OPTI 536 – Introduction to Image Science
Spring, 2025

Homework 2

Assigned: Feb. 19, 2025
Due: Mar. 5, 2025 at 11:59 pm

In this assignment, we will implement an angular spectrum propagator that can precisely simulate how light propagates through a variety of simulations. We will then use this propagator to analyze how multiple optical systems behave. For the duration of this assignment, **assume that t=0 for all plane waves and fields**.

Feel free to discuss the homework with others in the class or the instructors. You may use any books or websites you find, but please give a brief citation for any references you use. You must write up your final solutions independently without direct copying from these sources or from solutions by other students. **Please be aware that we implemented advanced measures to check for duplicated code (also for code downloaded from the internet)!**

Please submit your solutions to the D2L website by 11:59pm on the due date. This time, your submission should consist of a **pdf that includes all requested plots and the answers to the questions** PLUS **one code file for each problem**. Each code file should be a single file (e.g., ***.m-file**, ***.py** or ***.ipynb**-file) that automatically generates all requested plots/figures when executed. Again, make sure to include the requested plots and figures also in your “answer-pdf”.

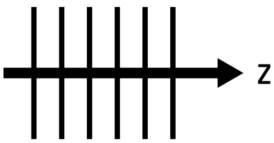
Please do not convert your pdf to grayscale this time (color is important) and do not lock/encrypt any of your submitted files.

Start with this homework EARLY, so that we have enough time to help you with any problems.

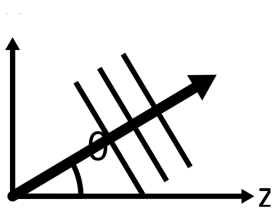
Problem 1 – Propagation of Plane Waves (10 points)

Before simulating arbitrary light fields, we will simplify the problem by propagating individual plane waves. In this problem, we will investigate how to do this. This problem does not require programming.

The formula for a phase of plane wave propagating at an angle θ from the z-axis with wavelength λ is given by $\varphi(x, z) = k[z\cos(\theta) - x\sin(\theta)]$.



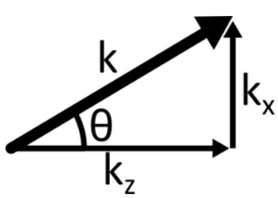
a) A λ=633 nm plane wave is propagating along the z-axis. At (x,y,z) = (0, 0, 0) the wave has a phase of φ = 0. What is the phase at (x,y,z) = (0, 0, 1 m)? Either unwrapped or wrapped phase are acceptable.



b) A λ=633 nm plane wave is propagating at an angle of 1 mrad from the z-axis. At (x,y,z) = (0, 0, 0) the wave has a phase of φ = 0. Either unwrapped or wrapped phase are acceptable.

What is the phase at:
i) (x,y,z) = (0, 0, 1 m)?
ii) (x,y,z) = (1 m, 0, 1 m)?
iii) (x,y,z) = (-1 m, 0, 1 m)?
iv) (x,y,z) = (2 m, 0, -1 m)?

c) In a 2D coordinate system, the wavenumber k of a plane wave is defined as $k^2 = k_x^2 + k_z^2$, where $k = \frac{2\pi}{\lambda} = 2\pi f_x$, k_x is the horizontal spatial frequency, and k_z is the longitudinal wavenumber.



Define the following quantities in terms of k_x and k_z:
i) $\cos(\theta)$
ii) $\sin(\theta)$
iii) The phase of a plane wave propagating at angle θ from the z-axis at the position (x,y,z) = $(x_0, 0, z_0)$.

d) A λ=633 nm plane wave propagates with $k_x = 24.815\frac{rad}{m}$ and $k_z = 9.926 * 10^6\frac{rad}{m}$.
At (x,y,z) = (0,0,0) the phase is 0. What angle θ from the z-axis is this plane wave propagating at?

e) A plane wave starts with a phase φ_0 at (x,y,z) = (0,0,0). Derive a formula for the phase at (x,y,z) = (0,0, *Δz*) in terms of *Δz*, k, and k_x (but not k_z).

Note that the electric field of a plane wave is given by $E(\boldsymbol{r}) = Ae^{j\varphi(\boldsymbol{r})}$, where A is a constant amplitude and $\varphi(\boldsymbol{r})$ is the phase calculated earlier in the problem. Since the amplitude is constant, we can calculate the complex electric field at any location in space using the methods in this problem.

Problem 2 – Fourier Analysis of Plane Waves (10 points)

Plane waves have a few interesting characteristics in the Fourier domain that are helpful for propagation. In this problem, we will discover these. This problem requires programming.

a) Using a programming language of your choice, do the following:

i) Plot the magnitude and phase of a normal-incidence λ=633 nm plane wave, at z=0, over the range x = [-1 cm, 1cm]. Only make a 1D plot – don't consider the y-axis for now.

ii) Plot the magnitude of the Fourier transform of the plane wave field. Make sure the units on the x-axis are correct. If you don't know how to get the x-axis units, refer to the "noisy signal" example in the following MATLAB documentation: https://www.mathworks.com/help/matlab/ref/fft.html.

iii) Plot the magnitude and phase of a λ=633 nm plane wave propagating at an angle of θ=100 µrad, at z=0, over the same range. Additionally, plot the Fourier transform magnitude.

iv) Plot the magnitude and phase of a λ=633 nm plane wave propagating at an angle of θ=5 mrad, at z=0, over the same range. Plot the Fourier transform magnitude.

v) What do you observe in the Fourier transform as the plane wave angle increases?

b) When two plane waves propagating at different angles interfere, they produce a sinusoidal interference pattern.

i) Using a programming language of your choice, plot the magnitude of an electric field following the formula $E(x) = \sin\left(\frac{2\pi}{1\ mm}x\right)$ over the range x = [-1 cm, 1 cm].

ii) Plot the Fourier transform magnitude of this sinusoidal field. What do you observe?

What is being observed in these parts is that every element of the Fourier transform of a field is the amplitude and phase of a single plane wave that makes up the field. Every optical field is made up of the interference of these plane waves.

Thus, to propagate an arbitrary light field forward, we can use the Fourier transform to find the magnitude and phase of the plane wave components, then propagate each plane wave forwards. The combination of these plane waves is the propagated version of the input field.

Problem 3 – Propagating Arbitrary Waves (30 points)

In this problem, we will put the theory from problems 1 and 2 into practice by implementing a program that can propagate arbitrary waves using the angular spectrum propagation algorithm. This problem requires programming –parts (b-d).

a) Given a field $E(x, y, z = z_0)$, the field at any z value can be found using the *angular spectrum propagation* algorithm, defined using the three step process below:

STEP 1: Obtain the Fourier Transform of the input field

$$F_E\left(k_x, k_y, z = z_0\right) = \mathcal{F}\{E(x, y, z = z_0)\}$$

STEP 2: Multiply the Fourier transform by the *kernel function*:

$$F_E\left(k_x, k_y, z = z_0 + \Delta z\right) = F_E\left(k_x, k_y, z = z_0\right) * e^{j\Delta z\sqrt{k^2 - k_x^2 - k_y^2}}$$

STEP 3: Obtain the result by taking the inverse Fourier transform:

$$E(x, y, z = z_0 + \Delta z) = \mathcal{F}^{-1}\{F_E\left(k_x, k_y, z = z_0 + \Delta z\right)\}$$

Using this information, answer the following questions:

i) Given the insight from problem 2, why do we take the Fourier transform in step 1?

ii) Given your answer to problems 1e and 2, what is the kernel function and why do we multiply the Fourier transform by it, and where does the phase of $\Delta z\sqrt{k^2 - k_x^2 - k_y^2}$ come from?

iii) Its time to put everything together. Describe in qualitative terms what we are doing and why this method can solve for the electric field at a different z-position.

b) Using a programming language of your choice, write a function that takes an 2D input electric field $E(x, y, z = 0)$ that is defined over a range $x \in [-x_{max}, x_{max}]$ and $y \in [-x_{max}, x_{max}]$ with a side length of N pixels in each direction. The program will additionally take the wavelength λ of the electric field as an input, and a propagation distance z.

Given these inputs, the function should calculate and return the magnitude and phase of the field $E(x, y, z)$ using the angular spectrum propagation algorithm. Report your code.

c) To validate your code, define the Gaussian field $E(x, y, 0) = e^{-\left(\frac{x^2+y^2}{(25\ mm)^2}\right)}$ with $x_{max} = 20\ cm$, λ=633 nm, and at least N = 1024 pixels per side. At a distance of z=5.373 km, this beam should have doubled in width, and the centroid should be in the center of the simulation space. Verify this.

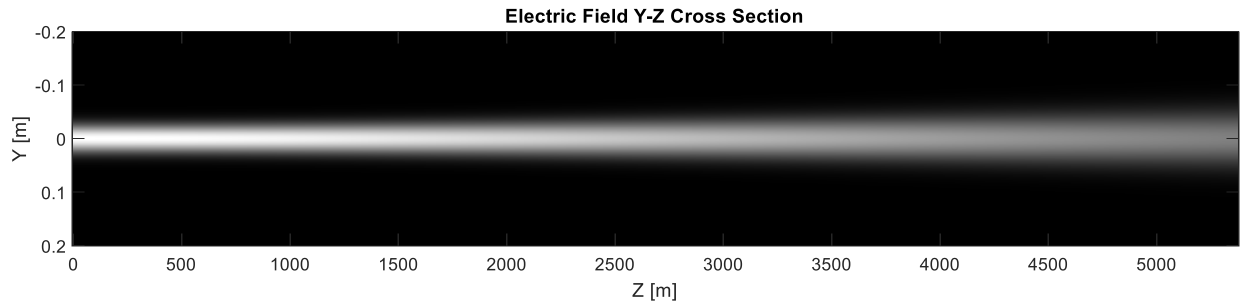
d) Simulate a field that has just passed through a two-slit mask. Assume that the slits are 250 µm apart, that the field has a wavelength of λ=633 nm, and that the interference pattern is observed at a distance of z=1 m. Set $x_{max} = 2\ cm$ and at least N = 512 pixels per side. Plot the magnitude of the observed field, and write a few words describing what you see.

Note: if the side length of your simulation is even and the simulation output is slightly off-center, you may have an off-by-one error when defining the spatial frequencies in the Fourier domain. If you encounter this, you can refer to the documentation linked in problem 2(a)(ii) to ensure that your frequencies are defined correctly.

Problem 4 – Cross-Sections and Zero Padding (20 points)

In this problem, we will discover common artifacts seen during angular spectrum propagation, as well as how to fix them. This problem requires programming.

a) A "cross-section" of a beam can be taken by propagating a field many times in a row, saving the central column of the field each step. For instance, the following cross-section of a Gaussian beam propagation was created using this method:



i) Using the Gaussian beam described in problem 3(c), simulate the cross-section of the bream propagation, and verify that it appears similar to the result shown above.

ii) Propagate 20 km instead of 5.375 km. Report the output field magnitude and cross section. Normalize the cross-section so the maximum field magnitude in each "slice" equals 1. Describe what you see.

b) Similar effects can be observed with angled beams, which provides more insight into the simulation result in part (a)(ii).

i) Simulate the Gaussian beam described in problem 3(c), but apply a linear phase so it propagates at an angle of 20 μrad. Propagate a distance of 5.375 km and report the field magnitude and cross-section. Verify that the beam centroid is at the expected location. Do not normalize the cross sections like you did in part (a)(ii).
HINT: To apply an angle to the beam, multiply the field by $e^{jk\sin(\theta)y}$.

ii) Repeat the simulation, propagating at an angle of 60 μrad instead. Report the field magnitude and cross-section. Describe what you see, and do your best to explain what might cause it.

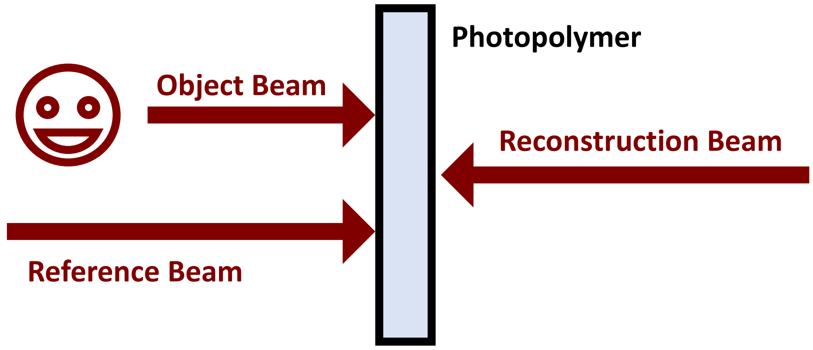
c) The artifacts seen in parts (a) and (b) can be fixed by adding zero-padding to the input field. By padding the input field with zeroes on each side before propagating, then removing the padding upon completion, many artifacts can be removed. Modify your propagation code to allow the user to specify a "zero-padding distance", in pixels, which will pad the input field during simulation. Report your code.

d) To validate your padded propagator, repeat the experiment in part (a)(ii) with 20cm padding on each side. Plotting cross-sections causes a windowing error with the padded propagator, so only propagate once and report the output field magnitude. Verify that the simulation artifacts have been removed.

Note: For a more advanced method of dealing with wrapping artifacts that isn't as reliant on padding, see: Kyoji Matsushima and Tomoyoshi Shimobaba, "Band-Limited Angular Spectrum Method for Numerical Simulation of Free-Space Propagation in Far and Near Fields," Opt. Express 17, 19662-19673 (2009)

Problem 5 – Inline Amplitude Holography (15 points)

In this problem, we will simulate the recording and reconstruction of a hologram. You may refer to the diagram below for an understanding of the experimental process. This problem requires programming.



a) Choose a grayscale image or make a drawing, and rescale the image to be 256x256 pixels. Report the image used.

b) Set this image to be the amplitude of an electric field with flat phase, assuming λ=633 nm and $x_{max} = 2\ cm$. Normalize the field, so that the maximum amplitude is 1. Then, propagate the field a distance of 500 mm. Show the amplitude and phase at this distance. We have just simulated the *object beam* in the above diagram. (Remember to pad your simulation if artifacts appear!)

c) The field interferes with a unit-amplitude plane wave (the *reference beam*) propagating parallel to the z-axis, resulting in the following field: $E_{interference}(x, y) = 1 + E_0(x, y)$, where $E_0(x, y)$ is the field that you simulated in part (b). Next, this interference pattern is recorded on a photopolymer, which causes the transmittance of the photopolymer to be $T(x, y) = \frac{|E_{interference}(x,y)|^2}{\max\left(|E_{interference}(x,y)|^2\right)}$, where $\max\left(|E_{interference}(x, y)|^2\right)$ is the maximum intensity of the interference pattern. Calculate the transmittance of the photopolymer, and report it as an image.

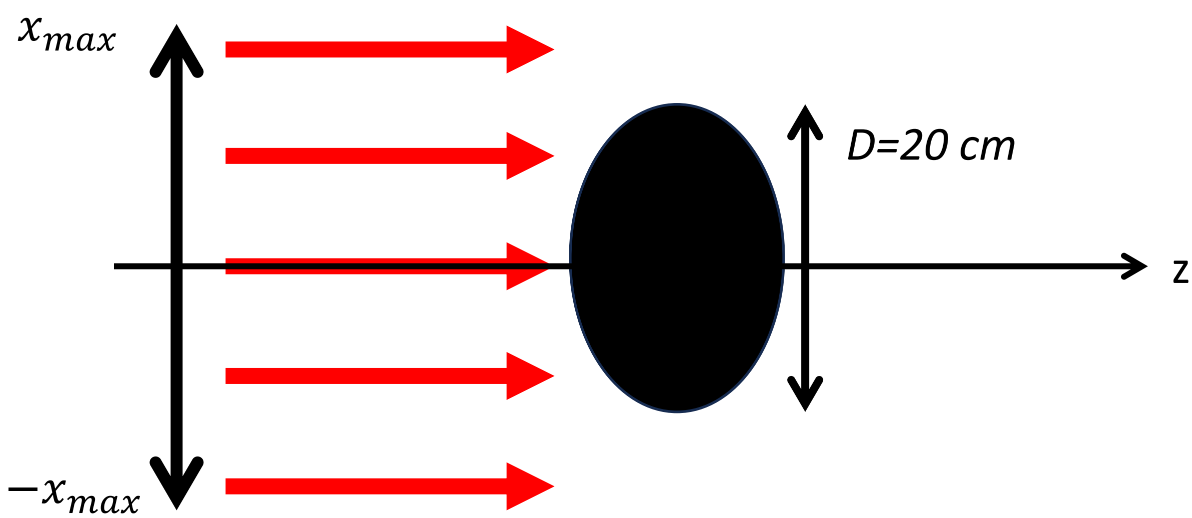
d) A unit-amplitude plane wave travelling in the opposite direction (The *reconstruction beam*) is sent through the photopolymer, which gives the plane wave an amplitude of $E(x, y) = T(x, y)$. Propagate this field 500 mm back to the starting plane, and report the field amplitude. Confirm that the reported field (the *object reconstruction*) looks similar to the object.

e) **Extra credit (2 points, does not need to be answered to get full points for this HW):**
Why is the reconstruction of your starting image not perfect? What changes would you make to the experiment to improve this reconstruction?

Problem 6 – Poisson's Spot (15 points)

In this problem, we will use the angular spectrum method to simulate the formation of Poisson's spot. This problem requires programming.

a) A normal-incidence plane wave is incident upon a circular occluder with diameter $D = 2\ cm$. Calculate the field directly after the occluder, and report the field magnitude. Set λ=633 nm, $x_{max} = 2\ cm$ and at least N = 512 pixels per side.



b) Propagate the field 5 meters, and confirm that a focus point appears in the center of the field, behind the occluder. Note: Remember to pad your simulation when performing your propagation!

c) **Extra credit (2 points, does not need to be answered to get full points for this HW)**: Explain why a focus point appears here!