

Homework 10 of Introduction to Analysis(II)

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1. (a) $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} 1 + 2h \sin\left(\frac{1}{h}\right) = 1.$
- (b) $\lim_{x \rightarrow 0} f(x) = \lim_{h \rightarrow \infty} \frac{1}{h} + 2 \frac{\sin(h)}{h^2} = 0.$ Thus, f is continuous on 0. And for any x close to 0, $|f(x+h) - f(x)| \leq |h| + 2|(x+h)^2| + 2|x^2| \rightarrow 0$ as $h, x \rightarrow 0$, f is continuous on a small interval I_1 .
- Since $f'(x) = 1 + 2(2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, there exists x in any interval contains 0 s.t. $f'(x) < 0$.
- Therefore, f is neither increasing nor decreasing and not one to one in any interval contains 0.
- Thus, f is not invertible near 0.
- (c) This is not contradict to inverse function theorem.

2.

$$\begin{aligned} \|f(x_1) - f(x_2) - (x_1 - x_2)\| &= \|x_1 + g(x_1) - (x_2 + g(x_2)) - (x_1 - x_2)\| \\ &= \|g(x_1) - g(x_2)\| \\ &\leq a\|x_1 - x_2\| \end{aligned}$$

Then, for any $x_1, x_2 \in \mathbb{R}^n$ and $x_1 \neq x_2$, $\|f(x_1) - f(x_2)\| \geq \|f(x_1) - f(x_2) - (x_1 - x_2)\| + \|x_1 - x_2\| \geq (1 - a)\|x_1 - x_2\| > 0$. Thus, f is one-to-one.

Since f is continuous, $f(\mathbb{R}^n)$ is closed. And for any $f(x)$, Thus, $f(\mathbb{R}^n)$ is both open and closed, that is, $f(\mathbb{R}^n) = \mathbb{R}^n$ and f is onto. Therefore, f is bijection.

3. Since $\|f(x) - f(y)\| \geq C\|x - y\|$ and $C > 0$,