Homework 12 of Introduction to Analysis(II)

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May 8, 2024

1. (a) Since E is Jordan region, $\text{Vol}(\partial E) = 0$.

   (b) Since $\text{cl}(E) = \text{int}(E) \cup \partial E$ and $\text{int}(E) \subseteq E \subseteq \text{cl}(E)$,

   $\text{Vol}(\text{cl}(E)) = \text{Vol}(\text{int}(E)) + \text{Vol}(\partial E) = \text{Vol}(\text{int}(E)) \leq \text{Vol}(E) \leq \text{Vol}(\text{cl}(E))$.

   Therefore, $\text{Vol}(\text{cl}(E)) = \text{Vol}(\text{int}(E)) = E$.

   (c)

   $(\implies)$ From (b), we know $\text{Vol}(\text{int}(E)) = \text{Vol}(E) > 0$, then we can find a set of rectangles R_n s.t. $\sum |R_n| > 0$ and $\cup R_n \subseteq \text{int}(E)$. Therefore, $\text{int}(E) \neq \emptyset$.

   $(\impliedby)$ Since $\text{int}(E)$ is non-empty, for any $x_0 \in \text{int}(E)$, there exists $\varepsilon > 0$ s.t. $D(x_0, \varepsilon) \subseteq \text{int}(E)$. Then, we can find a small rectangle R with each length is $\dfrac{\varepsilon}{2}$ and R is contained in $D(x_0, \varepsilon)$. Thus, $\text{Vol}(\text{int}(E)) > \left(\dfrac{\varepsilon}{2}\right)^2 > 0$.

   (d) Since f is continuous, for any $x_0 \in [a,b]$, we can find a sequence $x_k \to x_0$ s.t. $f(x_k) \to f(x_0)$. That is, $A = \{(x, f(x)) \mid x \in [a,b]\}$ is closed. And since $\partial A \subseteq A$, $\text{Vol}(\partial A) \leq \text{Vol}(A)$.

   Since f is continuous on $[a,b]$ is compact, we can find $M \in \mathbb{R}$ s.t. for all $\delta > 0$, $|x - y| < \delta$ implies that $|f(x) - f(y)| < M \cdot \delta$. Then, let $\varepsilon = M \cdot \delta$ find a finite sequence $\{x_i \mid x_i \in [a,b]\}_{i=1}^{N}$ s.t. $[a,b] \subseteq D(x_i, \delta)$. Therefore, for any $y = f(x)$, $y \in D(f(x_i), \varepsilon)$ for some i. Thus, $A = \{(x, f(x)) \mid x \in [a,b]\} \subseteq \displaystyle\bigcup_{i=1}^{N} D(x_i, \delta) \times D(f(x_i), \varepsilon)$ and $|D(x_i, \delta) \times D(f(x_i), \varepsilon)| = \delta \cdot \varepsilon < M \cdot \delta^2$. Since δ is