

Homework 2 of Introduction to Analysis(II)

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1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t.

$$\frac{\sum_{i=1}^N a_i - a}{N'} < \frac{\varepsilon}{2}. \text{ Thus, for any } n > N',$$

$$\begin{aligned} \left| \frac{\sum_{i=1}^n a_i}{n} - a \right| &\leq \left| \frac{\sum_{i=1}^N a_i - a}{n} \right| + \left| \frac{\sum_{i=N+1}^n a_i - a}{n - N'} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} b_n = a$.

2. Since $\lim_{n \rightarrow \infty} na_n = 0$, by 1., $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{na_n}{N} = 0$. Also, since $\lim_{n \rightarrow \infty} na_n = 0$, $\lim_{n \rightarrow \infty} a_n = 0$. Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $\left| \sum_{n=k+1}^{\infty} a_n \right| < \frac{\varepsilon}{3}$ for all $k > N$. Then, for $x \rightarrow 1^-$ s.t. $|f(x) - A| < \frac{\varepsilon}{3}$ and $\left| \sum_{n=1}^k a_n(1-x) \right| < \frac{\varepsilon}{3}$,

$$\begin{aligned} \left| \sum_{n=0}^N a_n - A \right| &= \left| \sum_{n=0}^N a_n(1-x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A) \right| \\ &\leq \left| \sum_{n=0}^N a_n(1-x) \right| + \left| \sum_{n=N+1}^{\infty} a_n x^n \right| + |f(x) - A| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3. Since $\lim_{x \rightarrow 1^-} \sum_{n=1}^{\infty} a_n x^n = A$, for any $\varepsilon > 0$, we can find a $x_1 \in (0, 1)$ s.t. $|\sum_{n=1}^{\infty} a_n x^n - A| < \frac{\varepsilon}{3}$ for all $x_1 < x < 1$. Then, there exists $N \in \mathbb{N}$ s.t. $|\sum_{n=N'+1}^{\infty} a_n x^n| < \frac{\varepsilon}{3}$ for all x and $N' > N$. And we have x_2 s.t. $|\sum_{n=1}^N a_n (1 - x^n)| < \frac{\varepsilon}{3}$ for all $x_2 < x < 1$.

Thus, for any $N' > N$ and $x > \max\{x_1, x_2\}$,

$$\begin{aligned} \left| \sum_{n=0}^N a_n - A \right| &\leq \left| \sum_{n=0}^{N'} a_n (1 - x^n) \right| + \left| \sum_{n=N'+1}^{\infty} a_n x^n \right| + \left| \sum_{n=1}^{\infty} a_n x^n - A \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus, $\sum_{n=1}^{\infty} a_n = A$.

4.

(\Rightarrow) Since $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=N+1}^{\infty} a_n \sin(nx)| < \varepsilon$ for all x .

And since we can find x s.t. $\sin(nx) = 1$ for all x and all n , we can rewrite it as $\lim_{N \rightarrow \infty} \sum_{n=N+1}^{\infty} a_n =$

$$\lim_{N \rightarrow \infty} \sum_{n=N+1}^{\infty} \frac{na_n}{n} = 0. \text{ By 1., } \lim_{n \rightarrow \infty} na_n = 0.$$

(\Leftarrow) Since $na_n \rightarrow 0$, we have

$$\left| \sum_{n=N+1}^{\infty} a_n \right| \rightarrow 0 \text{ as } N \rightarrow \infty. \text{ And since } |\sin(nx)| \leq 1 \text{ for all } x, \left| \sum_{n=N+1}^{\infty} a_n \sin(nx) \right| \leq \sum_{n=N+1}^{\infty} a_n \rightarrow 0.$$

By Cauchy Criterion, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly.