

## Homework 2 of Introduction to Analysis(II)

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1. For all  $\varepsilon > 0$ , we can find a  $N \in \mathbb{N}$  s.t.  $|a_n - a| < \frac{\varepsilon}{2}$  for all  $n > N$ . And we can find a  $N' > N$  s.t.

$$\frac{\sum_{i=1}^N a_i - a}{N'} < \frac{\varepsilon}{2}. \text{ Thus, for any } n > N',$$

$$\begin{aligned} \left| \frac{\sum_{i=1}^n a_i}{n} - a \right| &\leq \left| \frac{\sum_{i=1}^N a_i - a}{n} \right| + \left| \frac{\sum_{i=N+1}^n a_i - a}{n - N'} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} b_n = a$ .

2. Since  $\lim_{n \rightarrow \infty} na_n = 0$ , by 1.,  $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{na_n}{N} = 0$ . Also, since  $\lim_{n \rightarrow \infty} na_n = 0$ ,  $\lim_{n \rightarrow \infty} a_n = 0$ . Thus, for any  $\varepsilon > 0$ , there exists a  $N \in \mathbb{N}$  s.t.  $\left| \sum_{n=k+1}^{\infty} a_n \right| < \frac{\varepsilon}{3}$  for all  $k > N$ . Then, for  $x \rightarrow 1^-$  s.t.  $|f(x) - A| < \frac{\varepsilon}{3}$  and  $\left| \sum_{n=1}^k a_n(1-x) \right| < \frac{\varepsilon}{3}$ ,

$$\begin{aligned} \left| \sum_{n=0}^N a_n - A \right| &= \left| \sum_{n=0}^N a_n(1-x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A) \right| \\ &\leq \left| \sum_{n=0}^N a_n(1-x) \right| + \left| \sum_{n=N+1}^{\infty} a_n x^n \right| + |f(x) - A| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Therefore,  $\sum_{n=0}^{\infty} a_n = A$ .

3. Since  $\lim_{x \rightarrow 1^-} \sum_{n=1}^{\infty} a_n x^n = A$ , for any  $\varepsilon > 0$ , we can find a  $x_1 \in (0, 1)$  s.t.  $|\sum_{n=1}^{\infty} a_n x^n - A| < \frac{\varepsilon}{3}$  for all  $x_1 < x < 1$ . Then, there exists  $N \in \mathbb{N}$  s.t.  $|\sum_{n=N'+1}^{\infty} a_n x^n| < \frac{\varepsilon}{3}$  for all  $x$  and  $N' > N$ . And we have  $x_2$  s.t.  $|\sum_{n=1}^N a_n (1 - x^n)| < \frac{\varepsilon}{3}$  for all  $x_2 < x < 1$ .

Thus, for any  $N' > N$  and  $x > \max\{x_1, x_2\}$ ,

$$\begin{aligned} \left| \sum_{n=0}^N a_n - A \right| &\leq \left| \sum_{n=0}^{N'} a_n (1 - x^n) \right| + \left| \sum_{n=N'+1}^{\infty} a_n x^n \right| + \left| \sum_{n=1}^{\infty} a_n x^n - A \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} a_n = A$ .

4.

( $\implies$ ) Since  $\sum_{n=1}^{\infty} a_n \sin(nx)$  converges uniformly, there exists a  $N \in \mathbb{N}$  s.t.  $|\sum_{n=N+1}^{\infty} a_n \sin(nx)| < \varepsilon$  for all  $x$ . And assume we can find  $x$  s.t.  $\sin(nx) = 1$  for all  $n > N$ , we can rewrite it as  $\lim_{N \rightarrow \infty} \left| \sum_{n=N+1}^{\infty} a_n \right| =$

$$\lim_{N \rightarrow \infty} \sum_{n=N+1}^{\infty} \frac{na_n}{n} = 0. \text{ By 1., } \lim_{n \rightarrow \infty} na_n = 0.$$

( $\impliedby$ ) First, since  $\frac{1}{n} \rightarrow 0$  and  $\sum_{n=1}^m \sin(nx) = \frac{\sin(\frac{mx}{2}) \sin(\frac{(m+1)x}{2})}{\sin(\frac{x}{2})}$ . Since  $na_n \rightarrow 0$  and  $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$  is bdd (since it converges), by Dirichlet's test again,  $\sum_{n=1}^{\infty} na_n \frac{\sin(nx)}{n} = \sum_{n=1}^{\infty} a_n \sin(nx)$  converges.