

# Homework 1 of Computational Mathematics

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1.  $f(x) = x^3 + 2x + k$ , then  $f'(x) = 3x^2 + 2 > 0$  for all  $x$ . Thus, we assume there are two points  $a, b \in \mathbb{R}$  s.t.  $f(a) = f(b) = 0$ . By Rolle's Theorem, there exists a point  $c$  in  $[a, b]$  (or  $[b, a]$ ) s.t.  $f'(c) = 0$  (Contradiction).

And since  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  and  $f(x) \rightarrow -\infty$  as  $x \rightarrow -\infty$ , by IVT, there exists at least one  $x$  s.t.  $f(x) = 0$ . Thus, the graph of  $f(x)$  crosses the  $x$ -axis exactly once whatever  $k$  is.

2. By EVT, we know that the maximum occurs either  $f'(x) = 0$  or  $a, b$ .

(a)  $f'(x) = \frac{1}{3}(2 - e^x) = 0$  when  $x = \ln(2)$ . And since  $f'(x) > 0$  when  $x \in (0, \ln(2))$  and  $f'(x) < 0$  when  $x \in (\ln(2), 1)$ ,  $f(\ln(2)) = \frac{1}{3}(2 - 2 + 2\ln 2) = \frac{2\ln(2)}{3}$  is the maximum.

(b)  $f'(x) = \frac{4x^2 - 8x - (4x - 3)(2x - 2)}{x^4 - 4x^3 + 4x^2} = \frac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$  for  $x \in [0.5, 1]$ . Thus,  
 $f(0.5) = \frac{2 - 3}{0.25 - 1} = \frac{4}{3}$ .

(c)  $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$ ,  $x \approx 3.1311062779876$ . And then the maximum is about 4.981433957553.

(d)  $f'(x) = \sin(x-1)e^{-\cos(x-1)}$  since  $\sin(x) > 0$  for  $0 < x < 1$  and  $e^x$  is always positive, the maximum is  $f(2) = 1 + e^{-\cos(1)}$ .

3.  $f'(x) = e^x(\cos(x) - \sin(x))$ ,  $f''(x) = -2e^x \sin(x)$ , and  $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$ . Then,  $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x$  and  $R_2 = \frac{f^{(3)}(\xi(x))}{6}x^3 = \frac{-1}{3}x^3e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$ . Thus, we have  $f(x) = e^x \cos(x) = 1 + x - \frac{1}{3}x^3e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

(a)  $P_2(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$ . And

$$\begin{aligned} |f(\frac{1}{2}) - P_2(\frac{1}{2})| &= |R_2(\frac{1}{2})| \\ &= \frac{1}{3} \frac{1}{2^3} e^{\xi(\frac{1}{2})} (\sin(\xi(\frac{1}{2})) + \cos(\xi(\frac{1}{2}))) \\ &\leq \frac{1}{24} e^{\frac{1}{2}} (\sin(\frac{1}{2}) + \sin(\frac{1}{2})) \\ &\approx 0.09322200499 \end{aligned}$$

And the actual error is 0.05311086942.

(b) The bound is the maximum of  $|R_2(x)|$  for  $x \in [0, 1]$ . Thus, the bound  $= |R_2(x)| \leq \frac{1}{3} 1^3 e(\sqrt{2}) = 1.28141034272$ .

(c)  $\int_0^1 P_2(x) dx = \int_0^1 1 + x dx = 1 + \frac{1}{2} = \frac{3}{2}$ .

(d)

$$\begin{aligned} \left| \int_0^1 R_2(x) dx \right| &= \int_0^1 \frac{1}{3} x^3 e^{\xi(x)} (\sin(x) + \cos(x)) dx \\ &\leq \int_0^{\frac{\pi}{4}} \frac{1}{3} x^3 e^x (\sin(x) + \cos(x)) dx + \int_{\frac{\pi}{4}}^1 \frac{\sqrt{2}}{3} x^3 e^x dx \\ &\approx 0.08328 + 0.1809 \\ &= 0.26418 \end{aligned}$$

And the actual error is  $1.5 - 1.3780 = 0.122$ .

4.  $f(x) = \frac{1}{1-x}$ ,  $P_n(x) = \sum_{k=0}^n x^k$ . Then, the remainder is  $\frac{n!}{n! \cdot (1-\xi(x))^{n+1}} x^{n+1} < x^{n+1}$ . Thus, we only need to find the minimum  $n$  s.t.  $0.5^{n+1} < 10^{-6}$ . By taking log both side,  $(n+1)(-0.30102999566) < -6 \implies n \geq 19$ . Thus,  $n = 19$ .

5. Since the relative error is  $\left| \frac{p^* - p}{p} \right| \leq 10^{-4}$ ,  $p - p * 10^{-4} \leq p^* \leq p + p * 10^{-4}$ .

(a)  $[\pi - \pi * 10^{-4}, \pi + \pi * 10^{-4}]$

(b)  $[e - e * 10^{-4}, e + e * 10^{-4}]$

(c)  $[\sqrt{2} - \sqrt{2} * 10^{-4}, \sqrt{2} + \sqrt{2} * 10^{-4}]$

(d)  $[\sqrt[3]{7} - \sqrt[3]{7} * 10^{-4}, \sqrt[3]{7} + \sqrt[3]{7} * 10^{-4}]$

6. (a) Since  $e^0 - e^{-0} = 0$ ,  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} \stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{2e^x}{1} = 2$ .

(b)