

## Lec 1 on differentiaion (50/50)

*None of the following questions will be graded. Do them, but do not hand them in.*

**Completion of  $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$  = Completion of  $(X \times Y, \bar{\mathcal{A}} \otimes \bar{\mathcal{B}}, \bar{\mu} \times \bar{\nu})$**

Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be measure spaces. Let  $(X, \bar{\mathcal{A}}, \bar{\mu})$  and  $(Y, \bar{\mathcal{B}}, \bar{\nu})$  be their completions, respectively. Then, the completion of  $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$  is same as the completion of  $(X \times Y, \bar{\mathcal{A}} \otimes \bar{\mathcal{B}}, \bar{\mu} \times \bar{\nu})$ .

**Modified HL maximal inequality ( $\geq$  instead of  $>$ )**

Prove that there is a constant  $C_n > 0$  that only depends on  $n$  such that for every  $f \in L^1(\mathbb{R}^n)$  and  $\alpha > 0$ ,

$$m(\{x \in \mathbb{R}^n \mid Hf(x) \geq \alpha\}) \leq \frac{C_n}{\alpha} \int_{\mathbb{R}^n} |f(x)| dx$$

(Remark: We had  $Hf(x) > \alpha$  for the HL maximal inequality. Here we have  $Hf(x) \geq \alpha$ .)

**density of a mble set at a point:**  $D_E(x) = 1$  for a.e.  $x \in E$ ,  $0$  for a.e.  $x \in E^c$

For a Lebesgue measurable subset  $E$  of  $\mathbb{R}^n$ , the *density of  $E$  at  $x$*  is defined as

$$D_E(x) = \lim_{r \rightarrow 0} \frac{m(E \cap B(x, r))}{m(B(x, r))}$$

provided that the limit exists. Prove that  $D_E(x) = 1$  for a.e.  $x \in E$  and  $D_E(x) = 0$  for a.e.  $x \in E^c$ . Hint: ask Lebesgue.

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*Some of the following questions will be graded. Do them, and do hand them in.*

**An identity:**  $\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \frac{1}{4} \log(1 + s^{-2})$

Prove that  $\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \frac{1}{4} \log(1 + s^{-2})$  for  $s > 0$  by integrating the function  $e^{-2sx} \sin(2xy)$  with respect to  $x$  and  $y$  over suitable regions.

**Proof** For fixed  $x > 0$ , by FTC we have:

$$\sin^2(x) = \int_0^x \sin(2t) dt$$

We do change of variable  $t = xy$ . This is a valid diffeomorphism mapping  $y \in (0, 1)$  to  $t \in (0, x)$ .

Then by change of variable theorem we have:

$$\int_{(0,x)} \sin(2t) dt = \int_{(0,1)} x \sin(2xy) dy$$

Thus

$$\frac{\sin^2 x}{x} = \int_0^1 \sin(2xy) dy$$

Then we get:

$$\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \int_0^\infty e^{-2sx} \left[ \int_0^1 \sin(2xy) dy \right] dx$$

Consider the function

$$f(x, y) := e^{-2sx} \sin(2xy), \quad (x, y) \in (0, \infty) \times (0, 1)$$

$f$  is a composition of continuous functions, thus continuous. Note that it is also in  $L^1((0, \infty) \times (0, 1))$  since  $|f(x, y)|$  is bounded by  $g(x, y) := e^{-2sx}$ , which is  $L^1$  on the same domain (its integral is  $\frac{1}{2s}$ ), then by DCT,  $f \in L^1((0, \infty) \times (0, 1))$ .

Thus we can apply Fubini's theorem to switch the order of integration:

$$\int_0^\infty e^{-2sx} \left[ \int_0^1 \sin(2xy) dy \right] dx = \int_{(0,\infty) \times (0,1)} e^{-2sx} \sin(2xy) d(x \times y) \quad (1.1)$$

$$= \int_0^1 \left( \int_0^\infty e^{-2sx} \sin(2xy) dx \right) dy \quad (1.2)$$

Recall back in Calculus we use integration by part to get:

$$\int_0^\infty e^{-ax} \sin(bx) dx = \frac{b}{a^2 + b^2}$$

for  $a > 0$ . In our case,  $a = 2s$  and  $b = 2y$ . Thus

$$\int_0^\infty e^{-2sx} \sin(2xy) dx = \frac{2y}{(2s)^2 + (2y)^2} = \frac{y}{2(s^2 + y^2)}$$

Therefore we here get

$$\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \int_0^1 \left( \int_0^\infty e^{-2sx} \sin(2xy) dx \right) dy \quad (1.3)$$

$$= \int_0^1 \frac{y}{2(s^2 + y^2)} dy \quad (1.4)$$

$$= \frac{1}{2} \int_0^1 \frac{y}{s^2 + y^2} dy \quad (1.5)$$

By Calculus we have (by chain rule):

$$\int_0^1 \frac{y}{s^2 + y^2} dy = \left[ \frac{1}{2} \log(s^2 + y^2) \right]_0^1 = \frac{1}{2} \log\left(\frac{s^2 + 1}{s^2}\right) = \frac{1}{2} \log\left(1 + \frac{1}{s^2}\right)$$

Thus we conclude:

$$\int_0^\infty e^{-2sx} \frac{\sin^2 x}{x} dx = \frac{1}{2} \int_0^1 \frac{y}{s^2 + y^2} dy \quad (1.6)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \log\left(1 + \frac{1}{s^2}\right) \quad (1.7)$$

$$= \frac{1}{4} \log\left(1 + \frac{1}{s^2}\right) \quad (1.8)$$

as desired.

$E \in \mathcal{A} \otimes \mathcal{A} \implies \text{diagonal of } E \in \mathcal{A}$

(a) Prove that if  $E \in \mathcal{A} \otimes \mathcal{A}$ , then

$$\{x \in X : (x, x) \in E\} \in \mathcal{A}$$

(b) Using this fact, find an example of a subset  $E \subset \mathbb{R} \times \mathbb{R}$  such that  $E_x \in \mathcal{L}(\mathbb{R})$  for all  $x \in \mathbb{R}$  and  $E^y \in \mathcal{L}(\mathbb{R})$  for all  $y \in \mathbb{R}$ , but  $E \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ . Hint: ask Vitali.

**Proof of (a):**

We consider the map:

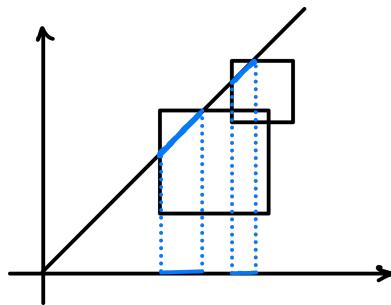
$$\phi : X \rightarrow X \times X \quad (1.9)$$

$$x \mapsto (x, x) \quad (1.10)$$

Then it suffices to show that  $\phi$  is  $(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ -measurable. Since if so, then for each  $E \in \mathcal{A} \otimes \mathcal{A}$ ,  $\phi^{-1}(E) = \{x \in X : (x, x) \in E\} \in \mathcal{A}$ , which is exactly what we want.

Let  $A \times B \in \mathcal{A} \otimes \mathcal{A}$  be a measurable rectangle, we discover that:

$$\phi^{-1}(A \times B) = \{x \in X : x \in A, x \in B\} = A \cap B \in \mathcal{A}$$



We first prove a lemma:

### Lemma 1.1

Suppose  $f : X \rightarrow Y \times Z$  is a function from a measurable space  $(X, \mathcal{A})$  to a product measure space  $(Y \times Z, \mathcal{B}_1 \otimes \mathcal{B}_2)$ .

Claim: If  $f^{-1}(B_1 \times B_2) \in \mathcal{A}$  for each measurable rectangle  $B_1 \times B_2 \in \mathcal{B}_1 \otimes \mathcal{B}_2$ , then  $f$  is an  $(\mathcal{A}, \mathcal{B}_1 \otimes \mathcal{B}_2)$ -measurable function.



#### Proof of Lemma:

Since  $f^{-1}(B \times C) \in \mathcal{A}$  for each measurable rectangle  $B_1 \times B_2 \in \mathcal{B}_1 \otimes \mathcal{B}_2$ , the preimage of any countable disjoint unions of measurable rectangles, is also in  $\mathcal{A}$ , since  $\mathcal{A}$  is an  $\sigma$ -algebra.

We want to show:  $f^{-1}(E) \in \mathcal{A}$  for any  $E \in \mathcal{B}_1 \otimes \mathcal{B}_2$ . It is equivalent to show that

$$\mathcal{B}_1 \otimes \mathcal{B}_2 \subset \mathcal{C} := \{E \in Y \times Z : \phi^{-1}(E) \in \mathcal{A}\}$$

Note that, it suffices to show that:  $\mathcal{C}$  is an  $\sigma$ -algebra. This is because we have shown

$$\{\text{all disjoint unions of measurable rectangles in } Y \times Z\} \subset \mathcal{C}$$

, and this is an algebra generating  $\mathcal{B}_1 \otimes \mathcal{B}_2$ . Thus, if  $\mathcal{C}$  is an  $\sigma$ -algebra, we must have  $\mathcal{B}_1 \otimes \mathcal{B}_2 \subset \mathcal{C}$ .

And since  $\{\text{all disjoint unions of measurable rectangles in } Y \times Z\}$  is an algebra, it suffices to show that  $\mathcal{C}$  is a monotone class, by the monotone class lemma.

Suppose  $E_1 \subseteq E_2 \subseteq \dots$  with each  $E_n \in \mathcal{C}$ , i.e.  $\phi^{-1}(E_n) \in \mathcal{A}$ . Since  $\{E_n\}$  is increasing, we have

$$\phi^{-1}(E_1) \subseteq \phi^{-1}(E_2) \subseteq \dots \subseteq \phi^{-1}(E_n) \subseteq \dots$$

Since  $\mathcal{A}$  is an  $\sigma$ -algebra, we have

$$\phi^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} \phi^{-1}(E_n) \in \mathcal{A}$$

Thus

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$$

This is dually true for decreasing intersection, **finishing the proof that  $\mathcal{C}$  is a monotone class thus  $\sigma$ -algebra, thus proving the lemma.**

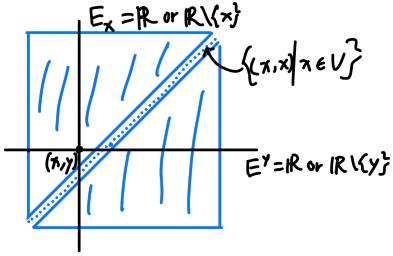
After we proved the Lemma, we return to the original statement, concluding that  $\phi$  is  $(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ -measurable, thus finishing the proof: if  $E \in \mathcal{A} \otimes \mathcal{A}$ , then

$$\{x \in X : (x, x) \in E\} \in \mathcal{A}$$

#### Sol. of (b):

Take a Vitali set  $V \subset \mathbb{R}$ , and consider:

$$E := \{(x, y) \in \mathbb{R}^2 : x \neq y\} \cup \{(x, x) : x \in V\}.$$



Then for any fixed  $x \in \mathbb{R}$ , we have:

$$E_x = \{y : (x, y) \in E\} = \begin{cases} \mathbb{R}, & x \in V \\ \mathbb{R} \setminus \{x\}, & x \notin V \end{cases}$$

And for any fixed  $y \in \mathbb{R}$ , we have:

$$E^y = \{x : (x, y) \in E\} = \begin{cases} \mathbb{R}, & y \in V \\ \mathbb{R} \setminus \{y\}, & y \notin V \end{cases}$$

Thus  $E_x \in \mathcal{L}(\mathbb{R})$  for all  $x \in \mathbb{R}$  and  $E^y \in \mathcal{L}(\mathbb{R})$  for all  $y \in \mathbb{R}$ .

However, we have  $E \notin \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ , since by (a) we have proved that if  $E \in \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$ , then

$$V = \{x \in \mathbb{R} : (x, x) \in E\} \in \mathcal{L}(\mathbb{R})$$

But it contradicts with the fact that  $V$  is not Lebesgue measurable.

Thus  $E$  satisfies our requirements.

(This happens since, as shown in class, the product measure space of two complete measure space is not necessarily complete. Here, the diagonal is a null set in  $\mathbb{R}^2$  and thus our Vitali portion is a subnull set, but  $\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R})$  is not complete (its completion is  $\mathcal{L}(\mathbb{R}^2)$ ).

**Too dense:**  $m(E \cap I) \leq \alpha m(I)$  for all  $I \implies m(E) = 0$  for mble  $E$

Prove that if  $E \subset \mathcal{L}(\mathbb{R})$  is a Lebesgue measurable subset such that

$$m(E \cap I) \leq 0.123m(I)$$

for all open intervals  $I \subset \mathcal{L}(\mathbb{R})$ , then  $m(E) = 0$ .

**Proof** Since  $E$  is Lebesgue measurable,  $m(E) = m^*(E)$ .

Let  $\epsilon > 0$ .

Then by definition of outer measure, we can pick open intervals seq  $\{I_k\}_{k=1}^\infty$  covering  $E$  s.t.

$$m(E) > \sum_{k=1}^{\infty} m(I_k) - \epsilon$$

Since  $E \subset \bigcup_k I_k$ , we have

$$E = (\bigcup_k I_k) \cap E \quad (1.11)$$

$$= \bigcup_k (I_k \cap E) \quad (1.12)$$

$$(1.13)$$

Thus

$$m(E) = m\left(\bigcup_k (I_k \cap E)\right) \leq \sum_k m(I_k \cap E) \quad \text{by ctbl subadditivity} \quad (1.14)$$

$$\leq 0.123 \sum_k m(I_k) \quad \text{by our requirement} \quad (1.15)$$

Thus we have:

$$\sum_k m(I_k) - \epsilon < 0.123 \sum_k m(I_k) \quad (1.16)$$

$$0.877 \sum_k m(I_k) < \epsilon \quad (1.17)$$

$$\sum_k m(I_k) < \frac{\epsilon}{0.877} \quad (1.18)$$

Thus

$$m(E) \leq \sum_k m(I_k) < \frac{\epsilon}{0.877}$$

Since  $\epsilon > 0$  is arbitrary, this proves that

$$m(E) = 0$$

## 给定任意 $0 < \alpha < 1$ , prescribe 出一个在 0 处 density 为 $\alpha/2$ 的集合

Let  $0 < \alpha < 1$ . Find an example of a Lebesgue measurable subset  $E$  of  $[0, \infty) \subset \mathcal{L}(\mathbb{R})$  whose density at 0 is  $\alpha/2$ . Hint: Consider  $E = \bigcup_{n=1}^{\infty} I_n$ . where  $I_n = (x_n, x_n + \delta_n)$  are disjoint small intervals accumulating at 0.

**Proof** Consider take

$$E := \bigcup_{n=1}^{\infty} \left( \frac{1}{n}, \frac{1}{n} + \frac{\alpha}{n(n-1)} \right)$$

as the union of a countable sequence of intervals drawing near 0.

Notice: There intervals are **mutually disjoint**, since

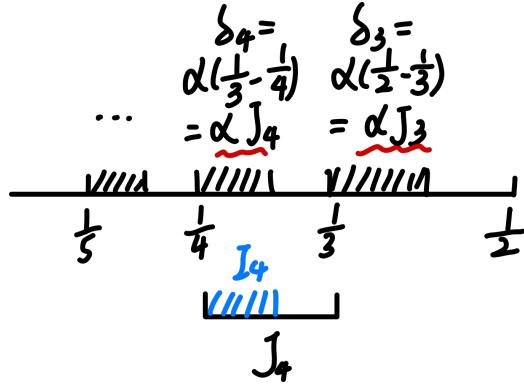
$$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > \frac{\alpha}{n(n-1)}$$

we thus have for  $n \geq 2$ ,

$$\frac{1}{n} + \frac{\alpha}{n(n-1)} < \frac{1}{n-1}$$

We use  $x_n := \frac{1}{n}$ ;  $I_n := (x_n, x_n + \delta_n)$  to denote each component interval;  $J_n := (x_n, x_{n-1})$  to denote the open interval where  $I_n$  is located at; and  $\delta_n := \frac{\alpha}{n(n-1)}$  to denote the length of each interval. Note that for each  $n$ ,

$$\delta_n = \alpha \left( \frac{1}{n-1} - \frac{1}{n} \right) = \alpha(x_{n-1} - x_n) = \alpha J_n$$



Now we show that this set has Lebesgue density  $\frac{\alpha}{2}$  at 0 below.

Let  $r > 0$  (WLOG  $r < 1$ ), then we have

$$\frac{1}{n+1} < r \leq \frac{1}{n} \quad \text{for some } n \in \mathbb{N}$$

Then for each  $k \geq n+2$ , we have  $\frac{1}{k} < \frac{1}{n+1} < r$ . Hence  $I_k$  is **entirely contained** in  $(0, r)$ :

$$\bigcup_{k=n+2}^{\infty} I_k \subseteq E \cap (-r, r)$$

We know that by telescoping,

$$\sum_{k=n+2}^{\infty} \frac{1}{k(k-1)} = \left( \frac{1}{n+1} - \frac{1}{n+2} \right) + \left( \frac{1}{n+2} - \frac{1}{n+3} \right) + \dots = \frac{1}{n+1}$$

Multiplying this by  $\frac{\alpha}{2}$  gives:

$$\sum_{k=n+2}^{\infty} \frac{\alpha}{k(k-1)} = \frac{\alpha}{n+1}$$

Thus by monotonicity of measure:

$$m(E \cap (-r, r)) \geq \frac{\alpha}{n+1}$$

And for each  $k \leq n$ ,  $I_k$  exceeds  $(0, r)$  on the right, thus we get dually:

$$m(E \cap (-r, r)) \leq \frac{\alpha}{n-1}$$

And we have:

$$\frac{2}{n+1} \leq m(-r, r) \leq \frac{2}{n}$$

since  $\frac{1}{n+1} \leq r \leq \frac{1}{n}$ .

Therefore we get:

$$\frac{\frac{\alpha}{n+1}}{\frac{2}{n}} \leq \frac{m(E \cap (-r, r))}{m((-r, r))} \leq \frac{\frac{\alpha}{n-1}}{\frac{2}{n+1}}$$

Further simplify:

$$\frac{n}{n+1} \cdot \frac{\alpha}{2} \leq \frac{m(E \cap (-r, r))}{m((-r, r))} \leq \frac{n+1}{n-1} \cdot \frac{\alpha}{2}$$

As  $r \rightarrow 0^+$ , we must have  $n \rightarrow \infty$ , and we know

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{\alpha}{2} = \lim_{n \rightarrow \infty} \frac{n+1}{n-1} \cdot \frac{\alpha}{2} = \frac{\alpha}{2}$$

Thus by **Squeeze Theorem**, we have:

$$\lim_{r \rightarrow 0^+} \frac{m(E \cap (-r, r))}{m((-r, r))} = \frac{\alpha}{2}$$

Hence by def,  $E$  indeed has Lebesgue density  $\alpha/2$  at 0.

(My note: The key point here is that, the harmonic seq shrinks very slowly in proportion as  $n$  grows,  $J_n$  almost have same length as  $J_{n+1}$  for large  $n$ , thus  $m(J_n)/m(\cup_{k>N} J_k) = 0$  as we knows, so that whether  $r$  lies in  $I_n$  or  $J_n \setminus I_n$  does not quite matter.)

On the other hand, the counterexample in class, using the geometric sequence as build block of  $J_n$ , fails since the length of  $J_n$  is too much compared to  $\cup_{k \geq n} J_k$ , actually  $m(J_n) = m(\cup_{k > n} J_k)$ , thus whether  $r$  lies in  $I_n$  or  $J_n \setminus I_n$  makes a lot difference, making the density at 0 undefined.)

**Sefs of complex numbers:**  $\ell^1 \subsetneq \bigcap_{1 < p < \infty} \ell^p$  and  $\bigcup_{1 < p < \infty} \ell^p \subsetneq \ell^\infty$

- (a) Prove that  $\ell^1 \subsetneq \bigcap_{1 < p < \infty} \ell^p$ .
- (b) Prove that  $\bigcup_{1 < p < \infty} \ell^p \subsetneq \ell^\infty$ .

**Proof of (a):**

We first want to show: for any  $1 < p < \infty$ , we have:

$$\ell^1 \subseteq \ell^p$$

Fix  $p > 1$ .

Let  $(x_n) \in \ell^1$ . By definition,

$$\sum_{n=1}^{\infty} |x_n| < \infty$$

We need to show that  $\sum_{n=1}^{\infty} |x_n|^p < \infty$ .

**Claim: There are at most finitely many  $n \in \mathbb{N}$  s.t.  $|x_n| \geq 1$ .**

Proof of Claim: Suppose for contradiction that there are infinitely many  $n \in \mathbb{N}$  s.t.  $|x_n| \geq 1$ , say, all terms in the subseqence  $\{x_{n_j}\}_{j=1}^{\infty}$  has  $|x_{n_j}| \geq 1$ . Then

$$\sum_{n=1}^{\infty} |x_n| \geq \sum_{j=1}^{\infty} |x_{n_j}| \geq \sum_{j=1}^{\infty} 1 = \infty$$

which contradicts with  $(x_n) \in \ell^1$ .

Thus, suppose only on the finite terms  $\{x_{n_j}\}_{j=1}^N$  we have  $|x_{n_j}| \geq 1$  (WLOG  $N \geq 1$ ). Then

$$\sum_{n=1}^{\infty} |x_n| = \sum_{j=1}^N |x_{n_j}| + \sum_{n \neq n_j \text{ for any } j} |x_n|$$

Since for  $n$  s.t.  $n \neq n_j$  for any subseq index  $j$ , we have  $|x_n| < 1$ , for these indexes we have:

$$|x_n|^p < |x_n| \quad \text{for any } p > 1$$

Thus we have

$$\sum_{n \neq n_j \text{ for any } j} |x_n|^p < \sum_{n \neq n_j \text{ for any } j} |x_n| < \infty$$

And also,

$$\sum_{j=1}^N |x_{n_j}|^p < \infty \quad \text{since only have finite terms}$$

Thus

$$\sum_{n=1}^{\infty} |x_n|^p = \sum_{j=1}^N |x_{n_j}|^p + \sum_{n \neq n_j \text{ for any } j} |x_n|^p < \infty$$

Thus

$$\ell^1 \subseteq \ell^p$$

Since  $p > 1$  is arbitrary, this proves that

$$\ell^1 \subseteq \bigcap_{1 < p < \infty} \ell^p$$

To show the strictness of the inclusion, we consider the **harmonic series**  $\sum_{n=1}^{\infty} \frac{1}{n}$ . We know that it diverges and for any  $p > 1$ , the  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  (absolutely for sure) converges, thus  $(\frac{1}{n}) \notin \ell^1$  but  $(\frac{1}{n}) \in \ell^p$  for every  $p > 1$ , showing that

$$\ell^1 \neq \bigcap_{1 < p < \infty} \ell^p$$

This finishes the proof that

$$\ell^1 \subsetneq \bigcap_{1 < p < \infty} \ell^p$$

### Proof of (b):

Fix  $p > 1$ .

Suppose sequence  $(x_n)$  belongs  $\ell^p$ , then

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

This implies that  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ , because if it did not, there would be infinitely many terms where  $|x_n|$  is bounded away from zero, leading to divergence of the sum.

Suppose for contradiction that

$$\sup_n |x_n| = \infty$$

Then there are infinitely many terms  $n$  s.t.  $|x_n| > 1$ , since otherwise, exists some  $N$  s.t. all  $|x_n| \leq 1$  for  $n \geq N$ , then  $\sup_n |x_n| \leq \max(1, \max_{1 \leq n \leq N-1} |x_n|) < \infty$ .

Suppose for the subseq  $\{x_{n_j}\}_{j=1}^{\infty}$  we have  $|x_{n_j}| > 1$ . Thus

$$\sum_{n=1}^{\infty} |x_n|^p \geq \sum_{j=1}^{\infty} |x_{n_j}|^p > \sum_{j=1}^{\infty} 1^p = \infty$$

which contradicts with  $\sum_{n=1}^{\infty} |x_n|^p < \infty$ . Therefore we have:

$$\sup_n |x_n| < \infty$$

This shows that

$$\ell^p \subseteq \ell^{\infty}$$

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Since  $p > 1$  is arbitrary, this proves that

$$\bigcup_{1 < p < \infty} \ell^p \subseteq \ell^\infty$$

Now we show the inclusion is strict. Consider the sequence  $x_n = 1$  for all  $n$ . Clearly,  $(x_n) \in \ell^\infty$  because it is bounded. However,  $x_n \notin \ell^p$  for any  $p > 1$ :

$$\sum_{n=1}^{\infty} |1|^p = \sum_{n=1}^{\infty} 1 = \infty$$

This shows

$$\bigcup_{1 < p < \infty} \ell^p \neq \ell^\infty$$

Thus we have

$$\bigcup_{1 < p < \infty} \ell^p \subsetneq \ell^\infty$$

*Nur für Verrückte*

(It's **really** not necessary to attempt these problems. Do not, under any circumstances, hand them in!)

## Prescribing a Lebesgue density, Season 2

Let  $0 < \alpha < 1$  and  $n \geq 1$ . Find an example of a Lebesgue measurable subset  $E$  of  $\mathcal{L}(\mathbb{R})^n$  whose density at 0 is  $\alpha$ . *Hint:* think spherically.