Homework 9 of Introduction to Analysis(II)

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1. By observe, $\Delta(x) = \det(D^2 f(x)) \geq 0$.

   (a) Since $\Delta(x) \geq 0$ and $f_{xx}(x) > 0$, $D^2 f(x)$ is positive semi-definite on $D(c,\delta)$. Thus, for any $x \in D(c,\delta)$,

$$
\begin{aligned}
Df(x) &= Df(x) - Df(c) \\
&= \int_0^1 D^2 f(c+t(x-c))(x-c)\,dt \\
&\geq \int_0^1 0(x-c)\,dt \\
&\geq 0
\end{aligned}
$$

   Then,

$$
\begin{aligned}
f(x) - f(c) &= \int_0^1 Df(c+t(x-c))(x-c)\,dt \\
&\geq \int_0^1 0(x-c)\,dt \\
&\geq 0
\end{aligned}
$$

   Threrfore, c is a local minimun point of f.

   (b) Since $f_{xx}(x) < 0$ and $\Delta(x) \geq 0$, $D^f(x)$ is negative semi-definite. Using the same argument as (a), we can get $f(x) - f(c) \leq 0$ for all $x \in \|x-c\| < \delta$. Thus, c is local maximun point of f.

2. Since $f_x(x_0) = f_y(x_0) = 0$ and $f \in C^2(V, \mathbb{R})$, $Df(x_0) = 0$ and x_0 is a critical point.

   If $f_{xy}(x_0) \neq 0$, $\det(D^2 f(x_0)) = -(f_{xy}(x_0))^2 < 0$, then $D^2 f(x_0)$ is not positive definite and negative definite. Thus, x_0 will be saddle point.

3. Since $D^r f(x)$ is continuous, $M = \sup\{\|D^r f(x)\| \mid x \in E \text{ and in neighberhood of } x_0\}$. Thus, $\lim\limits_{h \to 0} R_{r-1}(x_0, h) = \lim\limits_{h \to 0} \dfrac{D^r f(x_0)(h, h, \cdots, h)}{r!\|h\|^{r-1}} \leq \lim\limits_{h \to 0} \dfrac{M \cdot \|h\|^r}{r!\|h\|^{r-1}} = \lim\limits_{h \to 0} \dfrac{M}{r!}\|h\| = 0.$