Exercises(3)

March 5, 2024

1. (6 points) Let $\lim_{n\to\infty} a_n = a$. Let $b_n = (a_1 + \cdots + a_n)/n$. Show that $\lim_{n\to\infty} b_n = a$.

2. (8 points) Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges to $f(x)$ for $|x| < 1$ and that $\lim_{n\to\infty} n a_n = 0$. If $\lim_{x\to 1^-} f(x) = A$, show that the series $\sum_{n=0}^{\infty} a_n$ converges to A.(Hint: write $\sum_{n=0}^{N} a_n - A = \sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A)$ and using exercise 1.)

3. (8 points) Suppose that $a_n \geq 0$ and that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 1. Prove that if
$$A = \lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n,$$
then $\sum_{n=0}^{\infty} a_n$ converges to A.

4. (8 points) Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive terms. Prove that the series $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $\mathbb{R}$ if and only if $n a_n \to 0$ as $n \to \infty$.