Lec 1 on Carathéodory's and Hahn-Holmogrov Thm(40/40)

None of the following questions will be graded. Do them, but do not hand them in.

The Borel–Cantelli Lemma

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$, and suppose that

$$\sum_{i=1}^{\infty} \mu(A_i) < \infty.$$

(a) Prove that $\mu(\limsup_i A_i) = 0$, where

$$\limsup_i A_i = \{x \in X \mid x \in A_i \text{ for infinitely many } i\}.$$

(By the way, why is $\limsup_i A_i$ measurable?)

(b) Conversely, is it true that if $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$, and $\mu(\limsup_i A_i) = 0$, then $\sum_i \mu(A_i) < \infty$? Provide a proof or a counterexample. (Wrong) **Remark**

Theorem 1.1 (Borel–Cantelli Lemma)

一个 measure 和有限的 set seq, 其 lim sup (出现 infinitely many times 的元素) 是零测的.

其实这是 trivial 的, 因为如果出现 infinitely many times 的元素不是零测的, say $\mu(\limsup_i A_i) := k > 0$, 那么有 infinitely many 个 A_i 的测度大于等于 k, 那么 $\sum_{i=1}^{\infty} \mu(A_i) > k \times \infty$ 就一定不是有限的了.
其 application: 一个 prob space 中, a ctbl seq of 事件发生的概率的和如果收敛, 那么它们包含的任何事件发生无穷多次的概率为 0，意味着事件至多发生有限次 (almost surely).

The Completion of a Measure Space

Let $(X, \mathcal{A}, \mu)$ be a measure space, and set

$$\overline{\mathcal{A}} := \{E \cup F \mid E \in \mathcal{A} \text{ and } F \text{ is a } \mu\text{-subnull set}\}.$$

(a) Prove that $\overline{\mathcal{A}}$ is a σ-algebra. (b) Define $\overline{\mu}(A) := \mu(E)$ if $A = E \cup F \in \overline{\mathcal{A}}$. Prove that $\overline{\mu}$ is a well-defined measure on $\overline{\mathcal{A}}$. (c) Prove that $\overline{\mu}$ extends μ (i.e., $\overline{\mu}(A) = \mu(A)$ if $A \in \mathcal{A}$). (d) Prove that $\overline{\mu}$ **is the unique extension of** μ **to** $(X, \overline{\mathcal{A}})$. In other words, prove that if μ' is another measure on $(X, \overline{\mathcal{A}})$ that extends μ, then $\mu' = \overline{\mu}$. (e) Prove that $\overline{\mu}$ **is complete**. (f) Suppose $(X, \mathcal{A}', \mu')$ is another complete measure space that extends $(X, \mathcal{A}, \mu)$ (i.e., $\mathcal{A} \subset \mathcal{A}'$ and $\mu'|_{\mathcal{A}} = \mu$). Show that $\overline{\mathcal{A}} \subset \mathcal{A}'$ **and** $\mu'|_{\overline{\mathcal{A}}} = \overline{\mu}$. **Hint:** Start by reading Theorem 1.9 in Folland.

Proof 略.(嘻嘻)

The Hahn–Kolmogorov Extension as a Completion

Let $(X, \mathcal{A}_0, \mu_0)$ be a σ-finite measure pre-measure space, and $(X, \mathcal{A}, \mu)$ its Hahn – Kolmogorov extension. Prove that $(X, \mathcal{A}, \mu)$ is the completion of its restriction to the σ-algebra $\langle \mathcal{A}_0 \rangle$ generated by $\mathcal{A}_0$.

Proof Proved in lec notes.

Some of the following questions will be graded. Do them, and do hand them in.

$\mu(\emptyset)=0$ 的定义并非 redundant

Let $(X,\mathcal{A})$ be a measurable space. Is the condition $\mu(\emptyset)=0$ in the definition of a measure on $(X,\mathcal{A})$ redundant? In other words, if $\mu:\mathcal{A}\to[0,\infty]$ is a function such that

$$\mu\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i),$$

for any disjoint subsets $A_i\in\mathcal{A}$, $i\in\mathbb{N}$, does it follow that $\mu(\emptyset)=0$? If not, what can you say?

Proof It does not follow.

Counterexample: Consider $\mu(E)=\infty\ \ \forall E\in\mathcal{A}$.

This measure satisfies the countably disjoint additivity condition, since for every disjoint sequence of sets in $\mathcal{A}$, $\mu\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i)=\infty$ has infinite measure.

measurable set seq 的 limit 也 measurable (且如果 seq tail σ-finite $\implies$ limit commute)

Let $(X,\mathcal{A},\mu)$ be a measure space, and let $A_i\in\mathcal{A}, i\in\mathbb{N}$. Assume that the sets A_i converge to the set $A\subset X$ in the sense that: - If $x\in A$, then $x\in A_i$ for all but finitely many i; - If $x\notin A$, then $x\notin A_i$ for all but finitely many i.

(a) Prove that A is measurable, that is, $A\in\mathcal{A}$.

Proof Deduing from the conditions: If $x\in A$, then $x\in A_i$ for all but finitely many i; $\implies A\subset\liminf A_i$ If $x\notin A$, then $x\notin A_i$ for all but finitely many i. $\implies$ if $x\in A_i$ for all but finitely many i then $x\in A\implies\limsup A_i\subset A$ Thus

$$\limsup A_i\subset A\subset\liminf A_i \tag{1.1}$$

Claim1: For any sequence of sets $(A_i)_{i\in\mathbb{N}}$, we have

$$\liminf A_i\subset\limsup A_i$$

Proof of Claim 1: Follows trivially from the definition, since $x\notin A_i$ for all but finitely many $i\implies x\notin A_i$ for infinitely many i.

Combining claim (1) with (2.1) we have

$$\limsup A_i=A=\liminf A_i \tag{1.2}$$

Claim 2: For any sequence of sets $(A_i)_{i\in\mathbb{N}}$ in a σ-algebra, $\liminf_i Ai$ and $\limsup_i A_i$ is also in the σ-algebra.

Proof of Claim 2: This follows from the def and fact that union and intersection of a countable sequence sets in a σ-algebra is also in this σ-algebra. We have

Define for each $k \in \mathbb{N}$ $B_k := \bigcup_{i=k}^{\infty} A_i, \in \mathcal{A}$ since σ-algebra is close under ctbl union

$$\Rightarrow \limsup A_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i = \bigcap_{k=1}^{\infty} B_k, \in \mathcal{A}$$ since σ-algebra is closed under ctbl intersection

Similarly, $\liminf A_i = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} A_i \in \mathcal{A}$

This finishes the proof of Claim 2.

Combining claim 2 with (2.2), $A \in \mathcal{A}$, this finishes the proof.

(b) Prove that if there exists $n \geq 1$ such that $\mu\left(\bigcup_{i=n}^{\infty} A_i\right) < \infty$, then $\mu(A) = \lim_i \mu(A_i)$.

Proof

$$A = \limsup A_i = \liminf A_i$$
$$= \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} A_i$$

Define for each $m \in \mathbb{N}$ $B_m := \bigcup_{i=m}^{\infty} A_i$, $C_m := \bigcap_{i=m}^{\infty} A_i$

$$\Rightarrow A = \bigcap_{m=1}^{\infty} B_m = \bigcup_{m=1}^{\infty} C_m$$

and $B_{m+1} \subseteq B_m$, $C_m \subseteq C_{m+1}$ $\forall m \in \mathbb{N}$

By ctn from below / above property of measure

$$\Rightarrow \mu(A) = \mu\left(\bigcap_{m=1}^{\infty} B_m\right) = \mu\left(\bigcup_{m=1}^{\infty} C_m\right)$$
$$= \lim_{m \to \infty} \mu(B_m) = \lim_{m \to \infty} \mu(C_m) \quad \text{(equation 2.3)}$$

Fix $m \in \mathbb{N}$, we have $\mu(B_m) = \mu(\bigcup_{i=m}^{\infty} A_i) \geq \mu(A_n)$ for any $n \geq m$

so $\mu(B_m) \geq \sup_{n \geq m} \mu(A_n)$

Thus $\mu(A) = \lim_{m\to\infty} \mu(B_m) \geq \limsup \mu(A_i)$

For the same reason we have $\mu(A) = \lim_{m\to\infty} \mu(C_m) \leq \liminf \mu(A_i)$

Thus we have

$$\limsup \mu(A_i) \leq \mu(A) \leq \liminf \mu(A_i) \quad \text{(equation 2.4)}$$

Since for the numerical seq $(\mu(A_i))_{i\in\mathbb{N}}$, must have

$$\liminf (A_i) \leq \limsup (A_i)$$

Therefore

$$\limsup \mu(A_i) = \mu(A) = \liminf \mu(A_i)$$

Since liminf, limsup both exist and equal, this finishes the proof that $\lim_{i\to\infty} \mu(A_i) = \mu(A)$ $\square$

(c) Give an example showing that the condition in (b) is necessary.

Sol. Let μ be the Lebesgue measure defined on $\mathcal{B}(\mathbb{R})$. We set for each $i \in \mathbb{N}$ that

$$A_i = (i, i+1)$$

Since this is an interval, it is Lebesgue measurable. Note that no element of any A_i show up infinitely many times in the sequence. So

$$\liminf A_i = \limsup A_i = \varnothing$$

So $A = \varnothing$, we have $\lim_i \mu(A_i) = 0$. But we have $\lim_i \mu(A_i) = 1$ since it is true for every i.
In this case, $\mu(\bigcup_{i=1}^{\infty} A_i) = \infty$, which causes (b) to fail.
Hint: In analysis, it is often fruitful to use $\limsup$ and $\liminf$ to study limits.

measure space of two elements

Let X be a set with two elements, for example, $X = \{O, Q\}$.
(a) Find all σ-algebras on X.
Sol.

1. trivial σ-algebra:

$$\mathcal{A}_1 := \{\varnothing,\ X\}$$

2. power set:

$$\mathcal{A}_2 := \mathcal{P}(X) \ = \ \{\varnothing,\ \{O\},\ \{Q\},\ X\}$$

These are the only σ-algebras on X.

(b) Let $\mathcal{A}$ be a σ-algebra on X, and μ a measure on $(X, \mathcal{A})$. Is μ necessarily complete? Provide a proof or a counterexample.

Sol. It is not necessarily complete.

Cosider the trivial σ-algebra:$\mathcal{A}_1 := \{\varnothing,\ X\}$, and set μ as that $\mu(\varnothing) = \mu(X) = 0$. This makes X a null set, so $\{O\}, \{Q\}$ are subnull sets, but they are not measurable by μ.

(c) Find all outer measures μ^* on X. For each outer measure on X, find the σ-algebra of μ^*-measurable sets (see Carathéodory' s theorem).

Sol. Suppose μ^* is an outer measure on X. Since $\mathcal{P}(X)$ only has four elements: $\varnothing$, $\{O\}$, $\{Q\}$, X; and the outer measure of $\varnothing$ is 0, so we first parametrize μ^* by:

$$a := \mu^*(\{O\}), \quad b := \mu^*(\{Q\}), \quad c := \mu^*(X).$$

Then μ^* is well-defined iff it satisfies:

1. $a, b \le c$
2. $c = \mu^*(\{O\} \cup \{Q\}) \ \le\ \mu^*(\{O\}) + \mu^*(\{Q\}) \ =\ a + b.$

Any $(a, b, c) \in [0, \infty]^3$ satisfying

$$\max(a, b) \ \le\ c \ \le\ a + b,$$

can make μ^* a well-defined outer measure on X.

Therefore

$$S := \{\text{all } \sigma\text{-algebra on } X\} = \{\mu^* : \mathcal{P}(X) \to [0, \infty] \mid \max(\mu^*(\{O\}), \mu^*(\{Q\})) \le \mu^*(X) \le \mu^*(\{O\}) + \mu^*(\{Q\})\}$$

Now we specify the σ-algebra of μ^*-measurable sets for each $\mu^* \in S$.

By Carathéodory's criterion, a set $E \subset X$ is μ^*-measurable iff for all $A \subset X$,

$$\mu^*(A) \ =\ \mu^*(A \cap E) \ +\ \mu^*(A \cap E^c).$$

Note that $\varnothing$, X are always measurable since for any $A \subset X$, $A \cap \varnothing = \varnothing$, $A \cap (\varnothing)^c = A$; and $A \cap X = A$, $A \cap (X)^c = \varnothing$. So it suffices to check for $\{O\}, \{Q\}$. We first check for $\{O\}$. $\{O\}$ is μ^*-measurable iff $\mu^*(A) \ =\ \mu^*(A \cap \{O\}) \ +\ \mu^*(A \cap \{O\}^c)$ for any choice of A. There are only four possibilities for A: $\varnothing$, $\{O\}$, $\{Q\}$, X.

1. If $A = \varnothing$, both sides are 0, always stands.
2. If $A = \{O\}$, then $\mu^*(\{O\}) + \mu^*(\varnothing) = a + 0 = a$, always stands.
3. If $A = \{Q\}$, then $\mu^*(\varnothing) + \mu^*(\{Q\}) = 0 + b = b$, always stands.
4. If $A = X$, then $\mu^*(X) = c = \mu^*(\{O\}) + \mu^*(\{Q\}) = a + b$.

Therefore $\{O\}$ is μ^*-measurable iff $c = a + b$. For the same reasoning, $\{Q\}$ is μ^*-measurable iff $c = a + b$.

Thus we can conclude that:

1. If $c = a + b$, $\{\mu^*\text{-measurable sets}\} = \mathcal{P}(X)$.
2. otherwise, $\{\mu^*\text{-measurable sets}\} = \{\varnothing, X\}$.

(d) Find an example of a collection $\mathcal{E}$ of subsets of X with $\emptyset, X \in \mathcal{E}$ and a function $\rho : \mathcal{E} \to [0, \infty]$ with $\rho(\emptyset) = 0$ such that $\mathcal{E} \not\subset \mathcal{A}$, where $\mathcal{A}$ is the Carathéodory σ-algebra for the outer measure μ^* induced by

$(\mathcal{E}, \rho)$.

Sol. Consider $\mathcal{E} = \{\varnothing, X, \{O\}\}$, with ρ such that $\rho(\emptyset) = 0$, $\rho(X) = 1$, $\rho(\{O\}) = 1$.
The outer measure μ^* induced by μ^* is: $\mu^*(X) = 1$, $\mu^*(\{O\}) = 1$, $\mu^*(\{Q\}) = 1$. (the inf of length sum of sets covering $\{Q\}$ is 1, by taking $\{X\}$ as the covering.)
Since $c \neq a + b$, by (4), the Carathéodory σ-algebra by μ^* by $\mathcal{E}$ is $\{\varnothing, X\}$, so $\mathcal{E} \not\subset \mathcal{A}$.

Remark: The Hahn – Kolmogorov theorem states that if $\mathcal{E} = \mathcal{A}_0$ is an algebra and $\rho = \mu_0$ is a pre-measure, then $\mathcal{A}_0 \subset \mathcal{A}$. This exercise provides a counterexample when $\mathcal{E}$ and ρ are general. 、

Hahn – Kolmogorov Collapse (when μ_0 not σ-finite)

Let $X \subset \mathbb{R}$ be the set of dyadic rational numbers, that is, the set of numbers of the form $\frac{r}{2^n}$, where r and n are integers. Let $\mathcal{A}_0 \subset \mathcal{P}(X)$ be the collection of finite unions of intervals of the form $(a, b] \cap X$, where $-\infty \leq a < b \leq \infty$.

(a) Prove that $\mathcal{A}_0$ is an algebra.

Proof

1. $\varnothing \in \mathcal{A}_0$, since it is the empty union of intervals of the given form.
2. **Closed under complements**: Let $A \in \mathcal{A}_0$. Then A is a finite union of intervals of the form $(a_i, b_i] \cap X$. So
$$\begin{aligned} A^c \cap X &= X \setminus A && (1.3)\\ &= X \setminus \bigcup_{i=1}^{n} \big((a_i, b_i] \cap X\big) && (1.4)\\ &= X \cap \Big(\bigcap_{i=1}^{n} \big((a_i, b_i] \cap X\big)^c\Big) && (1.5)\\ &= X \cap \Big(\bigcap_{i=1}^{n} \big((-\infty, a_i] \cup (b_i, \infty]\big)\Big) && (1.6) \end{aligned}$$
Note that finite intersection of intervals of the form $(-\infty, a_i]$, $(b_i, \infty]$ is still of this form. Hence $A^c \cap X \in \mathcal{A}_0$.
3. **Closed under finite unions**: Suppose A_1 and A_2 are finite unions of intervals $\big((a_i, b_i] \cap X\big)$, then $A_1 \cup A_2$ is still a finite union of intervals of that form. (They either merge into one such interval, so are disjoint.) Hence $A_1 \cup A_2 \in \mathcal{A}_0$. The same reasoning extends to any finite union.

This finishes the proof that $\mathcal{A}_0$ is an algebra on X.

(b) Prove that the σ-algebra on X generated by $\mathcal{A}_0$ equals $\mathcal{P}(X)$.

Proof Since $< \mathcal{A}_0 > \subset \mathcal{P}(X)$, it suffices to show that $\mathcal{P}(X) \subset < \mathcal{A}_0 >$. Note that X is countable, so any set in $\mathcal{P}(X)$ is a countable union of singleton sets. Thus it suffices to show that any singleton set $\{x\}$ where $x \in X$ is in $< \mathcal{A}_0 >$, since if so, then any countable union of singleton sets from $\mathcal{P}(X)$ is also in $< \mathcal{A}_0 >$, with implies that $\mathcal{P}(X) \subset < \mathcal{A}_0 >$

Let $x \in X$. Then we have:

$$\{x\} = \bigcap_{n=1}^{\infty}((x - \frac{1}{2^n}, x] \cap X),$$

since x is in the RHS set, and for any $y < x$, we can find a $n \in \mathcal{B}$ such that $x - \frac{1}{2^n} > y$.

This finishes the proof that $< \mathcal{A}_0 >= \mathcal{P}(X)$.

(c) Define $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Prove that μ_0 is a pre-measure on $\mathcal{A}_0$

Proof It suffices to show the countable disjoint additivity.

Let $(A_i)_{i\in\mathcal{N}}$ be a sequence of disjoint sets in $\mathcal{A}_0$.

Case 1: all $A_i = \varnothing$, then $\sqcup_{i\in\mathcal{N}} A_i = \varnothing$, so $\mu_0(\sqcup_{i\in\mathcal{N}} A_i) = \sum_{i\in\mathcal{N}} \mu_0(A_i) = 0$.

Case 2: $A_k \neq \varnothing$ for some k, then $\mu_0(A_k) = \infty$ and $\sqcup_{i\in\mathcal{N}} A_i \neq \varnothing$. Thus $\sum_{i\in\mathcal{N}} \mu_0(A_i) \geq \mu_0(A_k) = \infty = \mu_0(\sqcup_{i\in\mathcal{N}} A_i)$.

The two cases cover all circumstances, finishing the proof.

(d) Prove that there exist infinitely many different measures μ on $\mathcal{P}(X)$ whose restriction to $\mathcal{A}_0$ equals μ_0.

Proof Given $n \in \mathbb{N}$, We define the "n-timed counting measure" on a σ-algebra S as:

$$\mu_{count_n}(E) := \begin{cases} n \times \#(E) & \text{, if } E \text{ is finite} \\ \infty & \text{, if } E \text{ is infinite} \end{cases}$$

Claim 1: For any set X and any σ-algebra S on X, the "n-timed counting measure" is a well-defined measure on S, for all $n \in \mathbb{N}$.

Proof of claim 1: $\mu_{count_n}(\varnothing) = 0$ since $\mathrm{card}(\varnothing) = 0$, and countable disjoint additivity trivially follows from the rule of counting.

Claim 2: for any $n \in \mathbb{N}$, $\mu_{count_n}(E)$ on $\mathcal{P}(X)$ restricted to $\mathcal{A}_0$ equals μ_0. Proof of claim 2: Let $E \in \mathcal{A}_0 \setminus \varnothing$, then E contains at least one interval of the form $(a, b] \cap X$, where $-\infty \leq a < b \leq \infty$. Sicne $a < b$, there are infinitely many elements in $(a, b] \cap X$, so $\mu_{count_n}(E) = \infty$.

This finishes the proof of the original statement.

(e) Explain why (d) does not contradict the uniqueness part of the Hahn – Kolmogorov theorem (see Theorem 1.14 in Folland).

Sol. This is because Hahn – Kolmogorov theorem requires μ_0 to be σ-finite to extend uniquely on $< \mathcal{A}_0 >$. But μ_0 here is not σ-finite.

Nur für Verrückte (Only for nuts)

(It's really not necessary to attempt these problems. Do not, under any circumstances, hand them in!)

1. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Define a morphism from $(X, \mathcal{A}, \mu)$ to $(Y, \mathcal{B}, \nu)$ to be a map $f : X \to Y$ that is measurable, that is, $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, and moreover measure preserving, in the sense that $\mu(f^{-1}(B)) = \nu(B)$ for all $B \in \mathcal{B}$.

(a) Prove that measure spaces with measure-preserving maps as morphisms form a category. Denote this category by C_3.

(b) Denote by C_1 the category of sets, and by C_2 the category of measurable spaces (see HW1). Consider the evident forgetful functors $C_3 \to C_2$ and $C_2 \to C_1$. Are these functors faithful? Are they full? Are they essentially surjective?