Homework 2 of Computational Mathematics

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1. $x^3 = x+1 \implies x^2 = 1+\frac{1}{x} \implies x = \sqrt{1+\frac{1}{x}} = g(x)$. $p_1 = g(p_0) = \sqrt{1+1} = \sqrt{2} \approx 1.414$. $p_2 = g(p_1) = \sqrt{1+\frac{1}{\sqrt{2}}} \approx 1.3065$. $p_3 = g(p_2) \approx 1.3172$. $p_4 = g(p_3) \approx 1.326$. $p_5 = g(p_4) \approx 1.324$ Then, p_4 is the answer that we want to find.

2. Let $f(x) = x^3+x-4$, $f'(x) = 3x^2+1 < 49$ for all $x \in [1,4]$. Thus, for $|x-y| < \frac{10^{-3}}{49} \approx 2.0409e-5$, $|f(x)-f(y)| < 10^3$. Find n s.t. $3 \cdot 2^{-n} < 2.0409e-5$, $n > -\log_2(\frac{2.0409e-5}{3}) \approx 17.1653$. Thus, the bound of the number of iteration is 18. Then, by python code below, the root is about 1.3787.

```
HW2 > HW2.py > ...
a = [1]
b = [4]

def f(x):
    return x**3 + x - 4

while 1:
    mid = (a[-1] + b[-1])/2
    val = f(mid)
    print(mid, val)

    if abs(val) < 0.0001:
        break
    elif val > 0:
        b.append(mid)
    else:
        a.append(mid)
```

```
PROBLEMS   OUTPUT   DEBUG CONSOLE   TERMINAL   PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> &
2.5 14.125
1.75 3.109375
1.375 -0.025390625
1.5625 1.377197265625
1.46875 0.637176513671875
1.421875 0.2965202331542969
1.3984375 0.13326025009155273
1.38671875 0.05336350202560425
1.380859375 0.013844214379787445
1.3779296875 -0.005808685906231403
1.37939453125 0.004008884658105671
1.378662109375 -0.0009021193400258198
1.3790283203125 0.0015528278327110456
1.37884521484375 0.0003252155581766057
1.378753662109375 -0.0002884865606631592 5
1.3787994384765625 1.8355831034710945e-05
```

3. In this question, we let $p_{n+1} = g(p_n)$ and $p = \sqrt[3]{21}$. And by $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$,

$$\frac{|p_{n+1}-p|}{|p_n-p|} \approx \frac{\lambda|p_n-p|^\alpha}{\lambda|p_{n-1}-p|^\alpha} \approx \left|\frac{p_n-p}{p_{n-1}-p}\right|^\alpha, \text{ then } \alpha \approx \frac{\ln|(p_{n+1}-p)/(p_n-p)|}{\ln|(p_n-p)/(p_{n-1}-p)|}.$$

Then, by observing the function, we get b is quadratic convergence (Newton's method). In the meanwhile, though $\sqrt[3]{21}$ is a fixed point of c, but it will diverges or converges to 0 for any open interval contains $\sqrt[3]{21}$.

Thus, by result of iteration with python below, the order of speed of convergence is b > d > a.

```python

def problem_3():
    import math
    p = 21**(1.0/3)
    def a(x):
        return (20*(x**3)+21)/(21*(x**2))

    def b(x):
        return x- (x**3 - 21)/(3*(x**2))

    def c(x):
        return x - (x**4 - 21*x)/(x**2 - 21)

    def d(x):
        return math.sqrt(21/x)

    def Alpha(f, x):
        return math.log(abs((f(x) - p)/(x - p)))

    functions = {"a": a, "b": b, "c":c, "d": d}

    for i in functions.keys():
        f = functions[i]
        p_now = 1
        for k in range(20):
            p_now = f(p_now)
            # print(k+1, "err:", abs(p_now - p))
        try:
            alpha = Alpha(f, f(p_now))/Alpha(f, p_now)
            print(i, alpha)
        except:
            print(i, "Cannot compute alpha by this method.")

        print("---")

def problem_4():
    def g(x):
```

```
PROBLEMS    OUTPUT    DEBUG CONSOLE    TERMINAL    PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/Users/9ryan/AppDa
a 0.9982564193669963
---
b Cannot compute alpha by this method.
---
c Cannot compute alpha by this method.
---
d 1.0000015723442375
---
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

4. $|g'(x)| = |-2^{-x}\ln(2)|$ is continuous and decreasing on $\mathbb{R}$. Thus, $g'(\frac{1}{3}) \approx 0.55015$. And since $g(\frac{1}{3}) \approx 0.79370 > \frac{1}{3}$ and $g(1) = \frac{1}{2} < 1$. By Theorem 2.3, $g(x)$ has unique fixed point on $[\frac{1}{3}, 1]$.

```python
def problem_4():
    def g(x):
        return 2**(-x)

    x = 0.3334
    step = 0
    while abs(x-g(x)) >= 10**(-4):
        x = g(x)
        step += 1
        print("step:", step, ",value:", x)


if __name__ == "__main__":
    problem_4()
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS

```
step: 1 ,value: 0.79366385007938
step: 2 ,value: 0.5768772000497738
step: 3 ,value: 0.6704133579003061
step: 4 ,value: 0.6283266346632733
step: 5 ,value: 0.6469263427213542
step: 6 ,value: 0.6386394847182052
step: 7 ,value: 0.6423183934819865
step: 8 ,value: 0.6406825519733249
step: 9 ,value: 0.6414094204320899
step: 10 ,value: 0.6410863425561439
step: 11 ,value: 0.6412299238405336
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

By the result from python above, we need 12 steps to let error less than 10^{-4}. Since $|g'(x)| \leq \ln(2)2^{-x} \leq 0.551$, we take $k = 0.551$ and $p_0 = 1$. $|p_n - p| \leq \dfrac{k^n}{1-k}|p_1 - p| \approx k^n \cdot 1.113585746 \leq 10^{-4}$. Then, $n \geq 15.633566$, that is, the bound is 16 times of iteration.

5. (a) Let x_* be nonzero fixed point of g, $x_* = 2x_* - Ax_*^2 \implies x_* = Ax_*^2 \implies x^*A = 1$. Thus, $x_* = \dfrac{1}{A}$.

   (b) Since $g'(x) = 2 - 2Ax$ and we need $|g'(x)| < k < 1$,

$$
\begin{aligned}
|2-2Ax| < k &\iff -k < 2-2Ax < k \\
&\iff \frac{k-2}{2A} < x < \frac{k+2}{2A} \\
&\overset{k=1-\varepsilon}{\iff} \frac{1}{2A} + \frac{\varepsilon}{2A} < x < \frac{3}{2A} - \frac{\varepsilon}{2A}
\end{aligned}
$$

   Thus, $x \in \left(\dfrac{1}{2A} + \dfrac{3}{2A}\right)$.

6. Let $g(x) = \dfrac{x}{2} + \dfrac{A}{2x}$, and $g'(x) = \dfrac{1}{2} - \dfrac{A}{2x^2}$. For $x \in (\sqrt{A}, \infty)$, $|g'(x)| \leq \dfrac{1}{2}$. And since $x > \sqrt{A}$, $\dfrac{x}{2} + \dfrac{A}{2x} - \sqrt{A} = \dfrac{x^2 + A - 2Ax}{2x} = \dfrac{(x-\sqrt{A})^2}{2x} > 0$. Thus, if $x > \sqrt{A}$, $g(x) > \sqrt{A}$. By Corollary 2.5, $g(x)$ converges to $\sqrt{A}$ for all $(\sqrt{A}, \infty)$.

Then, for $x \in (0, \sqrt{A})$, $\dfrac{x}{2} + \dfrac{A}{2x} - \sqrt{A} = \dfrac{(A-x)^2}{2x} > 0$. Thus, for $0 < x < \sqrt{A}$, $g(x) > \sqrt{A}$, and then by the argument above, it will converge to $\sqrt{A}$, too.

Therefore, for $x_0 > 0$, x_n will converge to $\sqrt{A}$.

7. (a) $p_2 = p_1 - \dfrac{f(p_1)[p_0 - p_1]}{f(p_0) - f(p_1)} = -\dfrac{-1}{-1+\cos(-1)-1} = \dfrac{1}{-2+\cos(1)} \approx -0.6850733573260451.$

$p_3 = p_2 - \dfrac{f(p_2)[p_1 - p_2]}{f(p_1) - f(p_2)} \approx -1.252076488909229.$

(b) $p_2 = p_1 - f(p_1) \cdot \dfrac{p_0 - p_1}{f(p_0) - f(p_1)} \approx -0.6850733573260451$. Since $f(p_2) < 0$, $f(p_2) \cdot f(p_0) < 0$.

Then, $p_3 = p_2 - f(p_2) \cdot \dfrac{p_0 - p_2}{f(p_0) - f(p_2)} \approx -0.8413551256656522.$

8. From the question, we can rewrite it as $f(i) = 1000(1-(1+i)^{-30\cdot 12}) - 135,000i = 0$. After testing, we know that $f(0.002) > 0$ and $f(0.01) < 0$.

```python

def problem_8():
    def secant_method(f, x_1, x_2):
        return x_1 - (f(x_1)*(x_2 - x_1))/(f(x_2) - f(x_1))

    f = lambda x: 1000*(1-(1+x)**(-360))-135000*x
    x = [0.002, 0.01]
    for _ in range(8):
        p = secant_method(f, x[-1], x[-2])
        print(p, f(p))
        x.append(p)

if __name__ == "__main__":
    problem_8()
```

```
PROBLEMS   OUTPUT   DEBUG CONSOLE   TERMINAL   PORTS

PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數> & C:/Users/9ryan/AppData/Loc
0.005130556113250995 148.91891659914575
0.006507247379340629 24.713056780832744
0.006781165610560291 -3.2322577565320216
0.006749483221502495 0.04480741496536211
0.0067499164158026734 7.675006418139674e-05
0.006749917159088972 -1.8349055608268827e-09
0.006749917159071203 1.1368683772161603e-13
0.0067499171590712035 0.0
PS C:\Users\9ryan\OneDrive - 國立陽明交通大學\HW\計數>
```

Then, by secant method and python, $i \approx 0.0067499171590712035$. Thus, the interest rate the borrower can afford is about $0.0067499171590712035 \cdot 12 = 0.080999005908854442 = 8.0999005908854442\%$.

9. (a) $\displaystyle\lim_{n\to\infty} \frac{|1/(n+1)^k|}{|1/n^k|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1$ for all k. Thus, p_n converges linearly.

(b) $\displaystyle\lim_{n\to\infty} \frac{10^{-2^{n+1}}}{|10^{-2^n}|^2} = \lim_{n\to\infty} 10^{-2^{n+1}+2^n\cdot 2} = 1$. Thus, p_n converges linearly.

10. (a)

i	p_i	j	$\hat{p}_j$
0	0.5		
1	0.8775825618903727161162815826038		
2	0.63901249416525922865672299864983	0	0.7313851863825818
3	0.80268510068233485562943472868 63	1	0.7360866917130169
4	0.69477802678800619881166704153443	2	0.7376528713963997
5	0.76819583128201604851502640571344	3	0.7384692208762632
6	0.71916544594241901793885065455766	4	0.7387980651735903

(b)

i	p_0	p_1	p_2	$\hat{p}_i$
0	0.5	0.8775825618903728	0.6390124941652592	0.7313851863825818
1	0.7313851863825818	0.7442499490045668	0.7355962089933913	0.7390763403695223
2	0.7390763403695223	0.7390910561531825	0.7390811434398971	0.7390851332036612
3	0.7390851332036612	0.7390851332229068	0.7390851332099427	0.7390851332151607