Homework 3: on Lebesgue-Stieljes measures(30/40)

None of the following questions will be graded. Do them, but do not hand them in.

Fun facts about increasing functions

Let $F\colon \mathbb{R} \to \mathbb{R}$ be an increasing function, that is, $F(x) \leq F(y)$ whenever $x \leq y$.

(a) Prove that the following limits exist (and make sure you understand the definitions):

   (i) $F(a-) := \lim_{x\to a-} F(x) \in \mathbb{R}$ and $F(a+) := \lim_{x\to a+} F(x) \in \mathbb{R}$ for $a \in \mathbb{R}$;

   (ii) $F(\infty) := \lim_{x\to\infty} F(x) \in (-\infty, \infty]$;

   (iii) $F(-\infty) := \lim_{x\to-\infty} F(x) \in [-\infty, \infty)$.

(b) Fix any $a \in \mathbb{R}$.

   (i) Prove that $F(a-) \leq F(a) \leq F(a+)$;

   (ii) Prove that F is continuous at a iff $F(a-) = F(a+)$.

   We say that a function is *left continuous* if $F(a-) = F(a)$ for every $a \in \mathbb{R}$. It is *right-continuous* if instead $F(a+) = F(a)$ for every $a \in \mathbb{R}$.

(c) If X is a metric space (or, more generally, a topological space), then a function $f\colon X \to \mathbb{R}$ is *upper semicontinuous* if the set $\{x \in X \mid f(x) < a\}$ is open for every $a \in \mathbb{R}$. It is *lower semicontinuous* if instead the set $\{x \in X \mid f(x) > a\}$ is open for every $a \in \mathbb{R}$. Prove that our function $F\colon \mathbb{R} \to \mathbb{R}$ is right-continuous (resp. left continuous) iff it is upper semicontinuous (resp. lower semicontinuous). Give an example showing that this is no longer true if F is not assumed increasing.

(d) Prove that the following are equivalent:

   (i) F is surjective;

   (ii) F is continuous, $F(\infty) = \infty$, and $F(-\infty) = -\infty$.

(e) Let $A \subset \mathbb{R}$ be the set of points where F fails to be continuous. Prove that A is a countable (i.e. empty, finite, or countably infinite) set. *Hint*: prove that for any integers $m, n \geq 1$, the set of points $x \in [-m, m]$ where $F(x+) - F(x-) \geq 1/n$ is finite.

Locally finite measures

If X is a metric space (or, more generally, a topological space), then a Borel measure μ on X is said to be *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset X$. Now let μ be a Borel measure on $\mathbb{R}$, that is $\mu\colon \mathcal{B}(\mathbb{R}) \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$ and is countably additive.

(a) Prove that the following are equivalent:

   (i) μ is locally finite;

   (ii) $\mu([-N, N]) < \infty$ for every $N \geq 0$;

   (iii) $\mu(I) < \infty$ for every bounded interval I.

(b) Prove that if μ is locally finite, then μ is σ-finite. Is the converse true? Give a proof or a counterexample.

Basic formulas for LS measures

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a distribution function, and $\mu = \mu_F$ the associated Lebesgue–Stieltjes measure. From its definition using h-intervals, it follows that $\mu((a, b]) = F(b) - F(a)$ for $-\infty < a < b < \infty$. Using this property together with basic general properties of (σ-finite) measures, we proved in class that $\mu((a, b)) = F(b-) - F(a)$ for $\infty < a \le b < \infty$. Using a similar strategy, prove the following:

(a) $\mu([a, b]) = F(b) - F(a-)$ for $-\infty < a \le b < \infty$;
(b) $\mu([a, b)) = F(b-) - F(a-)$ for $-\infty < a \le b < \infty$;
(c) $\mu(\{a\}) = F(a) - F(a-)$ for $-\infty < a < \infty$;
(d) $\mu((-\infty, b]) = F(b) - F(-\infty)$ for $-\infty < b < \infty$;
(e) $\mu((-\infty, b)) = F(b-) - F(-\infty)$ for $-\infty < b < \infty$;
(e) $\mu((a, \infty)) = F(\infty) - F(a)$ for $-\infty < a < \infty$;
(f) $\mu([a, \infty)) = F(\infty) - F(a-)$ for $-\infty < a < \infty$;
(g) $\mu([-\infty, \infty)) = F(\infty) - F(-\infty)$.

Vitali sets

For $x, y \in [-1, 1]$, write $x \sim y$ iff $x - y \in \mathbb{Q}$.

(a) Show that $\sim$ is an equivalence relation, i.e. show that (i) $x \sim x$, (ii) $x \sim y$ implies $y \sim x$, (iii) if $x \sim y$ and $y \sim z$, then $x \sim z$.
(b) The set $[-1, 1]$ is partitioned into equivalence classes. Let $V \subset [-1, 1]$ be a set containing exactly one element from each equivalence class. (Here, we use the Axiom of choice.) We call V a *Vitali set*. Let $\{r_1, r_2, \dots\} = [-2, 2] \cap \mathbb{Q}$. Define $V_i = r_i + V = \{r_i + x \mid x \in V\}$. Prove that the sets $V_1, V_2, \dots$ are mutually disjoint, and that

$$[-1, 1] \subset \bigcup_{i=1}^{\infty} V_i \subset [-3, 3].$$

Vitali sets, season 2

Let $V \subset [-1, 1]$ be a Vitali set (see above).

(a) Using the translation invariance of Lebesgue measure, prove V is not Lebesgue measurable.
(b) Prove that if E is a Lebesgue measurable set and satisfies $E \subset V$, then $m(E) = 0$.
(c) Using the technique in (a), prove the following statement: if $A \subset \mathbb{R}$ is any Lebesgue measurable set with $m(A) > 0$, then A contains a set which is not Lebesgue measurable.

The middle-thirds Cantor set

Let C be the middle-thirds Cantor set, defined as

$$C := \bigcap_{n=1}^{\infty} C_n,$$

where

$$C_n := \bigcup_{a_1,\dots,a_n \in \{0,2\}} [\sum_{i=1}^{n} \frac{a_i}{3^i}, \sum_{i=1}^{n} \frac{a_i}{3^i} + \frac{1}{3^n}]$$

(a) Set $C_0 = [0,1]$. Show that $C_n \subset C_{n-1}$ for all $n \geq 1$. Also prove that C_n is the union of 2^n *disjoint* closed intervals, that the set $U_n := C_{n-1} \setminus C_n$ is the union of the middle thirds open intervals of the disjoint closed intervals of C_{n-1}, and that

$$U_n = \bigcup_{a_1,\cdots,a_{n-1} \in \{0,2\}} (\sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{1}{3^n}, \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{2}{3^n}).$$

(We interpret this as the interval $(1/3, 2/3)$ when $n = 1$.) Thus, C is the set obtained by removing successive middle thirds of the remaining disjoint closed intervals starting with $[0,1]$. Sketch the first few sets C_n and U_n.

(b) Show that C is a compact set, and that $m(C) = 0$, where m denotes Lebesgue measure. Also show that C does not contain any non-empty open interval (a,b).

(c) Show that C equals the set of numbers $x \in [0,1]$ which have a base-3 expansion of the form $x = 0.a_1a_2a_3\cdots$ where a_i is either 0 or 2, i.e.

$$C = \{\sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i \in \{0,2\} \text{ for all } i \in \mathbb{N}\}.$$

(Note: A point may have two base-3 expansions such as $1/3 = 0.1000\ldots = 0.0222\ldots$; this number is in C since one of the expansions is of the desired form.)

(d) Show that $\frac{1}{4}, \frac{9}{13} \in C$ but $\frac{5}{36} \notin C$.

The Devil's Staircase: an increasing function build on Cantor set

Let C be the middle-thirds Cantor set, and define $F\colon C \to [0,1]$ by

$$F(x) = \sum_{i=1}^{\infty} \frac{a_i/2}{2^i} \tag{3.1}$$

for $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, $a_i \in \{0,2\}$.

(a) Prove that F is an increasing function, and that $F(C) = [0,1]$.

(b) Suppose that $x, y \in C$ and $x < y$. Prove that $F(x) = F(y)$ iff x and y are the endpoints of a removed open interval, that is, one of the 2^{n-1} disjoint open intervals whose union equals $U_n = C_{n-1} \setminus C_n$ for some $n \geq 1$.

(c) Prove that $F\colon C \to [0,1]$ extends uniquely to a continuous function which is constant on all the intervals in U_n, $n \geq 1$. Sketch the graph of F. *Hint*: to prove continuity, it suffices to show that $F([0,1]) = [0,1]$ (Why?)

(d) Prove that $F'(x) = 0$ for a.e. x. In other words, there exists a set $E \subset [0,1]$ such that $m(E) = 0$, and such that $\lim_{h\to 0}(F(x+h) - F(x))/h = 0$ for $x \in [0,1] \setminus E$.

(Remark 1: because of (c) and (d), the graph of F is called the *Devil's Staircase*; it is horizontal almost everywhere, and has no vertical jumps, but nevertheless climbs upwards.)

(Remark 2: the fact that $F(C) = [0,1]$ implies that C has the same cardinality as $[0,1]$, in particular the

Cantor set is uncountable.)

Some of the following questions will be graded. Do them, and do hand them in.

fun facts about distribution functions

(a) Let $A \subset \mathbb{R}$ be a countable set. Exhibit a distribution function F that is discontinuous at every point in A, but continuous everywhere else. Justify your answer. *Hint*: play around with the Heaviside function.

(b) Let $F\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Prove that there exists a unique distribution function G such that $G(x) = F(x)$ for all points x where F is continuous. *Hint*: there is a simple formula for G in terms of F.

Sol. of (a):

We list $A = \{a_n\}_{n=1}^{\infty}$ as a sequence to label its elements. Define:

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} H(x - a_n)$$

where $H(x)$ is the Heaviside function: $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$.

Claim 1.1 F **is non-decreasing**.

Proof: Suppose $y > x \in \mathbb{R}$, then $H(y - a_n) \geq H(x - a_n)$ for each $n \in \mathbb{N}$, so we have $F(y) \geq F(x)$.

Claim 1.2 F **is right continuous but not left continuous (thus discontinuous) at every** a_n**.**

Proof: Let $\epsilon > 0$.

We take $N \in \mathbb{N}$ s.t. $\sum_{k \geq N, n \in \mathbb{N}} \frac{1}{2^k} < \epsilon$.

Then we take $\delta > 0$ such that $a_1, a_2, \cdots, a_N \notin (a_n, a_n + \delta)$.(This can be done since there are only finite points here)

Thus $\forall y \in (a_n, a_n + \delta)$, we have $|F(y) - F(a_n)| < \epsilon$, since $F(y) < F(a_n) + \sum_{k \geq N, n \in \mathbb{N}} \frac{1}{2^k}$. Since ϵ is arbitrary, this finishes the proof that F is right continuous at a_n.

Also, $\forall y < a_n$, we have $F(y) < F(a_n) - \frac{1}{2^n}$, which means that $|F(y) - F(a_n)| > \frac{1}{2^n}$ for any y on the left, so F is not left continuous at a_n.

Claim 1.3: F **is continuous at every** $x \in \mathbb{R} \setminus A$**.**

Proof: This is similar to the proof in Claim 1.2.

Fix $x \in \mathbb{R} \setminus A$. Let $\epsilon > 0$.

We take $N \in \mathbb{N}$ s.t. $\sum_{k \geq N, n \in \mathbb{N}} \frac{1}{2^k} < \epsilon$.

Then we take $\delta > 0$ such that $a_1, a_2, \cdots, a_N \notin (a_n - \delta, a_n + \delta)$. This can be done since there are only finite points here.

Thus $\forall y \in (a_n - \delta, a_n + \delta)$, we have $|F(y) - F(a_n)| < \sum_{k \geq N, n \in \mathbb{N}} \frac{1}{2^k} < \epsilon$.

Since ϵ is arbitrary, this finishes the proof that F is continuous at x.

By claim 1.1, 1.2, 1.3, we have proved that F is a distribution function that is discontinuous at every point of A but continuous elsewhere.

Proof of (b):
Given an increasing function $F : \mathbb{R} \to \mathbb{R}$, we define $G : \mathbb{R} \to \mathbb{R}$ by:

$$G(x) = \lim_{y \to x^+} F(y).$$

We will show that this is the unique distribution function G such that $G(x) = F(x)$ for all points x where F is continuous.

Incresing: Since F is increasing, for any $x < y$ we have $F(x) \le F(y)$. Thus, for any $x < y$ we have:

$$G(x) = \lim_{z \to x^+} F(z) \le \lim_{z \to y^+} F(z) = G(y).$$

Thus, G is increasing.

Right-continuity: Since F is an increasing function, it can only have jump discontinuities, and the right limit exists for all X. By construction, G is right-ctn.

Above finishes the proof that G is a distribution function.

Agree with F at ctn point: $G(x) = F(x)$ where F is continuous at x, since $G(x) = \lim_{y \to x^+} F(y) = F(x)$ there.

It remains to show that it is unique.

Suppose R is another such function. It suffices to show: R agrees with G on discontinuous points of F.

Since R, G are right continuous, their right limit must exist at each point. Therefore, let x be an arbitrary point where F is discontinuous at x, **it suffices to show that there is a sequence $\{x_n\}$ approaching x, such that $\lim_n G(x_n) = \lim_n R(x_n)$.**

Since F is increasing, the points where F is disctn is at most countable. Therefore **the points where F is ctn, denote it as C, is dense in $\mathbb{R}$.** Thus we can pick a sequence $\{x_n\}$ in C approaching x, then $G(x_n) = R(x_n) = F(x_n)$ for each n, impling that $\lim_n G(x_n) = \lim_n R(x_n)$. This finishes the proof of uniqueness.

Finding intervals

Let $E \subset \mathbb{R}$ be a Lebesgue measurable subset with $m(E) > 0$. Prove that for every $\alpha \in (0, 1)$ there exists an (nonempty) bounded open interval I such that $m(E \cap I) \ge \alpha m(I)$. *Hint*: first reduce to the case when E is bounded, then use outer regularity.

Proof Let $\alpha \in (0, 1)$ be arbitrary and fix it.

We first consider the case that E is bounded. By outer regularity of Lebesgue measure, there exists an open set G such that

$$E \subset G \quad \text{and} \quad m(G) - m(E) \le (1/\alpha - 1)\, m(E)$$

since $\alpha \in (0, 1)$. Then we have:

$$m(G) \leq 1/\alpha\, m(E)$$

Note that in $\mathbb{R}$, an open set is just a countable disjoint union of open intervals. We write:

$$G = \bigsqcup_{i \in \mathbb{N}} I_i$$

Since $E \subset G$, we have:

$$E = \bigsqcup_{i \in \mathbb{N}} (I_i \cap E)$$

Thus

$$m(E) = \sum_{i \in \mathbb{N}} m(I_i \cap E) \geq \alpha \sum_{i \in \mathbb{N}} m(I_i)$$

So **there must exist some** i **such that** $m(I_i \cap E) \geq \alpha\, m(I_i)$**, otherwise contradicting with the ineq above.**
This finishes the proof of the bounded case.
The we consider the case when E is unbounded. We can write

$$E = \bigsqcup_{n \in \mathbb{Z}} (E \cap (n, n+1])$$

where each $E_n := E \cap (n, n+1]$ is bounded.
We apply the case where E is bounded, confirming that there is some interval I such that $m(E_1 \cap I) \geq \alpha m(I)$.
By monotonicity of measure, we have $m(E \cap I) \geq m(E_1 \cap I) \geq \alpha m(I)$.

So many differences

Let $E \subset \mathbb{R}$ be a Lebesgue measurable subset with $m(E) > 0$.

(a) Prove that the set

$$E - E := \{x - y \mid x, y \in E\} \subset \mathbb{R}$$

contains a nonempty open interval centered at the origin. *Hint*: use the previous exercise with α large enough, together with the translation invariance of Lebesgue measure.

(b) Prove that there exists $\epsilon > 0$ such that $E \times E \subset \mathbb{R}^2$ intersects every line $y = x + t$ with $|t| < \epsilon$.

(c) Let $C \subset \mathbb{R}$ be the middle-third Cantor set (so $m(C) = 0$). Does $C - C$ contain a nonempty open interval centered at the origin?

Proof of (a):

(a) Since $E \in \mathcal{L}$, we can choose open interval $I = (a, b)$ s.t. $m(E \cap I) \geq \frac{3}{4} m(I)$

Let $\varepsilon := \frac{1}{2}(b-a) = \frac{1}{2} m(I)$

Consider the interval $(-\varepsilon, \varepsilon)$

Suppose for contradiction that $(-\varepsilon, \varepsilon) \not\subset E - E$

$\Rightarrow \exists \delta \in (0, \varepsilon)$ s.t. $\delta \notin E - E$

$\Rightarrow \forall e \in E$, $e - \delta \notin E \Rightarrow (E + \delta) \cap E = \emptyset$

$$\Rightarrow \delta + m(I) = m(a, b+\delta) = m(I \cup I+\delta)$$
$$\geq m\big(E \cap (I \cup I+\delta)\big)$$
$$\geq m\big((E\cap I) \cup ((E+\delta)\cap(I+\delta))\big)$$
$$= m(E\cap I) + m((E\cap I)+\delta) \quad \text{since}$$

$E+\delta$ is disjoint with E

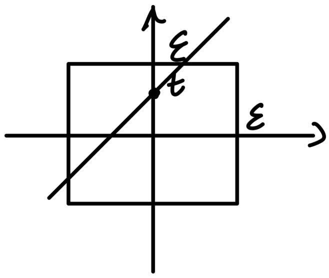
By translation invariance of Lebesgue measure : $m(E\cap I) = m((E\cap I)+\delta)$

$$\Rightarrow \delta + m(I) \geq 2m(E\cap I) \geq \tfrac{3}{2}m(I)$$
$$\Rightarrow \delta \geq \tfrac{1}{2}m(I), \text{ contradicting the fact that } \delta < \tfrac{1}{2}m(I)$$

Therefore we must have $(-\varepsilon, \varepsilon) \subseteq E-E$

Proof of (b):

Consider taking ϵ as the one in (a) where the interval contained in $E-E$ is $(-\epsilon, \epsilon)$, then the box $(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ is contained in $E\times E$. It trivially follows that $E\times E \subset \mathbb{R}^2$ intersects every line $y = x+t$ with $|t| < \epsilon$, since the intercept of this line with y-axis is below ϵ and above $-\epsilon$.



Sol. of (c): $C - C$ contain a nonempty open interval centered at the origin, and we will prove that one such interval is $(-1, 1)$.

Proof Recall the balanced ternary representation of $[-1/2, 1/2]$: $\forall x \in [-1/2, 1/2]$, there is a seq of $(a_n)_{n\in\mathbb{N}}$ in $\{-1, 0, 1\}$ s,t,

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \qquad a_n \in \{-1, 0, 1\},$$

Thus every $x \in [-1, 1]$ can be halved, ternary expanded and then doubled to recover:

$$x = 2\sum_{n=1}^{\infty}\frac{a_n}{3^n} = \sum_{n=1}^{\infty}\frac{2a_n}{3^n}, \qquad a_n \in \{-1,0,1\} \tag{3.2}$$

$$= \sum_{n=1}^{\infty}\frac{b_n}{3^n}, \qquad b_n \in \{-2,0,2\} \tag{3.3}$$

And by the problem "The middle-thirds Cantor set", we learned that

$$C = \{\sum_{i=1}^{\infty}\frac{a_i}{3^i} \mid a_i \in \{0,2\} \text{ for all } i \in \mathbb{N}\}.$$

Therefore we can write every number $x \in [-1, 1]$ into a difference of two $x, y \in C$, i.e. an element of $C - C$:

$$x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \qquad b_n \in \{-2, 0, 2\} \tag{3.4}$$

$$= \sum_{n=1}^{\infty} \frac{p_n - q_n}{3^n}, \qquad p_n, q_n \in \{-2, 0, 2\} \tag{3.5}$$

$$= \sum_{n=1}^{\infty} \frac{p_n}{3^n} - \sum_{n=1}^{\infty} \frac{q_n}{3^n}, \qquad p_n, q_n \in \{-2, 0, 2\} \tag{3.6}$$

since each series converges independently. Here we let $p_n = 2, q_n = 0$ if $b_n = 2$; $p_n = 0, q_n = 2$ if $b_n = -2$, $p_n = 0, q_n = 0$ if $b_n = 0$.

Thus $x \in C - C$, so $[-1, 1] \subset C - C$.

a holey set

Let $(x_n)_1^\infty$ be a countable dense sequence in $(0, 1)$. For each $t > 0$, consider the set

$$A_t := [0, 1] \setminus \bigcup_{n=1}^{\infty} (x_n - 2^{-n}t, x_n + 2^{-n}t).$$

(a) Prove that A_t is a compact (possibly empty) subset of $\mathbb{R}$. Also prove that A_t has empty interior, that is, A_t contains no nonempty open set.

(b) Prove that $t \mapsto m(A_t)$ is continuous.

(c) Prove that there exists $t > 0$ such that $m(A_t) = 597/2025$.

Proof **of a:**

cptness: $\bigcup_{n=1}^{\infty} (x_n - 2^{-n}t, x_n + 2^{-n}t)$ is a ctbl union of open sets, thus open

$A_t := [0,1] \setminus \bigcup_{n=1}^{\infty} (x_n - 2^{-n}t, x_n + 2^{-n}t)$ is a closed set minus an open set, thus closed

Since it is also bounded $\Rightarrow$ it is compact

Proof **of b:** Define for each $n \in \mathbb{N}$

$$I_n(t) = (x_n - 2^{-n}t, x_n + 2^{-n}t)$$

Then

$$A_t = [0, 1] \setminus \bigcup_{n=1}^{\infty} I_n(t)$$

So

$$m(A_t) = m([0, 1]) - m(\bigcup_{n=1}^{\infty} I_n(t)) = 1 - m(\bigcup_{n=1}^{\infty} I_n(t))$$

Thus it suffices to show $t \mapsto m(\bigcup_{n=1}^{\infty} I_n(t))$ is continuous.

Let $\epsilon > 0$. Let $t > 0$.

We consider $p \in (t, t + \epsilon/2)$:

By set inclusion relation and measure's property, we have:

$$m(\bigcup_{n=1}^{\infty} I_n(p)) - m(\bigcup_{n=1}^{\infty} I_n(t)) = m(\bigcup_{n=1}^{\infty} I_n(p) \setminus \bigcup_{n=1}^{\infty} I_n(t)) \tag{3.7}$$

Since

$$(\bigcup_{n=1}^{\infty} I_n(p)) \setminus (\bigcup_{n=1}^{\infty} I_n(t)) \subset \bigcup_{n=1}^{\infty} (I_n(p) \setminus I_n(t))$$

We have:

$$\begin{aligned} m(\bigcup_{n=1}^{\infty} I_n(p)) - m(\bigcup_{n=1}^{\infty} I_n(t)) &\leq m(\bigcup_{n=1}^{\infty} (I_n(p) \setminus I_n(t))) && (3.8) \\ &\leq \sum_{n=1}^{\infty} (m(I_n(p)) - m(I_n(t))) && (3.9) \\ &= \sum_{n=1}^{\infty} 2 \cdot 2^{-n}(p - t) && (3.10) \\ &= 2(p - t) && (3.11) \\ &\leq \epsilon && (3.12) \end{aligned}$$

Similarly for $p \in (t - \epsilon/2, t)$, we get the same bound. This finishes the proof pf (b).

Proof **of c:**

We use the same notation of $I_n(t)$ as in (b). We have:

$$m(\bigcup_{n=1}^{\infty} I_n(t)) \leq \sum_{n=1}^{\infty} m(I_n(t)) = \sum_{n=1}^{\infty} m(I_n(t)) = \sum_{n=1}^{\infty} 2 \cdot 2^{-n} t = 2t.$$

So by choosing $t := 1/6$, we have $m(A_t) = 1 - m(\bigcup_{n=1}^{\infty} I_n(t)) \geq 2/3$. And by choosing $t := 4$, $I_1(t)$ covers an interval of length 4, so $A_t = \varnothing$, $m(A_t) = 0$. By intermediate value theorem, there exists some $t \in (1/6, 4)$ such that $m(A_t) = 597/2025$.

a Cantor measure

(A Cantor measure.) Let $E \subset \mathbb{R}$ be a nonempty compact set with the following property: for every $x \in E$ and every $\epsilon > 0$, the set $(x - \epsilon, x) \cup (x, x + \epsilon)$ has nonempty intersection with both E and E^c. Prove that there exists a Borel measure μ on $\mathbb{R}$ with the following properties:

(i) if $I \subset \mathbb{R}$ is a nonempty open interval, then $\mu(I) > 0$ iff $I \cap E \neq \emptyset$.

(ii) $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$;

(iii) $\mu(\mathbb{R}) = 0.597597597\ldots$.

Hint: set $\mu = \mu_F$, where F is a distribution function whose graph is similar to the Devil's staircase above.

Proof Write $T := 0.597597597\ldots$

Since E is compact, E^c is open. Also, since E is compact, it takes min and max element.

Thus we consider $A := E^c \cap (\min E, \max E)$, this is an open set. We know any open set in $\mathbb{R}$ is a countable disjoint union of open intervals, so $A := E^c \cap (\min E, \max E) = \bigsqcup_{n=1}^{\infty} I_n$ for some disjoint intervals

$I_1 = (a_1, b_1), I_2 = (a_2, b_2), \cdots$.

Now we construct a function $G : A \to [0, T)$ by sending $G(x) = \sum_{b_i \leq a_N} \frac{T}{2^n}$, for $x \in I_N$.
This is an **increasing step function** since, each I_n is disjoint and on a fixed interval I_N, the number of b_i that its a_N surpasses is constant. And suppose $y > x$ is on I_M, we have must $G(y) \geq G(x)$ because he number of b_i that a_M surpasses is at least at many as that a_N surpasses.
And for each $x \in A$, we have $G(x) < T$, by geometric series
Then we construct F out of G, define:

$$F := \begin{cases} 0, & x \leq \min E \\ \inf \{G(y) \mid y \geq x, y \in A\}, & x \in (\min E, \max E) \\ T, & x \geq \max E \end{cases}$$

F **is increasing:** It is constant on $(-\infty, \min E) \cup (\max E, \infty)$ and is the infimum of $G(y)$ with $y \geq x$ on $(\min E, \max E)$. Since G is increasing, F is also increasing.
F **is right continuous:** It suffices to prove the right-continuity of F on $x \in E^c \cap (\min E, \max E)$.
Fix $x_0 \in E^c \cap (\min E, \max E)$.
Let $\epsilon > 0$.
Let $k \in \mathbb{N}$ such that $\epsilon > \frac{T}{2^k+1}$.
We define for each y, $S_y := \{b_i \mid x_0 \leq b_i \leq y\}$ as the set of all b_i (right endpoint of I_k) that is witin x_0 and y.
Note that $I_z \subset I_y$ for all $y > z$.
Consider $y_1 := \min(\{b_1, \cdots, b_k\} \setminus S_x)$.
Then for all $y \in (x_0, y_1)$, we have:

$$F(y) \leq F(x_0) + \sum_{i=k}^{\infty} \frac{T}{2^i} \leq F(x_0) + \epsilon$$

By defining $\delta := y_1 - x_0$, we have shown the right continuity of F.
(By dual reason, we can prove that F is left continuous. So F is actually continuous.) Above, we have shown that F is a distribution function.

Now let μ_F be the Lebesgue-Stieljes measure associated with F. We will prove for the three properties above:

(i) Let $\{x\}$ be a singleton set in $\mathbb{R}$, for each $n \in \mathbb{N}$, we can construct an h-intervals seq of covering of $\{x\}$ by $(x - 1/n, x]$ as the first covering set and $\varnothing$ as all other covering sets.
Then by the definition of μ_F, we have:

$$\mu_F(\{x\}) = \inf_{n \in \mathbb{N}} (F(x) - F(x - 1/n))$$

By continuity, it shows that $\mu_F(\{x\}) = 0$.

(ii)

$$\mu_F(\mathbb{R}) = \lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x) = T - 0 = T = 0.597597597 \cdots$$

(iii) Let $I = (a, b)$ be a nonempty open interval.
Suppose $\mu(I) > 0$, then $F(b) - F(a) > 0$, so by definition of G, some must be at least two different

intervals I_{n_1}, I_{n_2} in A such that for some $x, y \in (a, b)$, $x \in I_{n_1}$ and $y \in I_{n_2}$, thus $\exists$ some $e \in E$ such that $e \in (x, y)$. Thus $I \cap E \neq \emptyset$.

Suppose $I \cap E \neq \emptyset$. Let $e \in E \cap I$. Since $\forall \epsilon > 0$, $(x - \epsilon, x) \cup (x, x + \epsilon)$ has nonempty intersection with both E and E^c, e has some open neighborhood $B_\epsilon(e) \subset I$, intersecting two different I_N, $I_M \subset A$. Take $n \in I_N$, $m \in I_m$. Then $F(m) - F(n) = G(m) - G(n) > 0$, so $\mu_F(E) \geq F(m) - F(n) > 0$ by monotinicity of measure.

This finishes the proof.

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