

# Homework 8 of Introduction to Analysis(II)

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April 14, 2024

1.  $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{x \cdot 0(x^2 - 0^2)/(x^2 + 0^2)}{x} = 0$  exists. Also, we can easily get that  $f_y(0,0) = 0$ . Then,  $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y \rightarrow 0} \frac{y(0^4 + 4 \cdot 0^2 y^2 - y^4)/(0^2 + y^2)^2}{y} = \lim_{y \rightarrow 0} \frac{-y^4}{y^4} = -1$ . And,  $f_{yx}(0,0) = 1$ . Therefore,  $f_{xy}(0,0) \neq f_{yx}(0,0)$ .

2. Since  $Df$  is continuous on  $S$ , for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $\|f(x) - f(y)\| < \frac{\varepsilon}{\|b-a\|}$  if  $\|x-y\| < \delta$ . And since  $S$  is a closed line in  $R^p$ , we can find a sequence  $\{x_k\}_{k=1}^n$  s.t.  $D(x_k, \frac{\delta}{2}) \supseteq S$  and  $\|x_{k+1} - a\| > \|x_k - a\|$ . Let  $x_0 = a$  and  $x_{n+1} = b$  and  $x_k = a + t_k(b-a)$ . Then, by MCT,  $Df(x_k) - Df(x_{k-1}) = Df(c_k)(x_k - x_{k-1})$  for  $c_k$  on the line between  $x_k, x_{k-1}$ . Thus,

$$\begin{aligned} |f(b) - f(a) - \int_0^1 Df(a+t(b-a))(b-a)dt| &\leq \sum_{k=1}^{n+1} \|f(x_k) - f(x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\ &= \sum_{k=1}^{n+1} \|Df(c_k)(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\ &= \sum_{k=1}^{n+1} \|\int_{t_{k-1}}^{t_k} Df(c_k)(b-a) - Df(a+t(b-a))(b-a)dt\| \\ &= \sum_{k=1}^{n+1} \frac{\varepsilon(t_k - t_{k-1})}{\|(b-a)\|} \\ &= \varepsilon \end{aligned}$$

Therefore,  $f(b) - f(a) = \int_0^1 Df(tb + (1-t)a)(b-a) dt$ .

$$3. \lim_{u \rightarrow 0} \frac{\|g(x+u) - g(x) - Dg(x)(u)\|}{\|u\|} = \lim_{u \rightarrow 0} \frac{\|g(x) + g(u) + B(x,u) + B(u,x) - g(x) - Dg(x)(u)\|}{\|u\|} = 0.$$

4. For any  $(x_0, y_0) \in \mathbb{R}^2$  and tiny  $h, k \in \mathbb{R}$ , since  $\frac{\partial^2 f}{\partial x \partial y}$  exists and continuous,  $\frac{\partial f}{\partial x}(x_0, y_0)$ ,  $\frac{\partial f}{\partial x}(x_0 + h, y_0)$ ,  $\frac{\partial f}{\partial x}(x_0, y_0 + k)$ ,  $\frac{\partial f}{\partial x}(x_0 + h, y_0 + k)$  are all exists.