

Homework 12 of Introduction to Analysis(II)

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1. (a) Since E is Jordan region, $\text{Vol}(\partial E) = 0$.

(b) Since $\text{cl}(E) = \text{int}(E) \cup \partial E$ and $\text{int}(E) \subseteq E \subseteq \text{cl}(E)$,

$$\text{Vol}(\text{cl}(E)) = \text{Vol}(\text{int}(E)) + \text{Vol}(\partial E) = \text{Vol}(\text{int}(E)) \leq \text{Vol}(E) \leq \text{Vol}(\text{cl}(E)).$$

Therefore, $\text{Vol}(\text{cl}(E)) = \text{Vol}(\text{int}(E)) = \text{Vol}(E)$.

(c)

(\implies) From (b), we know $\text{Vol}(\text{int}(E)) = \text{Vol}(E) > 0$, then we can find a set of rectangles R_n s.t.

$$\sum |R_n| > 0 \text{ and } \cup R_n \subseteq \text{int}(E). \text{ Therefore, } \text{int}(E) \neq \emptyset.$$

(\impliedby) Since $\text{int}(E)$ is non-empty, for any $x_0 \in \text{int}(E)$, there exists $\varepsilon > 0$ s.t. $D(x_0, \varepsilon) \subseteq \text{int}(E)$. Then,

we can find a small rectangle R with each length is $\frac{\varepsilon}{2}$ and R is contained in $D(x_0, \varepsilon)$. Thus,

$$\text{Vol}(\text{int}(E)) > \left(\frac{\varepsilon}{2}\right)^2 > 0.$$

(d) Since f is continuous, for any $x_0 \in [a, b]$, we can find a sequence $x_k \rightarrow x_0$ s.t. $f(x_k) \rightarrow f(x_0)$. That

is, $A = \{(x, f(x)) \mid x \in [a, b]\}$ is closed. And since $\partial A \subseteq A$, $\text{Vol}(\partial A) \leq \text{Vol}(A)$.

Since f is continuous on $[a, b]$ is compact, for all $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|x - y| < \delta$

implies That $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Then, we can find a finite sequence $\{x_i \mid x_i \in [a, b]\}_{i=1}^N$ s.t.

$[a, b] \subseteq D(x_i, \delta)$. Therefore, for any $y = f(x)$, $y \in D(f(x_i), \varepsilon)$ for some i . Thus, $A = \{(x, f(x)) \mid$

$x \in [a, b]\} \subseteq \bigcup_{i=1}^N D(x_i, \delta) \times D(f(x_i), \varepsilon)$ and $|D(x_i, \delta) \times D(f(x_i), \varepsilon)| = \delta \cdot \varepsilon < M \cdot \delta^2$. Since δ is

arbitrary,