Homework 2 of Computational Mathematics

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1. $x^3 = x+1 \implies x^2 = 1+\frac{1}{x} \implies x = \sqrt{1+\frac{1}{x}} = g(x)$. $p_1 = g(p_0) = \sqrt{1+1} = \sqrt{2} \approx 1.414$. $p_2 = g(p_1) = \sqrt{1+\frac{1}{\sqrt{2}}} \approx 1.3065$. $p_3 = g(p_2) \approx 1.3172$. $p_4 = g(p_3) \approx 1.326$. $p_5 = g(p_4) \approx 1.324$ Then, p_4 is the answer that we want to find.

2. Let $f(x) = x^3+x-4$, $f'(x) = 3x^2+1 < 49$ for all $x \in [1,4]$. Thus, for $|x-y| < \frac{10^{-3}}{49} \approx 2.0409e-5$, $|f(x)-f(y)| < 10^3$. Find n s.t. $3 \cdot 2^{-n} < 2.0409e-5$, $n > -\log_2(\frac{2.0409e-5}{3}) \approx 17.1653$. Thus, the bound of the number of iteration is 18. Then, by python code below, the root is about 1.3787.

```
HW2 > HW2.py > ...
a = [1]
b = [4]

def f(x):
    return x**3 + x - 4

while 1:
    mid = (a[-1] + b[-1])/2
    val = f(mid)
    print(mid, val)

    if abs(val) < 0.0001:
        break
    elif val > 0:
        b.append(mid)
    else:
        a.append(mid)
```

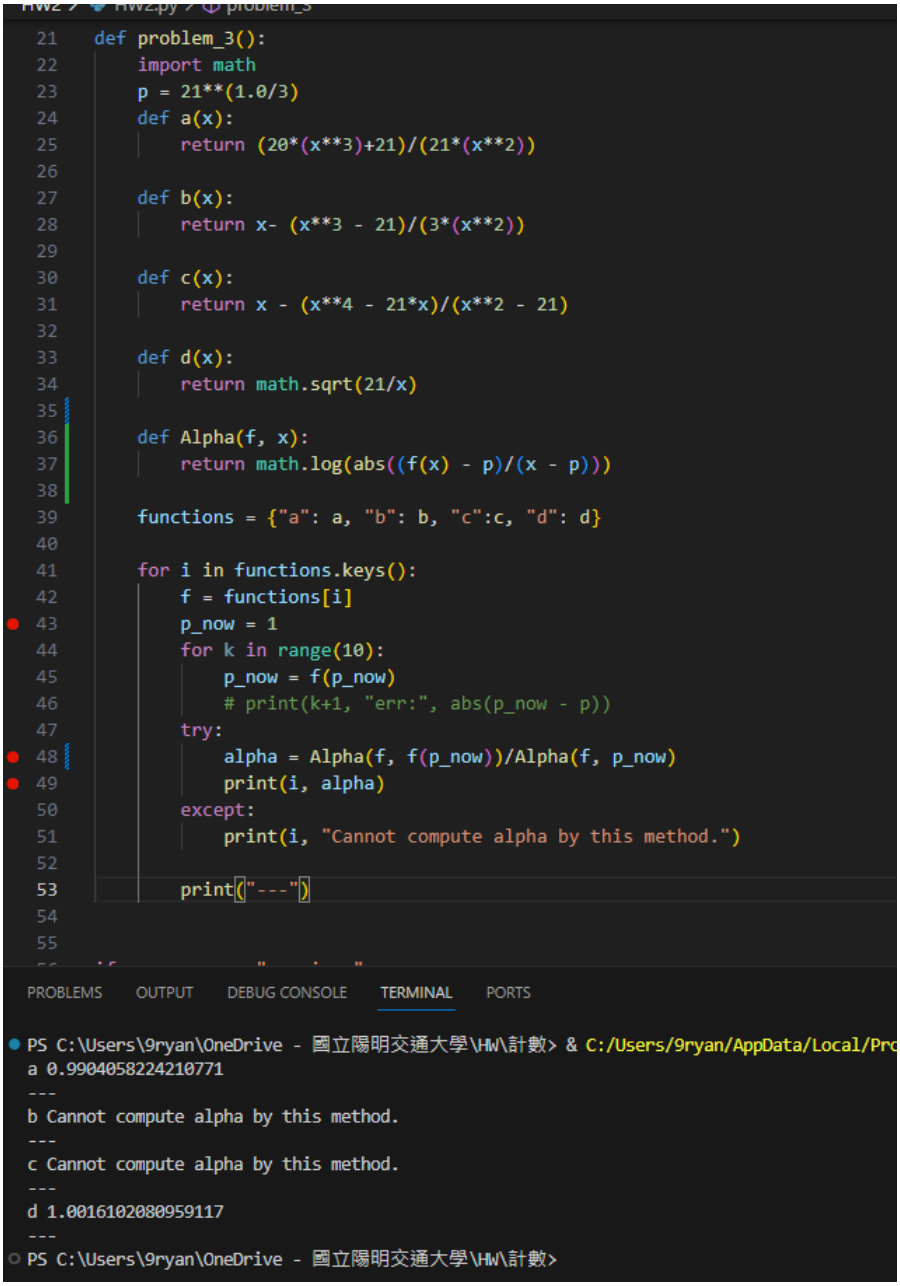
```
PROBLEMS   OUTPUT   DEBUG CONSOLE   TERMINAL   PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> &
2.5 14.125
1.75 3.109375
1.375 -0.025390625
1.5625 1.377197265625
1.46875 0.637176513671875
1.421875 0.2965202331542969
1.3984375 0.13326025009155273
1.38671875 0.05336350202560425
1.380859375 0.013844214379787445
1.3779296875 -0.005808685906231403
1.37939453125 0.004008884658105671
1.378662109375 -0.0009021193400258198
1.3790283203125 0.0015528278327110456
1.37884521484375 0.0003252155581776605
1.378753662109375 -0.00028848656066315925
1.3787994384765625 1.8355831034710945e-05
```

3. In this question, we let $p_{n+1} = g(p_n)$ and $p = \sqrt[3]{21}$. And by $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$,

$$\frac{|p_{n+1}-p|}{|p_n-p|} \approx \frac{\lambda|p_n-p|^\alpha}{\lambda|p_{n-1}-p|^\alpha} \approx \left|\frac{p_n-p}{p_{n-1}-p}\right|^\alpha, \text{ then } \alpha \approx \frac{\ln|(p_{n+1}-p)/(p_n-p)|}{\ln|(p_n-p)/(p_{n-1}-p)|}.$$

Then, running python and observe the function, we get a is linearly convergence, b is quadratic convergence (Newton's method), and d is suplinear converges. Meanwhile, though $\sqrt[3]{21}$ is a fixed point of c, but it will diverges or converges to 0 for any open interval contains $\sqrt[3]{21}$.

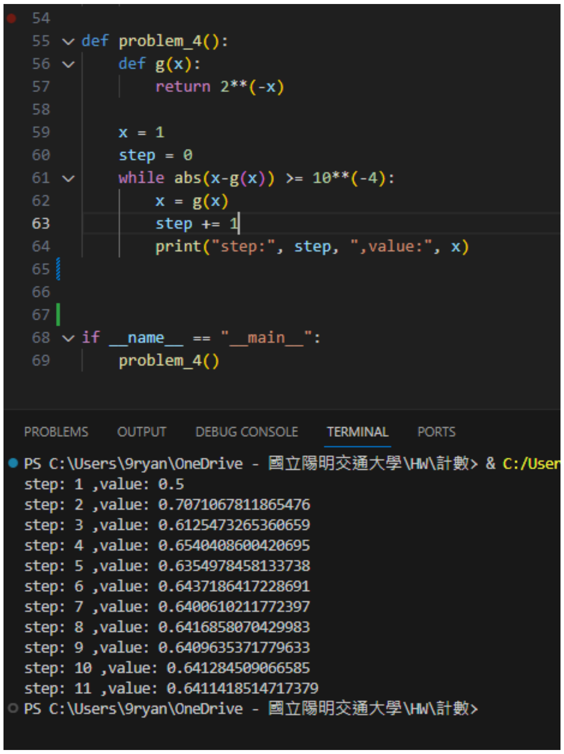
Thus, order of speed of convergence is b > d > a > c.



```python
def problem_3():
    import math
    p = 21**(1.0/3)
    def a(x):
        return (20*(x**3)+21)/(21*(x**2))

    def b(x):
        return x- (x**3 - 21)/(3*(x**2))

    def c(x):
        return x - (x**4 - 21*x)/(x**2 - 21)

    def d(x):
        return math.sqrt(21/x)

    def Alpha(f, x):
        return math.log(abs((f(x) - p)/(x - p)))

    functions = {"a": a, "b": b, "c":c, "d": d}

    for i in functions.keys():
        f = functions[i]
        p_now = 1
        for k in range(10):
            p_now = f(p_now)
            # print(k+1, "err:", abs(p_now - p))
        try:
            alpha = Alpha(f, f(p_now))/Alpha(f, p_now)
            print(i, alpha)
        except:
            print(i, "Cannot compute alpha by this method.")

        print("---")
```

```
PROBLEMS  OUTPUT  DEBUG CONSOLE  TERMINAL  PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/Users/9ryan/AppData/Local/Pr
a 0.9904058224210771
---
b Cannot compute alpha by this method.
---
c Cannot compute alpha by this method.
---
d 1.0016102080959117
---
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

4. $|g'(x)| = |-2^{-x}\ln(2)|$ is continuous and decreasing on $\mathbb{R}$. Thus, $g'(\frac{1}{3}) \approx 0.55015$. And since $g(\frac{1}{3}) \approx 0.79370 > \frac{1}{3}$ and $g(1) = \frac{1}{2} < 1$. By Theorem 2.3, $g(x)$ has unique fixed point on $[\frac{1}{3}, 1]$.



By the result from python above, we need 12 steps to let error less than 10^{-4}.