Single Variable Calculus

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Contents

1 Differentiation **2**

1.1 Limits 2

1.2 Continuity 3

1.3 Derivative 3

1.3.1 Geometric Definition 3

1.3.2 Physics Definition 4

1.3.3 Higher Derivative 5

1.4 Differentiation Rules 5

1.4.1 Power Rule 5

1.4.2 General Rules 6

1.4.3 Trigonometric Rules 8

1.4.4 Exponential and Logarithm 11

1.4.5 Hyperbolic Trigonometric 12

1.5 Implicit Differentiation 13

1.6 Inverse Function 14

1.6.1 Inverse Trigonometric Functions 15

2 Differentiation Application **16**

2.1 Approximation 17

2.1.1 Linear Approximation 17

2.1.2 Quadratic Approximation 18

2.2 Sketching Graph 19

2.3 Optimization 19

2.4 Newton's Method 20

2.5 Mean Value Theorem 21

3 Integration **22**

3.1 Indefinite Integral 22

3.2 Definite Integration 23

1 Differentiation

1.1 Limits

Limits is an important concept in differentiation and integration. Here is the general definition of limits.

Definition 1

If the left-side limit and right-side limit both **exist and equal** then the overall limit exist, write as

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L$$

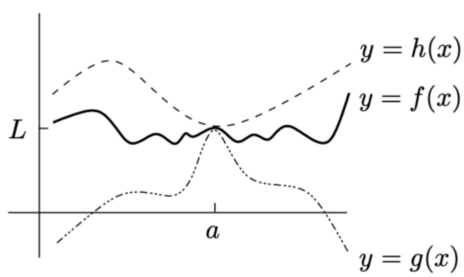
From the limit definition we can find more features about the asymptote.

Theorem 2

Asymptote when $x \to \infty$

- f has a right-side horizontal asymptote at $y = L$ means that $\lim_{x \to \infty} f(x) = L$
- f has a left-side horizontal asymptote at $y = M$ means that $\lim_{x \to -\infty} f(x) = M$

The squeeze principle says that, suppose that for all x near a, we have $g(x) \leq f(x) \leq h(x)$. That is $f(x)$ is squeezed between $g(x)$ and $h(x)$. Also let's suppose that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$. Then we can conclude that $\lim_{x \to a} f(x) = L$.



Theorem 3

Squeeze principle defines that if a function f is squeezed between two functions g and h that converge to the same limit L as $x \to a$, then f also converge to L as $x \to a$.

1.2 Continuity

Continuity is another important concept in differentiation.

Theorem 4

A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

From the theorem above we can know that

- the overall limit $\lim_{x \to a} f(x)$ must exist
- the function f is defined at point $x = a$, namely $f(a)$ exists.
- and $\lim_{x \to a} f(x) = f(a)$

If we know the function is a continuous function that will benefit us a lot.

Theorem 5

Intermediate Value Theorem : if f is continuous on $[a, b]$, and $f(a) < M$ and $f(b) > M$, then there is at least one number c in the interval (a, b) such that $f(c) = M$.

And knowing that the definition of continuity we can get another theorem

Theorem 6

If a function f is differentiable at x then it must be continuous at x.

1.3 Derivative

The derivative has different meaning in different fields. Here is the summery of the two main definition.

Geometric Definition

- $\frac{\Delta y}{\Delta x}$ is the slope of the secant line.
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of the tangent line.

Physics Definition

- $\frac{\Delta y}{\Delta x}$ is the the average change of rate.
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the the instantaneous rate of change or the speed.

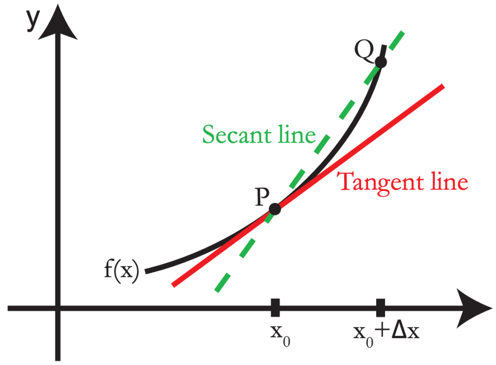
1.3.1 Geometric Definition

First let's review the definition of secant line and tangent line

- **Secant Line**: A secant line connects two distinct points on a curve and typically crosses through the curve.
- **Tangent Line**: A tangent line touches the curve at exactly one point.

Theorem 7

The derivative of $f(x)$ at $x = x_0$ is the slope of the **tangent** line to the graph of $f(x)$ at the point $(x_0, f(x_0))$.



We already know that the derivative of $f(x)$ at $x = x_0$ is the slope of the tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$. Now we need to find out the tangent line.

A secant line is a line that joins two points on a curve, and the tangent line equals the limit of secant lines PQ as $Q \to P$, here P is fixed and Q varies.

We start with a point $P(x_0, f(x_0))$ and move over a tiny horizontal distance Δx, so the point $Q = (x_0+\Delta x, f(x_0+\Delta x))$. And the slope of the secant line equals to the slope of the tangent line when Q **moves close to** P or Δx **is close to** 0.

$$m = \lim_{Q \to P} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Definition 8

The derivative of a function f at x is the following formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

1.3.2 Physics Definition

In geometric definition of derivative we think derivative as the slope of a tangent line. In physics definition we will think the derivative as change of rate.

Another way to think about $\frac{\Delta y}{\Delta x}$ is as the average change in y over an interval of size Δx. And y is often measuring average change in position or distance and x is measuring in time. In this case, the limit

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

measures **the instantaneous rate of change,** or **the speed**.

1.3.3 Higher Derivative

Since we've done with the definition of derivative. We can write higher derivative.

$f'(x)$	Df	$\frac{\mathrm{d}f}{\mathrm{d}x}$	$\frac{\mathrm{d}}{\mathrm{d}x}f$
$f''(x)$	D^2f	$\frac{\mathrm{d}^2f}{\mathrm{d}x^2}$	$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 f$
$f'''(x)$	D^3f	$\frac{\mathrm{d}^3f}{\mathrm{d}x^3}$	$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 f$
$f^{(n)}(x)$	D^nf	$\frac{\mathrm{d}^nf}{\mathrm{d}x^n}$	$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f$

Table 1: Higher Derivative Notation

Example 9

Example $D^n x^n$

Let's calculate the n^{th} derivative of x^n, let's start for a pattern

$$Dx^n = nx^{n-1} \tag{1}$$
$$D^2x^n = n(n-1)x^{n-2} \tag{2}$$
$$D^3x^n = n(n-1)(n-2)x^{n-3} \tag{3}$$
$$... \tag{4}$$
$$D^{n-1}x^n = (n(n-1)(n-2)...2)x \tag{5}$$

Each step we multiply the derivative by the power of x from the previous step, differentiation one more time we get

$$D^n x^n = (n(n-1)(n-2)...2 \cdot 1) \cdot 1$$

And because the number $n(n-1)(n-2)...2 \cdot 1$ is the factorial of n, written as $n!$.

$$D^n x^n = n!$$

1.4 Differentiation Rules

Differentials is a important concept in Calculus. Especially when move from differentiation to integration.

Theorem 10

Given a function $y = f(x)$, the differentials of y is

$$dy = f'(x)dx$$

1.4.1 Power Rule

The basic rule of differentiation rule is $\frac{\mathrm{d}}{\mathrm{d}x}x^n$ which is also called power rule. To get this formula we need binominal theorem.

Theorem 11

Binominal Theorem

$$(x+y)^n = x^n + nx^{n-1}y + nx^{n-2}y^2 + ... + nxy^{n-1} + y^n$$

Now we can derive the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \tag{6}$$

$$= \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1}\Delta x + O((\Delta x)^2)) - x^n}{\Delta x} \tag{7}$$

where $O((\Delta x)^2))$ is a shorthand for all of the terms with $(\Delta x)^2$,$(\Delta x)^3$ and so on up to $(\Delta x)^n$. Which the term from 2. Continue the simplification we can get

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + O((\Delta x)^2)}{\Delta x} \tag{8}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + O(\Delta x) \tag{9}$$

When we divide a term that contains Δx^2 by Δx, the Δx^2 becomes Δx and so our $O(\Delta x^2)$ becomes $O(\Delta x)$.When we take the limit as Δx approaches 0 we get

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

Definition 12

Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

1.4.2 General Rules

Derivative of a sum

The derivative of the sum of two functions is just the sum of their derivatives. Namely

$$(u+v)'(x) = u'(x) + v'(x)$$

Now we will prove that formula using the definition of derivative

$$(u+v)'(x) = \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \tag{10}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \tag{11}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x} \tag{12}$$

Definition 13

Derivative of a sum

$$\frac{\mathrm{d}}{\mathrm{d}x}(u+v) = \frac{\mathrm{d}}{\mathrm{d}x}u + \frac{\mathrm{d}}{\mathrm{d}x}v$$

Derivative of a constant multiple

The derivative of a constant multiplier is just the constant multiple times the derivative of the function

$$(C \cdot u(x))' = C \cdot u(x)'$$

Now we will prove that

$$
\begin{aligned}
(C \cdot u(x))' &= \lim_{\Delta x \to 0} \frac{C \cdot u(x+\Delta x) - C \cdot u(x)}{\Delta x} && (13)\\
&= \lim_{\Delta x \to 0} C \cdot \frac{u(x+\Delta x) - u(x)}{\Delta x} && (14)\\
&= C \cdot u(x)' && (15)
\end{aligned}
$$

Definition 14

Derivative of a constant multiple

$$\frac{\mathrm{d}}{\mathrm{d}x}(C \cdot u) = C \cdot \frac{\mathrm{d}}{\mathrm{d}x} u$$

Note that here C is not a function just a constant multiplier.

Similarly we can get the formula of multiplication, quotient, composite functions.

Definition 15

Product Rule : If $y = uv$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}$$

Proof of the **product rule** : Suppose u and v are two functions about x, and both of them are differentiable. We will proof this formula using the definition of the derivative.

$$
\begin{aligned}
(uv)' &= \lim_{\Delta x \to 0} \frac{uv(x+\Delta x) - uv(x)}{\Delta x} && (16)\\
&= \lim_{\Delta x \to} \frac{u(x+\Delta x)v(x+\Delta v) - u(x)v(x)}{\Delta x} && (17)
\end{aligned}
$$

We want our formula to appear in terms of u, v, u', and v'. And we also know the following equation equals to 0

$$u(x+\Delta x)v(x) - u(x+\Delta x)v(x) = 0$$

Adding zero to the numerator doesn't change the value of our expression, so

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x}$$

We then re-arrange that expression get:

$$(uv)' = \lim_{\Delta x \to 0} \left[\left(\frac{u(x+\Delta x) - u(x)}{\Delta x} \right) v(x) + u(x+\Delta x) \left(\frac{v(x+\Delta x) - v(x)}{\Delta x} \right) \right]$$

And finally because both u and v are differentiable so they must be continuous. So $\lim_{\Delta x \to 0} u(x+\Delta x) = u(x)$. Namely the final formula will be

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}$$

Definition 16

Quotient Rule: If $y = \frac{u}{v}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Definition 17

Chain Rule: If y is a function about u and u is a function about x, or $h(x) = f(g(x))$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{or} \quad h'(x) = f'(g(x))g'(x)$$

The Chain Rule is just like peel something from **outer** to **inner**.

1.4.3 Trigonometric Rules

Trigonometric Review

First in this section let's review some trigonometric function formulas.

Definition 18

$$\sin(A + B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

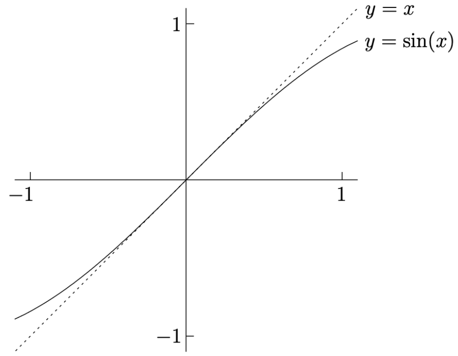
And according to the formula above, when $A = B = x$ we can get the double-angle formula of trigonometric functions. $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$, and according to the equation $\sin^2(x) + \cos^2(x) = 1$, we will replace it in the formula

Definition 19

$$\sin(2x) = 2\sin(x)\cos(x) \tag{18}$$
$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \tag{19}$$

The limit of trigonometric functions

When consider the limit if trigonometric functions we must take care of whether x is a small value or a big value. First let's consider $\lim_{x\to 0}\sin(x)$.



And we can see from figure that when x close to 0, $\sin(x)$ close to x. And when $x \to 0$, $\cos(x) \to 1$. And similarly we can get the limit of tangent when $x \to 0$.

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\frac{\sin(x)}{\cos(x)}}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = 1$$

Definition 20

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

Now let's find the value of $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} \tag{20}$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x} \cdot \frac{1}{1+\cos(x)} \tag{21}$$

$$= \lim_{x\to 0} \frac{\sin^2(x)}{x} \cdot \frac{1}{1+\cos(x)} \tag{22}$$

$$= x \cdot \frac{1}{1+\cos(x)} \tag{23}$$

$$= 0 \tag{24}$$

Definition 21

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

The derivative of trigonometric functions

To get the derivative of $\sin(x)$ and $\cos(x)$, we need the limit we get before. First let's find the derivative of $\sin(x)$

from the definition of the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x} \tag{25}$$
$$= \lim_{\Delta x\to 0}\frac{\sin(x)\cos(\Delta x)+\sin(\Delta x)\cos(x)-\sin(x)}{\Delta x} \tag{26}$$
$$= \lim_{\Delta x\to 0}\sin(x)\left(\frac{\cos(\Delta x)-1}{\Delta x}\right)+\lim_{\Delta x\to 0}\cos(x)\left(\frac{\sin(\Delta x)}{\Delta x}\right) \tag{27}$$
$$= \cos(x) \tag{28}$$

Definition 22

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$

The calculation of the derivative of $\cos(x)$ is very similar to the $\sin(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = \lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos(x)}{\Delta x} \tag{29}$$
$$= \lim_{\Delta x\to 0}\frac{\cos(x)\cos(\Delta x)-\sin(x)\sin(\Delta x)-\cos(x)}{\Delta x} \tag{30}$$
$$= \lim_{\Delta x\to 0}\cos(x)\left(\frac{\cos(\Delta x-1)}{\Delta x}\right)+\lim_{\Delta x\to 0}-\sin(x)\left(\frac{\sin(\Delta x)}{\Delta x}\right) \tag{31}$$
$$= -\sin(x) \tag{32}$$

Definition 23

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$

Definition 24

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$$

And here is a list of other derivative of trigonometric functions.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec(x) = \tan(x)\sec(x)$$

1.4.4 Exponential and Logarithm

First let's find the derivative of exponential function using the definition of derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} \tag{33}$$
$$= \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x} \tag{34}$$
$$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \tag{35}$$

Now we should define the function that

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

Using the definition of function $M(a)$ above we can say that $\frac{\mathrm{d}}{\mathrm{d}x}a^x = M(a)a^x$. And if we plug in $x = 0$, then $\frac{\mathrm{d}}{\mathrm{d}x}a^x = M(a)a^x = M(a)$. So we can think as $M(a)$ is the slope of the tangent of the graph of function $f(x) = a^x$ when $x = 0$. And we know $M(2) < 1$ and $M(4) > 1$.

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x M(a)$$

There must be a number a between 2 and 4 where $M(a) = 1$. Thus we can **define e to be the unique number such that** $M(e) = 1$. Then we can get $\frac{\mathrm{d}}{\mathrm{d}x}e^x = M(e)e^x = e^x$.

Now let's back to the origin problem the derivative of $f(x) = a^x$. There are two main method to get this. The first one is changing the base to e, and the second one is logarithmic differentiation.

Changing the base to e
We've already know that

$$a^x = e^{\ln(a^x)}$$

Now using this rule and chain rule we can get the differentiation of a^x

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{\ln(a^x)} = e^{\ln(a^x)}\frac{\mathrm{d}}{\mathrm{d}x}x\ln(a)$$

And because of $\ln(a)$ is just a constant value so $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$.

Logarithmic Differentiation If we want to differentiation a function like $u = a^x$ we can first differentiation $\ln(u)$ then get u' using the chain rule.

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(u) = \frac{\mathrm{d}\ln(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

And change the formula we can get

$$u' = u\ln(u)'$$

Definition 25

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$$

Once we definition of $M(e) = 1$, we can find the derivative of natural logarithm $\ln(x)$. First suppose $y = \ln(x)$, it's difficult to differentiate the formula directly, but we can do some algebraic operation. Let $e^y = e^{\ln(x)} = x$, then differentiate both sides.

$$\frac{\mathrm{d}}{\mathrm{d}x} e^y = \frac{\mathrm{d}}{\mathrm{d}x}(x) \tag{36}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} e^y = 1 \tag{37}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} \tag{38}$$

Definition 26

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$$

Let's dive more about the natural e. First find the value of

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

When consider a moving exponent we can using a logarithm to turn the exponent into a multiple.

$$\ln\left[\left(1 + \frac{1}{n}\right)^n\right] = n \ln\left(1 + \frac{1}{n}\right)$$

Let $\Delta x = \frac{1}{n}$, so $n = \frac{1}{\Delta x}$. And because $n \to \infty$ so $\Delta x \to 0$. Now we can change our formula to the definition of derivative

$$\lim_{\Delta x \to 0} \frac{\ln(1 + \Delta x) - \ln(1)}{\Delta x} = 1$$

Obviously it is the derivative of $\ln(x)$ when $x = 1$. $\frac{\mathrm{d}}{\mathrm{d}x} \ln(1) = 1$.

Now let's back to the limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} e^{n \ln\left(1+\frac{1}{n}\right)} = e^1 = e$$

Definition 27

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

1.4.5 Hyperbolic Trigonometric

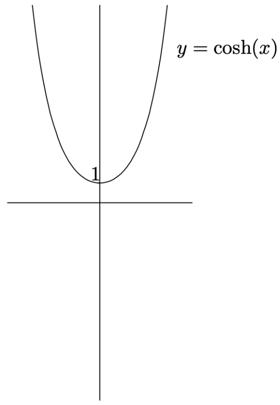
Definition 28

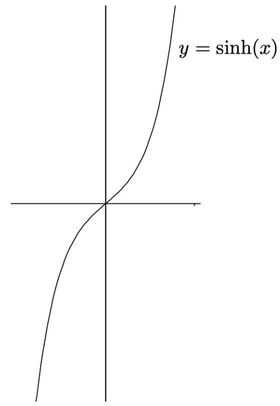
Hyperbolic sine and Hyperbolic cosine

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + x^{-x}}{2}$$

And here is another useful properties about hyperbolic trigonometric functions.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x)$$





Another important identity is that

$$\cosh^2(x) - \sin^2(x) = 1$$

Proof

$$\begin{aligned}
\cosh^2(x) - \sin^2(x) &= \left(\frac{e^x - e^{-x}}{2}\right)^2 - \left(\frac{e^x + e^{-x}}{2}\right)^2 &(39)\\
&= \frac{1}{4}(e^{2x} + 2e^x e^{-x} + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^x e^{-x} + e^{-2x}) &(40)\\
&= 1 &(41)
\end{aligned}$$

Theorem 29

Hyperbolic sine and hyperbolic cosine identity

$$\cosh^2(x) - \sin^2(x) = 1$$

Definition 30

$$\sinh(x + y) = \cosh(x)\sinh(y) + \sinh(x)\cosh(y)$$

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

1.5 Implicit Differentiation

Implicit Differentiation is a way of applying the chain rule. Implicit differentiation is used for solving some tricky problems that use the definition of derivative doesn't work.

$$\frac{\mathrm{d}}{\mathrm{d}x}y^n$$

We can differentiate it using chain rule. We know that y is a function of x. So we can apply the chain rule with outside function y^n and inside function y. Suppose $u = y^n$. According the chain rule

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}x}y^n = \left(\frac{\mathrm{d}}{\mathrm{d}y}y^n\right)\frac{\mathrm{d}y}{\mathrm{d}x} = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Example 31

Implicit Differentiation and the Second Derivative

Calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ using the implicit differentiation given the following equation

$$x^2 + 4y^2 = 1$$

First we need to get the first derivative, differentiating both sides give us:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(4y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}(0) \tag{42}$$

$$2x + 8y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{43}$$

And according we can get the first derivative easily

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{4y}$$

Now we need to differentiating this equation one more time to get the second derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x) + \frac{\mathrm{d}}{\mathrm{d}x}\left(8y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{44}$$

$$2 + \frac{\mathrm{d}}{\mathrm{d}x}(8y)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 8y\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{45}$$

$$2 + 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 8y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \tag{46}$$

Simplifying the above equation and we already know that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{4y}$ we can get

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{16y^3}$$

1.6 Inverse Function

Suppose that f is a differentiable function whose derivative is always positive which means that the function must be **increasing**.

And according to the definition of inverse function, if the graph of the origin function satisfy the horizontal line test then the origin function has its inverse function.

Theorem 32

Suppose function $f(x)$, $\forall x \in \mathbb{R}$ if $f'(x) > 0$, then $f^{-1}(x)$ exists.

Suppose function f has inverse function f^{-1}. What is the derivative of f^{-1} ? First Let $f^{-1}(x) = y$ so $f(y) = x$ we will using implicit differentiation to differentiate both sides.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(y)\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

Using the implicit differentiation rule, let $u = f(y)$

$$\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}f'(y)$$

Definition 33

Suppose f has inverse f^{-1}.

$$\frac{\mathrm{d}}{\mathrm{d}x}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

Example 34

We can solve the derivative of inverse function only use implicit differentiation, this will be useful when the inverse function is hard to solve. Suppose $f(x) = x^2$ solve

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{-1}(x)$$

First let $y = f^{-1}(x)$ we need find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Because $y = f^{-1}(x)$ so

$$f(y) = f(f^{-1}(x)) = x$$

And we will use the left-most and right-most then differentiate both sides. Second plug in y to the function $f(x) = x^2$ we get

$$f(y) = y^2 = x$$

Now it's time to differentiate this formula

$$\frac{\mathrm{d}}{\mathrm{d}x}y^2 = \frac{\mathrm{d}}{\mathrm{d}x}x \tag{47}$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \tag{48}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y} \tag{49}$$

$$\tag{50}$$

And because of $y^2 = x$ so $y = \sqrt{x}$, and so the replace y using x we will get

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{-1}(x) = \frac{1}{2\sqrt{x}}$$

1.6.1 Inverse Trigonometric Functions

Let's start from find the $\arctan(x)$ or $\tan^{-1}(x)$. Let $f(x) = \arctan(x)$ namely $y = \arctan(x)$ so that $x = \tan(y)$. Let's differentiate both sides

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(y) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

Using implicit differentiation we can get

$$\frac{dy}{dx}\frac{d}{dy}(\tan(y)) = 1$$

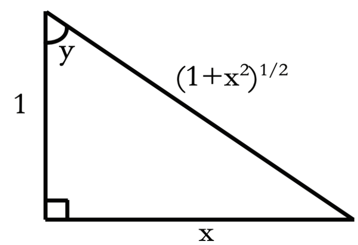
And we already that $\frac{d}{dx}\tan(x) = \sec^2(x)$ so

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} \tag{51}$$
$$= \cos^2(y) \tag{52}$$

So the result of $\frac{dy}{dx}$ will be $\cos^2(y)$, but this is the equation about y not x, we can replace all y using $y = \arctan(x)$ to get

$$\frac{dy}{dx} = \cos^2(\arctan(x))$$

And because $\tan(y) = x$ and $\arctan(x) = y$.



Finally we can get the formula below

$$\frac{dy}{dx} = \cos^2(y) \tag{53}$$
$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \tag{54}$$
$$= \frac{1}{1+x^2} \tag{55}$$

Definition 35

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Using some simple trigonometric we can get the derivative of $\arcsin(x)$ and $\arccos(x)$.

Definition 36

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

2 Differentiation Application

This Section introduce some useful techniques and applications of differentiation.

2.1 Approximation

Approximation is a technique used to estimate some value of x near the origin x_0, which include two specific methods, linear approximation and quadratic approximation.

2.1.1 Linear Approximation

The most important formula in linear approximation is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

If the tangent line approximates $f(x)$. It gives a good approximation near the tangent point x_0. However the approximation grows less accurate.

Definition 37

The Linear Approximation Formula

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Another way to interpret the formula for linear approximation involves the definition of the derivative:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

We can interpret this to mean that

$$\frac{\Delta f}{\Delta x} \approx f'(x_0) \quad \text{when} \quad \Delta x \approx 0$$

If we multiply both side with Δx for

$$\frac{\Delta f}{\Delta x} \approx f'(x_0) \tag{56}$$

$$\Delta x \cdot \frac{\Delta f}{\Delta x} \approx f'(x_0) \cdot \Delta x \tag{57}$$

$$f(x) - f(x_0) \approx f'(x_0) \cdot \Delta x \tag{58}$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{59}$$

Using this formula we can simply find the linear approximation near $x_0 = 0$.

Theorem 38

Linear Approximation at 0

$$\sin(x) \approx x \quad \text{and} \quad \cos(x) \approx 1 \quad \text{and} \quad e^x \approx 1 + x$$

$$\ln(1 + x) \approx x \quad \text{and} \quad (1 + x)^n \approx 1 + nx$$

Example 39

How should we use linear approximation. Suppose we want to know the value of $\ln(1.1)$, we can use the formula that $\ln(x + 1) \approx x$ when $x \approx 0$, so $\ln(1.1) \approx \ln(1 + 0.1) \approx 0.1$. This works because 0.1 is close enough to zero.

2.1.2 Quadratic Approximation

Quadratic approximation is an extension of linear approximation. We'll add one more term which is related to the second derivative. The formula for the quadratic approximation of a function $f(x)$ for values of x near x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Definition 40

Quadratic Approximation Formula

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

Ideally the quadratic approximation of a quadratic function should be identical to the original function, For instance consider

$$f(x) = a + bx + cx^2; \quad f'(x) = b + 2cx; \quad f''(x) = 2c.$$

Set the base point $x_0 = 0$, now we can find the values of a, b and c.

- $f(0) = a + b \cdot 0 + c \cdot 0 \implies a = f(0)$
- $f'(0) = b + 2 \cdot c \cdot 0 \implies b = f'(0)$
- $f''(0) = 2c \implies c = \frac{f''(0)}{2}$

This tell us that the coefficients of the quadratic approximation formula must be in order for the quadratic approximation of a quadratic function to equal that function.

Theorem 41

Quadratic Approximation at 0

$$\sin(x) \approx x \quad \text{and} \quad \cos(x) \approx 1 - \frac{1}{2}x^2 \quad \text{and} \quad e^x \approx 1 + x + \frac{1}{2}x^2$$

$$\ln(1 + x) \approx x - \frac{1}{2}x^2 \quad \text{and} \quad (1 + x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2$$

Example 42

The formula for an n^{th} degree polynomial approximation of a function $f(x)$ near $x = 0$. Suppose the approximation function is defined to be

$$A(x) = a_0 + a_1x^1 + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$$

We've already know that the n^{th} derivative of x^n is $n!$. Namely

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^n = f^{(n)}(x) = n!$$

And we also know the coefficient of approximation formula is same as the derivative of polynomial. So we can conclude that

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + ... + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n$$

Definition 43

Suppose $P(x)$ is defined to be the Taylor polynomial at $x = a$ (namely x near a). The n-th order Taylor polynomial $P(x)$ for $f(x)$ at $x = a$ has the form:

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

2.2 Sketching Graph

Suppose we have function f, the first principle is that if f' is positive, then f is increasing. And if f' is negative then f is decreasing.

Theorem 44

If $f'(x_0) = 0$ we call x_0 a **critical point** and $y_0 = f(x_0)$ is a critical value of f.

And if the slope increase from negative on the left to the positive on the right we say that the curves with this shape are **concave up**. Similarly if from positive on the left to the negative to the right we say that the shape are **concave down**.

Theorem 45

If $f''(x_0) = 0$ then it is an **inflection point** of the graph.

General Strategy for Curve Sketching

1. Plot discontinuities of f
2. Find endpoints where $x \to \pm\infty$
3. Find critical points namely solve $f'(x) = 0$
4. Decide whether $f'(x) < 0$ or $f'(x) > 0$ on each interval between critical points and discontinuities.
5. Decide whether $f''(x) < 0$ or $f''(x) > 0$ on each interval between critical points and discontinuities. This will tell whether the graph is concave up or concave down, Inflection points occur when $f''(x_0) = 0$.

2.3 Optimization

Maximum and Minimum

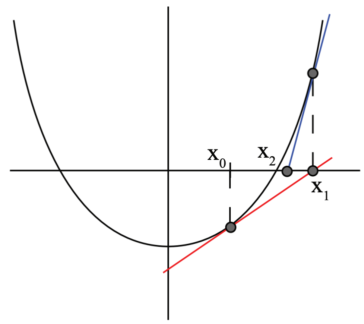
To find the Maximum and Minimum value of the function we need to find out the critical points and endpoints.

Theorem 46

Key to finding Maxima and Minima: Need to find the critical points, endpoints, and points of discontinuity.

2.4 Newton's Method

Newton's Method is a powerful tool for solving equations of the form $f(x) = 0$.



Example solve $x^2 = 5$. Using the Newton's Method we first need to transform the formula to $f(x) = x^2 - 5$. By finding the value of x for which $f(x) = 0$ we solve the equation $x^2 = 5$.

Our initial guess is to discover where the graph crosses the x-axis. We will guess that $x_0 = 2$, then $f(2) = -1$. We can use our initial guess (x_0, y_0) to get another point where crosses the x-axis $(x_1, 0)$. And using this new-get $(x_1, 0)$ to get (x_1, y_1) and using this point to get $(x_2, 0)$.

Continuing this process, we will get our estimate value more accurate. The equation for the tangent line is

$$y - y_0 = m(x - x_0)$$

When the tangent line intercepts the x-axis $y = 0$, and the x coordinate of that point is our new guess x_1.

$$-y_0 = m(x - x_0) \tag{60}$$

$$-\frac{y_0}{m} = x - x_0 \tag{61}$$

$$x = x_0 - \frac{y_0}{m} \tag{62}$$

And because of the x_0 is our last guess and $y_0 = f(x_0)$ and m is the slope of the tangent line namely $m = f'(x_0)$.

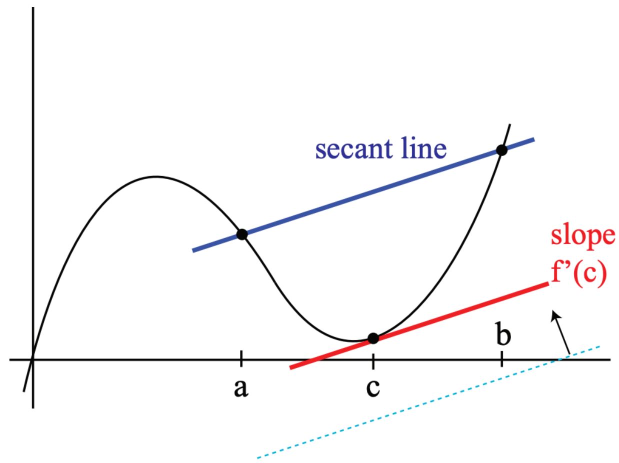
Definition 47

The point of Newton's Method is that we can improve our new guess by repeating this process. To get our $(n+1)^{th}$ guess we apply this formula to our last guess

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

2.5 Mean Value Theorem

Consider the graph of $f(x)$. Here $\frac{f(b)-f(a)}{b-a}$ is the slope of a secant line joining the points $(a, f(a))$ and $(b, f(b))$, and $f'(c)$ is the slope of the tangent line. We can see that somewhere between a and b there is a point on the graph $(c, f(c))$ whose tangent line has the same slope as the secant line.



The formal definition of Mean Value Theorem is

Definition 48

Provided that f is differentiable on $a < x < b$, and continuous on $a \leq x \leq b$.

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad (\text{ for some } c, a < c < b \text{ })$$

We now can use the Mean Value Theorem to do some proof. Now we can prove these below traits

- If $f' > 0$ then f is increasing
- If $f' < 0$ then f is decreasing
- If $f' = 0$ then f is constant

According to the Mean Value Theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c between a and b. For the purpose of this proof we will assume that $b > a$. We manipulate the equation to get

$$f(b) - f(a) = f'(c)(b - a) \tag{63}$$

$$f(b) = f(a) + f'(c)(b - a) \tag{64}$$

Since $a < b$ so $b - a > 0$ and the sign of $f'(c)(b - a)$ is completely determined by the sign of $f'(c)$. So we can get

- If $f'(c) > 0$ then $f(b) > f(a)$
- If $f'(c) < 0$ then $f(b) < f(a)$
- If $f'(c) = 0$ then $f(b) = f(a)$

Definition 49

The Mean Value Theorem tells that if f and f' are continuous on $[a, b]$ then for some value of c between a and b. Since f' is continuous $f'(c)$ must lie between the minimum and maximum values of $f'(x)$ on $[a, b]$.

$$\min_{a \leq x \leq b} f'(x) \leq \frac{f(b) - f(a)}{b - a} = f'(c) \leq \max_{a \leq x \leq b} f'(x)$$

The Mean Value Theorem says that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rearrange terms we can make this look very much like the linear approximation for $f(b)$ using the tangent line at $x = a$;

$$f(b) = f(a) + f'(c)(b - a)$$

The Difference between $f(b)$ and its n-th order Taylor polynomial at $x = a$ follows a similar patterns.

$$P(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Now we can guess that this form of difference for some c in (a, b).

$$f(b) - P(b) = \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}$$

3 Integration

3.1 Indefinite Integral

Suppose we have a function $G(x) = \int g(x)dx$ is the anti-derivative of g. This formula includes a differential dx. It also includes the symbol $\int$, called an integral sign. Another name for anti-derivative of g is the **indefinite integral** of g.

$$\int \sin(x)dx = -\cos(x) + C$$

This form is called the indefinite integral of $\sin(x)$ because the value C can be any constant. And here below is the list of formula about indefinite integral.

Definition 50

The indefinite integral of the power function, when $n = -1$ we need consider carefully. And the value c is any constant.

$$1. \int x^n dx = \frac{x^{n+1}}{n + 1} + c \quad \text{for } n \neq -1 \tag{65}$$

$$2. \int \frac{1}{x} dx = \ln|x| + c \tag{66}$$

Definition 51

The indefinite integral of trigonometric function. Where c is any constant.

$$1. \int \sin(x)dx = -\cos(x) + c \tag{67}$$

$$2. \int \sec^2(x)dx = \tan(x) + c \tag{68}$$

$$3. \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + c \tag{69}$$

$$4. \int \frac{1}{1+x^2}dx = \tan^{-1}(x) + c \tag{70}$$

We already know the relation between differentials and indefinite integral. The differentials formula is $dy = f'(x)dx$. And the indefinite integral is just another way to represent this formula

$$dy = f'(x)dx \quad \text{and} \quad y = \int f'(x)dx$$

Which the indefinite integral is just a reverse process to get the anti-derivative of the function y.

3.2 Definite Integration