

# Homework 10 of Introduction to Analysis(II)

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1. (a)  $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} 1 + 2h \sin\left(\frac{1}{h}\right) = 1.$   
(b)  $\lim_{x \rightarrow 0} f(x) = \lim_{h \rightarrow \infty} \frac{1}{h} + 2 \frac{\sin(h)}{h^2} = 0.$  Thus,  $f$  is continuous on 0. And for any  $x$  close to 0,  $|f(x+h) - f(x)| \leq |h| + 2|(x+h)^2| + 2|x^2| \rightarrow 0$  as  $h, x \rightarrow 0$ ,  $f$  is continuous on a small interval  $I_1$ .  
Since  $f'(x) = 1 + 2(2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}))$ , there exists  $x$  in any interval contains 0 s.t.  $f'(x) < 0$ .  
Therefore,  $f$  is neither increasing nor decreasing and not one to one in any interval contains 0.  
Thus,  $f$  is not invertible near 0.  
(c) This is not contradict to inverse function theorem.

2.

$$\begin{aligned} \|f(x_1) - f(x_2) - (x_1 - x_2)\| &= \|x_1 + g(x_1) - (x_2 + g(x_2)) - (x_1 - x_2)\| \\ &= \|g(x_1) - g(x_2)\| \\ &\leq a\|x_1 - x_2\| \end{aligned}$$

Then, for any  $x_1, x_2 \in \mathbb{R}^n$  and  $x_1 \neq x_2$ ,  $\|f(x_1) - f(x_2)\| \geq \|f(x_1) - f(x_2) - (x_1 - x_2)\| + \|x_1 - x_2\| \geq (1 - a)\|x_1 - x_2\| > 0$ . Thus,  $f$  is one-to-one.

And we can get  $g$  is a contraction mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , we can find an unique fixed point  $x^*$  s.t.

$g(x^*) = x^*$  and  $f(x^*) = 2x^*$ . Then, for any  $y \in \mathbb{R}^n$ ,

3. Since  $\|f(x) - f(y)\| \geq C\|x - y\|$  and  $C > 0$ ,