

Homework 5 of Introduction to Analysis(II)

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1. (a) If $x \in A$, $d(x, A) = \|x - x\| = 0$ and $d(x, B) = k > 0$, $\phi(x) = \frac{0}{0+k} = 0$. If $x \in B$, $d(x, A) = l > 0$ and $d(x, B) = 0$, $\phi(x) = \frac{l}{l+0} = 1$. If $x \notin A \wedge x \notin B$, $d(x, A) = l$ and $d(x, B) = k$, $\phi(x) = \frac{l}{l+k} < 1$ and is positive.

And for $x, y \in A$ and $\varepsilon > 0$, if $\|x - y\| < \delta = \varepsilon$, $|d(x, A) - d(y, A)| < \delta = \varepsilon$. Thus, $d(x, A)$ is continuous. Using the same way, $d(x, B)$ is also continuous. Since $\phi(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$ is a continuous function divided by a continuous function which is greater than 0, $\phi(x)$ is also continuous.

- (b) Let $\phi(x) = (b - a) \frac{d(x, A)}{d(x, A) + d(x, B)} + a$. From (a), we can get continuous function $\phi(x \in A) = (b - a) \cdot 0 + a = a$, $\phi(x \in B) = (b - a) \cdot 1 + a = b$, and $a \leq \phi(x) \leq b$ for all $x \in A$.

2. If f has more than one fixed point, there exists $x, y \in S$ s.t. $d(f^n(x), f^n(y)) = d(x, y)$ for all $n \in \mathbb{N}$ (contradiction to $a_n \rightarrow 0$). Then, we want to show that f has fixed point. For any $x_0 \in S$ and $\varepsilon > 0$, we let $x_k = f^k(x)$ and we can find a $N \in \mathbb{N}$ s.t. $a_n \leq \alpha < \frac{\varepsilon}{d(x_1, x_0)}$ for all $n > N$. Then, for any $m > n > N$,
$$d(x_m, x_n) \leq \sum_{k=0}^{m-n-1} d(x_{n+k+1}, x_{n+k}) \leq \sum_{k=0}^{m-n-1} a_{n+k} \cdot d(x_1, x_0).$$
 Thus, $x_n \rightarrow x^* \in S$ by S is complete, and x^* is a fixed point of f .

3. Since $T(u)(t) = \int_a^t u(s) ds$, $(T^m(u)(t))' = (T(T^{m-1}(u))(t))' = \left(\int_a^t T^{m-1}(u)(s) ds \right)' = T^{m-1}(u)(t)$

by the FTC. And since $T^m(u)(a) = 0$ for all $m \in \mathbb{N}$, we can get

$$\begin{aligned}
T^m(u)(t) &= T(T^{m-1}(u))(t) \\
&= \int_a^t T^{m-1}(u)(s) \, ds \\
&= - \int_a^t T^{m-1}(u)(s) \, d(t-s) \\
&= -(t-s)T^{m-1}(u)(s) \Big|_a^t + \int_a^t (t-s) \, d(T^{m-1}(u)(s)) \\
&= \int_a^t (t-s)T^{m-2}(u)(s) \, ds \\
&= \int_a^t T^{m-2}(u)(s) \, d\left(\frac{-(t-s)^2}{2}\right) \\
&= \int_a^t \frac{(t-s)^2}{2!} T^{m-3}(u)(s) \, ds \\
&\vdots \\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} \, d(T(u)(s)) \\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} u(s) \, ds
\end{aligned}$$

Thus, $T^m(u) = \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} u(s) \, ds \leq \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} M \, ds \leq \int_a^t \frac{M \cdot (t-a)^{m-1}}{(m-1)!} \, ds$ with M is upper bound of u . Then, taking $a_n = \frac{(t-a)^n}{(n-1)!} M$. Therefore, by 2., T has unique fixed point, and $T(0) = 0$ trivially.

4. We want to proof $T(f)(x) = \int_0^x f(t) \, dt$ has unique fixed point. Since f is on $[0, 1]$, $T(|f|)(x) \leq |f(x)|$ for all f and $x \in [0, 1]$. Thus, by 3., T has unique fixed point and $T(0) = 0$. Therefore, $\int_0^1 f(x)x^n \, dx = (n-1)!T^n(f)(1) = 0$ for all x implies that $f(x) = 0$ in $[0, 1]$.