Homework 2 of Computational Mathematics

AM15 黃琦翔 111652028

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1. $x^3 = x + 1 \implies x^2 = 1 + \frac{1}{x} \implies x = \sqrt{1 + \frac{1}{x}} = g(x)$. $p_1 = g(p_0) = \sqrt{1+1} = \sqrt{2} \approx 1.414$. $p_2 = g(p_1) = \sqrt{1 + \frac{1}{\sqrt{2}}} \approx 1.3065$. $p_3 = g(p_2) \approx 1.3172$. $p_4 = g(p_3) \approx 1.326$. $p_5 = g(p_4) \approx 1.324$ Then, p_4 is the answer that we want to find.

2. Let $f(x) = x^3 + x - 4$, $f'(x) = 3x^2 + 1 < 49$ for all $x \in [1,4]$. Thus, for $|x-y| < \frac{10^{-3}}{49} \approx 2.0409e-5$, $|f(x) - f(y)| < 10^3$. Find n s.t. $3 \cdot 2^{-n} < 2.0409e-5$, $n > -\log_2(\frac{2.0409e-5}{3}) \approx 17.1653$. Thus, the bound of the number of iteration is 18. Then, by python code below, the root is about 1.3787.

```
HW2 > HW2.py > ...
a = [1]
b = [4]

def f(x):
    return x**3 + x - 4

while 1:
    mid = (a[-1] + b[-1])/2
    val = f(mid)
    print(mid, val)

    if abs(val) < 0.0001:
        break
    elif val > 0:
        b.append(mid)
    else:
        a.append(mid)

PROBLEMS    OUTPUT    DEBUG CONSOLE    TERMINAL    PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> &
2.5 14.125
1.75 3.109375
1.375 -0.025390625
1.5625 1.377197265625
1.46875 0.637176513671875
1.421875 0.2965202331542969
1.3984375 0.13326025009155273
1.38671875 0.05336350202560425
1.380859375 0.013844214379787445
1.3779296875 -0.005808685906231403
1.37939453125 0.004008884658105671
1.378662109375 -0.0009021193400258198
1.3790283203125 0.0015528278327110456
1.37884521484375 0.0003252155581776057
1.378753662109375 -0.0002884865606631592 5
1.3787994384765625 1.835583103471094 5e-05
```

3. In this question, we let $p_{n+1} = g(p_n)$ and $p = \sqrt[3]{21}$. And using the fact that $\frac{|p_{n+1} - p|}{|p_n - p|} \approx \frac{\lambda |p_n - p|^\alpha}{\lambda |p_{n-1} - p|^\alpha} \approx \left| \frac{p_n - p}{p_{n-1} - p} \right|^\alpha$, then $\alpha \approx \frac{\ln |(p_{n+1} - p)/(p_n - p)|}{\ln |(p_n - p)/(p_{n-1} - p)|}$

(a) Suppose $p_n \to p$ linearly. $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} g'(p_n) = \lim_{n\to\infty} (\frac{20p_n^3+21}{21p_n^2})' = \frac{20}{21} - \frac{2}{p^3} = \frac{20}{21} - \frac{2}{21} = \frac{6}{7}$ is constant. Thus, it truely converges linearly.

(b) $\lim_{n\to\infty} g'(p_n) = 1 - \frac{1}{3} - \frac{14}{p^3} = \frac{2}{3} - \frac{2}{3} = 0$. Thus, it converges sublinearly.

(c) $\lim_{n\to\infty} g'(p_n) = 1 - \frac{2p^5 - 84p^3 + 21p^2 + 441}{(p^2-21)^2}$

(d) $\lim_{n\to\infty} |g'(p_n)| = |\frac{-21 \cdot \frac{1}{2}}{p^{3/2}}| = \frac{1}{2}$. Then, $\lim_{n\to\infty} \frac{|g''(p)|}{2} =$