Homework 8 of Introduction to Analysis(II)

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1. $f_x(0,0) = \lim_{x\to 0} \dfrac{f(x,0)-f(0)}{x} = \lim_{x\to 0} \dfrac{x\cdot 0(x^2-0^2)/(x^2+0^2)}{x} = 0$ exists. Also, we can esaily get that $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y}(0,0) = \lim_{y\to 0} \dfrac{y(0^4+4\cdot 0^2y^2-y^4)/(0^2+y^2)^2}{y} = \lim_{y\to 0} \dfrac{-y^4}{y^4} = -1$. And, $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_y(0,0)$.

2. Since Df is continuous on S, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x)-f(y)\| < \dfrac{\varepsilon}{\|b-a\|}$ if $\|x-y\| < \delta$. And since S is a closed line in R^p, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k, \dfrac{\delta}{2}) \supseteq S$ and $\|x_{k+1}-a\| > \|x_k-a\|$. Let $x_0 = a$ and $x_{n+1} = b$ and $x_k = a+t_k(b-a)$. Then, by MCT, $Df(x_k)-Df(x_{k-1}) = Df(c_k)(x_k-x_{k-1})$ for c_k on the line between x_k, x_{k-1}. Thus,

$$
\begin{aligned}
|f(b)-f(a)-\int_0^1 Df(a+t(b-a))(b-a)dt| &\leq \sum_{k=1}^{n+1} \|f(x_k)-f(x_{k-1})-\int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \|Df(c_k)(x_k-x_{k-1}) \\
&- \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)\ dt\| \\
&= \sum_{k=1}^{n+1} \|\int_{t_{k-1}}^{t_k} Df(c_k)(b-a)-Df(a+t(b-a))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \frac{\varepsilon(t_k-t_{k-1})}{\|(b-a)\|} \\
&= \varepsilon
\end{aligned}
$$

Therefore, $f(b) = f(a) = \int_0^1 Df(tb+(1-t)a)(b-a)\ dt$.

3. $\lim\limits_{u\to 0}\frac{\|g(x+u)-g(x)-Dg(x)(u)\|}{\|u\|}=\lim\limits_{u\to 0}\frac{\|g(x)+g(u)+B(x,u)+B(u,x)-g(x)-Dg(x)(u)\|}{\|u\|}=0.$

4. For any $(x_0,y_0)\in\mathbb{R}^2$ and $\varepsilon>0$, we can find small enough $h,k\in\mathbb{R}$. And since $\frac{\partial^2 f}{\partial x\partial y}$ exists and continuous, $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial x}(x_0,y_0+k)$, $\frac{\partial f}{\partial x}(x_0+h,y_0)$, $\frac{\partial f}{\partial x}(x_0+h,y_0+k)$ are all exists and

   $|f_x(x_0,y_0+k)-f_x(x_0,y_0)|<\varepsilon|k|$.

   And since $f\in C^1(\mathbb{R}^2,\mathbb{R})$, $|f_x(x_0,y_0)|<|\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}|+\varepsilon$.

   Then,

$$|f_y(x_0+h,y_0)|$$