Homework 3 of Introduction to Analysis(II)

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1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \dfrac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t. $\dfrac{\sum_{i=1}^{N} a_i - a}{N'} < \dfrac{\varepsilon}{2}$. Thus, for any $n > N'$,

$$
\begin{aligned}
|\frac{\sum_{i=1}^{n} a_i}{n} - a| &\leq |\frac{\sum_{i=1}^{N} a_i - a}{n}| + |\frac{\sum_{i=N+1}^{n} a_i - a}{n - N'}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} b_n = a$.

2. Since $\lim_{n\to\infty} na_n = 0$, by 1., $\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \dfrac{na_n}{N} = 0$ Also, since $\lim_{n\to\infty} na_n = 0$, $\lim_{n\to\infty} a_n = 0$. Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=k+1}^{\infty} a_n| < \dfrac{\varepsilon}{3}$ for all $k > N$. Then, for $x \to 1^-$ s.t. $|f(x) - A| < \dfrac{\varepsilon}{3}$ and $|\sum_{n=1}^{k} a_n(1-x)| < \dfrac{\varepsilon}{3}$,

$$
\begin{aligned}
|\sum_{n=0}^{N} a_n - A| &= |\sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A)| \\
&\leq |\sum_{n=0}^{N} a_n(1-x)| + |\sum_{n=N+1}^{\infty} a_n x^n| + |f(x) - A| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}
$$

Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3. Since $\lim_{x\to 1^-}\sum_{n=1}^{\infty} a_n x^n = A$, for any $\varepsilon > 0$, we can find a $x_1 \in (0,1)$ s.t. $|\sum_{n=1}^{\infty} a_n x^n - A| < \frac{\varepsilon}{3}$ for all $x_1 < x < 1$. Then, there exists $N \in \mathbb{N}$ s.t. $|\sum_{n=N'+1}^{\infty} a_n x^n| < \frac{\varepsilon}{3}$ for all x and $N' > N$. And we have x_2 s.t. $|\sum_{n=1}^{N} a_n(1-x^n)| < \frac{\varepsilon}{3}$ for all $x_2 < x < 1$.

   Thus, for any $N' > N$ and $x > \max\{x_1, x_2\}$,

$$\begin{aligned}
|\sum_{n=0}^{N} a_n - A| &\le |\sum_{n=0}^{N'} a_n(1-x^n)| + |\sum_{n=N'+1}^{\infty} a_n x^n| + |\sum_{n=1}^{\infty} a_n x^n - A| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

   Thus, $\sum_{n=1}^{\infty} a_n = A$.

4.

($\Longrightarrow$) Since $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=N}^{2N-1} a_n \sin(nx)| < \varepsilon$ for all x. Let $x = \frac{1}{2N}$, then $\sin(nx) = \sin(\frac{n}{2N}) \in [\sin(\frac{1}{2}), 1)$. Thus, $\sum_{n=N}^{2N-1} a_n \sin(nx) > \sum_{n=N}^{2N-1} a_n \sin(\frac{1}{2}) \ge \sum_{n=N}^{2N-1} a_{2n} \sin(\frac{1}{2}) = \frac{2n}{2}\sin(\frac{1}{2}) a_{2n}$. Therefore, $2na_{2n} < \frac{2}{\sin(\frac{1}{2})}\varepsilon$. Using the similar way, we can get $(2n-1)a_{2n-1} \to 0$ as $n \to \infty$ also. Hence, we get $na_n \to 0$ as $n \to \infty$.

($\Longleftarrow$) Since $na_n \to 0$ as $n \to \infty$, we can find an $N \in \mathbb{N}$ s.t. $na_n < \frac{\varepsilon}{\pi}$ for all $n > N$. And since $|\sin(nx)|$ is periodic function, we only need to check the series converges on $[0, \pi]$. Then, we want to proof $|\sum_{k=n}^{n+p} a_k \sin(kx)|$ uniformly converges for $n > N$ and $p \in N$. We seperate the interval into $[0, \frac{\pi}{n+p}]$, $[\frac{\pi}{n+p}, \frac{\pi}{n}]$, $[\frac{\pi}{n}, \pi]$ three interval. First one,

$$\begin{aligned}
|\sum_{k=n}^{n+p} a_k \sin(kx)| &\le \sum_{k=n}^{n+p} a_k \cdot k \cdot x \\
&\le \frac{k=n}{n+p} k \cdot k \cdot \frac{\pi}{n+p} \\
&\le (p+1)\frac{\varepsilon}{\pi} \cdot \frac{\pi}{n+p} \\
&< \varepsilon.
\end{aligned}$$

And the second interval, let $m = [\frac{\pi}{x}]$. We want to show that for any $n \in \mathbb{N}$ and $x \in [\frac{\pi}{n+p}, \pi]$, $\sum_{k=1}^{n} a_k \sin(n_k) \leq \frac{1}{\sin(\frac{x}{2})}$. Since $2\sin(\frac{x}{2})\sin(kx) = \cos((k-\frac{1}{2})x) - \cos((k+\frac{1}{2})x)$,

$$\begin{aligned}
|\sum_{k=1}^{n} \sin(kx)| &= \frac{1}{2\sin(\frac{x}{2})}|\sum_{k=1}^{n} \cos((k-\frac{1}{2})x) - \cos((k+\frac{1}{2})x)| \\
&= \frac{1}{2\sin(\frac{x}{2})}|\cos(\frac{1}{2}x) - \cos((n-\frac{1}{2})x)| \\
&\leq \frac{1}{\sin(\frac{x}{2})}.
\end{aligned}$$

Then, we want to show $|\sum_{k=m+1}^{n+p} a_k \sin(kx)| \leq \frac{2a_{m+1}}{\sin(\frac{x}{2})}$. Using Abel's formula,

$$\begin{aligned}
|\sum_{k=m+1}^{n+p} a_k \sin(kx)| &\leq a_{n+p}|\sum_{k=m+1}^{n+p} sin(kx)| + \sum_{k=m+1}^{n+p-1} |\sum_{j=m+1}^{k} sin(jx)|(a_k - a_{k-1}) \\
&\leq a_{n+p}\frac{2}{\sin(\frac{x}{2})} + \frac{2}{\sin(\frac{x}{2})}\sum_{k=m+1}^{n+p-1}(a_k - a_{k-1}) \\
&= \frac{2}{\sin(\frac{x}{2})}(a_{n+p} + a_{m+1} - a_{n+p}) \\
&= \frac{2a_{m+1}}{\sin(\frac{x}{2})}
\end{aligned}$$

Thus,

$$\begin{aligned}
|\sum_{k=n}^{n+p} a_k \sin(kx)| &\leq \sum_{k=n}^{m} a_k \sin(kx) + |\sum_{k=m+1}^{n+p} a_k \sin(kx)| \\
&\leq \sum_{k=n}^{m} ka_k \cdot x + \frac{2a_{m+1}}{\sin(\frac{x}{2})} \\
&\leq (m-n+1)\frac{\varepsilon}{\pi}x + 2a_{m+1}\frac{\pi}{x} \\
&\leq m\frac{x}{\pi}\varepsilon + 2a_{m+1}(m+1) \\
&\leq \varepsilon + 2\frac{\varepsilon}{\pi}
\end{aligned}$$

And the third one,

$$\begin{aligned}
|\sum_{k=n}^{n+p} a_k \sin(kx)| &\leq \frac{2a_n}{\sin(\frac{x}{2})} \\
&\leq \frac{2\pi}{2x} a_n \\
&\leq n a_n \\
&< \frac{\varepsilon}{\pi}
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly.