Homework 14 of Introduction to Analysis(II)

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1. Let $A_M = \{x \in E \mid f_M \text{ is discontinuous at } x\}$ and $A = \{x \in E \mid f \text{ is discontinuous at } x\}$. For any $M \in \mathbb{N}$, if x is a point that f_M is discontinuous at x, That means $f(x) \leq M$ and f is discontinuous at x. That is, $A_M \subseteq A$ for all M implies that $\cup A_m \subseteq A$. And for any $x \in A$, there exists a $N \in \mathbb{N}$ s.t. $f(x) < N$. Then, $x \in A_N$. Therefore, $A = \cup A_M$.

2. First, want to proof E contains finite union of intervals. Since f is integrable, we can find some $x \in E$ and $U \subseteq E$ for U is neighborhood of x. That is, we can find union of open intervals $\cup_N I_n$ in E with $|I_1| \geq |I_2| \geq \cdots$, and we let $L_N = \sum_{n=1}^{N} |I_n|$.

   We want to show that there exists finite N_0 s.t. $L_{N_0} \geq \frac{\alpha}{4M}$. Since there exists a partition P s.t. $\int_0^1 f(x)\,dx - L(f,P) \leq \frac{\alpha}{4}$ implies that $L(f,P) \geq \frac{3\alpha}{4}$, Then, if $\sup L_n \leq \frac{\alpha}{4M}$,

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\begin{aligned}
L(f,P) &\leq (1 - \sup L_n) \cdot \frac{\alpha}{2} + \sup L_n \cdot M \\
&\leq (1 - \frac{\alpha}{8M})\frac{\alpha}{2} + \frac{\alpha}{4} \\
&= \frac{3}{4}\alpha - \frac{\alpha^2}{8M}
\end{aligned}
$$

   Since $M > 0$ and $\alpha > 0$, it cause contradiction to $L(f,P) \geq \frac{3\alpha}{4}$. Thus, $\sup L_n > \frac{\alpha}{4M}$ and we can find N_0 such that $L_{N_0} \geq \frac{\alpha}{4M}$.

3. Let $A = \{x \in E \mid f(x) \neq 0\}$. If A is empty, then A is measure zero.

   Suppose A is non-empty. Then, for a large enough $N \in \mathbb{N}$, $A_N = \{x \in E \mid f(x) > \frac{1}{N}\}$. Using the same argument of 1. , we can have $A_N \to A$ as $N \to \infty$. Since $\int_E f(x)\,dx = 0$, $\int_{A_N} f(x)\,dx = 0$. Thus, for any $\varepsilon > 0$, there exists rectangles such that $\frac{1}{N}\sum|R_i| \leq (L)\int_{A_N} f(x)\,dx \leq \frac{\varepsilon}{N}$. Therefore, $\sum|R_i| < N \cdot \frac{\varepsilon}{N} = \varepsilon$ and A_N is measure zero.

   By the theorem that countable set of measure zero is also measure zero, we can have A is measure zero.