Lec 1 on σ-algebra (39/40)

Borel vs Open

Let X be a metric space such that every subset of X is Borel set. Does it follow that every subset of X is open? Give a proof or a counterexample.

Sol. It is not true.

Every subset of X is Borel set $\Leftrightarrow \mathcal{P}(X) \subset \mathcal{B}_X$. And We know $\mathcal{B}_X \subset \mathcal{P}(X)$, so it is equivalent to saying that $\mathcal{B}_X = \mathcal{P}(X)$.

So consider this counterexample: $\mathbb{Q}$ with the Euclidean metric.

Claim: every singleton set in $\mathbb{Q}$ is closed, thus in $\mathcal{B}_{\mathbb{Q}}$. This is because this only sequence in a singleton set is the point itself repeating, thus converging to itself, in the singleton set. This proves the claim.

And since $\mathbb{Q}$ is countable, every subset of $\mathbb{Q}$ is a countable union of singleton sets, thus by property of σ-algebra, every subset of $\mathbb{Q}$ is in $\mathcal{B}_{\mathbb{Q}}$. Thus:

$$\mathcal{B}_{\mathbb{Q}} = \mathcal{P}(\mathbb{Q})$$

But clearly, **not every subset in $\mathbb{Q}$ is open.** Consider any singleton set, $\{1\}$ as an example. Any open ball centered at 1 is not contained in $\{1\}$, thus contradicting the statement.

Restriction of a σ-algebra to a Subset

Let X be a set, and $Y \subset X$ a subset.

(a) Given a σ-algebra $\mathcal{A}$ on X, prove that

$$\mathcal{A}|_Y := \{E \cap Y \mid E \in \mathcal{A}\}$$

is a σ-algebra on Y.

(b) Given a σ-algebra $\mathcal{B}$ on Y, prove that there exists a σ-algebra $\mathcal{A}$ on X such that $\mathcal{A}|_Y = \mathcal{B}$.

(c) Is the σ-algebra $\mathcal{A}$ in (b) unique? Give a proof or a counterexample.

Remark 这表示任何一个 measurable space 都可以对其中的一个 subspace 取一个 submeasurable space

Proof

(a)
1. Since $\emptyset \in \mathcal{A}$, $\emptyset \cap Y = \emptyset$, we have $\emptyset \in \mathcal{A}|_Y$
2. Let $F \in \mathcal{A}|_Y$, we must have $E \in \mathcal{A}$ s.t. $E \cap Y = F$. Since $E \in \mathcal{A}$, we have $X \setminus E \in \mathcal{A}$, so $X \setminus E \cap Y \in \mathcal{A}|_Y$. Since $E \cap Y = F$ and $Y = (E \cap Y) \sqcup ((X \setminus E) \cap Y)$, it implies $(X \setminus E) \cap Y = Y \setminus F$, therefore $Y \setminus F \in \mathcal{A}|_Y$.
3. Let $F_1, F_2, \cdots$ be a sequence of subsets in $\mathcal{A}|_Y$. Then for each $i \in \mathbb{N}$, we have $F_i = E_i \cap Y$ for some $E_i \in \mathcal{A}$. Then $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty}(E_i \cap Y) = (\bigcup_{i=1}^{\infty} E_i) \cap Y \in \mathcal{A}|_Y$ since $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

(b) Let $\mathcal{B}$ be a σ-algebra on Y.

prove that there exists a σ-algebra $\mathcal{A}$ on X such that $\mathcal{A}|_Y = \mathcal{B}$. Consider let

$$\mathcal{A} := \{\, E \subset X \mid E \cap Y \in \mathcal{B}\}$$

Then

$$\mathcal{A}|_Y = \{E \cap Y \mid E, Y \subset X, E \cap Y \in \mathcal{B}\} = \mathcal{B}$$

We then prove that this is a σ-algebra on X.

1. $\emptyset \cap Y = \emptyset$ so $\emptyset \in \mathcal{A}$.
2. **Closed under complement**: Let $E \in \mathcal{A}$, we have $E \cap Y \in \mathcal{B}$, so $Y \setminus (E \cap Y) = Y \setminus E \in \mathcal{B}$. Then $(X \setminus E) \cap Y = Y \setminus E \in \mathcal{B}$, so $X \setminus E \in \mathcal{A}$.
3. **Closed under countable union**: Let $E_1, E_2, \cdots$ be a sequence in $\mathcal{A}$, then $E_n \cap Y \in \mathcal{B}$. for each n. Hence
$$\left(\bigcup_{n=1}^{\infty} E_n\right) \cap Y \;=\; \bigcup_{n=1}^{\infty}(E_n \cap Y) \;\in\; \mathcal{B},$$
since $\mathcal{B}$ is a σ-algebra on Y. Therefore, $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$.

(c) This is not unique.

Counterexample:

$$X = \{0, 1, 2\}, Y = \{0\} \subset X$$

Consider

$$A_1 := \mathcal{P}(X), A_2 := \{\emptyset, \{0\}, \{1, 2\}, X\}$$

are valid σ-algebra on X.

Then we have $A_1|_Y = A_2|_Y = \{\emptyset, \{0\}\}$, while A_1 is different from A_2.

Invariance Properties of the Borel σ-algebra on $\mathbb{R}^n$

(a) Prove that $\mathcal{B}(\mathbb{R}^n)$ is translation invariant, i.e., if $A \subset \mathbb{R}^n$ is a Borel measurable set, then

$$t + A := \{t + x \mid x \in A\}$$

is a Borel measurable set for every $t \in \mathbb{R}^n$. (Hint: For any fixed t, show that $A = \{B \subset \mathbb{R}^n : t + B \in \mathcal{B}(\mathbb{R}^n)\}$ is a σ-algebra.)

(b) Prove that $\mathcal{B}(\mathbb{R}^n)$ is scaling invariant, i.e., if $A \subset \mathbb{R}^n$ is a Borel measurable set, then

$$\lambda A = \{\lambda x \mid x \in A\}$$

is a Borel measurable set for every $\lambda \in \mathbb{R}$.

(1)

Proof

Fix $t \in \mathbb{R}^n$. Define

$$\mathcal{A} := \{\, B \subseteq \mathbb{R}^n : t + B \in \mathcal{B}(\mathbb{R}^n)\}.$$

We want to show that $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$. We first show that $\mathcal{A}$ is a σ-algebra.

1. $\emptyset \in \mathcal{A}$ since $t + \emptyset = \emptyset \in \mathcal{B}(\mathbb{R}^n)$.
2. $\mathcal{A}$ is closed under complement: Let $B \in \mathcal{A}$, then $t + B \in \mathcal{B}(\mathbb{R}^n)$. The complement $(t + B)^c$ is also in

$\mathcal{B}(\mathbb{R}^n)$. Observe

$$t + B^c = t + \mathbb{R}^n \setminus B = (t + \mathbb{R}^n) \setminus (t + B) = \mathbb{R}^n \setminus (t + B) = (t + B)^c$$

Since $t + B$ is Borel, its complement is Borel, hence $t + B^c$ is Borel, so $B^c \in \mathcal{A}$.

3. $\mathcal{A}$ is closed under countable unions: Let $B_k \in \mathcal{A}$ for $k = 1, 2, \ldots$, then $t + B_k \in \mathcal{B}(\mathbb{R}^n)$. Thus

$$t + \bigcup_{k=1}^{\infty} B_k \;=\; \bigcup_{k=1}^{\infty} (t + B_k) \;\in\; \mathcal{B}(\mathbb{R}^n).$$

Hence $\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}$. These three properties show that $\mathcal{A}$ is a σ-algebra.

Since $t+U$ is open if U is open in $\mathbb{R}^n$, $\mathcal{A}$ contains all open sets. Since $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra containing all open sets in $\mathbb{R}^n$, we have:$\mathcal{B}(\mathbb{R}^n) \;\subseteq\; \mathcal{A}$ Hence suppose $A \in \mathcal{B}(\mathbb{R}^n)$, then $A \in \mathcal{A}$, so $t + A \in \mathcal{B}(\mathbb{R}^n)$. This completes the proof of translation invariance.

(2)

Proof Fix $\lambda \in \mathbb{R}$. Case 1: $\lambda = 0$, then $\lambda A = \{0\}$ if $A \neq \emptyset$, and $\lambda A = \emptyset$ otherwise. Both $\{0\}$(closed set) and $\emptyset$ is Borel set.

Case 2: $\lambda \neq 0$. We define

$$\mathcal{A} := \{\, B \subseteq \mathbb{R}^n : \lambda B \in \mathcal{B}(\mathbb{R}^n) \}.$$

We want to show that $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$. We first show that $\mathcal{A}$ is a σ-algebra.

1. $\emptyset \in \mathcal{A}$ since $\lambda\emptyset = \emptyset$.

2. $\mathcal{A}$ is closed under complement: Let $B \in \mathcal{A}$, then $\lambda B \in \mathcal{B}(\mathbb{R}^n)$, then $(\lambda B)^c$ is also in $\mathcal{B}(\mathbb{R}^n)$. Observe $(\lambda B)^c = \lambda B^c$, so $\lambda B^c \in \mathcal{B}(\mathbb{R}^n)$, therefore $B^c \in \mathcal{A}$. 3. $\mathcal{A}$ is closed under countable unions: Let $B_k \in \mathcal{A}$ for $k = 1, 2, \ldots$, then $\lambda B_k \in \mathcal{B}(\mathbb{R}^n)$. Thus

$$\lambda \bigcup_{k=1}^{\infty} B_k \;=\; \bigcup_{k=1}^{\infty} (\lambda B_k) \;\in\; \mathcal{B}(\mathbb{R}^n).$$

Hence $\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}$. These three properties show that $\mathcal{A}$ is a σ-algebra.

Since $\lambda \neq 0$, λU is open iff U is open in $\mathbb{R}^n$, thus $\mathcal{A}$ contains all open sets, so $\mathcal{B}(\mathbb{R}^n) \;\subseteq\; \mathcal{A}$,

Hence if $A \in \mathcal{B}(\mathbb{R}^n)$, we have $A \in \mathcal{A}$, therefore $\lambda A \in \mathcal{B}(\mathbb{R}^n)$. This completes the proof of translation invariance.

Hex and Such

Let $A \subset [0, 1]$ be the set of real numbers in $[0, 1]$ having a hexadecimal expansion with the digit 5 appearing infinitely many times, and the ‘digit’ E appearing at most finitely many times. Prove that A is a Borel set. (Hint: see p. 2 of Folland’s book.)

Proof Define:

$$B := \{x \in [0, 1] \mid \text{the digit '5' appears infinitely many times in the hex expansion of } x\}.$$

$$C := \{x \in [0, 1] \mid \text{the digit 'E' appears at most finitely many times in the hex expansion of } x\}.$$

Then clearly

$$A = B \cap C.$$

Hence **it suffices to show that** B **and** C **are Borel sets**, since intersection of two Borel sets is a Borel set. And thus it **suffices to show that** B^c **and** C **are Borel sets**. Note

$$B^c = \{x \in [0,1] \mid \text{the digit '5' appears at most finitely many times in the hex expansion of } x\}$$

, so the proof for B^c and C are about the same. We now show B^c is a Borel set: We define

$$C_{d_1 d_2 \cdots d_n} \; := \; \{\, x \in [0,1] : \text{the first } n \text{ hexadecimal digits of } x \text{ are } d_1, d_2, \ldots, d_n \,\},$$

where each d_i is one of the 16 hexadecimal digits $\{0, 1, 2, \ldots, 9, A, B, C, D, E, F\}$. Then the set contains all real numbers between $\frac{d_1 d_2 \cdots d_n}{16^n}$ and $\frac{d_1 d_2 \cdots d_n + 1}{16^n}$, so actually it is an interval:

$$C_{d_1 d_2 \cdots d_n} = \left[\frac{d_1 d_2 \cdots d_n}{16^n}, \; \frac{d_1 d_2 \cdots d_n + 1}{16^n} \right)$$

Since it is an interval, it is a Borel set on $[0,1]$. And we define:

$$D_N = \{x : \text{from digit } N \text{ onward, there are no '5's}\}.$$

Then we have

$$B^c = \bigcup_{N=1}^{\infty} D_N,$$

So it suffices to prove that each D_N is Borel set, since a countable union of Borel sets is Borel set.
Claim : any D_N **is a Borel set.** To prove this, we fix an N and define for each $n \geq N$

$$E_n \; = \; \{x \in [0,1] : d_n(x) \neq 5\}.$$

Then we have

$$E_n = \bigcup_{d_i \in \{1, \cdots, F\} \forall 1 \leq i \leq n, d_n \neq 5} C_{d_1 d_2 \cdots d_n}$$

Thus **each** E_n **is a Borel set** since it is a finite union of Borel set, which shows that D_N is Borel set, since

$$D_N \; = \; \bigcap_{k=N}^{\infty} E_k.$$

This finishes the proof that B^c is a Borel set, and by a similar argument, C is a Borel set, and thus $A = B \cap C$ is a Borel set.

Admissible Annuli generating $\mathcal{B}(\mathbb{R}^n)$

Define an admissible annulus in $\mathbb{R}^2$ to be a set of the form

$$\{(x, y) \in \mathbb{R}^2 \mid r^2 < (x-a)^2 + (y-b)^2 < R^2\},$$

where $a, b \in \mathbb{Q}$, $r, R \in \mathbb{Q}_{>0}$, and $r < R$.

(a) Prove that there are only countably many admissible annuli.

(b) Prove that every open subset of $\mathbb{R}^2$ is a countable union of (not necessarily disjoint) admissible annuli.

(c) Prove that the Borel σ-algebra on $\mathbb{R}^2$ is generated by the collection of admissible annuli.

(1)

Proof Let
$$A := \{\text{all admissible annulis in } \mathbb{R}^2\}$$

And we define
$$f : \mathbb{Q}^4 \to A \tag{1.1}$$
$$(a, b, r, R) \mapsto \{(x, y) \in \mathbb{R}^2 \mid r^2 < (x-a)^2 + (y-b)^2 < R^2\} \tag{1.2}$$

Since a Annuli defined by this (a, b, r, R) is unique, this is a well-defined function; and since every admissible annulis can be defined by an element of $\mathbb{Q}^4$, this map is surjective. Therefore $\text{card}(A) \leq \text{card}(\mathbb{Q}^4)$, so A is countable.

(2)

Proof **Claim 1: every open set in $\mathbb{R}^2$ is a countable union of open balls, each centered at some $q \in \mathbb{Q}^2$.**

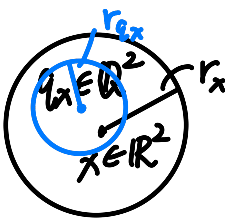
Proof for Claim 1:

Let U be an open set in $\mathbb{R}^2$. Define
$$\mathbb{Q}_U := U \cap \mathbb{Q}^2$$

By definition, every point in U have an open ball centered at it that is completely contained in U, so we pick such ball $B_{r_x}(x)$ for each $x \in U$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, for each $x \in U$ and each corresponding r_x, we can find a rational point $q_x \in \mathbb{Q}^2$ such that $|q_x - x| < \frac{r_x}{3}$. (Or more generally, as small as we wish.)

Let $r_{q_x} > 0$ be chosen so that $r_{q_x} = \frac{r_x}{3}$, Then observe that $x \in B(q_x, r_{q_x})$
$$B(q_x, r_{q_x}) \subsetneq B(x, r_x) \subset U$$

which follows from the triangle inequality.



For each $q \in \mathbb{Q}_U$, we define:
$$r_{q,sup} := \sup\{r_{q_x} \mid q \text{ is chosen by } x\}$$

Now we have:
$$U \subset \bigcup_{q \in U_q} B_{r_{q,sup}}(q)$$

This is because for each each $x \in U$, $x \in B_{r_{q_x}}(q_x) \subset B_{r_{q_x,sup}}(q_x)$

And we also have the other direction:
$$\bigcup_{q \in U_q} B_{r_{q,sup}}(q) \subseteq U$$

since every $B_{r_q}(q)$ is guaranteed to be the subset of some ball around some $x \in U$. All togethe we have
$$U = \bigcup_{q \in U_q} B_{r_{q,sup}}(q)$$

This finishes the proof of claim 1.

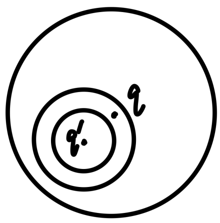
Claim 2: every open ball centered at some $q \in \mathbb{Q}^2$ is a countable union of admissible annulises with the same center, together with another admissible annulis whose center is also rational. Proof for Claim 2: Let $q = (a, b) \in \mathbb{Q}^2$.

We have

$$B(q, R) \setminus \{q\} = \bigcup_{n=1}^{\infty} \left\{ (x, y) : (R - \tfrac{1}{n})^2 < (x-a)^2 + (y-b)^2 < R^2 \right\}$$

-1, 这里写的略有问题, 因为 R 不一定是 rational 的, 不过我们可以用 density of $\mathbb{Q}$ in $\mathbb{R}$ 来写. It remains to cover the center. Let $q' := (a', b') \in \mathbb{Q}^2$ such that $R/6 < |q' - q| < R/3$, $r' := R/6$ and $R' := R/2$. Then the annuli $A(a', b', r', R')$ defined by the four parameters is contained in the $B(q, R)$ and it covers $\{q\}$. Therefore

$$B(q, R) = \left(\bigcup_{n=1}^{\infty} \left\{ (x, y) : (R - \tfrac{1}{n})^2 < (x-a)^2 + (y-b)^2 < R^2 \right\} \right) \cup A(a', b', r', R')$$



This finishes the proof of Claim 2.

Combining Claim 1 and Claim 2, we can conclude that **every open subset of $\mathbb{R}^2$ is a countable union of admissible annuli.**

(3)

Proof

As defined,

$$\mathcal{B}(\mathbb{R}^2) = < \mathcal{T}_{metric} > = < \{\text{all open sets in } \mathbb{R}^2\} >$$

Let

$$A := \{\text{all admissible annulis in } \mathbb{R}^2\}$$

Every admissible annuli is open in $\mathbb{R}^2$, so

$$A \subset \{\text{all open sets in } \mathbb{R}^2\}$$

and since $\mathcal{B}(\mathbb{R}^2)$ is a σ-algebra, we have

$$< A > \subset < \{\text{all open sets in } \mathbb{R}^2\} > = \mathcal{B}(\mathbb{R}^2)$$

by the proposition proved in class. And by (2), any open set is a countable union of admissible annulis, therefore every open set is in $< A >$ since any countable union of sets in a σ-algebra is still in the set. So

$$\{\text{all open sets in } \mathbb{R}^2\} \subset < A >$$

This finishes the proof that

$$< A > = < \{\text{all open sets in } \mathbb{R}^2\} > = \mathcal{B}(\mathbb{R}^2)$$

Nur für Verrückte

(It's really not necessary to attempt these problems. Do not hand them in!)

(1) Let X be a set, and define two operations on $\mathcal{P}(X)$:

- The "product" of two subsets $E, F \subset X$ is the intersection $E \cap F$.
- The "sum" of two sets $E, F \subset X$ is the symmetric difference $E \Delta F$.

(a) Prove that these operations endow $\mathcal{P}(X)$ with the structure of a commutative ring. What are the additive and multiplicative units? Prove that this ring is idempotent.

(b) Let us say that a nonempty subset $A \subset \mathcal{P}(X)$ is a ring if it is closed under differences and finite unions. In other words, if $E, F \in A$, then $E \setminus F \in A$ and $E \cup F \in A$. Prove that a subset $A \subset \mathcal{P}(X)$ is an algebra iff it is a ring containing X.

(c) Prove that a nonempty subset $A \subset \mathcal{P}(X)$ is a ring iff it is a subring of $\mathcal{P}(X)$. Also prove that it is an algebra iff it is a subring containing the multiplicative identity.

(2) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Say that a map $f : X \to Y$ is measurable (with respect to the σ-algebras $\mathcal{A}$ and $\mathcal{B}$) if $f^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{B}$.

(a) Prove that measurable spaces with measurable maps as morphisms form a category.

(b) Try convincing an analyst that (a) is useful.