Homework 4 of Introduction to Analysis(II)

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1. Since B_x are bounded on $\mathbb{R}^n$, B_x is compact for all $x \in A$. Thus, B is pointwise compact.

   And since $\mathbb{R}^n$ is complete, B is complete, too. Thus, B is closed.

   Then, by Arzela-Ascoli Theorem, B is compact. Therefore, B is sequentially compact.

2. First, we want to show that B is closed. For any sequence $f_k \in B$ which converges to f, since $f_k(0) = 0$ for all k, we can get $f(0) = 0$. Then, assume there exists an $x_0, x_1 \in (0,1)$ s.t. $\dfrac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} = \alpha > 1$. Take $\varepsilon = \dfrac{\alpha - 1}{3}$, there exists $N \in \mathbb{N}$ s.t. $|f(x) - f_k(x)| < \varepsilon$ for all x and $k > N$. Thus, $|\dfrac{f_k(x_0) - f_k(x_1)}{x_0 - x_1}| = \dfrac{|f_k(x_0) - f(x_0) + f(x_0) - f(x_1) + f(x_1) - f_k(x_1)|}{|x_0 - x_1|} > \alpha - \dfrac{2\varepsilon}{|x_0 - x_1|} > 1$ which causes contradiction to $|f_k'(x)| \leq 1$ for all $x \in (0,1)$. Thus, $\dfrac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} \leq 1 \implies f \in B$. Hence, B is closed.

   Then, since $|f'(x)| \leq 1$ for all $f \in B$ and $x \in (0,1)$. For any $\varepsilon > 0$, we take $\delta < \varepsilon$, for any $x, y \in [0,1]$ s.t. $|x - y| < \varepsilon$, $|f(x) - f(y)| \leq 1 \cdot |x - y| = \delta < \varepsilon$. Thus, B is equicontinuous.

   Last, for any $x \in [0,1]$, we want to proof B_x is compact. For $x = 0$, $B_0 = \{0\}$ obviously compact. For $x > 0$, since $|f'| \leq 1$, we can easily get $B_x = [-x, x]$ by $f(x) = ax$ for all $|a| \leq 1$. Thus, B is pointwise compact.

   Therefore, by Arzela-Ascoli Theorem, B is compact.

3. For any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x-y\| < \delta \implies \|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3}$. And since K is compact, we can find finite $p \in N$ and $\{x_k\}_{k=1}^p$ s.t. $K \subseteq \bigcup_{k=1}^{p} D(x_k, \delta)$. Also, for all x_k, we can find a $N_{x_k} \in \mathbb{N}$ s.t. $|f_n(x_k) - f_m(x_k)| < \frac{\varepsilon}{3}$ for all $n, m > N_{x_k}$.

   Then, we take $N = \max_{x \in [1,p]} \{N_{x_k}\}$, and for any $x \in K$, we can find a $k \in [1, p]$ s.t. $x \in D(x_k, \delta)$. Thus,

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\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}
$$

   Thus, by Cauchy Criterion, $f_k \to f$ uniformly.

4. Since $\mathfrak{F}$ is equicontinuous, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x-y\| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ for all $f \in \mathfrak{F}$. For all $x, y \in D$ and $\|x-y\| < \delta$, we want to show $|f^*(x) - f^*(y)| < \varepsilon$. First, if $f^*(x) > f^*(y)$, we can find a $f \in \mathfrak{F}$ s.t. $f^*(x) - f(x) < \frac{\varepsilon}{2}$. Then, $f^*(y) > f^*(x) - \frac{\varepsilon}{2} - |f(x) - f(y)| > f^*(x) - \varepsilon$. For the other case, just change x, y and get the same result. Thus, f^* is continuous.