

Homework 2: on Carathéodory's and Hahn-Holmogrov Thm(40/40)

None of the following questions will be graded. Do them, but do not hand them in.

The Borel–Cantelli Lemma

Let (X, \mathcal{A}, μ) be a measure space. Let $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$, and suppose that

$$\sum_{i=1}^{\infty} \mu(A_i) < \infty.$$

(a) Prove that $\mu(\limsup_i A_i) = 0$, where

$$\limsup_i A_i = \{x \in X \mid x \in A_i \text{ for infinitely many } i\}.$$

(By the way, why is $\limsup_i A_i$ measurable?)

(b) Conversely, is it true that if $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$, and $\mu(\limsup_i A_i) = 0$, then $\sum_i \mu(A_i) < \infty$? Provide a proof or a counterexample. (Wrong) **Remark**

Theorem 2.1 (Borel–Cantelli Lemma)

一个 measure 和有限的 set seq, 其 \limsup (出现 infinitely many times 的元素) 是零测的.



其实这是 trivial 的, 因为如果出现 infinitely many times 的元素不是零测的, say $\mu(\limsup_i A_i) := k > 0$, 那么有 infinitely many 个 A_i 的测度大于等于 k , 那么 $\sum_{i=1}^{\infty} \mu(A_i) > k \times \infty$ 就一定不是有限的了.
其 application: 一个 prob space 中, a ctbl seq of 事件发生的概率的和如果收敛, 那么它们包含的任何事件发生无穷多次的概率为 0, 意味着事件至多发生有限次 (almost surely).

The Completion of a Measure Space

Let (X, \mathcal{A}, μ) be a measure space, and set

$$\overline{\mathcal{A}} := \{E \cup F \mid E \in \mathcal{A} \text{ and } F \text{ is a } \mu\text{-subnull set}\}.$$

(a) Prove that $\overline{\mathcal{A}}$ is a σ -algebra. (b) Define $\overline{\mu}(A) := \mu(E)$ if $A = E \cup F \in \overline{\mathcal{A}}$. Prove that $\overline{\mu}$ is a well-defined measure on $\overline{\mathcal{A}}$. (c) Prove that $\overline{\mu}$ extends μ (i.e., $\overline{\mu}(A) = \mu(A)$ if $A \in \mathcal{A}$). (d) Prove that $\overline{\mu}$ is the unique extension of μ to $(X, \overline{\mathcal{A}})$. In other words, prove that if μ' is another measure on $(X, \overline{\mathcal{A}})$ that extends μ , then $\mu' = \overline{\mu}$. (e) Prove that $\overline{\mu}$ is complete. (f) Suppose (X, \mathcal{A}', μ') is another complete measure space that extends (X, \mathcal{A}, μ) (i.e., $\mathcal{A} \subset \mathcal{A}'$ and $\mu'|_{\mathcal{A}} = \mu$). Show that $\overline{\mathcal{A}} \subset \mathcal{A}'$ and $\mu'|_{\overline{\mathcal{A}}} = \overline{\mu}$. **Hint:** Start by reading Theorem 1.9 in Folland.

Proof 略.(嘻嘻)

The Hahn–Kolmogorov Extension as a Completion

Let $(X, \mathcal{A}_0, \mu_0)$ be a σ -finite measure pre-measure space, and (X, \mathcal{A}, μ) its Hahn – Kolmogorov extension. Prove that (X, \mathcal{A}, μ) is the completion of its restriction to the σ -algebra $\langle \mathcal{A}_0 \rangle$ generated by \mathcal{A}_0 .

Proof Proved in lec notes.

Some of the following questions will be graded. Do them, and do hand them in.

$\mu(\emptyset) = 0$ 的定义并非 redundant

Let (X, \mathcal{A}) be a measurable space. Is the condition $\mu(\emptyset) = 0$ in the definition of a measure on (X, \mathcal{A}) redundant? In other words, if $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a function such that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i),$$

for any disjoint subsets $A_i \in \mathcal{A}$, $i \in \mathbb{N}$, does it follow that $\mu(\emptyset) = 0$? If not, what can you say?

Proof It does not follow.

Counterexample: Consider $\mu(E) = \infty \quad \forall E \in \mathcal{A}$.

This measure satisfies the countably disjoint additivity condition, since for every disjoint sequence of sets in \mathcal{A} , $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \infty$ has infinite measure.

measurable set seq 的 limit 也 measurable (且如果 seq tail σ -finite \Rightarrow limit commute)

Let (X, \mathcal{A}, μ) be a measure space, and let $A_i \in \mathcal{A}$, $i \in \mathbb{N}$. Assume that the sets A_i converge to the set $A \subset X$ in the sense that: - If $x \in A$, then $x \in A_i$ for all but finitely many i ; - If $x \notin A$, then $x \notin A_i$ for all but finitely many i .

(a) Prove that A is measurable, that is, $A \in \mathcal{A}$.

Proof Deduing from the conditions: If $x \in A$, then $x \in A_i$ for all but finitely many i ; $\Rightarrow A \subset \liminf A_i$. If $x \notin A$, then $x \notin A_i$ for all but finitely many i . \Rightarrow if $x \in A_i$ for all but finitely many i then $x \in A$ $\Rightarrow \limsup A_i \subset A$ Thus

$$\limsup A_i \subset A \subset \liminf A_i \tag{2.1}$$

Claim1: For any sequence of sets $(A_i)_{i \in \mathbb{N}}$, we have

$$\liminf A_i \subset \limsup A_i$$

Proof of Claim 1: Follows trivially from the definition, since $x \notin A_i$ for all but finitely many $i \Rightarrow x \notin A_i$ for infinitely many i .

Combining claim (1) with (2.1) we have

$$\limsup A_i = A = \liminf A_i \tag{2.2}$$

Claim 2: For any sequence of sets $(A_i)_{i \in \mathbb{N}}$ in a σ -algebra, $\liminf_i A_i$ and $\limsup_i A_i$ is also in the σ -algebra.

Proof of Claim 2: This follows from the def and fact that union and intersection of a countable sequence sets in a σ -algebra is also in this σ -algebra. We have

Define for each $k \in \mathbb{N}$ $B_k := \bigcup_{i=k}^{\infty} A_i$, $\in \mathcal{A}$ since σ -algebra is closed under countable union
 $\Rightarrow \limsup A_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i = \bigcap_{k=1}^{\infty} B_k \in \mathcal{A}$ since σ -algebra is closed under countable intersection
Similarly, $\liminf A_i = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} A_i \in \mathcal{A}$

This finishes the proof of Claim 2.

Combining claim 2 with (2.2), $A \in \mathcal{A}$, this finishes the proof.

(b) Prove that if there exists $n \geq 1$ such that $\mu(\bigcup_{i=n}^{\infty} A_i) < \infty$, then $\mu(A) = \lim_i \mu(A_i)$.

Proof

$$\begin{aligned} A &= \limsup A_i = \liminf A_i \\ &= \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i = \bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} A_i \end{aligned}$$

Define for each $m \in \mathbb{N}$ $B_m := \bigcup_{i=m}^{\infty} A_i$, $C_m := \bigcap_{i=m}^{\infty} A_i$

$$\Rightarrow A = \bigcap_{m=1}^{\infty} B_m = \bigcup_{m=1}^{\infty} C_m$$

and $B_{m+1} \subseteq B_m$, $C_m \subseteq C_{m+1} \quad \forall m \in \mathbb{N}$

By ctn from below/above property of measure

$$\begin{aligned} \Rightarrow \mu(A) &= \mu\left(\bigcap_{m=1}^{\infty} B_m\right) = \mu\left(\bigcup_{m=1}^{\infty} C_m\right) \\ &= \lim_{m \rightarrow \infty} \mu(B_m) = \lim_{m \rightarrow \infty} \mu(C_m) \quad (\text{equation 2.3}) \end{aligned}$$

Fix $m \in \mathbb{N}$, we have $\mu(B_m) = \mu(\bigcup_{i=m}^{\infty} A_i) \geq \mu(A_n)$ for any $n \geq m$

$$\text{so } \mu(B_m) \geq \sup_{n \geq m} \mu(A_n)$$

$$\text{Thus } \mu(A) = \lim_{m \rightarrow \infty} \mu(B_m) \geq \limsup \mu(A_i)$$

For the same reason we have $\mu(A) = \lim_{m \rightarrow \infty} \mu(C_m) \leq \liminf \mu(A_i)$

Thus we have

$$\limsup \mu(A_i) \leq \mu(A) \leq \liminf \mu(A_i) \quad (\text{equation 2.4})$$

Since for the numerical seq, $(\mu(A_i))_{i \in \mathbb{N}}$, must have

$$\text{Therefore } \liminf(A_i) \leq \limsup(A_i)$$

$$\text{Underline } \limsup \mu(A_i) = \mu(A) = \liminf \mu(A_i)$$

Since \liminf, \limsup both exist and equal, this finishes the proof

$$\text{that } \lim_{i \rightarrow \infty} \mu(A_i) = \mu(A)$$

□

(c) Give an example showing that the condition in (b) is necessary.

Sol. Let μ be the Lebesgue measure defined on $\mathcal{B}(\mathbb{R})$. We set for each $i \in \mathbb{N}$ that

$$A_i = (i, i+1)$$

Since this is an interval, it is Lebesgue measurable. Note that no element of any A_i show up infinitely many times in the sequence. So

$$\liminf A_i = \limsup A_i = \emptyset$$

So $A = \emptyset$, we have $\lim_i \mu(A_i) = 0$. But we have $\lim_i \mu(A_i) = 1$ since it is true for every i .

In this case, $\mu(\bigcup_{i=1}^{\infty} A_i) = \infty$, which causes (b) to fail.

Hint: In analysis, it is often fruitful to use \limsup and \liminf to study limits.

measure space of two elements

Let X be a set with two elements, for example, $X = \{O, Q\}$.

(a) Find all σ -algebras on X .

Sol.

1. trivial σ -algebra:

$$\mathcal{A}_1 := \{\emptyset, X\}$$

2. power set:

$$\mathcal{A}_2 := \mathcal{P}(X) = \{\emptyset, \{O\}, \{Q\}, X\}$$

These are the only σ -algebras on X .

(b) Let \mathcal{A} be a σ -algebra on X , and μ a measure on (X, \mathcal{A}) . Is μ necessarily complete? Provide a proof or a counterexample.

Sol. It is not necessarily complete.

Consider the trivial σ -algebra: $\mathcal{A}_1 := \{\emptyset, X\}$, and set μ as that $\mu(\emptyset) = \mu(X) = 0$. This makes X a null set, so $\{O\}, \{Q\}$ are subnull sets, but they are not measurable by μ .

(c) Find all outer measures μ^* on X . For each outer measure on X , find the σ -algebra of μ^* -measurable sets (see Carathéodory's theorem).

Sol. Suppose μ^* is an outer measure on X . Since $\mathcal{P}(X)$ only has four elements: $\emptyset, \{O\}, \{Q\}, X$; and the outer measure of \emptyset is 0, so we first parametrize μ^* by:

$$a := \mu^*(\{O\}), \quad b := \mu^*(\{Q\}), \quad c := \mu^*(X).$$

Then μ^* is well-defined iff it satisfies:

1. $a, b \leq c$
2. $c = \mu^*(\{O\} \cup \{Q\}) \leq \mu^*(\{O\}) + \mu^*(\{Q\}) = a + b$.

Any $(a, b, c) \in [0, \infty]^3$ satisfying

$$\max(a, b) \leq c \leq a + b,$$

can make μ^* a well-defined outer measure on X .

Therefore

$$S := \{\text{all } \sigma\text{-algebra on } X\} = \{\mu^* : \mathcal{P}(X) \rightarrow [0, \infty] \mid \max(\mu^*(\{O\}), \mu^*(\{Q\})) \leq \mu^*(X) \leq \mu^*(\{O\}) + \mu^*(\{Q\})\}$$

Now we specify the σ -algebra of μ^* -measurable sets for each $\mu^* \in S$.

By Carathéodory's criterion, a set $E \subset X$ is μ^* -measurable iff for all $A \subset X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Note that \emptyset, X are always measurable since for any $A \subset X$, $A \cap \emptyset = \emptyset$, $A \cap (\emptyset)^c = A$; and $A \cap X = A$, $A \cap (X)^c = \emptyset$. So it suffices to check for $\{O\}, \{Q\}$. We first check for $\{O\}$. $\{O\}$ is μ^* -measurable iff $\mu^*(A) = \mu^*(A \cap \{O\}) + \mu^*(A \cap \{O\}^c)$ for any choice of A . There are only four possibilities for A : $\emptyset, \{O\}, \{Q\}, X$.

1. If $A = \emptyset$, both sides are 0, always stands.
2. If $A = \{O\}$, then $\mu^*(\{O\}) + \mu^*(\emptyset) = a + 0 = a$, always stands.
3. If $A = \{Q\}$, then $\mu^*(\emptyset) + \mu^*(\{Q\}) = 0 + b = b$, always stands.
4. If $A = X$, then $\mu^*(X) = c = \mu^*(\{O\}) + \mu^*(\{Q\}) = a + b$.

Therefore $\{O\}$ is μ^* -measurable iff $c = a + b$. For the same reasoning, $\{Q\}$ is μ^* -measurable iff $c = a + b$.

Thus we can conclude that:

1. If $c = a + b$, $\{\mu^*\text{-measurable sets}\} = \mathcal{P}(X)$.
2. otherwise, $\{\mu^*\text{-measurable sets}\} = \{\emptyset, X\}$.

(d) Find an example of a collection \mathcal{E} of subsets of X with $\emptyset, X \in \mathcal{E}$ and a function $\rho : \mathcal{E} \rightarrow [0, \infty]$ with $\rho(\emptyset) = 0$ such that $\mathcal{E} \not\subseteq \mathcal{A}$, where \mathcal{A} is the Carathéodory σ -algebra for the outer measure μ^* induced by

(\mathcal{E}, ρ) .

Sol. Consider $\mathcal{E} = \{\emptyset, X, \{O\}\}$, with ρ such that $\rho(\emptyset) = 0$, $\rho(X) = 1$, $\rho(\{O\}) = 1$.

The outer measure μ^* induced by μ^* is: $\mu^*(X) = 1$, $\mu^*(\{O\}) = 1$, $\mu^*(\{Q\}) = 1$. (the inf of length sum of sets covering $\{Q\}$ is 1, by taking $\{X\}$ as the covering.)

Since $c \neq a + b$, by (4), the Carathéodory σ -algebra by μ^* by \mathcal{E} is $\{\emptyset, X\}$, so $\mathcal{E} \not\subset \mathcal{A}$.

Remark: The Hahn – Kolmogorov theorem states that if $\mathcal{E} = \mathcal{A}_0$ is an algebra and $\rho = \mu_0$ is a pre-measure, then $\mathcal{A}_0 \subset \mathcal{A}$. This exercise provides a counterexample when \mathcal{E} and ρ are general. .

Hahn – Kolmogorov Collapse (when μ_0 not σ -finite)

Let $X \subset \mathbb{R}$ be the set of dyadic rational numbers, that is, the set of numbers of the form $\frac{r}{2^n}$, where r and n are integers. Let $\mathcal{A}_0 \subset \mathcal{P}(X)$ be the collection of finite unions of intervals of the form $(a, b] \cap X$, where $-\infty \leq a < b \leq \infty$.

(a) Prove that \mathcal{A}_0 is an algebra.

Proof

1. $\emptyset \in \mathcal{A}_0$, since it is the empty union of intervals of the given form.

2. **Closed under complements:** Let $A \in \mathcal{A}_0$. Then A is a finite union of intervals of the form $(a_i, b_i] \cap X$. So

$$A^c \cap X = X \setminus A \quad (2.3)$$

$$= X \setminus \bigcup_{i=1}^n ((a_i, b_i] \cap X) \quad (2.4)$$

$$= X \cap \left(\bigcap_{i=1}^n ((a_i, b_i] \cap X)^c \right) \quad (2.5)$$

$$= X \cap \left(\bigcap_{i=1}^n ((-\infty, a_i] \cup (b_i, \infty)) \right) \quad (2.6)$$

Note that finite intersection of intervals of the form $(-\infty, a_i]$, $(b_i, \infty]$ is still of this form. Hence $A^c \cap X \in \mathcal{A}_0$.

3. **Closed under finite unions:** Suppose A_1 and A_2 are finite unions of intervals $((a_i, b_i] \cap X)$, then $A_1 \cup A_2$ is still a finite union of intervals of that form. (They either merge into one such interval, so are disjoint.) Hence $A_1 \cup A_2 \in \mathcal{A}_0$. The same reasoning extends to any finite union.

This finishes the proof that \mathcal{A}_0 is an algebra on X .

(b) Prove that the σ -algebra on X generated by \mathcal{A}_0 equals $\mathcal{P}(X)$.

Proof Since $\mathcal{A}_0 \subset \mathcal{P}(X)$, it suffices to show that $\mathcal{P}(X) \subset \mathcal{A}_0$. Note that X is countable, so any set in $\mathcal{P}(X)$ is a countable union of singleton sets. Thus it suffices to show that any singleton set $\{x\}$ where $x \in X$ is in \mathcal{A}_0 , since if so, then any countable union of singleton sets from $\mathcal{P}(X)$ is also in \mathcal{A}_0 , with implies

that $\mathcal{P}(X) \subset \langle \mathcal{A}_0 \rangle$

Let $x \in X$. Then we have:

$$\{x\} = \bigcap_{n=1}^{\infty} \left(\left(x - \frac{1}{2^n}, x \right] \cap X \right),$$

since x is in the RHS set, and for any $y < x$, we can find a $n \in \mathbb{N}$ such that $x - \frac{1}{2^n} > y$.

This finishes the proof that $\langle \mathcal{A}_0 \rangle = \mathcal{P}(X)$.

(c) Define $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$ **by** $\mu_0(\emptyset) = 0$ **and** $\mu_0(A) = \infty$ **for** $A \neq \emptyset$. **Prove that** μ_0 **is a pre-measure on** \mathcal{A}_0

Proof It suffices to show the countable disjoint additivity.

Let $(A_i)_{i \in \mathcal{N}}$ be a sequence of disjoint sets in \mathcal{A}_0 .

Case 1: all $A_i = \emptyset$, then $\sqcup_{i \in \mathcal{N}} A_i = \emptyset$, so $\mu_0(\sqcup_{i \in \mathcal{N}} A_i) = \sum_{i \in \mathcal{N}} \mu_0(A_i) = 0$.

Case 2: $A_k \neq \emptyset$ for some k , then $\mu_0(A_k) = \infty$ and $\sqcup_{i \in \mathcal{N}} A_i \neq \emptyset$. Thus $\sum_{i \in \mathcal{N}} \mu_0(A_i) \geq \mu_0(A_k) = \infty = \mu_0(\sqcup_{i \in \mathcal{N}} A_i)$.

The two cases cover all circumstances, finishing the proof.

(d) Prove that there exist infinitely many different measures μ on $\mathcal{P}(X)$ whose restriction to \mathcal{A}_0 equals μ_0 .

Proof Given $n \in \mathbb{N}$, We define the "n-timed counting measure" on a σ -algebra S as:

$$\mu_{count_n}(E) := \begin{cases} n \times \#(E) & , \text{ if } E \text{ is finite} \\ \infty & , \text{ if } E \text{ is infinite} \end{cases}$$

Claim 1: For any set X and any σ -algebra S on X , the "n-timed counting measure" is a well-defined measure on S , for all $n \in \mathbb{N}$.

Proof of claim 1: $\mu_{count_n}(\emptyset) = 0$ since $\text{card}(\emptyset) = 0$, and countable disjoint additivity trivially follows from the rule of counting.

Claim 2: for any $n \in \mathbb{N}$, $\mu_{count_n}(E)$ on $\mathcal{P}(X)$ restricted to \mathcal{A}_0 equals μ_0 . Proof of claim 2: Let $E \in \mathcal{A}_0 \setminus \emptyset$, then E contains at least one interval of the form $(a, b] \cap X$, where $-\infty \leq a < b \leq \infty$. Since $a < b$, there are infinitely many elements in $(a, b] \cap X$, so $\mu_{count_n}(E) = \infty$.

This finishes the proof of the original statement.

(e) Explain why (d) does not contradict the uniqueness part of the Hahn – Kolmogorov theorem (see Theorem 1.14 in Folland).

Sol. This is because Hahn – Kolmogorov theorem requires μ_0 to be σ -finite to extend uniquely on $\langle \mathcal{A}_0 \rangle$. But μ_0 here is not σ -finite.

Nur für Verrückte (Only for nuts)

(It's really not necessary to attempt these problems. Do not, under any circumstances, hand them in!)

1. Let (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) be measure spaces. Define a morphism from (X, \mathcal{A}, μ) to (Y, \mathcal{B}, ν) to be a map $f : X \rightarrow Y$ that is measurable, that is, $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, and moreover measure preserving, in the sense that $\mu(f^{-1}(B)) = \nu(B)$ for all $B \in \mathcal{B}$.

(a) Prove that measure spaces with measure-preserving maps as morphisms form a category. Denote this category by C_3 .

(b) Denote by C_1 the category of sets, and by C_2 the category of measurable spaces (see HW1). Consider the evident forgetful functors $C_3 \rightarrow C_2$ and $C_2 \rightarrow C_1$. Are these functors faithful? Are they full? Are they essentially surjective?