

Homework 2 of Introduction to Analysis(II)

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1. Suppose $f_k(x) = \sum_{n=1}^k \frac{x}{n^\alpha(1+nx^2)}$. First, we observe at the function f_k , a single term of f_k we call $f_{k,n} = \frac{x}{n^\alpha(1+nx^2)}$.
 $f'_{k,n}(x) = \frac{n^\alpha(1+nx^2) - xn^\alpha(2nx)}{n^{2\alpha}(1+nx^2)^2} = \frac{1-nx^2}{n^\alpha(1+nx^2)^2}$ would equal to 0 at $x = \frac{1}{\sqrt{n}}$. Thus, the maximum of $|f_{k,n}| = \frac{\frac{1}{\sqrt{n}}}{n^\alpha(1+n \cdot \frac{1}{n})} = \frac{1}{n^{\alpha+1/2}}$. By p -test and $\alpha > \frac{1}{2}$, the $f_k(x)$ converges at $x = \frac{1}{\sqrt{n}}$. And since $f_{k,n}(\frac{1}{\sqrt{n}})$ is greater than any else $f_{k,n}(x)$, we can easily know that $f_k(x)$ converges uniformly on \mathbb{R} .
2. Since $f_k \rightarrow f$ uniformly and f_k are continuous, f is continuous. Then, for any $\varepsilon > 0$, we have $\delta > 0$ s.t. if $|y - y'| < \delta$ then $|f(y) - f(y')| < \frac{\varepsilon}{2}$ for all $y, y' \in \mathbb{R}$. Since $x_k \rightarrow x$, there exists $N_1 \in \mathbb{N}$ s.t. $|x_k - x| < \delta$ for all $k > N_1$. Also we have $N_2 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ for all $k > N_2$.
Then, take $N = \max\{N_1, N_2\}$, we can get $|f_k(x_k) - f(x)| \leq |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $k > N$. Thus, $\lim_{k \rightarrow \infty} f_k(x_k) = f(x)$.
3. Since f_n are continuous and converges uniformly, f is continuous(also integrable on $[0, 1]$). And since f is continuous, we can find the maximum of $|f|$ on $[0, 1]$ which is called as M . For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{4}$ for all $k > N_1$. And there exists $N_2 \in \mathbb{N}$ s.t. $N_2 > \frac{M}{2\varepsilon}$. Then, take

$$n > N = \max\{N_1, N_2\}$$

$$\begin{aligned}\left|\int_0^{1-\frac{1}{n}}f_n(x)dx-\int_0^1f(x)dx\right|&\leq\left|\int_0^{1-\frac{1}{n}}f_n(x)-f(x)\,dx\right|+\left|\int_{1-\frac{1}{n}}^1f(x)\,dx\right|\\&\leq\int_0^1|f_k(x)-f(x)|dx+\frac{1}{n}\cdot M\\&\leq 1\cdot 2\frac{\varepsilon}{4}+\frac{\varepsilon}{2}\\&=\varepsilon\end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \int_0^{1-\frac{1}{n}} f_n(x) dx = \int_0^1 f(x) dx.$$

4.