

Homework 2 of Computational Mathematics

AM15 黃琦翔 111652028

March 25, 2024

1. $x^3 = x + 1 \implies x^2 = 1 + \frac{1}{x} \implies x = \sqrt{1 + \frac{1}{x}} = g(x)$. $p_1 = g(p_0) = \sqrt{1+1} = \sqrt{2} \approx 1.414$. $p_2 = g(p_1) = \sqrt{1 + \frac{1}{\sqrt{2}}} \approx 1.3065$. $p_3 = g(p_2) \approx 1.3172$. $p_4 = g(p_3) \approx 1.326$. $p_5 = g(p_4) \approx 1.324$. Then, p_4 is the answer that we want to find.

2. Let $f(x) = x^3 + x - 4$, $f'(x) = 3x^2 + 1 < 49$ for all $x \in [1, 4]$. Thus, for $|x - y| < \frac{10^{-3}}{49} \approx 2.0409e-5$, $|f(x) - f(y)| < 10^3$. Find n s.t. $3 \cdot 2^{-n} < 2.0409e-5$, $n > -\log_2\left(\frac{2.0409e-5}{3}\right) \approx 17.1653$. Thus, the bound of the number of iteration is 18. Then, by python code below, the root is about 1.3787.

```
HW2 > HW2.py > ...
1  a = [1]
2  b = [4]
3
4  def f(x):
5      return x**3 + x - 4
6
7  while 1:
8      mid = (a[-1] + b[-1])/2
9      val = f(mid)
10     print(mid, val)
11
12     if abs(val) < 0.0001:
13         break
14     elif val > 0:
15         b.append(mid)
16     else:
17         a.append(mid)
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
PS C:\Users\ryan\OneDrive - 國立陽明交通大學\HW\計數> &
2.5 14.125
1.75 3.109375
1.375 -0.025390625
1.5625 1.377197265625
1.46875 0.637176513671875
1.421875 0.2965202331542969
1.3984375 0.13326025009155273
1.38671875 0.05336350202560425
1.380859375 0.013844214379787445
1.3779296875 -0.005808685906231403
1.37939453125 0.004008884658105671
1.378662109375 -0.0009021193400258198
1.3790283203125 0.0015528278327110456
1.37884521484375 0.00032521555817766057
1.378753662109375 -0.00028848656066315925
1.3787994384765625 1.8355831034710945e-05
```

3. In this question, we let $p_{n+1} = g(p_n)$ and $p = \sqrt[3]{21}$. And by $\lim_{n \rightarrow \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$,

$$\frac{|p_{n+1} - p|}{|p_n - p|} \approx \frac{\lambda |p_n - p|^\alpha}{\lambda |p_{n-1} - p|^\alpha} \approx \left| \frac{p_n - p}{p_{n-1} - p} \right|^\alpha, \text{ then } \alpha \approx \frac{\ln |(p_{n+1} - p)/(p_n - p)|}{\ln |(p_n - p)/(p_{n-1} - p)|}.$$

Then, by observing the function, we get b is quadratic convergence (Newton's method). In the meanwhile, though $\sqrt[3]{21}$ is a fixed point of c, but it will diverges or converges to 0 for any open interval contains $\sqrt[3]{21}$.

For a,

$$\lim_{n \rightarrow \infty} \frac{p_{n+1} - \sqrt[3]{21}}{p_n - \sqrt[3]{21}} = \lim_{n \rightarrow \infty} \frac{20p_n^3 + 21 - 21 \cdot \sqrt[3]{21} p_n^2}{21p_n^3 - 21\sqrt[3]{21} p_n^2}$$

Thus, order of speed of convergence is $b > d > a$.

```

21 def problem_3():
22     import math
23     p = 21**(1.0/3)
24     def a(x):
25         return (20*(x**3)+21)/(21*(x**2))
26
27     def b(x):
28         return x - (x**3 - 21)/(3*(x**2))
29
30     def c(x):
31         return x - (x**4 - 21*x)/(x**2 - 21)
32
33     def d(x):
34         return math.sqrt(21/x)
35
36     def Alpha(f, x):
37         return math.log(abs((f(x) - p)/(x - p)))
38
39     functions = {"a": a, "b": b, "c": c, "d": d}
40
41     for i in functions.keys():
42         f = functions[i]
43         p_now = 1
44         for k in range(10):
45             p_now = f(p_now)
46             # print(k+1, "err:", abs(p_now - p))
47         try:
48             alpha = Alpha(f, f(p_now))/Alpha(f, p_now)
49             print(i, alpha)
50         except:
51             print(i, "Cannot compute alpha by this method.")
52
53     print("----")
54
55

```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> C:/Users/9ryan/AppData/Local/Pro
a 0.9904058224210771
---
b Cannot compute alpha by this method.
---
c Cannot compute alpha by this method.
---
d 1.0016102080959117
---
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>

```

4. $|g'(x)| = |-2^{-x} \ln(2)|$ is continuous and decreasing on \mathbb{R} . Thus, $g'(\frac{1}{3}) \approx 0.55015$. And since $g(\frac{1}{3}) \approx 0.79370 > \frac{1}{3}$ and $g(1) = \frac{1}{2} < 1$. By Theorem 2.3, $g(x)$ has unique fixed point on $[\frac{1}{3}, 1]$.

```
54
55 v def problem_4():
56 v     def g(x):
57         return 2**(-x)
58
59     x = 1
60     step = 0
61 v     while abs(x-g(x)) >= 10**(-4):
62         x = g(x)
63         step += 1
64         print("step:", step, ",value:", x)
65
66
67
68 v if __name__ == "__main__":
69     problem_4()
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/User
step: 1 ,value: 0.5
step: 2 ,value: 0.7071067811865476
step: 3 ,value: 0.6125473265360659
step: 4 ,value: 0.6540408600420695
step: 5 ,value: 0.6354978458133738
step: 6 ,value: 0.6437186417228691
step: 7 ,value: 0.6400610211772397
step: 8 ,value: 0.6416858070429983
step: 9 ,value: 0.6409635371779633
step: 10 ,value: 0.641284509066585
step: 11 ,value: 0.6411418514717379
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

By the result from python above, we need 12 steps to let error less than 10^{-4} .