0.1 Use FTC and Tonelli for series

Let $g_k, k = 1, 2, \ldots$, be a sequence of functions that are absolutely continuous on the interval $[a, b]$. Suppose that there is a $c \in [a, b]$, such that the series $\sum_{k=1}^{\infty} g_k(c)$ is convergent, and

$$\sum_{k=1}^{\infty} \int_a^b |g_k'(x)| \, dx < \infty$$

(a) Show that $\sum_{k=1}^{\infty} g_k(x)$ is convergent for all $x \in [a, b]$. (b) Let $f(x) = \sum_{k=1}^{\infty} g_k(x)$. Show that f is absolutely continuous on $[a, b]$ and

$$f'(x) = \sum_{k=1}^{\infty} g_k'(x) \quad \text{for almost every } \in [a, b]$$

0.2 Use FTC and Holder

Let $f : [0, 1] \to R$ be absolutely continuous, satisfy $f(0) = 0$ and $f' \in L^2([0, 1])$. Show that

$$\lim_{x \to 0+} x^{-1/2} f(x)$$

exists and determine the value of this limit.

0.3 Use density of compactly supported continuous functions in a suitable space

Let f be a real Lebesgue measurable function on the interval $[0,1]$ such that $\|f\|_\infty < \infty$. Show that for any $\varepsilon, \delta > 0$, there is a continuous function g on $[0,1]$ such that $m\{x \in [0,1] : |f(x) - g(x)| > \varepsilon\} < \delta$.

0.4 Use one of the convergence theorems

Let A be a sequence of measurable subsets of $[0,1]$ such that $\inf m\left(A_n\right) > 0$, where m stands for the Lebesgue measure. (a) Prove that there exists $x \in [0,1]$ which belongs to infinitely many of the sets A_n. (b) Does there necessarily exist a point which belongs to any of the sets A_n, except finitely many?

0.5 How can we recover E from its indicator function

Let $E \subset \mathbb{R}^1$. Show that the characteristic function $\chi_E(x)$ is the limit of a sequence of continuous functions if and only if E is both F_σ and G_δ.

0.6 be an artisan

Let $f : [0,1] \to \mathbb{R}$ be a positive function of bounded variation. (a) Show that if $\inf(f) > 0$, then the function $g(x) = 1/f(x)$ is also of bounded variation on $[0,1]$. (b) Give an example of a positive function $f : [0,1] \to \mathbb{R}$ of bounded variation such that $g(x) = 1/f(x)$ is integrable but not of bounded variation.

0.7 Use a suitable theorem allowing you to differentiate $\exp(g)$ under the integral sign

Let f be a real Lebesgue measurable function on the interval $[0,1]$ such that $\|f\|_\infty < \infty$. For $\alpha \in \mathbb{R}$ define a function $g(\alpha)$ by

$$g(\alpha) = \log\left[\int_0^1 \exp[\alpha f(x)]dx\right]$$

(a) Prove that the function $g(\cdot)$ is twice continuously differentiable and that $g''(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$, i.e. the function $g(\cdot)$ is convex. (b) Prove that if f is a non-constant function, i.e. $m\{x \in [0,1] : |f(x) - c| \neq 0\} > 0$ for all constants $c \in \mathbb{R}$, then $g''(\alpha) > 0, \alpha \in \mathbb{R}$.

0.8 Use DCT

Let

$$f \in L_1([0,1], dx)$$

Find:

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log\left(1 + e^{nf(x)}\right) dx$$

0.9 Use Egoroff and Hölder

Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R}^n), 1 < p < \infty$, which converge almost everywhere to a function $f \in L^p(\mathbb{R}^n)$, and suppose that there is a constant M such that $\|f_n\|_p \leq M$ for all n. Show that for every $g \in L^q(\mathbb{R}^n), q$ the conjugate of p,

$$\int fg = \lim_{n\to\infty} \int f_n g$$

Is the statement true for $p = 1$? (Hint: you may want to use Egorov's Theorem.)

0.10 Read up on HL

Let $f(\cdot)$ be a locally integrable function on $\mathbb{R}^n$ and Mf the corresponding Hardy-Littlewood maximal function

$$Mf(x) = \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)| dy, \quad x \in \mathbb{R}^n$$

where $B(x,R)$ denotes the ball centered at x with radius R. a) Show that if f is integrable on $\mathbb{R}^n$ then $\sup_{\lambda>0} \lambda m \{x \in \mathbb{R}^n : |f(x)| > \lambda\} < \infty$. b) Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Show that Mf is not integrable on $\mathbb{R}^n$, but $\sup_{\lambda>0} \lambda m \{x \in \mathbb{R}^n : Mf(x) > \lambda\} < \infty$.

0.11 Use density of such functions g somewhere, and then Hölder.

Fix $1 < p < \infty$. Let $f \in L^p(E)$, where E is a measurable subset of $\mathbb{R}^d$. Assume that

$$\int_E f(x) g(x) dx = 0$$

for all compactly supported continuous functions $g : \mathbb{R}^d \to \mathbb{R}$. Is $f(x) = 0$ for almost every x in E ? If your answer is positive, prove it. Otherwise, given a counterexample.

0.12 Fubini and Tonelli

Suppose that $f(x), x > 0$, is a real valued Lebesgue measurable square integrable function. (a) Prove that for any $\alpha > 0$, the inequality $2|f(z)||f(y)| \le \alpha f(z)^2 + f(y)^2/\alpha$ holds for all $z, y, \alpha > 0$. (b) Express the double integral

$$\int_0^\infty \int_0^\infty \frac{|f(z)||f(y)|}{y+z} dzdy$$

as an integral over the region $\{0 < z < y < \infty\}$. (c) Show using your work from (a) and (b) that $|f(z)||f(y)|/(y+z), y, z > 0$, is integrable and

$$\int_0^\infty \int_0^\infty \frac{|f(z)||f(y)|}{y+z} dzdy \le 4 \int_0^\infty f(x)^2 dx$$

Hint: Use the inequality in (a) with $\alpha = (z/y)^{1/2}$.

0.13 Try a very nice function f first

Let $\{f_n(x)\}$ be a sequence of continuous, strictly positive functions on $\mathbb{R}$ which converges uniformly to the function $f(x)$. Suppose that all the functions $\{f_n\}, f$ are integrable. Is

$$\lim_{n\to\infty} \int f_n(x)dx = \int f(x)dx$$

Justify your answer.

0.14 Use Lebesgue. Can you get the same equality for more sets E?

Let $f \in L_1([0,1], dx)$ be a function such that $\int_E f(x)dx = 0$ for any measurable set $E \subset [0,1]$ of Lebesgue measure .99. Prove that $f = 0$ a.e.

0.15 Lebesgue

Let $f \in L^2(I)$, for any finite interval $I \subset \mathbb{R}$. Assume that

$$\int_{-a}^{a} |t||f(x+t)|dt \geq \frac{2}{\sqrt{3}}a^2$$

for all $a > 0$ and $x \in \mathbb{R}$. Show that $|f(x)| \geq 1$ for a.e. $x \in \mathbb{R}$.

0.16 Integration can be a trick to prove that a nonnegative function can't be identically zero.

Let f and g be nonnegative functions in $L^1(\mathbb{R})$. Suppose that each function is positive on some set of positive measure. (However, there need not be a single set of positive measure where both functions are positive.) Prove that the convolution

$$h(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

is positive on some set of positive measure.

0.17 Check what happens on some set $\{f < c\}$ with $c < ||f||_\infty$

Let E be a measurable subset of $\mathbb{R}$ such that $m(E) < \infty$. Let $f \in L^\infty(E)$ with $\|f\|_\infty > 0$. Show that

$$\lim_{n\to\infty} \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} = \|f\|_\infty$$

Here $\|f\|_n := \|f\|_{L^n(E)}$, $\|f\|_{n+1} := \|f\|_{L^{n+1}(E)}$.

0.18 Use distribution functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function which has the property that

$$m(|f| > \alpha) \leq \frac{1}{1+\alpha^3} \quad \text{for } \alpha > 0$$

(a) Show that $|f|^p$ is integrable for $p < 3$. (b) Give an example of a function satisfying the above for which $|f|^3$ is not integrable.