Homework 4: on measurable functions(36/40)

None of the following questions will be graded. Do them, but do not hand them in.

One with Vitali.

Let $(X, \mathcal{A})$ be a measurable space, and $E \subset X$ a subset. Prove that $E \in \mathcal{A}$ iff the function χ_E is measurable. Use this to construct a function $f\colon \mathbb{R} \to \mathbb{R}$ that is not Lebesgue measurable.

Truncations in L^+: 通过 $\int f_n$ 或者 $\int_{X_n} f$ 的极限 (bounded function / subset) 得到 $\int_X f$

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f\colon X \to [0, \infty]$ a measurable function.

(a) (Horizontal truncation) Suppose that $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_1 \subset X_2 \subset \cdots$ with $X_n \in \mathcal{A}$. Prove that

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_{X_n} f \, d\mu$$

(b) (Vertical truncation) Prove that

$$\int f \, d\mu = \lim_{n\to\infty} \int \min\{f, n\} \, d\mu.$$

(c) Explain the terminology "horizontal truncation" and "vertical truncation".

Disregarding null sets.

Let $(X, \mathcal{A}, \mu)$ be a *complete* measure space.

(a) Let $f\colon X \to \overline{\mathbb{R}}$ and $g\colon X \to \overline{\mathbb{R}}$ be functions such that $f = g$ μ-a.e.
   (i) Prove that f is measurable (i.e. $\mathcal{A}$-measurable) iff g is measurable.
   (ii) Prove the same statement when f and g are $\mathbb{C}$-valued, rather than $\overline{\mathbb{R}}$-valued.
   (iii) Give examples showing that the condition that μ be complete is necessary.

(b) Let $f_n : X \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, and $f\colon X \to \overline{\mathbb{R}}$ be functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in X$.
   (i) Prove that if f_n is measurable for all n, then so is f.
   (ii) Prove the same statement when f_n and f are $\mathbb{C}$-valued, rather than $\overline{\mathbb{R}}$-valued.
   (iii) Give examples showing that the condition that μ be complete is necessary.

Hint: this is Proposition 2.11 of [Folland].

Measurable functions and completions.

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(X, \bar{\mathcal{A}}, \bar{\mu})$ be its completion. Suppose that $f\colon X \to \overline{\mathbb{R}}$ is $\bar{\mathcal{A}}$-measurable. Prove that there is an $\mathcal{A}$-measurable function $g\colon X \to \overline{\mathbb{R}}$ such that $g = f$ $\bar{\mu}$-a.e., and hence $\int g \, d\mu = \int f \, d\bar{\mu}$. *Hint*: this is Proposition 2.12 of [Folland].

Measurability on subsets.

Let $(X, \mathcal{A})$ be a measurable space, and $Y \subset X$ a nonempty subset. We say that a function $g \colon Y \to \overline{\mathbb{R}}$ is *$\mathcal{A}$-measurable on Y* if g is $\mathcal{A}|_Y$-measurable, where the σ-algebra $\mathcal{A}|_Y$ on Y is defined as in HW1.

(a) Prove that if $f \colon X \to \overline{\mathbb{R}}$ is measurable and $Y \subset X$, then $g = f|_Y$ is $\mathcal{A}$-measurable on Y.

(b) Prove that if g is $\mathcal{A}$-measurable on Y and $Y \in \mathcal{A}$, then g can be extended to an $\mathcal{A}$-measurable function f on X. Is the extension unique?

(c) Let $f \colon X \to \overline{\mathbb{R}}$ be any function, and set $Y = f^{-1}(\mathbb{R})$. Prove that f is measurable iff $f^{-1}(\{\infty\}) \in \mathcal{A}$, $f^{-1}(\{-\infty\}) \in \mathcal{A}$, and $f|_Y \colon Y \to \mathbb{R}$ is $\mathcal{A}$-measurable on Y.

Suprema of uncountable families.

Construct (using the Axiom of Choice, if needed) an *uncountable* family $(f_\alpha)_\alpha$ of real-valued Borel measurable functions on $\mathbb{R}$ such that the function $\sup_\alpha f_\alpha$ is not Lebesgue measurable, let alone Borel measurable.

Increasing functions again.

Let $f \colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Prove that f is Borel measurable. Use this to give an example of a function $f \colon \mathbb{R} \to \mathbb{R}$ that cannot be written as a difference between increasing functions.

Lebesgue but not Borel.

Let $F \colon [0, 1] \to [0, 1]$ be the function from HW3, whose graph is the Devil's Staircase. Define $G(x) = F(x) + x$.

(a) Prove that $G : [0, 1] \to [0, 2]$ is an increasing homeomorphism. In other words, G is increasing, bijective, and both G and G^{-1} are continuous.

(b) Let C be the middle-thirds Cantor set, and set $K := G(C)$. Prove that $m(K) = 1$.

(c) Since $m(K) > 0$, we know from HW3 that there is a set $A \subset K$ that is not Lebesgue measurable. Prove that $B = G^{-1}(A)$ is Lebesgue measurable but not Borel measurable.

Measurability and absolute values.

Let $(X, \mathcal{A})$ be a measure space. Suppose that $f \colon X \to \mathbb{C}$ is a measurable function. Prove that the function $|f| \colon X \to \mathbb{R}$ is also measurable. Is the converse true?

Some of the following questions will be graded. Do them, and do hand them in. You may use the results from the exercises above.

Measurability of limit loci.

Let $(X, \mathcal{A})$ be a measurable space. For each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}$ be a measurable function. Consider the set

$$E := \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ converges to a real number}\}.$$

Prove that E is a measurable set in two ways:

(i) by expressing E in terms of the functions $g(x) = \limsup_{n\to\infty} f_n(x)$ and $h(x) = \liminf_{n\to\infty} f_n(x)$;

(ii) by expressing E in terms of the sets

$$E_{i,j,k} = \{x \mid |f_j(x) - f_k(x)| < \tfrac{1}{i}\},$$

where $i, j, k \in \mathbb{N}$. *Hint*: a sequence $(a_n)_n$ of real numbers converges iff it is a Cauchy sequence, i.e. for every $\epsilon > 0$ there is n such that for every $j, k \geq n$, $|a_j - a_k| < \epsilon$.

Hint: note that $\pm\infty$ are not real numbers, and please avoid considering $\infty - \infty$; you may want to prove a lemma to the effect that if $g, h : X \to \overline{\mathbb{R}}$ are measurable functions, then the set

$$\{x \in X \mid g(x) = h(x) \in \overline{\mathbb{R}}\}$$

is measurable; to do this, you may want to consider functions like $\max\{g, \kappa\}$, $\min\{h, \kappa\}$ and $\min\{g, -\kappa\}$, $\min\{h, -\kappa\}$ for large real constants $\kappa > 0$.

Proof of method (i):

Define:

$$g(x) := \limsup_{n\to\infty} f_n(x) \quad \text{and} \quad h(x) := \liminf_{n\to\infty} f_n(x)$$

Since each f_n is measurable function, by proposition in lecture (sequential preservation of measurability), g, h **are measurable.**

And as we know, for any real sequence (a_n),

$$\lim_{n\to\infty} a_n \text{ exists (as a real number)} \quad \Longleftrightarrow \quad \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n \in \mathbb{R}$$

Thus, for each $x \in X$ we have:

$$x \in E \quad \Longleftrightarrow \quad \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x) \in \mathbb{R}$$

Thus, we can write E as:

$$E = \{x \in X \mid g(x) = h(x) \in \mathbb{R}\}$$

Note: here we want to have a difference function of the two functions, but it is undefined on $\infty - \infty$ type of points. So actually it is not valid to take the difference for functions mapping to $\overline{\mathbb{R}}$. This is why we use the following method instead:

For each $n \in \mathbb{N}$, we define:

$$g_n(x) := \min\{\max\{g(x), -n\}, n\} \quad \text{and} \quad h_n(x) := \min\{\max\{h(x), -n\}, n\}$$

Notice that, **each** g_n, h_n **is measurable**, since g, h are measurable and constant function is measurable and we have proved in lecture that taking the max, min of two measurable functions is measurable.

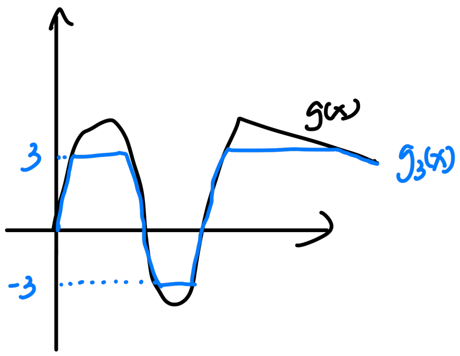
Claim 1.1:

$$g(x) = h(x) \in \mathbb{R} \quad \iff \quad \exists N_0 > 0,\ \forall n \geq N_0, \quad g_n(x) = h_n(x)$$

proof of claim 1.1: Suppose $g(x) = h(x) \in \mathbb{R}$. Let $M := \max\{|g(x)|, |h(x)|\} < \infty$, then for any $n > M$, we have $g_n(x) = g(x), h_n(x) = h(x)$, so $g_n(x) = h_n(x)$.

Suppose $\exists N_0 > 0,\ \forall n \geq N_0, \quad g_n(x) = h_n(x)$, Then it is clear that

$$g(x) = g_{N_0}(x) = h_{N_0}(x) = h(x) < \infty$$



proof of remaining: Therefore we have:

$$E = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{x \in X \mid g_n(x) = h_n(x)\} \tag{4.1}$$

Foe each $n \in \mathbb{N}$, we define

$$E_n := \{x \in X \mid g_n(x) = h_n(x)\}$$

Since each g_n, h_n is measurable and real-valued (finite), $g_n - h_n$ is measurable and $|g_n - h_n|$ is measurable, so we have for each $m \in \mathbb{N}$,

$$\{x \in X : |g_{\kappa_n}(x) - h_{\kappa_n}(x)| < 1/m\} = |g_n - h_n|^{-1}([0, 1/m)) \in \mathcal{A}$$

Thus

$$E_n = \bigcap_{m \in \mathbb{N}} |g_n - h_n|^{-1}([0, 1/m)) \in \mathcal{A}$$

is a measurable set. Thus E is a countable union of countable intersections of mea surable sets, then measurable.

Proof of method (ii):

Recall: **a seq of real numbers converges iff it is a Cauchy.** Now we fix an arbitrary $i \in \mathbb{N}$ and let $\epsilon = 1/i$. Define:

$$E_{i,j,k} = \{x \in X : |f_j(x) - f_k(x)| < 1/i\}$$

Since each f_j is measurable, the function $x \mapsto |f_j(x) - f_k(x)|$ is measurable (since each term in the sequence maps to $\mathbb{R}$ but not $\overline{\mathbb{R}}$), and hence **each** $E_{i,j,k} = |f_j(x) - f_k(x)|^{-1}([0, 1/i))$ **is measurable.**

For each i, consider the set of $x \in X$ for which the sequence $(f_n(x))$ satisfies the Cauchy condition with respect to $\epsilon = 1/i$. That is,

$$E_i = \left\{x \in X : \exists N \in \mathbb{N} \text{ s.t. } \forall j, k \geq N,\ |f_j(x) - f_k(x)| < \frac{1}{i}\right\}$$

We can write E_i as

$$E_i = \bigcup_{N=1}^{\infty} \bigcap_{j,k \geq N} E_{i,j,k}$$

Since countable unions and intersections of measurable sets are measurable, E_i **is measurable.**

Now, since $(f_n(x))$ converges in $\mathbb{R}$ **iff it is Cauchy, i.e. it is in** E_i **for each** $i \in \mathbb{N}$, we have:

$$E = \bigcap_{i=1}^{\infty} E_i = \bigcap_{i=1}^{\infty} \Big(\bigcup_{N=1}^{\infty} \bigcap_{j,k \geq N} E_{i,j,k} \Big)$$

This is a countable intersection of measurable sets, and therefore E is measurable.

Measurability of continuity loci.

Let (X, d) be a metric space, and $f\colon X \to \mathbb{C}$ any function. Prove that the set of points $x \in X$ such that f is continuous at x is a G_δ-set, and in particular a Borel set. *Hint*: consider sets of the form

$$\{x \in X \mid |f(y) - f(z)| \leq \tfrac{1}{n} \text{ whenever } \max\{d(y,x), d(z,x)\} \leq \delta\}$$

and show off your skills with quantifiers.

Proof Recall: $f\colon X \to \mathbb{C}$ **from a metric space** is continuous at $x \in X$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ such that$|f(y) - f(x)| < \varepsilon$ whenever $d(y,x) < \delta$. We can easily check that, **this condition is equivalent to**: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(y) - f(z)| < \varepsilon \quad \forall y, z \in B_\delta(x)$, by the relation of diameter and radius of the open ball).

Thus we have:

$$x \in C \iff \forall n \in \mathbb{N},\ \exists m \in \mathbb{N} \text{ s.t.} y, z \text{ with } d(y,x) < \tfrac{1}{m} \text{ and } d(z,x) < \tfrac{1}{m},\ |f(y) - f(z)| < \tfrac{1}{n}$$

In other words, x **is a continuity point iff it belongs to:**

$$C = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} U_{n,m}.$$

where

$$U_{n,m} = \Big\{ x \in X \mid y, z \in B_{\frac{1}{m}}(x) \implies |f(y) - f(z)| < \tfrac{1}{n} \Big\}$$

Claim: $U_{n,m}$ **is open.**

Proof of Claim:

Let $x \in U_{n,m}$. WTS: $\exists$ an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U_{n,m}$.

Consider: $\varepsilon = \frac{1}{2m}$.

Let $y \in B_\varepsilon(x)$. Take any two points $z, w \in X$ satisfying

$$d(z,y) < \tfrac{1}{2m} \quad \text{and} \quad d(w,y) < \tfrac{1}{2m}$$

Then by the triangle inequality, we have:

$$d(z,x) \leq d(z,y) + d(y,x) < \tfrac{1}{2m} + \tfrac{1}{2m} = \tfrac{1}{m}$$

Similarly, $d(w,x) < \frac{1}{m}$. Since $x \in U_{n,m}$, it follows that

$$|f(z) - f(w)| < \frac{1}{n}$$

Thus, the condition defining $U_{n,m}$ holds for y, meaning $y \in U_{n,m}$. This proves that $B_\varepsilon(x) \subset U_{n,m}$, thus $U_{n,m}$ is open since x is arbitrary.

Therefore:

$$C = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} U_{n,m}$$

is G_δ since each $\bigcup_{m=1}^{\infty} U_{n,m}$ is a union of open sets, thus open; and C is thus a countable intersection of open sets, namely a G_δ-set. (thus Borel).

Measurability of differentiability loci.

Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Let us say (as usual) that f is ***differentiable*** at x if there exists $\lambda \in \mathbb{R}$ such that $\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = \lambda$.

We also declare f to be ***strongly differentiable*** at x if there exists $\lambda \in \mathbb{R}$ with the following property: for each $\epsilon > 0$ there exists $\delta > 0$ such that if $|y - x| \le \delta$ and $|z - x| \le \delta$, then $|f(y) - f(z) - \lambda(y-z)| \le \epsilon |y - z|$.

(a) Does f being differentiable at x imply that f is strongly differentiable at x? Give a proof or a counterexample.

(b) Prove that the set of points $x \in \mathbb{R}$ at which f is strongly differentiable is a Borel set. *Hint*: consider sets of the form

$$E_{\lambda,m,n} := \{x \in \mathbb{R} \mid |f(y) - f(z) - \lambda(y-z)| \le \tfrac{1}{n}|y - z| \text{ whenever } \max\{|y-x|, |z-x| \le \tfrac{1}{m}\}.$$

(c) *Extra credit*: is the set of points $x \in \mathbb{R}$ at which f is differentiable a Borel set?

Sol. of (a): No. Consider the following counterexample:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We know that

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin(1/x)}{x} = x \sin(1/x)$$

Note $|x \sin(1/x)| \le |x|$, so when $x \to 0$ we have:

$$\lim_{x \to 0} x \sin(1/x) = 0$$

Thus f is differentiable at 0 and $f'(0) = 0$.

Lemma 4.1

$f : \mathbb{R} \to \mathbb{R}$ is strongly differentiable at $x \implies$ it is differentiable at x, and λ is uniquely equal to the derivative at x. ♡

Proof of lemma 4.1:

Suppose $f : \mathbb{R} \to \mathbb{R}$ is strongly differentiable at x, so for any $\epsilon > 0$, there exists $\delta > 0$ s.t. for all $y, z \in B_\delta(x)$, we have:

$$\Big|f(y) - f(z) - \lambda(y-z)\Big| \le \epsilon\,|y-z|.$$

Suppose $y \neq z$, then dividing by $|y-z|$ on both sides, we have

$$\Big|\frac{f(y)-f(x)}{y-x} - \lambda\Big| \le \epsilon$$

Since ϵ is arbitrary, this proves that

$$f'(x) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = \lambda$$

Now we go back to the counterexample. Suppose for contradiction that f is strongly differentiable at 0, then $\lambda = 0$, so for all $\epsilon > 0$, there exist $\delta > 0$ s.t. for all $y, z \in B_\delta(0)$, we have

$$|f(y) - f(z)| \le \epsilon\,|y-z|$$

Consider $\epsilon = \frac{1}{4}$. Let $\delta > 0$. Take $n \in \mathbb{N}$ s.t.

$$\frac{1}{(2n+\frac{3}{2})\pi} < \delta$$

and then take

$$y_n := \frac{1}{\left(2n+\frac{1}{2}\right)\pi}, \quad z_n := \frac{1}{\left(2n+\frac{3}{2}\right)\pi}$$

Note that each $|y_n|, |z_n| < \delta$. And we have

$$\sin\Big[\Big(2n+\tfrac{1}{2}\Big)\pi\Big] = (-1)^n, \quad \sin\Big[\Big(2n+\tfrac{3}{2}\Big)\pi\Big] = -(-1)^n$$

Thus

$$f(y_n) - f(z_n) = (-1)^n\left[y_n^2 + z_n^2\right]$$

while

$$y_n - z_n = \frac{1}{\left(2n+\frac{1}{2}\right)\pi} - \frac{1}{\left(2n+\frac{3}{2}\right)\pi} = \frac{1}{\pi\left(2n+\frac{1}{2}\right)\left(2n+\frac{3}{2}\right)}$$

Taking limit of this behavior (increasing n), we get the sequential limit of $\frac{|f(y_n)-f(z_n)|}{|y_n-z_n|}$ indexing over n is $\frac{\frac{1}{2\pi^2 n^2}}{\frac{1}{4\pi n^2}} = \frac{2}{\pi}$. By taking large enough n, we can alwasy get $\frac{|f(y_n)-f(z_n)|}{|y_n-z_n|}$ to be arbitrarily close to $\frac{2}{\pi} > \frac{1}{4}$. This shows that f is not strongly differentiable at 0.

Proof of (b):

Let $f : \mathbb{R} \to \mathbb{R}$ be any a function.Denote

$$E := \{x \in \mathbb{R} \mid f \text{ is strongly differentiable at } x\}$$

WTS: E is a Borel set.

Set for each $\lambda \in \mathbb{R}, m, n \in \mathbb{N}$:

$$E_{\lambda,m,n} := \{x \in \mathbb{R} \mid |f(y) - f(z) - \lambda(y-z)| \le \tfrac{1}{n}|y-z| \quad \forall y, z \in B_{\frac{1}{m}}(x)\}$$

where $B_{\frac{1}{m}}(x)$ denote the open ball centered at x with radius $\frac{1}{m}$.

Then by the definition of strongly differentiable, we have:

$$E = \bigcup_{\lambda\in\mathbb{R}}\bigcap_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}} E_{\lambda,m,n}$$

Claim 3.1: Each $E_{\lambda,m,n}$ is open.

Proof of Claim 3.1: Let $x \in E_{\lambda,m,n}$. Then

$$\forall y, z \in B_{1/m}(x), \quad \big|f(y) - f(z) - \lambda(y-z)\big| \leq \frac{1}{n}|y-z|$$

In particular, the inequality holds for all $y, z \in B_{1/(2m)}(x)$. Now consider $B_{1/(2m)}(x)$, let $x' \in B_{1/(2m)}(x)$, then for every $y \in B_{1/(2m)}(x')$, we have

$$|y-x| \leq |y-x'| + |x'-x| < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}$$

so $B_{1/(2m)}(x') \subset B_{1/m}(x)$. Hence the inequality holds for all $y, z \in B_{1/(2m)}(x')$. This confirms that every $x \in E_{\lambda,m,n}$ has a neighborhood contained in $E_{\lambda,m,n}$, proving that $E_{\lambda,m,n}$ is open.

Now that each $E_{\lambda,m,n}$ is open, we have $\bigcup_{m\in\mathbb{N}} E_{\lambda,m,n}$ is each for each λ, n; thus each for each λ, $G_\lambda := \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} E_{\lambda,m,n}$ is a G_δ set.

$$E = \bigcup_{\lambda\in\mathbb{R}} G_\lambda$$

is a union of G_δ sets.

(I do not now how to deal with it then, it might be that we somehow reduce it to countable union of G_δ sets, getting something like $E = \bigcup_{\lambda\in\mathbb{Q}} G_\lambda$ using the density of $\mathbb{Q}$ in $\mathbb{R}$, thus confirming that it is Borel.) -2. 这里的正解是: 要利用 density of $\mathbb{Q}$ in $\mathbb{R}$ 的话, 只需要考虑交换 set operation 的顺序就好了. 我们会发现其实:

$$E = \bigcap_{n\in\mathbb{N}} \bigcup_{\lambda\in\mathbb{Q}} \bigcup_{m\in\mathbb{N}} E_{\lambda,m,n}$$

就这么简单。。

Proof of extra credit: yes. 这个解法非常麻烦. 需要再多考虑两层. 令 $E_{\lambda,k,l,m,n}$ 表示 the set of points x s.t.

$$|f(y) - f(z) - \lambda(y-z)| \leq \frac{1}{n}|y-z|$$

whenever

$$\frac{1}{2^{l+1}}(1+\frac{1}{2^k}) \leq |y-x| \leq \frac{1}{2^{l-1}}(1-\frac{1}{2^k}) \quad \text{and} \quad |z-x| \leq \frac{1}{2^m}(1-\frac{1}{2^k})$$

Claim:

$$f \text{ is differentiable at x iff } x \in E := \bigcap_{n\in\mathbb{N}} \bigcup_{\lambda\in\mathbb{Q}} \bigcup_{l\in\mathbb{N}} \bigcap_{r\geq l} \bigcup_{m\geq 1} \bigcup_{k\in\mathbb{N}} E_{\lambda,k,r,m,n}$$

decreasing MCT: 成立当且仅当 integral 的 limit 是 finite 的

Let $(f_n)_1^\infty$ be a *decreasing* sequence of non-negative measurable functions on a measure space.

(a) Prove that if $\lim_n \int f_n < \infty$, then $\lim_n \int f_n = \int \lim_n f_n$.

(b) Give an example of a decreasing sequence $(f_n)_n$ of nonnegative measurable functions such that $\lim_n \int f_n \neq \int \lim_n f_n$.

Hint: use MCT correctly.

Proof of (a):

Since (f_n) is a decreasing sequence, i.e. for every $x \in X$ we have

$$f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots$$

We can define the function

$$g_n(x) = f_1(x) - f_n(x)$$

for each $n \in \mathbb{N}$. Then for the seq $(g_n(x))$ we have:

- non-negatice: $g_n(x) \geq 0 \ \ \forall x$ because $f_1(x) \geq f_n(x)$.
- increasing in n:

$$g_n(x) = f_1(x) - f_n(x) \leq f_1(x) - f_m(x) = g_m(x) \ \ \forall m \geq n, \forall x$$

since (f_n) is decreasing.

Define $f(x) := \lim_n f_n(x) \in \overline{\mathbb{R}}$ for each $x \in X$.

Since $f_n(x)$ decreases to $f(x) := \lim_{n\to\infty} f_n(x)$, we have

$$\lim_{n\to\infty} g_n(x) = f_1(x) - \lim_{n\to\infty} f_n(x) = f_1(x) - f(x)$$

Now we **apply MCT to the increasing sequence** (g_n). We have:

$$\lim_{n\to\infty} \int g_n \, d\mu = \int \Big(\lim_{n\to\infty} g_n \Big) \, d\mu = \int (f_1 - f) \, d\mu$$

And since $\lim_{n\to\infty} \int f_n \, d\mu < \infty$, we have

$$\lim_{n\to\infty} \int g_n \, d\mu = \int f_1 \, d\mu - \int f \, d\mu$$

Also, because of $\lim_{n\to\infty} \int f_n \, d\mu < \infty$, $\int f_n$ **is eventually finite**. Say, it is finite after $n \geq N \in \mathbb{N}$. We only need to consider $n \geq N$ when considering the limit behavior.

Then for each $n \geq N$,

$$\int g_n \, d\mu = \int (f_1 - f_n) \, d\mu = \int f_1 \, d\mu - \int f_n \, d\mu$$

-2. 这里注意，我们既然知道 f_1 的 integral 未必 finite，就不能这么定义 g_n. 正解是取 N s.t. $\int f_N$ finite，然后定义 $g_n := f_N - f_n$. Taking the limit as $n \to \infty$, have

$$\lim_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} \left(\int f_1 \, d\mu - \int f_n \, d\mu \right) = \int f_1 \, d\mu - \lim_{n\to\infty} \int f_n \, d\mu$$

by linearity of numerical sequence.

Thus, combining with the result from MCT we have:

$$\int f_1 \, d\mu - \lim_{n\to\infty} \int f_n \, d\mu = \int f_1 \, d\mu - \int f \, d\mu$$

Rearrange to get:

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu,$$

which is exactly what we wanted to prove.

Sol. of (b):

Consider defining $(f_n : \mathbb{R} \to \mathbb{R})_{n\in\mathbb{N}}$ with

$$f_n(x) = \chi_{[n,\infty)}(x)$$

Note that:

- f_n **is a decreasing seq**: For each n and every $x \in \mathbb{R}$,
$$f_{n+1}(x) = \chi_{[n+1,\infty)}(x) \leq \chi_{[n,\infty)}(x) = f_n(x)$$
since $[n+1, \infty) \subset [n, \infty)$.
- (f_n) **the pointwise limit**:
$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in \mathbb{R}$$
since for each x there exists an N (any integer greater than x) such that for all $n \geq N$, $x < n$ and hence $f_n(x) = 0$.
- For each n,
$$\int_{\mathbb{R}} f_n \, d\lambda = \int_n^\infty 1 \, dx = \infty$$
But on the other hand
$$\int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n\right) d\lambda = \int_{\mathbb{R}} 0 \, d\lambda = 0$$

Then we have the decreasing seq of function with
$$\lim_{n\to\infty} \int f_n \, d\lambda = \infty \quad \text{while} \quad \int \left(\lim_{n\to\infty} f_n\right) d\lambda = 0$$
This shows that in the absence of the finiteness assumption, the limit and integration need not commute.

Vitali meet Cantor.

Construct a function $f\colon [0,1] \to [0,1]$ such that:

(a) f fails to be Lebesgue measurable;

(b) there exists a compact subset $K \subset (0,1)$ of positive Lebesgue measure such that f is differentiable at every point $x \in K$.

Hint: use the function $g(x) = \inf\{|x-y| \mid y \in K\}$; then square this with the title of the problem.

Sol. Let V be a Vitali set on $[0,1]$, C be the fat Cantor set on $[0,1]$ by recursively taking away the middle open subinterval of length $\frac{1}{4^n}$ on the nth recursion. We consider the function:
$$f(x) = \chi_V \cdot d(x,C)^2$$
where
$$d(x,C) := \{\inf\{|x-y| \mid y \in C\}$$
By Hw3, we know V is not Lebesgue measurable, and C is compact with positive Lebesgue measure $\frac{1}{2}$.

And since $f^{-1}(\{1\}) = V$, mapping a not measurable set to a measurable set, χ_V **is not measurable function.**

And since the distance function $d(x,C)$ is a continuous function of $[0,1]$, it is measurable, by the result proved in class that a continuous funciton on a topological space is measurable.

Lemma 4.2

The product of a measurable $f : \mathbb{R} \to \mathbb{R}_{>0}$ and a not measurable $g : \mathbb{R} \to \mathbb{R}$ is not measurable.

Proof of Lemma 4.2: f measurable $\implies$ $1/f$ measurable. Suppose for contradiction that fg is measurable, then $g = \frac{1}{f}(fg)$ is the product of two measurable functions, thus measurable, contradicting the fact that g is not measurable. Thus fg is not measurable.

Claim 5.1: f **is not measurable. Proof of claim 5.1:** Thus on the open set $A = [0,1] \setminus C$, $d(x,C)^2$ is positive, so $\chi_V|_A d(x,C)^2|_A$ is not measurable since it is a product of measurable and not measurable function by lemma 4.2. Thus f **is not measurable,** otherwise its restriction on A should also be measurable.

Claim 5.2: f **is differentiable on** C**. Proof of claim 5.2:** Fix $x \in C$, then $f(x) = 0$. We want to show:$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)}{h}$ exists Let $h > 0$. Case 1: $x + h \notin V$, then $\chi_V(x+h) = 0$, so we have $f(x+h) = \chi_V(x+h)\, d(x+h,C)^2 = 0$, then $\frac{f(x+h)}{h} = 0$. Case 2: $x + h \in V$, we have:

$$d(x+h,C) = \inf_{y\in C} |(x+h) - y| \leq |(x+h) - x| = |h|$$

So

$$\left|\frac{f(x+h)}{h}\right| = \frac{d(x+h,C)^2}{|h|} \leq \frac{|h|^2}{|h|} = |h|$$

Therefore for all cases we have:

$$\left|\frac{f(x+h)-f(x)}{h}\right| = \left|\frac{f(x+h)}{h}\right| \leq |h|$$

This confirms that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0$$

This finishes the proof of required properties of f.

4.0.1 harder Vitali meet Cantor (extra credit)

We change the requirement of (a) to be: ”the restriction of f to any open interval $I \subset [0,1]$ fails to be Lebesgue measurable”. Then how can we make the construction?

Sol. I don’t know.

官方答案：我在前一问给出的

$$f(x) = \chi_V \cdot d(x,C)^2$$

这个函数，同样也是满足这一问的答案.（对于 C，不仅可以选择 fat Cantor set，实际上任何 choice of compact nowhere dense set 都可以.）