Homework 12 of Introduction to Analysis(II)

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1. (a) Since E is Jordan region, E is bounded and for all $\varepsilon^* > 0$, we can find a grid g s.t. $\mathrm{Vol}(\partial E, g) < \varepsilon^*$.

   i. If $x \in \partial\,\mathrm{int}(E)$, $D(x,\varepsilon) \cap \mathrm{int}(E) \neq \emptyset$ and $D(x,\varepsilon) \cap (M \setminus \mathrm{int}(E)) \neq \emptyset$.

   Thus, $D(x,\varepsilon) \cap E \neq \emptyset$ and $D(x,\varepsilon) \cap (M \setminus \mathrm{int}(E)) \neq \emptyset$.

   If $D(x,\varepsilon) \cap (M \setminus E) \neq \emptyset$, then $x \in \partial E$.

   If $D(x,\varepsilon) \cap (M \setminus E) = \emptyset$, $D(x,\varepsilon) \subseteq E$. Then, $D(x,\varepsilon) \cap (E \setminus \mathrm{int}(E)) \neq \emptyset$, that is, $D(x,\varepsilon) \cap \partial E \neq \emptyset$. If $x \in \mathrm{int}(E)$, then exists $\varepsilon > 0$ s.t. $D(x,\varepsilon) \subseteq \mathrm{int}(E) \cap \partial E = \emptyset$. Thus, $x \in \partial E$.

   Therefore, $x \in \partial\,\mathrm{int}(E) \implies x \in \partial E$. Then, $\mathrm{Vol}(\partial\,\mathrm{int}(E)) \leq \mathrm{Vol}(\partial E) = 0$.

   (b) Since $\mathrm{cl}(E) = \mathrm{int}(E) \cup \partial E$ and $\mathrm{int}(E) \subseteq E \subseteq \mathrm{cl}(E)$,

   $\mathrm{Vol}(\mathrm{cl}(E)) \leq \mathrm{Vol}(\mathrm{int}(E)) + \mathrm{Vol}(\partial E) = \mathrm{Vol}(\mathrm{int}(E)) \leq \mathrm{Vol}(E) \leq \mathrm{Vol}(\mathrm{cl}(E))$.

   Therefore, $\mathrm{Vol}(\mathrm{cl}(E)) = \mathrm{Vol}(\mathrm{int}(E)) = E$.

   (c)

   $(\implies)$ From (b), we know $\mathrm{Vol}(\mathrm{int}(E)) = \mathrm{Vol}(E) > 0$, then we can find a set of rectangles R_n s.t. $\sum |R_n| > 0$ and $\cup R_n \subseteq \mathrm{int}(E)$. Therefore, $\mathrm{int}(E) \neq \emptyset$.

   $(\impliedby)$ Since $\mathrm{int}(E)$ is non-empty, for any $x_0 \in \mathrm{int}(E)$, there exists $\varepsilon > 0$ s.t. $D(x_0,\varepsilon) \subseteq \mathrm{int}(E)$. Then, we can find a small rectangle R with each length is $\dfrac{\varepsilon}{2}$ and R is contained in $D(x_0,\varepsilon)$. Thus, $\mathrm{Vol}(\mathrm{int}(E)) > \left(\dfrac{\varepsilon}{2}\right)^2 > 0$.

   (d) Since f is continuous, for any $x_0 \in [a,b]$, we can find a sequence $x_k \to x_0$ s.t. $f(x_k) \to f(x_0)$. That is, $A = \{(x, f(x)) \mid x \in [a,b]\}$ is closed. And since $\partial A \subseteq A$, $\mathrm{Vol}(\partial A) \leq \mathrm{Vol}(A)$.

Since f is continuous on $[a,b]$ is compact, for all $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|x-y| < \delta$ implies That $|f(x) - f(y)| < \dfrac{\varepsilon}{b-a}$. Then, we can find a finite increasing sequence $\{x_i \mid x_i \in [a,b]\}_{i=1}^{N}$ s.t. $[a,b] \subseteq D(x_i, \dfrac{\delta}{2})$. Therefore, for any $y = f(x)$, $y \in D(f(x_i), \dfrac{\varepsilon}{b-a})$ for some i.
Then, take $u_0 = a$, $u_i \in D(x_i, \dfrac{\delta}{2}) \cap D(x_{i+1}, \dfrac{\delta}{2})$, $u_N = b$, and we can get $[a,b] = \cup[u_i, u_{i+1}]$. Thus, $A \subseteq \bigcup_{i=0}^{N} [u_i, u_{i+1}] \times D(\xi_i, \dfrac{\varepsilon}{b-a})$ for some $\xi_i \in \{f(x) \mid x \in [u_i, u_i = 1]\}$ with the sum of the rectangles is $\dfrac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$. Hence, $\mathrm{Vol}(A) = 0$ and $\mathrm{Vol}(\partial A) = 0$.

(e) Yes. Since f is integrable, for any $\varepsilon > 0$, we can find an partition P s.t. $|U(f,P) - L(f,P)| < \varepsilon$. That is, $\sum_{i=0}^{N} \left(\sup_{x \in [x_i, x_{i+1}]} \{f(x)\} - \inf_{x \in [x_i, x_{i+1}]} \{f(x)\} \right) \cdot (x_{i+1} - x_i) < \varepsilon$, and each one of the summation is a rectangle that contains all $(x, f(x))$ in the intevral. Thus, $\mathrm{Vol}(A) = 0$ and $\mathrm{Vol}(\partial A) = 0$.

2. (a) For $x \in \partial E_1 \cap E_2$, $x \in \mathrm{cl}(E_1 \cap E_2) \cap \mathrm{cl}(M \setminus E_1 \cap E_2)$. Then, if $E_1 \cap E_2$ is not Jordan region, $\mathrm{Vol}(E_1 \cap E_2) = a > 0$.