Homework 11 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

May 3, 2024

1. For $f \in \mathscr{C}^1(E, \mathbb{R}^n)$ with E is open subset of $\mathbb{R}^n$ and $Jf(x_0) \neq 0$ for some $x_0 \in E$ and $y_0 = f(x_0)$. Then, let $h(x,y) = f(x) - y \in \mathscr{C}^1(E \times \mathbb{R}^n \to \mathbb{R}^n)$ and $h(x_0, y_0) = f(x_0) - y_0 = y_0 - y_0 = 0$. We also can get $\dfrac{\partial h}{\partial x} = Df$. Then, by Implicit Function Theorem, there exists open set $W \subseteq \mathbb{R}^n$ with $y_0 \in W$, and unique $g \in \mathscr{C}^1(W, \mathbb{R}^n)$ with $g(y_0) = x_0$ s.t. $0 = h(g(y), y) = f(g(y)) - y \implies y = f(g(y))$ for all $y \in W$ and g is locally inverse function of f. And we can get $[Df^{-1}(y)] = [Dg(y)] = -[Df(g(y))]^{-1}[Dg(y)]_y = [Df(g(y))]^{-1} = [Df(f^{-1}(y))]^{-1}$ by Implicit Function Theorem, too.

2. Suppose $f \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R})$, $Df(x_0, y_0) \neq 0$ for some x_0, y_0 (or f is constant function and not one-to-one). Then, suppose $\dfrac{\partial f}{\partial x} \neq 0$ for neighborhood of (x_0, y_0), and let $h(x,y) = f(x,y) - f(x_0, y_0)$ with $\dfrac{\partial h}{\partial x} \neq 0$. by Implicit Function Theorem, there is a neighborhood $U \subseteq \mathbb{R}^2$ and $W \subseteq \mathbb{R}$ s.t. $(x_0, y_0) \in U$ and $y_0 \in W$ and a function $g : W \to \mathbb{R}^2$ s.t. $h(g(y), y) = f(g(y), y) - f(x_0, y_0) = 0$. Then, $f(g(y), y) = f(x_0, y_0)$ for $y \in W$ and f is not one-to-one. If $\dfrac{\partial f}{\partial x} = 0$, then $\dfrac{\partial f}{\partial y} \neq 0$ and use the same argument can get the same result.

3. (a) $\nabla f(x,y,z) = (y, x, 0)$ And $\nabla g_1(x,y,z) = (2x, 2y, 2z)$, $\nabla g_2(x,y,z) = (1,1,1)$.

Thus, we have

$$\begin{cases} y &= 2ax+b \\ x &= 2ay+b \\ 0 &= 2az+b \\ 1 &= x^2+y^2+z^2 \\ 0 &= x+y+z \end{cases}$$

Then, $a = \dfrac{-1}{2}$, $b = 0$, $(x,y,z) = (\pm\dfrac{1}{\sqrt{2}}, \mp\dfrac{1}{\sqrt{2}}, 0)$.

(b) $\nabla f(x,y,z,w) = (3,1,0,1)$, $\nabla g_1(x,y,z,w) = (6x,1,12z^2,0)$ and $\nabla g_2(x,y,z,w) = (-3x^2,0,12z^3,1)$.

Then, we have

$$\begin{cases} 3 &= 6cx-3dx^2 \\ 1 &= c \\ 0 &= 12cz^2+12dz^3 \\ 1 &= d \\ 1 &= 3x^2+y+4z^3 \\ 0 &= -x^3+3z^4+w \end{cases}$$

Thus, $c = 1, d = 1$ and $(x,y,z,w) = (1,2,-1,-2)$ or $(1,-2,0,1)$.

4. We show that for any $\varepsilon > 0$, we can not find $\phi : D(0,\varepsilon) \to \mathbb{R}$ s.t. $F(,\phi(x)) = (\phi(x))^2 - x$ for all $x \in D(0,\varepsilon)$.

$$\phi(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$