

# Homework 1 of Computational Mathematics

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1.  $f(x) = x^3 + 2x + k$ , then  $f'(x) = 3x^2 + 2 > 0$  for all  $x$ . Thus, we assume there are two points  $a, b \in \mathbb{R}$  s.t.  $f(a) = f(b) = 0$ . By Rolle's Theorem, there exists a point  $c$  in  $[a, b]$  (or  $[b, a]$ ) s.t.  $f'(c) = 0$  (Contradiction).

And since  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  and  $f(x) \rightarrow -\infty$  as  $x \rightarrow -\infty$ , by IVT, there exists at least one  $x$  s.t.  $f(x) = 0$ . Thus, the graph of  $f(x)$  crosses the  $x$ -axis exactly once whatever  $k$  is.

2. By EVT, we know that the maximum occurs either  $f'(x) = 0$  or  $a, b$ .

(a)  $f'(x) = \frac{1}{3}(2 - e^x) = 0$  when  $x = \ln(2)$ . And since  $f'(x) > 0$  when  $x \in (0, \ln(2))$  and  $f'(x) < 0$  when  $x \in (\ln(2), 1)$ ,  $f(\ln(2)) = \frac{1}{3}(2 - 2 + 2\ln 2) = \frac{2\ln(2)}{3}$  is the maximum.

(b)  $f'(x) = \frac{4x^2 - 8x - (4x - 3)(2x - 2)}{x^4 - 4x^3 + 4x^2} = \frac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$  for  $x \in [0.5, 1]$ . Thus,  $f(0.5) = \frac{2 - 3}{0.25 - 1} = \frac{4}{\frac{3}{4}}$ .

(c)  $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$ ,  $x \approx 3.1311062779876$  and  $f(x) \approx 4.981433957553$ . And  $|f(2)| \approx 2.61457$ ,  $|f(4)| \approx 37.1640$ . Then,  $\max |f(x)| \approx 37.1640$

(d)  $f'(x) = \sin(x-1)e^{-\cos(x-1)}$  since  $\sin(x) > 0$  for  $0 < x < 1$  and  $e^x$  is always positive, the maximum is  $f(2) = 1 + e^{-\cos(1)}$ .

3.  $f'(x) = e^x(\cos(x) - \sin(x))$ ,  $f''(x) = -2e^x \sin(x)$ , and  $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$ . Then,  $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x$  and  $R_2 = \frac{f^{(3)}(\xi(x))}{6}x^3 = \frac{-1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$ . Thus, we have  $f(x) = e^x \cos(x) = 1 + x - \frac{1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

(a)  $P_2(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$ . And

$$\begin{aligned} |f(\frac{1}{2}) - P_2(\frac{1}{2})| &= |R_2(\frac{1}{2})| \\ &= \frac{1}{3} \frac{1}{2^3} e^{\xi(\frac{1}{2})} (\sin(\xi(\frac{1}{2})) + \cos(\xi(\frac{1}{2}))) \\ &\leq \frac{1}{24} e^{\frac{1}{2}} (\sin(\frac{1}{2}) + \sin(\frac{1}{2})) \\ &\approx 0.09322200499 \end{aligned}$$

And the actual error is 0.05311086942.

(b) The bound is the maximum of  $|R_2(x)|$  for  $x \in [0, 1]$ . Thus, the bound  $= |R_2(x)| \leq \frac{1}{3} 1^3 e(\sqrt{2}) = 1.28141034272$ .

$$(c) \int_0^1 P_2(x) dx = \int_0^1 1+x dx = 1 + \frac{1}{2} = \frac{3}{2}.$$

(d)

$$\begin{aligned} \left| \int_0^1 R_2(x) dx \right| &= \int_0^1 \frac{1}{3} x^3 e^{\xi(x)} (\sin(x) + \cos(x)) dx \\ &\leq \int_0^{\frac{\pi}{4}} \frac{1}{3} x^3 e^x (\sin(x) + \cos(x)) dx + \int_{\frac{\pi}{4}}^1 \frac{\sqrt{2}}{3} x^3 e^x dx \\ &\approx 0.08328 + 0.1809 \\ &= 0.26418 \end{aligned}$$

And the actual error is  $1.5 - 1.3780 = 0.122$ .

4.  $f(x) = \frac{1}{1-x}$ ,  $P_n(x) = \sum_{k=0}^n x^k$ . Then, the remainder is  $\frac{n!}{n! \cdot (1-\xi(x))^{n+1}} x^{n+1} < \frac{x^{n+1}}{1-x}$ . Thus, we only need to find the minimum  $n$  s.t.  $\frac{0.5^{n+1}}{1-0.5} = 0.5^n < 10^{-6}$ . By taking log both side,  $n(-0.30102999566) < -6 \implies n \geq 20$ . Thus,  $n = 20$ .

5. Since the relative error is  $\left| \frac{p^* - p}{p} \right| \leq 10^{-4}$ ,  $p - p \times 10^{-4} \leq p^* \leq p + p \times 10^{-4}$ .

- (a)  $[\pi - \pi \times 10^{-4}, \pi + \pi \times 10^{-4}]$   
 (b)  $[e - e \times 10^{-4}, e + e \times 10^{-4}]$   
 (c)  $[\sqrt{2} - \sqrt{2} \times 10^{-4}, \sqrt{2} + \sqrt{2} \times 10^{-4}]$   
 (d)  $[\sqrt[3]{7} - \sqrt[3]{7} \times 10^{-4}, \sqrt[3]{7} + \sqrt[3]{7} \times 10^{-4}]$

6. (a) Since  $e^0 - e^{-0} = 0$ ,  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{2e^x}{1} = 2$ .

(b)  $f(0.1) = \frac{0.111 \times 10^1 - 0.905}{0.1} = 0.205 \times 10^2 = .205 \times 10^1$ .

(c)  $M_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = ((\frac{1}{6}x + \frac{1}{2})x + 1)x + 1$   
 , then  $f(x) = \frac{(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3) - (1-x+\frac{1}{2}x^2-\frac{1}{6}x^3)}{x} = \frac{2x+\frac{1}{3}x^3}{x} = 2 + \frac{1}{3}x^2$ . Thus,  $f(0.1) = 2 + .333 \times .01 = 2 + .333e - 2 = 2^x$

(d) The relative error of (b) =  $\frac{2.05 - f(.1)}{f(.1)} = .232937 \times 10^{-1}$ . And the relative error of (c) =  $\frac{2 - f(.1)}{f(.1)} = .166 \times 10^{-2}$ .

7. (a)  $= (-1)^0 \times (2^{1024+8+2-1023}) \times (1 + 2^{-1} + 2^{-4} + 2^{-7} + 2^{-8}) = 2^1 1 + 2^1 0 + 2^7 + 2^4 + 2^3$   
 $= 2048 + 1024 + 128 + 16 + 8 = 3224 = .3224 \times 10^4$ .  
 (b) Observe it and we get it is  $-1 \times$  the number of (a), then it is  $-.3224 \times 10^4$ .  
 (c)  $= (2^{1024-1-1023}) \times (1 + 2^{-2} + 2^{-4} + 2^{-7} + 2^{-8}) = .132421875 \times 10^1$ .  
 (d)  $= .132421875 \times 10^1 + 2^{-52}$ .

8. (a)

$$\begin{aligned}
 m &= \frac{.1013}{.1130} = .8965 \\
 d_1 &= -.6099e1 + .8965 \times .6990e1 = .168 \\
 f_1 &= .1422e2 - .8965 \times .1420e2 = .1490e1 \\
 y &= \frac{.1490e1}{.168} = .8869e1 \\
 x &= \frac{.1420e2 + .6990e1 \times .8969e1}{.1130e1} = .6742e2
 \end{aligned}$$

(b)

$$\begin{aligned}
 m &= \frac{-.1811e2}{.8110e1} = -.2233e1 \\
 d_1 &= .1122e3 + .2233e1 \times .1220e2 = .1394e3 \\
 f_1 &= -.1376 - .2233e1 \times .1370 = -.4435 \\
 y &= \frac{-.4435}{.1394e3} = -.3181e-2 \\
 x &= \frac{-.1370 + .1220e2 \times .3181e-2}{.8110e1} = -.1211e-1
 \end{aligned}$$

9. The upper bound of length is 3.5, 4.5, 5.5 and the lower bound of length is 2.5, 3.5, 4.5. Thus, the upper bound of volume is  $3.5 \cdot 4.5 \cdot 5.5 = .83e2$ , the lower bound of volume is  $.39e2$ . The upper bound of surface area is  $32 + 38 + 50 = .12e3$ , the lower bound of surface area is  $17 + 22 + 32 = .71e2$ .

10. (a)  $\sin\left(\frac{1}{n^2}\right) = 0 + \frac{1}{n^2} - \frac{1}{6}\left(\frac{1}{n^2}\right)^3 + \cdots = 0 + O\left(\frac{1}{n^2}\right).$

(b)  $\left(\sin\left(\frac{1}{n}\right)\right)^2 = \left(0 + O\left(\frac{1}{n}\right)\right)^2 = O\left(\frac{1}{n^2}\right).$

(c)  $\frac{\sin(h)}{h} = 1 + \frac{1}{2} \cdot \left(-\frac{1}{3}\right)h^2 + \cdots = O(h^2).$

(d)  $\frac{1 - e^h}{h} = -1 + \left(-\frac{1}{2}\right)h + \cdots = O(h).$

11. (a)  $F(x) = c_1F_1(x) + c_2F_2(x) = c_1L_1 + O(x^\alpha) + c_2L_2 + O(x^\beta) = c_1L_1 + c_2L_2 + O^{x^\gamma}.$

(b)  $G(x) = F_1(c_1x) + F_2(c_2x) = L_1 + O(x^\alpha) + L_2 + O(x^\beta) = L_1 + L_2 + O(x^\gamma).$