Homework 9: on signed measure (50/50)

Three real Banach spaces and a fake one

(a) Let

$$\ell_0^\infty := \{a = (a_1, a_2, \cdots) \mid a_i \in \mathbb{R}, \lim_{n\to\infty} a_n = 0\}.$$

Prove that $(\ell_0^\infty, \|\cdot\|_\infty)$, where $\|a\|_\infty = \sup_n |a_n|$, is a Banach space.

(b) Let

$$C_b^0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and bounded}\}.$$

Prove that $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$, where $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$, is a Banach space.

(c) Let

$$C_0^0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}, \lim_{x\to\pm\infty} f(x) = 0\}.$$

Prove that $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$, where $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$, is a Banach space.

(d) Recall that

$$C_c^0(\mathbb{R}) = \{f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } f = 0 \text{ outside a bounded set}\}.$$

Show that $(C_c^0(\mathbb{R}), \|\cdot\|_\infty)$, where $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$, is not a Banach space.

Proof of (a): Since we showed in class that

$$\ell^\infty = L^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{counting})$$

and L^∞ spaces are Banach, ℓ^∞ is Banach.

Thus it suffices to show that ℓ_0^∞ is closed in ℓ^∞, since a closed subset of a complete metric space is complete.

Let $(a^{(k)})_{k=1}^\infty$ be a sequence in ℓ_0^∞ converging in norm to $a \in \ell^\infty$, i.e.,

$$\|a^{(k)} - a\|_\infty \to 0$$

Let $\varepsilon > 0$.

Since $\|a^{(k)} - a\|_\infty \to 0$, there exists K such that for all $k \geq K$,

$$\|a^{(k)} - a\| = \sup_n |a_n^{(k)} - a_n| < \frac{\varepsilon}{2}$$

This implies that

$$\forall n, |a_n^{(K)} - a_n| < \varepsilon$$

Since $a^{(K)} \in \ell_0^\infty$, $a_n^{(K)} \to 0$ as $n \to \infty$. Thus there exists $N \in \mathbb{N}$ s.t. for all $n \geq N$,

$$|a_n^{(K)}| \leq \frac{\varepsilon}{2}$$

Then for all $n \geq N$, we have:

$$|a_n| \leq |a_n - a_n^{(K)}| + |a_n^{(K)}| < \varepsilon$$

This shows that

$$\lim_{n\to\infty} |a_n| < \epsilon$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$\lim_{n\to\infty} a_n = 0$$

Hence $a \in \ell_0^\infty$. So ℓ_0^∞ is closed in ℓ^∞, thus itself Banach.

Proof of (b): Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy seq in $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$, then

$$\forall\varepsilon > 0, \exists N \in \mathbb{N} \ \ s.t. \|f_n - f_m\|_\infty = \sup_{x\in\mathbb{R}} |f_n(x) - f_m(x)| < \varepsilon$$

In particular, for each fixed $x \in \mathbb{R}$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, hence converges (since $\mathbb{R}$ is complete). So we can define the pointwise limit:

$$f(x) := \lim_{n\to\infty} f_n(x)$$

Claim 1: $f_n \to f$ in $\|\cdot\|_\infty$.

Let $\varepsilon > 0$.

Since (f_n) is Cauchy in $\|\cdot\|_\infty$, there exists N such that:

$$\|f_n - f_m\|_\infty < \varepsilon, \quad \forall n, m \geq N$$

Fix $m \geq N$, and let $n \to \infty$. For each x, we get:

$$|f_n(x) - f_m(x)| < \varepsilon \ \ \forall n \implies \lim_{n\to\infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

Since this is true for each $x \in \mathbb{R}$, we obtain:

$$\|f - f_m\|_\infty \leq \varepsilon, \quad \text{for all } m \geq N$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\lim_{n\to\infty} \|f - f_n\|_\infty = 0$$

Claim 2: $f \in (C_b^0(\mathbb{R}), \|\cdot\|_\infty)$.

Since $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, it also implies that the convergence is uniform.

We know the uniform limit of continuous functions is continuous, so f is continuous. It remains to show f is bounded, and this directly follows from the uniform convergence. We take $\varepsilon = 1$. We have proved that there exists N s.t. for all $m \geq N$,

$$\|f - f_m\|_\infty \leq 1$$

Thus

$$\sup_{x\in\mathbb{R}} |f(x)| \leq \sup_{x\in\mathbb{R}} |f_N(x)| + 1$$

Since $f_n \in C_b^0(\mathbb{R})$), it is bounded, thus

$$\sup_{x\in\mathbb{R}} |f(x)| < \infty$$

showing that the limit function is bounded. This finishes the proof that $f \in C_b^0(\mathbb{R})$. Thus, every Cauchy seq in $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$ converges in $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$, i.e. it is Banach.

Proof of (c): Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy seq in $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$, then for each fixed $x \in \mathbb{R}$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, so for the same reason as (b), we can define the pointwise limit:

$$f(x) := \lim_{n\to\infty} f_n(x)$$

And for the same reason as (b), we get

$$f_n \to f \text{ in } \|\cdot\|_\infty$$

which also implies that the pointwise convergence is uniform. Since each f_n is continuous, the uniform limit f is continuous.

Thus it suffices to show that $\lim_{x\to\pm\infty} f(x) = 0$.

Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists N such that for all $n \geq N$, $\|f_n - f\|_\infty < \epsilon/2$. Also, since $f_N \in C_0^0(\mathbb{R})$, there exists $M > 0$ such that $|f_N(x)| < \epsilon/2$ for all $|x| > M$.

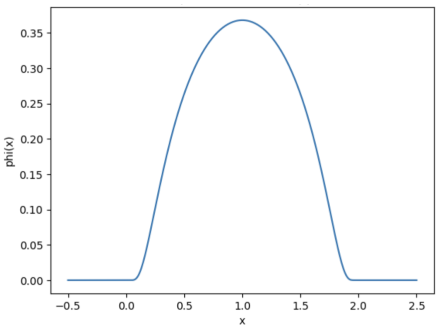
Then for $|x| > M$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < \epsilon/2 + \epsilon/2 < \epsilon$$

So $\lim_{x\to\pm\infty} f(x) = 0$, i.e., $f \in C_0^0(\mathbb{R})$. Thus, every Cauchy seq in $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$ converges in $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$, i.e. it is Banach.

Proof of (d): We consider a continuous (smooth actually) function $\phi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{supp}(\phi) = [0, 2]$ (here we take the closure):

$$\phi(x) := \begin{cases} \exp\Big(-\frac{1}{x\,(2-x)}\Big), & 0 < x < 2, \\ 0, & \text{otherwise} \end{cases}$$



This function reaches its maximum at $x = 1$,

$$\|\phi\|_\infty = \frac{1}{e}$$

For each integer $n \geq 1$, define

$$\phi_n(x) = \phi\big(x - n\big)$$

Then each ϕ_n is also continuous, and $\mathrm{supp}(\phi_n) = [n, n+2]$.

Consider the sequence $(S_N)_1^\infty$, defined as:

$$S_N(x) := \sum_{n=1}^{N} 2^{-n}\,\phi_n(x)$$

Then each $S_N \in C_c^0(\mathbb{R})$, since finite sum of continuous functions is also continuous, and $\mathrm{supp}(S_N) = [1, N+2]$, thus each $S_N \in C_c^0(\mathbb{R})$.

Claim: $(S_N)_1^\infty$ **is Cauchy in the sup norm.**

This is because for each (WLOG) $M > N \in \mathbb{N}$,

$$\begin{aligned}\|S_M - S_N\|_\infty &= \Big\| \sum_{n=N+1}^{M} \frac{1}{2^n}\phi_n \Big\|_\infty \\ &\le \sum_{n=N+1}^{M} \frac{1}{2^n}\|\phi\|_\infty \\ &\le \sum_{n=N+1}^{\infty} \frac{1}{2^n}\|\phi\|_\infty \\ &= \sum_{n=N+1}^{\infty} \frac{1}{2^n e} = \frac{1}{2^N e} \xrightarrow{N\to\infty} 0\end{aligned}$$

Thus for arbitrary $\varepsilon > 0$, exists $K \in \mathbb{N}$ s.t. for all $M, N \ge K$, $\|S_M - S_N\|_\infty < \varepsilon$. And by same reason as (b), (c), $(S_N)_1^\infty$ converges by $\|\cdot\|_\infty$ into its pointwise limit:

$$S(x) := \sum_{n=1}^{\infty} 2^{-n}\,\phi_n(x)$$

But $S(x)$ does not have compact support, $\operatorname{supp}(S) = [0,\infty)$. So $S \notin C_c^0(\mathbb{R})$. This serves as a counterexample showing that $C_c^0(\mathbb{R})$ is not Banach.

$\nu^+(E), \nu^-(E), |\nu|(E)$ 的 formula from original ν

Let ν be a signed measure on $(X, \mathcal{A})$, and $E \in \mathcal{A}$. Prove the following statements:

(i) $\nu^+(E) = \sup\{\nu(F) \mid: F \in \mathcal{A}, F \subset E\}$, and $\nu^-(E) = -\inf\{\nu(F) \mid F \in \mathcal{A}, F \subset E\}$;

(ii) $|\nu|(E) = \sup\{\sum_{i=1}^N |\nu(E_i)| \mid N \in \mathbb{N},\ E = \bigcup_{i=1}^N E_i \text{ disjoint union}\}$;

(iii) $|\nu|(E) \ge |\nu(E)|$. In the case ν finite, it achieves equality iff E is positive or negative for ν.

Proof of (i): By the Hahn decomposition theorem, we can take a Hahn decomposition $X = P \sqcup N$ where

$$\nu(A) \ge 0 \quad \text{for all } A \subset P, \qquad \nu(B) \le 0 \quad \text{for all } B \subset N$$

Fix $E \in \mathcal{A}$. By Jordan decomposition we have

$$\nu^+(E) = \nu(E \cap P)$$

Fix $F \subset E$, we have:

$$F = (F \cap P) \sqcup (F \cap N)$$

Since $\nu(F \cap N) \le 0$, we have:

$$\nu(F) \le \nu\big(F \cap P\big) \ \le\ \nu\big(E \cap P\big) = \nu^+(E)$$

Since F is arbitrary, this shows:

$$\sup\{\nu(F) \mid F \subset E\} \le \nu^+(E)$$

On the other hand, taking $F = E \cap P \subset E$, we get

$$\nu(F) = \nu\big(E \cap P\big) = \nu^+(E)$$

Hence

$$\sup\{\nu(F) \mid F \subset E\} \ge \nu^+(E)$$

Combining both inequalities gives

$$\nu^+(E) = \sup\{\nu(F) \mid F \subset E\}$$

Similarly, since $\nu(F \cap P) \geq 0$ and $\nu(F) = \nu(F \cap P) + \nu\big(F \cap N\big)$, we have $\nu(F) \geq \nu\big(F \cap N\big)$. And Since $\nu\big(E \cap N\big) = \nu(F \cap N) + \nu((E \setminus F) \cap N)$ with $\nu((E \setminus F) \cap N) \leq 0$, we get $\nu\big(F \cap N\big) \geq \nu\big(E \cap N\big)$. Putting it together:

$$\nu(F) \geq \nu\big(F \cap N\big) \geq \nu\big(E \cap N\big) = -\nu^-(E)$$

Since F is arbitrary, this shows:

$$\inf\{\nu(F) \mid F \subset E\} \geq -\nu^-(E)$$

On the other hand, taking $F = E \cap N \subset E$, we get

$$\nu(F) = \nu\big(E \cap N\big) = -\nu^-(E)$$

Hence

$$\inf\{\nu(F) \mid F \subset E\} \leq -\nu^-(E)$$

Combining both inequalities gives

$$\nu^-(E) = -\inf\{\nu(F) \mid F \subset E\}$$

Proof of (ii): Let $E \in \mathcal{A}$. By def of total variation measure,

$$|\nu|(E) = \nu^+(E) + \nu^-(E)$$

One direction of the equality is easy. Take a Hahn decomposition $X = P \sqcup N$ where

$$\nu(A) \geq 0 \quad \text{for all } A \subset P, \qquad \nu(B) \leq 0 \quad \text{for all } B \subset N$$

Then by Jordan decomposition, we have:

$$\nu^+(E) = \nu(E \cap P), \quad \nu^-(E) = -\nu(E \cap N)$$

So by taking $E_1 := E \cap P$, $E_2 := E \cap N$, we have:

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E_1) + \nu(E_2)$$

This shows that

$$|\nu|(E) \leq \sup\Big\{\sum |\nu(E_i)|\Big\}$$

And for the other direction, for any disjoint measurable partition $E = \bigcup_{i=1}^{N} E_i$, we have

$$|\nu(E_i)| = \big|\nu^+(E_i) - \nu^-(E_i)\big| \leq \nu^+(E_i) + \nu^-(E_i) = |\nu|(E_i)$$

Therefore

$$\sum_{i=1}^{N} \big|\nu(E_i)\big| \leq \sum_{i=1}^{N} |\nu|(E_i) = |\nu|\Big(\bigcup_{i=1}^{N} E_i\Big) = |\nu|(E)$$

since $|\nu|$ is a p.m. and the E_i's are disjoint. Thus

$$\sup\Big\{\sum_{i=1}^{N} |\nu(E_i)|\Big\} \leq |\nu|(E)$$

Combining the two inequalities gives

$$|\nu|(E) = \sup\left\{\sum_{i=1}^{N} |\nu(E_i)| \;\middle|\; N \in \mathbb{N}, E = \bigcup_{i=1}^{N} E_i \text{ disjoint}\right\}$$

proving the statement.

Proof of (iii): Let $E \in \mathcal{A}$. The ineq $|\nu|(E) \geq |\nu(E)|$ follows from triangular ineq on $\mathbb{R}$:

$$|\nu(E)| = \left|\nu^+(E) - \nu^-(E)\right| \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$$

Now we assume ν is finite (i.e. $|\nu|(X) < \infty$). The equality condition $|\nu(E)| = |\nu|(E)$ is detailedly:

$$\left|\nu^+(E) - \nu^-(E)\right| = \nu^+(E) + \nu^-(E)$$

Since $|\nu|(X) < \infty$, $\nu^+(E) < \infty$ and $\nu^-(E) < \infty$.

Case 1: $\nu^+(E) \geq \nu^-(E)$, then

$$\begin{aligned}
\left|\nu^+(E) - \nu^-(E)\right| = \nu^+(E) + \nu^-(E) &\iff \nu^+(E) - \nu^-(E) = \nu^+(E) + \nu^-(E) \\
&\iff -\nu^-(E) = \nu^-(E) \\
&\iff \nu^-(E) = 0 \\
&\iff E \subset P
\end{aligned}$$

Case 2: $\nu^+(E) < \nu^-(E)$, then

$$\begin{aligned}
\left|\nu^+(E) - \nu^-(E)\right| = \nu^+(E) + \nu^-(E) &\iff \nu^-(E) - \nu^+(E) = \nu^+(E) + \nu^-(E) \\
&\iff -\nu^+(E) = \nu^+(E) \\
&\iff \nu^+(E) = 0 \\
&\iff E \subset N
\end{aligned}$$

Therefore the equality condition implies that E must be positive or negative for ν; and in converse, if E is neither positive nor negative set, in either case it implies $|\nu(E)| \neq |\nu|(E)$, thus when ν finite, $|\nu(E)| = |\nu|(E)$ iff E is positive or negative for ν.

Signed integrals

Let ν be a signed measure on $(X, \mathcal{A})$.

(i) Prove that $\int g \, d|\nu| = \int g \, d\nu^+ + \int g \, d\nu^-$ for $g \in L^+(|\nu|)$ or $g \in L^1(|\nu|)$.

(ii) Define $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$. Prove that $L^1(\nu) = L^1(|\nu|)$.

(iii) Define $\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-$ for $f \in L^1(\nu)$. Prove that if $f \in L^1(\nu)$, then

$$\left|\int f \, d\nu\right| \leq \int |f| \, d|\nu|$$

(iv) Suppose that ν is a finite measure (i.e. $\nu^\pm(X) < \infty$.) Prove that if $E \in \mathcal{A}$, then

$$|\nu|(E) = \sup\left\{\left|\int_E f \, d\nu\right| \mid \|f\|_\infty \leq 1\right\}.$$

Proof of (i): Take a Hahn decomposition $X = P \sqcup N$.

Then by Jordan decomposition,

$$\nu^+(E) = \nu(E \cap P), \quad \nu^-(E) = -\nu(E \cap N), \quad \forall E \subset X$$

and therefore P is null set of ν^- and N is null set of ν^+. So on P, $|\nu| = \nu^+ + \nu^- = \nu^+$; on N, $|\nu| = \nu^+ + \nu^- = \nu^-$ Thus, suppose $g \in L^+(|\nu|)$,

$$\begin{aligned}\int g\,d|\nu| = \int_X g\,d|\nu| &= \int_P g\,d|\nu| + \int_N g\,d|\nu| \quad \text{since } X = P \sqcup N \\ &= \int_P g\,d\nu^+ + \int_N g\,d\nu^- \quad \text{since } |\nu| = \nu^+, \nu^- \text{ on } P, N \\ &= \int g\,d\nu^+ + \int g\,d\nu^- \quad \text{since } N, P \text{ is null for } \nu^+, \nu^-\end{aligned}$$

Suppose $g \in L^1(|\nu|)$, then

$$\begin{aligned}\int g\,d|\nu| = \int_X g\,d|\nu| &= \int_X g^+\,d|\nu| - \int_X g^-\,d|\nu| \quad \text{by def} \\ &= \left(\int_P g^+\,d\nu^+ + \int_N g^+\,d\nu^-\right) - \left(\int_P g^-\,d\nu^+ + \int_N g^-\,d\nu^-\right) \quad \text{since } X = P \sqcup N \\ &= (\int_P g^+\,d\nu^+ - \int_P g^-\,d\nu^+\Big) + \left(\int_N g^+\,d\nu^- - \int_N g^-\,d\nu^-\right) \\ &= \int_P g\,d\nu^+ + \int_N g\,d\nu^- \quad \text{since } g \in L^1(|\nu|) \\ &= \int g\,d\nu^+ + \int g\,d\nu^- \quad \text{since } N, P \text{ is null for } \nu^+, \nu^-\end{aligned}$$

This finishes the proof.

Proof of (ii): WTS: $L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|)$.

($\Rightarrow$): Suppose $f \in L^1(|\nu|)$, i.e. $\int |f|\,d|\nu| < \infty$.

Let ϕ be arbitrary positive-valued simple function:

$$\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$$

then

$$\int \phi\,d|\nu| = \sum_{i=1}^{n} a_j |\nu|(E_j)$$

Since $\nu^-(E_j), \nu^+(E_j) \le \nu^+(E_j) + \nu^-(E_j) = |\nu|(E_j)$ for each j, we have

$$\int \phi\,d\nu^+, \int \phi\,d\nu^- \le \int \phi\,d|\nu|$$

Since ϕ is arbitrary, we have

$$\int |f|\,d\nu^+ = \sup\{\int \phi\,d\nu^+ : 0 \le \phi \le |f|, \phi \text{ simple}\} \le \sup\{\int \phi\,d|\nu| : 0 \le \phi \le |f|, \phi \text{ simple}\} = \int |f|\,d|\nu|$$

Same for ν^-. This shows that

$$\int |f|\,d\nu^+, \int |f|\,d\nu^- \le \int |f|\,d|\nu| < \infty$$

i.e. $f \in L^1(\nu^+)$ and $f \in L^1(\nu^-)$, so $f \in L^1(\nu^+) \cap L^1(\nu^-)$.

Thus

$$L^1(|\nu|) \subset L^1(\nu^+) \cap L^1(\nu^-)$$

($\Leftarrow$): Suppose $f \in L^1(\nu^+) \cap L^1(\nu^-)$, i.e.

$$\int |f|\,d\nu^+ < \infty, \quad \int |f|\,d\nu^- < \infty$$

Since $|f|$ is non-negative and measurable, we have $|f| \in L^+(|\nu|)$. Thus by (i) we have:

$$\int |f|\, d|\nu| = \int |f|\, d\nu^+ + \int |f|\, d\nu^- < \infty$$

So $f \in L^1(|\nu|)$.

This shows that:

$$L^1(\nu^+) \cap L^1(\nu^-) \subset L^1(|\nu|)$$

Combining both direction, we finished the proof that:

$$L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|)$$

Proof of (iii): Suppose $f \in L^1(\nu)$, then

$$\begin{aligned}
\left|\int f\, d\nu\right| &= \left|\int f\, d\nu^+ - \int f\, d\nu^-\right| \quad \text{by def} \\
&\leq \left|\int f\, d\nu^+\right| + \left|\int f\, d\nu^-\right| \quad \text{by tri ineq} \\
&\leq \int |f|\, d\nu^+ + \int |f|\, d\nu^- \quad \text{by property of } L^1 \text{ integration} \\
&= \int |f|\, d|\nu| \quad \text{from (i)}
\end{aligned}$$

Therefore,

$$\left|\int f\, d\nu\right| \leq \int |f|\, d|\nu|$$

Proof of (iv): Suppose that ν is a finite measure (i.e. $\nu^{\pm}(X) < \infty$), let $E \in \mathcal{A}$.

We denote:

$$S := \sup\left\{\left|\int_E f\, d\nu\right| \,\middle|\, \|f\|_\infty \leq 1\right\}$$

First we show $S \leq |\nu|(E)$:

For any bounded measurable f with $\|f\|_\infty \leq 1$,

$$\begin{aligned}
\left|\int_E f\, d\nu\right| &\leq \int_E |f|\, d|\nu| \quad \text{by (iii)} \\
&\leq \int_E 1\, d|\nu| \quad \text{by linearity of integration} \\
&= |\nu|(E)
\end{aligned}$$

So by taking the supremum over such f, we get:

$$S \leq |\nu|(E)$$

Next we will show $|\nu|(E) \leq S$:

We take a Hahn decomposition, getting $X = P \sqcup N$ where

$$\nu^+(B) = \nu(P \cup B) \geq 0, \nu^-(B) = -\nu(P \cup B) \leq 0, \quad \text{for all } B \subset X$$

Then

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N)$$

Now define:

$$f := \chi_P - \chi_N$$

Then f is measurable since P, N are measurable. And $\|f\|_\infty \leq 1$ since $f(x) \in \{-1, 1\}\, \forall x \in X$ Compute:
$$\int_E f\, d\nu = \int_{E\cap P} 1\, d\nu - \int_{E\cap N} 1\, d\nu = \nu(E\cap P) - \nu(E\cap N) = \nu^+(E) + \nu^-(E) = |\nu|(E)$$
Thus
$$|\nu|(E) = \left|\int_E f\, d\nu\right| \leq S$$
Combining both inequalities, we get:
$$|\nu|(E) = S$$

finite signed measures on $(X, \mathcal{A})$ 是一个 NVM

Let $(X, \mathcal{A})$ be a measurable space.

(a) Let λ, μ be finite *positive* measures on $(X, \mathcal{A})$. Let $\nu = \lambda - \mu$. Prove that
$$\nu^+(E) \leq \lambda(E), \qquad \nu^-(E) \leq \mu(E), \qquad |\nu|(E) \leq \lambda(E) + \mu(E)$$
for every $E \in \mathcal{A}$.

(b) Let ν and κ be finite *signed* measures on $(X, \mathcal{A})$ (i.e. $\nu(E), \kappa(E) \in \mathbb{R}$ for all $E \in \mathcal{A}$). Show that
$$|\nu + \kappa|(E) \leq |\nu|(E) + |\kappa|(E)$$
for every $E \in \mathcal{A}$.

(c) Let $\mathcal{M}$ be the collection of finite signed measure ν on $(X, \mathcal{A})$. For $\nu \in \mathcal{M}$, define
$$\|\nu\| = |\nu|(X)$$
Prove that $\|\cdot\|$ is a norm on $\mathcal{M}$ with an appropriate definition of the sum of two signed measures and the multiplication of a signed measure by a (real) scalar.

(d) Suppose $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Compute $\|\delta_x - \delta_y\|$ for $x, y \in \mathbb{R}$.

Remark: the norm on $\mathcal{M}$ is called the *the total variation norm*.

Proof of (a):

Recall in problem 2 we get:
$$\nu^+(E) = \sup\{\nu(F) : F \subset E, F \in \mathcal{A}\}, \quad \nu^-(E) = -\inf\{\nu(F) : F \subset E, F \in \mathcal{A}\}$$
Claim 1: $\nu^+(E) \leq \lambda(E)$.

Let $F \subset E$, $F \in \mathcal{A}$. Then:
$$\nu(F) = \lambda(F) - \mu(F) \leq \lambda(F) \leq \lambda(E)$$
since $F \subset E$ and λ is positive. Taking the sup over all such F, we get
$$\nu^+(E) = \sup_{F\subset E} \nu(F) \leq \lambda(E)$$
Claim 2: $\nu^-(E) \leq \mu(E)$.

Similarly as Claim 1, for any $F \subset E$, since λ and μ are p.m., we have
$$\nu(F) = \lambda(F) - \mu(F) \geq -\mu(F) \geq -\mu(E) \implies -\nu(F) \leq \mu(E)$$
Taking the inf over $F \subset E$, we get
$$\nu^-(E) = -\inf_{F\subset E} \nu(F) \leq \mu(E)$$

Claim 3: $|\nu|(E) \leq \lambda(E) + \mu(E)$**.**

This is just combining the two ineqs:

$$|\nu|(E) = \nu^+(E) + \nu^-(E) \leq \lambda(E) + \mu(E)$$

Proof of (b):

Let $E \in \mathcal{A}$. WTS: $|\nu + \kappa|(E) \leq |\nu|(E) + |\kappa|(E)$.

Recall in problem 2 we showed that for a signed measure σ and a measurable set E , we have:

$$|\sigma|(E) = \sup \left\{ \sum_{i=1}^{n} |\sigma(E_i)| : E = \bigsqcup_{i=1}^{N} E_i \right\}$$

Let $\{E_i\}_{i=1}^n$ be any finite measurable partition of E. Then for each E_i:

$$|(\nu + \kappa)(E_i)| = |\nu(E_i) + \kappa(E_i)| \leq |\nu(E_i)| + |\kappa(E_i)| \quad \text{(by tri ineq on } \mathbb{R})$$

Summing over the partition, we have:

$$\sum_{i=1}^{n} |(\nu + \kappa)(E_i)| \leq \sum_{i=1}^{n} |\nu(E_i)| + \sum_{i=1}^{n} |\kappa(E_i)|$$

Now take the supremum over all such partitions of E:

$$\begin{aligned}
|\nu + \kappa|(E) &= \sup \left\{ \sum_{i=1}^{n} |(\nu + \kappa)(E_i)| : E = \bigsqcup_{i=1}^{N} E_i \right\} \\
&\leq \sup \left\{ \sum_{i=1}^{n} |\nu(E_i)| + \sum_{i=1}^{n} |\kappa(E_i)| : E = \bigsqcup_{i=1}^{N} E_i \right\} \\
&\leq \sup \left\{ \sum_{i=1}^{n} |\nu(E_i)| : E = \bigsqcup_{i=1}^{N} E_i \right\} + \sup \left\{ \sum_{i=1}^{n} |\kappa(E_i)| : E = \bigsqcup_{i=1}^{N} E_i \right\} \\
&= |\nu|(E) + |\kappa|(E)
\end{aligned}$$

Since measurable E is arbitrary, this finishes the proof.

Proof of (c):

$$\mathcal{M} := \{\text{all finite signed measures on } (X, \mathcal{A})\}$$

and for $\nu \in \mathcal{M}$, we define:

$$\|\nu\| := |\nu|(X)$$

WTS: $\|\cdot\|$ is a norm on $\mathcal{M}$.

1. **Positive Definiteness**:
   Let $\nu \in \mathcal{M}$. Since $|\nu|$ is a positive measure, $\|\nu\| = |\nu|(X) \geq 0$.
   Since $|\nu|$ is a positive measure, $\|\nu\| = |\nu|(X) \geq 0$.
   Suppose $|\nu|(X) = 0$, then X is a $|\nu|$-null set, so $|\nu|(E) = 0$ for all $E \in \mathcal{A}$. Thus $\nu = 0$.
   And suppose $\nu = 0$, then $|\nu| = 0$ also, so $|\nu|(X) = 0$.
   Thus, $\|\nu\| = 0$ iff $\nu = 0$. This finishes the proof of positive definiteness.
2. **Absolute Homogeneity**:

Since for any measurable set E:

$$\begin{aligned}|a\nu|(E) &= \sup\left\{\sum_{i=1}^{n}|(a\nu)(E_i)| : E = \bigsqcup_{i=1}^{N} E_i\right\} \\ &= \sup\left\{\sum_{i=1}^{n}|a||\nu(E_i)| : E = \bigsqcup_{i=1}^{N} E_i\right\} \\ &= |a|\sup\left\{\sum_{i=1}^{n}|\nu(E_i)| : E = \bigsqcup_{i=1}^{N} E_i\right\} \\ &= |a|\cdot|\nu|(E)\end{aligned}$$

We have:

$$\|a\nu\| = |a\nu|(X) = |a|\cdot|\nu|(X) = |a|\cdot\|\nu\|$$

finishing the proof of absolute homogeneity.

3. **Triangle Inequality**:

Recall we just proved in (b) that for any measurable E:

$$|\nu+\kappa|(E) \leq |\nu|(E) + |\kappa|(E)$$

Thus

$$\|\nu+\kappa\| = |\nu+\kappa|(X) \leq |\nu|(X) + |\kappa|(X) = \|\nu\| + \|\kappa\|$$

finishing the proof of triangle inequality.

So we can conclude that $\|\nu\| := |\nu|(X)$ defines a norm on $\mathcal{M}$, with the standard definitions of addition and scalar multiplication of signed measures.

Proof of (d)

Suppose $(X,\mathcal{A}) = (\mathbb{R},\mathcal{B}(\mathbb{R}))$. Compute $\|\delta_x - \delta_y\|$ for $x, y \in \mathbb{R}$.

Recall def: For any Borel set $A \subset \mathbb{R}$,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise}\end{cases}$$

So we define the signed measure $\nu := \delta_x - \delta_y$ as:

$$\nu(A) = \delta_x(A) - \delta_y(A)$$

If $x = y$, then $\delta_x = \delta_y$, then $\nu = 0$, so $\|\nu\| = 0$. This is the trivial case. if $x \neq y$: We first compute the Jordan decomposition.

We know that $\nu^+(E) = \sup\{\nu(F) \mid: F \in \mathcal{A}, F \subset E\}$, and $\nu^-(E) = -\inf\{\nu(F) \mid F \in \mathcal{A}, F \subset E\}$. For any $E \ni x$, we have

$$\nu^+(E) = \nu(\{x\}) = 1$$

In other cases, we have:

$$\nu^+(E) = \nu(E \setminus \{y\}) = 0$$

For any $E \ni y$, we have

$$\nu^{-}(y) = -\nu(\{y\}) = 1$$

In other cases, we have:

$$\nu^{-}(E) = -\nu(E \setminus \{x\}) = 0$$

And we thus discover that:

$$\nu^{+} = \delta_x, \quad \nu^{-} = \delta_y$$

So

$$\|\nu\| = |\nu|(\mathbb{R}) = \delta_x(\mathbb{R}) + \delta_y(\mathbb{R}) = 1 + 1 = 2$$

Thus we can conclude that

$$\|\nu\| = \begin{cases} 2 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

and more: finite signed measures on $(X, \mathcal{A})$ 组成一个 real Banach space

Prove that the normed vector space $\mathcal{M}$ in the previous problem is in fact a Banach space.

Proof In problem 4 we have shown that on $(\mathcal{M}, \|\cdot\|)$ is a normed vector space, where

$$\mathcal{M} := \{\text{all finite signed measures on } (X, \mathcal{A})\}$$

and

$$\|\nu\| := |\nu|(X)$$

Now we prove that the NVM $(\mathcal{M}, \|\cdot\|)$ is complete, i.e. it is a Banach space.

Let (ν_n) be a Cauchy sequence in $\mathcal{M}$. We have

$$|\nu_n(B) - \nu_m(B)| = |(\nu_n - \nu_m)(B)| \leq \|\nu_n - \nu_m\| \quad \text{for all } B \in \mathcal{A}$$

In particular, $(\nu_n(B))_n$ is a Cauchy sequence for all $B \in \mathcal{A}$. For each $B \in \mathcal{A}$, this is a Cauchy seq in $\mathbb{R}$, thus converges. So we can get:

$$\nu(B) := \lim_n \nu_n(B)$$

as the pointwise limit (by a point we mean a set).

Claim 1: $\nu \in \mathcal{M}$.

Since for all n, $\nu_n(\varnothing) = 0$, we have:

$$\nu(\varnothing) := \lim_n \nu_n(\varnothing) = 0$$

For a countable disjoint union of measurable sets $E = \bigsqcup_{i=1}^{\infty} E_i$,

$$\lim_n \nu_n(E) = \lim_n \sum_i \nu_n(E_i)$$

is the limit of a finite sum of numerical sequences in $\mathbb{R}$. So we can exchange the order of taking limit and sum. Then we get:

$$\nu(E) = \lim_n \nu_n(E) = \lim_n \sum_i \nu_n(E_i) = \sum_i \lim_n \nu_n(E_i) = \sum_i \nu(E_i)$$

And notice, for each measurable set $B \in \mathcal{A}$, **since $(\nu_n(B))_n$ is a Cauchy sequence in $\mathbb{R}$, it is bounded**, thus does not admit $\infty, -\infty$ values. verifying that ν **is a valid signed measure.**
Also, this means that taking Hahn Decomposition $X = P \sqcup N$ by ν, we have

$$\nu^+(X) = \nu(P), \quad \nu^-(X) = -\nu(N)$$

Since $\nu(P), \nu(N)$ are bounded, we have: Thus

$$|\nu|(X) = \nu^+(X) + \nu^-(X) < \infty$$

This verifies that ν is a finite s.m.
Claim 2: $\nu_n \to \nu$ in $\|\cdot\|$. Fix $\varepsilon > 0$. There exists N such that $\|\nu_n - \nu_m\| < \varepsilon/2$ for all $m, n \geq N$. Thus for all $n \geq N$ we have:

$$|(\nu_n - \nu)(B)| = \lim_m |(\nu_n - \nu_m)(B)| \leq \varepsilon/2, \quad \forall B \in \mathcal{A},\ \forall n \geq N$$

Notice that

$$\nu^+(B) = \sup\{\nu(C) \mid C \in \mathcal{A},\ C \subset B\}$$

and

$$\nu^-(B) = -\inf\{\nu(C) \mid C \in \mathcal{A},\ C \subset B\} = \sup\{-\nu(C) \mid C \in \mathcal{A},\ C \subset B\}$$

It follows that

$$(\nu_n - \nu)^+(X) = \sup\{(\nu_n - \nu)(B) \mid B \in \mathcal{A}\} \leq \varepsilon/2, \quad \forall n \geq N$$

Similarly,

$$(\nu_n - \nu)^-(X) = \sup\{-(\nu_n - \nu)(B) \mid B \in \mathcal{A}\} \leq \varepsilon/2, \quad \forall n \geq N$$

Thus

$$|\nu_n - \nu|(X) = (\nu_n - \nu)^+(X) + (\nu_n - \nu)^-(X) \leq \varepsilon$$

This holds for all $n \geq N$. And since $\varepsilon > 0$ is arbitrary, this proves that

$$\lim_{n \to \infty} \|\nu_n - \nu\| = 0$$

As a result, $\nu_n \to \nu$ in $\|\cdot\|$, completeing the proof.

Nur für Verrückte

(It's **really** not necessary to attempt these problems. Do not, under any circumstances, hand them in!) Does there exist a signed Borel measure ν on $\mathbb{R}$ with the property that for every $\alpha \in \mathbb{R}$ there exists a Borel set $E \subset \mathbb{R}$ with $\nu(E) = \alpha$.