Homework 1 of Computational Mathematics

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1. $f(x) = x^3 + 2x + k$, then $f'(x) = 3x^2 + 2 > 0$ for all x. Thus, we assume there are two points $a, b \in \mathbb{R}$ s.t. $f(a) = f(b) = 0$. By Rolle's Theorem, there exists a point c in $[a,b]$(or $[b,a]$) s.t. $f'(c) = 0$(Contradiction).

   And since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, by IVT, there eists at least one x s.t. $f(x) = 0$. Thus, the graph of $f(x)$ crosses the x-axis exactly once whatever k is.

2. By EVT, we know that the maximun occurs either $f'(x) = 0$ or a, b.

   (a) $f'(x) = \frac{1}{3}(2 - e^x) = 0$ when $x = \ln(2)$. And since $f'(x) > 0$ when $x \in (0, \ln(2))$ and $f'(x) < 0$ when $x \in (\ln(2), 1)$, $f(\ln(2)) = \frac{1}{3}(2 - 2 + 2\ln 2) = \frac{2\ln(2)}{3}$ is the maximun.

   (b) $f'(x) = \frac{4x^2 - 8x - (4x-3)(2x-2)}{x^4 - 4x^3 + 4x^2} = \frac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$ for $x \in [0.5, 1]$. Thus, $f(0.5) = \frac{2-3}{0.25-1} = \frac{4}{3}$.

   (c) $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$, $x \approx 3.1311062779876$ and $f(x) \approx 4.981433957553$. And $|f(2)| \approx 2.61457$, $|f(4)| \approx 37.1640$. Then, $\max|f(x)| \approx 37.1640$

   (d) $f'(x) = \sin(x-1)e^{-\cos(x-1)}$ since $\sin(x) > 0$ for $0 < x < 1$ and e^x is always positive, the maximun is $f(2) = 1 + e^{-\cos(1)}$.

3. $f'(x) = e^x(\cos(x) - \sin(x))$, $f''(x) = -2e^x\sin(x)$, and $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$. Then, $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x$ and $R_2 = \frac{f^{(3)}(\xi(x))}{6}x^3 = \frac{-1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$. Thus, we have $f(x) = e^x\cos(x) = 1 + x - \frac{1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

   (a) $P_2(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$. And

$$\begin{aligned}
|f(\tfrac{1}{2}) - P_2(\tfrac{1}{2})| &= |R_2(\tfrac{1}{2})| \\
&= \frac{1}{3}\frac{1}{2^3}e^{\xi(\frac{1}{2})}(\sin(\xi(\tfrac{1}{2})) + \cos(\xi(\tfrac{1}{2}))) \\
&\leq \frac{1}{24}e^{\frac{1}{2}}(\sin(\tfrac{1}{2}) + \sin(\tfrac{1}{2})) \\
&\approx 0.09322200499
\end{aligned}$$

   And the actual error is 0.05311086942.

   (b) The bound is the maximun of $|R_2(x)|$ for $x \in [0,1]$. Thus, the bound$= |R_2(x)| \leq \frac{1}{3}1^3 e(\sqrt{2}) = 1.28141034272$.

(c) $\int_0^1 P_2(x)\, dx = \int_0^1 1+x\, dx = 1+\frac{1}{2} = \frac{3}{2}$.

(d)

$$\begin{aligned}\int_0^1 R_2(x)\, dx| &= \int_0^1 \frac{1}{3}x^3 e^{\xi(x)}(\sin(x)+\cos(x))\, dx \\ &\leq \int_0^{\frac{\pi}{4}} \frac{1}{3}x^3 e^x(\sin(x)+\cos(x))\, dx + \int_{\frac{\pi}{4}}^1 \frac{\sqrt{2}}{3}x^3 e^x\, dx \\ &\approx 0.08328+0.1809 \\ &= 0.26418\end{aligned}$$

And the actual error is $1.5-1.3780 = 0.122$.

4. $f(x) = \frac{1}{1-x}$, $P_n(x) = \sum_{k=0}^{n} x^k$. Then, the remainder is $\frac{n!}{n!\cdot(1-\xi(x))^{n+1}}x^{n+1} < x^{n+1}$. Thus, we only need to find the minimun n s.t. $0.5^{n+1} < 10^{-6}$. By taking log both side, $(n+1)(-0.30102999566) < -6 \implies n \geq 19$. Thus, $n = 19$.

5. Since the relative error is $\left|\frac{p^*-p}{p}\right| \leq 10^{-4}$, $p - p\times 10^{-4} \leq p^* \leq p + p\times 10^{-4}$.

   (a) $[\pi - \pi\times 10^{-4}, \pi+\pi\times 10^{-4}]$
   (b) $[e - e\times 10^{-4}, e+e\times 10^{-4}]$
   (c) $[\sqrt{2}-\sqrt{2}\times 10^{-4}, \sqrt{2}+\sqrt{2}\times 10^{-4}]$
   (d) $[\sqrt[3]{7}-\sqrt[3]{7}\times 10^{-4}, \sqrt[3]{7}+\sqrt[3]{7}\times 10^{-4}]$

6. (a) Sicne $e^0 - e^{-0} = 0$, $\lim_{x\to 0}\frac{e^x - e^{-x}}{x} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{2e^x}{1} = 2$.

   (b) $f(0.1) = \frac{0.111\times 10^1 - 0.905}{0.1} = 0.021\times 10^2 = .21\times 10^1$.

   (c) $M_{3,+}(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6} = ((\frac{1}{6}x+\frac{1}{2})x+1)x+1$ and $M_{3,-} = 1-x+\frac{x^2}{2}-\frac{x^3}{6} = ((-\frac{1}{6}x+\frac{1}{2})x-1)x+1$. Then, $M_{3,+}(0.1) = ((.167\times 10^{-1}+.5)\cdot 0.1+1)\cdot 0.1+1 = .111\times 10^1$. $M_{3,-}(0.1) = ((-.167\times 10^{-1}+.5)\cdot 0.1-1)\cdot 0.1+1 = .905$.
   Thus, $f(0.1) = \frac{.111\cdot 10^1 - .905}{0.1} = .205\times 10^1$.

   (d) The relative error of (b) $= \frac{2.1-f(.1)}{f(.1)} = .4825203972\times 10^{-1}$. And the relative error of (c) $= \frac{2.050 f(.1)}{f(.1)} = .2329365782\times 10^{-1}$.

7. (a) $= (-1)^0 \times (2^{1024+8+2-1023}) \times (1+2^{-1}+2^{-4}+2^{-7}+2^{-8}) = 2^11+2^10+2^7+2^4+2^3$
   $= 2048+1024+128+16+8 = 3224 = .3224\times 10^4$.

   (b) Observe it and we get it is $-1\times$ the number of (a), then it is $-.3224\times 10^4$.

   (c) $= (2^{1024-1-1023})\times(1+2^{-2}+2^{-4}+2^{-7}+2^{-8}) = .132421875\times 10^1$.

   (d) $= .132421875\times 10^1 + 2^{-52}$.

8. (a)

$$\begin{aligned}
m &= \frac{.1013}{.1130} = .8965 \\
d_1 &= -.6099e1 + .8965 \times .6990e1 = .168 \\
f_1 &= .1422e2 - .8965 \times .1420e2 = .1490e1 \\
y &= \frac{.1490e1}{.168} = .8869e1 \\
x &= \frac{.1420e2 + .6990e1 \times .8969e1}{.1130e1} = .6742e2
\end{aligned}$$

(b)

$$\begin{aligned}
m &= \frac{-.1811e2}{.8110e1} = -.2233e1 \\
d_1 &= .1122e3 + .2233e1 \times .1220e2 = .1394e3 \\
f_1 &= -.1376 - .2233e1 \times .1370 = -.4435 \\
y &= \frac{-.4435}{.1394e3} = -.3181e-2 \\
x &= \frac{-.1370 + .1220e2 \times .3181e-2}{.8110e1} = -.1211e-1
\end{aligned}$$

9. The upper bound of length is 3.5, 4.5, 5.5 and the lower bound of length is 2.5, 3.5, 4.5. Thus, the upper bound of volume is $3.5 \cdot 4.5 \cdot 5.5 = .83e2$, the lower bound of volume is $.39e2$. The upper bound of surface area is $32+38+50 = .12e3$, the lower bound of surface area is $17+22+32 = .71e2$.

10. (a) $\sin\left(\frac{1}{n^2}\right) = 0 + \frac{1}{n^2} - \frac{1}{6}\left(\frac{1}{n^2}\right)^3 + \cdots = 0 + O\left(\frac{1}{n^2}\right)$.

(b) $(\sin\left(\frac{1}{n}\right))^2 = (0 + O\left(\frac{1}{n}\right))^2 = O\left(\frac{1}{n^2}\right)$.

(c) $\frac{\sin(h)}{h} = 1 + \frac{1}{2}\cdot(-\frac{1}{3})h^2 + \cdots = O(h^2)$.

(d) $\frac{1-e^h}{h} = -1 + (-\frac{1}{2})h + \cdots = O(h)$.

11. (a) $F(x) = c_1F_1(x) + c_2F_2(x) = c_1L_1 + O(x^\alpha) + c_2L_2 + O(x^\beta) = c_1L_1 + c_2L_2 + O^{x^\gamma}$.

(b) $G(x) = F_1(c_1x) + F_2(c_2x) = L_1 + O(x^\alpha) + L_2 + O(x^\beta) = L_1 + L_2 + O(x^\gamma)$.