Homework 13 of Introduction to Analysis(II)

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1. (a) For any $\varepsilon > 0$, we can find an open cube $D(x_0, \delta)$ that $R_j \subseteq D(x_0, \delta)$ and

   (b) Since for any E_i is measure zero, we can find $\{R_{i,j}\}_j$ s.t. $\sum_j |R_{i,j}| < \frac{\varepsilon}{2^i}$. Then, let $R_{1,1} = R_1$, $R_{1,2} = R_2$, $R_{2,1} = R_3$, $R_{1,3} = R_4$, $R_{2,2} = R_5$ etc. And we can get $\{R_k\}_k$ contains $\bigcup_i E_i$ and $\sum_k |R_k| < \varepsilon(\frac{1}{2} + \frac{1}{4} + \cdots) = \varepsilon$. Therefore, union of countable measure zero sets is measure zero.

   (c) Since A is compact and (a), we can find finite open cubes which covers A and volume of their union is less than ε. Thus, take closure on each open cube and we can find a retangle covers A with volume less than ε. Therefore, volume of A is 0.

2. For any grid g, $U(1,g) = L(1,g)$. Then, 1 is integrable. And for any $\varepsilon > 0$, $1 - \varepsilon \leq 1 \leq 1 + \varepsilon$ and $(1-\varepsilon)\,\mathrm{Vol}(E) \leq \int_E 1\; dE \leq (1+\varepsilon)\,\mathrm{Vol}(E)$. Thus, $\int_E 1\; dE = 1$.

3. (a) If x is rational, $G(x,t) = \int_0^t 1\; dy = t$. IF x is irrational, $G(x,t) = \int_0^t 2y\; dy = t^2$.
   Since $t \in [0,1]$, $t^2 \leq t$. Then, (U)$\int_0^1 G(x,t)\; dx = t$ and (L)$\int_0^1 G(x,t)\; dx = t^2$.
   Therefore, $\int_0^1 (\int_0^1 f(x,y)\; dy)\; dx = \int_0^1 G(x,1)\; dx = \int_0^1 1\; dx = 1$.

   (b) As $y \to 0$, (U)$\int_0^1 f(x,y)\; dx = 1$ but (L)$\int_0^1 f(x,y)\; dx = 2y \to 0$. Thus, $\int_Q f(X)\; dx$ doesn't exists.