Homework 8 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

April 14, 2024

1. $f_x(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0)}{x} = \lim\limits_{x \to 0} \dfrac{x \cdot 0(x^2 - 0^2)/(x^2 + 0^2)}{x} = 0$ exists. Also, we can easily get that

   $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y}(0,0) = \lim\limits_{y \to 0} \dfrac{y(0^4 + 4 \cdot 0^2 y^2 - y^4)/(0^2 + y^2)^2}{y} = \lim\limits_{y \to 0} \dfrac{-y^4}{y^4} = -1$. And,

   $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_y(0,0)$.

2. Since Df is continuous on S, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x) - f(y)\| < \dfrac{\varepsilon}{\|b - a\|}$ if $\|x - y\| <$

   δ. And since S is a closed line in R^p, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k, \dfrac{\delta}{2}) \supseteq S$ and $\|x_{k+1} - a\| >$

   $\|x_k - a\|$. Let $x_0 = a$ and $x_{n+1} = b$ and $x_k = a + t_k(b - a)$. Then, by MCT, $Df(x_k) - Df(x_{k-1}) =$

   $Df(c_k)(x_k - x_{k-1})$ for c_k on the line between x_k, x_{k-1}. Thus,

$$
\begin{aligned}
\left| f(b) - f(a) - \int_0^1 Df(a + t(b-a))(b-a)dt \right| &\leq \sum_{k=1}^{n+1} \| f(x_k) - f(x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a + t(b-a))(b-a)dt \| \\
&= \sum_{k=1}^{n+1} \| Df(c_k)(x_k - x_{k-1}) \\
&\quad - \int_{t_{k-1}}^{t_k} Df(a + t(b-a))(b-a)\, dt \| \\
&= \sum_{k=1}^{n+1} \| \int_{t_{k-1}}^{t_k} Df(c_k)(b-a) - Df(a + t(b-a))(b-a)dt \| \\
&= \sum_{k=1}^{n+1} \frac{\varepsilon(t_k - t_{k-1})}{\|(b-a)\|} \\
&= \varepsilon
\end{aligned}
$$

   Therefore, $f(b) = f(a) = \int_0^1 Df(tb + (1-t)a)(b-a)\, dt$.

3. $\displaystyle \lim_{u \to 0} \frac{\|g(x+u) - g(x) - Dg(x)(u)\|}{\|u\|} = \lim_{u \to 0} \frac{\|g(x) + g(u) + B(x,u) + B(u,x) - g(x) - Dg(x)(u)\|}{\|u\|} = 0.$

4. For any $(x_0, y_0) \in \mathbb{R}^2$ and $\varepsilon > 0$, we can find small enough $h, k \in \mathbb{R}$. And since $\dfrac{\partial^2 f}{\partial x \partial y}$ exists and continuous, $\dfrac{\partial f}{\partial x}(x_0, y_0), \dfrac{\partial f}{\partial x}(x_0, y_0 + k), \dfrac{\partial f}{\partial x}(x_0 + h, y_0), \dfrac{\partial f}{\partial x}(x_0 + h, y_0 + k)$ are all exists and

$|f_x(x_0, y_0 + k) - f_x(x_0, y_0)| < \varepsilon |k|.$

And since $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $\displaystyle |f_x(x_0, y_0)| < |\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}| + \varepsilon.$

Then,

$$\left| \frac{f_y(x_0 + h, y_0) - f_y(x_0, y_0)}{h} \right| \leq \frac{1}{h} \left(\left| \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k} - \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} \right| + 2\varepsilon \right)$$

$$=$$