Exercises(13)

May 14, 2024

1. A set $A \subseteq \mathbb{R}^n$ has measure zero if for each $\epsilon > 0$ there is a sequence $\{R_j\}$ of rectangles whose union contains A and such that $\sum_j |R_j| < \epsilon$.

   (a) (4 points) Show that, in the definition of "measure zero" given above, we can require the rectangles to be open cubes.

   (b) (3 points) Show that the union of a countable family of sets with measure zero has measure zero.

   (c) (3 points) Show that every compact set A with measure zero also has $\mathrm{Vol}(A) = 0$.

2. (10 points) Let $E \subseteq \mathbb{R}^n$ be a Jordan region. Show that

$$\mathrm{Vol}(E) = \int_E 1 dx.$$

3. Define f on the square $Q = [0, 1] \times [0, 1]$ as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } x \text{ is irrational.} \end{cases}$$

   (a) (6 points) Prove that $G(x, t) \equiv \int_0^t f(x, y)dy$ exists for $0 \le t \le 1$ and that

$$\text{(U)} \int_0^1 G(x, t)dx = t \quad \text{and} \quad \text{(L)} \int_0^1 G(x, t)dx = t^2.$$

   This shows that $\int_0^1 (\int_0^1 f(x, y)dy)dx = 1$.

   (b) (4 points) Prove that the double integral $\int_Q f(X)dX$ does not exist.