Homework 1
Due date: 03/14/2024

1.

Use the Intermediate Value Theorem 1.11 and Rolle's Theorem 1.7 to show that the graph of $f(x) = x^3 + 2x + k$ crosses the x-axis exactly once, regardless of the value of the constant k.

2.

Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

a. $f(x) = (2 - e^x + 2x)/3, \quad [0, 1]$

b. $f(x) = (4x - 3)/(x^2 - 2x), \quad [0.5, 1]$

c. $f(x) = 2x\cos(2x) - (x - 2)^2, \quad [2, 4]$

d. $f(x) = 1 + e^{-\cos(x-1)}, \quad [1, 2]$

3.

Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula, and compare it to the actual error.

b. Find a bound for the error $|f(x) - P_2(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

c. Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.

d. Find an upper bound for the error in (c) using $\int_0^1 |R_2(x)\,dx|$, and compare the bound to the actual error.

4.

Let $f(x) = (1 - x)^{-1}$ and $x_0 = 0$. Find the nth Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

5.

Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for each value of p.

a. π

b. e

c. $\sqrt{2}$

d. $\sqrt[3]{7}$

6.

Let

$$f(x) = \frac{e^x - e^{-x}}{x}.$$

a. Find $\lim_{x \to 0}(e^x - e^{-x})/x$.

b. Use three-digit rounding arithmetic to evaluate $f(0.1)$.

c. Replace each exponential function with its third Maclaurin polynomial, and repeat part (b).

d. The actual value is $f(0.1) = 2.003335000$. Find the relative error for the values obtained in parts (b) and (c).

7. Use the 64-bit long real format to find the decimal equivalent of the following floating-point machine numbers

a.	0	10000001010	1001001100000000000000000000000000000000000000000000
b.	1	10000001010	1001001100000000000000000000000000000000000000000000
c.	0	01111111111	0101001100000000000000000000000000000000000000000000
d.	0	01111111111	0101001100000000000000000000000000000000000000000001

8.

The two-by-two linear system

$$ax + by = e,$$
$$cx + dy = f,$$

where a, b, c, d, e, f are given, can be solved for x and y as follows:

$$\begin{aligned} \text{set } m &= \frac{c}{a}, \quad \text{provided } a \neq 0; \\ d_1 &= d - mb; \\ f_1 &= f - me; \\ y &= \frac{f_1}{d_1}; \\ x &= \frac{(e - by)}{a}. \end{aligned}$$

Solve the following linear systems using four-digit rounding arithmetic.

a. $1.130x - 6.990y = 14.20$
$1.013x - 6.099y = 14.22$

b. $8.110x + 12.20y = -0.1370$
$-18.11x + 112.2y = -0.1376$

9. Suppose one calculates in two-digit rounding arithmetic.

A rectangular parallelepiped has sides of length 3 cm, 4 cm, and 5 cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

10. Find the rate of convergence of the following as $n \to \infty$ or $h \to 0$

a. $\lim_{n\to\infty} \sin\left(\frac{1}{n^2}\right)$ **b.** $\lim_{n\to\infty} \left(\sin\left(\frac{1}{n}\right)\right)^2$

c. $\lim_{h\to 0} \frac{\sin(h)}{h}$ **d.** $\lim_{h\to 0} \frac{1-e^h}{h}$

11.

Suppose that as x approaches zero,

$$F_1(x) = L_1 + O(x^\alpha) \quad \text{and} \quad F_2(x) = L_2 + O(x^\beta).$$

Let c_1 and c_2 be nonzero constants, and define

$$F(x) = c_1F_1(x) + c_2F_2(x) \quad \text{and}$$
$$G(x) = F_1(c_1x) + F_2(c_2x).$$

Show that if γ = minimum $\{\alpha, \beta\}$, then as x approaches zero,

a. $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$

b. $G(x) = L_1 + L_2 + O(x^\gamma)$.