Homework 2 of Introduction to Analysis(II)

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February 27, 2024

1. Suppose $f_k(x) = \sum_{n=1}^{k} \frac{x}{n^\alpha(1+nx^2)}$ and $E_I = [-L, L]$ for $L \in \mathbb{N}$. Then, we want to proof that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|f_k(x) - f_l(x)| < \varepsilon$ for all $k, l > N$ and all $x \in I_L$.

   First, suppose that $l > k > N$, then

$$
\begin{aligned}
|f_k(x) - f_l(x)| &= \sum_{n=k}^{l} \frac{x}{n^\alpha(1+nx^2)} \\
&\leq \sum_{n=k}^{l} \frac{L}{n^\alpha(1+nL^2)}
\end{aligned}
$$

2. Since $f_k \to f$ uniformly and f_k are continuous, f is continuous. Then, for any $\varepsilon > 0$, we have $\delta > 0$ s.t. if $|y - y'| < \delta$ then $|f(y) - f(y')| < \frac{\varepsilon}{2}$ for all $y, y' \in \mathbb{R}$. Since $x_k \to x$, there exists $N_1 \in \mathbb{N}$ s.t. $|x_k - x| < \varepsilon$ for all $k > N_1$. Also we have $N_2 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ for all $k > N_2$.

   Then, take $N = \max\{N_1, N_2\}$, we can get $|f_k(x_k) - f(x)| \leq |f_k(x_k) - f_k(x)| + |f_k(x) - f(x)| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $k > N$. Thus, $\lim_{k\to\infty} f_k(x_k) = f(x)$.