Homework 2 of Introduction to Analysis(II)

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1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t. $\frac{\sum_{i=1}^{N} a_i - a}{N'} < \frac{\varepsilon}{2}$. Thus, for any $n > N'$,

$$\begin{aligned} \left|\frac{\sum_{i=1}^{n} a_i}{n} - a\right| &\leq \left|\frac{\sum_{i=1}^{N} a_i - a}{n}\right| + \left|\frac{\sum_{i=N+1}^{n} a_i - a}{n - N'}\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

   Therefore, $\lim_{n\to\infty} b_n = a$.

2. Since $\lim_{n\to\infty} na_n = 0$, by 1., $\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{na_n}{N} = 0$ Also, since $\lim_{n\to\infty} na_n = 0$, $\lim_{n\to\infty} a_n = 0$. Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=0}^{k} a_n| < \frac{\varepsilon}{2}$ for all $k > N$. Then, for $x \to 1^-$ s.t. $|f(x) - A| < \frac{\varepsilon}{2}$,

$$\begin{aligned} \sum_{n=0}^{N} a_n - A &= \sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A) \\ &\leq \frac{\varepsilon}{2} - 0 + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

   Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3. We have $\lim_{x \to 1^-} f(x) = A$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{N} a_n(1-x^n) - \sum_{n=N+1}^{\infty} a_n x^n + f(x)$$

$$\to 0 - 0 + A \text{ as } x \to 1^- \text{ and } N \to \infty$$

4.

$(\implies)$ Since $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=N+1}^{\infty} a_n \sin(nx)| < \varepsilon$ for all x. And since we can find x s.t. $\sin(nx) = 1$ for all x and all n, we can rewrite it as $\lim_{N \to \infty} \sum_{n=N+1}^{\infty} a_n| = \lim_{N \to \infty} \sum_{n=N+1}^{\infty} \frac{na_n}{n} = 0$. By 1., $\lim_{n \to \infty} na_n = 0$.

$(\impliedby)$ Since $na_n \to 0$, we have $\sum_{n=N+1}^{\infty} a_n \to 0$ as $N \to \infty$. And since $|\sin(nx)| \leq 1$ for all x, $|\sum_{n=N+1}^{infty} a_n sin(nx)| \leq \sum_{n=N+1}^{\infty} a_n \to 0$. By Cauchy Criterion, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly.