Can we explain cosmic birefringence without a new light field beyond Standard Model?
Yu-Cheng QIU
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Can we explain cosmic birefringence without a new light field beyond Standard Model?

Yu-Cheng QIU

Tsung-Dao Lee Institute

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with Yuichiro Nakai, Ryo Namba, Ippei Obata, Ryo Saito

CMB polarization

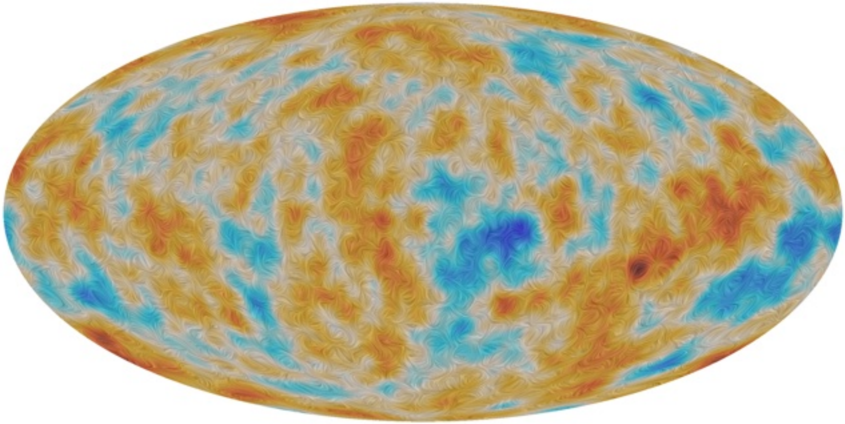


Figure: Copyright: ESA and the Planck Collaboration

Isotropic Cosmic Birefringence

$$C_l^{EB,obs} \neq 0 \implies \beta \neq 0$$

- $\beta = 0.35^\circ \pm 0.14^\circ$ (2.4σ) (Minami and Komatsu 2011.11254)
- $\beta = 0.34^\circ \pm 0.09^\circ$ (3.6σ) (Eskilt and Komatsu 2205.13962)
- $\cdots$ $(5\sigma?)$

1. Isotropic
2. Frequency-blind

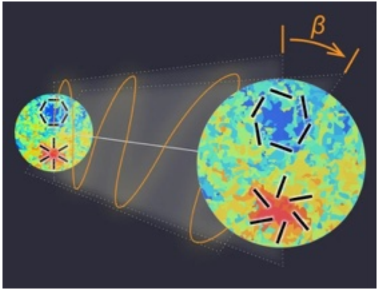


Figure: Credit: Yuto Minami

Axion (ALP) Explanation

Consider Axion-photon coupling (Carroll and Field,1991)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\theta(t)F_{\mu\nu}\tilde{F}^{\mu\nu}$$

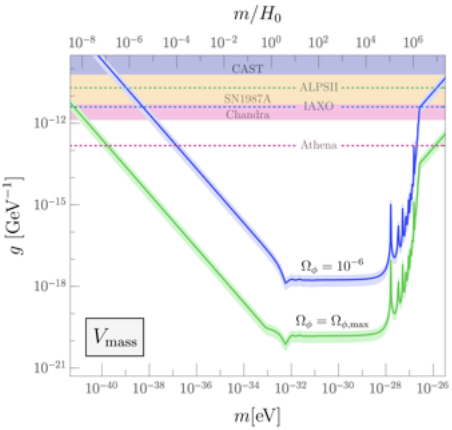
This modifies equation of motion of photon (Choosing $A^0 = 0$ and $\nabla \cdot \vec{A} = 0$),

$$\left(\frac{\partial^2}{\partial\eta^2} - \nabla^2 + \dot{\theta}\nabla\times\right)\vec{A} = 0 \quad \Longrightarrow \quad \omega_\pm^2 = k^2 \mp k\dot{\theta}$$

In the limit $\frac{|\dot{\omega}_\pm|}{\omega_\pm^2} \ll 1$, WKB approximation gives

$$A_\pm \propto e^{-i\int d\eta k} \exp\left(\pm i \int d\eta \frac{\dot{\theta}}{2}\right) \Longrightarrow \beta = \frac{1}{2}\int_{\eta_{\mathrm{LSS}}}^{\eta_0} d\eta\dot{\theta} = \frac{1}{2}\left[\theta(\eta_0) - \theta(\eta_{\mathrm{LSS}})\right]$$

Axion (ALP) Explanation



Ultra-light Axions!

Figure: Fujita et al. 2011.11894

SM?

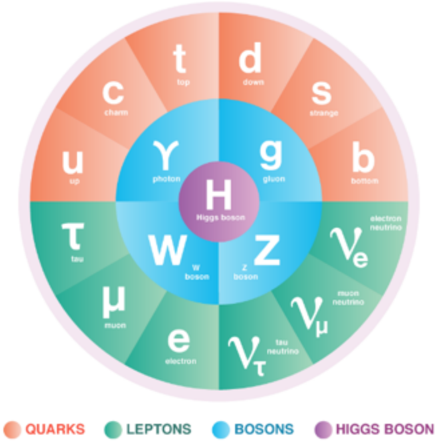
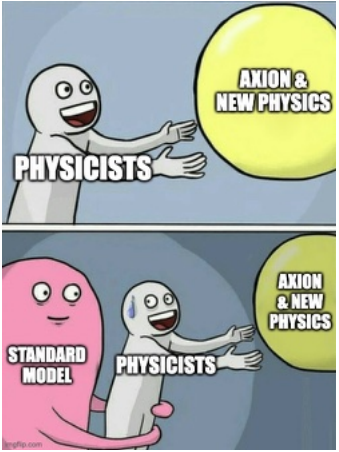


Figure: Artwork by Sandbox Studio, Chicago.



EFT!

Standard Model $\supset$ SMEFT + Standard Cosmology

What is ICB phenomenologically?

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Relevant Operators
Possible ICB
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Summary

- ICB is a propagating effect. Relevant operators are quadratic in $\vec{E}$ and $\vec{B}$. In the vacuum, only $F_{\mu\nu}F^{\mu\nu}$ is relevant. Therefore, we need medium.
- Observed isotropy $\implies$ medium is homogeneous.
- Derivative on $\vec{E}$ and $\vec{B}$ shall lead to frequency-dependent β, which do not fit the observation.

$$\mathcal{L} \sim c_{\mathrm{EE}}(t)\vec{E}\cdot\vec{E} + c_{\mathrm{BB}}(t)\vec{B}\cdot\vec{B} + c_{\mathrm{EB}}(t)\vec{E}\cdot\vec{B}$$

- $\vec{E}\cdot\vec{B}$-term violates parity, produces ICB.

Any operator that relevant to ICB should be reduced to $c_{\mathrm{EB}}(t)\vec{E}\cdot\vec{B}$ in a cosmological background.

$\tilde{\mathcal{O}} F \tilde{F}$?

- Of course, $\tilde{\mathcal{O}} F_{\mu\nu} \tilde{F}^{\mu\nu} \rightarrow \tilde{\mathcal{O}} \vec{E} \cdot \vec{B}$. Is it the only one?
- $\mathcal{J}_{\mu\nu\alpha\beta} F^{\mu\nu} \tilde{F}^{\alpha\beta}$ under the cosmological background,
  $\mathcal{J}$ could be formed by $g_{\mu\nu}$ and $u_\mu \propto \nabla_\mu t$, t is cosmic time.

$$\mathcal{J}_{\mu\nu\alpha\beta} = \frac{1}{2} \left(g_{\mu\alpha} J_{\nu\beta} + g_{\mu\beta} J_{\nu\alpha} - g_{\nu\alpha} J_{\mu\beta} - g_{\nu\beta} J_{\mu\alpha} \right) \ , \quad J_{\mu\nu} = f(g_{\mu\nu}, u_\mu)$$

  This reduce to

$$J_{\alpha\beta} F^{\alpha\mu} \tilde{F}^{\beta}{}_{\mu} \rightarrow \tilde{\mathcal{O}} F_{\mu\nu} \tilde{F}^{\mu\nu} \ , \quad \tilde{\mathcal{O}} = \frac{J_\mu{}^\mu}{4} = \frac{\mathcal{J}_{\alpha\beta}{}^{\alpha\beta}}{6}$$

- $J_\mu K^\mu$, where $K^\mu = 2 A_\nu \tilde{F}^{\mu\nu}$.
  It is not $U(1)$ invariant unless $\nabla_{[\mu} J_{\nu]} = 0$, which means that $J_\mu \propto \nabla_\mu \tilde{\mathcal{O}}$.
  This makes $J_\mu K^\mu \rightarrow \tilde{\mathcal{O}} F_{\mu\nu} \tilde{F}^{\mu\nu}$.

$\tilde{\mathcal{O}}F\tilde{F}$?

- $J_{\mu\nu}F^{\mu\nu}$ may give rise to P-violating effect in the loop level.

  Formally, $\delta J_{\mu\nu} = \hat{K}_{\mu\nu\alpha\beta}F^{\alpha\beta} \;\rightarrow\; F^{\mu\nu}\hat{K}_{\mu\nu\alpha\beta}F^{\alpha\beta}$.

  If $\hat{K}_{\mu\nu\alpha\beta} \supset \tilde{\mathcal{O}}_\epsilon \epsilon_{\mu\nu\alpha\beta}/2$, then $F^{\mu\nu}\hat{K}_{\mu\nu\alpha\beta}F^{\alpha\beta} \;\rightarrow\; \tilde{\mathcal{O}}_\epsilon F_{\mu\nu}\tilde{F}^{\mu\nu}$.

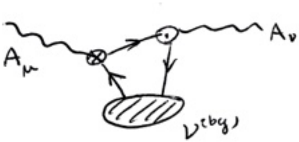
  However, $\tilde{\mathcal{O}}_\epsilon$ is non-local, produces frequency-dependent ICB angle.

- Cosmic Magnetic field breaks parity.

  However, it will produces Anisotropic Cosmic birefringence angle.

Dipole operator $J_{\mu\nu}F^{\mu\nu}$

$$J_{\mu\nu}F^{\mu\nu}\ ,\quad J_{\mu\nu}=\bar{\nu}\lambda\sigma_{\mu\nu}\nu\ ,\quad \lambda=\underbrace{\mathfrak{M}}_{\text{magnetic}}+\underbrace{i\mathfrak{E}\gamma^5}_{\text{electric}}$$



$$F^{\mu\nu}\hat{K}_{\mu\nu\alpha\beta}F^{\alpha\beta}\to\tilde{\mathcal{O}}_\epsilon F_{\mu\nu}\tilde{F}^{\mu\nu}\ ,\quad \tilde{\mathcal{O}}_\epsilon\propto\epsilon^{\mu\nu\alpha\beta}\hat{K}_{\mu\nu\alpha\beta}$$

$$\hat{K}_{\mu\nu\alpha\beta}\propto\bar{\nu}^{(\mathrm{bg})}\sigma_{\mu\nu}\lambda\frac{1}{i\not{\partial}-m}\sigma_{\alpha\beta}\lambda\nu^{(\mathrm{bg})}$$

$$\begin{aligned}\tilde{\mathcal{O}}_\epsilon&\propto\bar{\nu}^{(\mathrm{bg})}\epsilon^{\mu\nu\alpha\beta}\sigma_{\mu\nu}\lambda(i\not{\partial}-m)^{-1}\sigma_{\alpha\beta}\lambda\nu^{(\mathrm{bg})}\\&=-4im\bar{\nu}^{(\mathrm{bg})}\lambda\gamma^5(\partial^2+m^2)^{-1}\lambda\nu^{(\mathrm{bg})}\end{aligned}$$

Using $\epsilon^{\mu\nu\alpha\beta}\sigma_{\mu\nu}=-2i\gamma^5\sigma^{\alpha\beta}$, $\sigma^{\mu\nu}\sigma_{\mu\nu}=12$ and $\sigma^{\mu\nu}\gamma^\alpha\sigma_{\mu\nu}=0$.

Dipole operator $J_{\mu\nu}F^{\mu\nu}$

$$\langle \tilde{\mathcal{O}}_\epsilon \rangle \propto m \left\langle \bar{\nu}^{(\mathrm{bg})} \lambda \gamma^5 (\partial^2 + m^2)^{-1} \lambda \nu^{(\mathrm{bg})} \right\rangle$$

$\langle \bar{\nu}\gamma^5 \nu\rangle = 0 \neq \langle \bar{\nu}\nu\rangle$, this implies that

$$\langle \tilde{\mathcal{O}}_\epsilon \rangle \propto m(\mathfrak{M}\mathfrak{E} + \mathfrak{E}\mathfrak{M}) \int \frac{d^3 p_\nu}{(2\pi)^3} \frac{m}{E_\nu} \frac{n(p_\nu) + \bar{n}(p_\nu)}{(p_\nu + p_\gamma)^2 - m^2}$$

Let $p_\nu = (E_\nu, \mathbf{p}_\nu)$ and $p_\gamma = (\omega, \omega \mathbf{n})$, where $|\mathbf{n}| = 1$.

$$(p_\mu + p_\gamma)^2 - m^2 = 2\omega(E_\nu - \mathbf{n}\cdot\mathbf{p}_\nu)$$

$$\beta \propto \langle \tilde{\mathcal{O}}_\epsilon \rangle\big|_{\mathrm{LSS}} \propto \frac{1}{\omega}, \quad \text{Frequency-dependent ICB}$$

ICB from $\overline{B}$

$$\frac{1}{2}\int d^4k_1 d^4k_2 A_\mu(k_1)\Pi^{\mu\nu}(k_1,k_2)A_\nu(k_2)$$

- Introduce helicity basis with respect to **k**, $\{\epsilon^\mu_{\mathrm{L}}, \epsilon^\mu_+, \epsilon^\mu_-\}$.
  So $\epsilon^\mu_{\mathrm{L},\pm}u_\mu = 0$, $k_\mu\epsilon^\mu_\pm = 0 \neq k_\mu\epsilon^\mu_{\mathrm{L}}$.
- Including the background, one has $k_1^\mu + k_2^\mu \propto u^\mu \neq 0$.
  Expand $k_1^\mu = \omega_1 u^\mu + |\mathbf{k}|\epsilon^\mu_{\mathrm{L}}$ and $k_2^\mu = \omega_2 u^\mu - |\mathbf{k}|\epsilon^\mu_{\mathrm{L}}$.
- With the background unit vector $u_\mu \propto \nabla_\mu t$, one has $\overline{B}^\mu = u_\nu \tilde{F}^{(0)\mu\nu}$.
- One could expand $\overline{B}^\mu = \overline{B}_{\mathrm{L}}\epsilon^\mu_{\mathrm{L}} + \overline{B}_+\epsilon^\mu_+ + \overline{B}\epsilon^\mu_-$.
- $F_{\mu\nu}\tilde{F}^{\mu\nu} \implies \Pi^{\mu\nu} \supset \epsilon^{\mu\nu\alpha\beta}(k_1)_\alpha(k_2)_\beta \propto \epsilon^{\mu\nu\alpha\beta}u_\alpha(\epsilon_{\mathrm{L}})_\beta \propto \epsilon^{*\mu}_+\epsilon^\nu_+ - \epsilon^{*\nu}_-\epsilon^\mu_-$
  $(F^{(0)}_{\alpha\beta}F^{(0)\alpha\beta})(F_{\mu\nu}\tilde{F}^{\mu\nu}) = |\overline{B}|^2 F\tilde{F}$ term gives ICB if $|\overline{B}|$ is uniform.

ACB from Cosmic Magnetic field $\overline{B}$

$$(F^{(0)}_{\alpha\beta}F^{\alpha\beta})(F^{(0)}_{\mu\nu}\tilde{F}^{\mu\nu}) = (\overline{B}_\alpha u_\beta F^{\alpha\beta})(\overline{B}_\mu u_\nu \tilde{F}^{\mu\nu})$$

- $F^{\alpha\beta} \to k_1^\alpha A^\beta(k_1) - k_1^\beta A^\alpha(k_1)$ and $\tilde{F}^{\mu\nu} \to \epsilon^{\mu\nu\alpha\beta}(k_2)_\alpha A_\beta(k_2)$.
- $$\overline{B}_\alpha u_\beta F^{\alpha\beta} \to \left[(\overline{B}\cdot k_1)u^\mu - (u\cdot k_1)\overline{B}^\mu\right] A_\mu(k_1) \propto (\overline{B}_+\epsilon_+^\mu + \overline{B}_-\epsilon_-^\mu)A_\mu(k_1) + \cdots$$
- $\overline{B}_\mu u_\nu \tilde{F}^{\mu\nu} \propto (\overline{B}_+\epsilon_+^\nu - \overline{B}_-\epsilon_-^\nu)A_\nu(k_2)$
- $(F^{(0)}_{\alpha\beta}F^{\alpha\beta})(F^{(0)}_{\mu\nu}\tilde{F}^{\mu\nu}) \supset C\overline{B}_+\overline{B}_-(\epsilon_+^{*\mu}\epsilon_+^\nu - \epsilon_-^{*\mu}\epsilon_-^\nu)A_\mu A_\nu$.

This gives ACB due to the dependence on components of $\overline{B}$.

Relavant Operator $\tilde{O}F\tilde{F}$!

To explain the observed frequency-independent ICB, only CS-type operator $\tilde{O}F_{\mu\nu}\tilde{F}^{\mu\nu}$ should be considered.

SMEFT and LEFT

$$\mathcal{L}_{\rm CS} = \frac{\alpha}{8\pi} \sum_a \frac{\tilde{\mathcal{O}}_a}{\Lambda_a^n} F_{\mu\nu} \tilde{F}^{\mu\nu} \,, \qquad n = \dim[\tilde{\mathcal{O}}_a]$$

Building Blocks:

$$H(\dim 1)\,, \quad D_\mu(\dim 1)\,, \quad \psi(\dim 3/2)\,, \quad X_{\mu\nu}(\dim 2)$$

- $n = 2$: $H^\dagger H$
- $n = 3$: (LEFT) $\mathcal{C}^{ij} \bar{e}^i P_{\rm L} e^j + \text{h.c.}, \quad (e \to \nu, d, u)$.
  One does not have hypercharge singlet in SM.
  Therefore, one has to go down to $SU(3)_{\rm c} \times U(1)_{\rm EM}$. (Low-energy EFT)
- $n = 4$: $\sum_{X=F,Z,W,G} X_{\alpha\beta} X^{\alpha\beta} + X_{\alpha\beta} \tilde{X}^{\alpha\beta}$

ICB from $\tilde{\mathcal{O}} F \tilde{F}$?

$$\frac{\alpha}{8\pi}\tilde{\mathcal{O}} F_{\mu\nu}\tilde{F}^{\mu\nu} \xrightarrow{\text{Cosmos bg.}} \frac{1}{4}\phi_{\tilde{\mathcal{O}}} F_{\mu\nu}\tilde{F}^{\mu\nu} \,, \quad \phi_{\tilde{\mathcal{O}}} = \frac{\alpha}{2\pi}\langle\tilde{\mathcal{O}}\rangle$$

The ICB effect is given by (same as the axion)

$$\beta = \frac{1}{2}\left[\phi_{\tilde{\mathcal{O}}}(t_{\text{LSS}}) - \phi_{\tilde{\mathcal{O}}}(t_0)\right]$$

$n = 2$ ICB

$$\frac{\alpha}{8\pi}\frac{H^\dagger H}{\Lambda_H}F_{\mu\nu}\tilde{F}^{\mu\nu} \implies \phi_{\tilde{\mathcal{O}}_2} = \frac{\alpha v^2}{2\pi\Lambda_H^2}\,, \quad v \equiv \langle H \rangle$$

$$\beta \simeq \frac{\alpha v_0^2}{2\pi\Lambda_H^2}\delta v\,, \quad \delta v \equiv \frac{v - v_0}{v_0}$$

- CMB gives $\frac{\Delta m_e}{m_e} < (4 \pm 11) \times 10^{-3}$, this indicates $\delta v \lesssim 10^{-3}$–$10^{-2}$.
- Collider gives $\Lambda_H > 1\,\mathrm{TeV}$.

$$\beta < (4 \times 10^{-5})^\circ \ll \beta_{\mathrm{obs}}$$

$n = 3$ ICB

Charged particle are suppressed by baryon-to-photon ratio $\eta \sim 10^{-10}$.
Thus, the only possible candidate here is Cosmic Neutrino Background(CνB).
Assume Dirac neutrino.

$$\frac{\alpha}{8\pi}\frac{\tilde{\mathcal{O}}_\nu}{\Lambda_\nu}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \quad \tilde{\mathcal{O}}_\nu = \frac{(\tilde{\mathcal{C}}_\nu^\dagger + \tilde{\mathcal{C}}_\nu)^{ij}}{2}\bar{\nu}^i\nu^j + \frac{(\tilde{\mathcal{C}}_\nu^\dagger - \tilde{\mathcal{C}}_\nu)^{ij}}{2}\bar{\nu}^i\gamma^5\nu^j$$

- $\langle\bar{\nu}^i\gamma^5\nu^j\rangle = 0$ and $\langle\bar{\nu}^i\nu^j\rangle = \delta^{ij}\mathcal{F}(t)$,

$$\mathcal{F}(t) = \int \frac{d^3p}{(2\pi)^3}\frac{m_i}{E_{\mathbf{p}}}\left[n^i(p,t) + \bar{n}^i(p,t)\right]\,,$$

  where m_i is the i-th neutrino mass and n^i is phase-space number density. ($\bar{n}$ is that for anti-neutrino.)
- This indicates that

$$\phi_{\tilde{\mathcal{O}}_\nu}(t) = \frac{\alpha}{4\pi}\frac{\mathrm{tr}\left[(\tilde{\mathcal{C}}_\nu^\dagger + \tilde{\mathcal{C}}_\nu)\mathcal{F}(t)\right]}{\Lambda_\nu^3}$$

Majorana case with an extra $1/2$.

$n = 3$ ICB

- Due to cosmic expansion, $\phi_{\tilde{\mathcal{O}}_\nu}(t_0) \ll \phi_{\tilde{\mathcal{O}}_\nu}(t_{\mathrm{LSS}})$.
- At recombination $T_{\mathrm{LSS}} \sim 0.3\,\mathrm{eV}$,

$$\mathcal{F}(t_{\mathrm{LSS}}) \simeq 0.5 \frac{m_i}{T_{\mathrm{LSS}}} \left(N^i + \bar{N}^i\right) \ , \quad m_i \ll T_{\mathrm{LSS}}$$

where N and $\bar{N}$ are number density for neutrino and anti-neutrino.

- Possible ICB angle is

$$\beta \simeq \frac{1}{2}\phi_{\tilde{\mathcal{O}}_\nu}(t_{\mathrm{LSS}}) \simeq 0.008^\circ \frac{\alpha}{137^{-1}} \sum_i \frac{m_i}{T_{\mathrm{LSS}}} \left(\tilde{\mathcal{C}}_\nu + \tilde{\mathcal{C}}_\nu^\dagger\right) \frac{N^i + \bar{N}^i}{\Lambda_\nu^3} \ ,$$

$n = 3$ ICB

$$\beta \simeq 0.008^\circ \frac{\alpha}{137^{-1}} \sum_i \frac{m_i}{T_{\rm LSS}} \left(\tilde{\mathcal{C}}_\nu + \tilde{\mathcal{C}}_\nu^\dagger\right) \frac{N^i + \bar{N}^i}{\Lambda_\nu^3} ,$$

- CνB neutrino number density today is $\sim 56\,\mathrm{cm}^{-3}$. Tracing back to LSS gives $N^{1/3} \sim \mathcal{O}(10^{-10})\,\mathrm{GeV}$.
- Collider gives $\Lambda_\nu \simeq 10^{-2}$–$10^2\,\mathrm{GeV}$.
- Taking $\tilde{\mathcal{C}} \sim \mathcal{O}(1)$, $m_i \sim 0.1\,\mathrm{eV}$ and $T_{\rm LSS} \sim 0.3\,\mathrm{eV}$, one has

$$\beta < (10^{-27})^\circ \ll \beta_{\rm obs}$$

$n = 4$ ICB

$$\frac{\alpha}{8\pi}\frac{F_{\alpha\beta}F^{\alpha\beta}+F_{\alpha\beta}\tilde{F}^{\alpha\beta}}{\Lambda_F^4}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

Cosmic Background magnetic field. Let $F \to F^{(0)} + F$.
Then, quadratic terms are

- $(F^{(0)}_{\alpha\beta}F^{\alpha\beta})(F^{(0)}_{\mu\nu}\tilde{F}^{\mu\nu}) \propto \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} \quad \Longrightarrow \quad$ Anistropic CB
- $(F^{(0)}_{\mu\nu}\tilde{F}^{\mu\nu})^2 \propto \vec{E}_{\parallel}\cdot\vec{E}_{\parallel}$
- CS-type term $(F^{(0)}_{\alpha\beta}F^{(0)\alpha\beta})(F_{\mu\nu}\tilde{F}^{\mu\nu}) \propto \vec{E}\cdot\vec{B} \quad \Longrightarrow \quad$ ICB

They are of the same order!

Possible New Light Field

$$\frac{\alpha}{8\pi}\frac{\Phi^\dagger\Phi}{\Lambda^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad \text{for a scalar } \Phi$$
$$\frac{\alpha}{8\pi}\frac{\bar{\chi}\chi}{\Lambda^3}F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad \text{for a fermion } \chi$$

$\langle\Phi^\dagger\Phi\rangle$ and $\langle\bar{\chi}\chi\rangle$ should be time-dependent backgrounds:

- Classical fields : similar to Axion
- Pair condensates : effectively the same to Axion & Require exotic cosmological scenario.
- Particles : like CνB.

Possible New Light Field

For a single particle, $E_{\mathbf{p}} \geq m$. In the cosmological background

$$\langle \Phi^\dagger \Phi \rangle \lesssim \frac{\rho}{m^2} \,, \quad \langle \bar{\chi}\chi \rangle \lesssim \frac{\rho}{m}$$

At the LSS, one should have $\rho < \rho_{\mathrm{c,LSS}} \simeq (3 \times 10^{-13}\,\mathrm{TeV})^4$. Therefore,

$$m \lesssim 10^{-14}\,\mathrm{eV} \left(\frac{|\beta|}{0.3^\circ}\right)^{-1/2} \left(\frac{\Lambda}{\mathrm{TeV}}\right)^{-1} \quad (\mathrm{Scalar})$$

$$m \lesssim 10^{-40}\,\mathrm{eV} \left(\frac{|\beta|}{0.3^\circ}\right)^{-1} \left(\frac{\Lambda}{\mathrm{TeV}}\right)^{-3} \quad (\mathrm{Fermion})$$

Takeaway

- From EFT point of view, only CS-type operator could give frequency-blind ICB.
- SM particle could not give observed ICB under standard cosmology.
- If a new field is responsible to ICB under standard cosmology, it should be light.

Thank You!

$\langle \bar{\nu}\nu \rangle$

$$\nu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left[a^s_{\mathbf{p}} u^s(p) e^{-ipx} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ipx} \right]$$

$$\bar{\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left[b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ipx} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ipx} \right]$$

with $\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs}$.
Dirac equation gives

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \beta}\xi^s \\ \sqrt{p \cdot \bar{\beta}}\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \beta}\eta^s \\ -\sqrt{p \cdot \bar{\beta}}\eta^s \end{pmatrix}, \quad \xi^\dagger \xi = \eta^\dagger \eta = 1\,,$$

where $\beta^\mu = (1, \vec{\beta})$, $\bar{\beta}^\mu = (1, -\vec{\beta})$ and $\vec{\beta}$ are Pauli matrices.
Using $\bar{u}^s u^r = -\bar{v}^s v^r = 2m\delta^{sr}$ and $\bar{u}^s \gamma^5 u^r = \bar{v}^s \gamma^5 v^r = 0$, one has

$$\langle \bar{\nu}\gamma^5 \nu \rangle = 0\,, \quad \langle \bar{\nu}\nu \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_{\mathbf{p}}} [n(p) + \bar{n}(p)]$$