# Single Variable Calculus

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# 1 Differentiation

## 1.1 Limits

Limits is an important concept in differentiation and integration. Here is the general definition of limits.

**Definition 1**

If the left-side limit and right-side limit both **exist and equal** then the overall limit exist, write as

$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

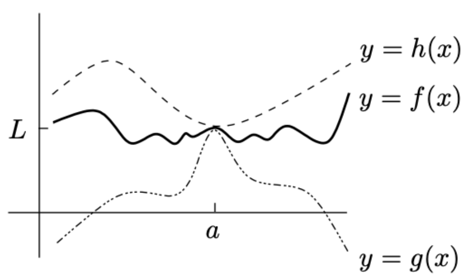
From the limit definition we can find more features about the asymptote.

**Theorem 2**

Asymptote when $x \to \infty$

- $f$ has a right-side horizontal asymptote at $y = L$ means that $\lim_{x\to\infty} f(x) = L$
- $f$ has a left-side horizontal asymptote at $y = M$ means that $\lim_{x\to-\infty} f(x) = M$

The squeeze principle says that, suppose that for all $x$ near $a$, we have $g(x) \leq f(x) \leq h(x)$. That is $f(x)$ is squeezed between $g(x)$ and $h(x)$. Also let's suppose that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$. Then we can conclude that $\lim_{x\to a} f(x) = L$.




**Theorem 3**

**Squeeze principle** defines that if a function $f$ is squeezed between two functions $g$ and $h$ that converge to the same limit $L$ as $x \to a$, then $f$ also converge to $L$ as $x \to a$.

## 1.2 Continuity

Continuity is another important concept in differentiation.

**Theorem 4**

A function $f$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

From the theorem above we can know that

- the overall limit $\lim_{x\to a} f(x)$ must exist
- the function $f$ is defined at point $x = a$, namely $f(a)$ exists.
- and $\lim_{x\to a} f(x) = f(a)$

If we know the function is a continuous function that will benefit us a lot.

**Theorem 5**

**Intermediate Value Theorem** : if $f$ is continuous on $[a, b]$, and $f(a) < M$ and $f(b) > M$, then there is at least one number $c$ in the interval $(a, b)$ such that $f(c) = M$.

And knowing that the definition of continuity we can get another theorem

**Theorem 6**

If a function $f$ is differentiable at $x$ then it must be continuous at $x$.

## 1.3 Derivative

The derivative has different meaning in different fields. Here is the summery of the two main definition.

Geometric Definition

- $\frac{\Delta y}{\Delta x}$ is the slope of the secant line.
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of the tangent line.

Physics Definition

- $\frac{\Delta y}{\Delta x}$ is the the average change of rate.
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the the instantaneous rate of change or the speed.

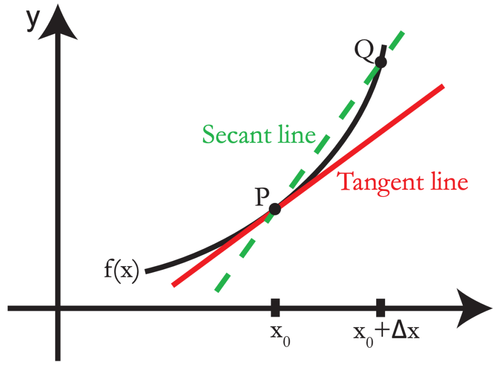
### 1.3.1 Geometric Definition

First let's review the definition of secant line and tangent line

- **Secant Line**: A secant line connects two distinct points on a curve and typically crosses through the curve.
- **Tangent Line**: A tangent line touches the curve at exactly one point.

**Theorem 7**

The derivative of $f(x)$ at $x = x_0$ is the slope of the **tangent** line to the graph of $f(x)$ at the point $(x_0, f(x_0))$.




We already know that the derivative of $f(x)$ at $x = x_0$ is the slope of the tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$. Now we need to find out the tangent line.

A secant line is a line that joins two points on a curve, and the tangent line equals the limit of secant lines $PQ$ as $Q \to P$, here $P$ is fixed and $Q$ varies.

We start with a point $P(x_0, f(x_0))$ and move over a tiny horizontal distance $\Delta x$, so the point $Q = (x_0 + \Delta x, f(x_0 + \Delta x))$. And the slope of the secant line equals to the slope of the tangent line when $Q$ **moves close to** $P$ or $\Delta x$ **is close to** $0$.

$$m = \lim_{Q \to P} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

**Definition 8**

The derivative of a function $f$ at $x$ is the following formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

### 1.3.2 Physics Definition

In geometric definition of derivative we think derivative as the slope of a tangent line. In physics definition we will think the derivative as change of rate.

Another way to think about $\frac{\Delta y}{\Delta x}$ is as the average change in $y$ over an interval of size $\Delta x$. And $y$ is often measuring average change in position or distance and $x$ is measuring in time. In this case, the limit

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

measures **the instantaneous rate of change,** or **the speed**.

### 1.3.3 Higher Derivative

Since we've done with the definition of derivative. We can write higher derivative.

| | | | |
|---|---|---|---|
| $f'(x)$ | $Df$ | $\frac{df}{dx}$ | $\frac{d}{dx}f$ |
| $f''(x)$ | $D^2f$ | $\frac{d^2f}{dx^2}$ | $\left(\frac{d}{dx}\right)^2 f$ |
| $f'''(x)$ | $D^3f$ | $\frac{d^3f}{dx^3}$ | $\left(\frac{d}{dx}\right)^3 f$ |
| $f^{(n)}(x)$ | $D^nf$ | $\frac{d^nf}{dx^n}$ | $\left(\frac{d}{dx}\right)^n f$ |

Table 1: Higher Derivative Notation

**Example 9**

Example $D^n x^n$

Let's calculate the $n^{th}$ derivative of $x^n$, let's start for a pattern

$$Dx^n = nx^{n-1} \tag{1}$$

$$D^2x^n = n(n-1)x^{n-2} \tag{2}$$

$$D^3x^n = n(n-1)(n-2)x^{n-3} \tag{3}$$

$$... \tag{4}$$

$$D^{n-1}x^n = (n(n-1)(n-2)...2)x \tag{5}$$

Each step we multiply the derivative by the power of $x$ from the previous step, differentiation one more time we get

$$D^n x^n = (n(n-1)(n-2)...2 \cdot 1) \cdot 1$$

And because the number $n(n-1)(n-2)...2 \cdot 1$ is the factorial of $n$, written as $n!$.

$$D^n x^n = n!$$

## 1.4 Differentiation Rules

### 1.4.1 Power Rule

The basic rule of differentiation rule is $\frac{d}{dx}x^n$ which is also called power rule. To get this formula we need binominal theorem.

**Theorem 10**

Binominal Theorem

$$(x+y)^n = x^n + nx^{n-1}y + nx^{n-2}y^2 + ... + nxy^{n-1} + y^n$$

Now we can derive the formula

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \tag{6}$$

$$= \lim_{\Delta x \to 0} \frac{(x^n + nx^{n-1}\Delta x + O((\Delta x)^2)) - x^n}{\Delta x} \tag{7}$$

where $O((\Delta x)^2))$ is a shorthand for all of the terms with $(\Delta x)^2$,$(\Delta x)^3$ and so on up to $(\Delta x)^n$. Which the term from 2. Continue the simplification we can get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}x^n &= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + O((\Delta x)^2)}{\Delta x} && (8)\\ &= \lim_{\Delta x \to 0} nx^{n-1} + O(\Delta x) && (9)\end{aligned}$$

When we divide a term that contains $\Delta x^2$ by $\Delta x$, the $\Delta x^2$ becomes $\Delta x$ and so our $O(\Delta x^2)$ becomes $O(\Delta x)$.When we take the limit as $\Delta x$ approaches 0 we get

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

**Definition 11**

Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

### 1.4.2 General Rules

**Derivative of a sum**

The derivative of the sum of two functions is just the sum of their derivatives. Namely

$$(u + v)'(x) = u'(x) + v'(x)$$

Now we will prove that formula using the definition of derivative

$$\begin{aligned}(u+v)'(x) &= \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} && (10)\\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} && (11)\\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x} && (12)\end{aligned}$$

**Definition 12**

Derivative of a sum

$$\frac{\mathrm{d}}{\mathrm{d}x}(u + v) = \frac{\mathrm{d}}{\mathrm{d}x}u + \frac{\mathrm{d}}{\mathrm{d}x}v$$

**Derivative of a constant multiple**

The derivative of a constant multiplier is just the constant multiple times the derivative of the function

$$(C \cdot u(x))' = C \cdot u(x)'$$

Now we will prove that

$$
\begin{aligned}
(C \cdot u(x))' &= \lim_{\Delta x \to 0} \frac{C \cdot u(x+\Delta x) - C \cdot u(x)}{\Delta x} && (13)\\
&= \lim_{\Delta x \to 0} C \cdot \frac{u(x+\Delta x) - u(x)}{\Delta x} && (14)\\
&= C \cdot u(x)' && (15)
\end{aligned}
$$

**Definition 13**

Derivative of a constant multiple

$$\frac{\mathrm{d}}{\mathrm{d}x}(C \cdot u) = C \cdot \frac{\mathrm{d}}{\mathrm{d}x} u$$

Note that here $C$ is not a function just a constant multiplier.

Similarly we can get the formula of multiplication, quotient, composite functions.

**Definition 14**

**Product Rule** : If $y = uv$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}$$

Proof of the **product rule** : Suppose $u$ and $v$ are two functions about $x$, and both of them are differentiable. We will proof this formula using the definition of the derivative.

$$
\begin{aligned}
(uv)' &= \lim_{\Delta x \to 0} \frac{uv(x+\Delta x) - uv(x)}{\Delta x} && (16)\\
&= \lim_{\Delta x \to} \frac{u(x+\Delta x)v(x+\Delta v) - u(x)v(x)}{\Delta x} && (17)
\end{aligned}
$$

We want our formula to appear in terms of $u$, $v$, $u'$, and $v'$. And we also know the following equation equals to 0

$$u(x+\Delta x)v(x) - u(x+\Delta x)v(x) = 0$$

Adding zero to the numerator doesn't change the value of our expression, so

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x}$$

We then re-arrange that expression get:

$$(uv)' = \lim_{\Delta x \to 0} \left[ \left( \frac{u(x+\Delta x) - u(x)}{\Delta x} \right) v(x) + u(x+\Delta x) \left( \frac{v(x+\Delta x) - v(x)}{\Delta x} \right) \right]$$

And finally because both $u$ and $v$ are differentiable so they must be continuous. So $\lim_{\Delta x \to 0} u(x+\Delta x) = u(x)$. Namely the final formula will be

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} v + u \frac{\mathrm{d}v}{\mathrm{d}x}$$

**Definition 15**

**Quotient Rule**: If $y = \frac{u}{v}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}u}{\mathrm{d}x}v - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

**Definition 16**

**Chain Rule**: If $y$ is a function about $u$ and $u$ is a function about $x$, or $h(x) = f(g(x))$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{or} \quad h'(x) = f'(g(x))g'(x)$$

The Chain Rule is just like peel something from **outer** to **inner**.

### 1.4.3 Trigonometric Rules

**Trigonometric Review**

First in this section let's review some trigonometric function formulas.

**Definition 17**

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

And according to the formula above, when $A = B = x$ we can get the double-angle formula of trigonometric functions. $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$, and according to the equation $\sin^2(x) + \cos^2(x) = 1$, we will replace it in the formula

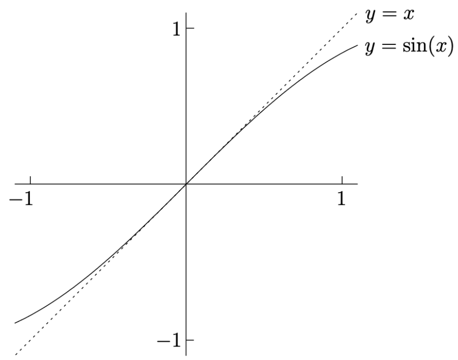
**Definition 18**

$$\sin(2x) = 2\sin(x)\cos(x) \tag{18}$$

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \tag{19}$$

**The limit of trigonometric functions**

When consider the limit if trigonometric functions we must take care of whether $x$ is a small value or a big value. First let's consider $\lim_{x\to 0}\sin(x)$.




And we can see from figure that when $x$ close to 0, $\sin(x)$ close to $x$. And when $x \to 0$, $\cos(x) \to 1$. And similarly we can get the limit of tangent when $x \to 0$.

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\frac{\sin(x)}{\cos(x)}}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = 1$$

**Definition 19**

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{\tan(x)}{x} = 1$$

Now let's find the value of $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} \tag{20}$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x} \cdot \frac{1}{1+\cos(x)} \tag{21}$$

$$= \lim_{x\to 0} \frac{\sin^2(x)}{x} \cdot \frac{1}{1+\cos(x)} \tag{22}$$

$$= x \cdot \frac{1}{1+\cos(x)} \tag{23}$$

$$= 0 \tag{24}$$

**Definition 20**

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$$

### The derivative of trigonometric functions

To get the derivative of $\sin(x)$ and $\cos(x)$, we need the limit we get before. First let's find the derivative of $\sin(x)$

from the definition of the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} \tag{25}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x) - \sin(x)}{\Delta x} \tag{26}$$

$$= \lim_{\Delta x \to 0} \sin(x)\left(\frac{\cos(\Delta x) - 1}{\Delta x}\right) + \lim_{\Delta x \to 0} \cos(x)\left(\frac{\sin(\Delta x)}{\Delta x}\right) \tag{27}$$

$$= \cos(x) \tag{28}$$

**Definition 21**

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$

The calculation of the derivative of $\cos(x)$ is very similar to the $\sin(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = \lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos(x)}{\Delta x} \tag{29}$$

$$= \lim_{\Delta x \to 0} \frac{\cos(x)\cos(\Delta x) - \sin(x)\sin(\Delta x) - \cos(x)}{\Delta x} \tag{30}$$

$$= \lim_{\Delta x \to 0} \cos(x)\left(\frac{\cos(\Delta x - 1)}{\Delta x}\right) + \lim_{\Delta x \to 0} -\sin(x)\left(\frac{\sin(\Delta x)}{\Delta x}\right) \tag{31}$$

$$= -\sin(x) \tag{32}$$

**Definition 22**

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$

**Definition 23**

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$$

## 1.5 Implicit Differentiation

Implicit Differentiation is a way of applying the chain rule. Implicit differentiation is used for solving some tricky problems that use the definition of derivative doesn't work.

$$\frac{\mathrm{d}}{\mathrm{d}x}y^n$$

We can differentiate it using chain rule. We know that $y$ is a function of $x$. So we can apply the chain rule with outside function $y^n$ and inside function $y$. Suppose $u = y^n$. According the chain rule

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}x}y^n = \left(\frac{\mathrm{d}}{\mathrm{d}y}y^n\right)\frac{\mathrm{d}y}{\mathrm{d}x} = ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$$

**Example 24**

Implicit Differentiation and the Second Derivative

Calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ using the implicit differentiation given the following equation

$$x^2 + 4y^2 = 1$$

First we need to get the first derivative, differentiating both sides give us:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(4y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}(0) \tag{33}$$

$$2x + 8y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \tag{34}$$

And according we can get the first derivative easily

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{4y}$$

Now we need to differentiating this equation one more time to get the second derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x) + \frac{\mathrm{d}}{\mathrm{d}x}\left(8y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{35}$$

$$2 + \frac{\mathrm{d}}{\mathrm{d}x}(8y)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 8y\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0 \tag{36}$$

$$2 + 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 8y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \tag{37}$$

Simplifying the above equation and we already know that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{4y}$ we can get

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{16y^3}$$

## 1.6 Inverse Function

Suppose that $f$ is a differentiable function whose derivative is always positive which means that the function must be **increasing**.

And according to the definition of inverse function, if the graph of the origin function satisfy the horizontal line test then the origin function has its inverse function.

**Theorem 25**

Suppose function $f(x)$, $\forall x \in \mathbb{R}$ if $f'(x) > 0$, then $f^{-1}(x)$ exists.

Suppose function $f$ has inverse function $f^{-1}$. What is the derivative of $f^{-1}$ ? First Let $f^{-1}(x) = y$ so $f(y) = x$ we will using implicit differentiation to differentiate both sides.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(y)\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

Using the implicit differentiation rule, let $u = f(y)$

$$\frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}f'(y)$$

**Definition 26**

Suppose $f$ has inverse $f^{-1}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

**Example 27**

We can solve the derivative of inverse function only use implicit differentiation, this will be useful when the inverse function is hard to solve. Suppose $f(x) = x^2$ solve

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{-1}(x)$$

First let $y = f^{-1}(x)$ we need find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Because $y = f^{-1}(x)$ so

$$f(y) = f(f^{-1}(x)) = x$$

And we will use the left-most and right-most then differentiate both sides. Second plug in $y$ to the function $f(x) = x^2$ we get

$$f(y) = y^2 = x$$

Now it's time to differentiate this formula

$$\frac{\mathrm{d}}{\mathrm{d}x}y^2 = \frac{\mathrm{d}}{\mathrm{d}x}x \tag{38}$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \tag{39}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2y} \tag{40}$$

$$\tag{41}$$

And because of $y^2 = x$ so $y = \sqrt{x}$, and so the replace $y$ using $x$ we will get

$$\frac{\mathrm{d}}{\mathrm{d}x}f^{-1}(x) = \frac{1}{2\sqrt{x}}$$

### 1.6.1 Inverse Trigonometric Functions

Let's start from find the $\arctan(x)$ or $\tan^{-1}(x)$. Let $f(x) = \arctan(x)$ namely $y = \arctan(x)$ so that $x = \tan(y)$. Let's differentiate both sides

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan(y) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

Using implicit differentiation we can get

$$\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}(\tan(y)) = 1$$

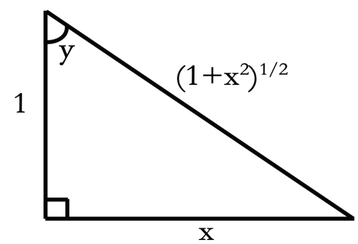
And we already that $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x)$ so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2(y)} \tag{42}$$
$$= \cos^2(y) \tag{43}$$

So the result of $\frac{\mathrm{d}y}{\mathrm{d}x}$ will be $\cos^2(y)$, but this is the equation about $y$ not $x$, we can replace all $y$ using $y = \arctan(x)$ to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2(\arctan(x))$$

And because $\tan(y) = x$ and $\arctan(x) = y$.




Finally we can get the formula below

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2(y) \tag{44}$$
$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 \tag{45}$$
$$= \frac{1}{1+x^2} \tag{46}$$

**Definition 28**

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$$

Using some simple trigonometric we can get the derivative of $\arcsin(x)$ and $\arccos(x)$.

**Definition 29**

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$