ECE 661 Computer Vision: HW1

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1 Problem 1

For a 2D point $(x, y) \in \mathbb{R}^2$, the homogeneous coordinate representation is the 3D vector $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \in \mathbb{R}^3$.

Then the 3D vector $\begin{pmatrix} kx \\ ky \\ k \end{pmatrix}$ for any non-zero value k represents the same point as well. So all the following points in the representational space $\mathbb{R}^3$ are the homogeneous coordinates of the origin $(0, 0)$ in the physical space $\mathbb{R}^2$

$$\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} \text{ for any non-zero value } k \in \mathbb{R}$$

2 Problem 2

No. The points at infinity in the physical plane $\mathbb{R}^2$ are NOT the same. However, the points form a line named *line at infinity* $\mathbf{l}_\infty = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The reason is that the other two coordinates are not constrained while the last coordinate of ideal points $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$ is pinpointed.

For example, the two ideal points

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are different points in the physical plane $\mathbb{R}^2$ since the points approaches infinity along with different directions.

3 Problem 3

The degenerate conic is represented by $\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$ where $\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$ and $\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$.

Then $\mathbf{l}\mathbf{m}^T = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{pmatrix}$.

In column space, the column vectors include $m_i \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, i = 1, 2, 3$.

In row space, the row vectors include $l_j \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix}, j = 1, 2, 3$.

It's clear to see that dimensionalities of both column space and row space are 1. Therefore, the rank of the matrix $\mathbf{l}\mathbf{m}^T$ is 1. Similarly, the rank of $\mathbf{m}\mathbf{l}^T$ is also 1.

Since $rank(\mathbf{A} + \mathbf{B}) \leqslant rank(\mathbf{A}) + rank(\mathbf{B})$ for matrices $\mathbf{A}$ and $\mathbf{B}$, we have

$$rank(\mathbf{l}) \leqslant rank(\mathbf{l}\mathbf{m}^T) + rank(\mathbf{m}\mathbf{l}^T) = 1 + 1 = 2 \tag{1}$$

That is, the matrix rank of a degenerate conic can never exceed 2.

4 Problem 4

a. (a) Compute line $\mathbf{l}_1$

$$\mathbf{l}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} \tag{2}$$

(b) Compute line $\mathbf{l}_2$

$$\mathbf{l}_2 = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 13 \end{pmatrix} \tag{3}$$

(c) Compute the intersection of two lines

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 4 \\ 13 \end{pmatrix} = \begin{pmatrix} 39 \\ 65 \\ 23 \end{pmatrix} \tag{4}$$

Therefore, the intersection point is $(39/23, 65/23)$. The computation takes the above three steps.

b. If the second line passed through $(-7, -5)$ and $(7, 5)$

$$\mathbf{l}_2 = \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 14 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix} \tag{5}$$

Therefore, the intersection point is $(0, 0)$. It will take only two step since both line go across the origin.

5 Problem 5

First compute line $\mathbf{l}_1$

$$\mathbf{l}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \tag{6}$$

Then compute line $\mathbf{l}_2$

$$\mathbf{l}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix} \tag{7}$$

Then compute the intersection of two lines

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix} = \begin{pmatrix} 75 \\ -45 \\ 0 \end{pmatrix} \tag{8}$$

From the result, the intersection point is an Ideal Point. Therefore, the two lines are parallel.

6 Problem 6

From the description, the representation of the ellipse is given by

$$\frac{(x-3)^2}{a^2} + \frac{(y-2^2)}{b^2} = 1 \tag{9}$$

where $a = 1$ and $b = 1/2$. Equivalently, we have

$$x^2 + 4y^2 - 6x - 16y + 24 = 0 \tag{10}$$

Therefore, the HC representation is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 24 \end{pmatrix} \tag{11}$$

The tangent to $\mathbf{C}$ at origin point $\mathbf{p}$ is

$$\mathbf{l} = \mathbf{C} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 24 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 24 \end{pmatrix} \tag{12}$$

Then we have

$$\mathbf{l} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 24 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -24 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} \tag{13}$$

$$\mathbf{l} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -8 \\ 24 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \tag{14}$$

Therefore, the intersection points of the polar line with $\mathbf{x}-$ and $\mathbf{y}-$axes are $(8,0)$ and $(0,3)$.

7 Problem 7

The two lines are

$$\mathbf{l}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \mathbf{l}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \tag{15}$$

Then

$$\mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix} \tag{16}$$

Therefore, the intersection point of two lines is $(1/2, -1/3)$.