Homework 1 of Computational Mathematics

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1. $f(x) = x^3 + 2x + k$, then $f'(x) = 3x^2 + 2 > 0$ for all x. Thus, we assume there are two points $a, b \in \mathbb{R}$ s.t. $f(a) = f(b) = 0$. By Rolle's Theorem, there exists a point c in $[a,b]$(or $[b,a]$) s.t. $f'(c) = 0$(Contradiction).

   And since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, by IVT, there eists at least one x s.t. $f(x) = 0$. Thus, the graph of $f(x)$ crosses the x-axis exactly once whatever k is.

2. By EVT, we know that the maximun occurs either $f'(x) = 0$ or a, b.

   (a) $f'(x) = \frac{1}{3}(2 - e^x) = 0$ when $x = \ln(2)$. And since $f'(x) > 0$ when $x \in (0, \ln(2))$ and $f'(x) < 0$ when $x \in (\ln(2), 1)$, $f(\ln(2)) = \frac{1}{3}(2 - 2 + 2\ln 2) = \frac{2\ln(2)}{3}$ is the maximun.

   (b) $f'(x) = \frac{4x^2 - 8x - (4x-3)(2x-2)}{x^4 - 4x^3 + 4x^2} = \frac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$ for $x \in [0.5, 1]$. Thus, $f(0.5) = \frac{2-3}{0.25-1} = \frac{4}{3}$.

   (c) $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$, $x \approx 3.1311062779876$. And then the maximun is about 4.981433957553.

   (d) $f'(x) = \sin(x-1)e^{-\cos(x-1)}$ since $\sin(x) > 0$ for $0 < x < 1$ and e^x is always positive, the maximun is $f(2) = 1 + e^{-\cos(1)}$.

3. $f'(x) = e^x(\cos(x) - \sin(x))$, $f''(x) = -2e^x\sin(x)$, and $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$. Then, $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x$ and $R_2 = \frac{f^{(3)}(\xi(x))}{6}x^3 = \frac{-1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$. Thus, we have $f(x) = e^x\cos(x) = 1 + x - \frac{1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

   (a) $P_2(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$. And

$$
\begin{aligned}
|f(\tfrac{1}{2}) - P_2(\tfrac{1}{2})| &= |R_2(\tfrac{1}{2})| \\
&= \frac{1}{3}\frac{1}{2^3}e^{\xi(\frac{1}{2})}(\sin(\xi(\tfrac{1}{2})) + \cos(\xi(\tfrac{1}{2}))) \\
&\leq \frac{1}{24}e^{\frac{1}{2}}(\sin(\tfrac{1}{2}) + \sin(\tfrac{1}{2})) \\
&\approx 0.09322200499
\end{aligned}
$$

   And the actual error is 0.05311086942.

(b) The bound is the maximun of $|R_2(x)|$ for $x \in [0,1]$. Thus, the bound$= |R_2(x)| \leq \frac{1}{3}1^3 e(\sqrt{2}) =$ 1.28141034272.

(c) $\int_0^1 P_2(x)\,dx = \int_0^1 1+x\,dx = 1+\frac{1}{2} = \frac{3}{2}$.

(d)

$$\begin{aligned}
\int_0^1 R_2(x)\,dx| &= \int_0^1 \frac{1}{3}x^3 e^{\xi(x)}(\sin(x)+\cos(x))\,dx \\
&\leq \int_0^{\frac{\pi}{4}} \frac{1}{3}x^3 e^x(\sin(x)+\cos(x))\,dx + \int_{\frac{\pi}{4}}^1 \frac{\sqrt{2}}{3}x^3 e^x\,dx \\
&\approx 0.08328 + 0.1809 \\
&= 0.26418
\end{aligned}$$

And the actual error is $1.5 - 1.3780 = 0.122$.

4. $f(x) = \frac{1}{1-x}$, $P_n(x) = \sum_{k=0}^{n} x^k$. Then, the remainder is $\frac{n!}{n!\cdot(1-\xi(x))^{n+1}}x^{n+1} < x^{n+1}$. Thus, we only need to find the minimun n s.t. $0.5^{n+1} < 10^{-6}$. By taking log both side, $(n+1)(-0.30102999566) < -6 \implies n \geq 19$. Thus, $n = 19$.

5. Since the relative error is $\left|\frac{p^*-p}{p}\right| \leq 10^{-4}$, $p - p\times 10^{-4} \leq p^* \leq p + p\times 10^{-4}$.

   (a) $[\pi - \pi\times 10^{-4}, \pi + \pi\times 10^{-4}]$
   (b) $[e - e\times 10^{-4}, e + e\times 10^{-4}]$
   (c) $[\sqrt{2} - \sqrt{2}\times 10^{-4}, \sqrt{2} + \sqrt{2}\times 10^{-4}]$
   (d) $[\sqrt[3]{7} - \sqrt[3]{7}\times 10^{-4}, \sqrt[3]{7} + \sqrt[3]{7}\times 10^{-4}]$

6. (a) Sicne $e^0 - e^{-0} = 0$, $\lim_{x\to 0}\frac{e^x - e^{-x}}{x} \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{2e^x}{1} = 2$.

   (b) $f(0.1) = \frac{0.111\times 10^1 - 0.905}{0.1} = 0.021\times 10^2$.

   (c) $M_{3,+}(x) = 1+x+\frac{x^2}{2}+\frac{x^3}{6} = ((\frac{1}{6}x+\frac{1}{2})x+1)x+1$ and $M_{3,-} = 1-x+\frac{x^2}{2}-\frac{x^3}{6} = ((-\frac{1}{6}x+\frac{1}{2})x-1)x+1$. Then, $M_{3,+}(0.1) = ((.167\times 10^{-1}+.5)\cdot 0.1+1)\cdot 0.1+1 = .111\times 10^1$. $M_{3,-}(0.1) = ((-.167\times 10^{-1}+.5)\cdot 0.1-1)\cdot 0.1+1 = .905$.

   Thus, $f(0.1) = \frac{.111\cdot 10^1 - .905}{0.1} = .205\times 10^1$.