量子计算与量子信息

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自定义：信息

不要以为抹消过去，重新来过，即可发生什么改变。——比企谷八幡

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2.1 Linear algebra

2.1.1 Bases and linear independence

The vector space $\mathbb{C}^n$:

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{2.1}$$

The vector space $\mathbb{C}^2$:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{2.2}$$

2.1.2 The Pauli matrices:

eigenvectors of σ_z:

$$|\uparrow\rangle \tag{2.3}$$

$$|\downarrow\rangle \tag{2.4}$$

eigenvectors of σ_x:

$$|\rightarrow\rangle \tag{2.5}$$

$$|\leftarrow\rangle \tag{2.6}$$

$$|\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \tag{2.7}$$

$$|\leftarrow\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} \tag{2.8}$$

定义 2.1 (Pauli operators)

1. σ_z:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.9}$$

eigenvectors:

$$+1: \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad -1: \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.10}$$

2. σ_x:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.11}$$

eigenvectors:

$$+1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad -1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.12}$$

3. σ_y:

$$\sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{2.13}$$

eigenvectors:

$$+1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \quad -1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \tag{2.14}$$

♣

2.1.3 Inner products

定义 2.2 (内积运算)

内积运算:

$$(\psi, \varphi) = c \tag{2.15}$$

内积满足的条件:

$$\begin{cases} (\psi,\varphi) = (\varphi,\psi)^* & \langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^* \\ (\psi,\varphi+\chi) = (\psi,\varphi)+(\psi,\chi) & \\ (\psi,\varphi a) = (\psi,\varphi)a & \\ (\psi,\varphi) \geqslant 0 & if\ (\psi,\varphi)=0,\ \varphi=0 \end{cases} \tag{2.16}$$

c^* 表示为 c 的复共轭。

♣

结论

1. 内积空间：具有加法、数乘、内积三种运算的空间
2. 希尔伯特空间：完全的内积空间
3. 完全的意思指在空间中任何在 Canchy 意义下收敛的序列 $\psi_1, \psi_2, \ldots,$ 的极限也在此空间中。

笔记 给定任何一个度量空间 (M, d)，一个序列:

$$x_1, x_2, x_3, \ldots \tag{2.17}$$

被称为柯西列，如果对于任何正实数 $r > 0$，存在一个正整数 N 使得对于所有的整数 $m, n > N$，都有:

$$d(x_m, x_n) < r \tag{2.18}$$

其中 $d(x_m, x_n)$ 表示 x 和 y 之间的距离。

笔记 [矢量空间的矩阵表示]

$$l = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}; m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} \tag{2.19}$$

$$l + m = \begin{pmatrix} l_1 + m_1 \\ l_2 + m_2 \\ l_3 + m_3 \\ l_4 + m_4 \end{pmatrix} \tag{2.20}$$

$$(l, m) = l_1^* m_1 + l_2^* m_2 + l_3^* m_3 + l_4^* m_4 \tag{2.21}$$

笔记

$$(f(x), g(x)) = \int_a^b f^*(x) g(x) dx \tag{2.22}$$

笔记

1. 内积对于右因子是线性的：

$$(\psi, \varphi a) = (\psi, \varphi) a \tag{2.23}$$

$$(\psi, \varphi_1 + \varphi_2) = (\psi, \varphi_1) + (\psi, \varphi_2) \tag{2.24}$$

2. 内积对于左因子是反线性的：

$$(\psi a, \varphi) = a^* (\psi, \varphi) \tag{2.25}$$

$$(\psi_1 + \psi_2, \varphi) = (\psi_1, \varphi) + (\psi_2, \varphi) \tag{2.26}$$

笔记 A 的外积表示：（利用完备性关系）

$$\begin{aligned} A &= \sum_{ij} |w_j\rangle \langle w_j | A | v_i \rangle \langle v_i | \\ &= \sum_{ij} \langle w_j | A | v_i \rangle |w_j\rangle \langle v_i | \end{aligned} \tag{2.27}$$

2.1.4 Eigenvectors and eigenvalues

A diagonal representation for an vector space V is a representation

$$A = \sum_i \lambda_i |i\rangle \langle i| \tag{2.28}$$

for example

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{2.29}$$

2.1.5 Adjoints and Hermitian operators

$$(|v\rangle, A|w\rangle) = (A^\dagger|v\rangle, |w\rangle) \tag{2.30}$$

$$|v\rangle^\dagger \equiv \langle v| \tag{2.31}$$

定理 2.1 (spectral decomposition)

An operator is a normal operator if and only if it is diagonaliable.

2.1.6 Tensor product

例题 2.1 If $|w\rangle$and $|v\rangle$are any two vectors, show that$(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

定义 2.3 (anti-linear)

additivity and conjugate homogeneity:

$$f(x+y) = f(x) + f(y) \tag{2.32}$$

$$f(sx) = \bar{s}x \tag{2.33}$$

例题 2.2 Show that any projector P satisfies the equation $P^2 = P$.

解

$$\begin{aligned} P^2 &= \sum_{ij} |i\rangle\langle i||j\rangle\langle j| \\ &= \sum_{ij} |i\rangle\delta_{ij}\langle j| \\ &= \sum_i |i\rangle\langle i| \\ &= P \end{aligned} \tag{2.34}$$

$U \equiv \sum_i |w_i\rangle\langle v_i|$ is a unitary operator

$\Longleftrightarrow$

$|v_i\rangle$and $|w_i\rangle$ are any two orthonormal basis set.

例题 2.3(2.18) Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

解 设：

$$U = \sum_i \lambda_i |v_i\rangle\langle v_i| \tag{2.35}$$

因为：

$$
\begin{aligned}
UU^\dagger &= \sum_i \lambda_i |v_i\rangle\langle v_i| \sum_i \lambda_i^\dagger |v_i\rangle\langle v_i| \\
&= \sum_i \lambda_i\lambda_i^\dagger |v_i\rangle\langle v_i| \\
&= I \\
&= \sum_i |v_i\rangle\langle v_i|
\end{aligned} \tag{2.36}
$$

故：

$$
\lambda_i\lambda_i^\dagger = 1 \tag{2.37}
$$

$$
||\lambda_i|| = 1 \tag{2.38}
$$

例题 2.4(2.20) Suppose $A'$and A'' are matrix representations of an operator$A$on a vector space V with respect to two different orthonormal bases,$|v_i\rangle$and $|w_i\rangle$. Then the elements ofA' and $A''$are $A'_{ij} = \langle v_i|A|v_j\rangle$and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

The spectral decomposition

定理 2.2 (The spectral decomposition)

Any normal operator M on a vector space V is diagonal with respect to some orthonormal basis for V.

Conversly, any diagonaliable operator is normal.

定义 2.4 (projector)

$$
P \equiv \sum_{i=1}^{k} |i\rangle\langle i| \tag{2.39}
$$

因为对任意的向量 $|v\rangle$ ，$|v\rangle\langle v|$ 是厄米的，因此 P 是厄米的：

$$
p^\dagger = P \tag{2.40}
$$

P 的正交补：（Q 是由 $|k+1\rangle, \ldots, |d\rangle$ 张成的向量空间上的投影）

$$
Q \equiv I - P \tag{2.41}
$$

证明 定理（2.2）的证明：

参考：https://zhuanlan.zhihu.com/p/353753888

例题 2.5(2.21) Repeat the proof of the spectral decomposition in Box 2.2 for the case when M is Hermitian, simplifying the proof wherever possible.

解

$$
M = QMQ + PMP \tag{2.42}
$$

$$\text{证明 Q 的幂等性：} Q^2 = (I-P)(I-P) = I^2 - 2P + P^2 = I - P = Q \tag{2.43}$$

$$\text{证明 Q 是厄米算符：} Q^\dagger = (I-P)^\dagger = I - P^\dagger = Q \tag{2.44}$$

所以：

$$PMP = (PMP)^\dagger \tag{2.45}$$

$$QMQ = (QMQ)^\dagger \tag{2.46}$$

$$\tag{2.47}$$

例题 2.6(2.22) Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

例题 2.7(2.22) Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

解

$$\begin{aligned}(|v_i\rangle, A|v_j\rangle) =& (|v_i\rangle, \lambda_j|v_j\rangle) \\ =& \lambda_j(|v_i\rangle, |v_j\rangle)\end{aligned} \tag{2.48}$$

$$\begin{aligned}(|v_i\rangle, A|v_j\rangle) =& (\lambda_i^\dagger|v_i\rangle, |v_j\rangle) \\ =& (\lambda_i|v_i\rangle, |v_j\rangle) \\ =& \lambda_i^\dagger(|v_i\rangle, |v_j\rangle) \\ =& \lambda_i(|v_i\rangle, |v_j\rangle)\end{aligned} \tag{2.49}$$

有：

$$\lambda_j(|v_i\rangle, |v_j\rangle) - \lambda_i(|v_i\rangle, |v_j\rangle) = 0 \tag{2.50}$$

$$(|v_i\rangle, |v_j\rangle) = 0 \tag{2.51}$$

Tensor products

- $V \otimes W$ is an mn dimensional vector space.
- We often use the abbreviated notations $|v\rangle|w\rangle$,or $|v, w\rangle$or even$|vw\rangle$ for the tensor product $V \otimes W$.
- if V is a two-dimensional vector space with basis vectors $|0\rangle$and $|1\rangle$ then $|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle$ is an element of $V \otimes V$
- $z(|v\rangle \otimes |w\rangle) = (z|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z|w\rangle)$
- $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle$
- $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$

Linear operator types on the spaces $V \otimes W$

- $(A \otimes B)(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$

- $(A \otimes B)(\sum_i a_i|v_i\rangle \otimes |w_i\rangle) \equiv \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$
- $(\sum_i c_i A_i \otimes B_i)(|v\rangle \otimes |w\rangle) \equiv \sum_i c_i A_i|v\rangle \otimes B_i|w\rangle$

定义 2.5 (Inner product on $V \otimes W$)

$$(\sum_i a_i|v_i\rangle \otimes |w_i\rangle, \sum_j b_j|v_j'\rangle \otimes |w_j'\rangle) \equiv \sum_i a_i^* b_j \langle v_i|v_j'\rangle\langle|w_i|w_j'\rangle) \tag{2.52}$$

From this inner product, the inner product space $V \otimes W$ inherits the other structure we are familiar with, such as notions of an adjoint, unitarity, normality, and Hermiticity. ♣

定义 2.6

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{21} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{bmatrix} \tag{2.53}$$

♣

例题 2.8(2.26) Let $|\psi\rangle = \frac{(|0\rangle+|1\rangle)}{\sqrt{2}}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product.

例题 2.9(2.27) Calculate the matrix representation of the tensor products of the Pauli operators (a). X and Z; (b). I and X; (c). X and I. Is the tensor product commutative?

例题 2.10(2.28) Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,

$$(A \otimes B)^* = A^* \otimes B^*; (A \otimes B)^T = A^T \otimes B^T; (A \otimes B)^\dagger = A^* \otimes B^\dagger. \tag{2.54}$$

例题 2.11(2.32) Show that the tensor product of two projectors is a projector.

Operator functions

$$A = \sum_a a|a\rangle\langle a| \tag{2.55}$$

$$f(A) = \sum_a f(a)|a\rangle\langle a| \tag{2.56}$$

2.1.7 The commutator and anti-commutator

定义 2.7

$$[A, B] = AB - BA \tag{2.57}$$

$$\{A, B\} = AB + BA \tag{2.58}$$

♣

定理 2.3 (Simultaneous diagonalization theorem)

A、B 是厄米算符，$[A,B]=0$ 当且仅当存在同一组正交基使得 A 和 B 可对角化。 ♡

问题 2.1(2.40) Verify the commutation relations

$$[X,Y]=2iZ;\quad [Y,Z]=2iX;\quad [Z,X]=2iY. \tag{2.59}$$

There is an elegant way of writing this using ϵ_{jkl}, the antisymmetric tensor on three indices.

问题 2.2(2.41) Verify the anti-commutation relations

$$\{\sigma_i,\sigma_j\}=0 \tag{2.60}$$

where $i\neq j$are both chosen from the set 1, 2, 3. Also verify that (i = 0, 1, 2, 3)

$$\sigma_i^2=I \tag{2.61}$$

问题 2.3(2.42) Verify that

$$AB=\frac{[A,B]+\{A,B\}}{2} \tag{2.62}$$

问题 2.4(2.42) Show that for $j,k=1,2,3,$

$$\sigma_j\sigma_k=\delta_{jk}I+i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l \tag{2.63}$$

问题 2.5(2.43) Suppose$[A,B]=0$,$\{A,B\}=0$, and A is invertible. Show that B must be 0.

问题 2.6(2.45) Show that $[A,B]^\dagger=[B^\dagger,A^\dagger]$.

问题 2.7(2.46) Show that $[A,B]=[B,A]$.

问题 2.8(2.47) Suppose A and B are Hermitian. Show that $i[A,B]$ is Hermitian.

2.1.8 The polar and singular value decompositions

定理 2.4

Let A be a linear operator on a vector space V . Then there exists unitary U and positive operators J and K such that

$$A=UJ=KU \tag{2.64}$$

where the unique positive operators J and K satisfying these equations are defined by $J\equiv\sqrt{A^\dagger A}$ and $K\equiv\sqrt{AA^\dagger}$. Moreover, if A is invertible then U is unique. ♡

证明

$$\begin{aligned}J\equiv&\sqrt{A^\dagger A}\\ =&\sum_i\lambda_i|i\rangle\langle i|\end{aligned} \tag{2.65}$$

defined

$$|\psi_i\rangle\equiv A|i\rangle \tag{2.66}$$

defined

$$|e_i\rangle \equiv \frac{|\psi_i\rangle}{\lambda_i} \tag{2.67}$$

$|e_i\rangle$is normalized and orthogonal defined

$$U = \sum_i |e_i\rangle\langle i| \tag{2.68}$$

$$UJ|i\rangle = \lambda_i|e_i\rangle = |\psi_i\rangle = A|i\rangle \tag{2.69}$$

$$A = UJ \tag{2.70}$$

定义 2.8

Let A is a square matirx. Then there exist unitary matrices $U$and V, and a diagonal matirx D with non-negative entries such that

$$A = UDV \tag{2.71}$$

♣

问题 2.9 What is the polar decomposition of a positive matrix P?Of a unitary matrix U? Of a Hermitian matrix, H?

解

- $$P = PI = IP \tag{2.72}$$
- $$U = UI = IU \tag{2.73}$$
- $$H = \sum_i \lambda_i|i\rangle\langle i| \tag{2.74}$$
  $$J = \sum_i |\lambda_i||i\rangle\langle i| \tag{2.75}$$
  $$U = \sum_i sgn(\lambda_i)|i\rangle\langle i| \tag{2.76}$$

定义 2.9

- normal operator:$NN^\dagger = N^\dagger N$
- unitary operators: $N^* = N^{-1}$
- Hermitian operators (i.e., self-adjoint operators): $N^* = N$
- Skew-Hermitian operators: $N^* = -N$
- positive operators: $N = MM^*$ for some M (so N is self-adjoint).

♣

问题 2.10 Express the polar decomposition of a normal matrix in the outer product representation.

解

$$A = \sum_i \lambda_i |i\rangle\langle i| (\text{谱分解}) \tag{2.77}$$

$$J = \sum_i |\lambda_i||i\rangle\langle i| \tag{2.78}$$

$$U = \sum_i sgn(\lambda_i)|i\rangle\langle i| \tag{2.79}$$

2.2 State space

公设 2.1

Associated to any isolated physical is a complex vector space with inner product(that is, a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system' s state space.

2.3 Evolution

公设 2.2

- The evolution of a closed quantum system is described by a unitary transformation. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2

$$|\psi'\rangle = U|\psi\rangle \tag{2.80}$$

  U is a unitary operator
- The time evolution of the state of a closed quantum system is described by the Schrodinger equation,

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle \tag{2.81}$$

  H is a fixed Hermitian operator known as the Hamiltonian of the closed system.

问题 2.11 Verify that the Hadamard gate H is unitary.

问题 2.12 Verify that $H_2 = I$.

问题 2.13 What are the eigenvalues and eigenvectors of H?

问题 2.14 Suppose A and B are commuting Hermitian operators. Prove that $exp(A)exp(B) = exp(A+B)$. (Hint: Use the results of Section 2.1.9.)

问题 2.15 Prove that $U(t_1, t_2)$ defined in Equation (2.91) is unitary.

问题 2.16 Use the spectral decomposition to show that$K \equiv ilog(U)$is Hermitian for any unitary U, and thus$U = exp(iK)$ for some Hermitian K.

量子测量由一组测量算子 $\{M_m\}$ 描述，这些算子作用在被测系统状态空间上，指标 m 表示实验中可能的测量结果，若在测量前，量子系统的最新状态是 $|\psi\rangle$，则结果 m 发生的可能性由：

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \tag{2.82}$$

给出，且测量后的系统的状态为：

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \tag{2.83}$$

测量算子满足完备性方程

$$\sum_m M_m^\dagger M_m = I \tag{2.84}$$

完备性方程表达了概率之和为 1 的事实：

$$1 = \sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle \tag{2.85}$$

该方程对所有的 $|\psi\rangle$ 成立，等价于完备性方程。

第 2 章 练习

1. **Show that $(1,1)$,$(1,2)$and $(2,1)$are linearly dependent.**

解

定义 2.10 (线性无关)

矢量空间中 n 个矢量的集合 ψ_i，若下式：

$$\sum_{i=1}^{n} \psi_i a_i = 0 \tag{2.86}$$

只有当全部复数 $a_i(i = 1,2,3,,n)$ 都为零时才成立，则这 n 个矢量 ψ_i 是线性无关。(如存在不全为零的 a_i，则线性相关。

$$\psi_1 a_1 + \psi_2 a_2 + \psi_3 a_3 = 0 \tag{2.87}$$

$$\Rightarrow \tag{2.88}$$

$$\psi_1 = -\frac{\psi_2 a_2 + \psi_3 a_3}{a_1} \tag{2.89}$$

即 ψ_i 是其余矢量的线性叠加) ♣

$$(2,1) = (1,1) + (1,2) \tag{2.90}$$

2. **Suppose$V$ is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from $V$to Vsuch that $A|0\rangle = |1\rangle$and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle$,$|1\rangle$, and the output basis $|0\rangle$, $|1\rangle$. Find input and output bases which give rise to a different matrix representation of A.**

解

$$A\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\quad A\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \tag{2.91}$$

$$A=\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}^{-1}\quad A=\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}^{-1} \tag{2.92}$$

得:

$$A=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{2.93}$$

笔记

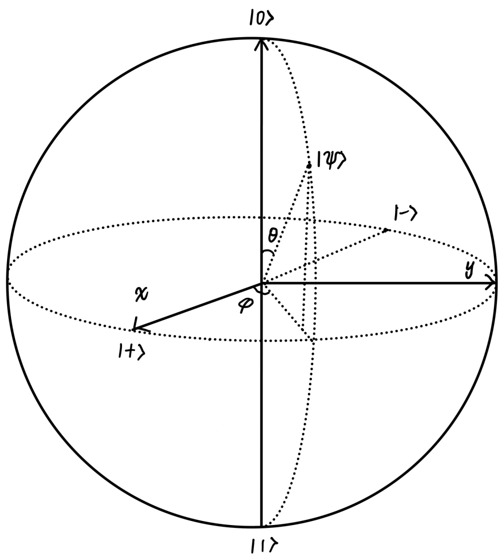

图 2.1

3. **Suppose A is a linear operator from vector space V to vector space W, and B is a linear operator from vector space W to vector space X. Let $|v_i\rangle$, $|w_j\rangle$, and $|x_k\rangle$ be bases for the vector spaces V, W, and X, respectively. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to the appropriate bases.**

解

$$\begin{aligned}BA|v_i\rangle =&B\sum_j A_{ij}|w_j\rangle\\ =&\sum_j A_{ij}B(|w_j\rangle)\end{aligned} \tag{2.94}$$

其中:

$$B|w_j\rangle=\sum_k B_{jk}|x_k\rangle \tag{2.95}$$

所以：

$$\begin{aligned} BA|v_i\rangle &= B\sum_j A_{ij}|w_j\rangle \\ &= \sum_j A_{ij}B(|w_j\rangle) \\ &= \sum_j \sum_k A_{ij}B_{jk}|x_k\rangle \\ &= \sum_k (\sum_j A_{ij}B_{jk})|x_k\rangle \end{aligned} \tag{2.96}$$

即：

$$(BA)_{ik} = \sum_j A_{ij}B_{jk} \tag{2.97}$$

4. **Show that the identity operator on a vector space V has a matrix representation which is one along the diagonal and zero everywhere else, if the matrix representation is taken with respect to the same input and output bases. This matrix is known as the identity matrix.**

解

$$I|v_i\rangle = v_i\rangle \tag{2.98}$$

只有：

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{2.99}$$

5. **The Pauli matrices(Figure 2.2 on page 65) can be considered as operators with respect to an orthonormal basis |0 , |1 for a two-dimensional Hilbert space. Express each of the Pauli operators in the outer product notation.**

解

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{2.100}$$

6. **Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y , and Z.**

   (a). X

   解

   I. **eigenvectors**

$$\lambda = 1, -1 \tag{2.101}$$

   II. **eigenvalues**

   A. $\lambda = 1$

$$X - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.102}$$

解得

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.103}$$

B. 归一化

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.104}$$

III. **diagonal representations**

X 的对角表示为:

$$\begin{aligned} X =& \sum_i \lambda_i |i\rangle\langle i| \\ =& |+\rangle\langle +| - |-\rangle\langle -| \end{aligned} \tag{2.105}$$

(b). Y

(c). Z

7. **Prove that the matirx**

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{2.106}$$

is not diagonaliable.

解

笔记 n 阶矩阵 A 可以对角化的充分必要条件是 A 有 n 个线性无关的特征向量。

8. **If** $|w\rangle$**and** $|v\rangle$**are any two vectors, show that**$(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$**.**

2.4 Application: superdense coding

第 3 章 3

第 4 章 4

第 5 章 5

附录 A 基本数学工具

本附录包括了计量经济学中用到的一些基本数学，我们扼要论述了求和算子的各种性质，研究了线性和某些非线性方程的性质，并复习了比例和百分数。我们还介绍了一些在应用计量经济学中常见的特殊函数，包括二次函数和自然对数，前 4 节只要求基本的代数技巧，第 5 节则对微分学进行了简要回顾；虽然要理解本书的大部分内容，微积分并非必需，但在一些章末附录和第 3 篇某些高深专题中，我们还是用到了微积分。

A.1 求和算子与描述统计量

求和算子是用以表达多个数求和运算的一个缩略符号，它在统计学和计量经济学分析中扮演着重要作用。如果 $\{x_i : i = 1, 2, \ldots, n\}$ 表示 n 个数的一个序列，那么我们就把这 n 个数的和写为：

$$\sum_{i=1}^{n} x_i \equiv x_1 + x_2 + \cdots + x_n \tag{A.1}$$