

Homework 1 of Computational Mathematics

AM15 黃琦翔 111652028

March 9, 2024

1. $f(x) = x^3 + 2x + k$, then $f'(x) = 3x^2 + 2 > 0$ for all x . Thus, we assume there are two points $a, b \in \mathbb{R}$ s.t. $f(a) = f(b) = 0$. By Rolle's Theorem, there exists a point c in $[a, b]$ (or $[b, a]$) s.t. $f'(c) = 0$ (Contradiction).

And since $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, by IVT, there exists at least one x s.t. $f(x) = 0$. Thus, the graph of $f(x)$ crosses the x -axis exactly once whatever k is.

2. By EVT, we know that the maximum occurs either $f'(x) = 0$ or a, b .

(a) $f'(x) = \frac{1}{3}(2 - e^x) = 0$ when $x = \ln(2)$. And since $f'(x) > 0$ when $x \in (0, \ln(2))$ and $f'(x) < 0$ when $x \in (\ln(2), 1)$, $f(\ln(2)) = \frac{1}{3}(2 - 2 + 2\ln 2) = \frac{2\ln(2)}{3}$ is the maximum.

(b) $f'(x) = \frac{4x^2 - 8x - (4x - 3)(2x - 2)}{x^4 - 4x^3 + 4x^2} = \frac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$ for $x \in [0.5, 1]$. Thus, $f(0.5) = \frac{2 - 3}{0.25 - 1} = \frac{4}{\frac{3}{4}}$.

(c) $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$, $x \approx 3.1311062779876$ and $f(x) \approx 4.981433957553$. And $|f(2)| \approx 2.61457$, $|f(4)| \approx 37.1640$. Then, $\max |f(x)| \approx 37.1640$.

(d) $f'(x) = \sin(x - 1)e^{-\cos(x-1)}$ since $\sin(x) > 0$ for $0 < x < 1$ and e^x is always positive, the maximum is $f(2) = 1 + e^{-\cos(1)}$.

3. $f'(x) = e^x(\cos(x) - \sin(x))$, $f''(x) = -2e^x \sin(x)$, and $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$. Then, $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x$ and $R_2 = \frac{f^{(3)}(\xi(x))}{6}x^3 = \frac{-1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$. Thus, we have $f(x) = e^x \cos(x) = 1 + x - \frac{1}{3}x^3 e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

(a) $P_2(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$. And

$$\begin{aligned} |f(\frac{1}{2}) - P_2(\frac{1}{2})| &= |R_2(\frac{1}{2})| \\ &= \frac{1}{3} \frac{1}{2^3} e^{\xi(\frac{1}{2})} (\sin(\xi(\frac{1}{2})) + \cos(\xi(\frac{1}{2}))) \\ &\leq \frac{1}{24} e^{\frac{1}{2}} (\sin(\frac{1}{2}) + \sin(\frac{1}{2})) \\ &\approx 0.09322200499 \end{aligned}$$

And the actual error is 0.05311086942.

(b) The bound is the maximum of $|R_2(x)|$ for $x \in [0, 1]$. Thus, the bound $= |R_2(x)| \leq \frac{1}{3} 1^3 e(\sqrt{2}) = 1.28141034272$.

$$(c) \int_0^1 P_2(x) dx = \int_0^1 1+x dx = 1 + \frac{1}{2} = \frac{3}{2}.$$

(d)

$$\begin{aligned} \left| \int_0^1 R_2(x) dx \right| &= \int_0^1 \frac{1}{3} x^3 e^{\xi(x)} (\sin(x) + \cos(x)) dx \\ &\leq \int_0^{\frac{\pi}{4}} \frac{1}{3} x^3 e^x (\sin(x) + \cos(x)) dx + \int_{\frac{\pi}{4}}^1 \frac{\sqrt{2}}{3} x^3 e^x dx \\ &\approx 0.08328 + 0.1809 \\ &= 0.26418 \end{aligned}$$

And the actual error is $1.5 - 1.3780 = 0.122$.

4. $f(x) = \frac{1}{1-x}$, $P_n(x) = \sum_{k=0}^n x^k$. Then, the remainder is $\frac{n!}{n! \cdot (1-\xi(x))^{n+1}} x^{n+1} < x^{n+1}$. Thus, we only need to find the minimum n s.t. $0.5^{n+1} < 10^{-6}$. By taking log both side, $(n+1)(-0.30102999566) < -6 \implies n \geq 19$. Thus, $n = 19$.

5. Since the relative error is $\left| \frac{p^* - p}{p} \right| \leq 10^{-4}$, $p - p \times 10^{-4} \leq p^* \leq p + p \times 10^{-4}$.

$$(a) [\pi - \pi \times 10^{-4}, \pi + \pi \times 10^{-4}]$$

$$(b) [e - e \times 10^{-4}, e + e \times 10^{-4}]$$

$$(c) [\sqrt{2} - \sqrt{2} \times 10^{-4}, \sqrt{2} + \sqrt{2} \times 10^{-4}]$$

$$(d) [\sqrt[3]{7} - \sqrt[3]{7} \times 10^{-4}, \sqrt[3]{7} + \sqrt[3]{7} \times 10^{-4}]$$

6. (a) Since $e^0 - e^{-0} = 0$, $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{2e^x}{1} = 2$.

$$(b) f(0.1) = \frac{0.111 \times 10^1 - 0.905}{0.1} = 0.205 \times 10^2 = .205 \times 10^1.$$

$$(c) M_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = ((\frac{1}{6}x + \frac{1}{2})x + 1)x + 1, \text{ then } f(x) = \frac{(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3) - (1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3)}{x} = \frac{2x + \frac{1}{3}x^3}{x} = 2 + \frac{1}{3}x^2.$$

$$\text{Thus, } f(0.1) = 2 + .333 * .01 = 2 + .333e - 2 = 2$$

$$(d) \text{ The relative error of (b)} = \frac{2.1 - f(.1)}{f(.1)} = .4825203972 \times 10^{-1}. \text{ And the relative error of (c)} = \frac{2.050f(.1)}{f(.1)} = .2329365782 \times 10^{-1}.$$

$$7. (a) = (-1)^0 \times (2^{1024+8+2-1023}) \times (1 + 2^{-1} + 2^{-4} + 2^{-7} + 2^{-8}) = 2^1 1 + 2^1 0 + 2^7 + 2^4 + 2^3 = 2048 + 1024 + 128 + 16 + 8 = 3224 = .3224 \times 10^4.$$

(b) Observe it and we get it is $-1 \times$ the number of (a), then it is $-.3224 \times 10^4$.

$$(c) = (2^{1024-1-1023}) \times (1 + 2^{-2} + 2^{-4} + 2^{-7} + 2^{-8}) = .132421875 \times 10^1.$$

$$(d) = .132421875 \times 10^1 + 2^{-52}.$$

8. (a)

$$\begin{aligned}
 m &= \frac{.1013}{.1130} = .8965 \\
 d_1 &= -.6099e1 + .8965 \times .6990e1 = .168 \\
 f_1 &= .1422e2 - .8965 \times .1420e2 = .1490e1 \\
 y &= \frac{.1490e1}{.168} = .8869e1 \\
 x &= \frac{.1420e2 + .6990e1 \times .8869e1}{.1130e1} = .6742e2
 \end{aligned}$$

(b)

$$\begin{aligned}
 m &= \frac{-.1811e2}{.8110e1} = -.2233e1 \\
 d_1 &= .1122e3 + .2233e1 \times .1220e2 = .1394e3 \\
 f_1 &= -.1376 - .2233e1 \times .1370 = -.4435 \\
 y &= \frac{-.4435}{.1394e3} = -.3181e-2 \\
 x &= \frac{-.1370 + .1220e2 \times .3181e-2}{.8110e1} = -.1211e-1
 \end{aligned}$$

9. The upper bound of length is 3.5, 4.5, 5.5 and the lower bound of length is 2.5, 3.5, 4.5. Thus, the upper bound of volume is $3.5 \cdot 4.5 \cdot 5.5 = .83e2$, the lower bound of volume is $.39e2$. The upper bound of surface area is $32 + 38 + 50 = .12e3$, the lower bound of surface area is $17 + 22 + 32 = .71e2$.

10. (a) $\sin\left(\frac{1}{n^2}\right) = 0 + \frac{1}{n^2} - \frac{1}{6}\left(\frac{1}{n^2}\right)^3 + \cdots = 0 + O\left(\frac{1}{n^2}\right).$

(b) $\left(\sin\left(\frac{1}{n}\right)\right)^2 = \left(0 + O\left(\frac{1}{n}\right)\right)^2 = O\left(\frac{1}{n^2}\right).$

(c) $\frac{\sin(h)}{h} = 1 + \frac{1}{2} \cdot \left(-\frac{1}{3}\right)h^2 + \cdots = O(h^2).$

(d) $\frac{1 - e^h}{h} = -1 + \left(-\frac{1}{2}\right)h + \cdots = O(h).$

11. (a) $F(x) = c_1F_1(x) + c_2F_2(x) = c_1L_1 + O(x^\alpha) + c_2L_2 + O(x^\beta) = c_1L_1 + c_2L_2 + O(x^\gamma).$

(b) $G(x) = F_1(c_1x) + F_2(c_2x) = L_1 + O(x^\alpha) + L_2 + O(x^\beta) = L_1 + L_2 + O(x^\gamma).$