# Homework 5 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

March 25, 2024

1. (a) If $x \in A$, $d(x,A) = \|x-x\| = 0$ and $d(x,B) = k > 0$, $\phi(x) = \dfrac{0}{0+k} = 0$. If $x \in B$, $d(x,A) = l > 0$ and $d(x,B) = 0$, $\phi(x) = \dfrac{l}{l+0} = 1$. If $x \notin A \wedge x \notin B$, $d(x,A) = l$ and $d(x,B) = k$, $\phi(x) = \dfrac{l}{l+k} < 1$ and is positive.

   And for $x,y \in A$ and $\varepsilon > 0$, if $\|x-y\| < \delta = \varepsilon$, $|d(x,A) - d(x,Y)| < \delta = \varepsilon$. Thus, $d(x,A)$ is continuous. Using the same way, $d(x,B)$ is also continuous. Since $\phi(x) = \dfrac{d(x,A)}{d(x,A)+d(x,B)}$ is a continuous function divided by a continuous function which is greater than 0, $\phi(x)$ is also continuous.

   (b) Let $\phi(x) = (b-a)\dfrac{d(x,A)}{d(x,A)+d(x,B)} + a$. From (a), we can get continuous function $\phi(x \in A) = (b-a)\cdot 0 + a = a$, $\phi(x \in B) = (b-a)\cdot 1 + a = b$, and $a \le \phi(x) \le b$ for all $x \in A$.

2. If $f$ has more than one fixed point, there exists $x,y \in S$ s.t. $d(f^n(x), f^n(y)) = d(x,y)$ for all $n \in \mathbb{N}$(contradiction to $a_n \to 0$). Then, we want to show that $f$ has fixed point. Since $a_n \to 0$, we can find an $N \in \mathbb{N}$ s.t. $d(f^N(x), f^N(y)) \le ad(x,y)$ for $a < 1$. Thus, $f^N$ is a contraction mapping. Take $x^*$ be fixed point of $f^N$, that is, $f^N(x^*) = x^*$. Then, since $f^{N+1}(x^*) = f(f^N(x^*)) = f(x^*)$ and $f^{N+1} = f^N(f(x^*))$, $f(x^*) = f^N(f(x^*))$ is fixed point of $f^N$. Since $f^N$ has unique fixed point, $f(x^*) = x^*$. Thus, $f$ has unique fixed point.

3. Since $T(u)(t) = \int_a^t u(s)\ ds$, $(T^m(u)(t))' = \big(T(T^{m-1}(u))(t)\big)' = \left(\int_a^t T^{m-1}(u)(s)ds\right)' = T^{m-1}(u)(t)$

by the FTC. And since $T^m(u)(a)=0$ for all $m\in\mathbb{N}$, we can get

$$\begin{aligned}
T^m(u)(t) &= T(T^{m-1}(u))(t)\\
&= \int_a^t T^{m-1}(u)(s)\ ds\\
&= -\int_a^t T^{m-1}(u)(s)\ d(t-s)\\
&= -(t-s)T^{m-1}(u)(s)\ |_a^t + \int_a^t (t-s)\ d(T^{m-1}(u)(s))\\
&= \int_a^t (t-s)T^{m-2}(u)(s)\\
&= \int_a^t T^{m-2}(u)(s)\ d(\frac{-(t-s)^2}{2})\\
&= \int_a^t \frac{(t-s)^2}{2!}T^{m-3}(u)(s)\ ds\\
&\vdots\\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!}\ d(T(u)(s))\\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!}u(s)\ ds
\end{aligned}$$

Thus, $T^m(u)=\dfrac{1}{(m-1)!}\displaystyle\int_a^t (t-s)^{m-1}u(s)ds \le \int_a^t \frac{(t-s)^{m-1}}{(m-1)!}M\ ds \le \int_a^t \frac{M\cdot(t-a)^{m-1}}{(m-1)!}\ ds$ with $M$ is upper bound of $u$. Then, taking $a_n=\dfrac{(b-a)^n}{(n-1)!}$, $d(T^n(u),T^n(v))\le a_n d(u,v)$ and $a_n\to 0$. Therefore, by 2., $T$ has unique fixed point, and $T(0)=0$ trivially.

4. We want to proof $T(f)(x)=\displaystyle\int_0^x f(t)\ dt$ has unique fixed point. Since $f$ is on $[0,1]$, $T(|f|)(x)\le|f(x)|$ for all $f$ and $x\in[0,1]$. Thus, by 3., $T$ has unique fixed point and $T(0)=0$. Therefore, $\displaystyle\int_0^1 f(x)x^n dx=(n-1)!T^n(f)(1)=0$ for all $x$ implies that $f(x)=0$ in $[0,1]$.