

# Homework 5 of Introduction to Analysis(II)

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1. (a) If  $x \in A$ ,  $d(x, A) = \|x - x\| = 0$  and  $d(x, B) = k > 0$ ,  $\phi(x) = \frac{0}{0+k} = 0$ . If  $x \in B$ ,  $d(x, A) = l > 0$  and  $d(x, B) = 0$ ,  $\phi(x) = \frac{l}{l+0} = 1$ . If  $x \notin A \wedge x \notin B$ ,  $d(x, A) = l$  and  $d(x, B) = k$ ,  $\phi(x) = \frac{l}{l+k} < 1$  and is positive. Thus,  $0 \leq \phi(x) \leq 1$  for all  $x \in \mathbb{R}^n$ .

- (b) Let  $\phi(x) = (b-a) \frac{d(x, A)}{d(x, A) + d(x, B)} + a$ . From (a), we can get  $\phi(x \in A) = (b-a) \cdot 0 + a = a$ ,  $\phi(x \in B) = (b-a) \cdot 1 + a = b$ , and  $a \leq \phi(x) \leq b$  for all  $x \in A$ .

2. If  $f$  has more than one fixed point, there exists  $x, y \in S$  s.t.  $d(f^n(x), f^n(y)) = d(x, y)$  for all  $n \in \mathbb{N}$  (contradiction to  $a_n \rightarrow 0$ ). Then, we want to show that  $f$  has fixed point. For any  $x_0 \in S$ , we let  $x_k = f^k(x)$  and we can find a  $N \in \mathbb{N}$  s.t.  $a_n < \varepsilon$  for all  $n > N$ . Then,  $d(x_{n+k}, x_n) \leq a_n d(x_k, x_0) < \varepsilon \cdot d(x_k, x_0)$  for any  $k$  and  $n > N$ . Thus,  $x_n \rightarrow x^* \in S$  by  $S$  is complete, and  $x^*$  is a fixed point of  $f$ .

3. Since  $T(u)(t) = \int_a^t u(s) ds$ ,

$$\begin{aligned} T^2(u)(t) &= T(T(u))(t) \\ &= \int_a^t T(u)(s) ds \\ &= \int_a^t \int_a^s u(x) dx ds \\ &= \int_a^t (t-s)u(s)ds \end{aligned}$$