Theorem 1. *Given a signed bipartite graph G, the problem of computing the maximum balanced (k, ϵ)-Bitruss is NP-hard.*

Proof. We prove the NP-hardness by reducing the maximum balanced (k, ϵ)-Bitruss problem to the 4-CNF problem, which is known to be NP-complete [10]. Suppose $\Psi = R_1 \wedge R_2 \wedge \cdot \wedge R_s$ is a 4-CNF formula such that $R_i = l_{i,-1} \vee l_{i,1} \vee l_{i,2} \vee l_{i,3}$, $i \in \{1, \ldots, s\}$. For each i we construct an unbalanced butterfly $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ where $\mathrm{sign}(e_{i,-1})$ = '-', and $\mathrm{sign}(e_{i,1}) = \mathrm{sign}(e_{i,1}) = \mathrm{sign}(e_{i,1})$ = '+'.

Let t be the number of compliment pairs in the formula Ψ. For the r-th pair of compliment literals $l_{i,p}$ and $l_{j,q}$ (i.e., $l_{i,p} = \neg l_{j,q}$) where $i, j \in \{1, \ldots, n\}$, $i \neq j$, $p, q \in \{-1, 1, 2, 3\}$, and $r \in \{1, \ldots, t\}$, we construct two butterflies $(e_{i,p}, e'_{r,1}, e'_{r,2}, e'_{r,0})$ and $(e_{j,q}, e'_{r,3}, e'_{r,4}, e'_{r,0})$ by adding 5 additional edges $e'_{r_0}, e'_{r,1}, e'_{r,2}, e'_{r,3}$, and $e'_{r,4}$. We require $\mathrm{sign}(e_{i,p}) = \mathrm{sign}(e'_{r,1})$, $\mathrm{sign}(e'_{j,q}) = \mathrm{sign}(e_{r,3})$, $\mathrm{sign}(e'_{r_0}) = \mathrm{sign}(e'_{r,2}) = \mathrm{sign}(e'_{r,4})$ = '+' such that both $(e_{i,p}, e'_{r,1}, e'_{r,2}, e'_{r,0})$ and $(e_{j,q}, e'_{r,3}, e'_{r,4}, e'_{r,0})$ are balanced.

Denote edges sets $E_M := \{e_{i,p} : i \in \{1, \ldots, s\}, p \in \{-1, 1, 2, 3\}\} \cup \{e'_{r,p} : r \in \{1, \ldots, t\}, p \in \{1, 2, 3, 4\}\}$ and $E_N := \{e'_{r,0} : r \in \{1, \ldots, t\}\}$. For each edge in E_M, we exclusively add $2k + 1$ edges to form a $(k + 1)$-bloom where every butterfly is balanced; for each edge in E_N, we exclusively add $2k - 1$ edges to form a k-bloom where every butterfly is balanced.

Let $G = (U, V, E)$ be the signed bipartite graph from the construction above. We have the following facts about G:

(1) There are only s unbalanced butterflies in G, namely, $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ for $i \in \{1, \ldots, s\}$;
(2) The balanced support of each edge in E_M is at least k; a removal of an edge in E_M does not cause the balanced support of any other edge in E_M drop below k;
(3) The balanced support of each edge in E_N is exactly $k + 1$.

Thus, a balanced $(k, 0)$-bitruss must by induced by removing one edge from the unbalanced butterfly $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ for each $i \in \{1, \ldots, s\}$. Consider the two balanced butterflies$(e_{i,p}, e'_{r,1}, e'_{r,2}, e'_{r,0})$ and $(e_{j,q}, e'_{r,3}, e'_{r,4}, e'_{r,0})$ for some $r \in \{1, \ldots, t\}$. If both $e_{i,p}$ and $e_{j,q}$ are removed, then $e'_{r,0}$ cannot be in a balanced $(k, 0)$-bitruss because $\mathrm{Sup}^+_G(e'_{r,0}) = k-1$. Therefore, to obtain the maximum balanced $(k, 0)$-bitruss, we should avoid removing a pair of edges corresponding to a pair of compliment literals in the formula Ψ, as it causes an additional loss of 1 edge.

We show that the transformation from Ψ to G is a reduction. Suppose Ψ has a satisfying assignment. Then in each clause $R_i = (l_{i,-1} \vee l_{i,1} \vee l_{i,2} \vee l_{i,3}$, at least literal $l_{i,p}$ that is assigned as "true" by such satisfying assignment. Let E_T denote a set of edges containing one edge in each unbalanced butterfly $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ that corresponds to a "true" literal, and hence the subgraph induced by $E \backslash E_T$ is the maximum balanced $(k, 0)$-bitruss. Note that no pair of edges in E_T correspond to a pair of compliment literals, so no edge in E_N fails to be in the maximum balanced $(k, 0)$-bitruss.

Conversely, suppose the subgraph induced by $E \backslash E_T$ is the maximum balanced $(k, 0)$-bitruss, where E_T is a set of edges containing one edge in each unbalanced butterfly $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ that corresponds to a "true" literal. Since no pair of edges in E_T correspond to a pair of compliment literals, otherwise some $e'_{r,0} \in N$ cannot be in the maximum balanced $(k, 0)$-bitruss, we can formulate a satisfying assignment by setting $l_{i,p} = 1$ for each $e_{i,p} \in E_T$. Therefore, the theorem holds.

□

Theorem 2. *Given a signed bipartite graph G, it is NP-hard to approximate the maximum balanced (k, ϵ)-bitruss within a factor of $|E|^{1-\delta}$, for any $\delta > 0$.*

Proof. This result can be proved by modifying the construction from the proof of NP-hardness together with an arbitrarily-large 5-bitruss.

Remark 1. *For an arbitrary $n > 0$, consider a bipartite graph G constructed by a sequence of 4-blooms as in Figure 4 :*

$$B_0 = (\{u_0, u_1\}, \{v_0, v_1, v_2, v_3\}),$$
$$B_1 = (\{u_2, u_3\}, \{v_2, v_3, v_4, v_5\}),$$
$$\cdots,$$
$$B_n = (\{u_{2n}, u_{2n+1}\}, \{v_{2n}, v_{2n+1}, v_{2n+2}, v_{2n+3}\})$$

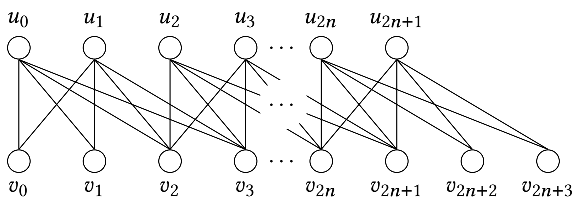

Figure 4: An arbitrarily-large 5-bitruss.

Then for an edge (u_i, v_j) in G, we observe that: (1) If $i \in \{2k, 2k+1\}$ and $j \in \{2k, 2k+1\}$ for some $0 < k \le n$, then (u_i, v_j) is contained in two 4-blooms, B_k and $(\{u_{2k-2}, u_{2k-1}, u_{2k}, u_{2k+1}\}, \{v_{2k}, v_{2k+1}\})$. The edge (u_i, v_j) gets 3 support from each bloom, while the two blooms intersect by a butterfly $[u_{2_k}, u_{2k+1}, v_{2k}, v_{2k+1}]$, so $\mathrm{Sup}((u_i, v_j)) = 5$; (2) if $i \in \{2k, 2k+1\}$ and $j \in \{2k+2, 2k+3\}$ for some $0 \le k < n$, then (u_i, v_j) is contained in two 4-blooms, B_k and $(\{u_{2k}, u_{2k+1}, u_{2k+2}, u_{2k+3}\}, \{v_{2k+2}, v_{2k+3}\})$. The edge (u_i, v_j) gets 3 support from each bloom, while the two blooms intersect by a butterfly $[u_{2_k}, u_{2k+1}, v_{2k+2}, v_{2k+3}]$, so $\mathrm{Sup}((u_i, v_j)) = 5$; (3) if $i \in \{0, 1\}$ and $j \in \{0, 1\}$, then (u_i, v_j) in the 4-bloom B_0 only, so $\mathrm{Sup}((u_i, v_j)) = 3$; (4) if $i \in \{2n, 2n+1\}$ and $j \in \{2n+2, 2n+3\}$, then (u_i, v_j) in the 4-bloom B_3 only, so $\mathrm{Sup}((u_i, v_j)) = 3$. By adding sufficient support to the edges in latter two cases, we get a 5-bitruss with $8n$ edges having support of exactly 5. If one of these $8n$ edges is removed, all other $8n-1$ edges have to be removed from the 5-bitruss, as their supports will decrease in cascade.

Suppose $\Psi = R_1 \wedge R_2 \wedge \cdot \wedge R_s$ is a 4-CNF formula such that $R_i = l_{i,-1} \vee l_{i,1} \vee l_{i,2} \vee l_{i,3}$, $i \in \{1, \ldots, s\}$. We construct s unbalanced butterflies same as we do in the last proof. For the r-th pair of compliment literals $l_{i,p}$ and $l_{j,q}$ where $i, j \in \{1, \ldots, n\}$, $i \neq j$, $p, q \in \{-1, 1, 2, 3\}$, and $r \in \{1, \ldots, t\}$, we add 12 additional edges to form a 7-bloom $(\{u_{r,0}, u_{r,1}\}, \{v_{r,0}, \ldots, u_{r,6}\})$ containing no unbalanced butterfly, in which $(u_{r,0}, v_{r,0}) = e_{i,p}$ and $(u_{r,0}, v_{r,0}) = e_{j,q}$. Then we construct the arbitrarily-large 5-bitruss as described in the above remark:

$$B_{r,0} = (\{u'_{r,0}, u'_{r,1}\}, \{v'_{r,0}, v'_{r,1}, v'_{r,2}, v'_{r,3}\}),$$
$$\cdots,$$
$$B_{r,n_r} = (\{u'_{r,2n_r}, u'_{r,2n_r+1}\}, \{v'_{r,2n_r}, v'_{r,2n_r+1}, v'_{r,2n_r+2}, v'_{r,2n_r+3}\}).$$

where $v'_0 = v_5$ and $v'_1 = v_6$, as shown in Figure 5.

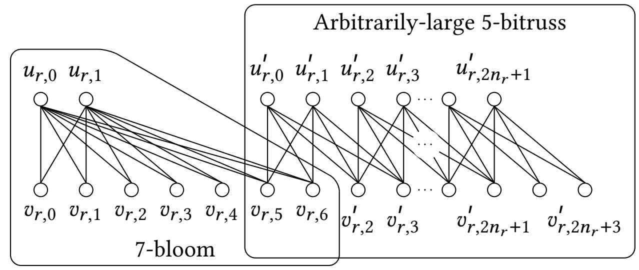

Figure 5: Partial construction of Theorem 2.

Moreover, we add $2t$ edges to form t additional butterflies:

$$[u'_{r,2n_r+1}, u'_{r+1,2n_{r+1}+1}, v'_{r,2n_r+3}, v'_{r+1,2n_{r+1}+3}]$$

for $r = 1, \ldots, t-1$, and

$$[u'_{t,2n_t+1}, u'_{1,2n_1+1}, v'_{r,2n_t+3}, v'_{1,2n_1+3}].$$

Lastly, let E_S be the set of the following edges:

(1) $(u'_{r,2n_r}, v'_{r,2n_r+2})$, $(u'_{r,2n_r+1}, v'_{r,2n_r+2})$, and $(u'_{r,2n_r}, v'_{r,2n_r+3})$ for each $r \in 1, \ldots, t$,
(2) $(u'_{r,2n_r+1}, v'_{r+1,2n_{r+1}+3})$, $(u'_{r+1,2n_{r+1}+1}, v'_{r+1,2n_r+3})$ for each $r \in 1, \ldots, t-1$, and
(3) $(u'_{t,2n_r+1}, v'_{1,2n_1+3})$, $(u'_{1,2n_1+1}, v'_{1,2n_1+3})$;

and for each edge in E_S, we exclusively add 11 edges to form a 6-bloom where every butterfly is balanced. Let E'_S denote the resulting edge set. Observe that $|E'_S| \le 5 \cdot 12t \le Cs^2$ for some constant C.

Let $G = (U, V, E)$ be the signed bipartite graph from the construction above. Since $E \backslash E'_S$ can be arbitrarily large, we let $|E \backslash E'_S| \ge (2Cs^2)^{\frac{1}{\delta}}$. We have the following facts about G:

(1) There are only s unbalanced butterflies in G, namely, $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ for $i \in \{1, \ldots, s\}$;
(2) The balanced support of each edge in E'_S is at least 5;
(3) For each $i \in \{1, \ldots, n\}$ and $p \in \{-1, 1, 2, 3\}$, $\mathrm{Sup}_G(e_{i,p}) = 6$, and the removal of $e_{i,p}$ does not cause the balanced support of any other edge drop below 5;
(4) However, if $e_{i,p}$ and $e_{j,q}$, a pair of edges corresponding to a pair of compliment literals, are removed together, then all edges except in E'_S fail to be in a balanced $(5, 0)$-bitruss.

For the reason similar to the last proof, the transformation from Ψ to G is a reduction. Suppose Ψ has a satisfying assignment. Then the subgraph induced by $E \backslash E_T$ is the maximum balanced $(5, 0)$-bitruss, where E_T a set of edges containing exactly one edge in each unbalanced butterfly $(e_{i,-1}, e_{i,1}, e_{i,2}, e_{i,3})$ such that the corresponding literal is assigned to "true". Moreover, if there is an $|E|^{1-\delta}$-approximation algorithm that computes the maximal balanced $(5, 0)$-bitruss, then it will give a result of size at least

$$\begin{aligned}
\frac{|E \backslash E_T|}{|E|^{1-\delta}} \ge \frac{|E \backslash E'_S|}{|E|^{1-\delta}} &\ge \frac{|E \backslash E'_S|}{(|E \backslash E_T| + Cs^2)^{1-\delta}} \\
&\ge \frac{|E \backslash E_T|}{(2|E \backslash E_T|)^{1-\delta}} > \frac{1}{2} \cdot \frac{|E \backslash E_T|}{|E \backslash E_T|^{1-\delta}} = \frac{|E \backslash E_T|^{\delta}}{2} \\
&\ge \frac{(2Cs^2)^{\frac{1}{\delta} \cdot \delta}}{2} = Cs^2.
\end{aligned}$$