

Homework 13 of Introduction to Analysis(II)

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1. (a) For any $\varepsilon > 0$, we can find an open cube $D(x_0, \delta)$ that $R_j \subseteq D(x_0, \delta)$ and
(b) Since for any E_i is measure zero, we can find $\{R_{i,j}\}_j$ s.t. $\sum_j |R_{i,j}| < \frac{\varepsilon}{2^i}$. Then, let $R_{1,1} = R_1$, $R_{1,2} = R_2, R_{2,1} = R_3, R_{1,3} = R_4, R_{2,2} = R_5$ etc. And we can get $\{R_k\}_k$ contains $\bigcup_i E_i$ and $\sum_k |R_k| < \varepsilon(\frac{1}{2} + \frac{1}{4} + \cdots) = \varepsilon$. Therefore, union of countable measure zero sets is measure zero.
(c) Since A is compact and (a), we can find finite open cubes which covers A and volume of their union is less than ε . Thus, take closure on each open cube and we can find a rectangle covers A with volume less than ε . Therefore, volume of A is 0.
2. For any grid g , $U(1, g) = L(1, g)$. Then, 1 is integrable. And for any $\varepsilon > 0$, $1 - \varepsilon \leq 1 \leq 1 + \varepsilon$ and $(1 - \varepsilon) \text{Vol}(E) \leq \int_E 1 \, dE \leq (1 + \varepsilon) \text{Vol}(E)$. Thus, $\int_E 1 \, dE = 1$.
3. (a) If x is rational, $G(x, t) = \int_0^t 1 \, dy = t$. If x is irrational, $G(x, t) = \int_0^t 2y \, dy = t^2$.
Since $t \in [0, 1]$, $t^2 \leq t$. Then, (U) $\int_0^1 G(x, t) \, dx = t$ and (L) $\int_0^1 G(x, t) \, dx = t^2$.
Therefore, $\int_0^1 (\int_0^1 f(x, y) \, dy) \, dx = \int_0^1 G(x, 1) \, dx = \int_0^1 1 \, dx = 1$.
(b)