

## Homework 5 of Introduction to Analysis(II)

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1. (a) If  $x \in A$ ,  $d(x, A) = \|x - x\| = 0$  and  $d(x, B) = k > 0$ ,  $\phi(x) = \frac{0}{0+k} = 0$ . If  $x \in B$ ,  $d(x, A) = l > 0$  and  $d(x, B) = 0$ ,  $\phi(x) = \frac{l}{l+0} = 1$ . If  $x \notin A \wedge x \notin B$ ,  $d(x, A) = l$  and  $d(x, B) = k$ ,  $\phi(x) = \frac{l}{l+k} < 1$  and is positive.

And for  $x, y \in A$  and  $\varepsilon > 0$ , if  $\|x - y\| < \delta = \varepsilon$ ,  $|d(x, A) - d(y, A)| < \delta = \varepsilon$ . Thus,  $d(x, A)$  is continuous. Using the same way,  $d(x, B)$  is also continuous. Since  $\phi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$  is a continuous function divided by a continuous function which is greater than 0,  $\phi(x)$  is also continuous.

- (b) Let  $\phi(x) = (b - a) \frac{d(x, A)}{d(x, A) + d(x, B)} + a$ . From (a), we can get continuous function  $\phi(x \in A) = (b - a) \cdot 0 + a = a$ ,  $\phi(x \in B) = (b - a) \cdot 1 + a = b$ , and  $a \leq \phi(x) \leq b$  for all  $x \in A$ .

2. If  $f$  has more than one fixed point, there exists  $x, y \in S$  s.t.  $d(f^n(x), f^n(y)) = d(x, y)$  for all  $n \in \mathbb{N}$  (contradiction to  $a_n \rightarrow 0$ ). Then, we want to show that  $f$  has fixed point. For any  $x_0 \in S$ , we let  $x_k = f^k(x)$  and we can find a  $N \in \mathbb{N}$  s.t.  $a_n \leq \alpha < 1$  for all  $n > N$ . Then,  $d(x_{n+k}, x_n) \leq \alpha^{n-N} d(x_{N+k}, x_N) < \alpha^{n-N} \cdot a_N \cdot d(x_k, x_0) \leq \alpha^{n-N} \cdot (a_N \cdot \left\lceil \frac{k}{N} \right\rceil \cdot \max_{i \in [0, N]} \{d(x_i, x_0)\})$  for any  $k$  and  $n > N$ . Thus,  $x_n \rightarrow x^* \in S$  by  $S$  is complete, and  $x^*$  is a fixed point of  $f$ .

3. Since  $T(u)(t) = \int_a^t u(s) ds$ ,  $(T^m(u)(t))' = (T(T^{m-1}(u))(t))' = \left( \int_a^t T^{m-1}(u)(s) ds \right)' = T^{m-1}(u)(t)$

by FTC. And since  $T^m(u)(a) = 0$  for all  $m \in \mathbb{N}$ , we can get

$$\begin{aligned}
T^m(u)(t) &= T(T^{m-1}(u))(t) \\
&= \int_a^t T^{m-1}(u)(s) \, ds \\
&= - \int_a^t T^{m-1}(u)(s) \, d(t-s) \\
&= -(t-s)T^{m-1}(u)(s) \Big|_a^t + \int_a^t (t-s) \, d(T^{m-1}(u)(s)) \\
&= \int_a^t (t-s)T^{m-2}(u)(s) \\
&= \int_a^t T^{m-2}(u)(s) \, d\left(\frac{-(t-s)^2}{2}\right) \\
&= \int_a^t \frac{(t-s)^2}{2!} T^{m-3}(u)(s) \, ds \\
&\vdots \\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} \, d(T(u)(s)) \\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} u(s) \, ds
\end{aligned}$$

Thus,  $T^m(u) = \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} u(s) \, ds \leq \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} M \, ds \leq \int_a^t \frac{M \cdot (t-a)^{m-1}}{(m-1)!} \, ds$  with  $M$  is upper bound of  $u$ . Then, taking  $a_n = \frac{(t-a)^n}{(n-1)!} M$ . Therefore, by 2.,  $T$  has unique fixed point, and  $T(0) = 0$  trivially.

4. We want to proof  $T(f)(x) = \int_0^x f(t) \, dt$  has unique fixed point. Since  $f$  is on  $[0, 1]$ ,  $T(|f|)(x) \leq |f(x)|$  for all  $f$  and  $x \in [0, 1]$ . Thus, by 3.,  $T$  has unique fixed point and  $T(0) = 0$ . Therefore,  $\int_0^1 f(x)x^n \, dx = (n-1)!T^n(f)(1) = 0$  for all  $x$  implies that  $f(x) = 0$  in  $[0, 1]$ .