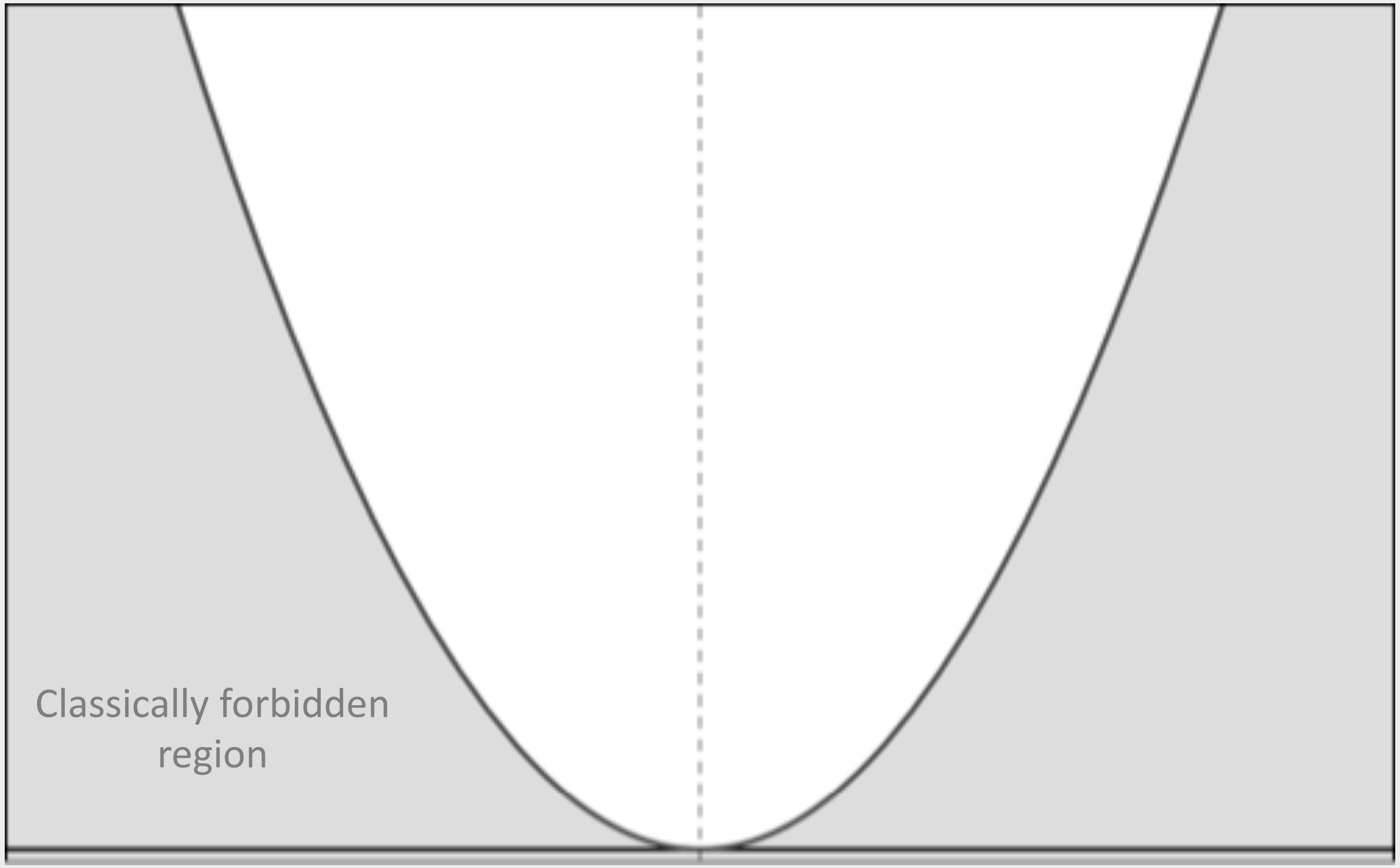




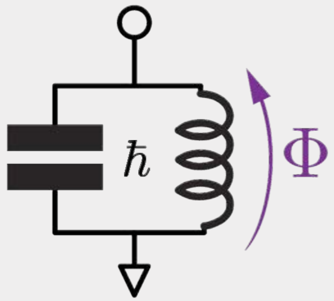

# Wavefunctions of the quantum oscillator

$$\mathcal{E}_{\text{ind}}\left(\Phi\right) = \frac{\Phi^2}{2L}$$

Energy




$\Phi$



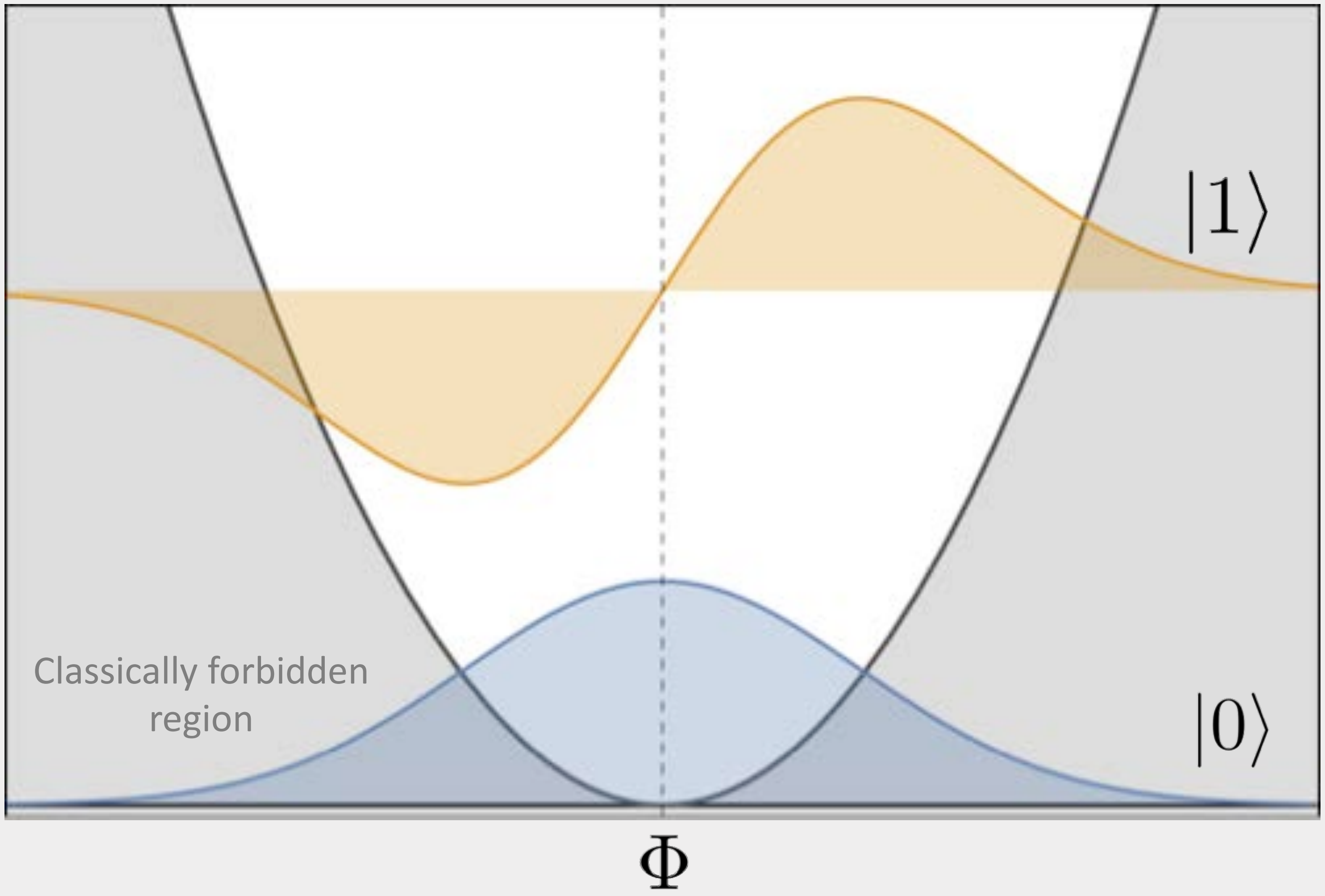

# Wavefunctions of the quantum oscillator

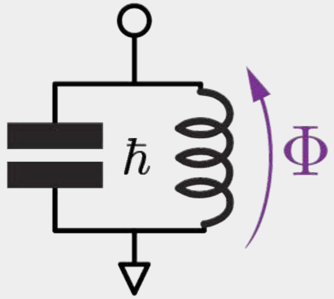
$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$

$$\psi_n(\Phi) \equiv \langle \Phi | n \rangle$$

Energy /

Scaled wavefunction amplitude



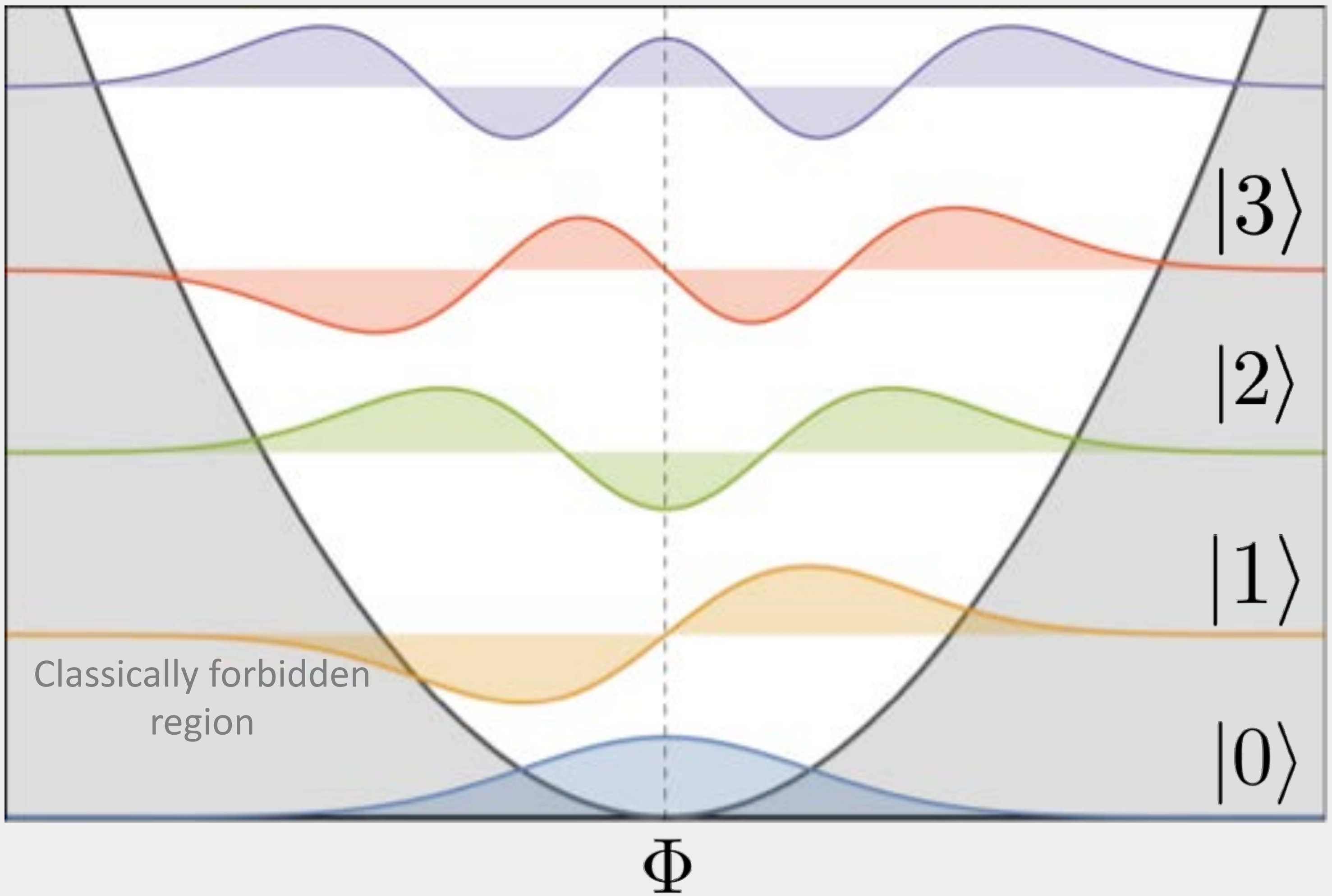


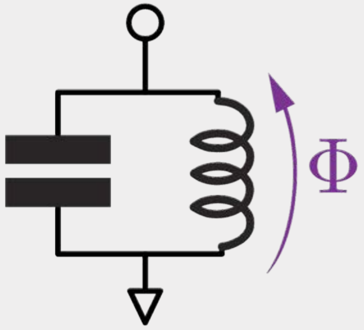

# Wavefunctions of the quantum oscillator

$$\mathcal{E}_{\mathrm{ind}}\left(\Phi\right) = \frac{\Phi^2}{2L} \qquad\qquad \psi_n\left(\Phi\right) \equiv \langle \Phi | n \rangle$$

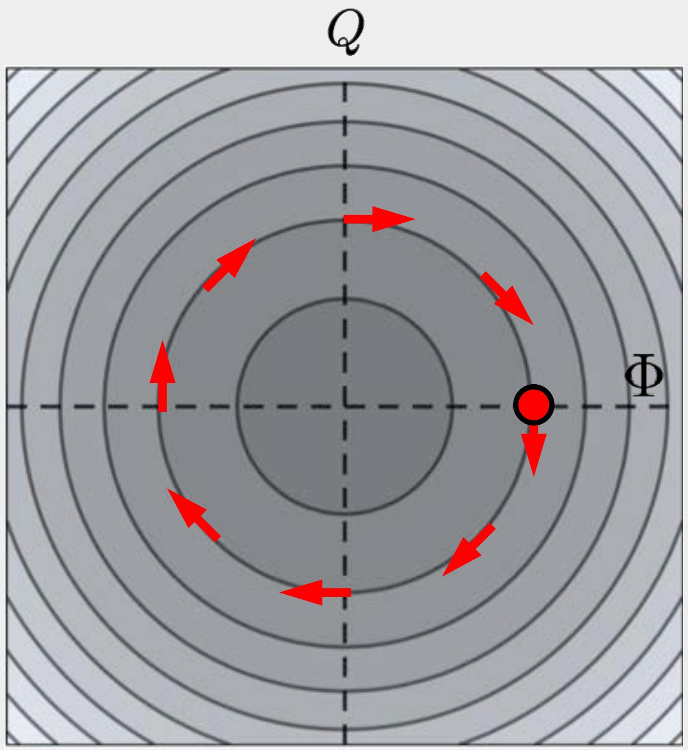
Energy /

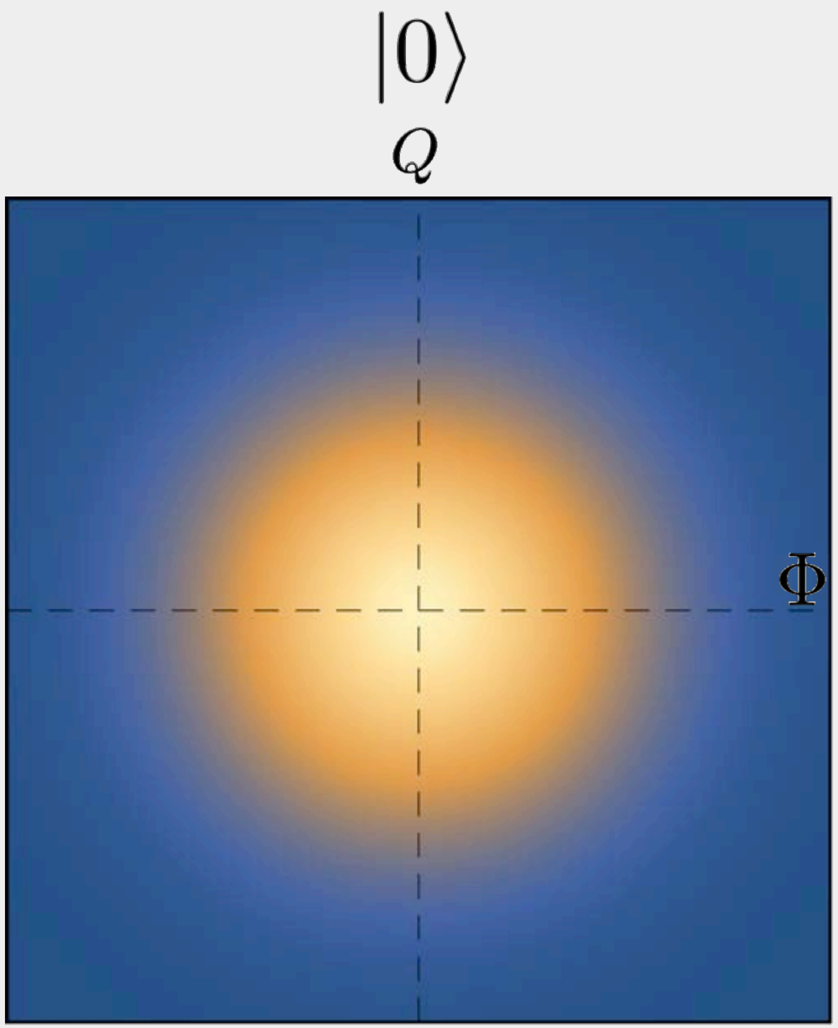
Scaled wavefunction amplitude

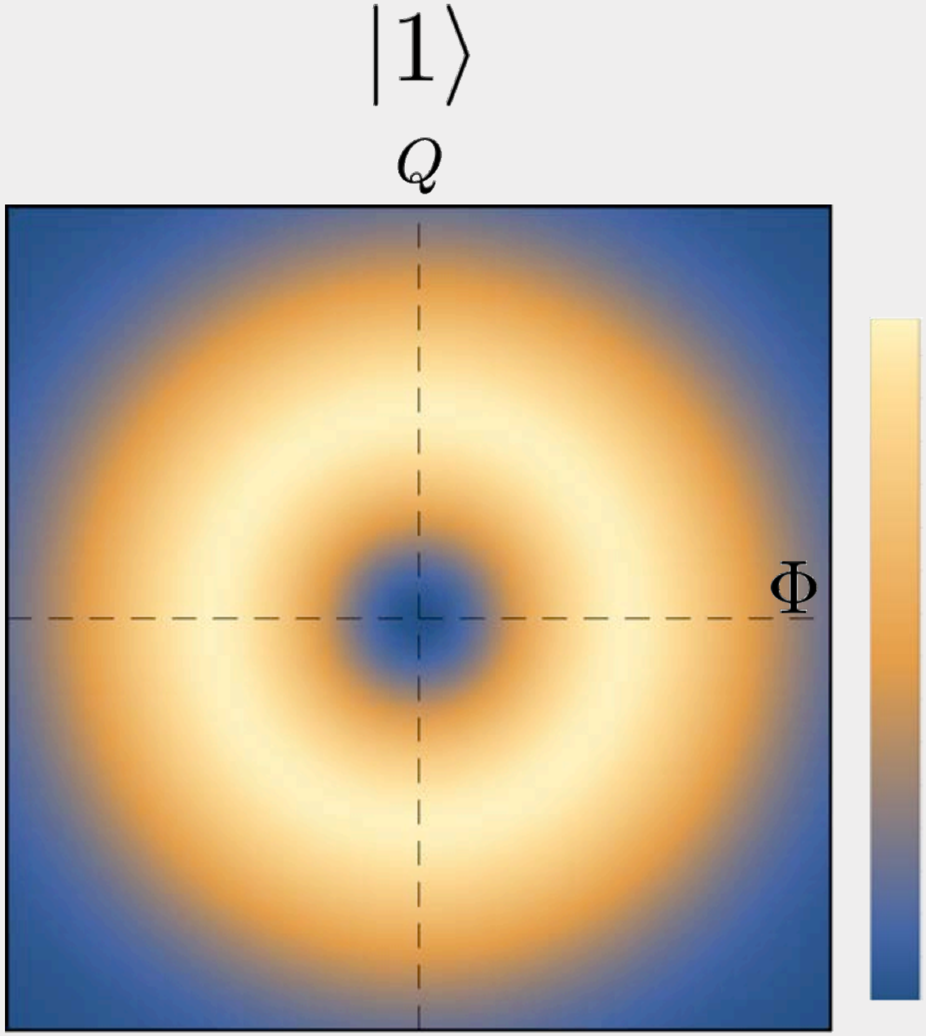





# Phase-space (Husimi Q function)

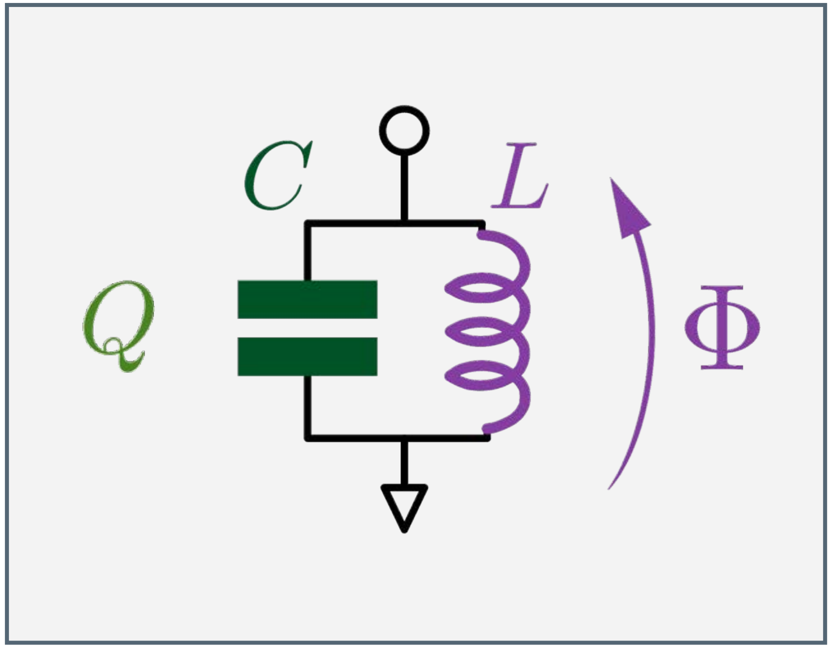



$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle$$

# Pop-up question




The flux and charge operators are Hermitian observables.

How can some expectations, such as

$$\left\langle 0 \middle| \hat{\Phi}\hat{Q} \middle| 0 \right\rangle = \frac{1}{4} i \, ,$$

be imaginary?

Or, others be negative…?

# Advanced questions

1. What is $\hat{\Phi}(t)$ in terms of $\hat{a}$ in the Heisenberg picture?

2. **Autocorrelation.** Find the autocorrelation operator $\hat{C}_{\Phi\Phi}(t) = \hat{\Phi}(t)\,\hat{\Phi}(0)$. What frequency components does it have?

   (a) What is the flux autocorrelation expectation value $C_{\Phi\Phi}(t) = \left\langle n \middle| \hat{\Phi}(t)\,\hat{\Phi}(0) \middle| n \right\rangle$ for the Fock state $|n\rangle$? Is it real? Why not? What is the frequency spectrum?

   (b) Repeat for a coherent state $|\alpha\rangle$.

3. **Thermal state.** The thermal state is $\hat{\rho}_{\mathrm{th}} = \exp\left[-\beta\hat{a}^{\dagger}\hat{a}\right] / \operatorname{Tr}\left[\exp\left(-\beta\hat{a}^{\dagger}\hat{a}\right)\right]$, where $\beta = \hbar\omega_0/k_{\mathrm{B}}T$, and $T$ is the temperature of the oscillator.

   (a) What is the mean and variance of the flux $\hat{\Phi}$ and charge $\hat{Q}$ operators?

   (b) How does the frequency spectrum of the autocorrelation $\left\langle \hat{C}_{\Phi\Phi}(t) \right\rangle = \operatorname{Tr}\left[\hat{\rho}_{\mathrm{th}}\hat{C}_{\Phi\Phi}(t)\right]$ change when with that for the state $|n\rangle$ and $|\alpha\rangle$?

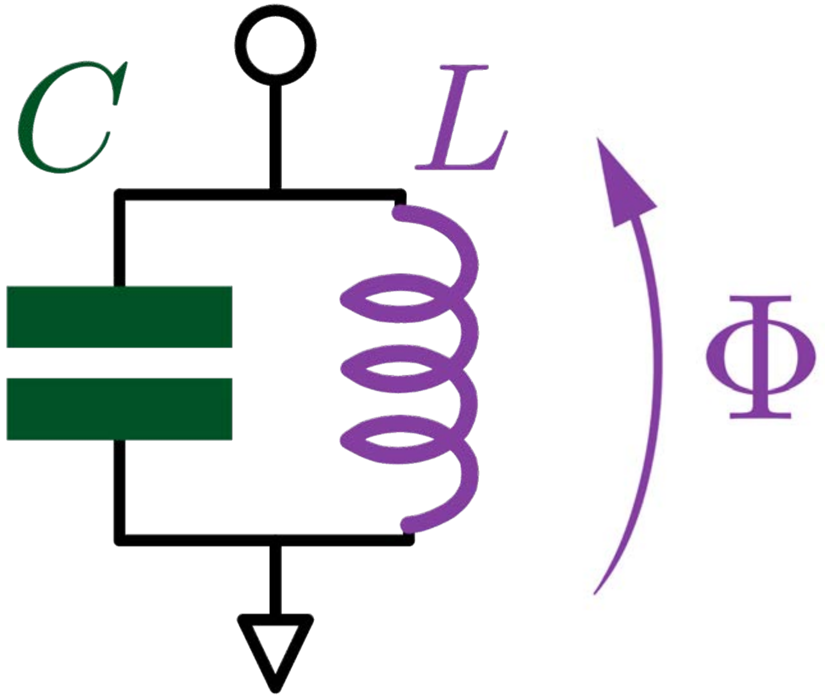
   (c) The spectrum is not symmetric in frequency. How can you interpret that positive and negative frequencies have different weights? How is this related to absorption and emission of the oscillator? (Consider Fermi's golden rule).

   (d) What happens to the spectrum in the limit of high temperature, $k_{\mathrm{B}}T \gg \hbar\omega_0$? How about low temperature, $k_{\mathrm{B}}T \gg \hbar\omega_0$?

   (e) How do the above conclusions change for charge; i.e., for $\left\langle \hat{C}_{QQ}(t) \right\rangle = \operatorname{Tr}\left[\hat{\rho}_{\mathrm{th}}\hat{C}_{QQ}(t)\right]$?

Will discuss answers on my blog sometime soon

# Quantum Harmonic Oscillator Applets

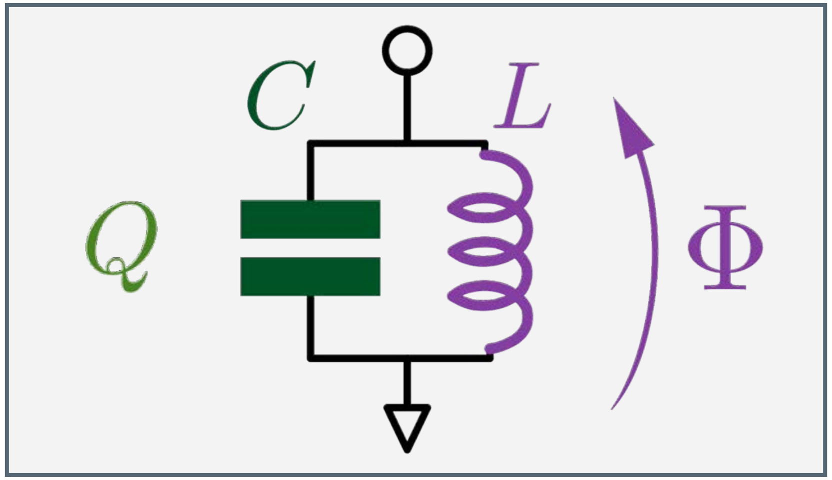



[Energy levels](https://www.st-andrews.ac.uk/physics/quvis) of SHO applet from
https://www.st-andrews.ac.uk/physics/quvis

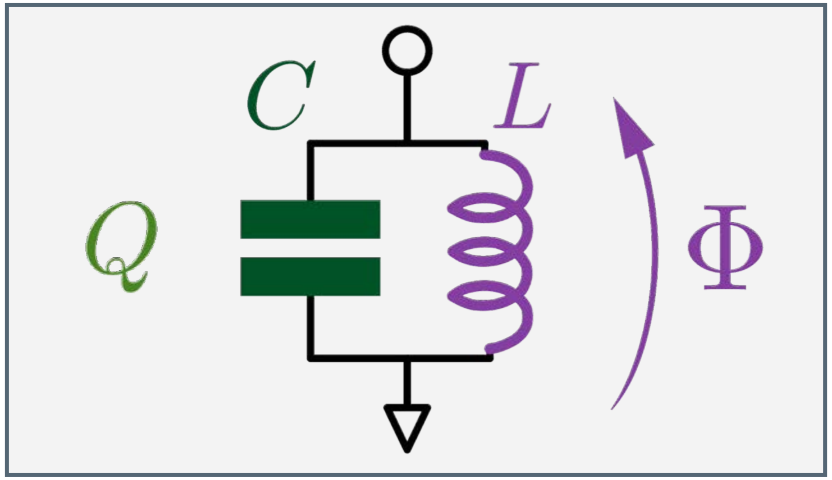
Wigner function
https://demonstrations.wolfram.com/WignerFunctionOfHarmonicOscillator/

Coherent states
https://demonstrations.wolfram.com/CoherentStatesOfTheHarmonicOscillator/

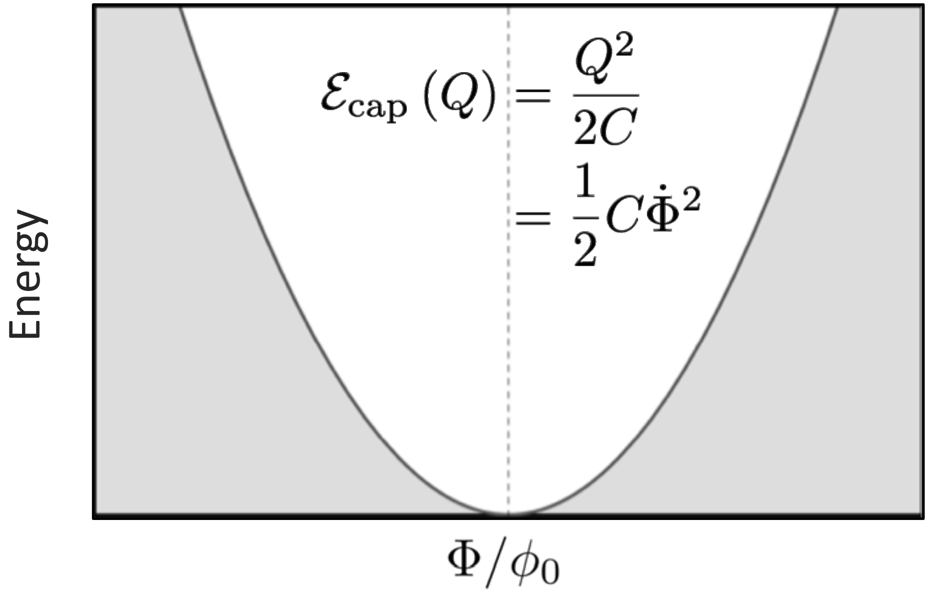
# Linear harmonic oscillator summary



# The LC quantum harmonic oscillator




$$\mathcal{E}_{\text{ind}}(\Phi) = \frac{\Phi^2}{2L}$$




$$\mathcal{E}_{\text{cap}}(Q) = \frac{Q^2}{2C} = \frac{1}{2} C \dot{\Phi}^2$$

Position: $\Phi \mapsto x$ Momentum: $Q \mapsto p$

Mass: $C \mapsto m$ Spring constant: $L^{-1} \mapsto k$

$$\hat{H}\left(\hat{\Phi}, \hat{Q}\right) = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \qquad \omega_0^2 = \frac{1}{LC}$$

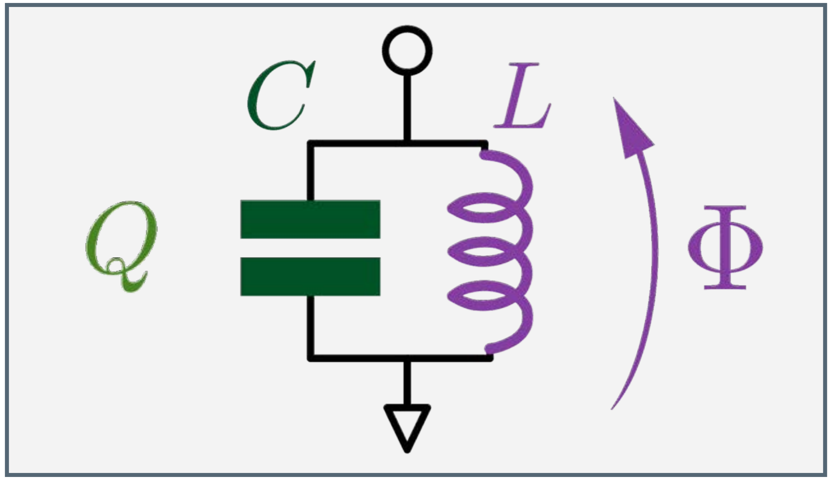
$$\hat{H}\left|n\right\rangle = \hbar\omega_0 \left(n + \frac{1}{2}\right)\left|n\right\rangle \qquad Z_0 = \frac{L}{C}$$

$$\hat{\Phi} = \Phi_{\text{ZPF}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \Phi_{\text{ZPF}} = \sqrt{\frac{\hbar}{2} Z_0} = \Phi_0 \sqrt{\frac{z_0}{2\pi}},$$

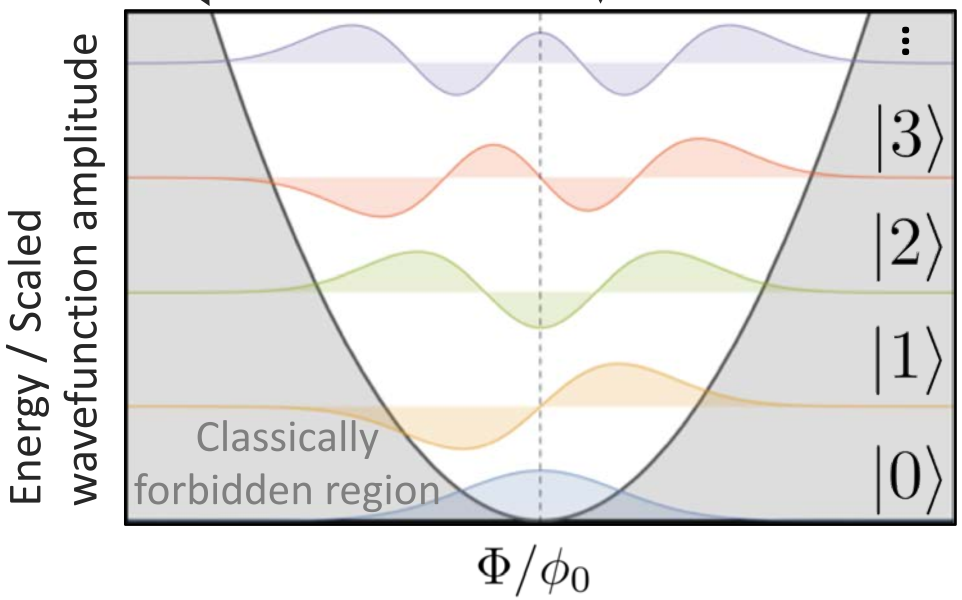
$$\hat{Q} = iQ_{\text{ZPF}}\left(\hat{a}^\dagger - \hat{a}\right) \qquad Q_{\text{ZPF}} = \sqrt{\frac{\hbar}{2} Z_0^{-1}} = (2e)\sqrt{\frac{1}{2\pi z_0}},$$

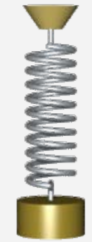
Graph: Minev; Spring-mass: Svjo

# The LC quantum harmonic oscillator




$$\mathcal{E}_{\text{ind}}\left(\Phi\right) = \frac{\Phi^2}{2L} \qquad \psi_n\left(\Phi\right) \equiv \langle\Phi|n\rangle$$



Position: $\Phi \mapsto x$ Momentum: $Q \mapsto p$

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$$\hat{H}\left(\hat{\Phi}, \hat{Q}\right) = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \qquad \omega_0^2 = \frac{1}{LC}$$

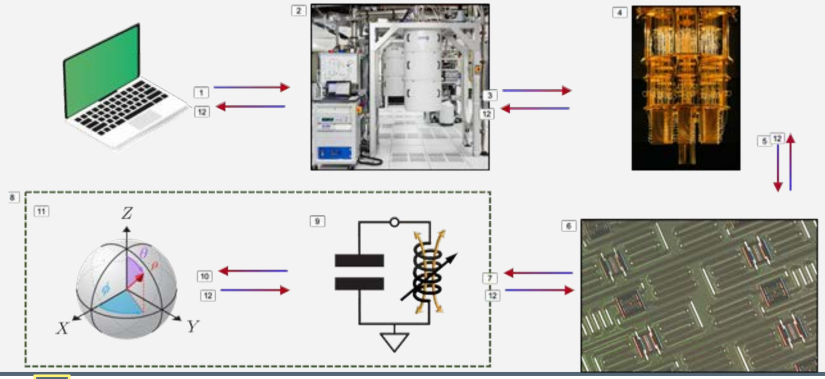
$$\hat{H}\left|n\right\rangle = \hbar\omega_0\left(n + \frac{1}{2}\right)\left|n\right\rangle \qquad \Phi_{\text{ZPF}}Q_{\text{ZPF}} = \frac{\hbar}{2} \qquad Z_0 = \frac{L}{C}$$

$$\hat{\Phi} = \Phi_{\text{ZPF}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \Phi_{\text{ZPF}} = \sqrt{\frac{\hbar}{2}Z_0} \qquad \left\langle 0\middle|\hat{\Phi}^2\middle|0\right\rangle = \Phi_{\text{ZPF}}^2$$

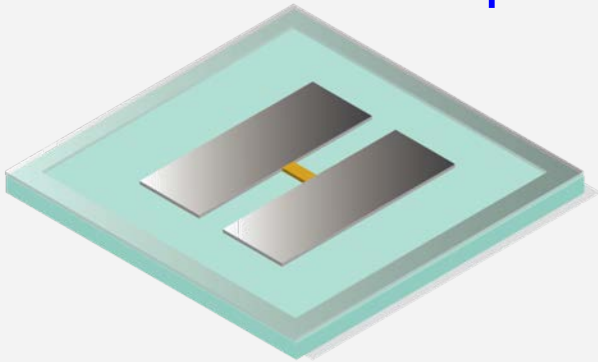
$$\hat{Q} = iQ_{\text{ZPF}}\left(\hat{a}^\dagger - \hat{a}\right) \qquad Q_{\text{ZPF}} = \sqrt{\frac{\hbar}{2}Z_0^{-1}} \qquad \left\langle 0\middle|\hat{Q}^2\middle|0\right\rangle = Q_{\text{ZPF}}^2$$

# The road behind and ahead

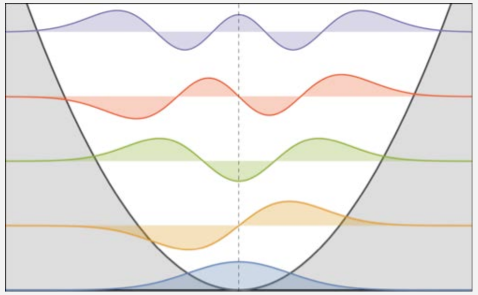
## Qubit in the cloud

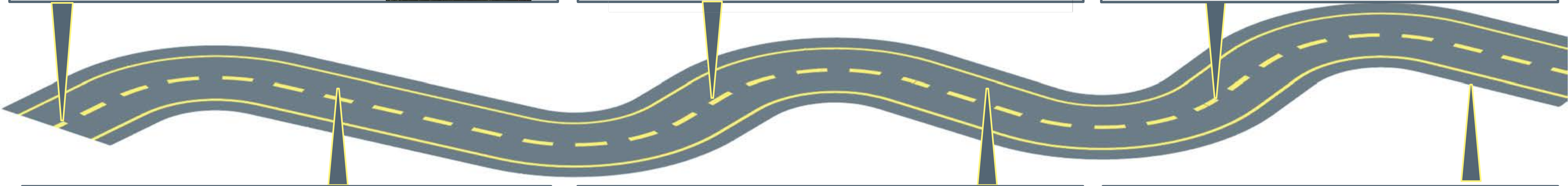


## cQED: Transmon qubit

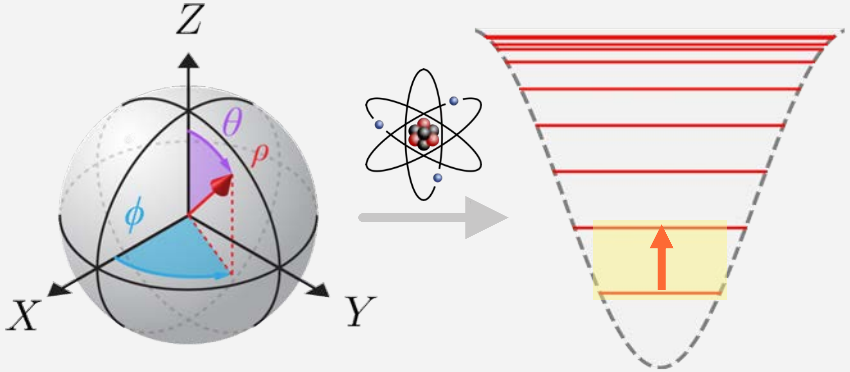


## Unveiling the quantum oscillator

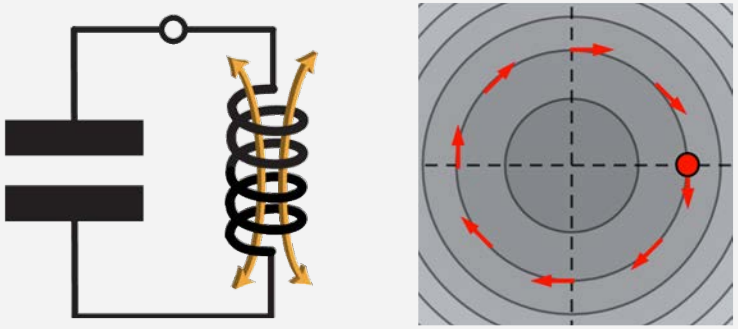




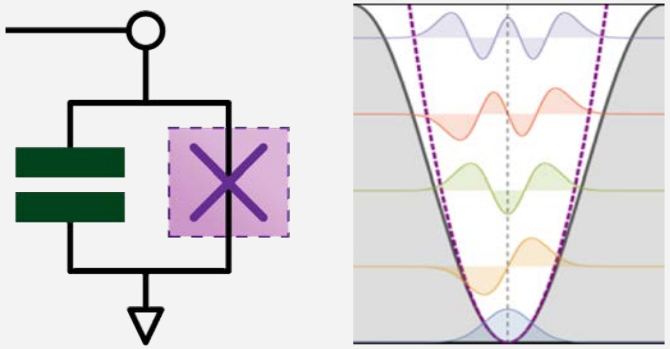
## Qubit from atom / oscillator




## Classical circuits & the LC



## Transmon qubit



# Next steps

Tightly integrated lab work with Dr. Nick Bronn and Co.!
More depth on qubit control
Run experiments on real devices

Check out references, problems given in the lecture, dangerous bends

Break away from the rules of today

Thank you!

**Zlatko K. Minev**