Homework 2 of Introduction to Analysis(II)

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1. Suppose $f_k(x) = \sum_{n=1}^{k} \frac{x}{n^\alpha(1+nx^2)}$. First, we observe at the funciton f_k, a single term of f_k we call $f_{k,n} = \frac{x}{n^\alpha(1+nx^2)}$.

   $f'_{k,n}(x) = \frac{n^\alpha(1+nx^2) - xn^\alpha(2nx)}{n^{2\alpha}(1+nx^2)^2} = \frac{1-nx^2}{n^\alpha(1+nx^2)^2}$ would equal to 0 at $x = \frac{1}{\sqrt{n}}$. Thus, the maximun of $|f_{k,n}| = \frac{\frac{1}{\sqrt{n}}}{n^\alpha(1+n\cdot\frac{1}{n})} = \frac{1}{n^{\alpha+1/2}}$. By p-test and $\alpha > \frac{1}{2}$, the $f_k(x)$ converges at $x = \frac{1}{\sqrt{n}}$. And since $f_{k,n}(\frac{1}{\sqrt{n}})$ is greater than any else $f_{k,n}(x)$, we can easily know that $f_k(x)$ converges uniformly on $\mathbb{R}$ by Weierstrass M-test.

2. Since $f_k \to f$ uniformly and f_k are continuous, f is continuous. Then, for any $\varepsilon > 0$, we have $\delta > 0$ s.t. if $|y-y'| < \delta$ then $|f(y)-f(y')| < \frac{\varepsilon}{2}$ for all $y, y' \in \mathbb{R}$. Since $x_k \to x$, there exists $N_1 \in \mathbb{N}$ s.t. $|x_k - x| < \delta$ for all $k > N_1$. Also we have $N_2 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ for all $k > N_2$.

   Then, take $N = \max\{N_1, N_2\}$, we can get $|f_k(x_k) - f(x)| \le |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $k > N$. Thus, $\lim_{k\to\infty} f_k(x_k) = f(x)$.

3. Since f_n are continuous and converges uniformly, f is continuous(also integrable on $[0,1]$). And since f is continuous, we can find the maximun of $|f|$ on $[0,1]$ which is called as M. For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{4}$ for all $k > N_1$. And there exists $N_2 \in \mathbb{N}$ s.t. $N_2 > \frac{M}{2\varepsilon}$. Then, take

$n > N = \max\{N_1, N_2\}$

$$\begin{aligned}
\left|\int_0^{1-\frac{1}{n}} f_n(x)dx - \int_0^1 f(x)dx\right| &\leq \left|\int_0^{1-\frac{1}{n}} f_n(x) - f(x)\ dx\right| + \left|\int_{1-\frac{1}{n}}^1 f(x)\ dx\right| \\
&\leq \int_0^1 |f_k(x) - f(x)|dx + \frac{1}{n}\cdot M \\
&\leq 1\cdot 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Thus, $\lim\limits_{n\to\infty}\int_0^{1-\frac{1}{n}} f_n(x)dx = \int_0^1 f(x)dx$.

4. Since $|f_k| \leq g$, we have $|f_k(x) - f(x)| \leq 2g(x)$. For any $\varepsilon > 0$, there exists larger enough n s.t. $\int_0^{\frac{1}{n}} 2g(x)\ dx$ and $\int_n^{\infty} 2g(x)\ dx$ less than $\frac{\varepsilon}{3}$. And there exists $N \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{3(n+1)}$ for all $k > N$.

   Then,

$$\begin{aligned}
\left|\int_0^{\infty} f_k(x)\ dx - \int_0^{\infty} f(x)\ dx\right| &\leq \int_0^{\infty} |f_k(x) - f(x)|dx \\
&= \int_0^{\frac{1}{n}} |f_k(x) - f(x)|\ dx + \int_{\frac{1}{n}}^{n} |f_k(x) - f(x)|\ dx + \int_n^{\infty} |f_k(x) - f(x)|\ dx \\
&\leq \int_0^{\frac{1}{n}} 2g(x)\ dx + \int_n^{\infty} 2g(x)\ dx + \int_{\frac{1}{n}}^{n} |f_k(x) - f(x)|\ dx \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

   Thus, $\lim\limits_{n\to\infty}\int_0^{\infty} f_n(x)\ dx = \int_0^{\infty} f(x)\ dx$.