

## Homework 9: on signed measure (50/50)

### Three real Banach spaces and a fake one

(a) Let

$$\ell_0^\infty := \{a = (a_1, a_2, \dots) \mid a_i \in \mathbb{R}, \lim_{n \rightarrow \infty} a_n = 0\}.$$

Prove that  $(\ell_0^\infty, \|\cdot\|_\infty)$ , where  $\|a\|_\infty = \sup_n |a_n|$ , is a Banach space.

(b) Let

$$C_b^0(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous and bounded}\}.$$

Prove that  $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$ , where  $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ , is a Banach space.

(c) Let

$$C_0^0(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous, } \lim_{x \rightarrow \pm\infty} f(x) = 0\}.$$

Prove that  $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$ , where  $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ , is a Banach space.

(d) Recall that

$$C_c^0(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous and } f = 0 \text{ outside a bounded set}\}.$$

Show that  $(C_c^0(\mathbb{R}), \|\cdot\|_\infty)$ , where  $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$ , is not a Banach space.

**Proof of (a):** Since we showed in class that

$$\ell^\infty = L^\infty(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{counting})$$

and  $L^\infty$  spaces are Banach,  $\ell^\infty$  is Banach.

Thus it suffices to show that  $\ell_0^\infty$  is closed in  $\ell^\infty$ , since a closed subset of a complete metric space is complete.

Let  $(a^{(k)})_{k=1}^\infty$  be a sequence in  $\ell_0^\infty$  converging in norm to  $a \in \ell^\infty$ , i.e.,

$$\|a^{(k)} - a\|_\infty \rightarrow 0$$

Let  $\varepsilon > 0$ .

Since  $\|a^{(k)} - a\|_\infty \rightarrow 0$ , there exists  $K$  such that for all  $k \geq K$ ,

$$\|a^{(k)} - a\|_\infty = \sup_n |a_n^{(k)} - a_n| < \frac{\varepsilon}{2}$$

This implies that

$$\forall n, |a_n^{(K)} - a_n| < \varepsilon$$

Since  $a^{(K)} \in \ell_0^\infty$ ,  $a_n^{(K)} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus there exists  $N \in \mathbb{N}$  s.t. for all  $n \geq N$ ,

$$|a_n^{(K)}| \leq \frac{\varepsilon}{2}$$

Then for all  $n \geq N$ , we have:

$$|a_n| \leq |a_n - a_n^{(K)}| + |a_n^{(K)}| < \varepsilon$$

This shows that

$$\lim_{n \rightarrow \infty} |a_n| < \varepsilon$$

Since  $\varepsilon > 0$  is arbitrary, this implies

$$\lim_{n \rightarrow \infty} a_n = 0$$

Hence  $a \in \ell_0^\infty$ . So  $\ell_0^\infty$  is closed in  $\ell^\infty$ , thus itself Banach.

**Proof of (b):** Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy seq in  $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$ , then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \|f_n - f_m\|_\infty = \sup_{x \in \mathbb{R}} |f_n(x) - f_m(x)| < \varepsilon$$

In particular, for each fixed  $x \in \mathbb{R}$ ,  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$ , hence converges (since  $\mathbb{R}$  is complete).

So we can define the pointwise limit:

$$f(x) := \lim_{n \rightarrow \infty} f_n(x)$$

**Claim 1:**  $f_n \rightarrow f$  in  $\|\cdot\|_\infty$ .

Let  $\varepsilon > 0$ .

Since  $(f_n)$  is Cauchy in  $\|\cdot\|_\infty$ , there exists  $N$  such that:

$$\|f_n - f_m\|_\infty < \varepsilon, \quad \forall n, m \geq N$$

Fix  $m \geq N$ , and let  $n \rightarrow \infty$ . For each  $x$ , we get:

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall n \implies \lim_{n \rightarrow \infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

Since this is true for each  $x \in \mathbb{R}$ , we obtain:

$$\|f - f_m\|_\infty \leq \varepsilon, \quad \text{for all } m \geq N$$

Since  $\varepsilon > 0$  is arbitrary, this shows that

$$\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$$

**Claim 2:**  $f \in (C_b^0(\mathbb{R}), \|\cdot\|_\infty)$ .

Since  $\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0$ , it also implies that the convergence is uniform.

We know the uniform limit of continuous functions is continuous, so  $f$  is continuous. It remains to show  $f$  is bounded, and this directly follows from the uniform convergence. We take  $\varepsilon = 1$ . We have proved that there exists  $N$  s.t. for all  $m \geq N$ ,

$$\|f - f_m\|_\infty \leq 1$$

Thus

$$\sup_{x \in \mathbb{R}} |f(x)| \leq \sup_{x \in \mathbb{R}} |f_N(x)| + 1$$

Since  $f_n \in C_b^0(\mathbb{R})$ , it is bounded, thus

$$\sup_{x \in \mathbb{R}} |f(x)| < \infty$$

showing that the limit function is bounded. This finishes the proof that  $f \in C_b^0(\mathbb{R})$ . Thus, every Cauchy seq in  $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$  converges in  $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$ , i.e. it is Banach.

**Proof of (c):** Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy seq in  $(C_b^0(\mathbb{R}), \|\cdot\|_\infty)$ , then for each fixed  $x \in \mathbb{R}$ ,  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$ , so for the same reason as (b), we can define the pointwise limit:

$$f(x) := \lim_{n \rightarrow \infty} f_n(x)$$

And for the same reason as (b), we get

$$f_n \rightarrow f \text{ in } \|\cdot\|_\infty$$

which also implies that the pointwise convergence is uniform. Since each  $f_n$  is continuous, the uniform limit  $f$  is continuous.

Thus it suffices to show that  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ .

Let  $\epsilon > 0$ . Since  $f_n \rightarrow f$  uniformly, there exists  $N$  such that for all  $n \geq N$ ,  $\|f_n - f\|_\infty < \epsilon/2$ . Also, since  $f_N \in C_0^0(\mathbb{R})$ , there exists  $M > 0$  such that  $|f_N(x)| < \epsilon/2$  for all  $|x| > M$ .

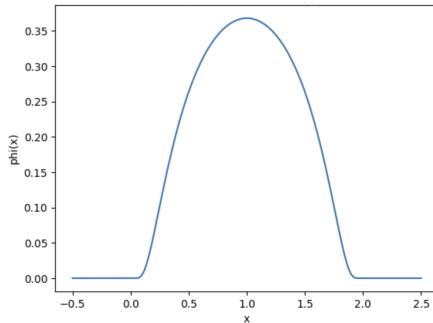
Then for  $|x| > M$ ,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < \epsilon/2 + \epsilon/2 < \epsilon$$

So  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ , i.e.,  $f \in C_0^0(\mathbb{R})$ . Thus, every Cauchy seq in  $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$  converges in  $(C_0^0(\mathbb{R}), \|\cdot\|_\infty)$ , i.e. it is Banach.

**Proof** of (d): We consider a continuous (smooth actually) function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  with  $\text{supp}(\phi) = [0, 2]$  (here we take the closure):

$$\phi(x) := \begin{cases} \exp\left(-\frac{1}{x(2-x)}\right), & 0 < x < 2, \\ 0, & \text{otherwise} \end{cases}$$



This function reaches its maximum at  $x = 1$ ,

$$\|\phi\|_\infty = \frac{1}{e}$$

For each integer  $n \geq 1$ , define

$$\phi_n(x) = \phi(x - n)$$

Then each  $\phi_n$  is also continuous, and  $\text{supp}(\phi_n) = [n, n+2]$ .

Consider the sequence  $(S_N)_1^\infty$ , defined as:

$$S_N(x) := \sum_{n=1}^N 2^{-n} \phi_n(x)$$

Then each  $S_N \in C_c^0(\mathbb{R})$ , since finite sum of continuous functions is also continuous, and  $\text{supp}(S_N) = [1, N+2]$ , thus each  $S_N \in C_c^0(\mathbb{R})$ .

**Claim:**  $(S_N)_1^\infty$  is Cauchy in the sup norm.

This is because for each (WLOG)  $M > N \in \mathbb{N}$ ,

$$\begin{aligned}\|S_M - S_N\|_\infty &= \left\| \sum_{n=N+1}^M \frac{1}{2^n} \phi_n \right\|_\infty \\ &\leq \sum_{n=N+1}^M \frac{1}{2^n} \|\phi\|_\infty \\ &\leq \sum_{n=N+1}^\infty \frac{1}{2^n} \|\phi\|_\infty \\ &= \sum_{n=N+1}^\infty \frac{1}{2^n e} = \frac{1}{2^N e} \xrightarrow{N \rightarrow \infty} 0\end{aligned}$$

Thus for arbitrary  $\varepsilon > 0$ , exists  $K \in \mathbb{N}$  s.t. for all  $M, N \geq K$ ,  $\|S_M - S_N\|_\infty < \varepsilon$ . And by same reason as (b), (c),  $(S_N)_1^\infty$  converges by  $\|\cdot\|_\infty$  into its pointwise limit:

$$S(x) := \sum_{n=1}^\infty 2^{-n} \phi_n(x)$$

But  $S(x)$  does not have compact support,  $\text{supp}(S) = [0, \infty)$ . So  $S \notin C_c^0(\mathbb{R})$ . This serves as a counterexample showing that  $C_c^0(\mathbb{R})$  is not Banach.

## $\nu^+(E), \nu^-(E), |\nu|(E)$ 的 formula from original $\nu$

Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ , and  $E \in \mathcal{A}$ . Prove the following statements:

- (i)  $\nu^+(E) = \sup\{\nu(F) \mid F \in \mathcal{A}, F \subset E\}$ , and  $\nu^-(E) = -\inf\{\nu(F) \mid F \in \mathcal{A}, F \subset E\}$ ;
- (ii)  $|\nu|(E) = \sup\{\sum_{i=1}^N |\nu(E_i)| \mid N \in \mathbb{N}, E = \bigcup_{i=1}^N E_i \text{ disjoint union}\}$ ;
- (iii)  $|\nu|(E) \geq |\nu(E)|$ . In the case  $\nu$  finite, it achieves equality iff  $E$  is positive or negative for  $\nu$ .

**Proof of (i):** By the Hahn decomposition theorem, we can take a Hahn decomposition  $X = P \sqcup N$  where

$$\nu(A) \geq 0 \quad \text{for all } A \subset P, \quad \nu(B) \leq 0 \quad \text{for all } B \subset N$$

Fix  $E \in \mathcal{A}$ . By Jordan decomposition we have

$$\nu^+(E) = \nu(E \cap P)$$

Fix  $F \subset E$ , we have:

$$F = (F \cap P) \sqcup (F \cap N)$$

Since  $\nu(F \cap N) \leq 0$ , we have:

$$\nu(F) \leq \nu(F \cap P) \leq \nu(E \cap P) = \nu^+(E)$$

Since  $F$  is arbitrary, this shows:

$$\sup\{\nu(F) \mid F \subset E\} \leq \nu^+(E)$$

On the other hand, taking  $F = E \cap P \subset E$ , we get

$$\nu(F) = \nu(E \cap P) = \nu^+(E)$$

Hence

$$\sup\{\nu(F) \mid F \subset E\} \geq \nu^+(E)$$

Combining both inequalities gives

$$\nu^+(E) = \sup\{\nu(F) \mid F \subset E\}$$

Similarly, since  $\nu(F \cap P) \geq 0$  and  $\nu(F) = \nu(F \cap P) + \nu(F \cap N)$ , we have  $\nu(F) \geq \nu(F \cap N)$ . And Since  $\nu(E \cap N) = \nu(F \cap N) + \nu((E \setminus F) \cap N)$  with  $\nu((E \setminus F) \cap N) \leq 0$ , we get  $\nu(F \cap N) \geq \nu(E \cap N)$ .

Putting it together:

$$\nu(F) \geq \nu(F \cap N) \geq \nu(E \cap N) = -\nu^-(E)$$

Since  $F$  is arbitrary, this shows:

$$\inf\{\nu(F) \mid F \subset E\} \geq -\nu^-(E)$$

On the other hand, taking  $F = E \cap N \subset E$ , we get

$$\nu(F) = \nu(E \cap N) = -\nu^-(E)$$

Hence

$$\inf\{\nu(F) \mid F \subset E\} \leq -\nu^-(E)$$

Combining both inequalities gives

$$\nu^-(E) = -\inf\{\nu(F) \mid F \subset E\}$$

**Proof of (ii):** Let  $E \in \mathcal{A}$ . By def of total variation measure,

$$|\nu|(E) = \nu^+(E) + \nu^-(E)$$

One direction of the equality is easy. Take a Hahn decomposition  $X = P \sqcup N$  where

$$\nu(A) \geq 0 \quad \text{for all } A \subset P, \quad \nu(B) \leq 0 \quad \text{for all } B \subset N$$

Then by Jordan decomposition, we have:

$$\nu^+(E) = \nu(E \cap P), \quad \nu^-(E) = -\nu(E \cap N)$$

So by taking  $E_1 := E \cap P, E_2 := E \cap N$ , we have:

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E_1) + \nu(E_2)$$

This shows that

$$|\nu|(E) \leq \sup\{\sum |\nu(E_i)|\}$$

And for the other direction, for any disjoint measurable partition  $E = \bigcup_{i=1}^N E_i$ , we have

$$|\nu(E_i)| = |\nu^+(E_i) - \nu^-(E_i)| \leq \nu^+(E_i) + \nu^-(E_i) = |\nu|(E_i)$$

Therefore

$$\sum_{i=1}^N |\nu(E_i)| \leq \sum_{i=1}^N |\nu|(E_i) = |\nu|\left(\bigcup_{i=1}^N E_i\right) = |\nu|(E)$$

since  $|\nu|$  is a p.m. and the  $E_i$ 's are disjoint. Thus

$$\sup\left\{\sum_{i=1}^N |\nu(E_i)|\right\} \leq |\nu|(E)$$

Combining the two inequalities gives

$$|\nu|(E) = \sup \left\{ \sum_{i=1}^N |\nu(E_i)| \mid N \in \mathbb{N}, E = \bigcup_{i=1}^N E_i \text{ disjoint} \right\}$$

proving the statement.

**Proof of (iii):** Let  $E \in \mathcal{A}$ . The ineq  $|\nu|(E) \geq |\nu(E)|$  follows from triangular ineq on  $\mathbb{R}$ :

$$|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$$

Now we assume  $\nu$  is finite (i.e.  $|\nu|(X) < \infty$ ). The equality condition  $|\nu(E)| = |\nu|(E)$  is detailedly:

$$|\nu^+(E) - \nu^-(E)| = \nu^+(E) + \nu^-(E)$$

Since  $|\nu|(X) < \infty$ ,  $\nu^+(E) < \infty$  and  $\nu^-(E) < \infty$ .

Case 1:  $\nu^+(E) \geq \nu^-(E)$ , then

$$\begin{aligned} |\nu^+(E) - \nu^-(E)| = \nu^+(E) + \nu^-(E) &\iff \nu^+(E) - \nu^-(E) = \nu^+(E) + \nu^-(E) \\ &\iff -\nu^-(E) = \nu^-(E) \\ &\iff \nu^-(E) = 0 \\ &\iff E \subset P \end{aligned}$$

Case 2:  $\nu^+(E) < \nu^-(E)$ , then

$$\begin{aligned} |\nu^+(E) - \nu^-(E)| = \nu^+(E) + \nu^-(E) &\iff \nu^-(E) - \nu^+(E) = \nu^+(E) + \nu^-(E) \\ &\iff -\nu^+(E) = \nu^+(E) \\ &\iff \nu^+(E) = 0 \\ &\iff E \subset N \end{aligned}$$

Therefore the equality condition implies that  $E$  must be positive or negative for  $\nu$ ; and in converse, if  $E$  is neither positive nor negative set, in either case it implies  $|\nu(E)| \neq |\nu|(E)$ , thus when  $\nu$  finite,  $|\nu(E)| = |\nu|(E)$  iff  $E$  is positive or negative for  $\nu$ .

## Signed integrals

Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ .

- (i) Prove that  $\int g d|\nu| = \int g d\nu^+ + \int g d\nu^-$  for  $g \in L^+(\nu)$  or  $g \in L^1(\nu)$ .
- (ii) Define  $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ . Prove that  $L^1(\nu) = L^1(|\nu|)$ .
- (iii) Define  $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$  for  $f \in L^1(\nu)$ . Prove that if  $f \in L^1(\nu)$ , then

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|$$

- (iv) Suppose that  $\nu$  is a finite measure (i.e.  $\nu^\pm(X) < \infty$ ) Prove that if  $E \in \mathcal{A}$ , then

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| \mid \|f\|_\infty \leq 1 \right\}.$$

**Proof of (i):** Take a Hahn decomposition  $X = P \sqcup N$ .

Then by Jordan decomposition,

$$\nu^+(E) = \nu(E \cap P), \quad \nu^-(E) = -\nu(E \cap N), \quad \forall E \subset X$$

and therefore  $P$  is null set of  $\nu^-$  and  $N$  is null set of  $\nu^+$ . So on  $P$ ,  $|\nu| = \nu^+ + \nu^- = \nu^+$ ; on  $N$ ,  $|\nu| = \nu^+ + \nu^- = \nu^-$ . Thus, suppose  $g \in L^+(\nu|)$ ,

$$\begin{aligned}\int g d|\nu| &= \int_X g d|\nu| = \int_P g d|\nu| + \int_N g d|\nu| \quad \text{since } X = P \sqcup N \\ &= \int_P g d\nu^+ + \int_N g d\nu^- \quad \text{since } |\nu| = \nu^+, \nu^- \text{ on } P, N \\ &= \int g d\nu^+ + \int g d\nu^- \quad \text{since } N, P \text{ is null for } \nu^+, \nu^-\end{aligned}$$

Suppose  $g \in L^1(|\nu|)$ , then

$$\begin{aligned}\int g d|\nu| &= \int_X g d|\nu| = \int_X g^+ d|\nu| - \int_X g^- d|\nu| \quad \text{by def} \\ &= \left( \int_P g^+ d\nu^+ + \int_N g^+ d\nu^- \right) - \left( \int_P g^- d\nu^+ + \int_N g^- d\nu^- \right) \quad \text{since } X = P \sqcup N \\ &= \left( \int_P g^+ d\nu^+ - \int_P g^- d\nu^+ \right) + \left( \int_N g^+ d\nu^- - \int_N g^- d\nu^- \right) \\ &= \int_P g d\nu^+ + \int_N g d\nu^- \quad \text{since } g \in L^1(|\nu|) \\ &= \int g d\nu^+ + \int g d\nu^- \quad \text{since } N, P \text{ is null for } \nu^+, \nu^-\end{aligned}$$

This finishes the proof.

**Proof of (ii):** WTS:  $L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|)$ .

( $\Rightarrow$ ): Suppose  $f \in L^1(|\nu|)$ , i.e.  $\int |f| d|\nu| < \infty$ .

Let  $\phi$  be arbitrary positive-valued simple function:

$$\phi = \sum_{j=1}^n a_j \chi_{E_j}$$

then

$$\int \phi d|\nu| = \sum_{i=1}^n a_i |\nu|(E_j)$$

Since  $\nu^-(E_j), \nu^+(E_j) \leq \nu^+(E_j) + \nu^-(E_j) = |\nu|(E_j)$  for each  $j$ , we have

$$\int \phi d\nu^+, \int \phi d\nu^- \leq \int \phi d|\nu|$$

Since  $\phi$  is arbitrary, we have

$$\int |f| d\nu^+ = \sup \{ \int \phi d\nu^+ : 0 \leq \phi \leq |f|, \phi \text{ simple} \} \leq \sup \{ \int \phi d|\nu| : 0 \leq \phi \leq |f|, \phi \text{ simple} \} = \int |f| d|\nu|$$

Same for  $\nu^-$ . This shows that

$$\int |f| d\nu^+, \int |f| d\nu^- \leq \int |f| d|\nu| < \infty$$

i.e.  $f \in L^1(\nu^+)$  and  $f \in L^1(\nu^-)$ , so  $f \in L^1(\nu^+) \cap L^1(\nu^-)$ .

Thus

$$L^1(|\nu|) \subset L^1(\nu^+) \cap L^1(\nu^-)$$

( $\Leftarrow$ ): Suppose  $f \in L^1(\nu^+) \cap L^1(\nu^-)$ , i.e.

$$\int |f| d\nu^+ < \infty, \quad \int |f| d\nu^- < \infty$$

Since  $|f|$  is non-negative and measurable, we have  $|f| \in L^+(\nu)$ . Thus by (i) we have:

$$\int |f| d|\nu| = \int |f| d\nu^+ + \int |f| d\nu^- < \infty$$

So  $f \in L^1(\nu)$ .

This shows that:

$$L^1(\nu^+) \cap L^1(\nu^-) \subset L^1(\nu)$$

Combining both direction, we finished the proof that:

$$L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$$

**Proof of (iii):** Suppose  $f \in L^1(\nu)$ , then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \quad \text{by def} \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \quad \text{by tri ineq} \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \quad \text{by property of } L^1 \text{ integration} \\ &= \int |f| d|\nu| \quad \text{from (i)} \end{aligned}$$

Therefore,

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|$$

**Proof of (iv):** Suppose that  $\nu$  is a finite measure (i.e.  $\nu^\pm(X) < \infty$ ), let  $E \in \mathcal{A}$ .

We denote:

$$S := \sup \left\{ \left| \int_E f d\nu \right| \mid \|f\|_\infty \leq 1 \right\}$$

**First we show  $S \leq |\nu|(E)$ :**

For any bounded measurable  $f$  with  $\|f\|_\infty \leq 1$ ,

$$\begin{aligned} \left| \int_E f d\nu \right| &\leq \int_E |f| d|\nu| \quad \text{by (iii)} \\ &\leq \int_E 1 d|\nu| \quad \text{by linearity of integration} \\ &= |\nu|(E) \end{aligned}$$

So by taking the supremum over such  $f$ , we get:

$$S \leq |\nu|(E)$$

**Next we will show  $|\nu|(E) \leq S$ :**

We take a Hahn decomposition, getting  $X = P \sqcup N$  where

$$\nu^+(B) = \nu(P \cup B) \geq 0, \nu^-(B) = -\nu(P \cup B) \leq 0, \quad \text{for all } B \subset X$$

Then

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N)$$

Now define:

$$f := \chi_P - \chi_N$$

Then  $f$  is measurable since  $P, N$  are measurable. And  $\|f\|_\infty \leq 1$  since  $f(x) \in \{-1, 1\} \forall x \in X$  Compute:

$$\int_E f d\nu = \int_{E \cap P} 1 d\nu - \int_{E \cap N} 1 d\nu = \nu(E \cap P) - \nu(E \cap N) = \nu^+(E) + \nu^-(E) = |\nu|(E)$$

Thus

$$|\nu|(E) = \left| \int_E f d\nu \right| \leq S$$

Combining both inequalities, we get:

$$|\nu|(E) = S$$

## finite signed measures on $(X, \mathcal{A})$ 是一个 NVM

Let  $(X, \mathcal{A})$  be a measurable space.

- (a) Let  $\lambda, \mu$  be finite *positive* measures on  $(X, \mathcal{A})$ . Let  $\nu = \lambda - \mu$ . Prove that

$$\nu^+(E) \leq \lambda(E), \quad \nu^-(E) \leq \mu(E), \quad |\nu|(E) \leq \lambda(E) + \mu(E)$$

for every  $E \in \mathcal{A}$ .

- (b) Let  $\nu$  and  $\kappa$  be finite *signed* measures on  $(X, \mathcal{A})$  (i.e.  $\nu(E), \kappa(E) \in \mathbb{R}$  for all  $E \in \mathcal{A}$ ). Show that

$$|\nu + \kappa|(E) \leq |\nu|(E) + |\kappa|(E)$$

for every  $E \in \mathcal{A}$ .

- (c) Let  $\mathcal{M}$  be the collection of finite signed measure  $\nu$  on  $(X, \mathcal{A})$ . For  $\nu \in \mathcal{M}$ , define

$$\|\nu\| = |\nu|(X)$$

Prove that  $\|\cdot\|$  is a norm on  $\mathcal{M}$  with an appropriate definition of the sum of two signed measures and the multiplication of a signed measure by a (real) scalar.

- (d) Suppose  $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . Compute  $\|\delta_x - \delta_y\|$  for  $x, y \in \mathbb{R}$ .

*Remark:* the norm on  $\mathcal{M}$  is called the *the total variation norm*.

### Proof of (a):

Recall in problem 2 we get:

$$\nu^+(E) = \sup\{\nu(F) : F \subset E, F \in \mathcal{A}\}, \quad \nu^-(E) = -\inf\{\nu(F) : F \subset E, F \in \mathcal{A}\}$$

**Claim 1:**  $\nu^+(E) \leq \lambda(E)$ .

Let  $F \subset E, F \in \mathcal{A}$ . Then:

$$\nu(F) = \lambda(F) - \mu(F) \leq \lambda(F) \leq \lambda(E)$$

since  $F \subset E$  and  $\lambda$  is positive. Taking the sup over all such  $F$ , we get

$$\nu^+(E) = \sup_{F \subset E} \nu(F) \leq \lambda(E)$$

**Claim 2:**  $\nu^-(E) \leq \mu(E)$ .

Similarly as Claim 1, for any  $F \subset E$ , since  $\lambda$  and  $\mu$  are p.m., we have

$$\nu(F) = \lambda(F) - \mu(F) \geq -\mu(F) \geq -\mu(E) \implies -\nu(F) \leq \mu(E)$$

Taking the inf over  $F \subset E$ , we get

$$\nu^-(E) = -\inf_{F \subset E} \nu(F) \leq \mu(E)$$

---

**Claim 3:**  $|\nu|(E) \leq \lambda(E) + \mu(E)$ .

This is just combining the two ineqs:

$$|\nu|(E) = \nu^+(E) + \nu^-(E) \leq \lambda(E) + \mu(E)$$

**Proof of (b):**

Let  $E \in \mathcal{A}$ . WTS:  $|\nu + \kappa|(E) \leq |\nu|(E) + |\kappa|(E)$ .

Recall in problem 2 we showed that for a signed measure  $\sigma$  and a measurable set  $E$ , we have:

$$|\sigma|(E) = \sup \left\{ \sum_{i=1}^n |\sigma(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\}$$

Let  $\{E_i\}_{i=1}^n$  be any finite measurable partition of  $E$ . Then for each  $E_i$ :

$$|(\nu + \kappa)(E_i)| = |\nu(E_i) + \kappa(E_i)| \leq |\nu(E_i)| + |\kappa(E_i)| \quad (\text{by tri ineq on } \mathbb{R})$$

Summing over the partition, we have:

$$\sum_{i=1}^n |(\nu + \kappa)(E_i)| \leq \sum_{i=1}^n |\nu(E_i)| + \sum_{i=1}^n |\kappa(E_i)|$$

Now take the supremum over all such partitions of  $E$ :

$$\begin{aligned} |\nu + \kappa|(E) &= \sup \left\{ \sum_{i=1}^n |(\nu + \kappa)(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} \\ &\leq \sup \left\{ \sum_{i=1}^n |\nu(E_i)| + \sum_{i=1}^n |\kappa(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} \\ &\leq \sup \left\{ \sum_{i=1}^n |\nu(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} + \sup \left\{ \sum_{i=1}^n |\kappa(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} \\ &= |\nu|(E) + |\kappa|(E) \end{aligned}$$

Since measurable  $E$  is arbitrary, this finishes the proof.

**Proof of (c):**

$$\mathcal{M} := \{\text{all finite signed measures on } (X, \mathcal{A})\}$$

and for  $\nu \in \mathcal{M}$ , we define:

$$\|\nu\| := |\nu|(X)$$

WTS:  $\|\cdot\|$  is a norm on  $\mathcal{M}$ .

### 1. Positive Definiteness:

Let  $\nu \in \mathcal{M}$ . Since  $|\nu|$  is a positive measure,  $\|\nu\| = |\nu|(X) \geq 0$ .

Since  $|\nu|$  is a positive measure,  $\|\nu\| = |\nu|(X) \geq 0$ .

Suppose  $|\nu|(X) = 0$ , then  $X$  is a  $|\nu|$ -null set, so  $|\nu|(E) = 0$  for all  $E \in \mathcal{A}$ . Thus  $\nu = 0$ .

And suppose  $\nu = 0$ , then  $|\nu| = 0$  also, so  $|\nu|(X) = 0$ .

Thus,  $\|\nu\| = 0$  iff  $\nu = 0$ . This finishes the proof of positive definiteness.

### 2. Absolute Homogeneity:

Since for any measurable set  $E$ :

$$\begin{aligned}
|a\nu|(E) &= \sup \left\{ \sum_{i=1}^n |(a\nu)(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} \\
&= \sup \left\{ \sum_{i=1}^n |a||\nu(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} \\
&= |a| \sup \left\{ \sum_{i=1}^n |\nu(E_i)| : E = \bigsqcup_{i=1}^N E_i \right\} \\
&= |a| \cdot |\nu|(E)
\end{aligned}$$

We have:

$$\|a\nu\| = |a\nu|(X) = |a| \cdot |\nu|(X) = |a| \cdot \|\nu\|$$

finishing the proof of absolute homogeneity.

### 3. Triangle Inequality:

Recall we just proved in (b) that for any measurable  $E$ :

$$|\nu + \kappa|(E) \leq |\nu|(E) + |\kappa|(E)$$

Thus

$$\|\nu + \kappa\| = |\nu + \kappa|(X) \leq |\nu|(X) + |\kappa|(X) = \|\nu\| + \|\kappa\|$$

finishing the proof of triangle inequality.

So we can conclude that  $\|\nu\| := |\nu|(X)$  defines a norm on  $\mathcal{M}$ , with the standard definitions of addition and scalar multiplication of signed measures.

### Proof of (d)

Suppose  $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . Compute  $\|\delta_x - \delta_y\|$  for  $x, y \in \mathbb{R}$ .

Recall def: For any Borel set  $A \subset \mathbb{R}$ ,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

So we define the signed measure  $\nu := \delta_x - \delta_y$  as:

$$\nu(A) = \delta_x(A) - \delta_y(A)$$

If  $x = y$ , then  $\delta_x = \delta_y$ , then  $\nu = 0$ , so  $\|\nu\| = 0$ . This is the trivial case. if  $x \neq y$ : We first compute the Jordan decomposition.

We know that  $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{A}, F \subset E\}$ , and  $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{A}, F \subset E\}$ . For any  $E \ni x$ , we have

$$\nu^+(E) = \nu(\{x\}) = 1$$

In other cases, we have:

$$\nu^+(E) = \nu(E \setminus \{y\}) = 0$$

For any  $E \ni y$ , we have

$$\nu^-(y) = -\nu(\{y\}) = 1$$

In other cases, we have:

$$\nu^-(E) = -\nu(E \setminus \{x\}) = 0$$

And we thus discover that:

$$\nu^+ = \delta_x, \quad \nu^- = \delta_y$$

So

$$\|\nu\| = |\nu|(\mathbb{R}) = \delta_x(\mathbb{R}) + \delta_y(\mathbb{R}) = 1 + 1 = 2$$

Thus we can conclude that

$$\|\nu\| = \begin{cases} 2 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

## and more: finite signed measures on $(X, \mathcal{A})$ 组成一个 real Banach space

Prove that the normed vector space  $\mathcal{M}$  in the previous problem is in fact a Banach space.

**Proof** In problem 4 we have shown that on  $(\mathcal{M}, \|\cdot\|)$  is a normed vector space, where

$$\mathcal{M} := \{\text{all finite signed measures on } (X, \mathcal{A})\}$$

and

$$\|\nu\| := |\nu|(X)$$

Now we prove that the NVM  $(\mathcal{M}, \|\cdot\|)$  is complete, i.e. it is a Banach space.

Let  $(\nu_n)$  be a Cauchy sequence in  $\mathcal{M}$ . We have

$$|\nu_n(B) - \nu_m(B)| = |(\nu_n - \nu_m)(B)| \leq \|\nu_n - \nu_m\| \quad \text{for all } B \in \mathcal{A}$$

In particular,  $(\nu_n(B))_n$  is a Cauchy sequence for all  $B \in \mathcal{A}$ . For each  $B \in \mathcal{A}$ , this is a Cauchy seq in  $\mathbb{R}$ , thus converges. So we can get:

$$\nu(B) := \lim_n \nu_n(B)$$

as the pointwise limit (by a point we mean a set).

**Claim 1:**  $\nu \in \mathcal{M}$ .

Since for all  $n$ ,  $\nu_n(\emptyset) = 0$ , we have:

$$\nu(\emptyset) := \lim_n \nu_n(\emptyset) = 0$$

For a countable disjoint union of measurable sets  $E = \bigsqcup_{i=1}^{\infty} E_i$ ,

$$\lim_n \nu_n(E) = \lim_n \sum_i \nu_n(E_i)$$

is the limit of a finite sum of numerical sequences in  $\mathbb{R}$ . So we can exchange the order of taking limit and sum.

Then we get:

$$\nu(E) = \lim_n \nu_n(E) = \lim_n \sum_i \nu_n(E_i) = \sum_i \lim_n \nu_n(E_i) = \sum_i \nu(E_i)$$

And notice, for each measurable set  $B \in \mathcal{A}$ , since  $(\nu_n(B))_n$  is a Cauchy sequence in  $\mathbb{R}$ , it is bounded, thus does not admit  $\infty, -\infty$  values. verifying that  $\nu$  is a valid signed measure.

Also, this means that taking Hahn Decomposition  $X = P \sqcup N$  by  $\nu$ , we have

$$\nu^+(X) = \nu(P), \quad \nu^-(X) = -\nu(N)$$

Since  $\nu(P), \nu(N)$  are bounded, we have: Thus

$$|\nu|(X) = \nu^+(X) + \nu^-(X) < \infty$$

This verifies that  $\nu$  is a finite s.m.

**Claim 2:**  $\nu_n \rightarrow \nu$  in  $\|\cdot\|$ . Fix  $\varepsilon > 0$ . There exists  $N$  such that  $\|\nu_n - \nu_m\| < \varepsilon/2$  for all  $m, n \geq N$ . Thus for all  $n \geq N$  we have:

$$|(\nu_n - \nu)(B)| = \lim_m |(\nu_n - \nu_m)(B)| \leq \varepsilon/2, \quad \forall B \in \mathcal{A}, \forall n \geq N$$

Notice that

$$\nu^+(B) = \sup\{\nu(C) \mid C \in \mathcal{A}, C \subset B\}$$

and

$$\nu^-(B) = -\inf\{\nu(C) \mid C \in \mathcal{A}, C \subset B\} = \sup\{-\nu(C) \mid C \in \mathcal{A}, C \subset B\}$$

It follows that

$$(\nu_n - \nu)^+(X) = \sup\{(\nu_n - \nu)(B) \mid B \in \mathcal{A}\} \leq \varepsilon/2, \quad \forall n \geq N$$

Similarly,

$$(\nu_n - \nu)^-(X) = \sup\{-(\nu_n - \nu)(B) \mid B \in \mathcal{A}\} \leq \varepsilon/2, \quad \forall n \geq N$$

Thus

$$|\nu_n - \nu|(X) = (\nu_n - \nu)^+(X) + (\nu_n - \nu)^-(X) \leq \varepsilon$$

This holds for all  $n \geq N$ . And since  $\varepsilon > 0$  is arbitrary, this proves that

$$\lim_{n \rightarrow \infty} \|\nu_n - \nu\| = 0$$

As a result,  $\nu_n \rightarrow \nu$  in  $\|\cdot\|$ , completing the proof.

*Nur für Verrückte*

(It's **really** not necessary to attempt these problems. Do not, under any circumstances, hand them in!) Does there exist a signed Borel measure  $\nu$  on  $\mathbb{R}$  with the property that for every  $\alpha \in \mathbb{R}$  there exists a Borel set  $E \subset \mathbb{R}$  with  $\nu(E) = \alpha$ .