

## Homework 2: on Carathéodory's and Hahn-Holmogrov

### Thm(40/40)

None of the following questions will be graded. Do them, but do not hand them in.

### The Borel–Cantelli Lemma

Let  $(X, \mathcal{A}, \mu)$  be a measure space. Let  $A_i \in \mathcal{A}$  for  $i \in \mathbb{N}$ , and suppose that

$$\sum_{i=1}^{\infty} \mu(A_i) < \infty.$$

(a) Prove that  $\mu(\limsup_i A_i) = 0$ , where

$$\limsup_i A_i = \{x \in X \mid x \in A_i \text{ for infinitely many } i\}.$$

(By the way, why is  $\limsup_i A_i$  measurable?)

(b) Conversely, is it true that if  $A_i \in \mathcal{A}$  for  $i \in \mathbb{N}$ , and  $\mu(\limsup_i A_i) = 0$ , then  $\sum_i \mu(A_i) < \infty$ ? Provide a proof or a counterexample. (Wrong) **Remark**

#### Theorem 2.1 (Borel–Cantelli Lemma)

一个 measure 和有限的 set seq, 其  $\limsup$  (出现 infinitely many times 的元素) 是零测的.



其实这是 trivial 的, 因为如果出现 infinitely many times 的元素不是零测的, say  $\mu(\limsup_i A_i) := k > 0$ , 那么有 infinitely many 个  $A_i$  的测度大于等于  $k$ , 那么  $\sum_{i=1}^{\infty} \mu(A_i) > k \times \infty$  就一定不是有限的了. 其 application: 一个 prob space 中, a ctbl seq of 事件发生的概率的和如果收敛, 那么它们包含的任何事件发生无穷多次的概率为 0, 意味着事件至多发生有限次 (almost surely).

### The Completion of a Measure Space

Let  $(X, \mathcal{A}, \mu)$  be a measure space, and set

$$\overline{\mathcal{A}} := \{E \cup F \mid E \in \mathcal{A} \text{ and } F \text{ is a } \mu\text{-subnull set}\}.$$

(a) Prove that  $\overline{\mathcal{A}}$  is a  $\sigma$ -algebra. (b) Define  $\overline{\mu}(A) := \mu(E)$  if  $A = E \cup F \in \overline{\mathcal{A}}$ . Prove that  $\overline{\mu}$  is a well-defined measure on  $\overline{\mathcal{A}}$ . (c) Prove that  $\overline{\mu}$  extends  $\mu$  (i.e.,  $\overline{\mu}(A) = \mu(A)$  if  $A \in \mathcal{A}$ ). (d) Prove that  $\overline{\mu}$  is the **unique extension of  $\mu$  to  $(X, \overline{\mathcal{A}})$** . In other words, prove that if  $\mu'$  is another measure on  $(X, \overline{\mathcal{A}})$  that extends  $\mu$ , then  $\mu' = \overline{\mu}$ . (e) Prove that  $\overline{\mu}$  is **complete**. (f) Suppose  $(X, \mathcal{A}', \mu')$  is another complete measure space that extends  $(X, \mathcal{A}, \mu)$  (i.e.,  $\mathcal{A} \subset \mathcal{A}'$  and  $\mu'|_{\mathcal{A}} = \mu$ ). Show that  $\overline{\mathcal{A}} \subset \mathcal{A}'$  and  $\mu'|_{\overline{\mathcal{A}}} = \overline{\mu}$ . **Hint:** Start by reading Theorem 1.9 in Folland.

**Proof** 略.(嘻嘻)

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## The Hahn–Kolmogorov Extension as a Completion

Let  $(X, \mathcal{A}_0, \mu_0)$  be a  $\sigma$ -finite measure pre-measure space, and  $(X, \mathcal{A}, \mu)$  its Hahn – Kolmogorov extension. Prove that  $(X, \mathcal{A}, \mu)$  is the completion of its restriction to the  $\sigma$ -algebra  $\langle \mathcal{A}_0 \rangle$  generated by  $\mathcal{A}_0$ .

**Proof** Proved in lec notes.

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Some of the following questions will be graded. Do them, and do hand them in.

### $\mu(\emptyset) = 0$ 的定义并非 redundant

Let  $(X, \mathcal{A})$  be a measurable space. Is the condition  $\mu(\emptyset) = 0$  in the definition of a measure on  $(X, \mathcal{A})$  redundant? In other words, if  $\mu : \mathcal{A} \rightarrow [0, \infty]$  is a function such that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),$$

for any disjoint subsets  $A_i \in \mathcal{A}, i \in \mathbb{N}$ , does it follow that  $\mu(\emptyset) = 0$ ? If not, what can you say?

**Proof** It does not follow.

Counterexample: Consider  $\mu(E) = \infty \quad \forall E \in \mathcal{A}$ .

This measure satisfies the countably disjoint additivity condition, since for every disjoint sequence of sets in  $\mathcal{A}$ ,  $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \infty$  has infinite measure.

### measurable set seq 的 limit 也 measurable (且如果 seq tail $\sigma$ -finite $\implies$ limit commute)

Let  $(X, \mathcal{A}, \mu)$  be a measure space, and let  $A_i \in \mathcal{A}, i \in \mathbb{N}$ . Assume that the sets  $A_i$  converge to the set  $A \subset X$  in the sense that: - If  $x \in A$ , then  $x \in A_i$  for all but finitely many  $i$ ; - If  $x \notin A$ , then  $x \notin A_i$  for all but finitely many  $i$ .

**(a) Prove that  $A$  is measurable, that is,  $A \in \mathcal{A}$ .**

**Proof** Deducing from the conditions: If  $x \in A$ , then  $x \in A_i$  for all but finitely many  $i$ ;  $\implies A \subset \liminf A_i$   
If  $x \notin A$ , then  $x \notin A_i$  for all but finitely many  $i$ .  $\implies$  if  $x \in A_i$  for all but finitely many  $i$  then  $x \in A \implies \limsup A_i \subset A$  Thus

$$\limsup A_i \subset A \subset \liminf A_i \quad (2.1)$$

**Claim1: For any sequence of sets  $(A_i)_{i \in \mathbb{N}}$ , we have**

$$\liminf A_i \subset \limsup A_i$$

Proof of Claim 1: Follows trivially from the definition, since  $x \notin A_i$  for all but finitely many  $i \implies x \notin A_i$  for infinitely many  $i$ .

Combining claim (1) with (2.1) we have

$$\limsup A_i = A = \liminf A_i \quad (2.2)$$

**Claim 2: For any sequence of sets  $(A_i)_{i \in \mathbb{N}}$  in a  $\sigma$ -algebra,  $\liminf_i A_i$  and  $\limsup_i A_i$  is also in the  $\sigma$ -algebra.**

Proof of Claim 2: This follows from the def and fact that union and intersection of a countable sequence sets in a  $\sigma$ -algebra is also in this  $\sigma$ -algebra. We have

Define for each  $k \in \mathbb{N}$   $B_k := \bigcup_{i=k}^{\infty} A_i \in \mathcal{A}$  since  $\sigma$ -algebra is close under ctbl union

$\Rightarrow \limsup A_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i = \bigcap_{k=1}^{\infty} B_k \in \mathcal{A}$  since  $\sigma$ -algebra is closed under ctbl intersection

Similarly,  $\liminf A_i = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} A_i \in \mathcal{A}$

This finishes the proof of Claim 2.

Combining claim 2 with (2.2),  $A \in \mathcal{A}$ , this finishes the proof.

(b) Prove that if there exists  $n \geq 1$  such that  $\mu(\bigcup_{i=n}^{\infty} A_i) < \infty$ , then  $\mu(A) = \lim_i \mu(A_i)$ .

**Proof**

$$\begin{aligned}
 A &= \limsup A_i = \liminf A_i \\
 &= \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} A_i \\
 \text{Define for each } m \in \mathbb{N} \quad B_m &:= \bigcup_{i=m}^{\infty} A_i, \quad C_m := \bigcap_{i=m}^{\infty} A_i \\
 \Rightarrow A &= \bigcap_{m=1}^{\infty} B_m = \bigcup_{m=1}^{\infty} C_m \\
 &\text{and } B_{m+1} \subseteq B_m, \quad C_m \subseteq C_{m+1} \quad \forall m \in \mathbb{N} \\
 \text{By ctn from below/above property of measure} \\
 \Rightarrow \mu(A) &= \mu\left(\bigcap_{m=1}^{\infty} B_m\right) = \mu\left(\bigcup_{m=1}^{\infty} C_m\right) \\
 &= \lim_{m \rightarrow \infty} \mu(B_m) = \lim_{m \rightarrow \infty} \mu(C_m) \quad (\text{equation 2.3})
 \end{aligned}$$

Fix  $m \in \mathbb{N}$ , we have  $\mu(B_m) = \mu(\bigcup_{i=m}^{\infty} A_i) \geq \mu(A_n)$  for any  $n \geq m$   
 so  $\mu(B_m) \geq \sup_{n \geq m} \mu(A_n)$

Thus  $\mu(A) = \lim_{m \rightarrow \infty} \mu(B_m) \geq \limsup \mu(A_i)$   
 For the same reason we have  $\mu(A) = \lim_{m \rightarrow \infty} \mu(C_m) \leq \liminf \mu(A_i)$   
 Thus we have

$$\limsup \mu(A_i) \leq \mu(A) \leq \liminf \mu(A_i) \quad (\text{equation 2.4})$$

Since for the numerical seq,  $(\mu(A_i))_{i \in \mathbb{N}}$ , must have

Therefore  $\liminf \mu(A_i) \leq \limsup \mu(A_i)$   
 $\limsup \mu(A_i) = \mu(A) = \liminf \mu(A_i)$

Since  $\liminf, \limsup$  both exist and equal, this finishes the proof  
 that  $\lim_{i \rightarrow \infty} \mu(A_i) = \mu(A)$   $\square$

(c) Give an example showing that the condition in (b) is necessary.

**Sol.** Let  $\mu$  be the Lebesgue measure defined on  $\mathcal{B}(\mathbb{R})$ . We set for each  $i \in \mathbb{N}$  that

$$A_i = (i, i + 1)$$

Since this is an interval, it is Lebesgue measurable. Note that no element of any  $A_i$  show up infinitely many times in the sequence. So

$$\liminf A_i = \limsup A_i = \emptyset$$

So  $A = \emptyset$ , we have  $\lim_i \mu(A_i) = 0$ . But we have  $\lim_i \mu(A_i) = 1$  since it is true for every  $i$ .

In this case,  $\mu(\bigcup_{i=1}^{\infty} A_i) = \infty$ , which causes (b) to fail.

**Hint:** In analysis, it is often fruitful to use  $\limsup$  and  $\liminf$  to study limits.

## measure space of two elements

Let  $X$  be a set with two elements, for example,  $X = \{O, Q\}$ .

(a) Find all  $\sigma$ -algebras on  $X$ .

**Sol.**

1. trivial  $\sigma$ -algebra:

$$\mathcal{A}_1 := \{\emptyset, X\}$$

2. power set:

$$\mathcal{A}_2 := \mathcal{P}(X) = \{\emptyset, \{O\}, \{Q\}, X\}$$

These are the only  $\sigma$ -algebras on  $X$ .

**(b) Let  $\mathcal{A}$  be a  $\sigma$ -algebra on  $X$ , and  $\mu$  a measure on  $(X, \mathcal{A})$ . Is  $\mu$  necessarily complete? Provide a proof or a counterexample.**

**Sol.** It is not necessarily complete.

Consider the trivial  $\sigma$ -algebra:  $\mathcal{A}_1 := \{\emptyset, X\}$ , and set  $\mu$  as that  $\mu(\emptyset) = \mu(X) = 0$ . This makes  $X$  a null set, so  $\{O\}, \{Q\}$  are subnull sets, but they are not measurable by  $\mu$ .

**(c) Find all outer measures  $\mu^*$  on  $X$ . For each outer measure on  $X$ , find the  $\sigma$ -algebra of  $\mu^*$ -measurable sets (see Carathéodory's theorem).**

**Sol.** Suppose  $\mu^*$  is an outer measure on  $X$ . Since  $\mathcal{P}(X)$  only has four elements:  $\emptyset, \{O\}, \{Q\}, X$ ; and the outer measure of  $\emptyset$  is 0, so we first parametrize  $\mu^*$  by:

$$a := \mu^*(\{O\}), \quad b := \mu^*(\{Q\}), \quad c := \mu^*(X).$$

Then  $\mu^*$  is well-defined iff it satisfies:

1.  $a, b \leq c$
2.  $c = \mu^*(\{O\} \cup \{Q\}) \leq \mu^*(\{O\}) + \mu^*(\{Q\}) = a + b$ .

Any  $(a, b, c) \in [0, \infty]^3$  satisfying

$$\max(a, b) \leq c \leq a + b,$$

can make  $\mu^*$  a well-defined outer measure on  $X$ .

Therefore

$$S := \{\text{all } \sigma\text{-algebra on } X\} = \{\mu^* : \mathcal{P}(X) \rightarrow [0, \infty] \mid \max(\mu^*(\{O\}), \mu^*(\{Q\})) \leq \mu^*(X) \leq \mu^*(\{O\}) + \mu^*(\{Q\})\}$$

Now we specify the  $\sigma$ -algebra of  $\mu^*$ -measurable sets for each  $\mu^* \in S$ .

By Carathéodory's criterion, a set  $E \subset X$  is  $\mu^*$ -measurable iff for all  $A \subset X$ ,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Note that  $\emptyset, X$  are always measurable since for any  $A \subset X$ ,  $A \cap \emptyset = \emptyset$ ,  $A \cap (\emptyset)^c = A$ ; and  $A \cap X = A$ ,  $A \cap (X)^c = \emptyset$ . So it suffices to check for  $\{O\}, \{Q\}$ . We first check for  $\{O\}$ .  $\{O\}$  is  $\mu^*$ -measurable iff  $\mu^*(A) = \mu^*(A \cap \{O\}) + \mu^*(A \cap \{O\}^c)$  for any choice of  $A$ . There are only four possibilities for  $A$ :  $\emptyset, \{O\}, \{Q\}, X$ .

1. If  $A = \emptyset$ , both sides are 0, always stands.
2. If  $A = \{O\}$ , then  $\mu^*(\{O\}) + \mu^*(\emptyset) = a + 0 = a$ , always stands.
3. If  $A = \{Q\}$ , then  $\mu^*(\emptyset) + \mu^*(\{Q\}) = 0 + b = b$ , always stands.
4. If  $A = X$ , then  $\mu^*(X) = c = \mu^*(\{O\}) + \mu^*(\{Q\}) = a + b$ .

Therefore  $\{O\}$  is  $\mu^*$ -measurable iff  $c = a + b$ . For the same reasoning,  $\{Q\}$  is  $\mu^*$ -measurable iff  $c = a + b$ .

Thus we can conclude that:

1. If  $c = a + b$ ,  $\{\mu^*\text{-measurable sets}\} = \mathcal{P}(X)$ .
2. otherwise,  $\{\mu^*\text{-measurable sets}\} = \{\emptyset, X\}$ .

**(d) Find an example of a collection  $\mathcal{E}$  of subsets of  $X$  with  $\emptyset, X \in \mathcal{E}$  and a function  $\rho : \mathcal{E} \rightarrow [0, \infty]$  with  $\rho(\emptyset) = 0$  such that  $\mathcal{E} \not\subset \mathcal{A}$ , where  $\mathcal{A}$  is the Carathéodory  $\sigma$ -algebra for the outer measure  $\mu^*$  induced by**

$(\mathcal{E}, \rho)$ .

**Sol.** Consider  $\mathcal{E} = \{\emptyset, X, \{O\}\}$ , with  $\rho$  such that  $\rho(\emptyset) = 0$ ,  $\rho(X) = 1$ ,  $\rho(\{O\}) = 1$ .

The outer measure  $\mu^*$  induced by  $\mu^*$  is:  $\mu^*(X) = 1$ ,  $\mu^*(\{O\}) = 1$ ,  $\mu^*(\{Q\}) = 1$ . (the inf of length sum of sets covering  $\{Q\}$  is 1, by taking  $\{X\}$  as the covering.)

Since  $c \neq a + b$ , by (4), the Carathéodory  $\sigma$ -algebra by  $\mu^*$  by  $\mathcal{E}$  is  $\{\emptyset, X\}$ , so  $\mathcal{E} \not\subset \mathcal{A}$ .

**Remark:** The Hahn – Kolmogorov theorem states that if  $\mathcal{E} = \mathcal{A}_0$  is an algebra and  $\rho = \mu_0$  is a pre-measure, then  $\mathcal{A}_0 \subset \mathcal{A}$ . This exercise provides a counterexample when  $\mathcal{E}$  and  $\rho$  are general. ,

## Hahn – Kolmogorov Collapse (when $\mu_0$ not $\sigma$ -finite)

Let  $X \subset \mathbb{R}$  be the set of dyadic rational numbers, that is, the set of numbers of the form  $\frac{r}{2^n}$ , where  $r$  and  $n$  are integers. Let  $\mathcal{A}_0 \subset \mathcal{P}(X)$  be the collection of finite unions of intervals of the form  $(a, b] \cap X$ , where  $-\infty \leq a < b \leq \infty$ .

**(a) Prove that  $\mathcal{A}_0$  is an algebra.**

**Proof**

1.  $\emptyset \in \mathcal{A}_0$ , since it is the empty union of intervals of the given form.
2. **Closed under complements:** Let  $A \in \mathcal{A}_0$ . Then  $A$  is a finite union of intervals of the form  $(a_i, b_i] \cap X$ .  
So

$$A^c \cap X = X \setminus A \quad (2.3)$$

$$= X \setminus \bigcup_{i=1}^n ((a_i, b_i] \cap X) \quad (2.4)$$

$$= X \cap \left( \bigcap_{i=1}^n ((a_i, b_i] \cap X)^c \right) \quad (2.5)$$

$$= X \cap \left( \bigcap_{i=1}^n ((-\infty, a_i] \cup (b_i, \infty]) \right) \quad (2.6)$$

Note that finite intersection of intervals of the form  $(-\infty, a_i]$ ,  $(b_i, \infty]$  is still of this form. Hence  $A^c \cap X \in \mathcal{A}_0$ .

3. **Closed under finite unions:** Suppose  $A_1$  and  $A_2$  are finite unions of intervals  $((a_i, b_i] \cap X)$ , then  $A_1 \cup A_2$  is still a finite union of intervals of that form. (They either merge into one such interval, so are disjoint.)  
Hence  $A_1 \cup A_2 \in \mathcal{A}_0$ . The same reasoning extends to any finite union.

**This finishes the proof that  $\mathcal{A}_0$  is an algebra on  $X$ .**

**(b) Prove that the  $\sigma$ -algebra on  $X$  generated by  $\mathcal{A}_0$  equals  $\mathcal{P}(X)$ .**

**Proof** Since  $\langle \mathcal{A}_0 \rangle \subset \mathcal{P}(X)$ , it suffices to show that  $\mathcal{P}(X) \subset \langle \mathcal{A}_0 \rangle$ . Note that  $X$  is countable, so any set in  $\mathcal{P}(X)$  is a countable union of singleton sets. Thus it suffices to show that any singleton set  $\{x\}$  where  $x \in X$  is in  $\langle \mathcal{A}_0 \rangle$ , since if so, then any countable union of singleton sets from  $\mathcal{P}(X)$  is also in  $\langle \mathcal{A}_0 \rangle$ , with implies

that  $\mathcal{P}(X) \subset \langle \mathcal{A}_0 \rangle$

Let  $x \in X$ . Then we have:

$$\{x\} = \bigcap_{n=1}^{\infty} \left( \left(x - \frac{1}{2^n}, x\right] \cap X \right),$$

since  $x$  is in the RHS set, and for any  $y < x$ , we can find a  $n \in \mathbb{N}$  such that  $x - \frac{1}{2^n} > y$ .

This finishes the proof that  $\langle \mathcal{A}_0 \rangle = \mathcal{P}(X)$ .

**(c) Define  $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$  by  $\mu_0(\emptyset) = 0$  and  $\mu_0(A) = \infty$  for  $A \neq \emptyset$ . Prove that  $\mu_0$  is a pre-measure on  $\mathcal{A}_0$**

**Proof** It suffices to show the countable disjoint additivity.

Let  $(A_i)_{i \in \mathbb{N}}$  be a sequence of disjoint sets in  $\mathcal{A}_0$ .

Case 1: all  $A_i = \emptyset$ , then  $\sqcup_{i \in \mathbb{N}} A_i = \emptyset$ , so  $\mu_0(\sqcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu_0(A_i) = 0$ .

Case 2:  $A_k \neq \emptyset$  for some  $k$ , then  $\mu_0(A_k) = \infty$  and  $\sqcup_{i \in \mathbb{N}} A_i \neq \emptyset$ . Thus  $\sum_{i \in \mathbb{N}} \mu_0(A_i) \geq \mu_0(A_k) = \infty = \mu_0(\sqcup_{i \in \mathbb{N}} A_i)$ .

The two cases cover all circumstances, finishing the proof.

**(d) Prove that there exist infinitely many different measures  $\mu$  on  $\mathcal{P}(X)$  whose restriction to  $\mathcal{A}_0$  equals  $\mu_0$ .**

**Proof** Given  $n \in \mathbb{N}$ , We define the "n-timed counting measure" on a  $\sigma$ -algebra  $S$  as:

$$\mu_{count_n}(E) := \begin{cases} n \times \#(E) & , \text{ if } E \text{ is finite} \\ \infty & , \text{ if } E \text{ is infinite} \end{cases}$$

**Claim 1: For any set  $X$  and any  $\sigma$ -algebra  $S$  on  $X$ , the "n-timed counting measure" is a well-defined measure on  $S$ , for all  $n \in \mathbb{N}$ .**

Proof of claim 1:  $\mu_{count_n}(\emptyset) = 0$  since  $\text{card}(\emptyset) = 0$ , and countable disjoint additivity trivially follows from the rule of counting.

**Claim 2: for any  $n \in \mathbb{N}$ ,  $\mu_{count_n}(E)$  on  $\mathcal{P}(X)$  restricted to  $\mathcal{A}_0$  equals  $\mu_0$ .** Proof of claim 2: Let  $E \in \mathcal{A}_0 \setminus \emptyset$ , then  $E$  contains at least one interval of the form  $(a, b] \cap X$ , where  $-\infty \leq a < b \leq \infty$ . Since  $a < b$ , there are infinitely many elements in  $(a, b] \cap X$ , so  $\mu_{count_n}(E) = \infty$ .

This finishes the proof of the original statement.

**(e) Explain why (d) does not contradict the uniqueness part of the Hahn – Kolmogorov theorem (see Theorem 1.14 in Folland).**

**Sol.** This is because Hahn – Kolmogorov theorem requires  $\mu_0$  to be  $\sigma$ -finite to extend uniquely on  $\langle \mathcal{A}_0 \rangle$ . But  $\mu_0$  here is not  $\sigma$ -finite.

## Nur für Verrückte (Only for nuts)

(It' s really not necessary to attempt these problems. Do not, under any circumstances, hand them in!)

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1. Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be measure spaces. Define a morphism from  $(X, \mathcal{A}, \mu)$  to  $(Y, \mathcal{B}, \nu)$  to be a map  $f : X \rightarrow Y$  that is measurable, that is,  $f^{-1}(B) \in \mathcal{A}$  for all  $B \in \mathcal{B}$ , and moreover measure preserving, in the sense that  $\mu(f^{-1}(B)) = \nu(B)$  for all  $B \in \mathcal{B}$ .

(a) Prove that measure spaces with measure-preserving maps as morphisms form a category. Denote this category by  $C_3$ .

(b) Denote by  $C_1$  the category of sets, and by  $C_2$  the category of measurable spaces (see HW1). Consider the evident forgetful functors  $C_3 \rightarrow C_2$  and  $C_2 \rightarrow C_1$ . Are these functors faithful? Are they full? Are they essentially surjective?