# Homework 10 of Introduction to Analysis(II)

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April 28, 2024

1. (a) $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} 1+2h\sin(\frac{1}{h}) = 1.$

   (b) $\lim_{x\to 0} f(x) = \lim_{h\to\infty} \frac{1}{h} + 2\frac{\sin(h)}{h^2} = 0$. Thus, $f$ is continuous on 0. And for any $x$ close to 0, $|f(x+h)-f(x)| \leq |h| + 2|(x+h)^2| + 2|x^2| \to 0$ as $h, x \to 0$, $f$ is continuous on a small interval $I_1$.

   Since $f'(x) = 1+2(2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, there exsits $x$ in any interval contains 0 s.t. $f'(x) < 0$. Therefore, $f$ is neither increasing nor decreasing and not one to one in any interval contains 0. Thus, $f$ is not invertible near 0.

   (c) This is not contradict to inverse function theorem.

2. 

$$\begin{aligned}\|f(x_1)-f(x_2)-(x_1-x_2)\| &= \|x_1+g(x_1)-(x_2+g(x_2))-(x_1-x_2)\| \\ &= \|g(x_1)-g(x_2)\| \\ &\leq a\|x_1-x_2\|\end{aligned}$$

   Then, for any $x_1, x_2 \in \mathbb{R}^n$ and $x_1 \neq x_2$, $\|f(x_1)-f(x_2)\| \geq \|f(x_1)-f(x_2)-(x_1-x_2)\| + \|x_1-x_2\| \geq (1-a)\|x_1-x_2\| > 0$. Thus, $f$ is one-to-one.

   Since $f$ is continuous, $f(\mathbb{R}^n)$ is closed. And for any $f(x)$, assume that for any $\varepsilon > 0, D(f(x),\varepsilon) \not\subseteq f(\mathbb{R}^n)$. Thus, $f(\mathbb{R}^n)$ is both open and closed, that is, $f(\mathbb{R}^n) = \mathbb{R}^n$ and $f$ is onto. Therefore, $f$ is bijection.

3. Since $\|f(x)-f(y)\| \geq C\|x-y\|$ and $C > 0$,