

Qubit reduction techniques for bosonic systems

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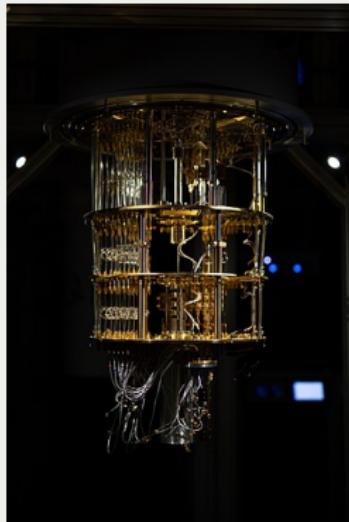


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Introduction

Quantum computing emerged and evolved with the promise of surpassing the constraints of classical computation.



Today, quantum computers allow for interesting experimentation, but they have several limitations; therefore, optimizing resources to execute quantum algorithms is crucial.

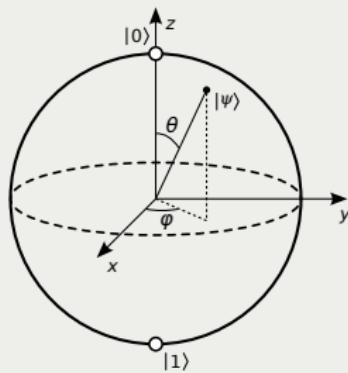
Several approaches have been developed in order to reduce the qubits requirements. This thesis introduces one such reduction method known as **Qubit Tapering**.

Quantum computing: qubits

A **qubit** is a two-level quantum system characterized by two states $|0\rangle$ and $|1\rangle$. A qubit can exist in any state of the form:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.



When dealing with two or more qubits, their collective state space is the tensor product (\otimes) of their respective Hilbert spaces.

In general, for a system of n qubits the base states are of the form $|x_1, x_2, \dots, x_n\rangle$, where $x \in \{0, 1\}$ and the dimension of the Hilbert space is 2^n .



The Stabilizer formalism

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

An operator given by the tensor product of single-qubit Pauli operators and the identity I , is referred to as a **Pauli string**.

The **Pauli group** P_n is the set of Pauli strings on n qubits, with the matrix multiplication as the group operation:

$$P_n = \{ \eta_i \mid \eta_i = \omega \sigma_1 \otimes \dots \otimes \sigma_n, \sigma_j \in \{I, \sigma_x, \sigma_y, \sigma_z\}, \omega \in \{+1, -1, +i, -i\} \}$$

A **stabilizer group** S_n is an abelian subgroup of the Pauli group P_n that does not contain $-I$.

Since every element is a Pauli operator, each stabilizer has eigenvalues ± 1 .

The stabilizer group can be described by its generators g_1, \dots, g_l .



Qubit Tapering

A n-qubit Hamiltonian can be expressed as

$$H = \sum h_i \eta_i \quad (1)$$

where h_i are real coefficients and η_i are elements of the Pauli group.

Consider a stabilizer group S_n which is a symmetry of the Hamiltonian such that

$$[g_j, \eta_i] = 0 \quad \forall i, j$$

There exist a unitary operator U_i which maps the symmetry generators to **single-qubit Pauli** stabilizers:

$$U_i g_i U_i^\dagger = \sigma_p^{(i)} \quad \forall i = 1, \dots, k$$

The Hamiltonian H can be transformed using the operators U_i :

$$\tilde{H} = U_i H U_i^\dagger$$

The transformed operators still commute: $[\sigma_p^{(i)}, U\eta_i U^\dagger] = 0 \quad \forall i$



Qubit Tapering

Therefore, each term of the transformed Hamiltonian $U\eta_i U^\dagger$ must act by the operator σ_p or I on the i^{th} qubit.

Example: if $n = 6$ and $\sigma_p^{(i)} = IXIIII$, then $U\eta_i U^\dagger$ will have the Pauli X or the identity I acting on the second qubit $\forall i$.

In each Pauli term of \tilde{H} , σ_p can be replaced one of its eigenvalues, +1 or -1, and **the associated qubit can be removed from the simulation**.

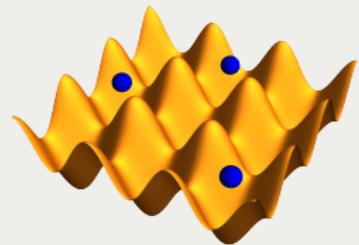
$$\begin{array}{ccc} g_i & \longrightarrow & G_p^{(i)} \\ & & \\ H = \sum_i h_i \hat{\eta}_i & \longrightarrow & U_i \hat{\eta}_i U_i^\dagger \end{array}$$

$G_p \rightarrow +1$
 $G_p \rightarrow -1$



The Bose-Hubbard model

Qubit tapering can be applied to a physical system described by the **Bose-Hubbard model**, which characterizes the motion of **bosons in a lattice**.



The second quantization Bose-Hubbard **Hamiltonian** is:

$$H_{BH}/\hbar = -t \sum_{\langle i,j \rangle} a_i^\dagger a_j + \frac{U}{2} \sum_i n_i(n_i - 1) - \mu \sum_i n_i$$

The Bose-Hubbard model **conserves the number of bosons of system**. Therefore, the Hamiltonian is invariant under the following transformation:

$$U_\alpha = e^{i\alpha \sum_k n_k}, \alpha \in [0, 2\pi) \Rightarrow H = U_\alpha H U_\alpha^\dagger$$

The **state of a system** of composed of N_b bosons in motion in a N-site lattice is of the form:

$$|n_1, \dots, n_2, \dots, n_N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_N\rangle$$

where n_i is the number of bosons localized in a site.

Qubit mappings

In order to describe a physical system on a quantum computer it is necessary to *describe its model using qubits*.

In general, there are various **encodings** for bosonic systems, with different properties.

Binary mapping = binary representation of the occupation number

$$|n_i\rangle = |010 \dots 110\rangle = |0\rangle_1 \otimes |1\rangle_2 \otimes \dots \otimes |1\rangle_{N_q-1} \otimes |0\rangle_{N_q}$$

M qubits correspond to at most $2^M - 1$ bosons in each site.

According to the encoding chosen the bosonic operators representations are obtained.

A class called **BoseHubbardHamiltonian** was implemented with the Python package Qiskit, in order to simulate a system described by the BoseHubbard model employing the binary encoding.

Is it possible to reduce the number of qubits utilized?



Application of the Tapering procedure

Consider a system comprising three sites, each described by two qubits.

$$|n_1\rangle = |q_1 q_2\rangle$$

1

$$|n_2\rangle = |q_3 q_4\rangle$$

2

$$|n_3\rangle = |q_5 q_6\rangle$$

3

$$|n_1 n_2 n_3\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle = |q_1 q_2\rangle \otimes |q_3 q_4\rangle \otimes |q_5 q_6\rangle$$

According to the binary mapping, the Hamiltonian of the system has the form:

$$0.933 + 0.000 j \quad \text{IXIXII} \quad +$$

$$0.483 + 0.000 j \quad \text{IXXXII} \quad +$$

$$0.483 + 0.000 j \quad \text{IXYYII} \quad +$$

...

It is now possible to apply Qubit Tapering.

The application of Qubit Tapering relies on **Symmer**, a Python package for reducing qubit requirements in quantum computing.

Results: symmetries of H and Tapering

The Tapering procedure allows to remove one qubit from the simulation.
The symmetry found is the following:

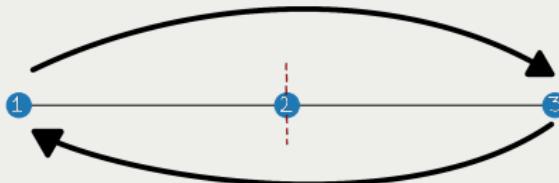
$$IZIZIZ$$

This operator represents the **parity of the total number of particles** in the system and it corresponds to

$$\alpha = \pi \quad \text{in} \quad U_\alpha = e^{i\alpha \sum_k n_k}.$$

This result is the same regardless of the lattice geometry or of the number of qubits describing each site.

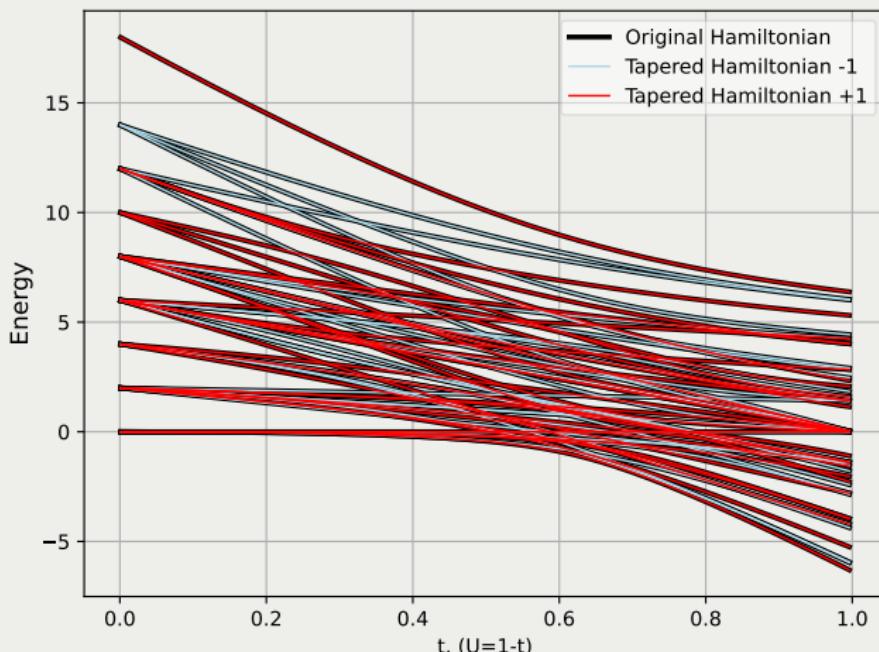
The Bose-Hubbard model for a specific lattice also possesses geometric symmetries in the system, however they are not suitable for Tapering, since they commute with H , but not with each of its terms.



Results: eigenvalues of the original and tapered Hamiltonian

To confirm the success of the Tapering procedure, the eigenvalues of the original Hamiltonian can be compared to those of the tapered Hamiltonian.

It is expected that the two coincide.



Conclusions

- Stabilizers are useful to remove certain qubits from simulations of physical systems;
- It is not enough for a symmetry operator to commute with the Hamiltonian, it must commute with all its Pauli strings;
- Qubit Tapering is a method that preserves the energy spectrum of the Hamiltonian;
- It could be valuable to search for **alternative mappings** beyond the unary and binary mappings presented in this thesis, seeking more efficient approaches;
- The Tapering procedure could be expanded to **qudits**.



Grazie per l'attenzione

Quantum computing: quantum gates

In quantum computing, data manipulation is performed by quantum logic gates. They can be represented by unitary matrices. Some single qubit-gates are:

$$\alpha |0\rangle + \beta |1\rangle \xrightarrow{X} \beta |0\rangle + \alpha |1\rangle$$

$$\alpha |0\rangle + \beta |1\rangle \xrightarrow{Z} \alpha |0\rangle - \beta |1\rangle$$

$$\alpha |0\rangle + \beta |1\rangle \xrightarrow{H} \alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

The stabilizer formalism and quantum error correction

Pauli error model:

$$X |\psi\rangle = \alpha |1\rangle + \beta |0\rangle$$

$$Z |\psi\rangle = \alpha |0\rangle - \beta |1\rangle$$

In order to identify and correct errors in a quantum circuit, the base states of logical qubits can be encoded using *redundant qubits*:

$$|\psi\rangle_e = \alpha |000\rangle + \beta |111\rangle$$

Suppose that an error occurs:

$$|\tilde{\psi}\rangle_e = E |\psi\rangle = \alpha |100\rangle + \beta |011\rangle$$

Stabilizers allow to detect errors without causing the collapse of the system's state:

- the common +1 eigenspace constitutes a stabilizer code;
- those stabilizers that **anticommute with the error operator E** are used to identify the error.

This latter property is employed to build the unitary transformations of the symmetry generators in Qubit Tapering.

Finding the unitary transformations of symmetry generators

The generators of the stabilizer group symmetry of H are transformed as:

$$U_i g_i U_i^\dagger = \sigma_p^{(i)} \quad \forall i = 1, \dots, k$$

The unitary U_i can be constructed as a **Clifford operator**:

$$U \gamma U^\dagger \in P_n \quad \forall \gamma \in P_n$$

Consider $\sigma_p^{(i)}$ such that it anti-commutes with g_i and commutes with all the other generators g_j $\forall j \neq i$:

$$\sigma_p^{(i)} g_j = (-1)^{\delta_{ij}} g_j \sigma_p^{(i)}$$

This allows to define the unitary Clifford operators

$$U_i = \frac{1}{\sqrt{2}} (\sigma_p^{(i)} + g_i) \quad i = 1, \dots, k$$

It can be shown that

$$U_i^2 = 1, \quad U_i \sigma_p^{(i)} U_i^\dagger = g_i, \quad U_j \sigma_p^{(i)} U_j^\dagger = \sigma_p^{(i)}, \quad \forall j \neq i$$

Finding the symmetries

Pauli strings can be represented as binary strings ($a_x|a_z$) and the corresponding operator $\eta(a_x|a_z)$ is obtained by

$$\eta(a_x|a_z) = \left(\prod_{i \in a_x} \sigma_x^i \right) \cdot \left(\prod_{j \in a_z} \sigma_z^j \right)$$

$$\eta(a_x|a_z) \eta(b_x|b_z) = (-1)^{a_x b_x + a_z b_z} \eta(b_x|b_z) \eta(a_x|a_z)$$

The two operators commute if $a_x b_x + a_z b_z = 0 \pmod{2}$.

All the Pauli strings that appear in the Hamiltonian can be represented as a binary matrix

$$G = \begin{pmatrix} G_x & G_z \end{pmatrix}$$

where the j^{th} line coincides with $(a_x|a_z)$ representing η_j .

$$\begin{pmatrix} a_x & a_z \\ b_x & b_z \end{pmatrix} \begin{pmatrix} s_x \\ s_z \end{pmatrix} = \begin{pmatrix} s_x a_x + s_z a_z \\ s_x b_x + s_z b_z \end{pmatrix}$$

The symmetry elements correspond to the kernel of G matrix.

Binary mapping: states

$$|n_i\rangle = |010 \dots 110\rangle = |0\rangle_1 \otimes |1\rangle_2 \otimes \dots \otimes |1\rangle_{N_q-1} \otimes |0\rangle_{N_q}$$

$$|0\rangle \leftrightarrow |0\rangle_1 \otimes |0\rangle_2 \otimes \dots \otimes |0\rangle_{N_q-1} \otimes |0\rangle_{N_q}$$

$$|1\rangle \leftrightarrow |0\rangle_1 \otimes |0\rangle_2 \otimes \dots \otimes |0\rangle_{N_q-1} \otimes |1\rangle_{N_q}$$

$$|2\rangle \leftrightarrow |0\rangle_1 \otimes |0\rangle_2 \otimes \dots \otimes |1\rangle_{N_q-1} \otimes |0\rangle_{N_q}$$

⋮

$$|2^{N_q} - 1\rangle \leftrightarrow |1\rangle_1 \otimes |1\rangle_2 \otimes \dots \otimes |1\rangle_{N_q-1} \otimes |1\rangle_{N_q}$$

Binary mapping: operators

$$a_i^\dagger = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \sqrt{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{2^{N_q} - 1} & 0 \end{pmatrix}$$

$$a_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \sqrt{2} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \sqrt{2^{N_q} - 1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$n_i = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 2^{N_q} - 1 \end{pmatrix}$$

Bose-Hubbard model quantum phase transition

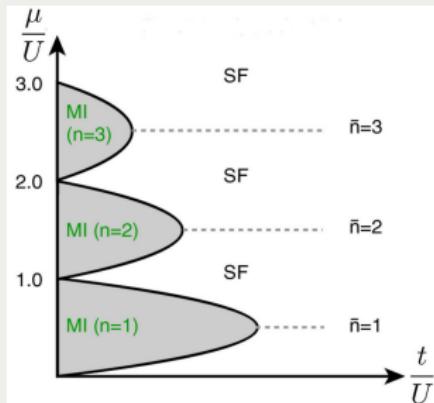
Competition kinetic vs interaction term \Rightarrow quantum phase transitions

■ Superfluid phase: $t \gg U$

- Eigenstates consists of single particle wavefunctions spread out over the entire lattice;
- In the ground state all particles occupy the same lowest momentum state;
- Adding a particle costs no energy and the energy spectrum is gapless.

■ Mott insulator: $t \ll U$

- Particles are localized in the sites of the lattice;
- The ground state consists of localized particles distributed in order to minimize the interaction term, depending of the parameters μ and U .
- Adding or removing a one boson from a lattice site requires a finite amount of energy, the energy spectrum is gapped.



Site exchange symmetry

SWAP operator for two qubits:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$SWAP |01\rangle = |10\rangle \quad SWAP |10\rangle = |01\rangle ,$$

$$SWAP = \frac{1}{2} (II + XX + ZZ + YY)$$

SWAP site 1 and site 3 in the three site lattice system:

$$SWAP_1 = \frac{1}{2} (IIIII + XIIIXI + YIIYI + ZIIIIZI)$$

$$SWAP_2 = \frac{1}{2} (IIIII + IXIIIX + IYIIIY + IZIIIIZ)$$

The Hamiltonian selected energy spectrum

When the bosonic system is mapped into the finite Hilbert space of qubits, a maximum number of bosons, denoted as $N_{b\max}$, is selected. Only those states for which the total number of particles is less than or equal to the limit $N_{b\max}$ accurately describe the physical system.

