

Homework 7: on differentiaion (50/50)

None of the following questions will be graded. Do them, but do not hand them in.

Completion of $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu) = \text{Completion of } (X \times Y, \bar{\mathcal{A}} \otimes \bar{\mathcal{B}}, \bar{\mu} \times \bar{\nu})$

Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be measure spaces. Let $(X, \bar{\mathcal{A}}, \bar{\mu})$ and $(Y, \bar{\mathcal{B}}, \bar{\nu})$ be their completions, respectively. Then, the completion of $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ is same as the completion of $(X \times Y, \bar{\mathcal{A}} \otimes \bar{\mathcal{B}}, \bar{\mu} \times \bar{\nu})$.

Modified HL maximal inequality (\geq instead of $>$)

Prove that there is a constant $C_n > 0$ that only depends on n such that for every $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$,

$$m(\{x \in \mathbb{R}^n \mid Hf(x) \geq \alpha\}) \leq \frac{C_n}{\alpha} \int_{\mathbb{R}^n} |f(x)| \, dx$$

(Remark: We had $Hf(x) > \alpha$ for the HL maximal inequality. Here we have $Hf(x) \geq \alpha$.)

density of a mble set at a point: $D_E(x) = 1$ for a.e. $x \in E$, 0 for a.e. $x \in E^c$

For a Lebesgue measurable subset E of \mathbb{R}^n , the *density of E at x* is defined as

$$D_E(x) = \lim_{r \rightarrow 0} \frac{m(E \cap B(x, r))}{m(B(x, r))}$$

provided that the limit exists. Prove that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$. *Hint:* ask Lebesgue.

Some of the following questions will be graded. Do them, and do hand them in.

An identity: $\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \frac{1}{4} \log(1 + s^{-2})$

Prove that $\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \frac{1}{4} \log(1 + s^{-2})$ for $s > 0$ by integrating the function $e^{-2sx} \sin(2xy)$ with respect to x and y over suitable regions.

Proof For fixed $x > 0$, by FTC we have:

$$\sin^2(x) = \int_0^x \sin(2t) dt$$

We do change of variable $t = xy$. This is a valid diffeomorphism mapping $y \in (0, 1)$ to $t \in (0, x)$.

Then by change of variable theorem we have:

$$\int_{(0,x)} \sin(2t) dt = \int_{(0,1)} x \sin(2xy) dy$$

Thus

$$\frac{\sin^2 x}{x} = \int_0^1 \sin(2xy) dy$$

Then we get:

$$\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \int_0^\infty e^{-2sx} \left[\int_0^1 \sin(2xy) dy \right] dx$$

Consider the function

$$f(x, y) := e^{-2sx} \sin(2xy), \quad (x, y) \in (0, \infty) \times (0, 1)$$

f is a composition of continuous functions, thus continuous. Note that it is also in $L^1((0, \infty) \times (0, 1))$ since $|f(x, y)|$ is bounded by $g(x, y) := e^{-2sx}$, which is L^1 on the same domain (its integral is $\frac{1}{2s}$), then by DCT, $f \in L^1((0, \infty) \times (0, 1))$.

Thus we can apply Fubini's theorem to switch the order of integration:

$$\int_0^\infty e^{-2sx} \left[\int_0^1 \sin(2xy) dy \right] dx = \int_{(0,\infty) \times (0,1)} e^{-2sx} \sin(2xy) d(x \times y) \quad (8.1)$$

$$= \int_0^1 \left(\int_0^\infty e^{-2sx} \sin(2xy) dx \right) dy \quad (8.2)$$

Recall back in Calculus we use integration by part to get:

$$\int_0^\infty e^{-ax} \sin(bx) dx = \frac{b}{a^2 + b^2}$$

for $a > 0$. In our case, $a = 2s$ and $b = 2y$. Thus

$$\int_0^\infty e^{-2sx} \sin(2xy) dx = \frac{2y}{(2s)^2 + (2y)^2} = \frac{y}{2(s^2 + y^2)}$$

Therefore we here get

$$\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \int_0^1 \left(\int_0^\infty e^{-2sx} \sin(2xy) dx \right) dy \quad (8.3)$$

$$= \int_0^1 \frac{y}{2(s^2 + y^2)} dy \quad (8.4)$$

$$= \frac{1}{2} \int_0^1 \frac{y}{s^2 + y^2} dy \quad (8.5)$$

By Calculus we have (by chain rule):

$$\int_0^1 \frac{y}{s^2 + y^2} dy = \left[\frac{1}{2} \log(s^2 + y^2) \right]_0^1 = \frac{1}{2} \log\left(\frac{s^2 + 1}{s^2}\right) = \frac{1}{2} \log\left(1 + \frac{1}{s^2}\right)$$

Thus we conclude:

$$\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \frac{1}{2} \int_0^1 \frac{y}{s^2 + y^2} dy \quad (8.6)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \log\left(1 + \frac{1}{s^2}\right) \quad (8.7)$$

$$= \frac{1}{4} \log\left(1 + \frac{1}{s^2}\right) \quad (8.8)$$

as desired.

$E \in \mathcal{A} \otimes \mathcal{A} \implies \text{diagonal of } E \in \mathcal{A}$

(a) Prove that if $E \in \mathcal{A} \otimes \mathcal{A}$, then

$$\{x \in X : (x, x) \in E\} \in \mathcal{A}$$

(b) Using this fact, find an example of a subset $E \subset \mathbb{R} \times \mathbb{R}$ such that $E_x \in \mathcal{L}(\mathbb{R})$ for all $x \in \mathbb{R}$ and $E^y \in \mathcal{L}(\mathbb{R})$ for all $y \in \mathbb{R}$, but $E \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$. *Hint: ask Vitali.*

Proof of (a):

We consider the map:

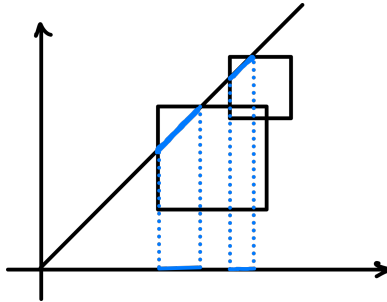
$$\phi : X \rightarrow X \times X \quad (8.9)$$

$$x \mapsto (x, x) \quad (8.10)$$

Then it suffices to show that ϕ is $(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ -measurable. Since if so, then for each $E \in \mathcal{A} \otimes \mathcal{A}$, $\phi^{-1}(E) = \{x \in X : (x, x) \in E\} \in \mathcal{A}$, which is exactly what we want.

Let $A \times B \in \mathcal{A} \otimes \mathcal{A}$ be a measurable rectangle, we discover that:

$$\phi^{-1}(A \times B) = \{x \in X : x \in A, x \in B\} = A \cap B \in \mathcal{A}$$



We first prove a lemma:

Lemma 8.1

Suppose $f : X \rightarrow Y \times Z$ is a function from a measurable space (X, \mathcal{A}) to a product measure space $(Y \times Z, \mathcal{B}_1 \otimes \mathcal{B}_2)$.

Claim: If $f^{-1}(B_1 \times B_2) \in \mathcal{A}$ for each measurable rectangle $B_1 \times B_2 \in \mathcal{B}_1 \otimes \mathcal{B}_2$, then f is an $(\mathcal{A}, \mathcal{B}_1 \otimes \mathcal{B}_2)$ -measurable function. 

Proof of Lemma:

Since $f^{-1}(B \times C) \in \mathcal{A}$ for each measurable rectangle $B \times C \in \mathcal{B}_1 \otimes \mathcal{B}_2$, the preimage of any countable disjoint unions of measurable rectangles, is also in \mathcal{A} , since \mathcal{A} is an σ -algebra.

We want to show: $f^{-1}(E) \in \mathcal{A}$ for any $E \in \mathcal{B}_1 \otimes \mathcal{B}_2$. It is equivalent to show that

$$\mathcal{B}_1 \otimes \mathcal{B}_2 \subset \mathcal{C} := \{E \in Y \times Z : \phi^{-1}(E) \in \mathcal{A}\}$$

Note that, it suffices to show that: \mathcal{C} is an σ -algebra. This is because we have shown

$$\{\text{all disjoint unions of measurable rectangles in } Y \times Z\} \subset \mathcal{C}$$

, and this is an algebra generating $\mathcal{B}_1 \otimes \mathcal{B}_2$. Thus, if \mathcal{C} is an σ -algebra, we must have $\mathcal{B}_1 \otimes \mathcal{B}_2 \subset \mathcal{C}$.

And since $\{\text{all disjoint unions of measurable rectangles in } Y \times Z\}$ is an algebra, it suffices to show that \mathcal{C} is a monotone class, by the monotone class lemma.

Suppose $E_1 \subseteq E_2 \subseteq \dots$ with each $E_n \in \mathcal{C}$, i.e. $\phi^{-1}(E_n) \in \mathcal{A}$. Since $\{E_n\}$ is increasing, we have

$$\phi^{-1}(E_1) \subseteq \phi^{-1}(E_2) \subseteq \dots \subseteq \phi^{-1}(E_n) \subseteq \dots$$

Since \mathcal{A} is an σ -algebra, we have

$$\phi^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} \phi^{-1}(E_n) \in \mathcal{A}$$

Thus

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$$

This is dually true for decreasing intersection, **finishing the proof that \mathcal{C} is a monotone class thus σ -algebra, thus proving the lemma.**

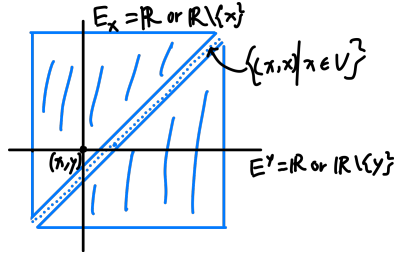
After we proved the Lemma, we return to the original statement, concluding that ϕ is $(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ -measurable, thus finishing the proof: if $E \in \mathcal{A} \otimes \mathcal{A}$, then

$$\{x \in X : (x, x) \in E\} \in \mathcal{A}$$

Sol. of (b):

Take a Vitali set $V \subset \mathbb{R}$, and consider:

$$E := \{(x, y) \in \mathbb{R}^2 : x \neq y\} \cup \{(x, x) : x \in V\}.$$



Then for any fixed $x \in \mathbb{R}$, we have:

$$E_x = \{y : (x, y) \in E\} = \begin{cases} \mathbb{R}, & x \in V \\ \mathbb{R} \setminus \{x\}, & x \notin V \end{cases}$$

And for any fixed $y \in \mathbb{R}$, we have:

$$E_y = \{x : (x, y) \in E\} = \begin{cases} \mathbb{R}, & y \in V \\ \mathbb{R} \setminus \{y\}, & y \notin V \end{cases}$$

Thus $E_x \in \mathcal{L}(\mathbb{R})$ for all $x \in \mathbb{R}$ and $E_y \in \mathcal{L}(\mathbb{R})$ for all $y \in \mathbb{R}$.

However, we have $E \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$, since by (a) we have proved that if $E \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$, then

$$V = \{x \in \mathbb{R} : (x, x) \in E\} \in \mathcal{L}(\mathbb{R})$$

But it contradicts with the fact that V is not Lebesgue measurable.

Thus E satisfies our requirements.

(This happens since, as shown in class, the product measure space of two complete measure space is not necessarily complete. Here, the diagonal is a null set in \mathbb{R}^2 and thus our Vitali portion is a subnull set, but $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ is not complete (its completion is $\mathcal{L}(\mathbb{R}^2)$.)

Too dense: $m(E \cap I) \leq \alpha m(I)$ for all $I \implies m(E) = 0$ for mble E

Prove that if $E \subset \mathcal{L}(\mathbb{R})$ is a Lebesgue measurable subset such that

$$m(E \cap I) \leq 0.123m(I)$$

for all open intervals $I \subset \mathcal{L}(\mathbb{R})$, then $m(E) = 0$.

Proof Since E is Lebesgue measurable, $m(E) = m^*(E)$.

Let $\epsilon > 0$.

Then by definition of outer measure, we can pick open intervals seq $\{I_k\}_{k=1}^{\infty}$ covering E s.t.

$$m(E) > \sum_{k=1}^{\infty} m(I_k) - \epsilon$$

Since $E \subset \bigcup_k I_k$, we have

$$E = \left(\bigcup_k I_k \right) \cap E \tag{8.11}$$

$$= \bigcup_k (I_k \cap E) \tag{8.12}$$

$$\tag{8.13}$$

Thus

$$m(E) = m\left(\bigcup_k (I_k \cap E)\right) \leq \sum_k m(I_k \cap E) \quad \text{by ctbl subadditivity} \quad (8.14)$$

$$\leq 0.123 \sum_k m(I_k) \quad \text{by our requirement} \quad (8.15)$$

Thus we have:

$$\sum_k m(I_k) - \epsilon < 0.123 \sum_k m(I_k) \quad (8.16)$$

$$0.877 \sum_k m(I_k) < \epsilon \quad (8.17)$$

$$\sum_k m(I_k) < \frac{\epsilon}{0.877} \quad (8.18)$$

Thus

$$m(E) \leq \sum_k m(I_k) < \frac{\epsilon}{0.877}$$

Since $\epsilon > 0$ is arbitrary, this proves that

$$m(E) = 0$$

给定任意 $0 < \alpha < 1$, prescribe 出一个在 0 处 density 为 $\alpha/2$ 的集合

Let $0 < \alpha < 1$. Find an example of a Lebesgue measurable subset E of $[0, \infty) \subset \mathcal{L}(\mathbb{R})$ whose density at 0 is $\alpha/2$. *Hint*: Consider $E = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (x_n, x_n + \delta_n)$ are disjoint small intervals accumulating at 0.

Proof Consider take

$$E := \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, \frac{1}{n} + \frac{\alpha}{n(n-1)} \right)$$

as the union of a countable sequence of intervals drawing near 0.

Notice: There intervals are **mutually disjoint**, since

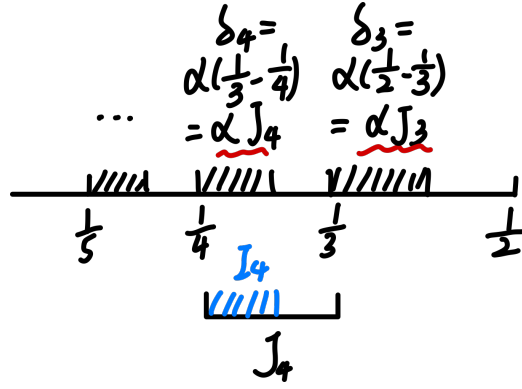
$$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > \frac{\alpha}{n(n-1)}$$

we thus have for $n \geq 2$,

$$\frac{1}{n} + \frac{\alpha}{n(n-1)} < \frac{1}{n-1}$$

We use $x_n := \frac{1}{n}$; $I_n := (x_n, x_n + \delta_n)$ to denote each component interval; $J_n := (x_n, x_{n-1})$ to denote the open interval where I_n is located at; and $\delta_n := \frac{\alpha}{n(n-1)}$ to denote the length of each interval. Note that for each n ,

$$\delta_n = \alpha \left(\frac{1}{n-1} - \frac{1}{n} \right) = \alpha(x_{n-1} - x_n) = \alpha J_n$$



Now we show that this set has Lebesgue density $\frac{\alpha}{2}$ at 0 below.

Let $r > 0$ (WLOG $r < 1$), then we have

$$\frac{1}{n+1} < r \leq \frac{1}{n} \quad \text{for some } n \in \mathbb{N}$$

Then for each $k \geq n+2$, we have $\frac{1}{k} < \frac{1}{n+1} < r$. Hence I_k is **entirely contained** in $(0, r)$:

$$\bigcup_{k=n+2}^{\infty} I_k \subseteq E \cap (-r, r)$$

We know that by telescoping,

$$\sum_{k=n+2}^{\infty} \frac{1}{k(k-1)} = \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \left(\frac{1}{n+2} - \frac{1}{n+3} \right) + \cdots = \frac{1}{n+1}$$

Multiplying this by $\frac{\alpha}{2}$ gives:

$$\sum_{k=n+2}^{\infty} \frac{\alpha}{k(k-1)} = \frac{\alpha}{n+1}$$

Thus by monotonicity of measure:

$$m(E \cap (-r, r)) \geq \frac{\alpha}{n+1}$$

And for each $k \leq n$, I_k exceeds $(0, r)$ on the right, thus we get dually:

$$m(E \cap (-r, r)) \leq \frac{\alpha}{n-1}$$

And we have:

$$\frac{2}{n+1} \leq m(-r, r) \leq \frac{2}{n}$$

since $\frac{1}{n+1} \leq r \leq \frac{1}{n}$.

Therefore we get:

$$\frac{\frac{\alpha}{n+1}}{\frac{2}{n}} \leq \frac{m(E \cap (-r, r))}{m((-r, r))} \leq \frac{\frac{\alpha}{n-1}}{\frac{2}{n+1}}$$

Further simplify:

$$\frac{n}{n+1} \cdot \frac{\alpha}{2} \leq \frac{m(E \cap (-r, r))}{m((-r, r))} \leq \frac{n+1}{n-1} \cdot \frac{\alpha}{2}$$

As $r \rightarrow 0^+$, we must have $n \rightarrow \infty$, and we know

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{\alpha}{2} = \lim_{n \rightarrow \infty} \frac{n+1}{n-1} \cdot \frac{\alpha}{2} = \frac{\alpha}{2}$$

Thus by **Squeeze Theorem**, we have:

$$\lim_{r \rightarrow 0^+} \frac{m(E \cap (-r, r))}{m((-r, r))} = \frac{\alpha}{2}$$

Hence by def, E indeed has Lebesgue density $\alpha/2$ at 0.

(My note: The key point here is that, the harmonic seq shrinks very slowly in proportion as n grows, J_n almost have same length as J_{n+1} for large n , thus $m(J_n)/m(\cup_{k \geq n} J_k) = 0$ as we know, so that whether r lies in I_n or $J_n \setminus I_n$ does not quite matter.

On the other hand, the counterexample in class, using the geometric sequence as build block of J_n , fails since the length of J_n is too much compared to $\cup_{k \geq n} J_k$, actually $m(J_n) = m(\cup_{k \geq n} J_k)$, thus whether r lies in I_n or $J_n \setminus I_n$ makes a lot difference, making the density at 0 undefined.)

Seqs of complex numbers: $\ell^1 \subsetneq \bigcap_{1 < p < \infty} \ell^p$ and $\bigcup_{1 < p < \infty} \ell^p \subsetneq \ell^\infty$

(a) Prove that $\ell^1 \subsetneq \bigcap_{1 < p < \infty} \ell^p$.

(b) Prove that $\bigcup_{1 < p < \infty} \ell^p \subsetneq \ell^\infty$.

Proof of (a):

We first want to show: for any $1 < p < \infty$, we have:

$$\ell^1 \subseteq \ell^p$$

Fix $p > 1$.

Let $(x_n) \in \ell^1$. By definition,

$$\sum_{n=1}^{\infty} |x_n| < \infty$$

We need to show that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Claim: There are at most finitely many $n \in \mathbb{N}$ s.t. $|x_n| \geq 1$.

Proof of Claim: Suppose for contradiction that there are infinitely many $n \in \mathbb{N}$ s.t. $|x_n| \geq 1$, say, all terms in the subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ has $|x_{n_j}| \geq 1$. Then

$$\sum_{n=1}^{\infty} |x_n| \geq \sum_{j=1}^{\infty} |x_{n_j}| \geq \sum_{j=1}^{\infty} 1 = \infty$$

which contradicts with $(x_n) \in \ell^1$.

Thus, suppose only on the finite terms $\{x_{n_j}\}_{j=1}^N$ we have $|x_{n_j}| \geq 1$ (WLOG $N \geq 1$). Then

$$\sum_{n=1}^{\infty} |x_n| = \sum_{j=1}^N |x_{n_j}| + \sum_{n \neq n_j \text{ for any } j} |x_n|$$

Since for n s.t. $n \neq n_j$ for any subseq index j , we have $|x_n| < 1$, for these indexes we have:

$$|x_n|^p < |x_n| \quad \text{for any } p > 1$$

Thus we have

$$\sum_{n \neq n_j \text{ for any } j} |x_n|^p < \sum_{n \neq n_j \text{ for any } j} |x_n| < \infty$$

And also,

$$\sum_{j=1}^N |x_{n_j}|^p < \infty \quad \text{since only have finite terms}$$

Thus

$$\sum_{n=1}^{\infty} |x_n|^p = \sum_{j=1}^N |x_{n_j}|^p + \sum_{n \neq n_j \text{ for any } j} |x_n|^p < \infty$$

Thus

$$\ell^1 \subseteq \ell^p$$

Since $p > 1$ is arbitrary, this proves that

$$\ell^1 \subseteq \bigcap_{1 < p < \infty} \ell^p$$

To show the strictness of the inclusion, we consider the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$. We know that it diverges and for any $p > 1$, the p -**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (absolutely for sure) converges, thus $(\frac{1}{n}) \notin \ell^1$ but $(\frac{1}{n}) \in \ell^p$ for every $p > 1$, showing that

$$\ell^1 \neq \bigcap_{1 < p < \infty} \ell^p$$

This finishes the proof that

$$\ell^1 \subsetneq \bigcap_{1 < p < \infty} \ell^p$$

Proof of (b):

Fix $p > 1$.

Suppose sequence (x_n) belongs ℓ^p , then

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

This implies that $x_n \rightarrow 0$ as $n \rightarrow \infty$, because if it did not, there would be infinitely many terms where $|x_n|$ is bounded away from zero, leading to divergence of the sum.

Suppose for contradiction that

$$\sup_n |x_n| = \infty$$

Then there are infinitely many terms n s.t. $|x_n| > 1$, since otherwise, exists some N s.t. all $|x_n| \leq 1$ for $n \geq N$, then $\sup_n |x_n| \leq \max(1, \max_{1 \leq n \leq N-1} |x_n|) < \infty$.

Suppose for the subseq $\{x_{n_j}\}_{j=1}^{\infty}$ we have $|x_{n_j}| > 1$. Thus

$$\sum_{n=1}^{\infty} |x_n|^p \geq \sum_{j=1}^{\infty} |x_{n_j}|^p > \sum_{j=1}^{\infty} 1^p = \infty$$

which contradicts with $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Therefore we have:

$$\sup_n |x_n| < \infty$$

This shows that

$$\ell^p \subseteq \ell^{\infty}$$

Since $p > 1$ is arbitrary, this proves that

$$\bigcup_{1 < p < \infty} \ell^p \subseteq \ell^\infty$$

Now we show the inclusion is strict. Consider the sequence $x_n = 1$ for all n . Clearly, $(x_n) \in \ell^\infty$ because it is bounded. However, $x_n \notin \ell^p$ for any $p > 1$:

$$\sum_{n=1}^{\infty} |1|^p = \sum_{n=1}^{\infty} 1 = \infty$$

This shows

$$\bigcup_{1 < p < \infty} \ell^p \neq \ell^\infty$$

Thus we have

$$\bigcup_{1 < p < \infty} \ell^p \subsetneq \ell^\infty$$

Nur für Verrückte

(It's **really** not necessary to attempt these problems. Do not, under any circumstances, hand them in!)

Prescribing a Lebesgue density, Season 2

Let $0 < \alpha < 1$ and $n \geq 1$. Find an example of a Lebesgue measurable subset E of $\mathcal{L}(\mathbb{R})^n$ whose density at 0 is α . *Hint*: think spherically.