

Homework 1 of Computational Mathematics

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1. $f(x) = x^3 + 2x + k$, then $f'(x) = 3x^2 + 2 > 0$ for all x . Thus, we assume there are two points $a, b \in \mathbb{R}$ s.t. $f(a) = f(b) = 0$. By Rolle's Theorem, there exists a point c in $[a, b]$ (or $[b, a]$) s.t. $f'(c) = 0$ (Contradiction).

And since $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$, by IVT, there exists at least one x s.t. $f(x) = 0$. Thus, the graph of $f(x)$ crosses the x -axis exactly once whatever k is.

2. By EVT, we know that the maximum occurs either $f'(x) = 0$ or a, b .

(a) $f'(x) = \frac{1}{3}(2 - e^x) = 0$ when $x = \ln(2)$. And since $f'(x) > 0$ when $x \in (0, \ln(2))$ and $f'(x) < 0$ when $x \in (\ln(2), 1)$, $f(\ln(2)) = \frac{1}{3}(2 - 2 + 2\ln 2) = \frac{2\ln(2)}{3}$ is the maximum.

(b) $f'(x) = \frac{4x^2 - 8x - (4x - 3)(2x - 2)}{x^4 - 4x^3 + 4x^2} = \frac{-4x^2 + 6x - 8}{x^4 - 4x^3 + 4x^2} < 0$ for $x \in [0.5, 1]$. Thus,
 $f(0.5) = \frac{2 - 3}{0.25 - 2} = \frac{4}{7}$.

(c) $f'(x) = 2\cos(2x) - 4x\sin(2x) - 2x + 4 = 0$

(d) $f'(x) = \sin(x - 1)e^{-\cos(x-1)}$ since $\sin(x) > 0$ for $0 < x < 1$ and e^x is always positive, the maximum is $f(2) = 1 + e^{-\cos(1)}$.

3. $f'(x) = e^x(\cos(x) - \sin(x))$, $f''(x) = -2e^x \sin(x)$, and $f^{(3)}(x) = -2e^x(\sin(x) + \cos(x))$. Then, $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + x$ and $R_2 = \frac{f^{(3)}(\xi(x))}{6}x^3 = \frac{-1}{3}x^3e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$. Thus, we have $f(x) = e^x \cos(x) = 1 + x - \frac{1}{3}x^3e^{\xi(x)}(\sin(\xi(x)) + \cos(\xi(x)))$

(a) $P_2(\frac{1}{2}) = 1 + \frac{1}{2} = \frac{3}{2}$. And

$$\begin{aligned} |f(\frac{1}{2}) - P_2(\frac{1}{2})| &= |R_2(\frac{1}{2})| \\ &= \frac{1}{3} \frac{1}{2^3} e^{\xi(\frac{1}{2})} (\sin(\xi(\frac{1}{2})) + \cos(\xi(\frac{1}{2}))) \\ &\leq \frac{1}{24} e^{\frac{1}{2}} (\sin(\frac{1}{2}) + \sin(\frac{1}{2})) \\ &\approx 0.09322200499 \end{aligned}$$

And the actual error is 0.05311086942 (calculated by the calculator of google).

- (b) The bound is the maximum of $|R_2(x)|$ for $x \in [0, 1]$. Thus, the bound $= |R_2(x)| \leq \frac{1}{3} 1^3 e(\sqrt{2}) = 1.28141034272$.

$$(c) \int_0^1 P_2(x) dx = \int_0^1 1 + x dx = 1 + \frac{1}{2} = \frac{3}{2}.$$

(d)

$$\begin{aligned} \left| \int_0^1 R_2(x) dx \right| &= \int_0^1 \frac{1}{3} x^3 e^{\xi(x)} (\sin(x) + \cos(x)) dx \\ &\leq \int_0^{\frac{\pi}{4}} \frac{1}{3} x^3 e^x (\sin(x) + \cos(x)) dx + \int_{\frac{\pi}{4}}^1 \frac{\sqrt{2}}{3} x^3 e^x dx \\ &\approx 0.08328 + 0.1809 \\ &= 0.26418 \end{aligned}$$

And the actual error is $1.5 - 1.3780 = 0.122$.

4. $f(x) = \frac{1}{1-x}$, $P_n(x) = \sum_{k=0}^n x^k$. Then, the remainder is $\frac{n!}{n! \cdot (1-\xi(x))^{n+1}} x^{n+1} < x^{n+1}$. Thus, we only need to find the minimum n s.t. $0.5^{n+1} < 10^{-6}$. By taking log both side, $(n+1)(-0.30102999566) < -6 \implies n \geq 19$. Thus, $n = 19$.

5.