Exercises(12)

May 7, 2024

1. Let E be a Jordan region in $\mathbb{R}^n$.

   (a) (3 points) Prove that $\operatorname{int}(E)$ and $\overline{E}$ are Jordan regions.

   (b) (3 points) Prove that $\operatorname{Vol}(\operatorname{int}(E)) = \operatorname{Vol}(\overline{E}) = \operatorname{Vol}(E)$.

   (c) (4 points) Prove that $\operatorname{Vol}(E) > 0$ if and only if $\operatorname{int}(E) \neq \emptyset$.

   (d) (4 points) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Prove that the graph of $y = f(x), x \in [a, b]$, is a Jordan region in $\mathbb{R}^2$.

   (e) (4 points) Does part (d) hold if continuous is replaced by integrable?

2. Suppose that E_1, E_2 are Jordan regions in $\mathbb{R}^n$.

   (a) (3 points) Prove that $E_1 \cap E_2$ and $E_1 \setminus E_2$ are Jordan regions.

   (b) (3 points) Prove that if E_1, E_2 are nonoverlapping, then

   $$\operatorname{Vol}(E_1 \cup E_2) = \operatorname{Vol}(E_1) + \operatorname{Vol}(E_2).$$

   (c) (3 points) If $E_2 \subseteq E_1$, prove that

   $$\operatorname{Vol}(E_1 \setminus E_2) = \operatorname{Vol}(E_1) - \operatorname{Vol}(E_2).$$

   (d) (3 points) Prove that

   $$\operatorname{Vol}(E_1 \cup E_2) = \operatorname{Vol}(E_1) + \operatorname{Vol}(E_2) - \operatorname{Vol}(E_1 \cap E_2).$$