Homework 2 of Computational Mathematics

AM15 黃琦翔 111652028

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1. $x^3 = x + 1 \implies x^2 = 1 + \dfrac{1}{x} \implies x = \sqrt{1 + \dfrac{1}{x}} = g(x)$. $p_1 = g(p_0) = \sqrt{1+1} = \sqrt{2} \approx 1.414$. $p_2 = g(p_1) = \sqrt{1 + \dfrac{1}{\sqrt{2}}} \approx 1.3065$. $p_3 = g(p_2) \approx 1.3172$. $p_4 = g(p_3) \approx 1.326$. $p_5 = g(p_4) \approx 1.324$ Then, p_4 is the answer that we want to find.

2. Let $f(x) = x^3 + x - 4$, $f'(x) = 3x^2 + 1 < 49$ for all $x \in [1,4]$. Thus, for $|x - y| < \dfrac{10^{-3}}{49} \approx 2.0409e-5$, $|f(x) - f(y)| < 10^3$. Find n s.t. $3 \cdot 2^{-n} < 2.0409e-5$, $n > -\log_2(\dfrac{2.0409e-5}{3}) \approx 17.1653$. Thus, the bound of the number of iteration is 18. Then, by python code below, the root is about 1.3787.

```
HW2 > HW2.py > ...
a = [1]
b = [4]

def f(x):
    return x**3 + x - 4

while 1:
    mid = (a[-1] + b[-1])/2
    val = f(mid)
    print(mid, val)

    if abs(val) < 0.0001:
        break
    elif val > 0:
        b.append(mid)
    else:
        a.append(mid)
```

```
PROBLEMS  OUTPUT  DEBUG CONSOLE  TERMINAL  PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> &
2.5 14.125
1.75 3.109375
1.375 -0.025390625
1.5625 1.377197265625
1.46875 0.637176513671875
1.421875 0.2965202331542969
1.3984375 0.13326025009155273
1.38671875 0.05336350202560425
1.380859375 0.013844214379787445
1.3779296875 -0.005808685906231403
1.37939453125 0.004008884658105671
1.378662109375 -0.0009021193400258198
1.3790283203125 0.0015528278327110456
1.37884521484375 0.00032521555817766057
1.378753662109375 -0.00028848656066315925
1.3787994384765625 1.8355831034710945e-05
```

3. In this question, we let $p_{n+1} = g(p_n)$ and $p = \sqrt[3]{21}$. And by $\lim_{n\to\infty} \dfrac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$,

$$\frac{|p_{n+1} - p|}{|p_n - p|} \approx \frac{\lambda |p_n - p|^\alpha}{\lambda |p_{n-1} - p|^\alpha} \approx \left| \frac{p_n - p}{p_{n-1} - p} \right|^\alpha, \text{ then } \alpha \approx \frac{\ln |(p_{n+1} - p)/(p_n - p)|}{\ln |(p_n - p)/(p_{n-1} - p)|}.$$

Then, by observing the function, we get b is quadratic convergence (Newton's method). In the meanwhile, though $\sqrt[3]{21}$ is a fixed point of c, but it will diverges or converges to 0 for any open interval contains $\sqrt[3]{21}$.

Thus, by result of iteration with python below, the order of speed of convergence is b > d > a.

```python
def problem_3():
    import math
    p = 21**(1.0/3)
    def a(x):
        return (20*(x**3)+21)/(21*(x**2))

    def b(x):
        return x- (x**3 - 21)/(3*(x**2))

    def c(x):
        return x - (x**4 - 21*x)/(x**2 - 21)

    def d(x):
        return math.sqrt(21/x)

    def Alpha(f, x):
        return math.log(abs((f(x) - p)/(x - p)))

    functions = {"a": a, "b": b, "c":c, "d": d}

    for i in functions.keys():
        f = functions[i]
        p_now = 1
        for k in range(20):
            p_now = f(p_now)
            # print(k+1, "err:", abs(p_now - p))
        try:
            alpha = Alpha(f, f(p_now))/Alpha(f, p_now)
            print(i, alpha)
        except:
            print(i, "Cannot compute alpha by this method.")

        print("---")

def problem_4():
    def g(x):
```

PROBLEMS   OUTPUT   DEBUG CONSOLE   TERMINAL   PORTS

```
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/Users/9ryan/AppDa
a 0.9982564193669963
---
b Cannot compute alpha by this method.
---
c Cannot compute alpha by this method.
---
d 1.0000015723442375
---
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

4. $|g'(x)| = |-2^{-x}\ln(2)|$ is continuous and decreasing on $\mathbb{R}$. Thus, $g'(\frac{1}{3}) \approx 0.55015$. And since $g(\frac{1}{3}) \approx 0.79370 > \frac{1}{3}$ and $g(1) = \frac{1}{2} < 1$. By Theorem 2.3, $g(x)$ has unique fixed point on $[\frac{1}{3}, 1]$.

```python
def problem_4():
    def g(x):
        return 2**(-x)

    x = 0.3334
    step = 0
    while abs(x-g(x)) >= 10**(-4):
        x = g(x)
        step += 1
        print("step:", step, ",value:", x)


if __name__ == "__main__":
    problem_4()
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
step: 1 ,value: 0.79366385007938
step: 2 ,value: 0.5768772000497738
step: 3 ,value: 0.6704133579003061
step: 4 ,value: 0.6283266346632733
step: 5 ,value: 0.6469263427213542
step: 6 ,value: 0.6386394847182052
step: 7 ,value: 0.6423183934819865
step: 8 ,value: 0.6406825519733249
step: 9 ,value: 0.6414094204320899
step: 10 ,value: 0.6410863425561439
step: 11 ,value: 0.6412299238405336
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

By the result from python above, we need 12 steps to let error less than 10^{-4}. Since $|g'(x)| \le \ln(2)2^{-x} \le 0.551$, we take $k = 0.551$ and $p_0 = 1$. $|p_n - p| \le \frac{k^n}{1-k}|p_1 - p| \approx k^n \cdot 1.113585746 \le 10^{-4}$. Then, $n \ge 15.633566$, that is, the bound is 16 times of iteration.

5. (a) Let x_* be nonzero fixed point of g, $x_* = 2x_* - Ax_*^2 \implies x_* = Ax_*^2 \implies x^*A = 1$. Thus, $x_* = \frac{1}{A}$.

   (b) Since $g'(x) = 2 - 2Ax$ and we need $|g'(x)| < k < 1$,

$$
\begin{aligned}
|2-2Ax| < k &\iff -k < 2 - 2Ax < k \\
&\iff \frac{k-2}{2A} < x < \frac{k+2}{2A} \\
&\overset{k=1-\varepsilon}{\iff} \frac{1}{2A} + \frac{\varepsilon}{2A} < x < \frac{3}{2A} - \frac{\varepsilon}{2A}
\end{aligned}
$$

   Thus, $x \in \left(\frac{1}{2A} + \frac{3}{2A}\right)$.

6. Let $g(x) = \frac{x}{2} + \frac{A}{2x}$, and $g'(x) = \frac{1}{2} - \frac{A}{2x^2}$. For $x \in (\sqrt{A}, \infty)$, $|g'(x)| \le \frac{1}{2}$. And since $x > \sqrt{A}$, $\frac{x}{2} + \frac{A}{2x} - \sqrt{A} = \frac{x^2 + A - 2Ax}{2x} = \frac{(x - \sqrt{A})^2}{2x} > 0$. Thus, if $x > \sqrt{A}$, $g(x) > \sqrt{A}$. By Corollary 2.5, $g(x)$ converges to $\sqrt{A}$ for all $(\sqrt{A}, \infty)$.

   Then, for $x \in (0, \sqrt{A})$, $\frac{x}{2} + \frac{A}{2x} - \sqrt{A} = \frac{(A-x)^2}{2x} > 0$. Thus, for $0 < x < \sqrt{A}$, $g(x) > \sqrt{A}$, and then by the argument above, it will converge to $\sqrt{A}$, too.

   Therefore, for $x_0 > 0$, x_n will converge to $\sqrt{A}$.

7. (a) $p_2 = p_1 - \dfrac{f(p_1)[p_0 - p_1]}{f(p_0) - f(p_1)} = -\dfrac{-1}{-1+\cos(-1)-1} = \dfrac{1}{-2+\cos(1)} \approx -0.6850733573260451.$

$p_3 = p_2 - \dfrac{f(p_2)[p_1 - p_2]}{f(p_1) - f(p_2)} \approx -1.252076488909229.$

(b) $p_2 = p_1 - f(p_1) \cdot \dfrac{p_0 - p_1}{f(p_0) - f(p_1)} \approx -0.6850733573260451$. Since $f(p_2) < 0$, $f(p_2) \cdot f(p_0) < 0$.

Then, $p_3 = p_2 - f(p_2) \cdot \dfrac{p_0 - p_2}{f(p_0) - f(p_2)} \approx -0.8413551256656522.$

8. From the question, we can rewrite it as $f(i) = 1000(1 - (1+i)^{-30\cdot 12}) - 135,000i = 0$. After testing, we know that $f(0.002) > 0$ and $f(0.01) < 0$.

```python

def problem_8():
    def secant_method(f, x_1, x_2):
        return x_1 - (f(x_1)*(x_2 - x_1))/(f(x_2) - f(x_1))

    f = lambda x: 1000*(1-(1+x)**(-360))-135000*x
    x = [0.002, 0.01]
    for _ in range(8):
        p = secant_method(f, x[-1], x[-2])
        print(p, f(p))
        x.append(p)

if __name__ == "__main__":
    problem_8()
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/Users/9ryan/AppData/Loc
0.005130556113250995 148.91891659914575
0.006507247379340629 24.713056780832744
0.006781165610560291 -3.2322577565320216
0.006749483221502495 0.04480741496536211
0.0067499164158026734 7.675006418139674e-05
0.006749917159088972 -1.8349055608268827e-09
0.006749917159071203 1.1368683772161603e-13
0.0067499171590712035 0.0
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

Then, by secant method and python, $i \approx 0.0067499171590712035$. Thus, the interest rate the borrower can afford is about $0.0067499171590712035 \cdot 12 = 0.080999005908854442 = 8.0999005908854442\%$.

9. (a) $\lim\limits_{n\to\infty} \dfrac{|1/(n+1)^k|}{|1/n^k|} = \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^k = 1$ for all k. Thus, p_n converges linearly.

(b) $\lim\limits_{n\to\infty} \dfrac{10^{-2^{n+1}}}{|10^{-2^n}|^2} = \lim\limits_{n\to\infty} 10^{-2^{n+1}+2^n\cdot 2} = 1$. Thus, p_n converges linearly.

10. (a)

i	p_i	j	$\hat{p}_j$
0	0.5		
1	0.8775825618		
2	0.6390124941	0	0.7313851863
3	0.8026851006	1	0.7360866917
4	0.6947780267	2	0.7376528713
5	0.7681958312	3	0.7384692208
6	0.7191654459	4	0.7387980651

(b) From 1., we let $g(x) = \sqrt{1 + \frac{1}{x}}$ and do iteration.

i	p_0	p_1	p_2	$\hat{p}_i$
0	2	1.22474487139	1.34777467735	1.33092441907
1	1.3309244190	1.32338863029	1.32500412497	1.32471893841

The answer above is given by python.

```python
def problem_10():
    def Aitkens_method(f, x_0, step):
        x = [x_0, f(x_0), f(f(x_0))]
        for i in range(step):
            p = x[-3] - ((x[-2]-x[-3])**2)/(x[-1] - 2*x[-2] + x[-3])
            print(f"p_{i}: {p}")
            x.append(f(x[-1]))

    def Steffensens_method(f, x_0):
        x = [x_0]
        step = 0
        while abs(x[-1]-f(x[-1])) >= 10**(-4):
            p_0 = x[-1]
            p_1 = f(p_0)
            p_2 = f(p_1)
            p = p_0 - ((p_1 - p_0)**2)/(p_2 - 2*p_1 + p_0)
            print(f"i: {step}, p_0: {p_0}, p_1: {p_1}, p_2: {p_2}, p: {p}")
            x.append(p)
            step += 1

    from math import cos, sqrt
    f = lambda x: cos(x)
    x_0 = 0.5
    print("Aitken\'s method: ")
    Aitkens_method(f, x_0, 5)
    print("Steffensen\'s method: ")
    g = lambda x: sqrt(1 + 1/x)
    y_0 = 2
    Steffensens_method(g, y_0)

if __name__ == "__main__":
    problem_10()
```

PROBLEMS 1 OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/Users/9ryan/AppData/Local/Programs/Python/Py
Aitken's method:
p_0: 0.7313851863825818
p_1: 0.7360866917130169
p_2: 0.7376528713963997
p_3: 0.7384692208762632
p_4: 0.7387980651735903
Steffensen's method:
i: 0, p_0: 2, p_1: 1.224744871391589, p_2: 1.3477746773580983, p: 1.330924419078614
i: 1, p_0: 1.330924419078614, p_1: 1.3233886302912288, p_2: 1.3250041249776752, p: 1.3247189384145663
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

11. (a)

$P(x) = x^3 - 5x^2 + 8x - 6$, $x_0 = 2$.		
Step	y	z
0	$a_0 = 1$	$a_0 = 1$
1	$1 \cdot 2 - 5 = -3$	$1 \cdot 2 + -3 = -1$
2	$-3 \cdot 2 + 8 = 2$	$-1 \cdot 2 - 2 = 0$
3	$2 \cdot 2 - 6 = -2$	

Then, $P(2) = -2$ and $P'(2) = 0$.

$P(x) = x^3 - 5x^2 + 8x - 6$, $x_0 = 4$.		
Step	y	z
0	$a_0 = 1$	$a_0 = 1$
1	$1 \cdot 4 - 5 = -1$	$1 \cdot 4 - 1 = 3$
2	$-1 \cdot 4 + 8 = 4$	$3 \cdot 4 + 4 = 16$
3	$4 \cdot 4 - 6 = 10$	

Then, $P(4) = 10$ and $P'(4) = 16$.

(b) $P(x) = x^3 - 5x^2 + 8x - 6$ and $P'(x) = 3x^2 - 10x + 8$. And since $P'(2) = 0$, Newton's method may not be usefully in $x_0 = 2$. Then, for $x_0 = 4$,

Newton's method for $P(x)$ with $x_0 = 4$.				
i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
0	4	10	16	3.375
1	3.375	2.490234375	8.421875	3.07931354359925
2	3.079313543599258	0.42222920359638394	5.65338026338887	3.00462739408783
3	3.004627394087836	0.02322272062870922	5.03708339103081	3.00001704344919
4	3.000017043449198	8.52184079072060e-05		

(c) $P(x) = (x-3)(x^2 - 2x + 2) = 0$. Then, $x = \dfrac{2 \pm 2i}{2}$.

(d) Let $h_1 = 1 - 0 = 1$, $h_2 = 2 - 1 = 2$. Then, $\delta_1 = \dfrac{P(1) - P(0)}{h_1} = \dfrac{-2-(-6)}{1} = 4$, $\delta_2 = \dfrac{P(2) - P(1)}{h_2} = \dfrac{-2-(-2)}{1} = 0$, $d = \dfrac{\delta_2 - \delta_1}{h_2 + h_1} = \dfrac{0-4}{2} = -2$.

Thus, $b = 0 + 1 \cdot (-2) = -2$ and $D = \sqrt{(-2)^2 - 4 \cdot (-2) \cdot (-2)} = 2\sqrt{3}i$. Since $|b - D| > |b + D|$, $E = -2 - 2\sqrt{3}i$. Therefore, $h = \dfrac{-2 \cdot (-2)}{-2 - 2\sqrt{3}i} = \dfrac{2}{-1 - \sqrt{3}i} = \dfrac{-1 + \sqrt{3}i}{2} \approx -0.5 + 0.866025403784438i$, and $p = 2 + h \approx 1.5 + 0.866025403784438i$.

(e)