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0.0.0. Definition. Partition

A *partition* of $[a, b]$ is a finite list of the form $x_0, x_1, \ldots, x_n$, s.t.

$$a = x_0 < x_1 < \cdots < x_n = b$$

0.0.1. Definition. Lower/Upper Darboux Sum

Recall

$$\inf_A f := \inf\{f(x) \mid x \in A\}$$

$$\sup_A f := \sup\{f(x) \mid x \in A\}$$

Suppose $f : [a, b] \to \mathbb{R}$ bounded and P a partition of $[a, b]$. The *lower Darbox sum* $L(f, P, [a, b])$ and the *upper Darboux sum* $U(f, P, [a, b])$ are defined by

$$L(f, P, [a, b]) := \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \inf_{[x_{j-1}, x_j]} f$$

$$U(f, P, [a, b]) := \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \sup_{[x_{j-1}, x_j]} f$$

0.0.2. Definition. Lower/Upper Darboux Integral

Suppose $f : [a, b] \to \mathbb{R}$ bounded. The *lower Darboux integral* $L(f, [a, b])$ and the *upper Darboux integral* $U(f, [a, b])$ are defined by

$$L(f, [a, b]) := \sup_{\mathbf{P}}(P \mapsto L(f, P, [a, b]))$$

$$U(f, [a, b]) := \inf_{\mathbf{P}}(P \mapsto U(f, P, [a, b]))$$

where $\mathbf{P} = \{P \mid P \text{ partition of } [a, b]\}$.

0.0.3. Definition. Riemann Integral

Suppose $f : [a, b] \to \mathbb{R}$ bounded. f *Riemann integrable* iff the lower and upper Darboux integrals equate. If f is Riemann integrable, the *Riemann integral* $\int_a^b f$ is defined by

$$\int_a^b f := L(f, [a, b]) = U(f, [a, b])$$

0.0.4. f continuous $\Rightarrow$ f Riemann integrable

Proof. Let $\varepsilon > 0$. Suppose $f : [a, b] \to \mathbb{R}$ bounded and continuous. Then it is uniformly continuous, i.e.

$$\forall x, x' \in [a, b]. \quad \exists \delta > 0. \quad |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$$

Let $n \in \mathbb{N}^{\geq 1}$ s.t. $n > \frac{b-a}{\delta} \Leftrightarrow \frac{b-a}{n} < \delta$. Let P be the equally spaced partition $x_0, x_1, ..., x_n$ of $[a, b]$ s.t.

$$\forall j \in \mathbb{N}^{\geq 1}. \quad x_j - x_{j-1} = \frac{b-a}{n}$$

Then

$$\begin{aligned} U(f, [a, b]) - L(f, [a, b]) &\leq U(f, P, [a, b]) - L(f, P, [a, b]) \\ &= \frac{b-a}{n} \sum_{j=1}^{n} \left(\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f \right) \\ &\leq (b-a)\varepsilon \end{aligned}$$

1. Measures

1.0. Outer Measure on $\mathbb{R}$

1.0.0. Definition. Length

The *length* ℓ of an open interval i is defined by

$$\ell := \begin{cases} b - a & \text{if } I = (a, b) \text{ for some } a, b \in \mathbb{R} \text{ s.t. } a < b \\ \infty & \text{if } I \in \{(-\infty, a), (a, \infty), (-\infty, \infty)\} \text{ for some } a \in \mathbb{R} \\ 0 & \text{if } I = \emptyset \end{cases}$$

1.0.1. Definition. Outer Measure

The *outer measure* $|A|$ of a set $A \subset \mathbb{R}$ is defined by

$$|A| := \inf\left\{ \sum_{k=0}^{\infty} \ell(I_k) \,\middle|\, I_0, I_1, ... \text{ open intervals s.t. } A \subset \bigcup_{k=0}^{\infty} I_k \right\}$$

Note that $r + \infty = \infty + \infty = \infty$ for $r \in \mathbb{R}$. Also recall

$$\sum_{k=0}^{\infty} x_k := \lim_{n \to \infty} \sum_{k=1}^{n} x_k$$

1.0.2. A countable $\Rightarrow$ A outer measure 0

Proof. Suppose $A = \{a_0, a_1, ...\} \subset \mathbb{R}$ countable. For $k \in \mathbb{N}$ and arbitrary $\varepsilon > 0$, let

$$I_k = \left(a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}\right)$$

Then $I_0, I_1, \ldots$ are open intervals whose union contains A, and

$$\sum_{k \in \mathbb{N}} \ell(I_k) = 2\varepsilon \Rightarrow |A| = 0$$