Homework 8 of Introduction to Analysis(II)

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1. $f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0)}{x} = \lim_{x\to 0} \frac{x\cdot 0(x^2-0^2)/(x^2+0^2)}{x} = 0$ exists. Also, we can esaily get that $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y\to 0} \frac{y(0^4+4\cdot 0^2y^2-y^4)/(0^2+y^2)^2}{y} = \lim_{y\to 0} \frac{-y^4}{y^4} = -1$. And, $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

2. Since Df is continuous on S, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x)-f(y)\| < \frac{\varepsilon}{\|b-a\|}$ if $\|x-y\| < \delta$. And since S is a closed line in R^p, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k, \frac{\delta}{2}) \supseteq S$ and $\|x_{k+1}-a\| > \|x_k - a\|$. Let $x_0 = a$ and $x_{n+1} = b$ and $x_k = a + t_k(b-a)$. Then, by MVT, $f(x_k) - f(x_{k-1}) = Df(c_k)(x_k - x_{k-1})$ for c_k on the line between x_k, x_{k-1}. Thus,

$$
\begin{aligned}
|f(b)-f(a)-\int_0^1 Df(a+t(b-a))(b-a)dt| &\leq \sum_{k=1}^{n+1} \|f(x_k)-f(x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \|Df(c_k)(x_k - x_{k-1}) \\
&\quad - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)\, dt\| \\
&= \sum_{k=1}^{n+1} \| \int_{t_{k-1}}^{t_k} Df(c_k)(b-a) - Df(a+t(b-a))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \varepsilon(t_k - t_{k-1}) \\
&= \varepsilon
\end{aligned}
$$

Therefore, $f(b)-f(a) = \int_0^1 Df(tb+(1-t)a)(b-a)\, dt$.

3. Since B is bilinear, $g(x+u) = B(x+u,x+u) = B(x,x+u)+B(u,x+u) = B(x,x)+B(x,u)+B(u,x)+B(u,u) = g(x)+g(u)+(B(u,x)+B(x,u))$. And since B is bilinear, $B(0,0) = 0$ and $g(0) = 0$.

   For any $x,u \in \mathbb{R}^p$ and any $\varepsilon > 0$, we can find an $c \in \mathbb{R}$ s.t. $\|g(x+u') - g(x) - Dg(x)(u')\| \leq \varepsilon\|u'\|$ where $u = cu'$. Thus,

$$
\begin{aligned}
\|B(x,u)+B(u,x)-Dg(x)(u)\| &= \|cB(x,u')+cB(u',x)-cDg(x)(u')\| \\
&\leq c(\|g(x)+g(u')+B(x,u')+B(u',x)-g(x)-Dg(x)(u')\|+\|g(u')\|) \\
&= c(\|g(x+u')-g(x)-Dg(x)(u')\|+\|g(u')\|) \\
&\leq c(\varepsilon\|u'\|+\|g(u')\|) \\
&= \varepsilon\|u\|+\|B(u,u')\|
\end{aligned}
$$

4. For any $x_0, y_0 \in \mathbb{R}$, $f(x_0,y_0), f_x(x_0,y_0), f_y(x_0,y_0), f_{xy}(x_0,y_0)$ are all continuous. Then,

$$
\begin{aligned}
\lim_{h\to 0} \frac{f_y(x_0+h,y_0)-f_y(x_0,y_0)}{h} &= \lim_{h,k\to 0} \frac{1}{h} \frac{f(x_0+h,y_0+k)-f(x_0+h,y_0)}{k} - \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k} \\
&= \lim_{h,k\to 0} \frac{1}{k} \frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h} - \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \\
&= \lim_{k\to 0} \frac{f_x(x_0,y_0+k)-f_x(x_0,y_0)}{k} \\
&= f_{xy}(x_0,y_0)
\end{aligned}
$$

   Thus, f_{yx} exists and $f_{yx} = f_{xy}$.