Exercises(7)

April 2, 2024

1. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$G(x,y) = \begin{cases} (x^2+y^2)\sin 1/(x^2+y^2) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Show that G is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives $\frac{\partial G}{\partial x_1}$ and $\frac{\partial G}{\partial x_2}$ are not bounded (and hence not continuous) on a neighborhood of $(0,0)$.

2. Let $G : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x,y) = \begin{cases} x^2+y^2 & \text{if both x, y are rational,} \\ 0 & \text{otherwise .} \end{cases}$$

   Show that F is differentiable at the point $(0,0)$.

3. Show that every continuous real-valued function f on $[0,\pi]$ is the uniform limit of a sequence of functions of the form

$$x \to a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx.$$

4. Use Exercise 3 to show that every continuous real-valued function f on $[0,\pi]$ with $f(0) = f(\pi)$ is the uniform limit of a sequence of functions of the form

$$x \to b_0 + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx.$$

   (Hint: If $f(0) = f(\pi) = 0$, first approximate f by a function g vanishing on some intervals $[0,\delta]$ and $[\pi-\delta,\pi]$. Then consider $h(x) = g(x)/\sin x$ for $x \in (0,\pi)$, $h(x) = 0$ for $x = 0, \pi$.)