

## Homework 1: on $\sigma$ -algebra (39/40)

### Borel vs Open

Let  $X$  be a metric space such that every subset of  $X$  is Borel set. Does it follow that every subset of  $X$  is open? Give a proof or a counterexample.

**Sol.** It is not true.

Every subset of  $X$  is Borel set  $\Leftrightarrow \mathcal{P}(X) \subset \mathcal{B}_X$ . And We know  $\mathcal{B}_X \subset \mathcal{P}(X)$ , so it is equivalent to saying that  $\mathcal{B}_X = \mathcal{P}(X)$ .

So consider this counterexample:  $\mathbb{Q}$  with the Euclidean metric.

Claim: every singleton set in  $\mathbb{Q}$  is closed, thus in  $\mathcal{B}_{\mathbb{Q}}$ . This is because this only sequence in a singleton set is the point itself repeating, thus converging to itself, in the singleton set. This proves the claim.

And since  $\mathbb{Q}$  is countable, every subset of  $\mathbb{Q}$  is a countable union of singleton sets, thus by property of  $\sigma$ -algebra, every subset of  $\mathbb{Q}$  is in  $\mathcal{B}_{\mathbb{Q}}$ . Thus:

$$\mathcal{B}_{\mathbb{Q}} = \mathcal{P}(\mathbb{Q})$$

But clearly, **not every subset in  $\mathbb{Q}$  is open**. Consider any singleton set,  $\{1\}$  as an example. Any open ball centered at 1 is not contained in  $\{1\}$ , thus contradicting the statement.

### Restriction of a $\sigma$ -algebra to a Subset

Let  $X$  be a set, and  $Y \subset X$  a subset.

(a) Given a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$ , prove that

$$\mathcal{A}|_Y := \{E \cap Y \mid E \in \mathcal{A}\}$$

is a  $\sigma$ -algebra on  $Y$ .

(b) Given a  $\sigma$ -algebra  $\mathcal{B}$  on  $Y$ , prove that there exists a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$  such that  $\mathcal{A}|_Y = \mathcal{B}$ .

(c) Is the  $\sigma$ -algebra  $\mathcal{A}$  in (b) unique? Give a proof or a counterexample.

**Remark** 这表示任何一个 measurable space 都可以对其中的一个 subspace 取一个 submeasurable space

#### Proof

- (a) 1. Since  $\emptyset \in \mathcal{A}$ ,  $\emptyset \cap Y = \emptyset$ , we have  $\emptyset \in \mathcal{A}|_Y$   
2. Let  $F \in \mathcal{A}|_Y$ , we must have  $E \in \mathcal{A}$  s.t.  $E \cap Y = F$ . Since  $E \in \mathcal{A}$ , we have  $X \setminus E \in \mathcal{A}$ , so  $X \setminus E \cap Y \in \mathcal{A}|_Y$ . Since  $E \cap Y = F$  and  $Y = (E \cap Y) \cup ((X \setminus E) \cap Y)$ , it implies  $(X \setminus E) \cap Y = Y \setminus F$ , therefore  $Y \setminus F \in \mathcal{A}|_Y$ .  
3. Let  $F_1, F_2, \dots$  be a sequence of subsets in  $\mathcal{A}|_Y$ . Then for each  $i \in \mathbb{N}$ , we have  $F_i = E_i \cap Y$  for some  $E_i \in \mathcal{A}$ . Then  $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (E_i \cap Y) = (\bigcup_{i=1}^{\infty} E_i) \cap Y \in \mathcal{A}|_Y$  since  $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$ .

(b) Let  $\mathcal{B}$  be a  $\sigma$ -algebra on  $Y$ .

prove that there exists a  $\sigma$ -algebra  $\mathcal{A}$  on  $X$  such that  $\mathcal{A}|_Y = \mathcal{B}$ . Consider let

$$\mathcal{A} := \{E \subset X \mid E \cap Y \in \mathcal{B}\}$$

Then

$$\mathcal{A}|_Y = \{E \cap Y \mid E, Y \subset X, E \cap Y \in \mathcal{B}\} = \mathcal{B}$$

We then prove that this is a  $\sigma$ -algebra on  $X$ .

1.  $\emptyset \cap Y = \emptyset$  so  $\emptyset \in \mathcal{A}$ .

2. **Closed under complement:** Let  $E \in \mathcal{A}$ , we have  $E \cap Y \in \mathcal{B}$ , so  $Y \setminus (E \cap Y) = Y \setminus E \in \mathcal{B}$ .

Then  $(X \setminus E) \cap Y = Y \setminus E \in \mathcal{B}$ , so  $X \setminus E \in \mathcal{A}$ .

3. **Closed under countable union:** Let  $E_1, E_2, \dots$  be a sequence in  $\mathcal{A}$ , then  $E_n \cap Y \in \mathcal{B}$ . for each  $n$ . Hence

$$\left( \bigcup_{n=1}^{\infty} E_n \right) \cap Y = \bigcup_{n=1}^{\infty} (E_n \cap Y) \in \mathcal{B},$$

since  $\mathcal{B}$  is a  $\sigma$ -algebra on  $Y$ . Therefore,  $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ .

(c) This is not unique.

Counterexample:

$$X = \{0, 1, 2\}, Y = \{0\} \subset X$$

Consider

$$A_1 := \mathcal{P}(X), A_2 := \{\emptyset, \{0\}, \{1, 2\}, X\}$$

are valid  $\sigma$ -algebra on  $X$ .

Then we have  $A_1|_Y = A_2|_Y = \{\emptyset, \{0\}\}$ , while  $A_1$  is different from  $A_2$ .

## Invariance Properties of the Borel $\sigma$ -algebra on $\mathbb{R}^n$

(a) Prove that  $\mathcal{B}(\mathbb{R}^n)$  is translation invariant, i.e., if  $A \subset \mathbb{R}^n$  is a Borel measurable set, then

$$t + A := \{t + x \mid x \in A\}$$

is a Borel measurable set for every  $t \in \mathbb{R}^n$ . (Hint: For any fixed  $t$ , show that  $A = \{B \subset \mathbb{R}^n : t + B \in \mathcal{B}(\mathbb{R}^n)\}$  is a  $\sigma$ -algebra.)

(b) Prove that  $\mathcal{B}(\mathbb{R}^n)$  is scaling invariant, i.e., if  $A \subset \mathbb{R}^n$  is a Borel measurable set, then

$$\lambda A = \{\lambda x \mid x \in A\}$$

is a Borel measurable set for every  $\lambda \in \mathbb{R}$ .

(1)

### Proof

Fix  $t \in \mathbb{R}^n$ . Define

$$\mathcal{A} := \{B \subseteq \mathbb{R}^n : t + B \in \mathcal{B}(\mathbb{R}^n)\}.$$

We want to show that  $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$ . We first show that  $\mathcal{A}$  is a  $\sigma$ -algebra.

1.  $\emptyset \in \mathcal{A}$  since  $t + \emptyset = \emptyset \in \mathcal{B}(\mathbb{R}^n)$ .

2.  $\mathcal{A}$  is closed under complement: Let  $B \in \mathcal{A}$ , then  $t + B \in \mathcal{B}(\mathbb{R}^n)$ . The complement  $(t + B)^c$  is also in

$\mathcal{B}(\mathbb{R}^n)$ . Observe

$$t + B^c = t + \mathbb{R}^n \setminus B = (t + \mathbb{R}^n) \setminus (t + B) = \mathbb{R}^n \setminus (t + B) = (t + B)^c$$

Since  $t + B$  is Borel, its complement is Borel, hence  $t + B^c$  is Borel, so  $B^c \in \mathcal{A}$ .

3.  $\mathcal{A}$  is closed under countable unions: Let  $B_k \in \mathcal{A}$  for  $k = 1, 2, \dots$ , then  $t + B_k \in \mathcal{B}(\mathbb{R}^n)$ . Thus

$$t + \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} (t + B_k) \in \mathcal{B}(\mathbb{R}^n).$$

Hence  $\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}$ . These three properties show that  $\mathcal{A}$  is a  $\sigma$ -algebra.

Since  $t + U$  is open if  $U$  is open in  $\mathbb{R}^n$ ,  $\mathcal{A}$  contains all open sets. Since  $\mathcal{B}(\mathbb{R}^n)$  is the smallest  $\sigma$ -algebra containing all open sets in  $\mathbb{R}^n$ , we have  $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{A}$ . Hence suppose  $A \in \mathcal{B}(\mathbb{R}^n)$ , then  $A \in \mathcal{A}$ , so  $t + A \in \mathcal{B}(\mathbb{R}^n)$ . This completes the proof of translation invariance.

(2)

**Proof** Fix  $\lambda \in \mathbb{R}$ . Case 1:  $\lambda = 0$ , then  $\lambda A = \{0\}$  if  $A \neq \emptyset$ , and  $\lambda A = \emptyset$  otherwise. Both  $\{0\}$  (closed set) and  $\emptyset$  is Borel set.

Case 2:  $\lambda \neq 0$ . We define

$$\mathcal{A} := \{B \subseteq \mathbb{R}^n : \lambda B \in \mathcal{B}(\mathbb{R}^n)\}.$$

We want to show that  $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$ . We first show that  $\mathcal{A}$  is a  $\sigma$ -algebra.

1.  $\emptyset \in \mathcal{A}$  since  $\lambda \emptyset = \emptyset$ .

2.  $\mathcal{A}$  is closed under complement: Let  $B \in \mathcal{A}$ , then  $\lambda B \in \mathcal{B}(\mathbb{R}^n)$ , then  $(\lambda B)^c$  is also in  $\mathcal{B}(\mathbb{R}^n)$ . Observe  $(\lambda B)^c = \lambda B^c$ , so  $\lambda B^c \in \mathcal{B}(\mathbb{R}^n)$ , therefore  $B^c \in \mathcal{A}$ . 3.  $\mathcal{A}$  is closed under countable unions: Let  $B_k \in \mathcal{A}$  for  $k = 1, 2, \dots$ , then  $\lambda B_k \in \mathcal{B}(\mathbb{R}^n)$ . Thus

$$\lambda \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} (\lambda B_k) \in \mathcal{B}(\mathbb{R}^n).$$

Hence  $\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}$ . These three properties show that  $\mathcal{A}$  is a  $\sigma$ -algebra.

Since  $\lambda \neq 0$ ,  $\lambda U$  is open iff  $U$  is open in  $\mathbb{R}^n$ , thus  $\mathcal{A}$  contains all open sets, so  $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{A}$ ,

Hence if  $A \in \mathcal{B}(\mathbb{R}^n)$ , we have  $A \in \mathcal{A}$ , therefore  $\lambda A \in \mathcal{B}(\mathbb{R}^n)$ . This completes the proof of translation invariance.

## Hex and Such

Let  $A \subset [0, 1]$  be the set of real numbers in  $[0, 1]$  having a hexadecimal expansion with the digit 5 appearing infinitely many times, and the ‘digit’ E appearing at most finitely many times. Prove that  $A$  is a Borel set. (Hint: see p. 2 of Folland’s book.)

**Proof** Define:

$$B := \{x \in [0, 1] \mid \text{the digit '5' appears infinitely many times in the hex expansion of } x\}.$$

$$C := \{x \in [0, 1] \mid \text{the digit 'E' appears at most finitely many times in the hex expansion of } x\}.$$

Then clearly

$$A = B \cap C.$$

Hence it suffices to show that  **$B$  and  $C$  are Borel sets**, since intersection of two Borel sets is a Borel set. And thus it suffices to show that  **$B^c$  and  $C$  are Borel sets**. Note

$$B^c = \{x \in [0, 1] \mid \text{the digit '5' appears at most finitely many times in the hex expansion of } x\}$$

, so the proof for  $B^c$  and  $C$  are about the same. We now show  $B^c$  is a Borel set: We define

$$C_{d_1 d_2 \dots d_n} := \{x \in [0, 1] : \text{the first } n \text{ hexadecimal digits of } x \text{ are } d_1, d_2, \dots, d_n\},$$

where each  $d_i$  is one of the 16 hexadecimal digits  $\{0, 1, 2, \dots, 9, A, B, C, D, E, F\}$ . Then the set contains all real numbers between  $\frac{d_1 d_2 \dots d_n}{16^n}$  and  $\frac{d_1 d_2 \dots d_n + 1}{16^n}$ , so actually it is an interval:

$$C_{d_1 d_2 \dots d_n} = \left[ \frac{d_1 d_2 \dots d_n}{16^n}, \frac{d_1 d_2 \dots d_n + 1}{16^n} \right)$$

Since it is an interval, it is a Borel set on  $[0, 1]$ . And we define:

$$D_N = \{x : \text{from digit } N \text{ onward, there are no '5's}\}.$$

Then we have

$$B^c = \bigcup_{N=1}^{\infty} D_N,$$

So it suffices to prove that each  $D_N$  is Borel set, since a countable union of Borel sets is Borel set.

**Claim : any  $D_N$  is a Borel set.** To prove this, we fix an  $N$  and define for each  $n \geq N$

$$E_n = \{x \in [0, 1] : d_n(x) \neq 5\}.$$

Then we have

$$E_n = \bigcup_{d_i \in \{1, \dots, F\} \forall 1 \leq i \leq n, d_i \neq 5} C_{d_1 d_2 \dots d_n}$$

Thus **each  $E_n$  is a Borel set** since it is a finite union of Borel set, which shows that  $D_N$  is Borel set, since

$$D_N = \bigcap_{k=N}^{\infty} E_k.$$

This finishes the proof that  $B^c$  is a Borel set, and by a similar argument,  $C$  is a Borel set, and thus  $A = B \cap C$  is a Borel set.

## Admissible Annuli generating $\mathcal{B}(\mathbb{R}^n)$

Define an admissible annulus in  $\mathbb{R}^2$  to be a set of the form

$$\{(x, y) \in \mathbb{R}^2 \mid r^2 < (x - a)^2 + (y - b)^2 < R^2\},$$

where  $a, b \in \mathbb{Q}$ ,  $r, R \in \mathbb{Q}_{>0}$ , and  $r < R$ .

- (a) Prove that there are only countably many admissible annuli.
  - (b) Prove that every open subset of  $\mathbb{R}^2$  is a countable union of (not necessarily disjoint) admissible annuli.
  - (c) Prove that the Borel  $\sigma$ -algebra on  $\mathbb{R}^2$  is generated by the collection of admissible annuli.
- (1)

**Proof** Let

$$A := \{\text{all admissible annulis in } \mathbb{R}^2\}$$

And we define

$$f : \mathbb{Q}^4 \rightarrow A \quad (1.1)$$

$$(a, b, r, R) \mapsto \{(x, y) \in \mathbb{R}^2 \mid r^2 < (x - a)^2 + (y - b)^2 < R^2\} \quad (1.2)$$

Since a Annuli defined by this  $(a, b, r, R)$  is unique, this is a well-defined function; and since every admissible annulis can be defined by an element of  $\mathbb{Q}^4$ , this map is surjective. Therefore  $\text{card}(A) \leq \text{card}(\mathbb{Q}^4)$ , so  $A$  is countable.

(2)

**Proof Claim 1:** every open set in  $\mathbb{R}^2$  is a countable union of open balls, each centered at some  $q \in \mathbb{Q}^2$ .

Proof for Claim 1:

Let  $U$  be an open set in  $\mathbb{R}^2$ . Define

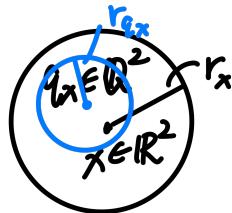
$$\mathbb{Q}_U := U \cap \mathbb{Q}^2$$

By definition, every point in  $U$  have an open ball centered at it that is completely contained in  $U$ , so we pick such ball  $B_{r_x}(x)$  for each  $x \in U$ . Since  $\mathbb{Q}^2$  is dense in  $\mathbb{R}^2$ , for each  $x \in U$  and each corresponding  $r_x$ , we can find a rational point  $q_x \in \mathbb{Q}^2$  such that  $|q_x - x| < \frac{r_x}{3}$ . (Or more generally, as small as we wish.)

Let  $r_{q_x} > 0$  be chosen so that  $r_{q_x} = \frac{r_x}{3}$ , Then observe that  $x \in B(q_x, r_{q_x})$

$$B(q_x, r_{q_x}) \subset B(x, r_x) \subset U$$

which follows from the triangle inequality.



For each  $q \in \mathbb{Q}_U$ , we define:

$$r_{q,sup} := \sup\{r_{q_x} \mid q \text{ is chosen by } x\}$$

Now we have:

$$U \subset \bigcup_{q \in U_q} B_{r_{q,sup}}(q)$$

This is because for each each  $x \in U$ ,  $x \in B_{r_{q_x}}(q_x) \subset B_{r_{q,sup}}(q_x)$

And we also have the other direction:

$$\bigcup_{q \in U_q} B_{r_{q,sup}}(q) \subseteq U$$

since every  $B_{r_q}(q)$  is guaranteed to be the subset of some ball around some  $x \in U$ . All togethe we have

$$U = \bigcup_{q \in U_q} B_{r_{q,sup}}(q)$$

This finishes the proof of claim 1.

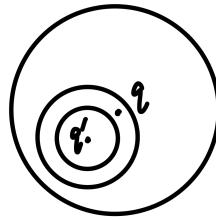
**Claim 2: every open ball centered at some  $q \in \mathbb{Q}^2$  is a countable union of admissible annulises with the same center, together with another admissible annulis whose center is also rational.** Proof for Claim 2: Let  $q = (a, b) \in \mathbb{Q}^2$ .

We have

$$B(q, R) \setminus \{q\} = \bigcup_{n=1}^{\infty} \left\{ (x, y) : (R - \frac{1}{n})^2 < (x - a)^2 + (y - b)^2 < R^2 \right\}$$

-1, 这里写的略有问题, 因为  $R$  不一定是 rational 的, 不过我们可以用 density of  $\mathbb{Q}$  in  $\mathbb{R}$  来写. It remains to cover the center. Let  $q' := (a', b') \in \mathbb{Q}^2$  such that  $R/6 < |q' - q| < R/3$ ,  $r' := R/6$  and  $R' := R/2$ . Then the annuli  $A(a', b', r', R')$  defined by the four parameters is contained in the  $B(q, R)$  and it covers  $\{q\}$ . Therefore

$$B(q, R) = \left( \bigcup_{n=1}^{\infty} \left\{ (x, y) : (R - \frac{1}{n})^2 < (x - a)^2 + (y - b)^2 < R^2 \right\} \right) \cup A(a', b', r', R')$$



This finishes the proof of Claim 2.

Combining Claim 1 and Claim 2, we can conclude that **every open subset of  $\mathbb{R}^2$  is a countable union of admissible annuli.**

(3)

### Proof

As defined,

$$\mathcal{B}(\mathbb{R}^2) = \langle \mathcal{T}_{metric} \rangle = \langle \text{all open sets in } \mathbb{R}^2 \rangle$$

Let

$$A := \{ \text{all admissible annulises in } \mathbb{R}^2 \}$$

Every admissible annuli is open in  $\mathbb{R}^2$ , so

$$A \subset \{ \text{all open sets in } \mathbb{R}^2 \}$$

and since  $\mathcal{B}(\mathbb{R}^2)$  is a  $\sigma$ -algebra, we have

$$\langle A \rangle \subset \langle \{ \text{all open sets in } \mathbb{R}^2 \} \rangle = \mathcal{B}(\mathbb{R}^2)$$

by the proposition proved in class. And by (2), any open set is a countable union of admissible annulises, therefore every open set is in  $\langle A \rangle$  since any countable union of sets in a  $\sigma$ -algebra is still in the set. So

$$\{ \text{all open sets in } \mathbb{R}^2 \} \subset \langle A \rangle$$

This finishes the proof that

$$\langle A \rangle = \langle \{ \text{all open sets in } \mathbb{R}^2 \} \rangle = \mathcal{B}(\mathbb{R}^2)$$

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## Nur für Verrückte

(It's really not necessary to attempt these problems. Do not hand them in!)

- (1) Let  $X$  be a set, and define two operations on  $\mathcal{P}(X)$ :
  - The “product” of two subsets  $E, F \subset X$  is the intersection  $E \cap F$ .
  - The “sum” of two sets  $E, F \subset X$  is the symmetric difference  $E \Delta F$ .
  - (a) Prove that these operations endow  $\mathcal{P}(X)$  with the structure of a commutative ring. What are the additive and multiplicative units? Prove that this ring is idempotent.
  - (b) Let us say that a nonempty subset  $A \subset \mathcal{P}(X)$  is a ring if it is closed under differences and finite unions. In other words, if  $E, F \in A$ , then  $E \setminus F \in A$  and  $E \cup F \in A$ . Prove that a subset  $A \subset \mathcal{P}(X)$  is an algebra iff it is a ring containing  $X$ .
  - (c) Prove that a nonempty subset  $A \subset \mathcal{P}(X)$  is a ring iff it is a subring of  $\mathcal{P}(X)$ . Also prove that it is an algebra iff it is a subring containing the multiplicative identity.
- (2) Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Say that a map  $f : X \rightarrow Y$  is measurable (with respect to the  $\sigma$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ ) if  $f^{-1}(E) \in \mathcal{A}$  for every  $E \in \mathcal{B}$ .
  - (a) Prove that measurable spaces with measurable maps as morphisms form a category.
  - (b) Try convincing an analyst that (a) is useful.