Homework 1 of Computational Mathematics

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1. $f(x) = x^3 + 2x + k$, then $f'(x) = 3x^2 + 2 > 0$ for all x. Thus, we assume there are two points $a, b \in \mathbb{R}$ s.t. $f(a) = f(b) = 0$. By Rolle's Theorem, there exists a point c in $[a,b]$(or $[b,a]$) s.t. $f'(c) = 0$(Contradiction).

   And since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, by IVT, there eists at least one x s.t. $f(x) = 0$. Thus, the graph of $f(x)$ crosses the x-axis exactly once whatever k is.

2. By EVT, we know that the maximun occurs either $f'(x) = 0$ or a, b.

   (a) $f'(x) = \dfrac{1}{3}(2 - e^x) = 0$ when $x = \ln(2)$. And since $f'(x) > 0$ when $x \in (0, \ln(2))$ and $f'(x) < 0$ when $x \in (\ln(2), 1)$, $f(\ln(2)) = \dfrac{1}{3}(2 - 2 + 2\ln 2) = \dfrac{2\ln(2)}{3}$ is the maximun.

   (b) $f'(x) = \dfrac{4x^2 - 8x - (4x-3)(2x-2)}{x^4 - 4x^3 + 4x^2} = \dfrac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$ for $x \in [0.5, 1]$. Thus, $f(0.5) = \dfrac{2-3}{0.25-2} = \dfrac{4}{7}$.

   (c) $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$

   (d) $f'(x) = \sin(x-1)e^{-\cos(x-1)}$ since $\sin(x) > 0$ for $0 < x < 1$ and e^x is always positive, the maximun is $f(2) = 1 + e^{-\cos(1)}$.

3. $f'(x) = e^x(\cos(x) - \sin(x))$, $f''(x) = -2e^x \sin(x)$, and $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$. Then, $P_2(x) = f(0) + f'(0)x + \dfrac{1}{2}f''(0)x^2 = 1 + x$ and $R_2 = \dfrac{f^{(3)}(\xi(x))}{6}x^3 = \dfrac{-1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$. Thus, we have $f(x) = e^x\cos(x) = 1 + x - \dfrac{1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

   (a) $P_2(\dfrac{1}{2}) = 1 + \dfrac{1}{2} = \dfrac{3}{2}$. And

$$
\begin{aligned}
|f(\tfrac{1}{2}) - P_2(\tfrac{1}{2})| &= |R_2(\tfrac{1}{2})| \\
&= \frac{1}{3}\frac{1}{2^3}e^{\xi(\frac{1}{2})}(\sin(\xi(\tfrac{1}{2})) + \cos(\xi(\tfrac{1}{2}))) \\
&\leq \frac{1}{24}e^{\frac{1}{2}}(\sin(\tfrac{1}{2}) + \sin(\tfrac{1}{2})) \\
&\fallingdotseq 0.09322200499
\end{aligned}
$$

   And the actual error is 0.05311086942(calculated by the calculator of google).

   (b) The bound is the maximun of $|R_2(x)|$ for $x \in [0,1]$. Thus, $|R_2(x)| \leq \dfrac{1}{3}1^3 e(\sqrt{2})$