

# Homework 14 of Introduction to Analysis(II)

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1. Let  $A_M = \{x \in E \mid f_M \text{ is discontinuous at } x\}$  and  $A = \{x \in E \mid f \text{ is discontinuous at } x\}$ . For any  $M \in \mathbb{N}$ , if  $x$  is a point that  $f_M$  is discontinuous at  $x$ , That means  $f(x) \leq M$  and  $f$  is discontinuous at  $x$ . That is,  $A_M \subseteq A$  for all  $M$  implies that  $\cup A_m \subseteq A$ .

And for any  $x \in A$ , there exists a  $N \in \mathbb{N}$  s.t.  $f(x) < N$ . Then,  $x \in A_N$ . Therefore,  $A = \cup A_M$ .

2. Let  $\int_0^1 f(x) dx = \alpha$  and  $\sup_{x \in [0,1]} f(x) = M$ .

By 1. ,

Since  $f$  is integrable, there exists a partition  $P$  s.t.  $\int_0^1 f(x) dx - L(f, P) \leq \frac{\alpha}{4}$ .

3. Let  $A = \{x \in E \mid f(x) \neq 0\}$ . If  $A$  is empty, then  $A$  is measure zero.

Suppose  $A$  is non-empty. Then, for a large enough  $N \in \mathbb{N}$ ,  $A_N = \{x \in E \mid f(x) > \frac{1}{N}\}$ . Using the same argument of 1. , we can have  $A_N \rightarrow A$  as  $N \rightarrow \infty$ . Since  $\int_E f(x) dx = 0$ ,  $\int_{A_N} f(x) dx = 0$ . Thus, for any  $\varepsilon > 0$ , there exists rectangles such that  $\frac{1}{N} \sum |R_i| \leq (L) \int_{A_N} f(x) dx \leq \varepsilon$ . Thus,  $\sum |R_i| < N \cdot \varepsilon$ .