Lecture 11

Weighted Graphs: Dijkstra and Bellman-Ford

Piazza Heroes

Name, Email	number of answers
Jabari Hastings	76
Ashish Paliwal	45
Michael Cooper	34
Adam Leon	26
Jiao Li	19
Logan Pearce	19
Pranav Jain	18
Xinlan Emily Hu	18
Trenton Chang	14
Xieyuan Zhang	14

Midterm Wednesday 2/20!

- Organization will probably be something like:
  - 1. Multiple choice
  - 2. Short answer
  - 3. Algorithm analysis
  - 4. Algorithm design
  - 5. Challenge problem (not worth very many points)

- Promise: one of the questions will be very close to a homework question.
- This may be a hard exam.
  - **If it is, that means it's okay if you don't get all the questions.**
  - (Please don't freak out).

Midterm Logistics

- Last names A-R: NVIDIA Auditorium (here)
- Last names S-Z: Oshman
  - McMurtry Art & Art History Building, Oshman Presentation Space, Room 102
- (Unless you have already arranged an alternate midterm).
- We will start PROMPTLY at 10:30, please be on time.
- We could not find enough space during this timeslot to seat everyone every-other seat.
  - Please be mindful of the honor code.

More announcements

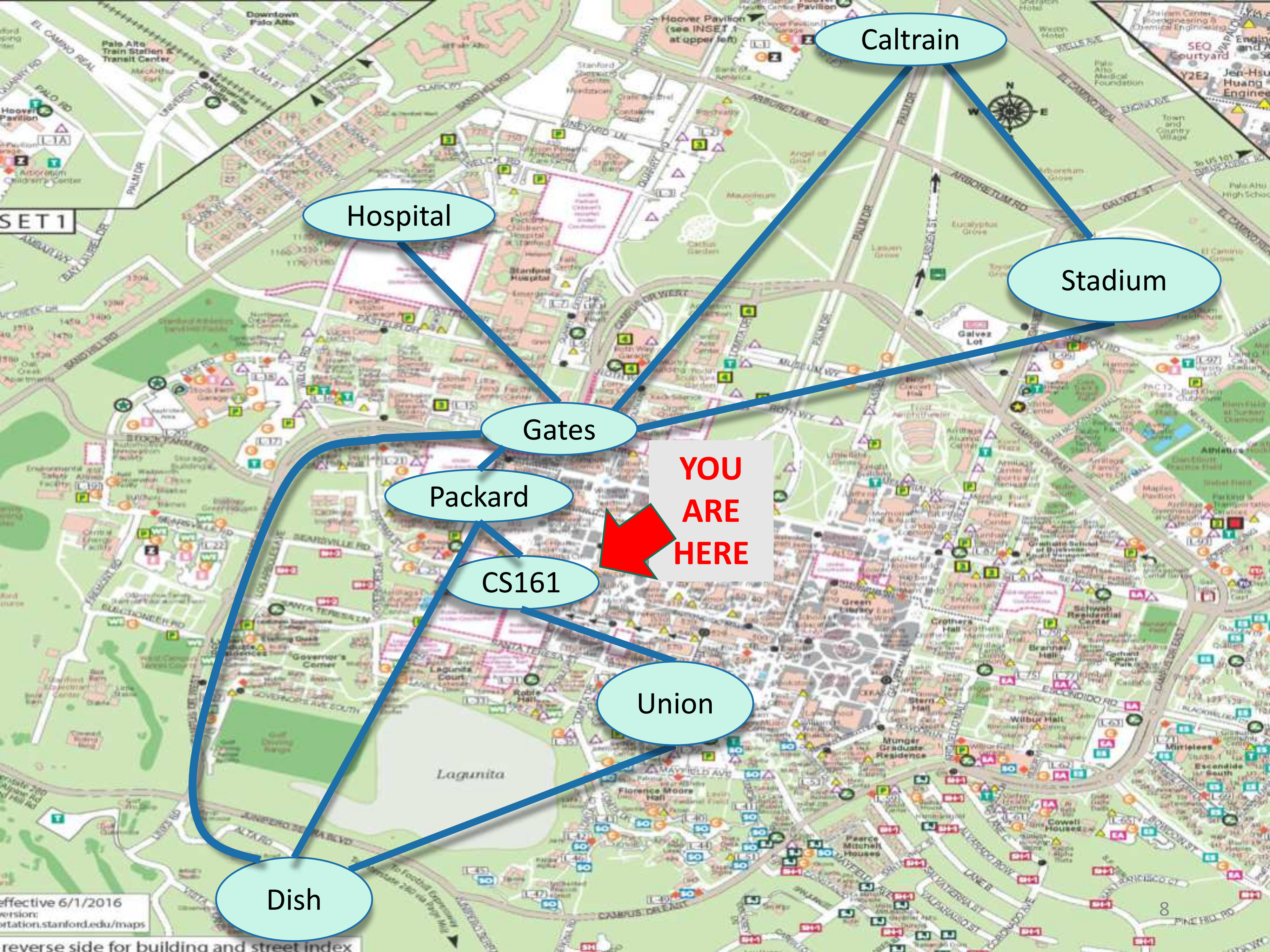
- HW4 due Friday.
- No new HW released Friday because of the midterm next week.
- Have a good long weekend!
  - Monday is President’s day

Previous two lectures

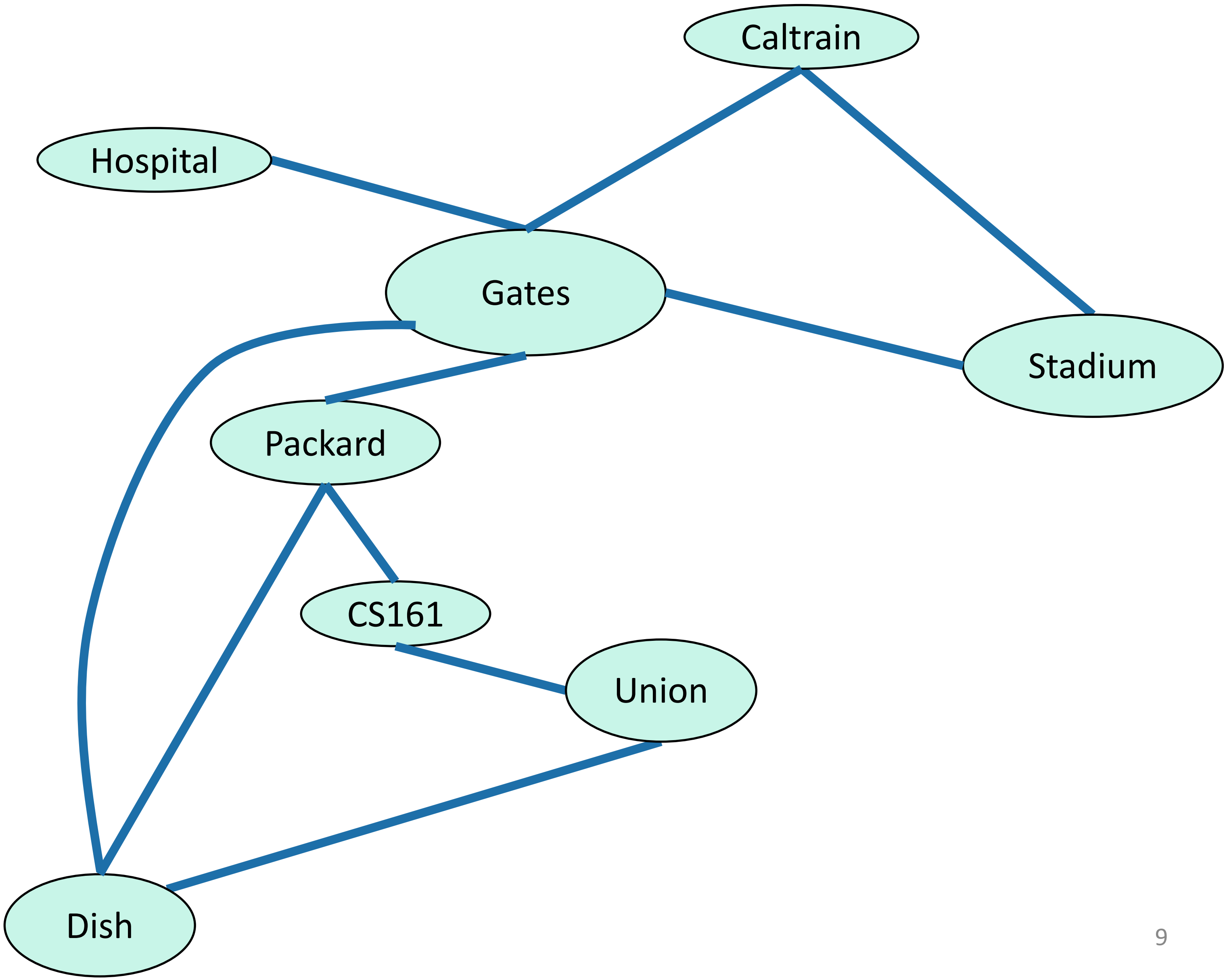
- Graphs!
- DFS
  - Topological Sorting
  - Strongly Connected Components
- BFS
  - Shortest Paths in unweighted graphs

Today

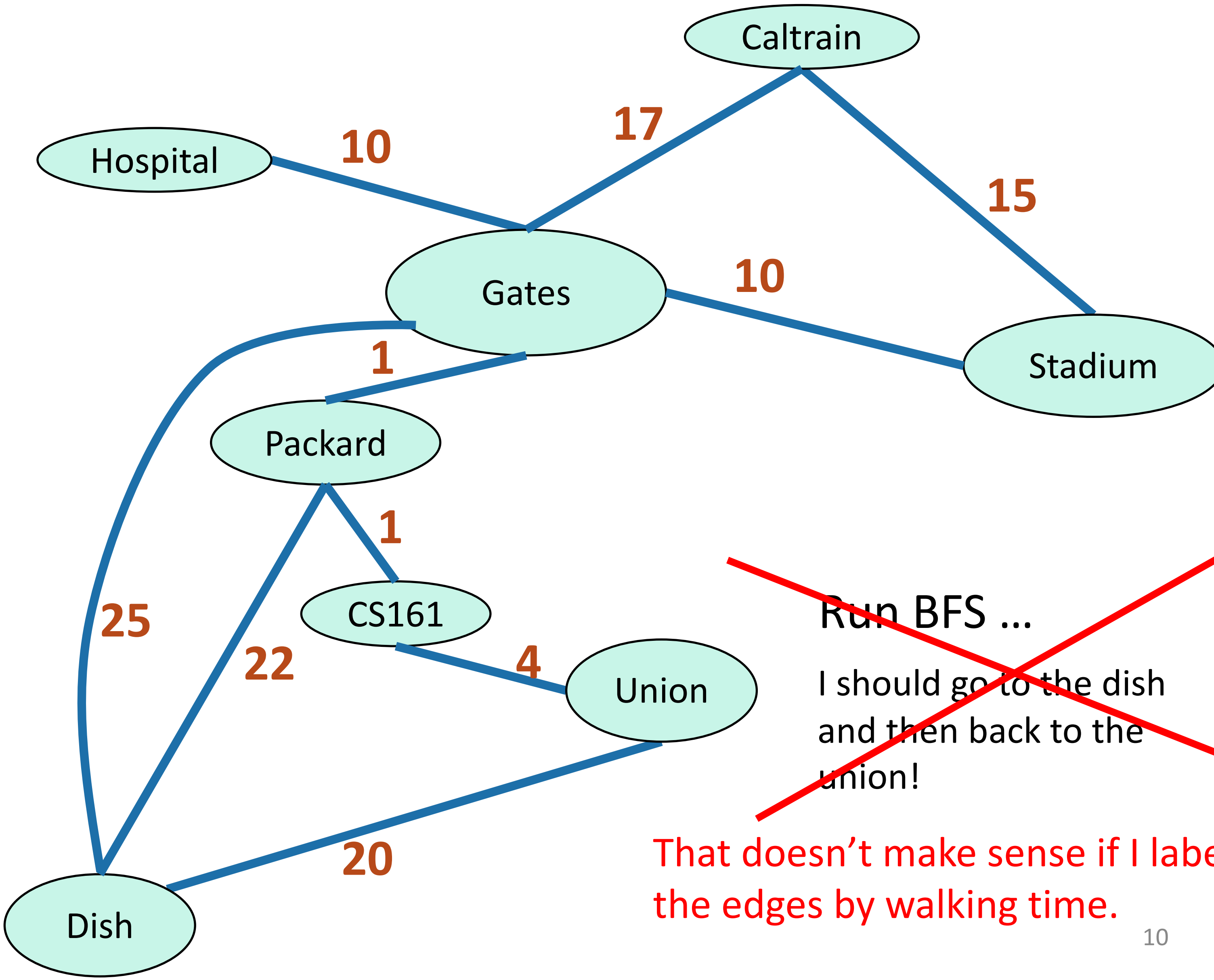
- What if the graphs are weighted?
- Part 1: Dijkstra!
  - This will take most of today's class
- Part 2: Bellman-Ford!
  - Real quick at the end!
  - We'll come back to Bellman-Ford in more detail, so today is just a taste.


Caltrain
Hospital
Stadium
Gates
Packard
YOU ARE HERE
CS161
Union
Dish
Lagunita

Just the graph



Shortest path from Gates to the Union?



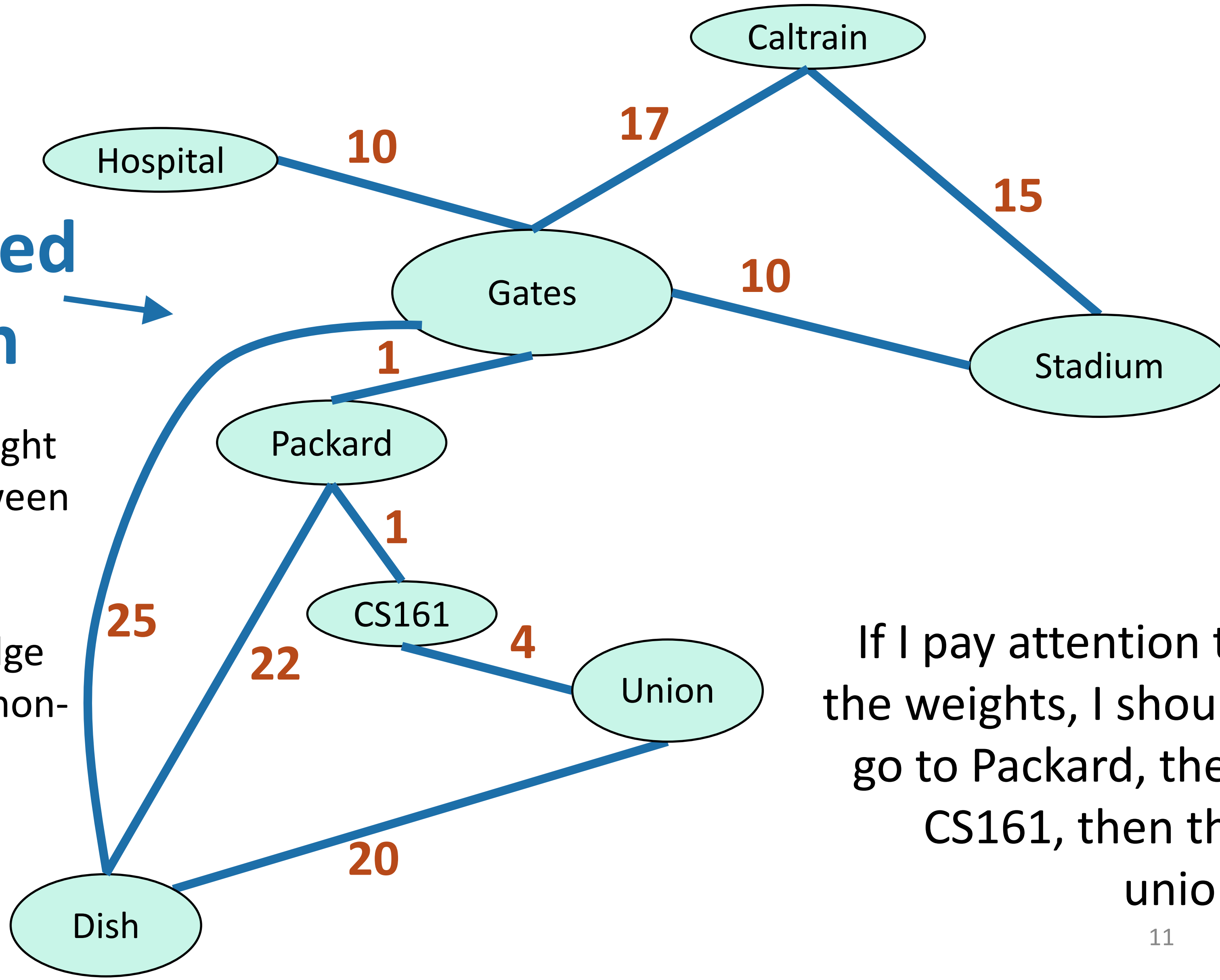
~~Run BFS …~~

~~I should go to the dish and then back to the union!~~

That doesn't make sense if I label the edges by walking time.

Shortest path from Gates to the Union?

weighted graph



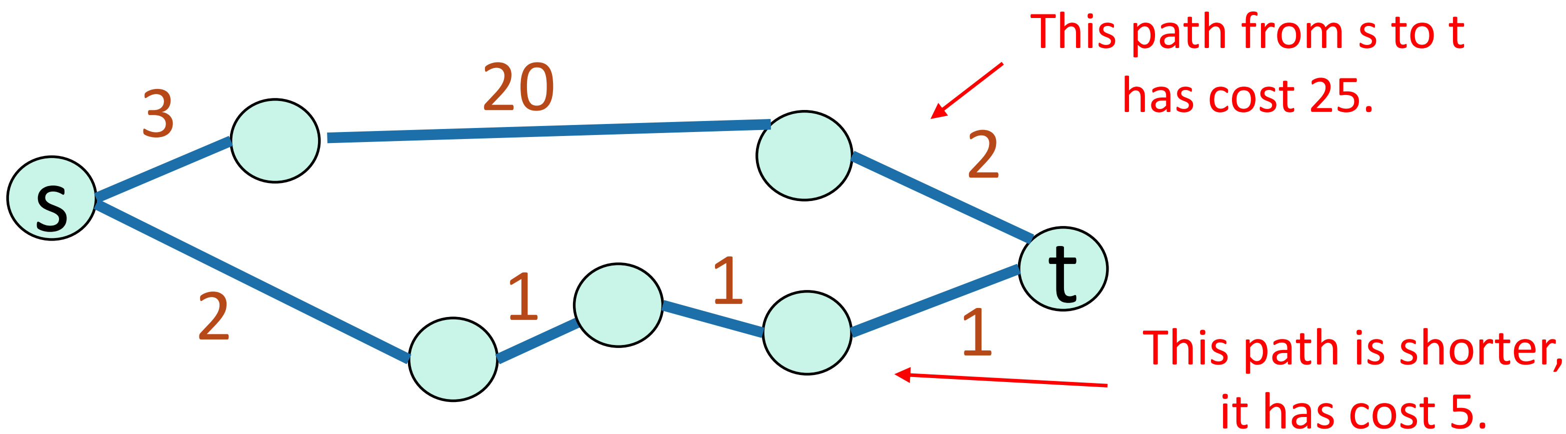
w(u,v) = weight of edge between u and v.

For now, edge weights are non-negative.

If I pay attention to the weights, I should go to Packard, then CS161, then the union.

Shortest path problem

- What is the shortest path between u and v in a weighted graph?
  - the **cost** of a path is the sum of the weights along that path
  - The **shortest path** is the one with the minimum cost.

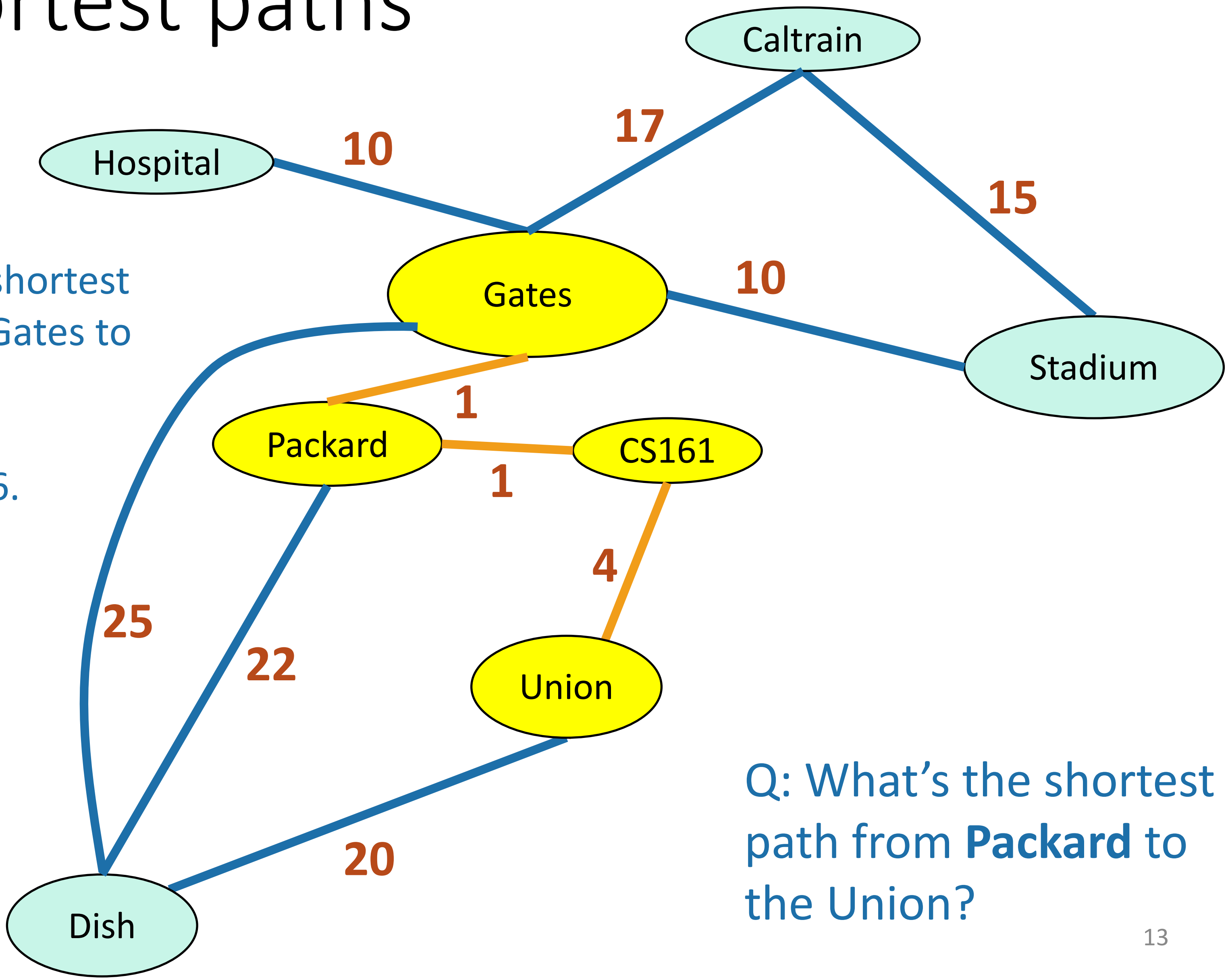


- The **distance** d(u,v) between two vertices u and v is the cost of the the shortest path between u and v.
- For this lecture **all graphs are directed**, but to save on notation I'm just going to draw undirected edges.

Shortest paths

This is the shortest path from Gates to the Union.

It has cost 6.

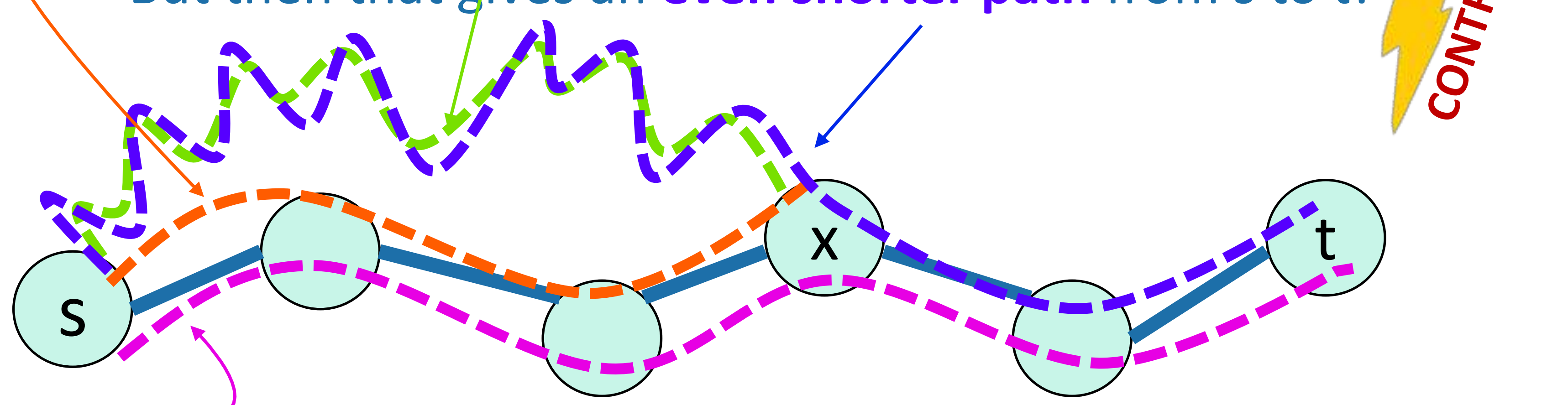


Q: What’s the shortest path from **Packard** to the Union?

Warm-up

- A sub-path of a shortest path is also a shortest path.

- Say **this** is a shortest path from s to t.
- Claim: **this** is a shortest path from s to x.
  - Suppose not, **this** one is a shorter path from s to x.
  - But then that gives an **even shorter path** from s to t!



Single-source shortest-path problem

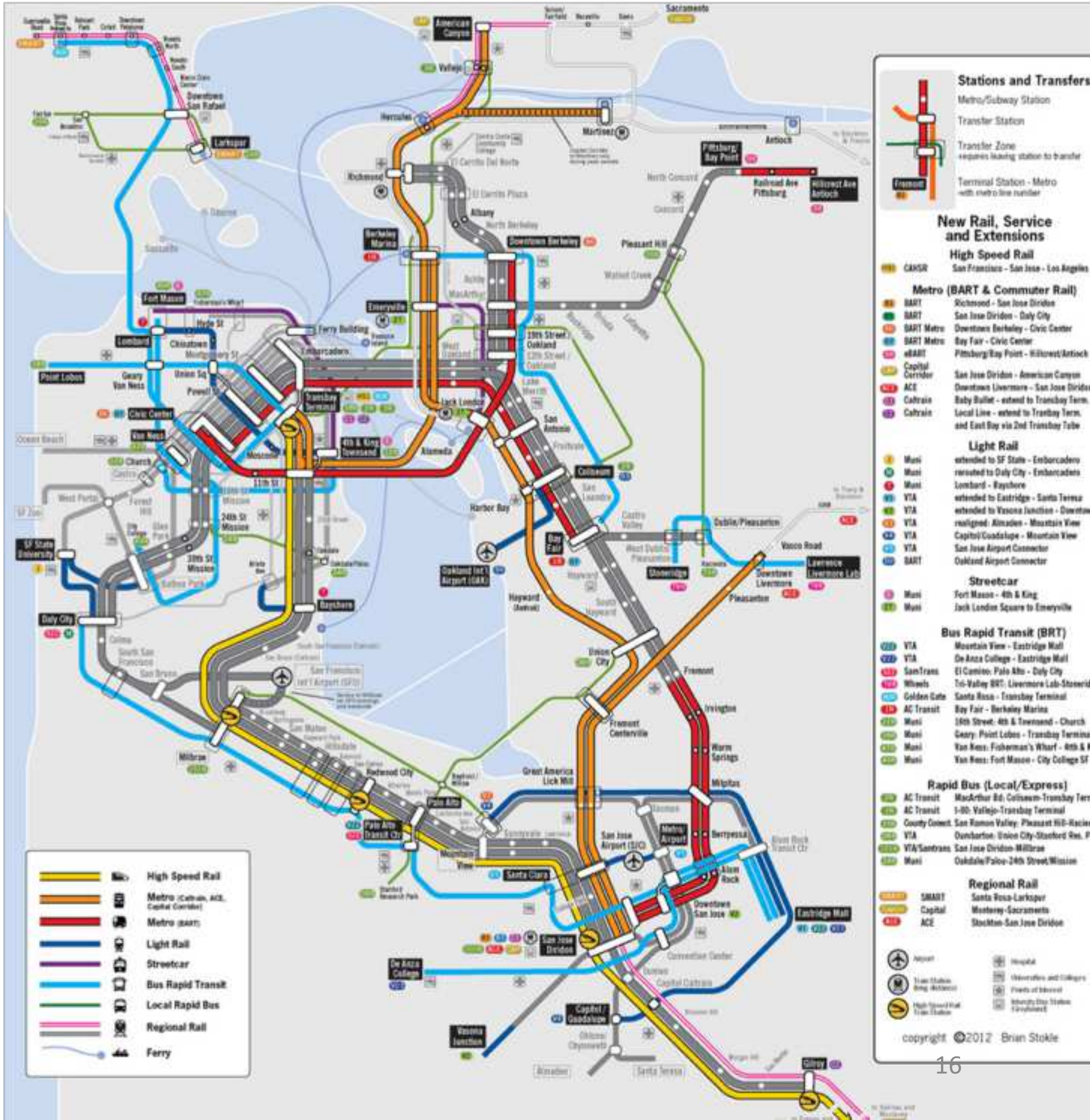
- I want to know the shortest path from one vertex (Gates) to all other vertices.

Destination	Cost	To get there
Packard	1	Packard
CS161	2	Packard-CS161
Hospital	10	Hospital
Caltrain	17	Caltrain
Union	6	Packard-CS161-Union
Stadium	10	Stadium
Dish	23	Packard-Dish

(Not necessarily stored as a table – how this information is represented will depend on the application)

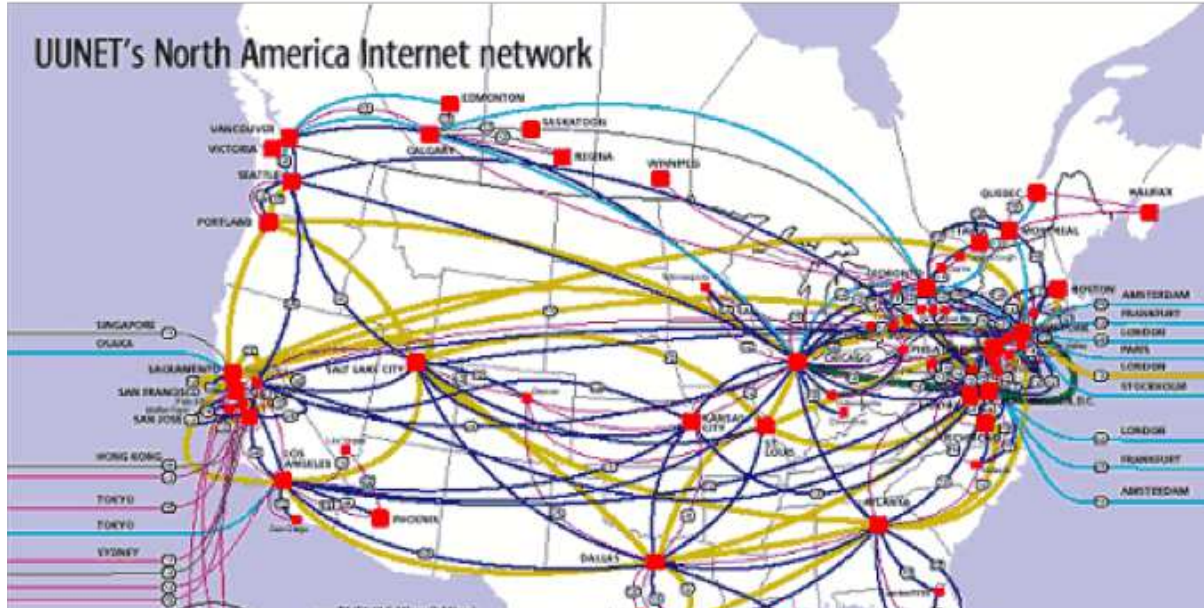
Example

- **"what is the shortest path from Palo Alto to [anywhere else]"** using BART, Caltrain, lightrail, MUNI, bus, Amtrak, bike, walking, uber/lyft.
- Edge weights have something to do with time, money, hassle.

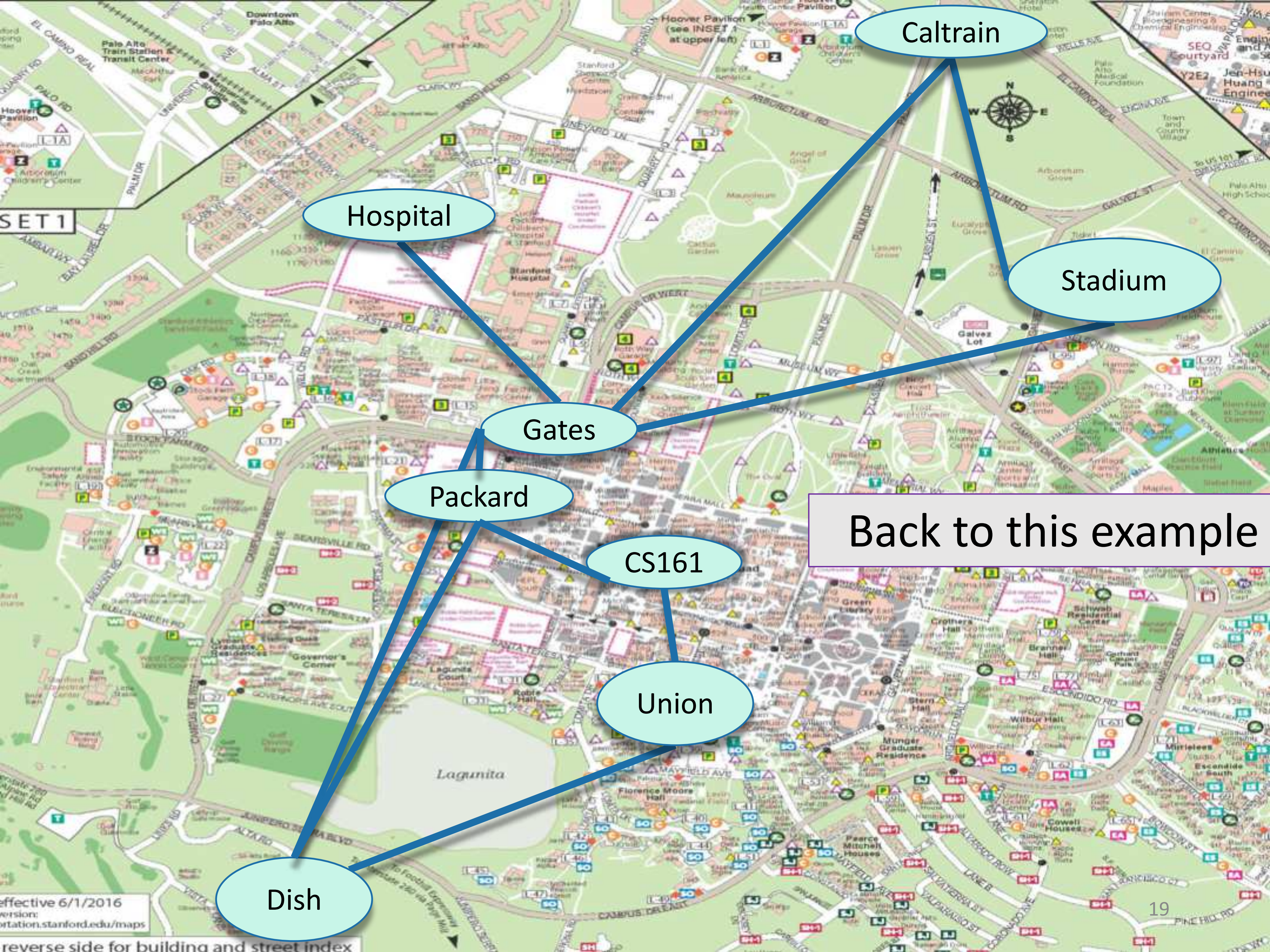


Example

- **Network routing**
- I send information over the internet, from my computer to to all over the world.
- Each path has a cost which depends on link length, traffic, other costs, etc..
- How should we send packets?



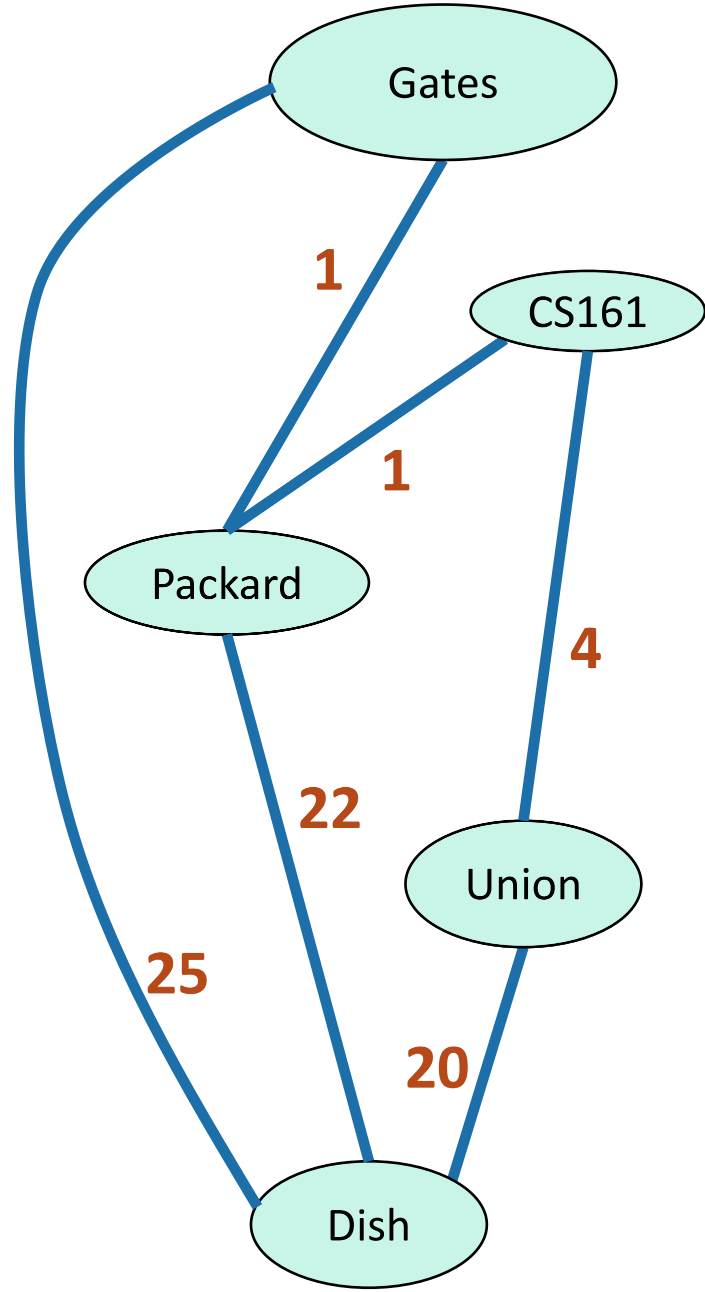

```
DN0a22a0e3:~ mary$ traceroute -a www.ethz.ch
traceroute to www.ethz.ch (129.132.19.216), 64 hops max, 52 byte packets
 1  [AS0] 10.34.160.2 (10.34.160.2)  38.168 ms  31.272 ms  28.841 ms
 2  [AS0] cwa-vrtr.sunet (10.21.196.28)  33.769 ms  28.245 ms  24.373 ms
 3  [AS32] 171.66.2.229 (171.66.2.229)  24.468 ms  20.115 ms  23.223 ms
 4  [AS32] hpr-svl-rtr-vlan8.sunet (171.64.255.235)  24.644 ms  24.962 ms  1
 5  [AS2152] hpr-svl-hpr2--stan-ge.cenic.net (137.164.27.161)  22.129 ms  4.
 6  [AS2152] hpr-lax-hpr3--svl-hpr3-100ge.cenic.net (137.164.25.73)  12.125 
 7  [AS2152] hpr-i2--lax-hpr2-r&e.cenic.net (137.164.26.201)  40.174 ms  38.
 8  [AS0] et-4-0-0.4079.sdn-sw.lasv.net.internet2.edu (162.252.70.28)  46.57
 9  [AS0] et-5-1-0.4079.rtsw.salt.net.internet2.edu (162.252.70.31)  30.424 
10  [AS0] et-4-0-0.4079.sdn-sw.denv.net.internet2.edu (162.252.70.8)  47.454
11  [AS0] et-4-1-0.4079.rtsw.kans.net.internet2.edu (162.252.70.11)  70.825 
12  [AS0] et-4-1-0.4070.rtsw.chic.net.internet2.edu (198.71.47.206)  77.937 
13  [AS0] et-0-1-0.4079.sdn-sw.ashb.net.internet2.edu (162.252.70.60)  77.68
14  [AS0] et-4-1-0.4079.rtsw.wash.net.internet2.edu (162.252.70.65)  71.565 
15  [AS21320] internet2-gw.mx1.lon.uk.geant.net (62.40.124.44)  154.926 ms  
16  [AS21320] ae0.mx1.lon2.uk.geant.net (62.40.98.79)  146.565 ms  146.604 m
17  [AS21320] ae0.mx1.par.fr.geant.net (62.40.98.77)  153.289 ms  184.995 ms
18  [AS21320] ae2.mx1.gen.ch.geant.net (62.40.98.153)  160.283 ms  160.104 m
19  [AS21320] swice1-100ge-0-3-0-1.switch.ch (62.40.124.22)  162.068 ms  160.
20  [AS559] swizh1-100ge-0-1-0-1.switch.ch (130.59.36.94)  165.824 ms  164.2
21  [AS559] swiez3-100ge-0-1-0-4.switch.ch (130.59.38.109)  164.269 ms  164.
22  [AS559] rou-gw-lee-tengig-to-switch.ethz.ch (192.33.92.1)  164.082 ms  1
23  [AS559] rou-fw-rz-rz-gw.ethz.ch (192.33.92.169)  164.773 ms  165.193 ms
```



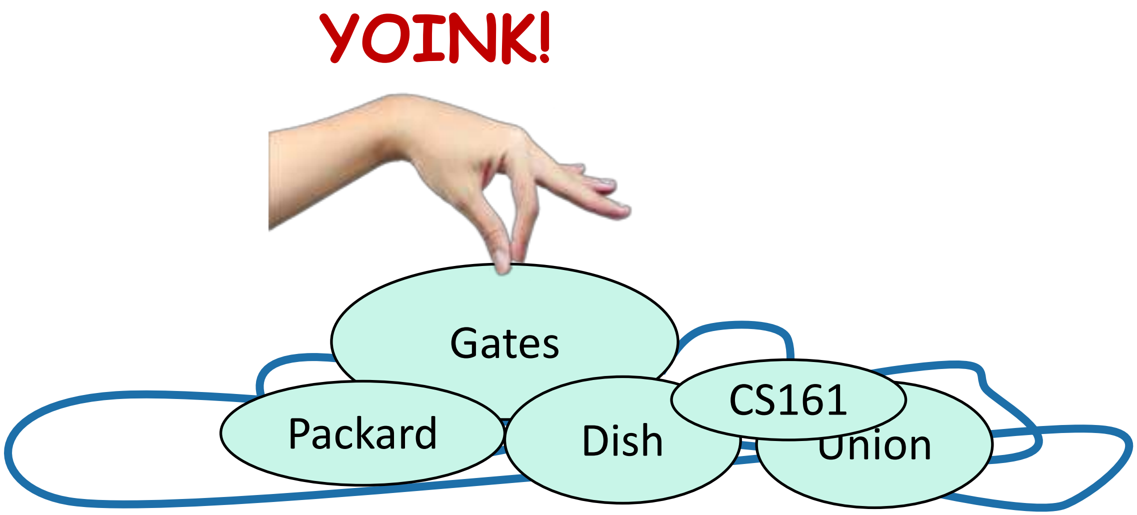
Back to this example

Dijkstra's algorithm

- Finds shortest paths from Gates to everywhere else.



Dijkstra
intuition



Dijkstra
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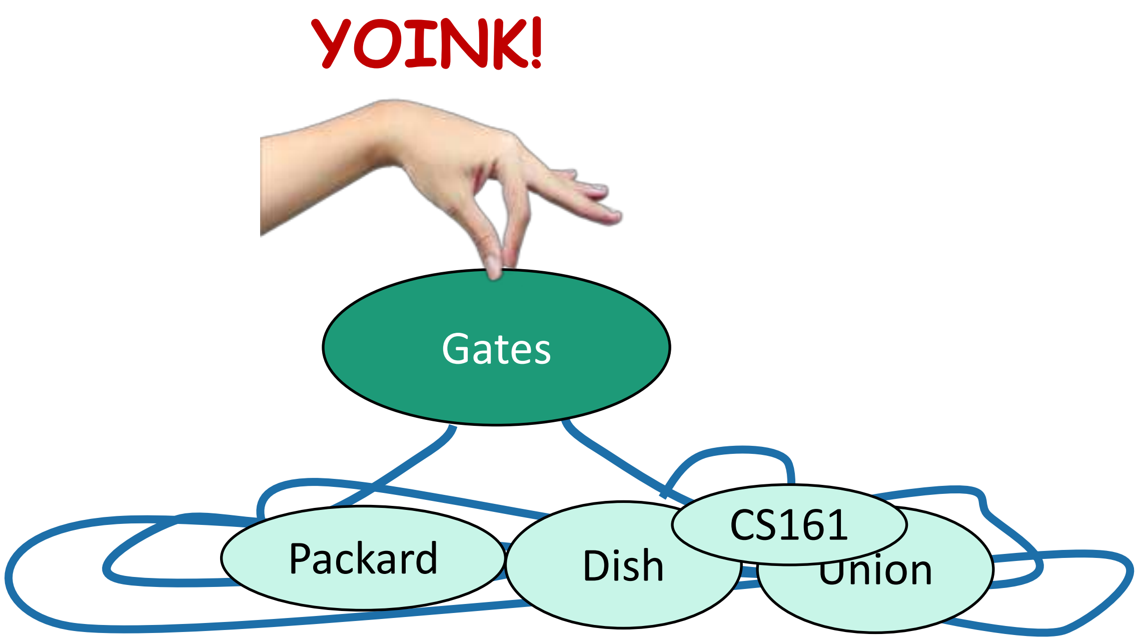
A vertex is done when it's not on the ground anymore.



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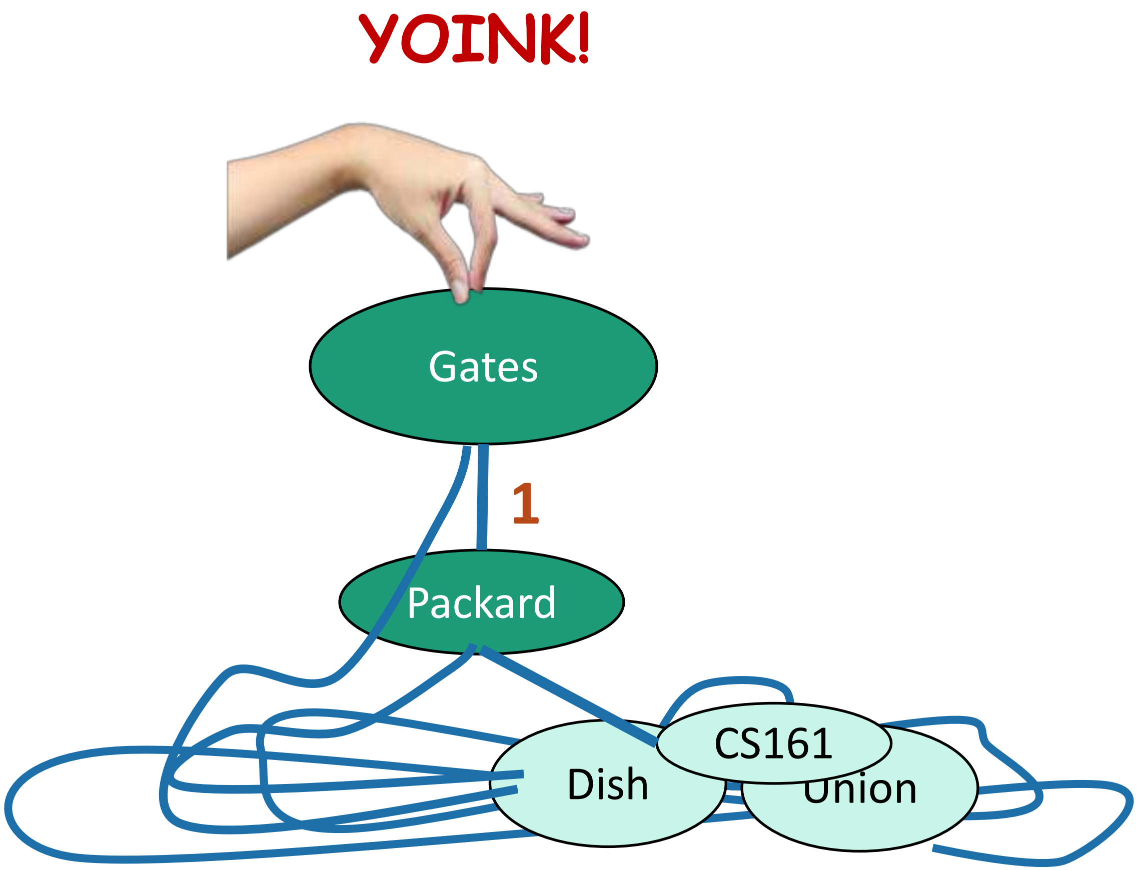

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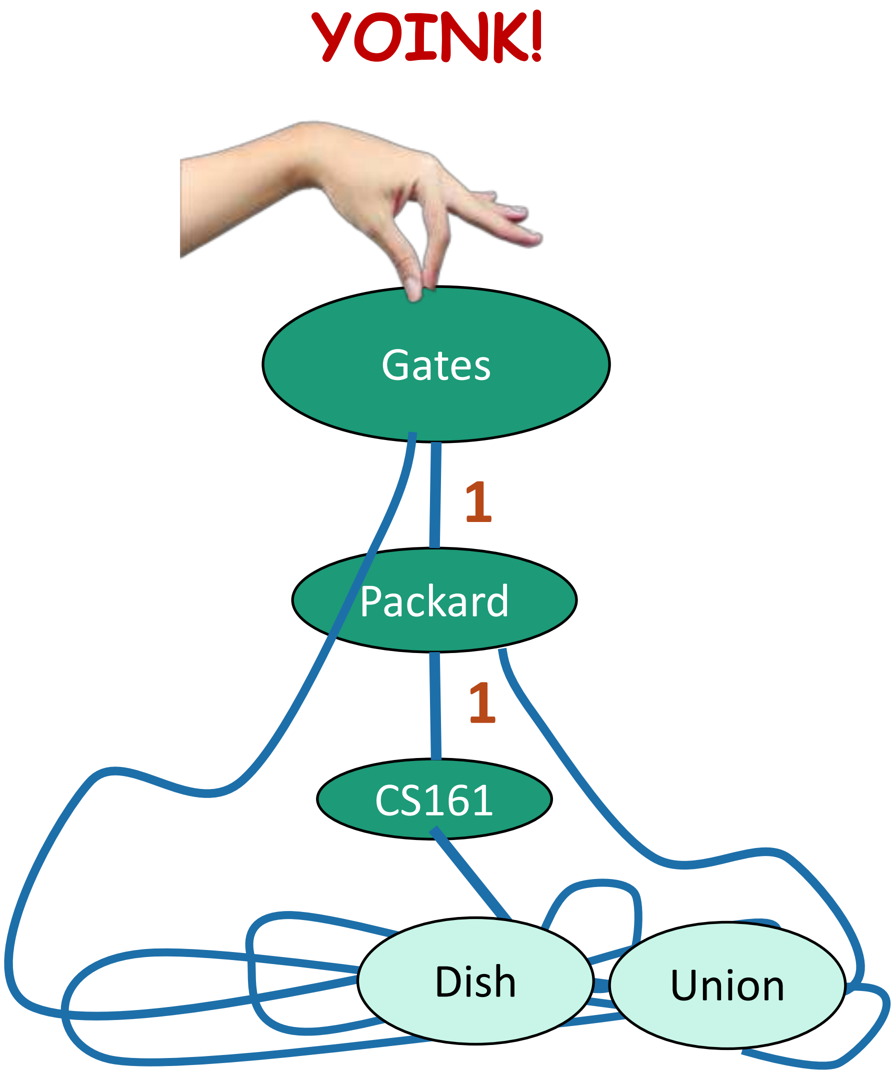

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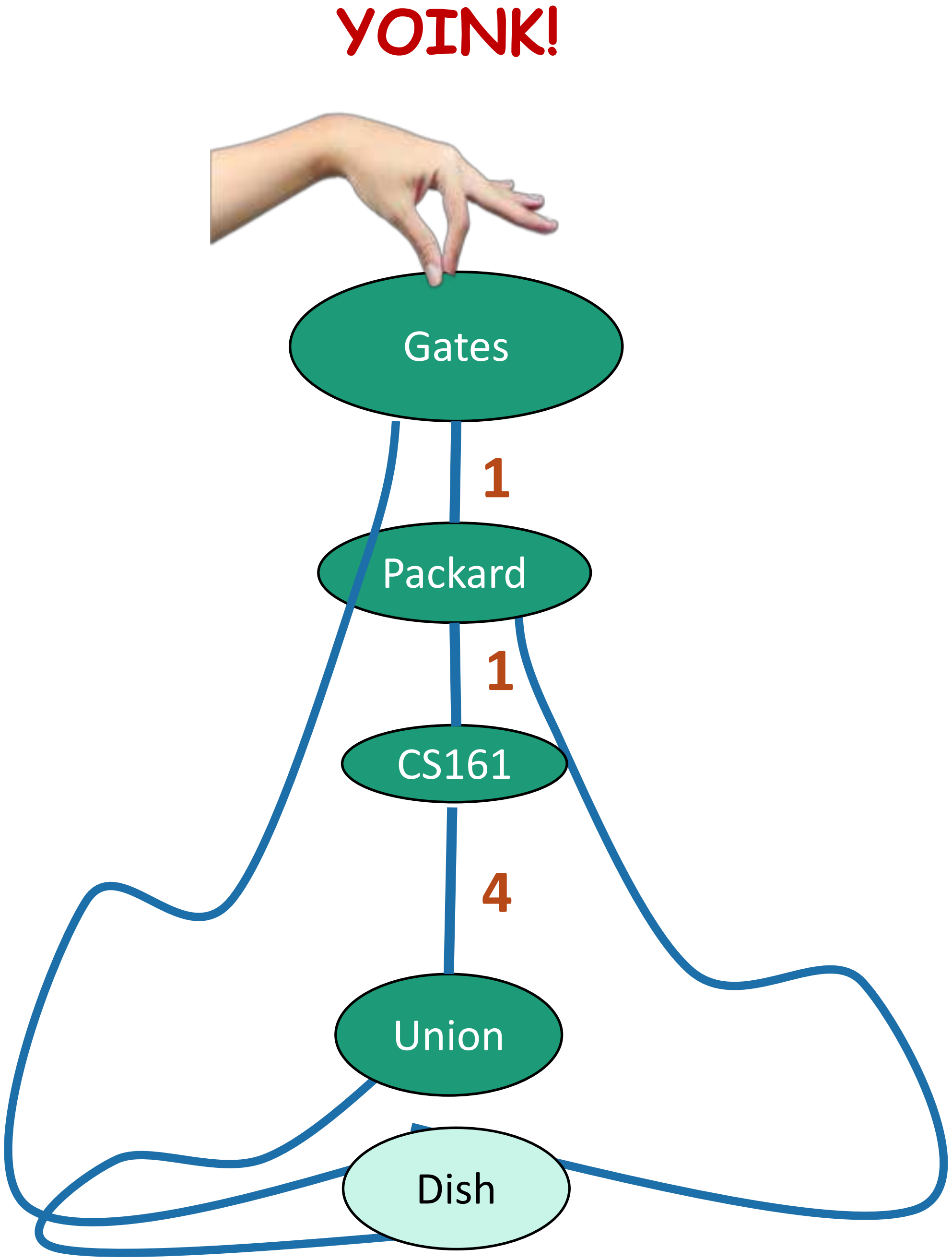

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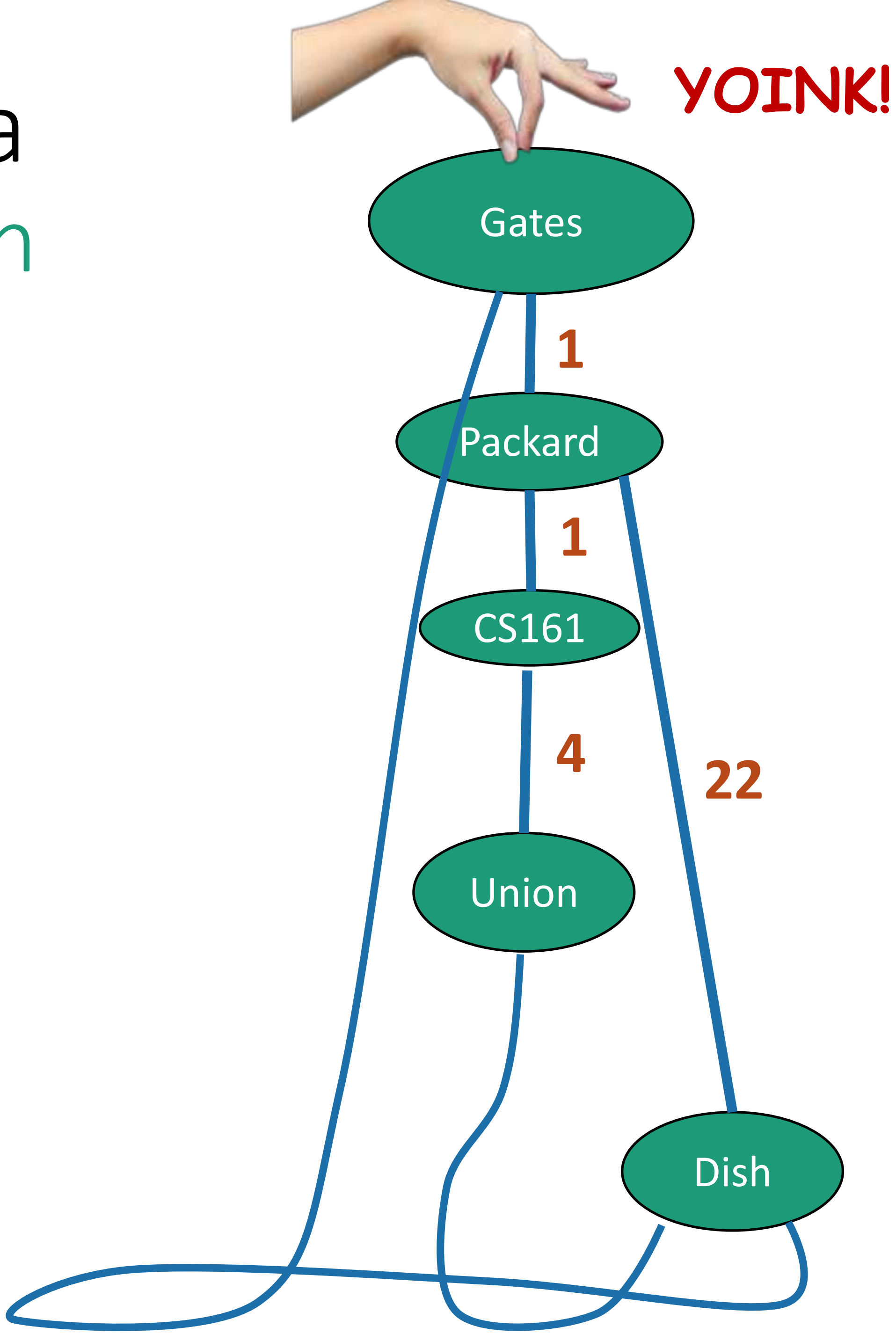

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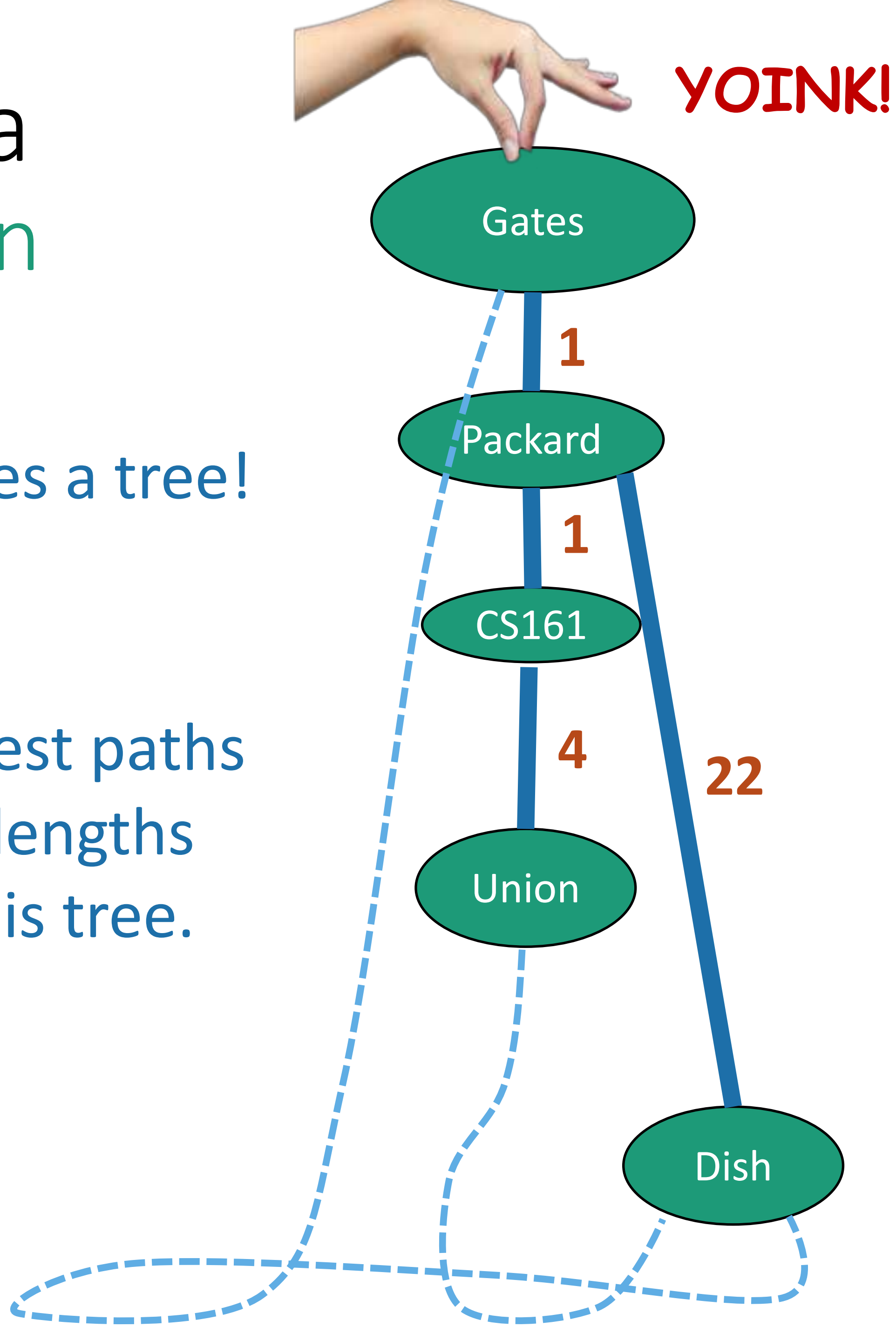
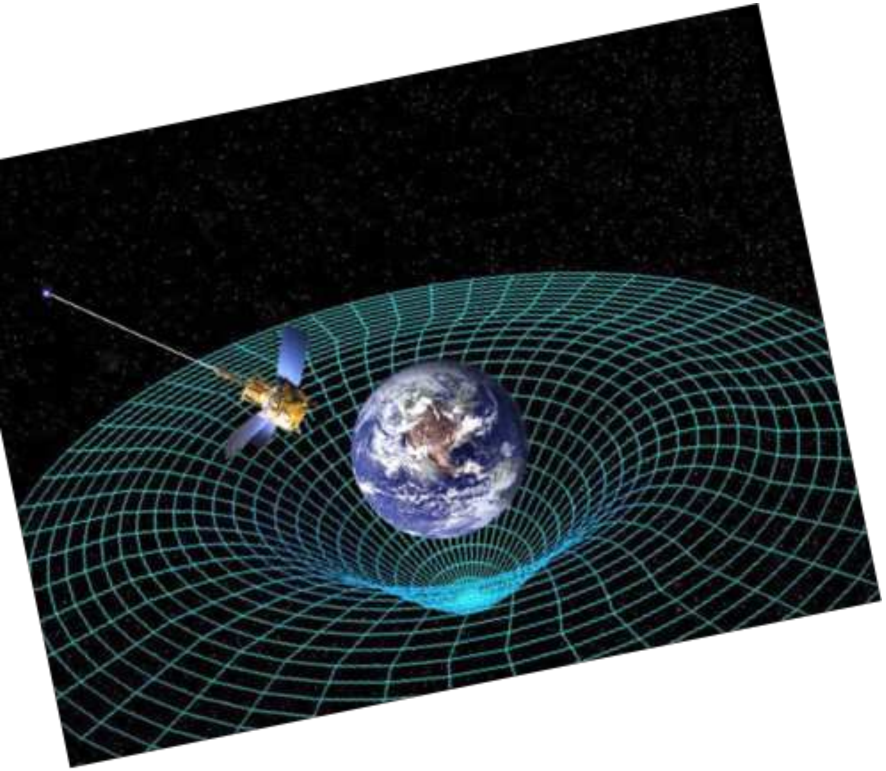


This creates a tree!

The shortest paths are the lengths along this tree.

How do we actually implement this?

- **Without** string and gravity?



Dijkstra by example

How far is a node from Gates?

I'm not sure yet

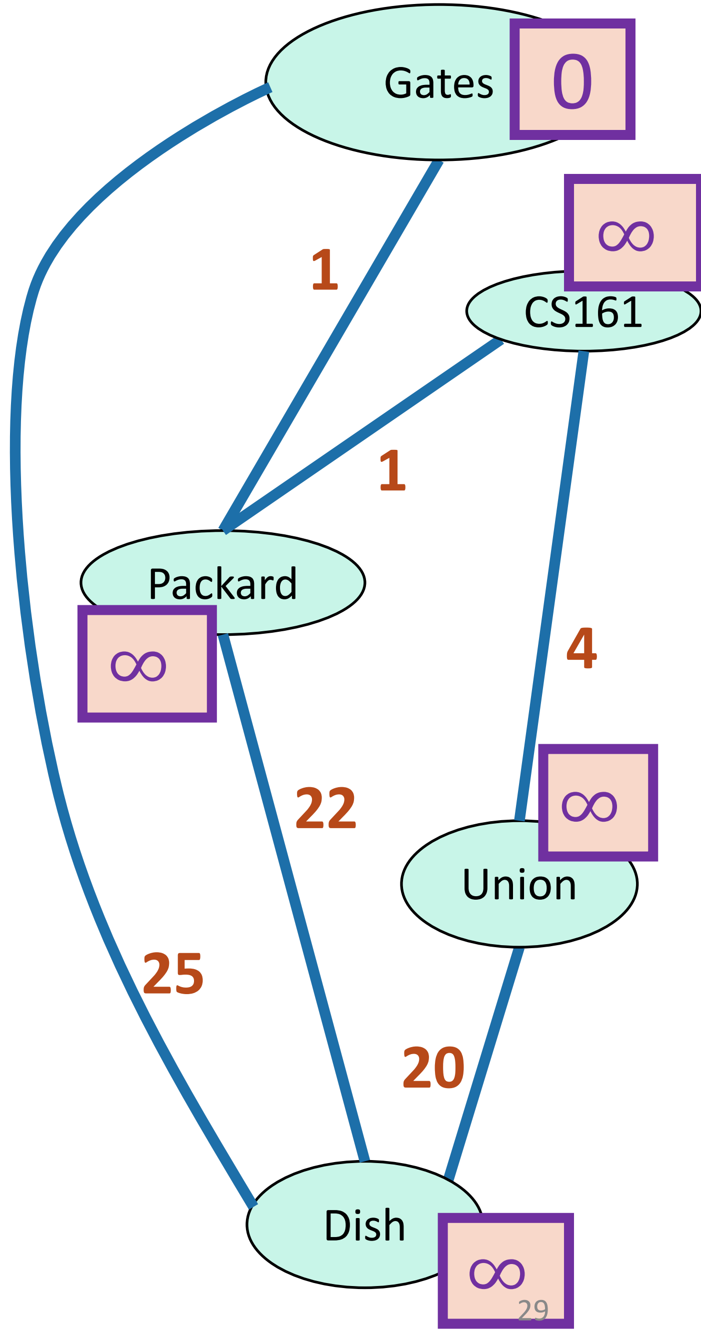
I'm sure

x

x = d[v] is my best **over-estimate** for dist(Gates,v).

Initialize d[v] = ∞
for all non-starting vertices
v, and d[Gates] = 0

- Pick the **not-sure** node u with the smallest estimate **d[u].**



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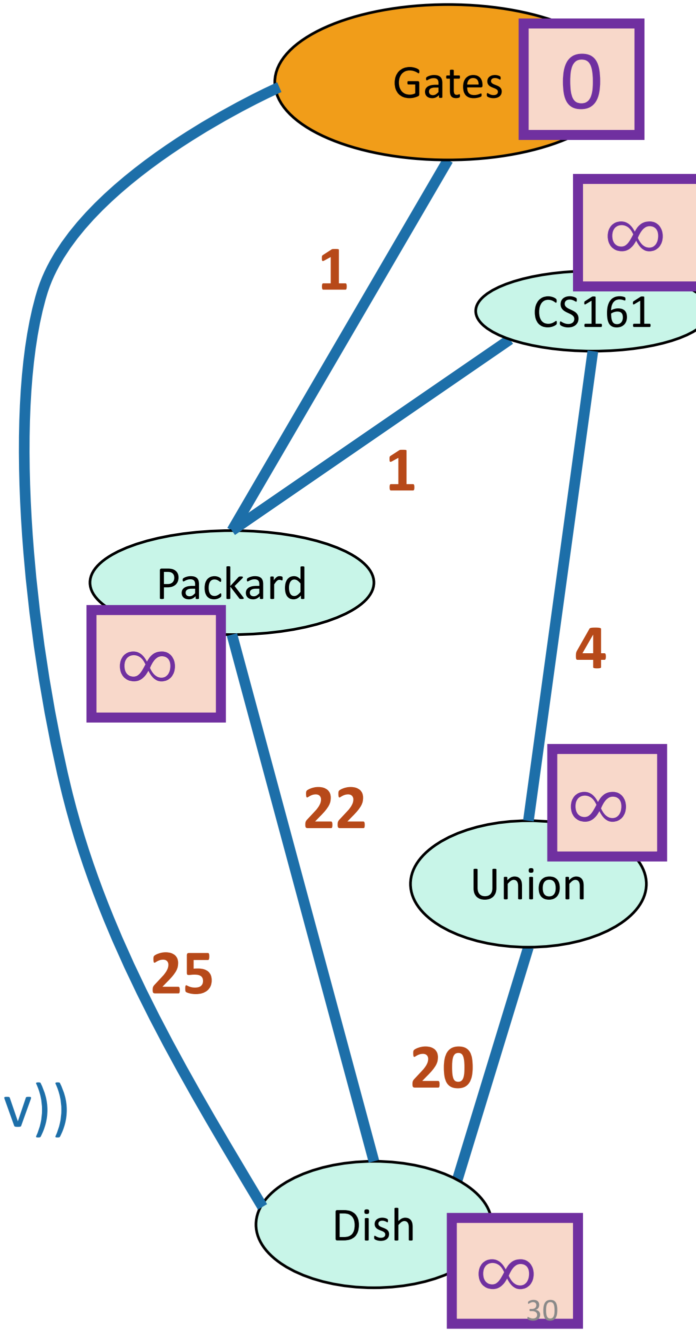

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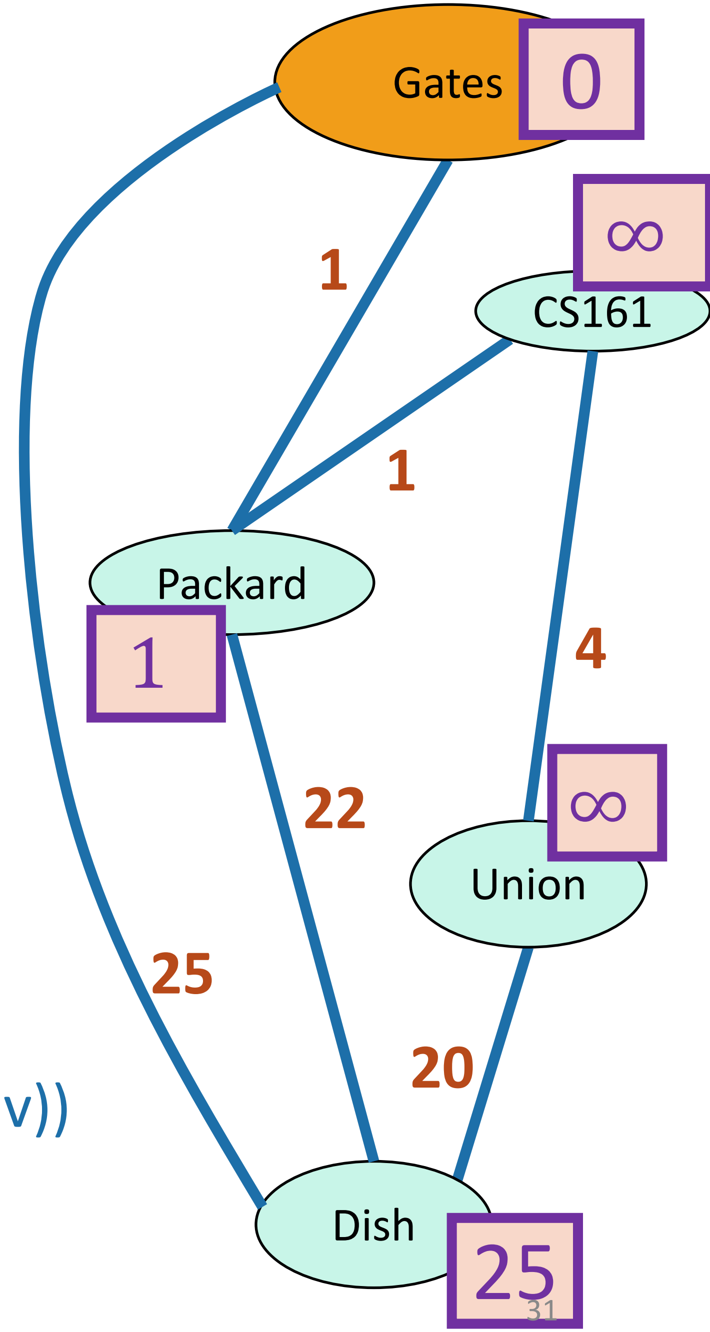

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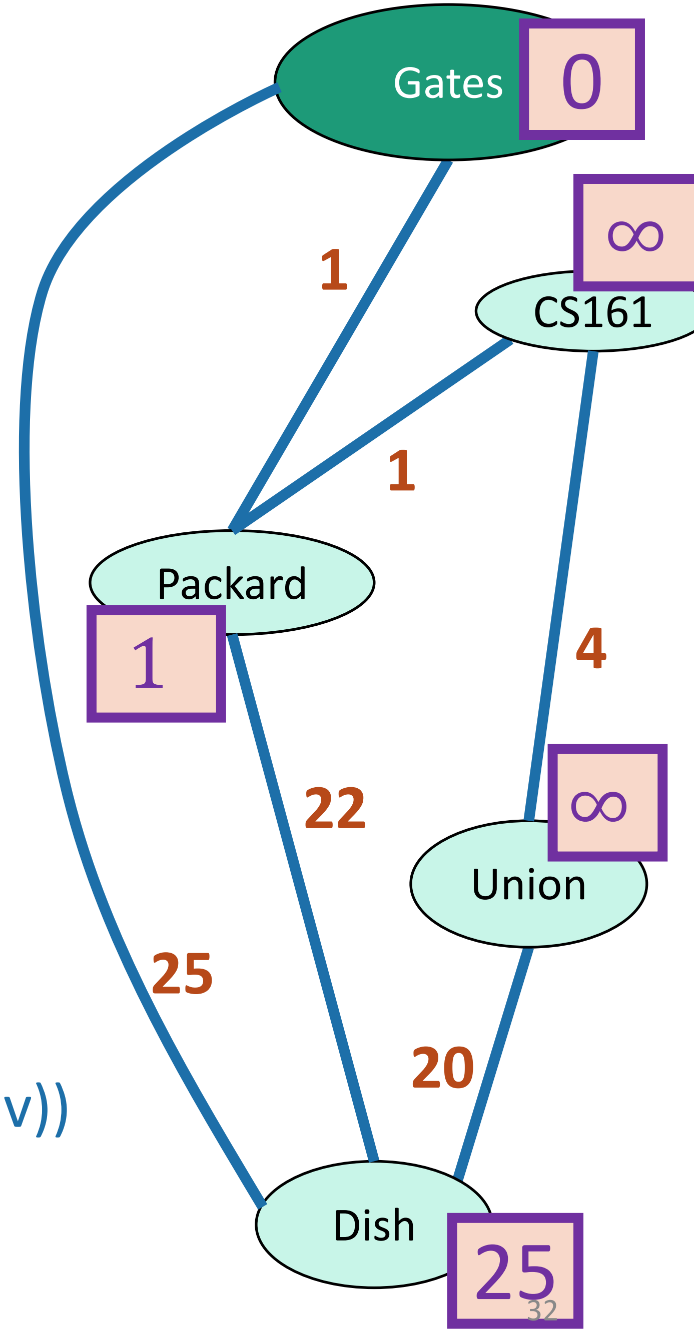

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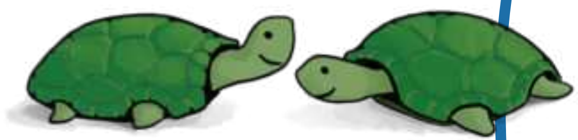
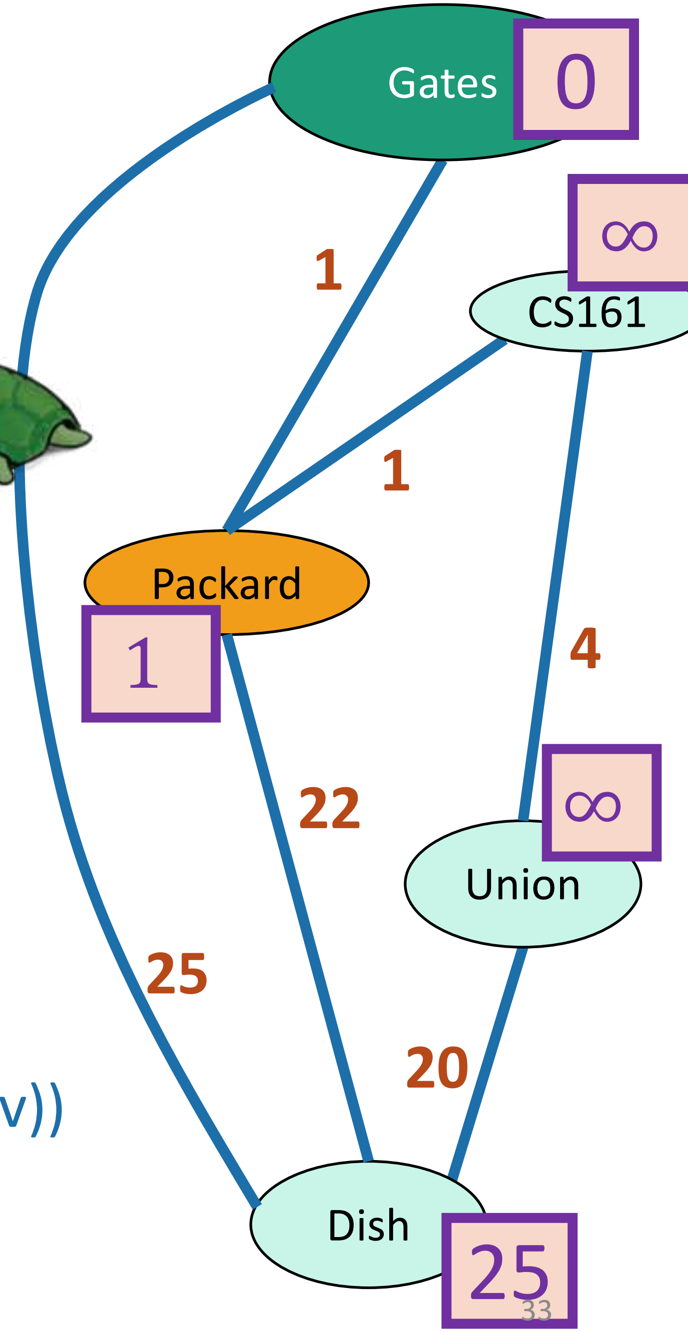
Packard has three neighbors. What happens when we update them?





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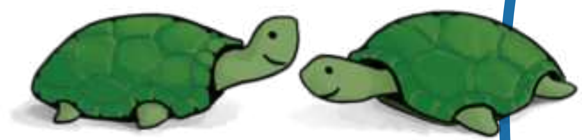

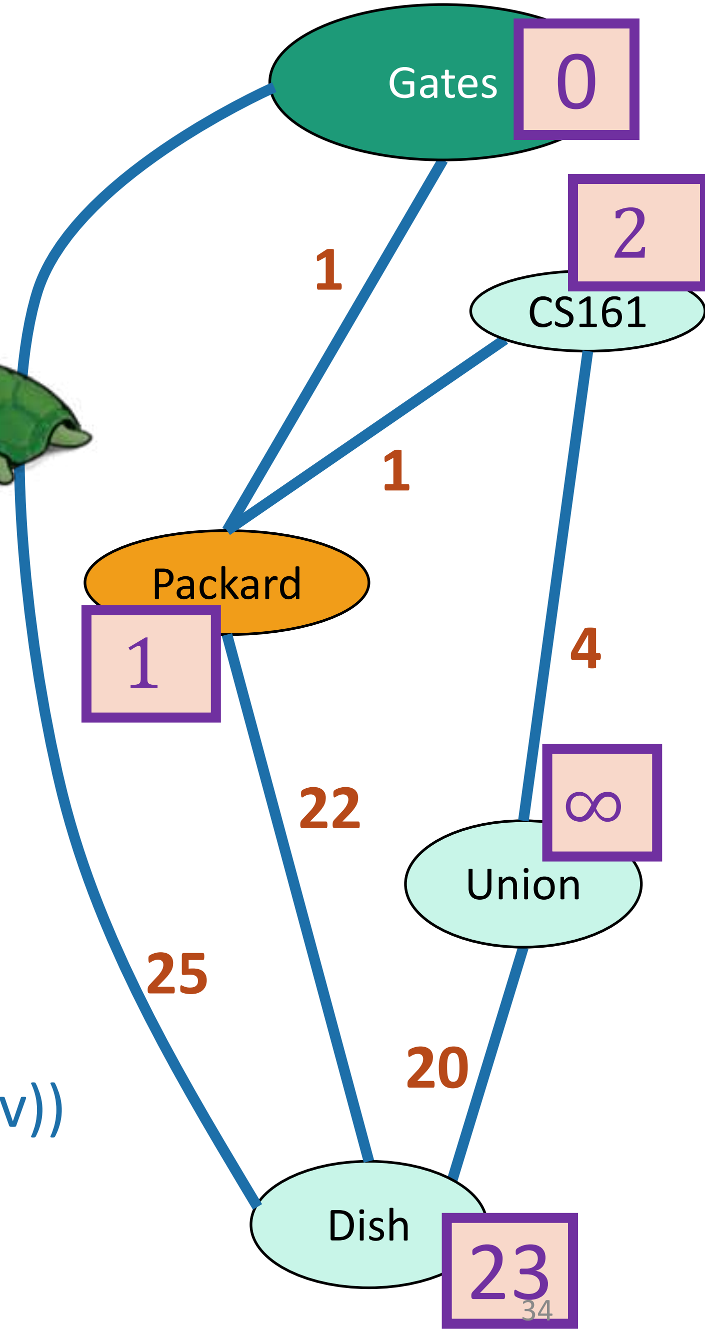

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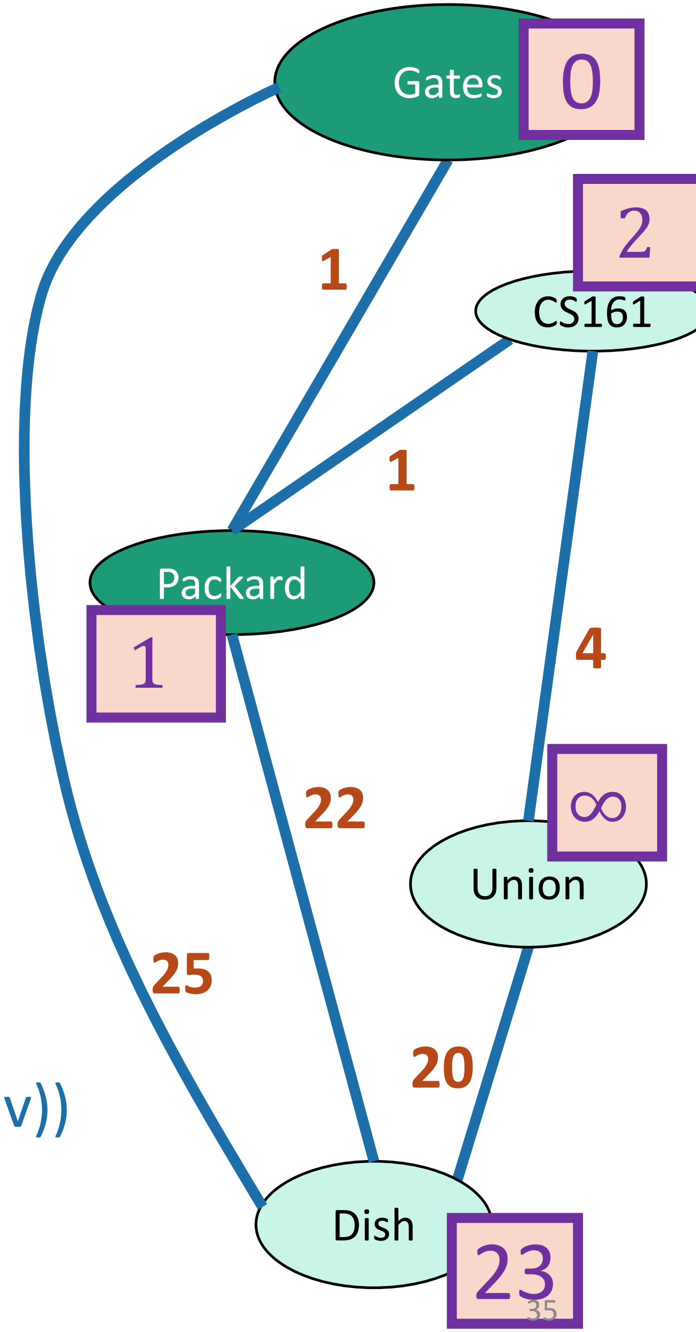

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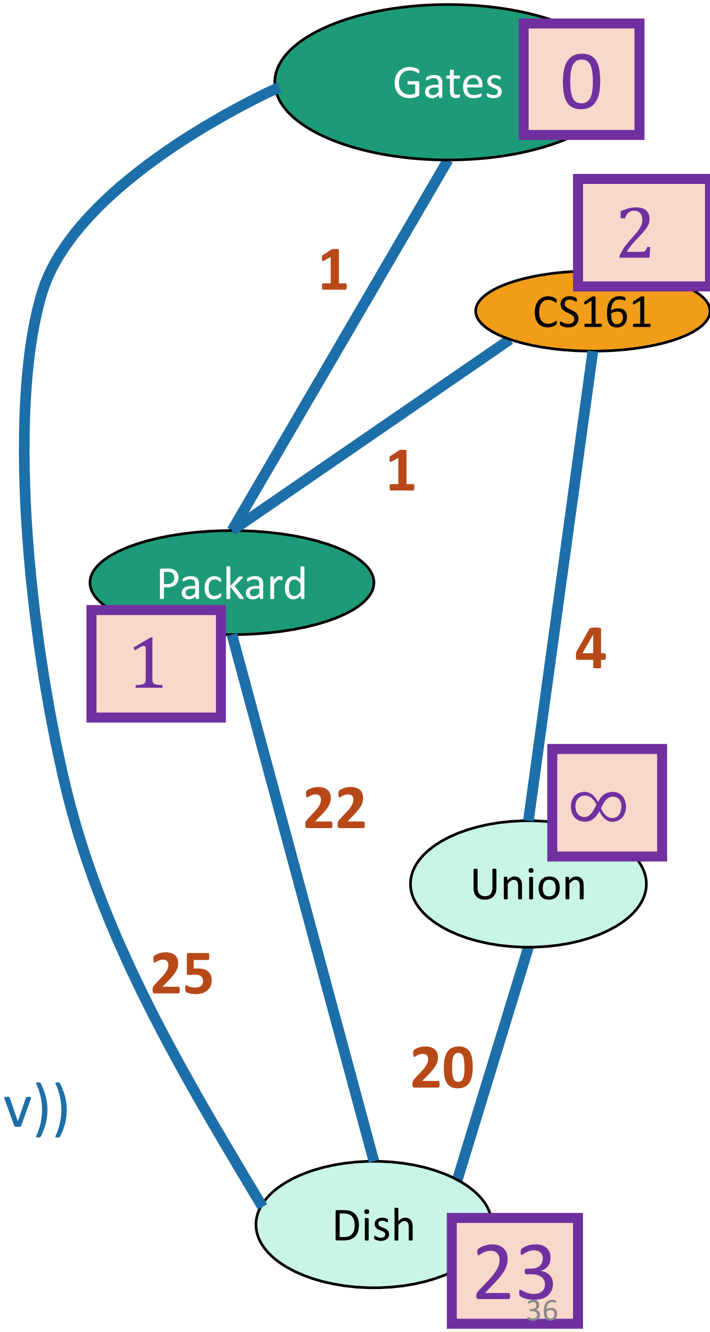

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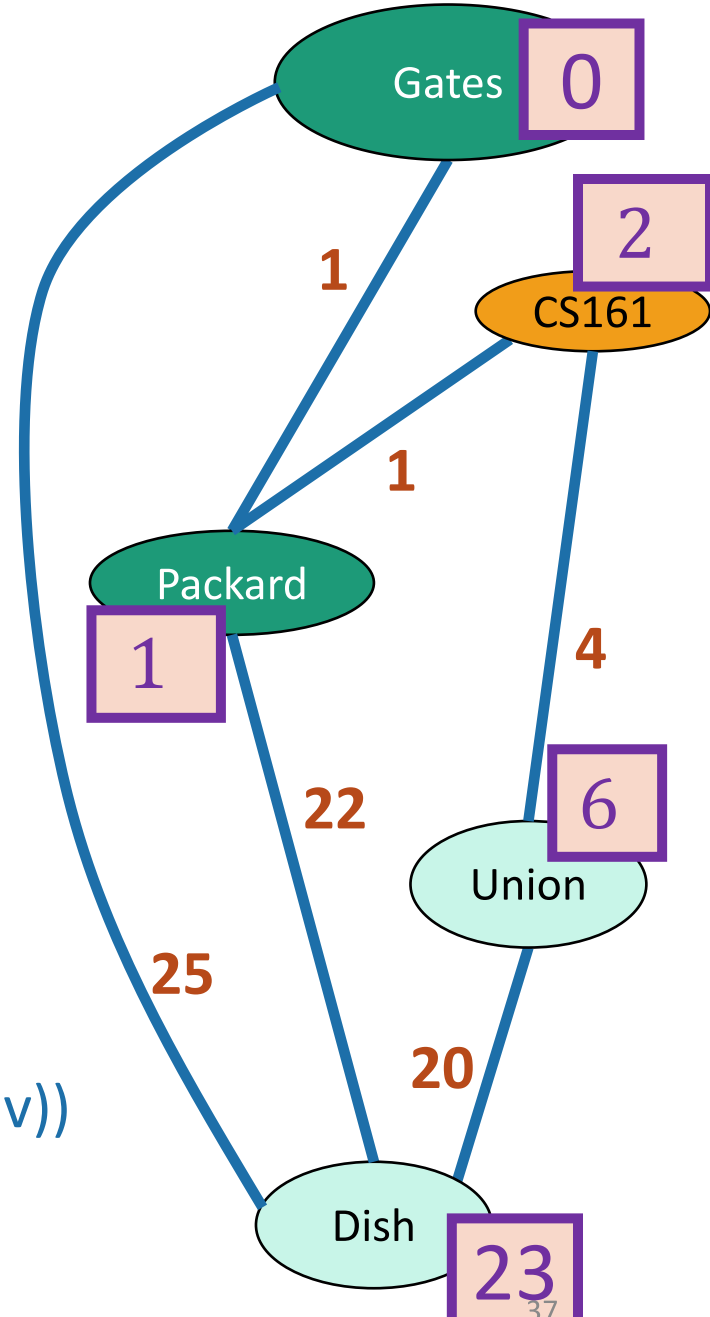

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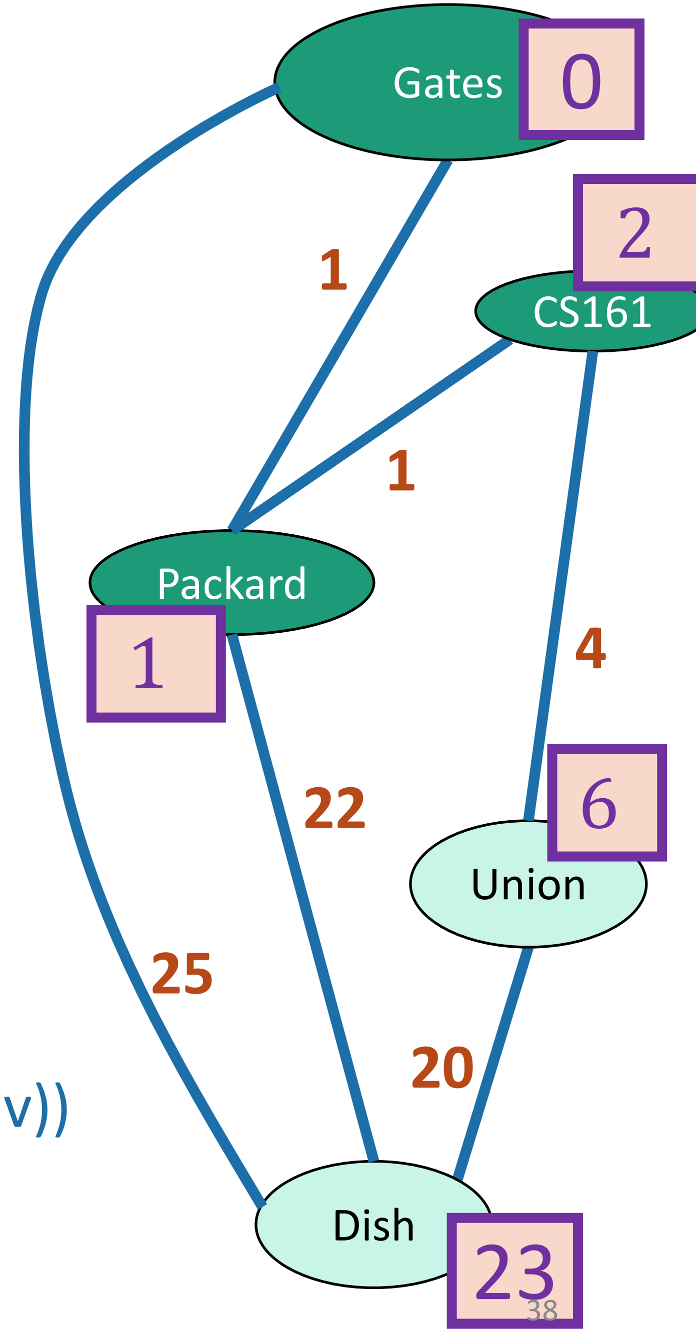

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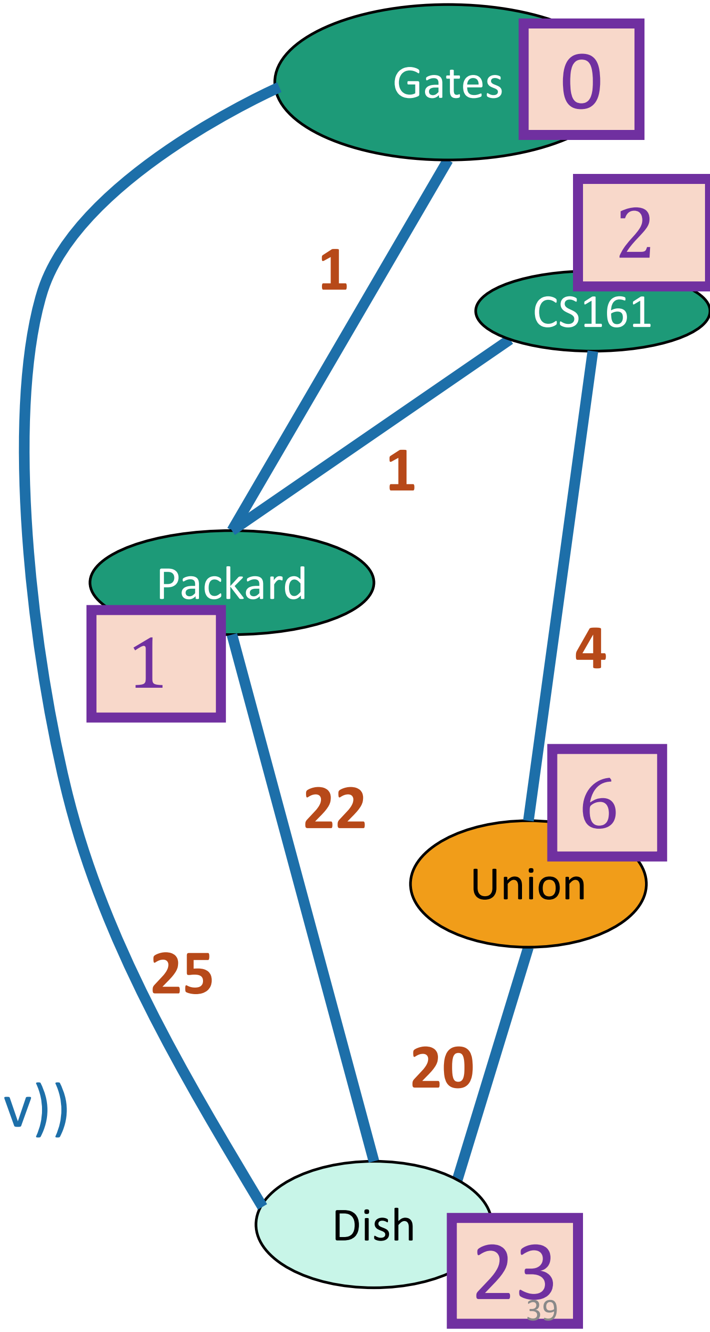

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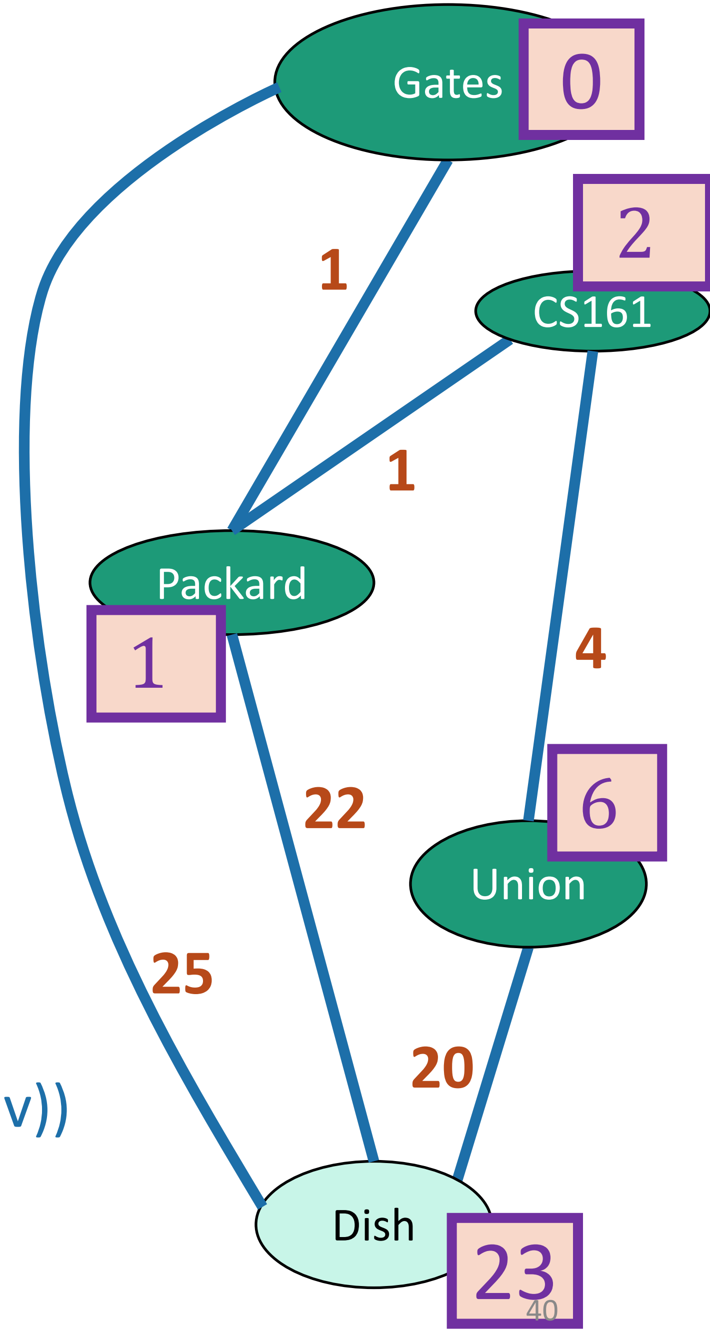

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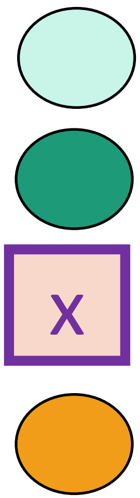

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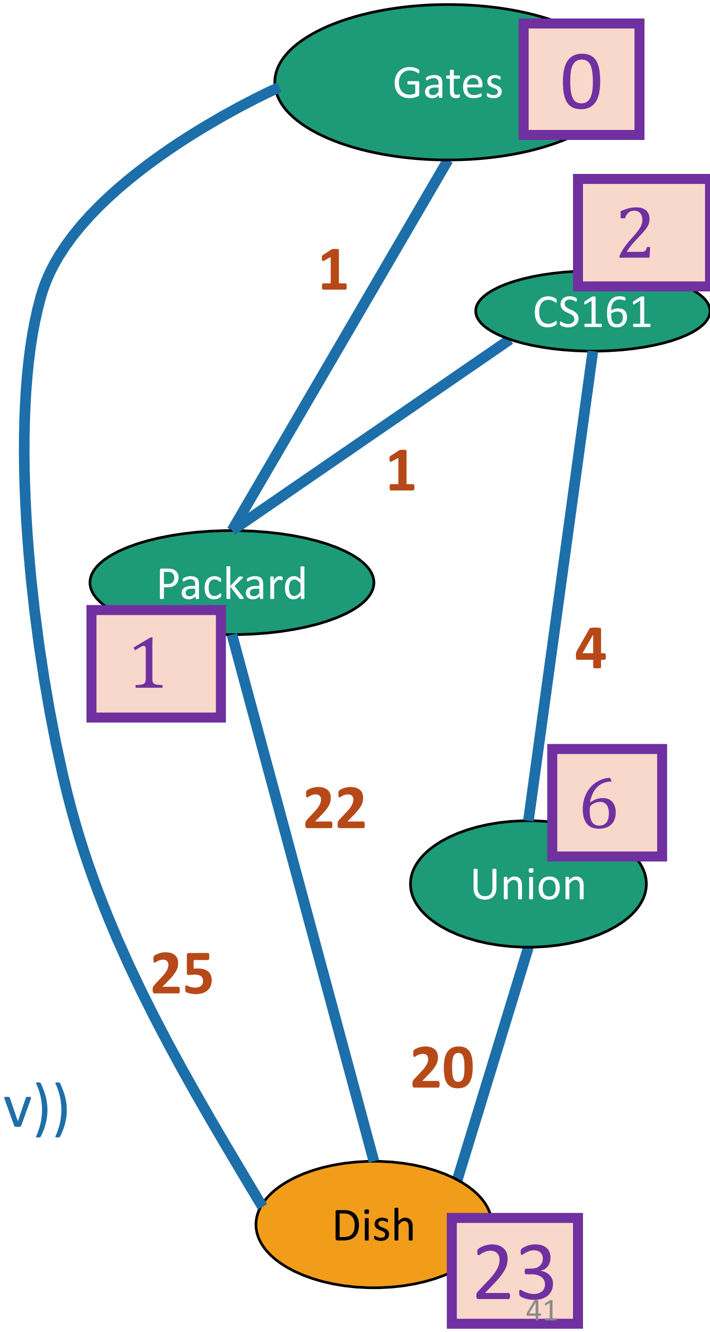

Dijkstra by example

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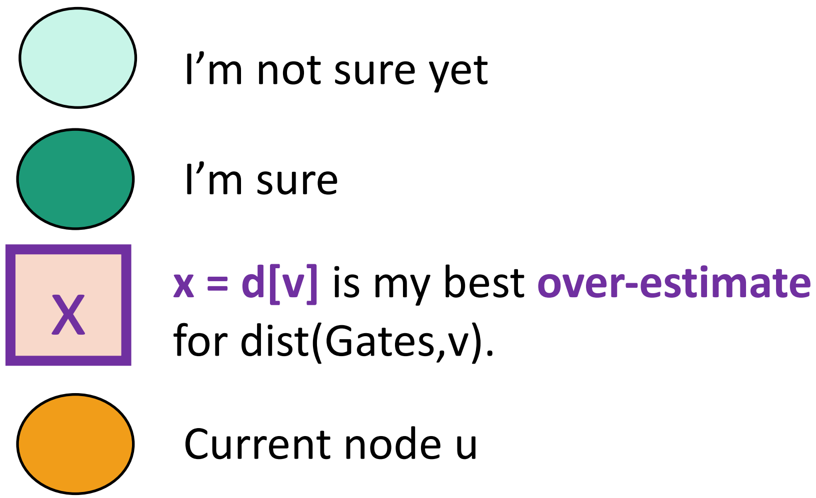

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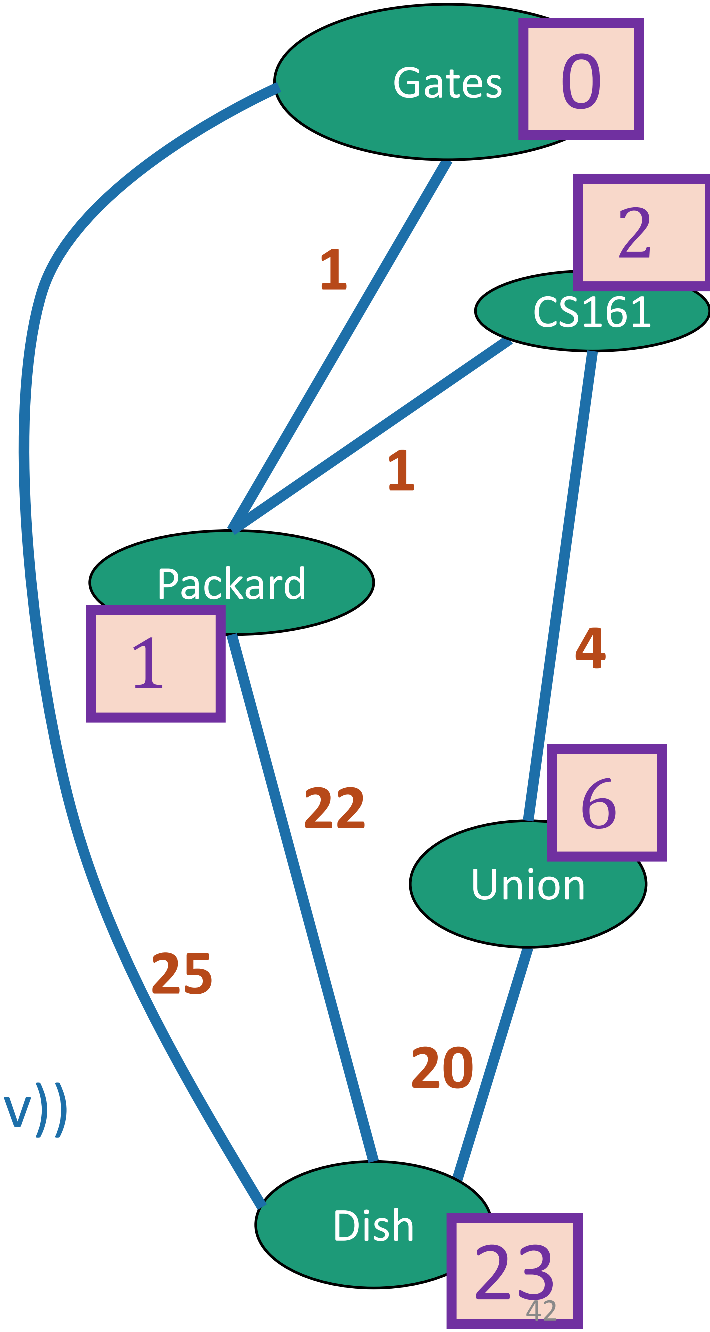

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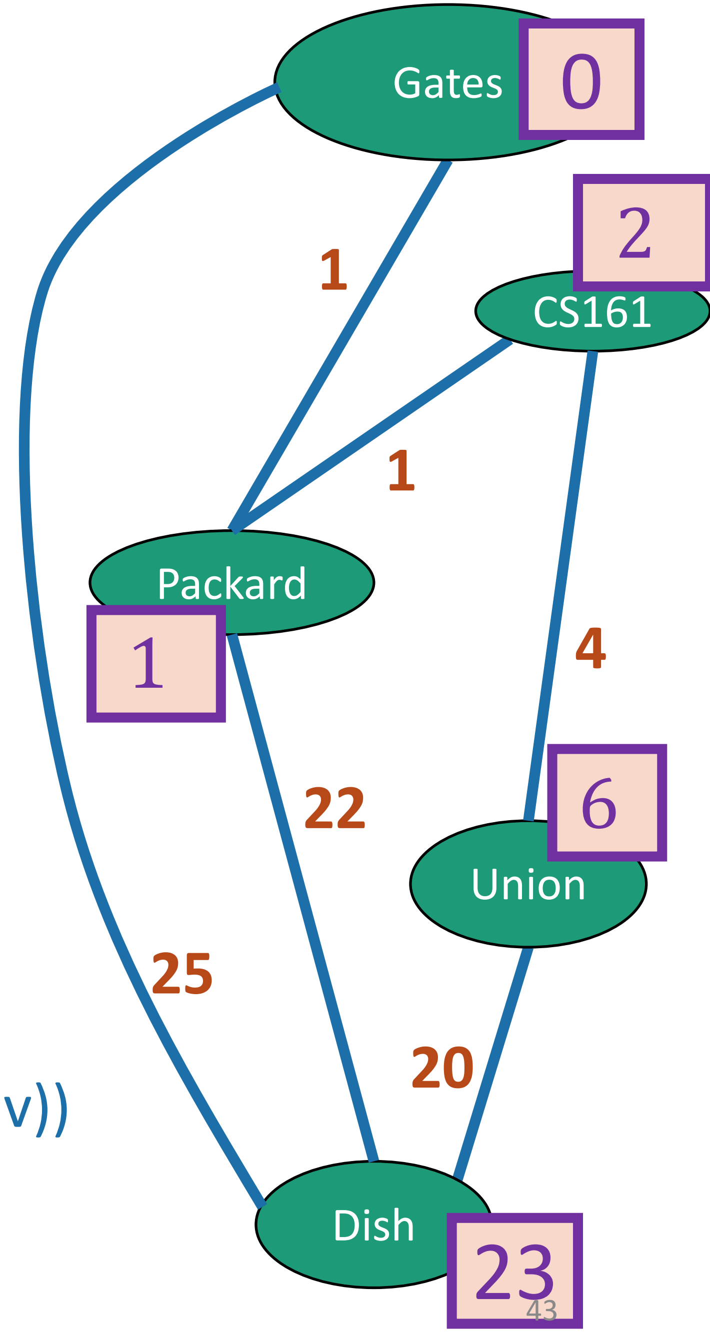
I'm sure

x — **x = d[v]** is my best **over-estimate** for dist(Gates,v).

Current node u

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] = min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.
- Repeat

- After all nodes are **sure**, say that d(Gates, v) = d[v] for all v



Dijkstra's algorithm

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now d(s, v) = d[v]

Lots of implementation details left un-explained.
We'll get to that!

See IPython Notebook for code!

As usual

- Does it work?
  - Yes.
- Is it fast?
  - Depends on how you implement it.

Why does this work?

- **Theorem**:
  - Suppose we run Dijkstra on G =(V,E), starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

Let's rename "Gates" to "s", our starting vertex.

- Proof outline:
  - **Claim 1**: For all v, **d[v] $\geq$ d(s,v).**
  - **Claim 2**: When a vertex v is marked **sure**, **d[v] = d(s,v).**

- **Claims 1 and 2** imply the **theorem.**
  - When v is marked **sure**, **d[v] = d(s,v).**
  - **d[v] $\geq$ d(s,v)** and never increases, so after v is **sure**, **d[v]** stops changing.
  - This implies that at any time *after* v is marked **sure**, **d[v] = d(s,v).**
  - All vertices are **sure** at the end, so all vertices end up with **d[v] = d(s,v).**

Claim 2

Claim 1 + def of algorithm

Next let's prove the claims!

Claim 1

d[v] ≥ d(s,v) for all v.

Intuition!

Informally:

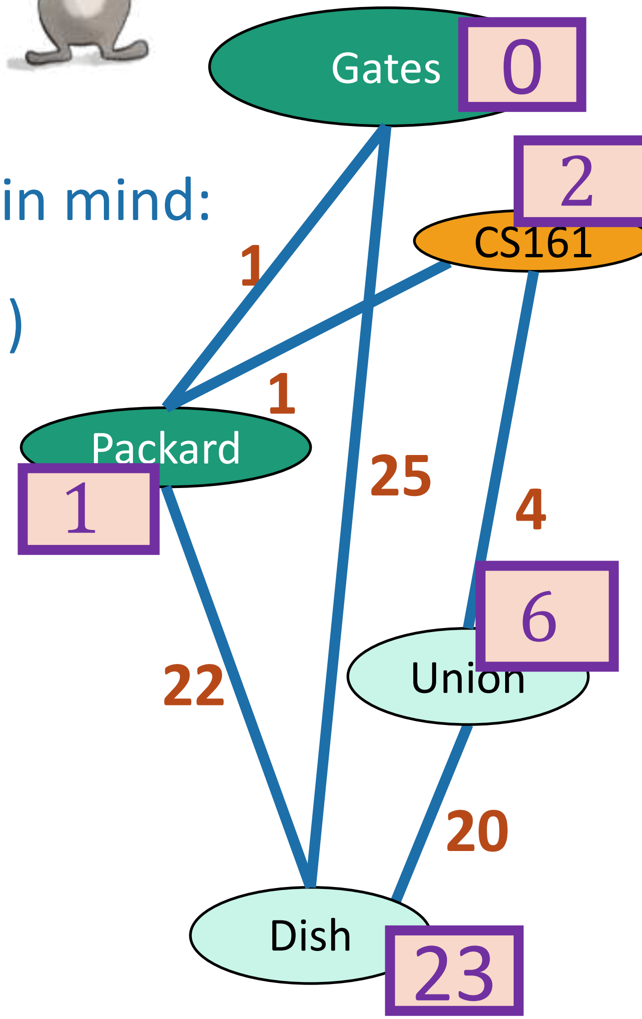
- Every time we update d[v], we have a path in mind:

$$d[v] \leftarrow \min(\, d[v]\, ,\, d[u] + \text{edgeWeight}(u,v)\,)$$

Whatever path we had in mind before

The shortest path to u, and then the edge from u to v.

- d[v] = length of the path we have in mind
  ≥ length of shortest path
  = d(s,v)



Formally:

- We should prove this by induction.
  - (See skipped slide or do it yourself)

Claim 1

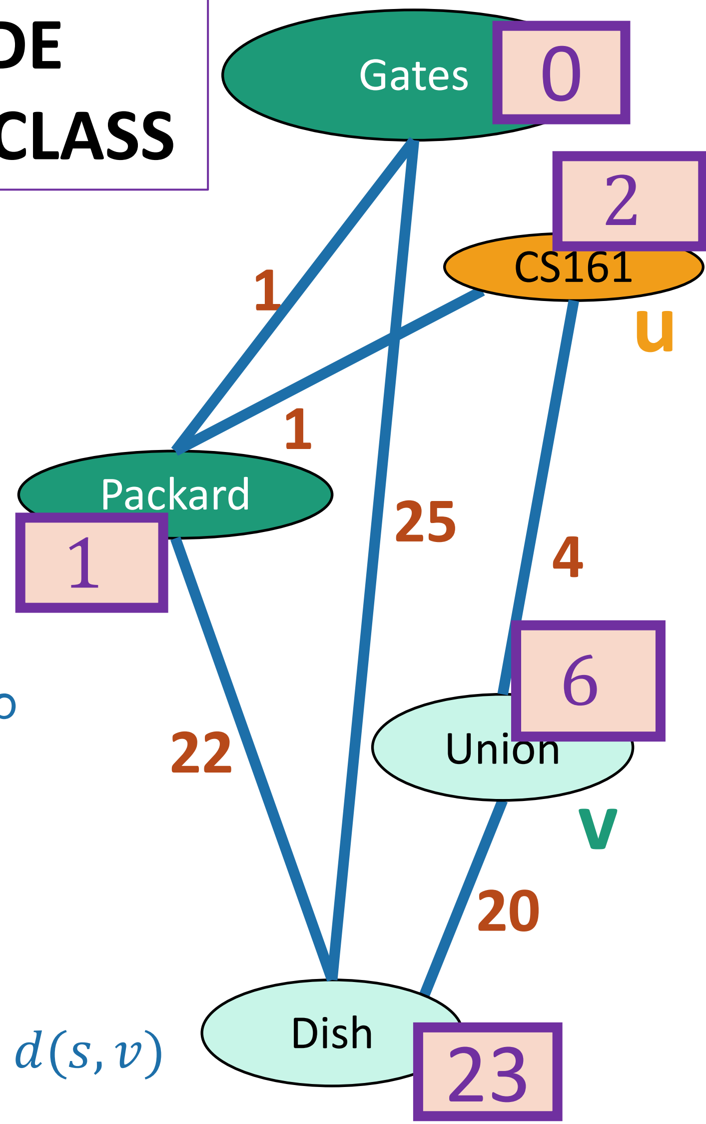
THIS SLIDE SKIPPED IN CLASS

d[v] $\geq$ d(s,v) for all v.

- Inductive hypothesis.
  - After t iterations of Dijkstra, d[v] $\geq$ d(s,v) for all v.
- Base case:
  - At step 0, $\mathrm{d}(\mathrm{s},\mathrm{s}) = 0$, and $d(s,v) \leq \infty$
- Inductive step: say hypothesis holds for t.
  - At step t+1:
    - Pick **u**; for each neighbor **v**:
    - d[v] ← min(d[v] , d[u] + w(u,v)) $\geq d(s,v)$

By induction, $d(s,v) \leq d[v]$

$\mathrm{d}(s,v) \leq d(s,u) + d(u,v)$
$\leq d[u] + w(u,v)$
using induction again for d[u]



So the inductive hypothesis holds for t+1, and Claim 1 follows.

Note, this slide changed since lecture: I proved the inductive step but didn't state the rest of the induction argument!

Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

- Inductive Hypothesis:
  - When we mark the t'th vertex v as sure, d[v] = dist(s,v).
- Base case:
  - The first vertex marked **sure** is s, and d[s] = d(s,s) = 0.
- Inductive step:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

> - Pick the **not-sure** node u with the smallest estimate **d[u].**
> - Update all u's neighbors v:
>   - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
> - Mark u as **sure**.
> - Repeat

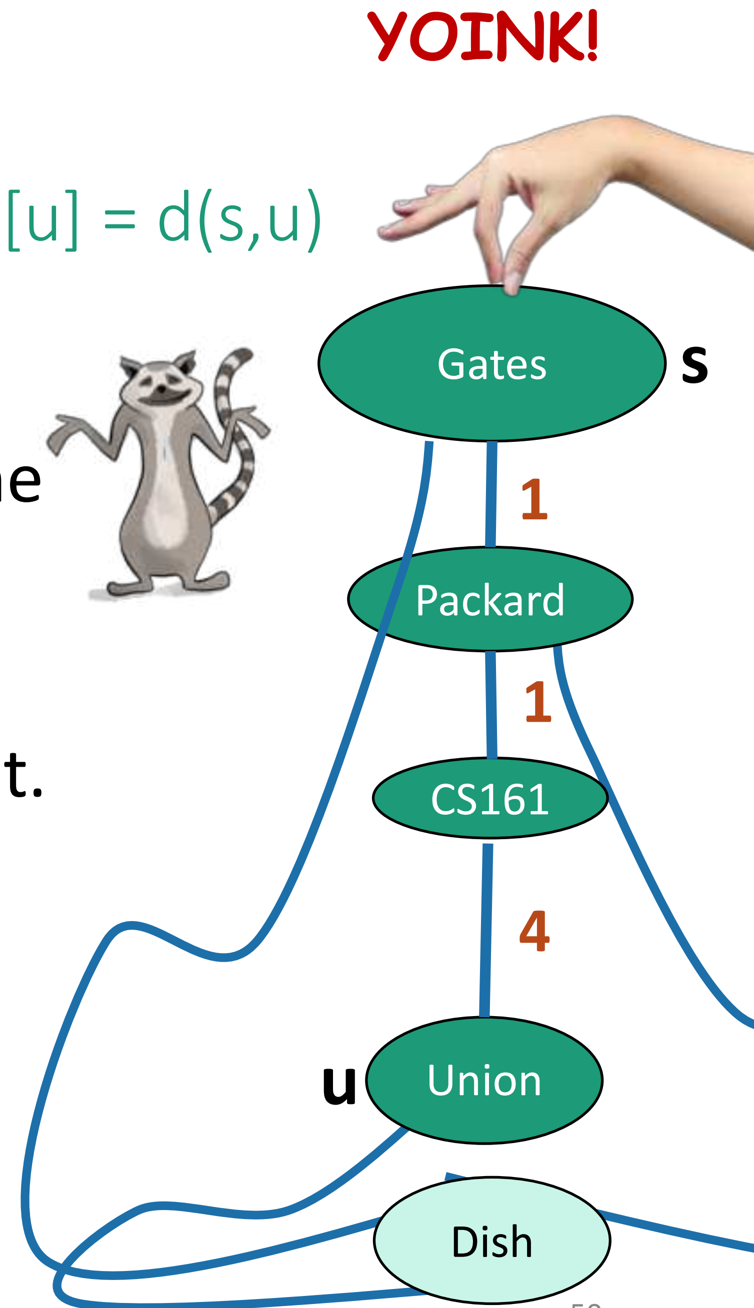
  - Assume by induction that every v already marked **sure** has d[v] = d(s,v).
  - Want to show that d[u] = d(s,u).

Intuition

When a vertex u is marked sure, d[u] = d(s,u)

- The first path that lifts **u** off the ground is the shortest one.

- But we should actually prove it.

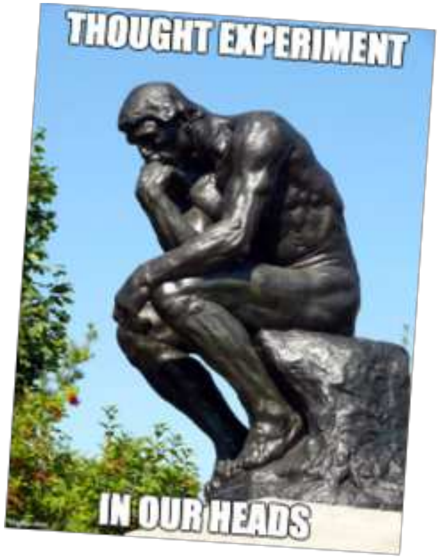


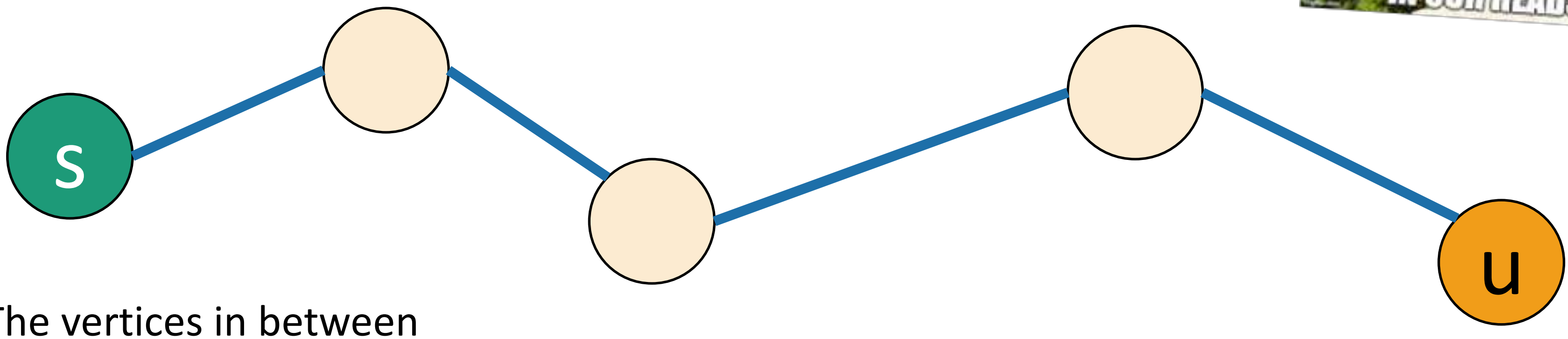
Claim 2

Inductive step

Temporary definition:
v is "good" means that d[v] = d(s,v)

- Want to show that u is good.
- Consider a **true** shortest path from s to u:





The vertices in between are beige because they may or may not be **sure.**

True shortest path.

Claim 2

Inductive step

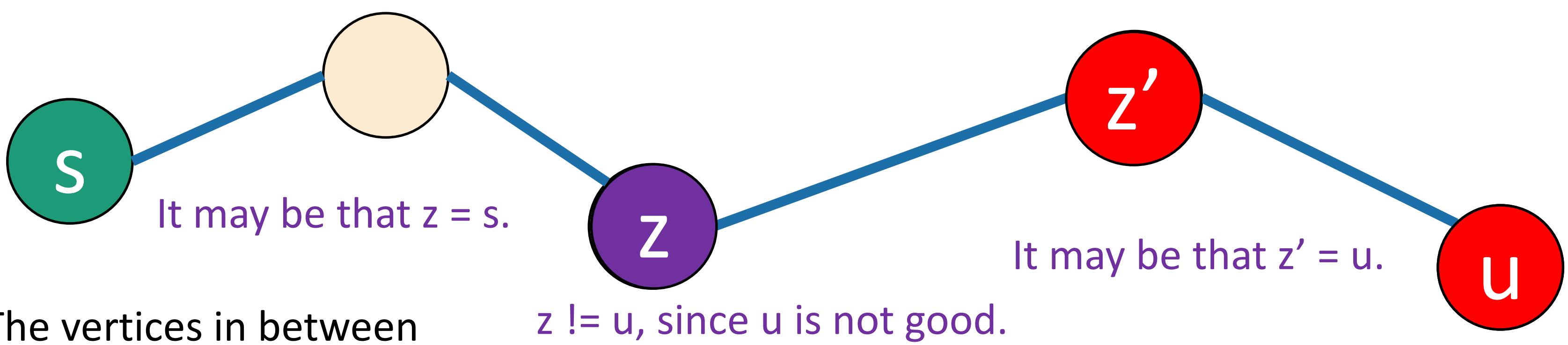
Temporary definition:

v is "good" means that d[v] = d(s,v)

means good means not good

"by way of contradiction"

- Want to show that u is good. BWOC, suppose u isn't good.
- Say z is the last good vertex before u.
- z' is the vertex after z.

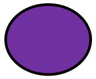


The vertices in between are beige because they may or may not be **sure.**

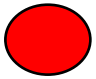
True shortest path.

Claim 2
Inductive step

Temporary definition:
v is "good" means that d[v] = d(s,v)



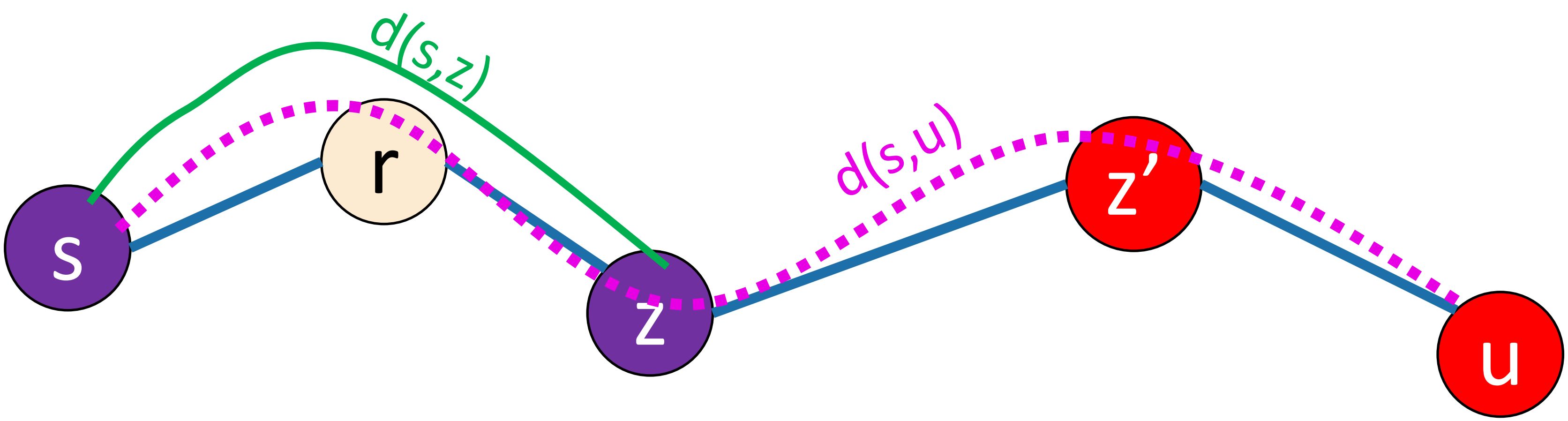
means good



means not good

- Want to show that u is good. BWOC, suppose u isn't good.

$$\underbrace{d[z] = d(s,z)}_{\text{z is good}} \leq \underbrace{d(s,z) \leq d(s,u)}_{\text{Subpaths of shortest paths are shortest paths.}} \leq d[u]$$



Claim 2

Inductive step

Temporary definition:

v is "good" means that d[v] = d(s,v)

means good   means not good

- Want to show that u is good. BWOC, suppose u isn't good.

$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

z is good

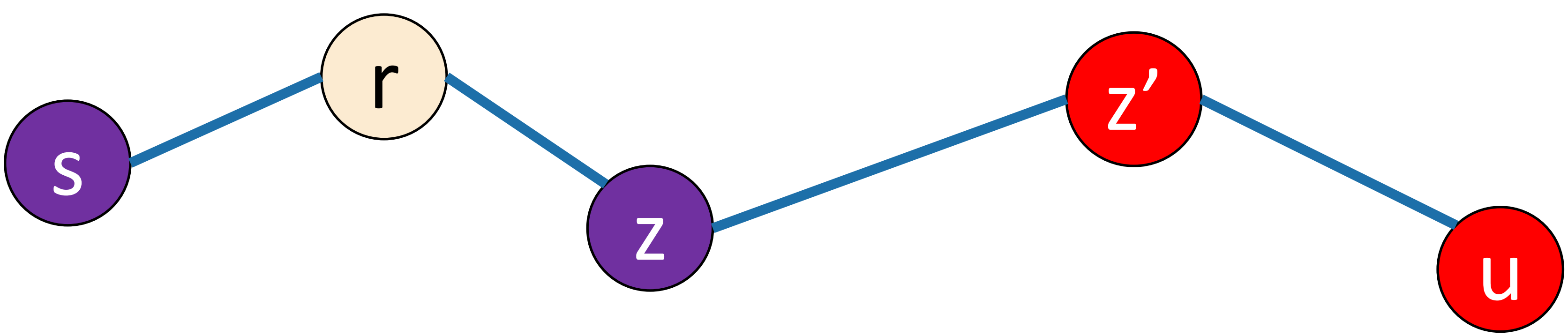
Subpaths of shortest paths are shortest paths.

Claim 1

- If $d[z] = d[u]$, then u is good.
- So $d[z] < d[u]$, so z is **sure.**

But u is not good!

We chose u so that d[u] was smallest of the unsure vertices.



Claim 2

Inductive step

Temporary definition:

v is “good” means that d[v] = d(s,v)

means good

means not good

- Want to show that u is good. BWOC, suppose u isn’t good.

$$d[z] = d(s,z) \leq d(s,u) \leq d[u]$$

z is good

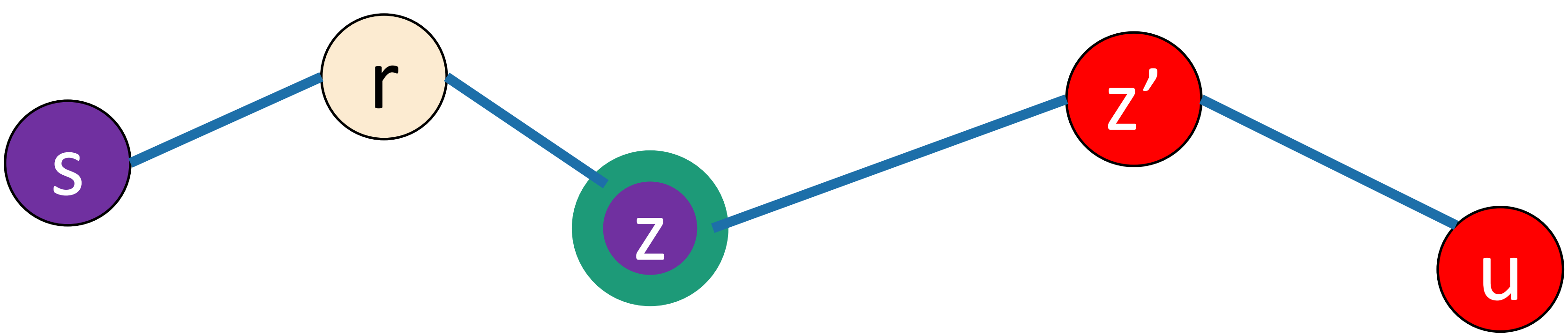
Subpaths of shortest paths are shortest paths.

Claim 1

But u is not good!

- If $d[z] = d[u]$, then u is good.
- So $d[z] < d[u]$, so z is **sure.**

We chose u so that d[u] was smallest of the unsure vertices.



Claim 2

Inductive step

Temporary definition:
v is “good” means that d[v] = d(s,v)

means good   means not good

- Want to show that u is good. BWOC, suppose u isn’t good.
- If z is sure then we’ve already updated z’:

$d[z'] \leftarrow min\{\, d[z'], d[z] + w(z,z')\}$

- $d[z'] \leq d[z] + w(z,z')$ def of update

$= \ d(s,z) + w(z,z')$ By induction when z was added to the sure list it had d(s,z) = d[z]

$= \ d(s,z')$ sub-paths of shortest paths are shortest paths

$\leq d[z']$ Claim 1

That is, the value of d[z] when z was marked sure…

So d(s,z’) = d[z’] and so z’ is good.

So u is good!

Back to this slide

Claim 2

When a vertex u is marked sure, d[u] = d(s,u)

- Inductive Hypothesis:
  - When we mark the t'th vertex v as sure, d[v] = dist(s,v).
- Base case:
  - The first vertex marked **sure** is s, and d[s] = d(s,s) = 0.
- Inductive step:
  - Suppose that we are about to add u to the **sure** list.
  - That is, we picked u in the first line here:

> - Pick the **not-sure** node u with the smallest estimate **d[u].**
> - Update all u's neighbors v:
>   - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
> - Mark u as **sure**.
> - Repeat

  - Assume by induction that every v already marked **sure** has d[v] = d(s,v).
  - Want to show that d[u] = d(s,u).

Conclusion: Claim 2 holds!

Why does this work?

Now back to this slide

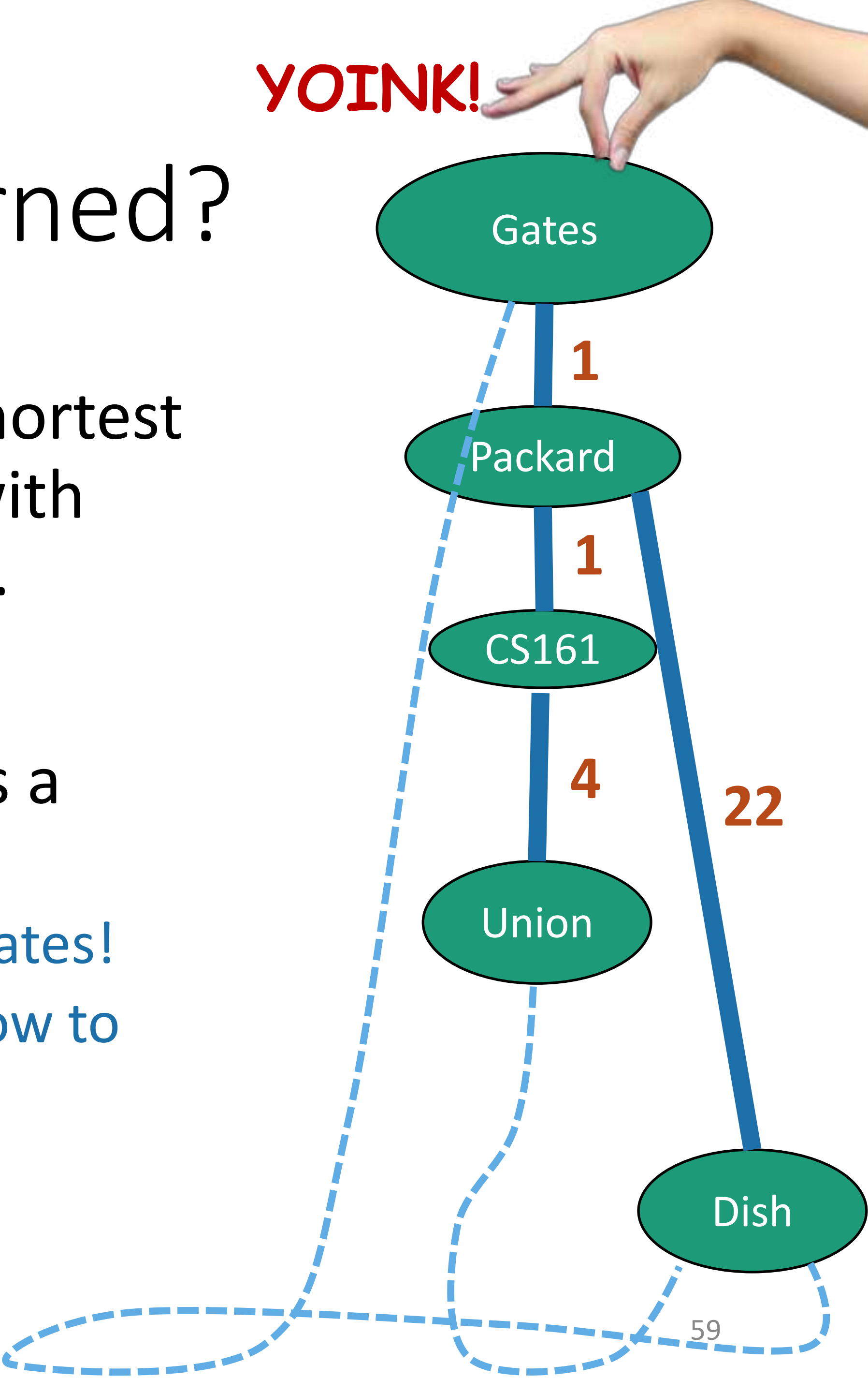
- **Theorem**:
  - Run Dijkstra on G =(V,E) starting from s.
  - At the end of the algorithm, the estimate **d[v]** is the actual distance d(s,v).

- Proof outline:
  - **Claim 1**: For all v, **d[v]** $\geq$ **d(s,v).**
  - **Claim 2**: When a vertex is marked **sure**, **d[v] = d(s,v).**

- **Claims 1 and 2** imply the **theorem.**

What have we learned?

- Dijkstra's algorithm finds shortest paths in weighted graphs with non-negative edge weights.
- Along the way, it constructs a nice tree.
  - We could post this tree in Gates!
  - Then people would know how to get places quickly.



As usual

- Does it work?
  - Yes.
- Is it fast?
  - Depends on how you implement it.

Running time?

Dijkstra(G,s):

- Set all vertices to **not-sure**
- d[v] = ∞ for all v in V
- d[s] = 0
- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.
- Now dist(s, v) = d[v]

- n iterations (one per vertex)
- How long does one iteration take?

Depends on how we implement it…

We need a data structure that:

- Stores unsure vertices v
- Keeps track of d[v]
- Can find u with minimum d[u]
  - `findMin()`
- Can remove that u
  - `removeMin(u)`
- Can update (decrease) d[v]
  - `updateKey(v,d)`

Just the inner loop:

- Pick the **not-sure** node u with the smallest estimate **d[u].**
- Update all u's neighbors v:
  - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
- Mark u as **sure**.

Total running time is big-oh of:

$$\sum_{u \in V} \left(T(\text{findMin}) + \left(\sum_{v \in u.neighbors} T(\text{updateKey}) \right) + T(\text{removeMin}) \right)$$

= n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`)

If we use an array

- T(findMin) = O(n)
- T(removeMin) = O(n)
- T(updateKey) = O(1)

- Running time of Dijkstra

  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  = $O(n^2) + O(m)$

  = $O(n^2)$

If we use a red-black tree

- T(findMin) = O(log(n))
- T(removeMin) = O(log(n))
- T(updateKey) = O(log(n))

- Running time of Dijkstra

  $= O(n(\, T(\texttt{findMin}) + T(\texttt{removeMin})\,) + m\, T(\texttt{updateKey}))$

  $= O(n\log(n)) + O(m\log(n))$

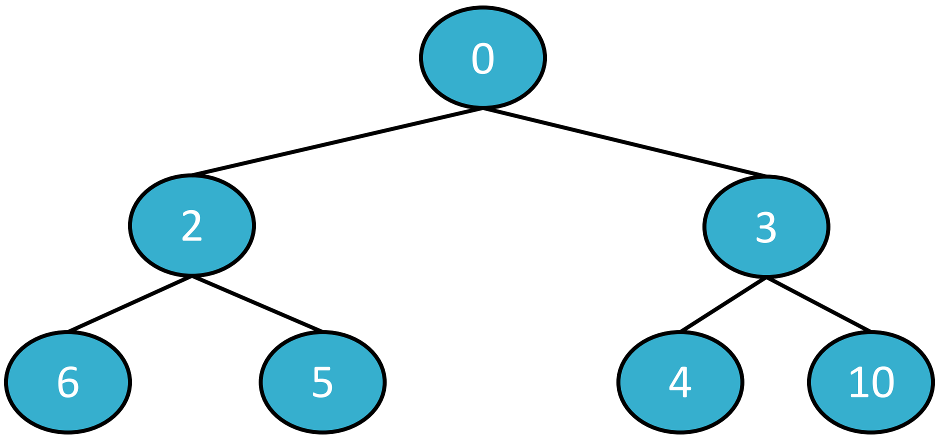
  $= O((n + m)\log(n))$

Better than an array if the graph is sparse!

aka if m is much smaller than n^2

Heaps support these operations

- T(findMin)
- T(removeMin)
- T(updateKey)



- A **heap** is a tree-based data structure that has the property that every node has a smaller key than its children.
- Not covered in this class – see CS166! (Or CLRS).
- But! We will use them.

Many heap implementations

Nice chart on Wikipedia:

Operation	Binary[7]	Leftist	Binomial[7]	Fibonacci[7][8]	Pairing[9]	Brodal[10][b]	Rank-pairing[12]	Strict Fibonacci[13]
find-min	$\Theta(1)$	$\Theta(1)$	$\Theta(\log n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
delete-min	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$[c]	$O(\log n)$[c]	$O(\log n)$	$O(\log n)$[c]	$O(\log n)$
insert	$O(\log n)$	$\Theta(\log n)$	$\Theta(1)$[c]	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
decrease-key	$\Theta(\log n)$	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$[c]	$o(\log n)$[c][d]	$\Theta(1)$	$\Theta(1)$[c]	$\Theta(1)$
merge	$\Theta(n)$	$\Theta(\log n)$	$O(\log n)$[e]	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Say we use a **Fibonacci Heap**

- T(findMin) = O(1) (amortized time*)
- T(removeMin) = O(log(n)) (amortized time*)
- T(updateKey) = O(1) (amortized time*)
- See CS166 for more! (or CLRS)

- Running time of Dijkstra

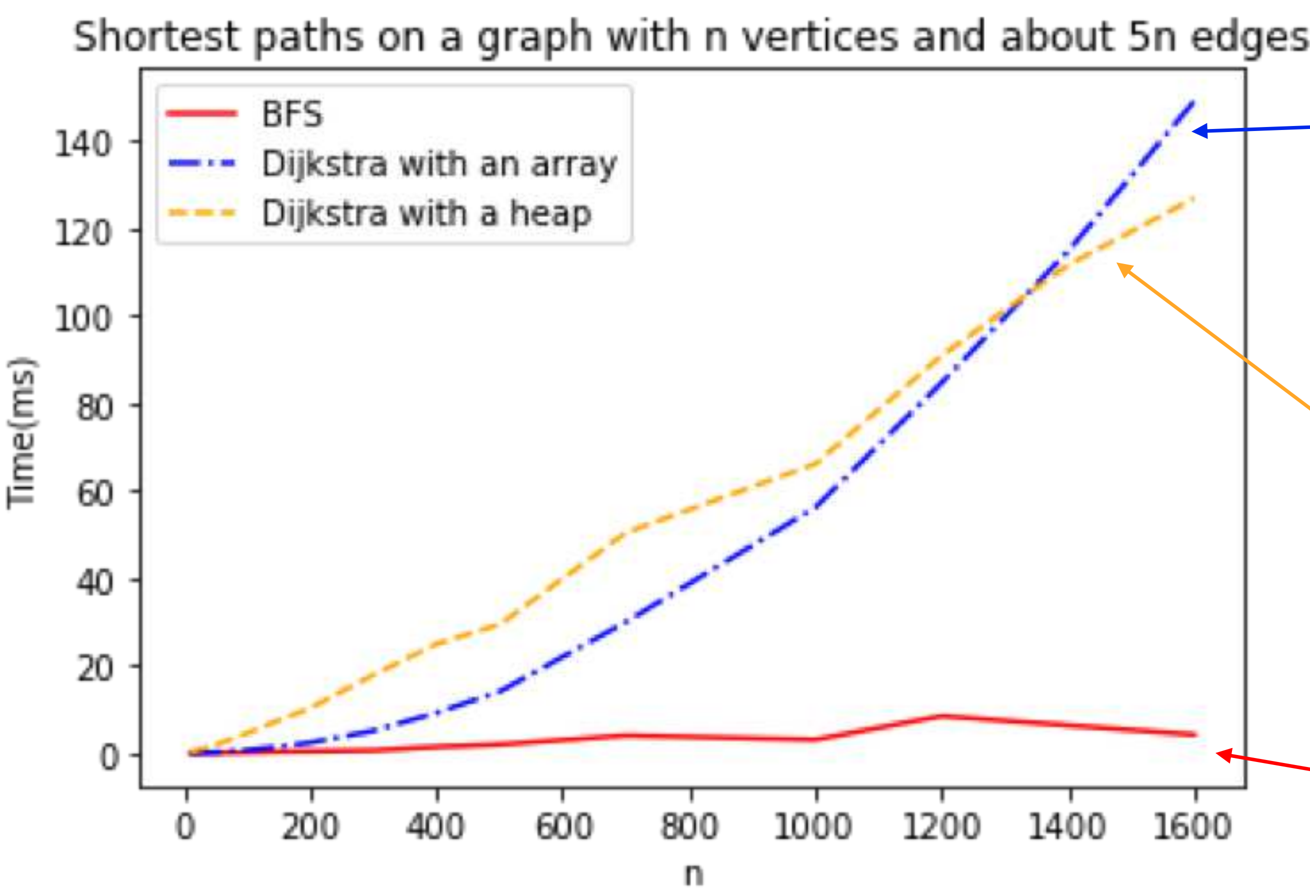
  = O(n(T(`findMin`) + T(`removeMin`)) + m T(`updateKey`))

  = O(nlog(n) + m) (amortized time)

*This means that any sequence of d `removeMin` calls takes time at most O(dlog(n)).
But a few of the d may take longer than O(log(n)) and some may take less time..

See IPython Notebook for Lecture 11
The heap is implemented using `heapdict`

In practice



Dijkstra using a Python list to keep track of vertices has quadratic runtime.

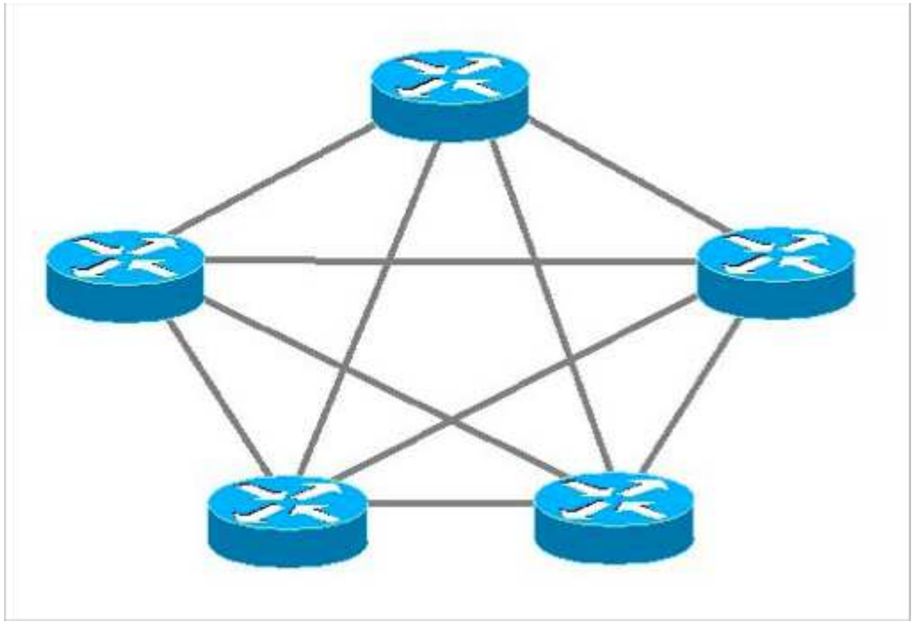
Dijkstra using a heap looks a bit more linear (actually nlog(n))

BFS is really fast by comparison! But it doesn't work on weighted graphs.

Dijkstra is used in practice

- eg, OSPF (Open Shortest Path First), a routing protocol for IP networks, uses Dijkstra.

But there are some things it's not so good at.



Dijkstra Drawbacks

- Needs non-negative edge weights.
- If the weights change, we need to re-run the whole thing.
    - in OSPF, a vertex broadcasts any changes to the network, and then every vertex re-runs Dijkstra’s algorithm from scratch.

Bellman-Ford algorithm

- (-) Slower than Dijkstra's algorithm

- (+) Can handle negative edge weights.
  - Can be useful if you want to say that some edges are actively good to take, rather than costly.
  - Can be useful as a building block in other algorithms.

- (+) Allows for some flexibility if the weights change.
  - We'll see what this means later

Today: *intro* to Bellman-Ford

- We'll see a definition by example.
- We'll come back to it next lecture with more rigor.
  - Don't worry if it goes by quickly today.
  - There are some skipped slides with pseudocode, but we'll see them again next lecture.

- Basic idea:
  - Instead of picking the u with the smallest d[u] to update, just update all of the u's simultaneously.

Bellman-Ford algorithm

SLIDE SKIPPED IN CLASS

Bellman-Ford(G,s):

- d[v] = ∞ for all v in V
- d[s] = 0
- **For** i=0,…,n-1:
  - **For** u in V:
    - **For** v in u.neighbors:
      - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))

Instead of picking u cleverly, just update for all of the u's.

Compare to Dijkstra:

- **While** there are **not-sure** nodes:
  - Pick the **not-sure** node u with the smallest estimate **d[u].**
  - **For** v in u.neighbors:
    - d[v] ← min(d[v] , d[u] + edgeWeight(u,v))
  - Mark u as **sure**.

For pedagogical reasons

which we will see next lecture

SLIDE SKIPPED IN CLASS

- We are actually going to change this to be less smart.
- Keep n arrays: $d^{(0)}, d^{(1)}, \ldots, d^{(n-1)}$

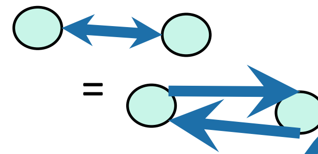
Bellman-Ford*(G,s):

- $d^{(0)}[v] = \infty$ for all v in V
- $d^{(0)}[s] = 0$
- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \mathrm{edgeWeight}(u,v))$
- Then $\mathrm{dist}(s,v) = d^{(n-1)}[v]$

Slightly different than the original Bellman-Ford algorithm, but the analysis is basically the same.

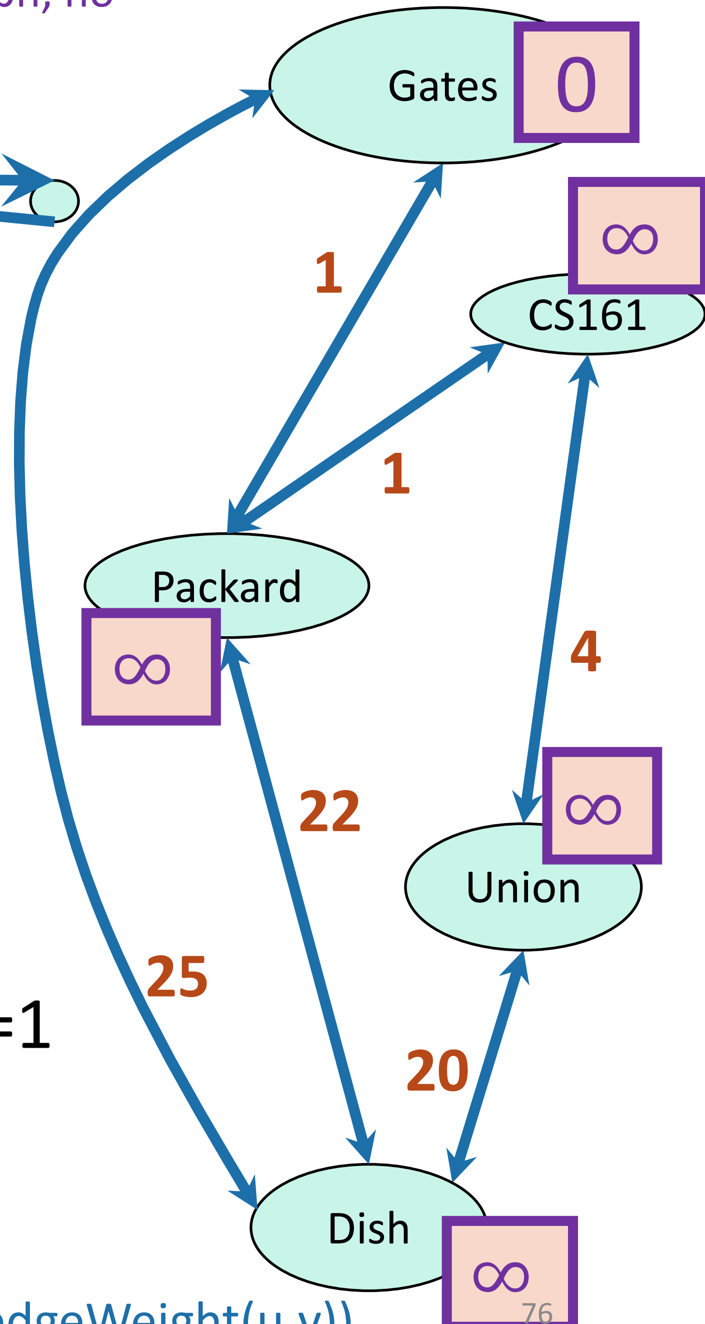
Bellman-Ford

Start with the same graph, no negative weights.



How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$					
$d^{(2)}$					
$d^{(3)}$					
$d^{(4)}$					

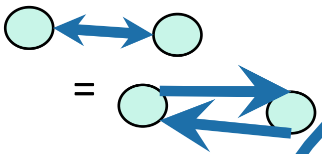


i=1

- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

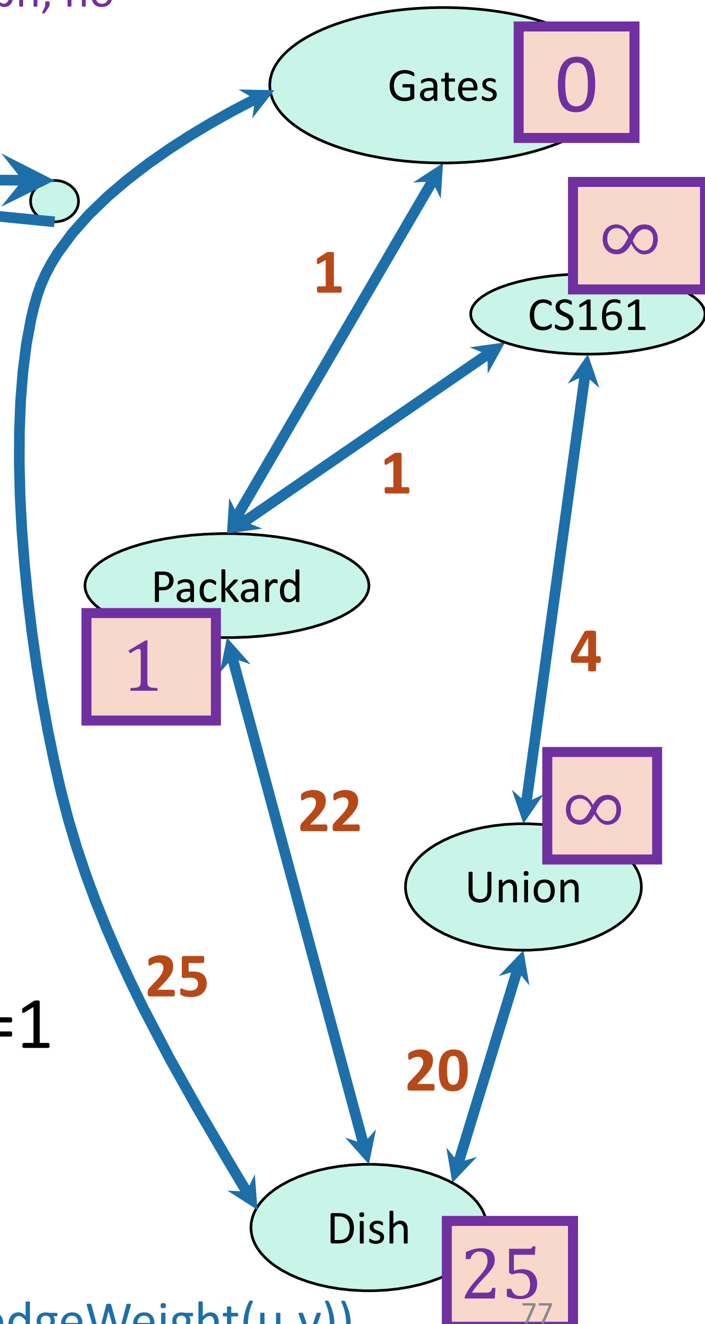
Bellman-Ford

Start with the same graph, no negative weights.



How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$					
$d^{(3)}$					
$d^{(4)}$					



i=1

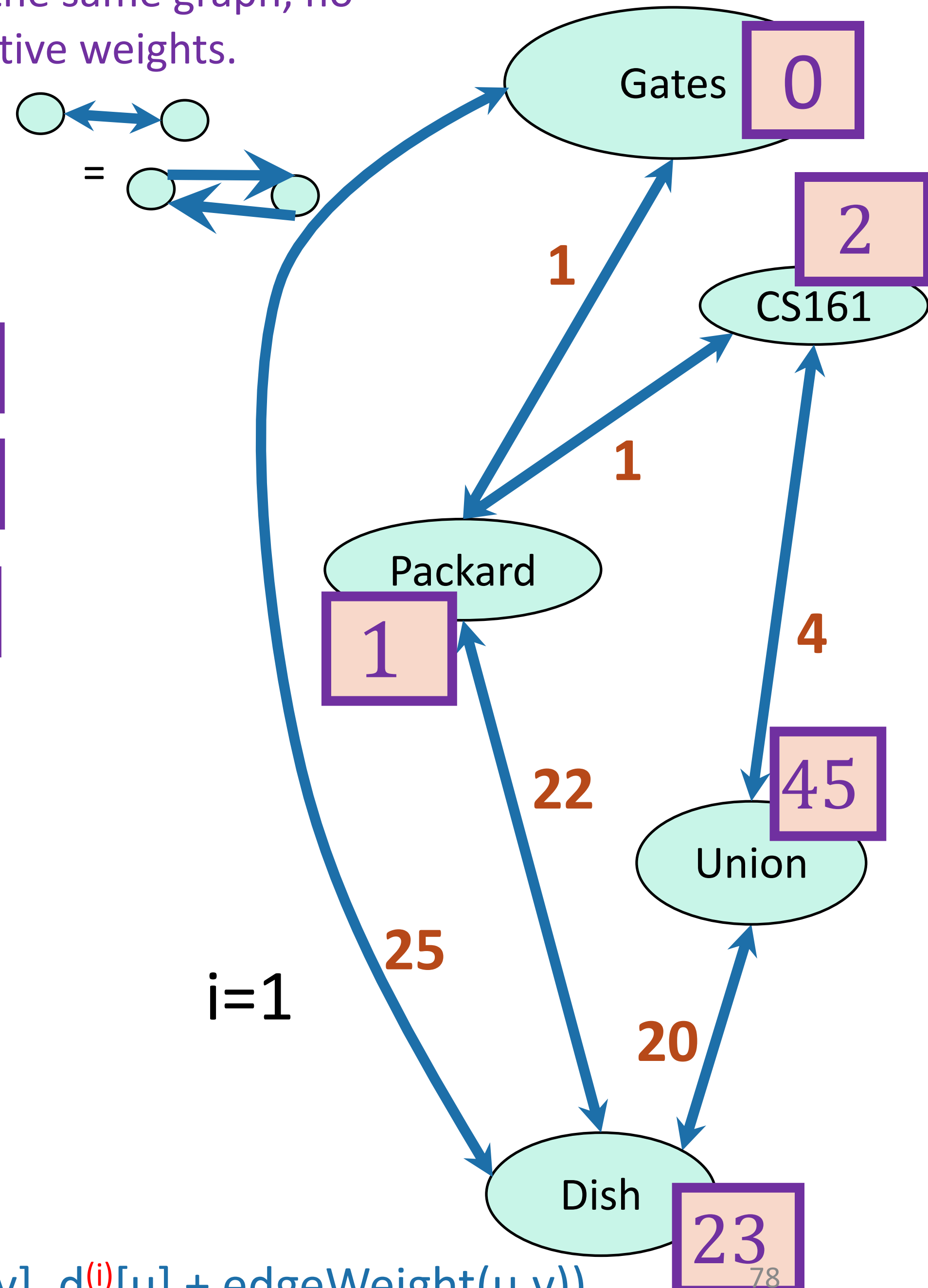
- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

Bellman-Ford

Start with the same graph, no negative weights.

How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$					
$d^{(4)}$					



i=1

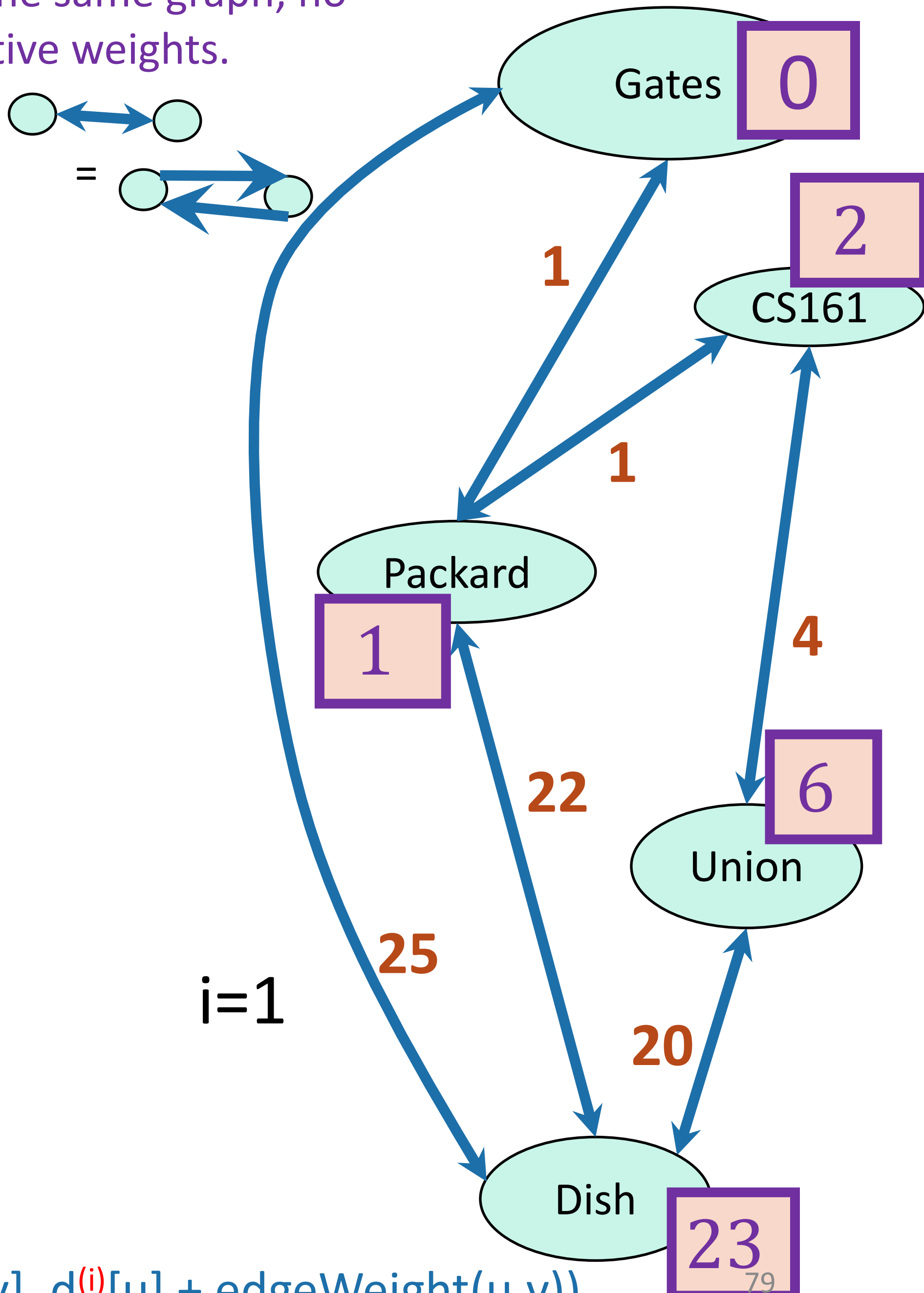
- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

Bellman-Ford

Start with the same graph, no negative weights.

How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$					

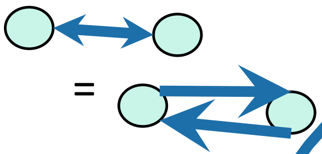


i=1

- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

Bellman-Ford

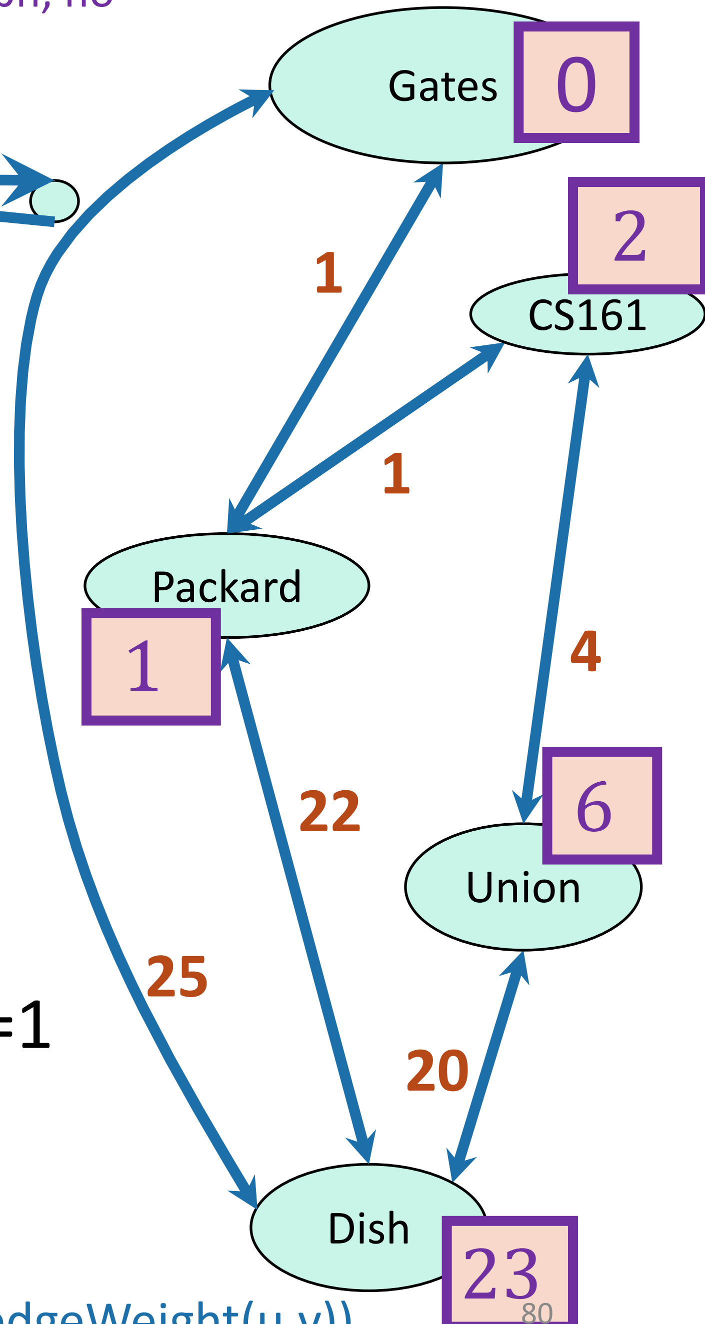
Start with the same graph, no negative weights.



How far is a node from Gates?

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$	0	1	2	6	23

These are the final distances!



i=1

- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \text{edgeWeight}(u,v))$

As usual

- Does it work?
  - Yes
  - Idea to the right.
  - (See hidden slides for details)
- Is it fast?
  - Not really…

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$	0	1	2	6	23

Idea: proof by induction.
Inductive Hypothesis:
$d^{(i)}[v]$ is equal to the cost of the shortest path between s and v **with at most i edges**.
Conclusion:
$d^{(n-1)}[v]$ is equal to the cost of the shortest simple path between s and v. **(Since all simple paths have at most n-1 edges).**

Skipped in class

Proof by induction

- **Inductive Hypothesis:**
  - After iteration i, for each v, $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v with at most i edges.
- **Base case:**
  - After iteration 0… 
- **Inductive step:**

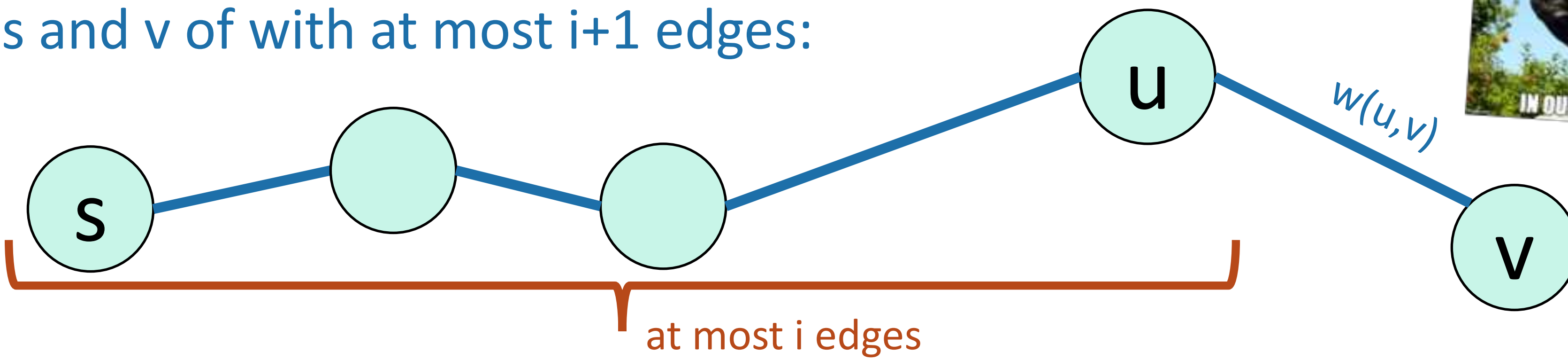
Inductive step

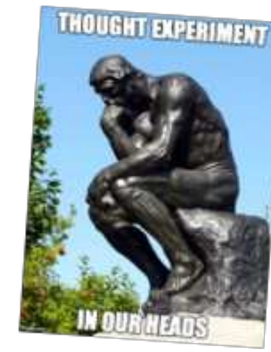
Hypothesis: After iteration i, for each v, $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v with at most i edges.

- Suppose the inductive hypothesis holds for i.
- We want to establish it for i+1.

Say this is the shortest path between s and v of with at most i+1 edges:

Let u be the vertex right before v in this path.





- By induction, $d^{(i)}[u]$ is the cost of a shortest path between s and u of i edges.
- By setup, $d^{(i)}[u] + w(u,v)$ is the cost of a shortest path between s and v of i+1 edges.
- In the i+1'st iteration, we ensure $\mathbf{d^{(i+1)}[v] <= d^{(i)}[u] + w(u,v)}$.
- So $d^{(i+1)}[v]$ <= cost of shortest path between s and v with i+1 edges.
- But $d^{(i+1)}[v]$ = cost of a particular path of at most i+1 edges >= cost of shortest path.
- So d[v] = cost of shortest path with at most i+1 edges.

Skipped in class

Proof by induction

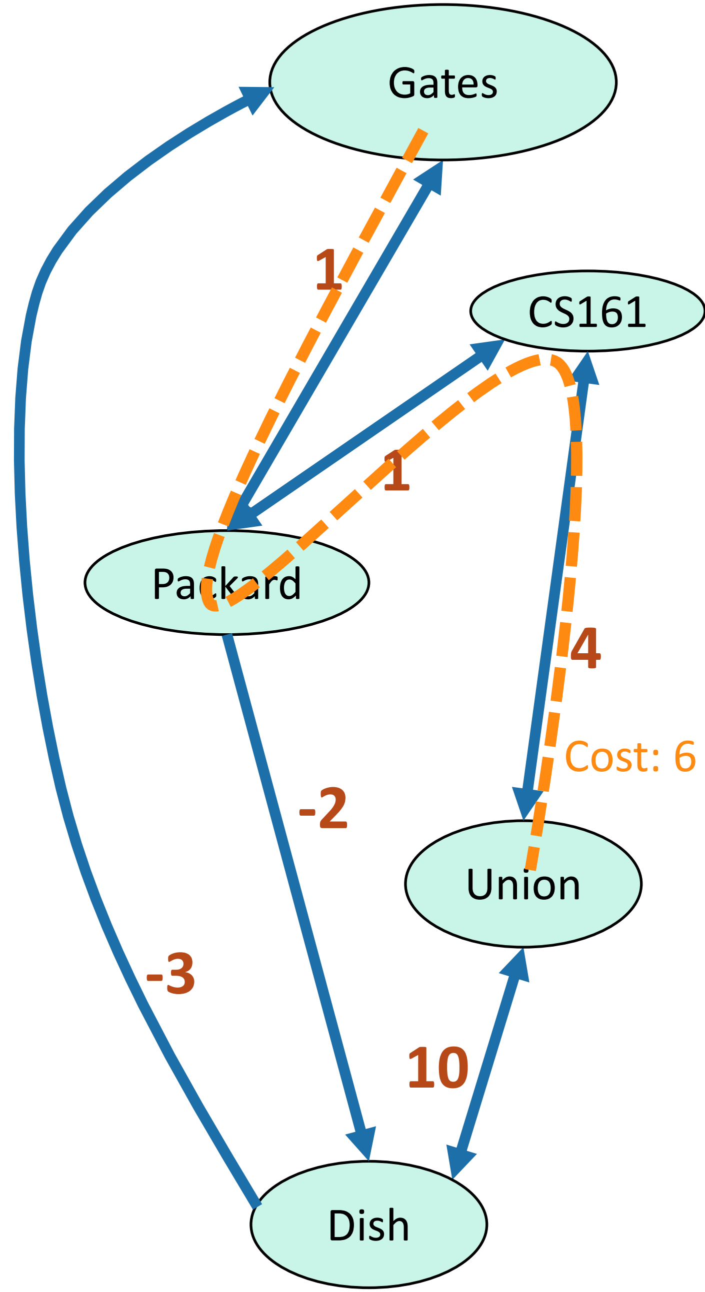
- **Inductive Hypothesis:**
  - After iteration i, for each v, $d^{(i)}[v]$ is equal to the cost of the shortest path between s and v of length at most i edges.
- **Base case:**
  - After iteration 0…
- **Inductive step:**
- **Conclusion:**
  - After iteration n-1, for each v, d[v] is equal to the cost of the shortest path between s and v of length at most n-1 edges.
  - **Aka, d[v] = d(s,v) for all v** as long as there are no cycles!

Pros and cons of Bellman-Ford

- Running time: O(mn) running time
  - For each of n steps we update m edges
  - Slower than Dijkstra
- However, it's also more flexible in a few ways.
  - Can handle negative edges
  - If we constantly do these iterations, any changes in the network will eventually propagate through.

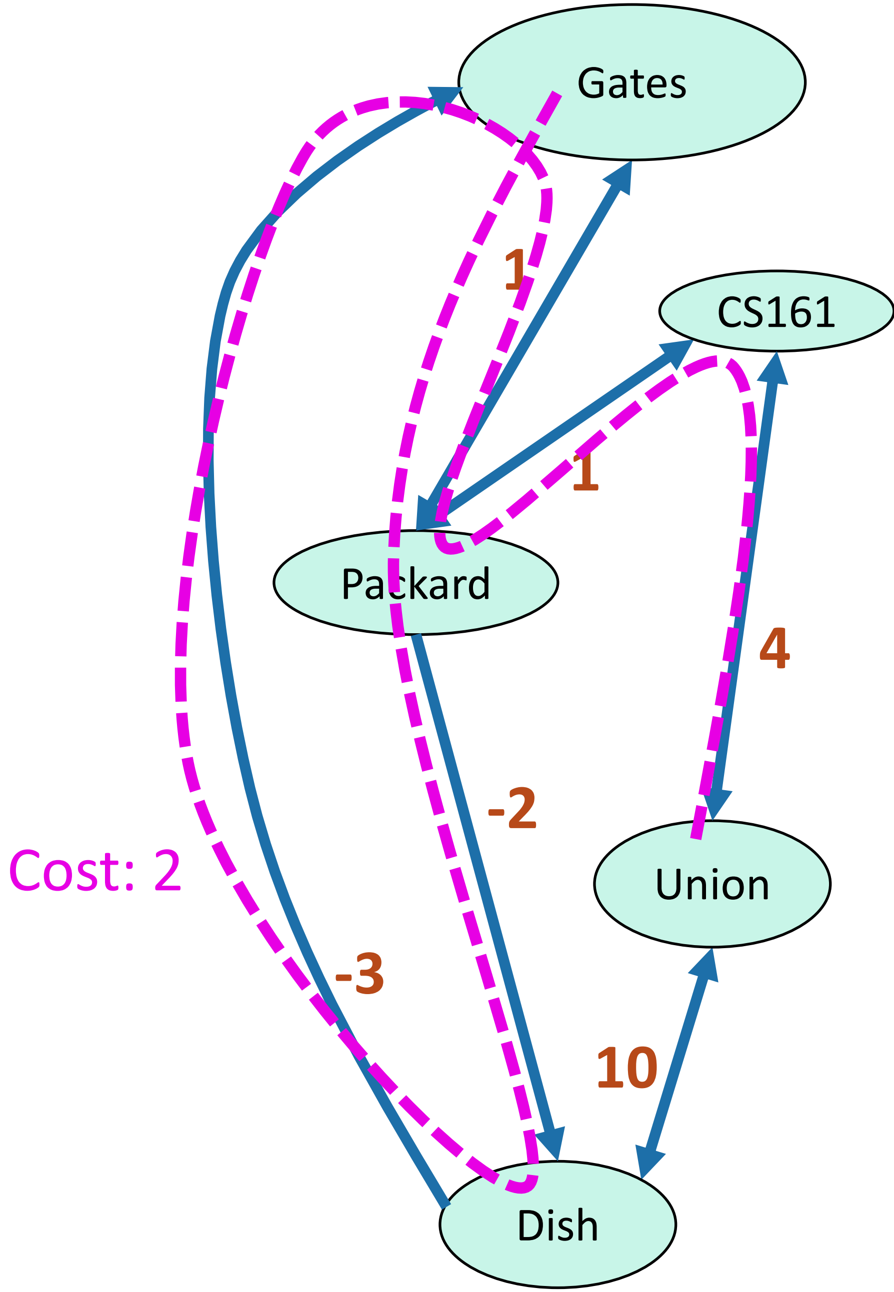
Wait a second…

- What is the shortest path from Gates to the Union?



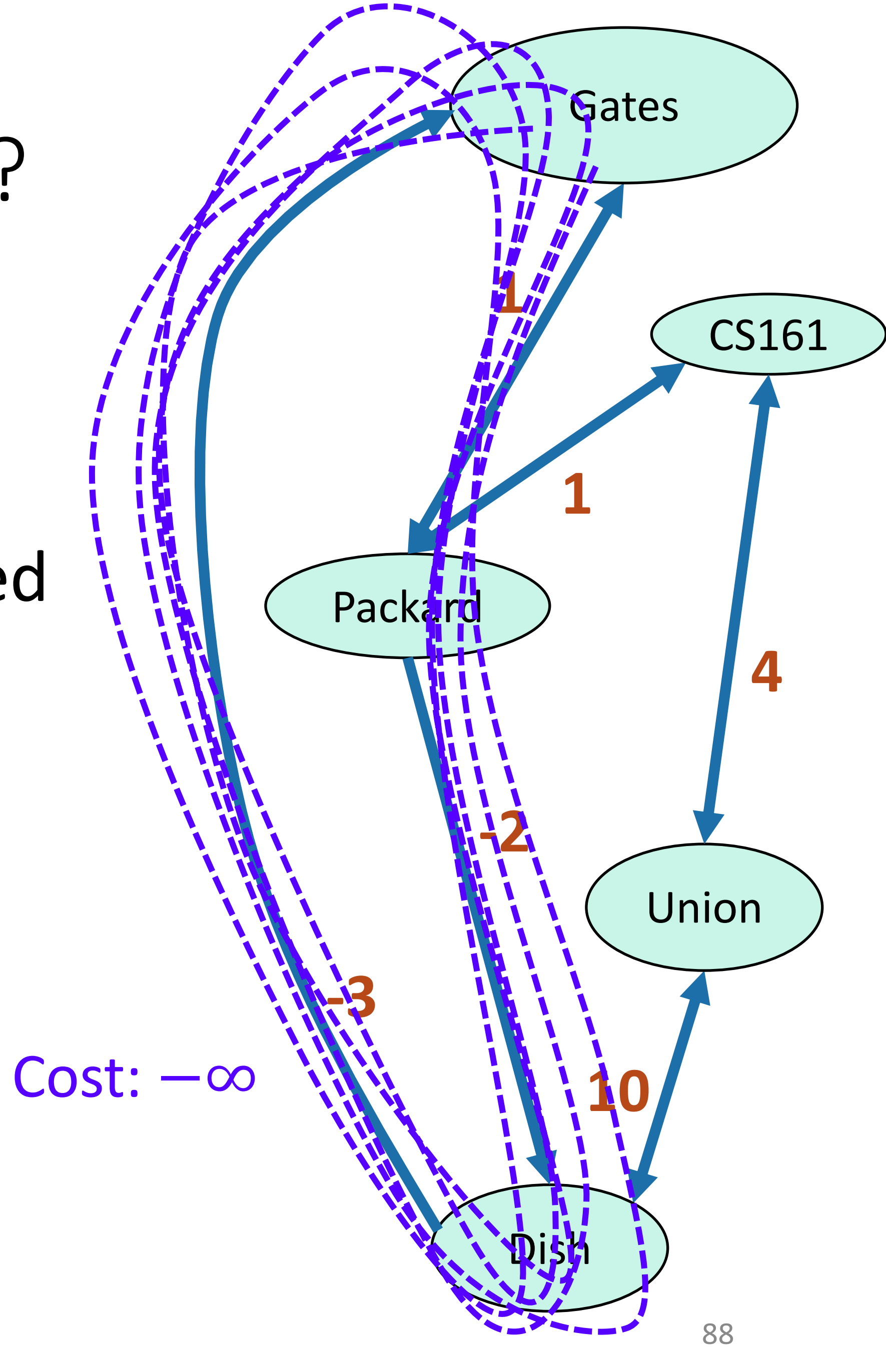
Wait a second…

- What is the shortest path from Gates to the Union?



Negative edge weights?

- What is the shortest path from Gates to the Union?
- Shortest paths aren't defined if there are negative cycles!



Bellman-Ford and negative edge weights

- B-F works with negative edge weights…as long as there are not negative cycles.
  - A negative cycle is a path with the same start and end vertex whose cost is negative.
- However, B-F can detect negative cycles.

Back to the correctness

- Does it work?
  - Yes
  - Idea to the right.

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**25**
$d^{(2)}$	0	1	**2**	**45**	**23**
$d^{(3)}$	0	1	2	**6**	23
$d^{(4)}$	0	1	2	6	23

Idea: proof by induction.

Inductive Hypothesis:

$d^{(i)}[v]$ is equal to the cost of the shortest path between s and v **with at most i edges**.

Conclusion:

$d^{(n-1)}[v]$ is equal to the cost of the shortest simple path between s and v. **(Since all simple paths have at most n-1 edges).**

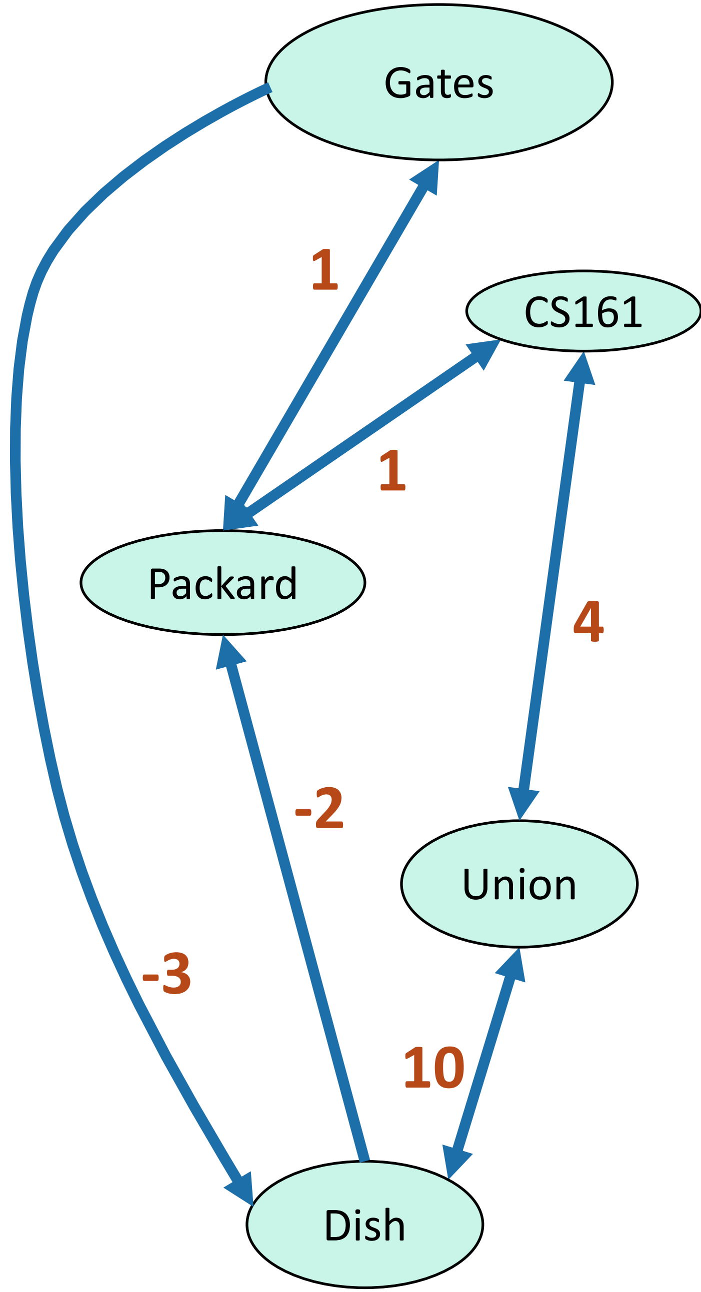
If there are negative cycles, then non-simple paths matter!

So the proof breaks for negative cycles.

B-F with negative cycles

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**-3**
$d^{(2)}$	0	**-5**	**2**	**7**	**-3**
$d^{(3)}$	**-4**	-5	**-4**	**6**	**-3**

This is not looking good!



- **For** i=0,…,n-2:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v] , d^{(i+1)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

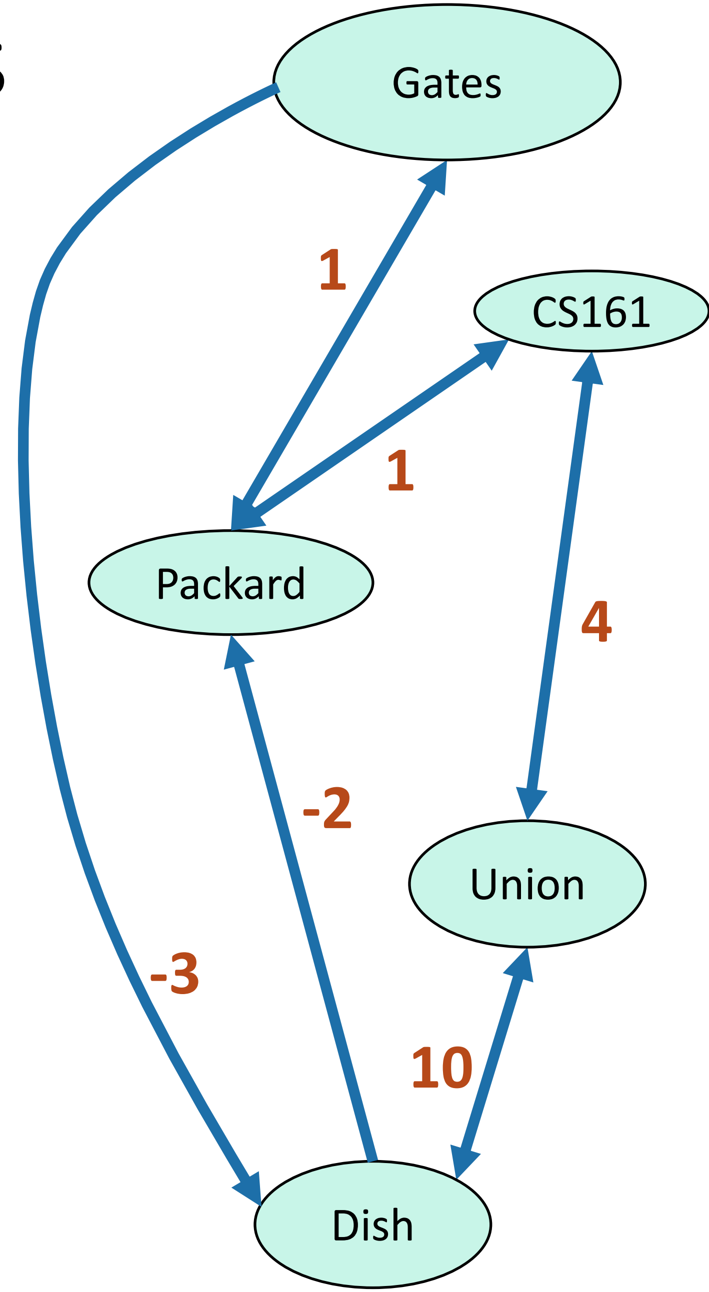
B-F with negative cycles

	Gates	Packard	CS161	Union	Dish
$d^{(0)}$	0	∞	∞	∞	∞
$d^{(1)}$	0	**1**	∞	∞	**-3**
$d^{(2)}$	0	**-5**	**2**	**7**	**-3**
$d^{(3)}$	**-4**	-5	**-4**	**6**	**-3**
$d^{(4)}$	-4	-5	-4	6	**-7**

But **we can tell** that it's not looking good:

	Gates	Packard	CS161	Union	Dish
$d^{(5)}$	-4	**-9**	-4	**3**	-7

Some stuff changed!



- **For** i=0,…,n-1:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v], d^{(i+1)}[v], d^{(i)}[u] + \text{edgeWeight}(u,v))$

How Bellman-Ford deals with negative cycles

- If there are no negative cycles:
  - Everything works as it should.
  - The algorithm stabilizes after n-1 rounds.
  - Note: Negative ***edges*** are okay!!
- If there are negative cycles:
  - Not everything works as it should…
    - Note: it couldn't possibly work, since shortest paths aren't well-defined if there are negative cycles.
  - The d[v] values will keep changing.
- Solution:
  - Go one round more and see if things change.
    - If so, return NEGATIVE CYCLE ☹
  - (Pseudocode on skipped slide)

Bellman-Ford algorithm

SLIDE SKIPPED IN CLASS

Bellman-Ford*(G,s):

- $d^{(0)}[v] = \infty$ for all v in V
- $d^{(0)}[s] = 0$
- **For** i=0,…,n-1:
  - **For** u in V:
    - **For** v in u.neighbors:
      - $d^{(i+1)}[v] \leftarrow \min(d^{(i)}[v]\ ,\ d^{(i+1)}[v],\ d^{(i)}[u] + \mathrm{edgeWeight}(u,v))$
- If $d^{(n-1)} \mathrel{!=} d^{(n)}$:
  - **Return** NEGATIVE CYCLE ☹
- Otherwise, $\mathrm{dist}(s,v) = d^{(n-1)}[v]$

Summary

It's okay if that went by fast, we'll come back to Bellman-Ford

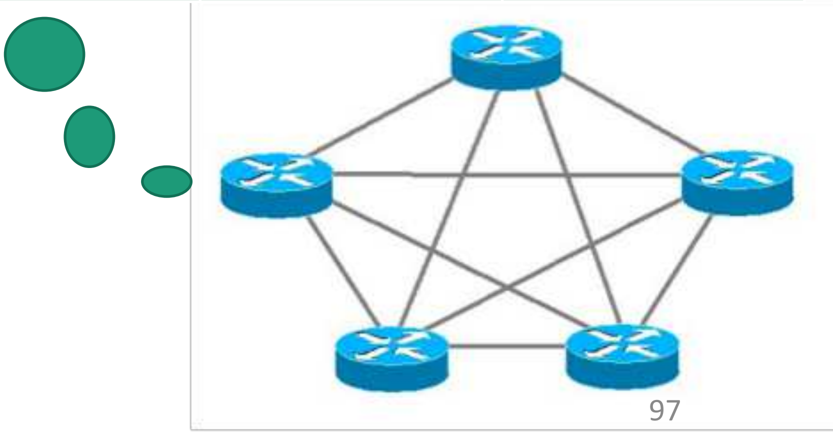
- The Bellman-Ford algorithm:
  - Finds shortest paths in weighted graphs with negative edge weights
  - runs in time $O(nm)$ on a graph G with n vertices and m edges.
- If there are no negative cycles in G:
  - the BF algorithm terminates with $d^{(n-1)}[v] = d(s,v)$.
- If there are negative cycles in G:
  - the BF algorithm returns `negative cycle`.

Bellman-Ford is also used in practice.

- eg, Routing Information Protocol (RIP) uses something like Bellman-Ford.
  - Older protocol, not used as much anymore.

- Each router keeps a **table** of distances to every other router.
- Periodically we do a Bellman-Ford update.
- This means that if there are changes in the network, this will propagate. (maybe slowly…)

Destination	Cost to get there	Send to whom?
172.16.1.0	34	172.16.1.1
10.20.40.1	10	192.168.1.2
10.155.120.1	9	10.13.50.0



Recap: shortest paths

- BFS:
  - (+) O(n+m)
  - (-) only unweighted graphs
- Dijkstra's algorithm:
  - (+) weighted graphs
  - (+) O(nlog(n) + m) if you implement it right.
  - (-) no negative edge weights
  - (-) very "centralized" (need to keep track of all the vertices to know which to update).
- The Bellman-Ford algorithm:
  - (+) weighted graphs, even with negative weights
  - (+) can be done in a distributed fashion, every vertex using only information from its neighbors.
  - (-) O(nm)

Next Time

- Midterm!

Before next time

- Study for midterm! ☺