#12 Testing Bell inequalities

"What can be more *fascinating* than experimental metaphysics?"
-Abner Shimony-

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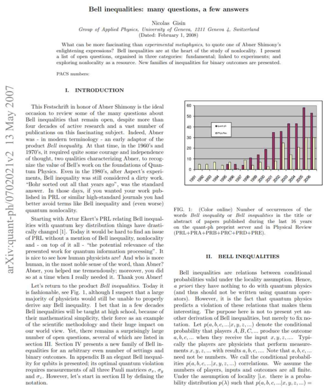
QISKIT code: https://github.com/Siheon-Park/test-bell-ineq

Motivation



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Bell inequalities: many questions, a few answers

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(Dated: February 1, 2008)

What can be more fascinating than *experimental metaphysics*, to quote one of Abner Shimony's enlightening expressions? Bell inequalities are at the heart of the study of nonlocality. I present a list of open questions, organised in three categories: fundamental; linked to experiments; and exploring nonlocality as a resource. New families of inequalities for binary outcomes are presented.

PACS numbers:

I. INTRODUCTION

This Festschrift in honor of Abner Shimony is the ideal occasion to review some of the many questions about Bell inequalities that remain open, despite more than four decades of active research and a vast number of publications on this fascinating subject. Indeed, Abner was - in modern terminology - an early adaptor of the product *Bell inequality*. At that time, in the 1960's and 1970's, it required quite some courage and independence of thought, two qualities characterizing Abner, to recognize the value of Bell's work on the foundations of Quantum Physics. Even in the 1980's, after Aspect's experiments, Bell inequality was still considered a dirty work. "Bohr sorted out all that years ago", was the standard answer. In those days, if you wanted your work published in PRL or similar high-standard journals you had better avoid terms like Bell inequality and (even worse) quantum nonlocality.

Starting with Artur Ekert's PRL relating Bell inequalities with quantum key distribution things have drastically changed [1]. Today it would be hard to find an issue of PRL without a mention of Bell inequality, nonlocality and - on top of it all - "the potential relevance of the presented work for quantum information processing". It is nice to see how human physicists are! And who is more human, in the most noble sense of the word, than Abner? Abner, you helped me tremendously; moreover, you did so at a time when I really needed it. Thank you Abner!

Let's return to the product *Bell inequalities*. Today it is fashionable, see Fig. 1, although I suspect that a large majority of physicists would still be unable to properly derive any Bell inequality. I bet that in a few decades Bell inequalities will be taught at high school, because of their mathematical simplicity, their force as an example of the scientific methodology and their huge impact on our world view. Yet, there remains a surprisingly large number of open questions, several of which are listed in section III. Section IV presents a new family of Bell inequalities for an arbitrary even number of settings and binary outcomes. In appendix B an elegant Bell inequality for qubits is presented; its optimal quantum violation requires measurements of all three Pauli matrices σ_x, σ_y and σ_z. However, let's start in section II by defining the notation.

FIG. 1: (Color online) Number of occurrences of the words *Bell inequality* or *Bell inequalities* in the title or abstract of papers published during the last 16 years on the quant-ph preprint server and in Physical Review (PRL+PRA+PRB+PRC+PRD+PRE).

II. BELL INEQUALITIES

Bell inequalities are relations between conditional probabilities valid under the locality assumption. Hence, *a priori* they have nothing to do with quantum physics (and thus should not be written using quantum operators). However, it is the fact that quantum physics predicts a violation of these relations that makes them interesting. The purpose here is not to present yet another derivation of Bell inequalities, but merely to fix notation. Let $p(a, b, c, ...|x, y, z, ...)$ denote the conditional probability that players $A, B, C, ...$ produce the outcome $a, b, c, ...$ when they receive the input $x, y, z, ...$ Typically the players are physicists that perform measurements $x, y, z, ...$ with results $a, b, c, ...$ Note that $a, b, c, ...$ need not be numbers. We call the conditional probabilities $p(a, b, c, ...|x, y, z, ...)$ correlations. We assume the numbers of players, inputs and outcomes are all finite. Under the assumption of locality [i.e. there is a probability distribution $p(\lambda)$ such that $p(a, b, c, ...|x, y, z, ...) =$

Article

Quantum theory based on real numbers can be experimentally falsified

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Although complex numbers are essential in mathematics, they are not needed to describe physical experiments, as those are expressed in terms of probabilities, hence real numbers. Physics, however, aims to explain, rather than describe, experiments through theories. Although most theories of physics are based on real numbers, quantum theory was the first to be formulated in terms of operators acting on complex Hilbert spaces[1,2]. This has puzzled countless physicists, including the fathers of the theory, for whom a real version of quantum theory, in terms of real operators, seemed much more natural[3]. In fact, previous studies have shown that such a 'real quantum theory' can reproduce the outcomes of any multipartite experiment, as long as the parts share arbitrary real quantum states[4]. Here we investigate whether complex numbers are actually needed in the quantum formalism. We show this to be case by proving that real and complex Hilbert-space formulations of quantum theory make different predictions in network scenarios comprising independent states and measurements. This allows us to devise a Bell-like experiment, the successful realization of which would disprove real quantum theory, in the same way as standard Bell experiments disproved local physics.

Without qualification, the question of whether complex numbers are necessary for natural sciences, and, more concretely, for physics, must be answered in the negative: physics experiments are described by the statistics they generate, that is, by probabilities, and hence real numbers, thus, there is no need for complex numbers. The question becomes meaningful, however, when considering a specific theoretical framework, designed to explain existing experiments and make predictions about future ones. Whether complex numbers are needed within a theory to correctly explain experiments, or whether real numbers only are sufficient, is not straightforward. Complex numbers are sometimes introduced in electromagnetism to simplify calculations: one might, for instance, regard the electric and magnetic fields as complex vector fields to describe electromagnetic waves. However, this is just a computational trick. We wonder whether the same can be said about complex numbers in quantum theory.

In its Hilbert space formulation, quantum theory is defined in terms of the following postulates[5,6]. (1) For every physical system S, there corresponds a Hilbert space $\mathcal{H}_S$ and its state is represented by a normalized vector ϕ in $\mathcal{H}_S$, that is, $\langle\phi|\phi\rangle = 1$. (2) A measurement Π in S corresponds to an ensemble $\{\Pi_r\}_r$ of projection operators, indexed by the measurement result r and acting on $\mathcal{H}_S$, with $\sum_r \Pi_r = \mathbb{1}_S$. (3) Born rule: if we measure Π when system S is in state ϕ, the probability of obtaining result r is given by $\Pr(r) = \langle\phi|\Pi_r|\phi\rangle$. (4) The Hilbert space $\mathcal{H}_{ST}$ corresponding to the composition of two systems S and T is $\mathcal{H}_S \otimes \mathcal{H}_T$. The operators used to describe measurements or transformations in system S act trivially on $\mathcal{H}_T$ and vice versa. Similarly, the state representing two independent preparations of the two systems is the tensor product of the two preparations.

This last postulate has a key role in our discussions: we remark that it even holds beyond quantum theory, specifically for space-like separated systems in some axiomatizations of quantum field theory[7–10] (Supplementary Information).

As originally introduced by Dirac and von Neumann[1,2], the Hilbert spaces $\mathcal{H}_S$ in postulate (1) are traditionally taken to be complex. We call the resulting postulate (1_c). The theory specified by postulates (1_c) and (2)–(4) is the standard formulation of quantum theory in terms of complex Hilbert spaces and tensor products. For brevity, we will refer to it simply as 'complex quantum theory'. Contrary to classical physics, complex numbers (in particular, complex Hilbert spaces) are thus an essential element of the very definition of complex quantum theory.

Owing to the controversy surrounding their irruption in mathematics and their almost total absence in classical physics, the occurrence of complex numbers in quantum theory worried some of its founders, for whom a formulation in terms of real operators seemed much more natural ("What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function." (Letter from Schrödinger to Lorentz, 6 June 1926; ref. [3])). This is precisely the question we address in this work: whether complex numbers can be replaced by real numbers in the Hilbert space formulation of quantum theory without limiting its predictions. The resulting 'real quantum theory', which has appeared in the literature under various names[11,12], obeys the same postulates (2)–(4) but assumes real

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Editors' Suggestion Featured in Physics

Ruling Out Real-Valued Standard Formalism of Quantum Theory

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Standard quantum theory was formulated with complex-valued Schrödinger equations, wave functions, operators, and Hilbert spaces. Previous work attempted to simulate quantum systems using only real numbers by exploiting an enlarged Hilbert space. A fundamental question arises: are the complex numbers really necessary in the standard formalism of quantum theory? To answer this question, a quantum game has been developed to distinguish standard quantum theory from its real-number analog, by revealing a contradiction between a high-fidelity multiqubit quantum experiment and players using only real-number quantum theory. Here, using superconducting qubits, we faithfully realize the quantum game based on deterministic entanglement swapping with a state-of-the-art fidelity of 0.952. Our experimental results violate the real-number bound of 7.66 by 43 standard deviations. Our results disprove the real-number formulation and establish the indispensable role of complex numbers in the standard quantum theory.

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Physicists use mathematics to describe nature. In classical physics, the real number appears complete to describe the physical reality in all classical phenomenon, whereas the complex number is only sometimes employed as a convenient mathematical tool. In quantum mechanics, the complex number was introduced as the first principle in Schrödinger's equation and Heisenberg's commutation relation [1,2]. The complex-valued wave function has been shown to represent the physical reality of quantum objects under certain physically plausible assumptions [3]. Experimentally, the real and imaginary parts of the wave function have been directly measured [4]. Today, quantum mechanics with complex-valued wave functions seems the most successful theory to describe nature.

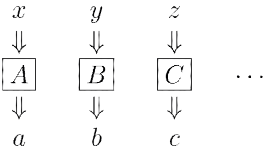
On the other hand, starting with von Neumann in 1936, many works [5–13] have shown that it is possible to universally simulate quantum systems using only real numbers by exploiting an enlarged Hilbert space in various alternative formalisms of quantum theory. For example, by adding an extra qubit $|\pm i\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$, a single-quantum system with a complex density matrix ρ and Hermitian operator H can be simulated through $tr(\rho H) = tr(\tilde{\rho}\tilde{H})$, where $\tilde{\rho}$ and $\tilde{H}$ are real and of the form:

$$\tilde{\rho} = (\rho \otimes |+i\rangle\langle +i| + \rho^* \otimes |-i\rangle\langle -i|)/2.$$
$$\tilde{H} = H \otimes |+i\rangle\langle +i| + H^* \otimes |-i\rangle\langle -i|.$$

Therefore, it is interesting to ask a fundamental question why the complex number is necessary in the standard formalism of quantum theory. The standard quantum theory is established by the following four axioms [14]: (1) A pure quantum system is described by a unit complex vector in a Hilbert space. (2) The state space of a composite quantum system is the tensor product of the state spaces of the component systems. (3) The dynamics of a closed quantum system is described by a unitary operator acting on the state vector. (4) A physical observable is described by a Hermitian operator, and the measurement outcome obeys the Born rule.

In this work, we intend to investigate the real-number formalism of standard quantum theory, which satisfies the formalism in the above four axioms but replaces the complex vectors and operators in the Hilbert space by corresponding real vectors and operators, respectively. In this formalism, the dimension of real Hilbert space is not restricted to be the same as the complex Hilbert space. A distinguishing feature of the standard quantum theory from other quantum theories is at the second axiom, where

Background of Bell inequality



$$p(a, b, c, \dots | x, y, z, \dots) \Rightarrow \text{correlation}$$

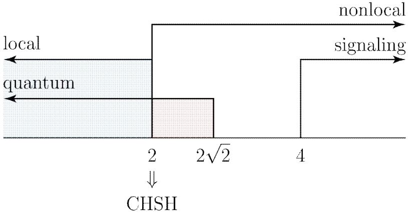
$$\exists p(\lambda) \text{ such that } p(a, b, c, \dots | x, y, z, \dots) = \sum_{\lambda} p(\lambda) \cdot p(a|x, \lambda) \cdot p(b|y, \lambda) \cdot p(c|z, \lambda) \cdot \dots$$

General Family of Bell inequality

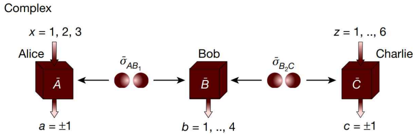
↓

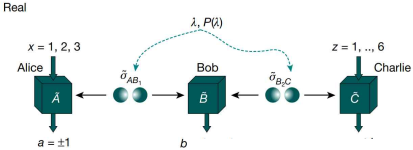
CHSH

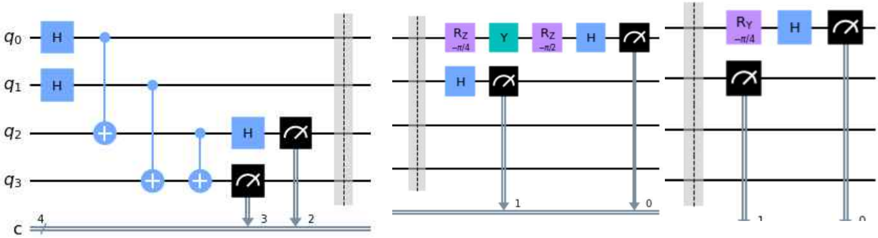
$$\sum_{a,b,c,\dots,x,y,z,\dots} C^{xyz\dots}_{abc\dots} p(a, b, c, \dots | x, y, z, \dots) \leq S_{lhv}$$



Entanglement Swapping







$$\mathcal{B} = \mathrm{Z}^{\mathrm{C}}(\mathrm{D}^{\mathrm{A}}_{\mathrm{z,x}} + \mathrm{E}^{\mathrm{A}}_{\mathrm{z,x}}) + \mathrm{X}^{\mathrm{C}}(\mathrm{D}^{\mathrm{A}}_{\mathrm{z,x}} - \mathrm{E}^{\mathrm{A}}_{\mathrm{z,x}}) + \mathrm{Z}^{\mathrm{C}}(\mathrm{D}^{\mathrm{A}}_{\mathrm{z,y}} + \mathrm{E}^{\mathrm{A}}_{\mathrm{z,y}}) - \mathrm{Y}^{\mathrm{C}}(\mathrm{D}^{\mathrm{A}}_{\mathrm{z,y}} - \mathrm{E}^{\mathrm{A}}_{\mathrm{z,y}}) + \mathrm{X}^{\mathrm{C}}(\mathrm{D}^{\mathrm{A}}_{\mathrm{x,y}} + \mathrm{E}^{\mathrm{A}}_{\mathrm{x,y}}) - \mathrm{Y}^{\mathrm{C}}(\mathrm{D}^{\mathrm{A}}_{\mathrm{x,y}} - \mathrm{E}^{\mathrm{A}}_{\mathrm{x,y}}).$$

$$\mathrm{CHSH}_3 := \mathrm{CHSH}(1,2;1,2) + \mathrm{CHSH}(1,3;3,4) + \mathrm{CHSH}(2,3;5,6) \le 6,$$

Maximally Violated of CHSH3

- Dotted

Complex Quantum Bound

(Maximally violated)

- Green

Real Quantum Bound

- Red

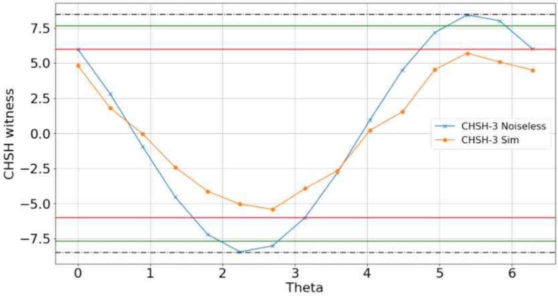
Classical Bound

Result

We performed $CHSH_3$ inequality violation experiment on various IBMQ quantum devices. To reproduce the experiment, run `Entanglement_Swapping_CHSH_3_with_2_graphs.ipynb` notebook with your choice of IBMQ devices. The meaningful result against quantum volume of the devices are shown in the table.

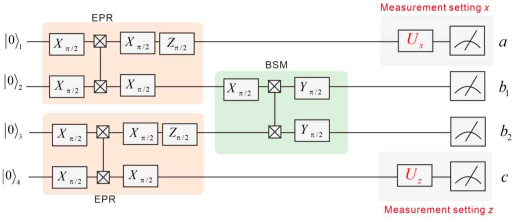
Device	QuantumVolume	# qubits	CHSH_3
ibmq_quito	16	5	1.769776616219977
ibm_lagos	32	7	5.812749912639812

Although we did not achieved violation due to noise, as QV improves, the bound becomes larger. It seems that $CHSH$ of specific measurment basis is responsible for the inviolation. We also presents experiment result figure performed on ibm_lagos.



Green line represents 7.66 and red line represents 6. Dashed black line is $6\sqrt{2}$. label 'CHSH-Sim' is the experiment result.

PRL Data comparison



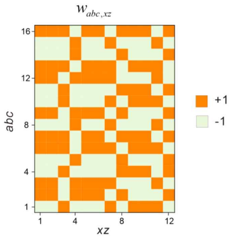
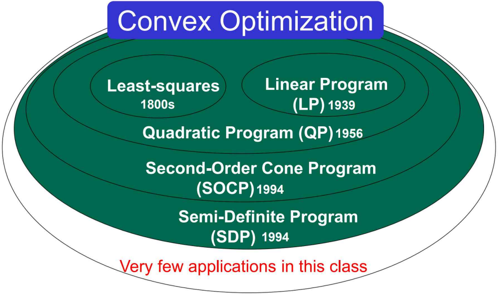


Figure S2. The weights W for the game score.

CHSH weight							abs									
	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
xz	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1
	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1
	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1

Conclusion

- used as a nice benchmark method
- Quantum Algorithm to solve SDP
- Readout error correction had minimal effect



THANK YOU!

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