

Homework 11 of Introduction to Analysis(II)

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1. For $f \in \mathcal{C}^1(E, \mathbb{R}^n)$ with E is open subset of \mathbb{R}^n and $Jf(x_0) \neq 0$ for some $x_0 \in E$ and $y_0 = f(x_0)$. Then, let $h(x, y) = f(x) - y \in \mathcal{C}^1(E \times \mathbb{R}^n \rightarrow \mathbb{R}^n)$ and $h(x_0, y_0) = f(x_0) - y_0 = y_0 - y_0 = 0$. We also can get $\frac{\partial h}{\partial x} = Df$. Then, by Implicit Function Theorem, there exists open set $W \subseteq \mathbb{R}^n$ with $y_0 \in W$, and unique $g \in \mathcal{C}^1(W, \mathbb{R}^n)$ with $g(y_0) = x_0$ s.t. $0 = h(g(y), y) = f(g(y)) - y \implies y = f(g(y))$ for all $y \in W$ and g is locally inverse function of f . And we can get $[Df^{-1}(y)] = [Dg(y)] = -[Df(g(y))]^{-1}[Dg(y)]_y = [Df(g(y))]^{-1} = [Df(f^{-1}(y))]^{-1}$ by Implicit Function Theorem, too.
2. Suppose $f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$, $Df(x_0, y_0) \neq 0$ for some x_0, y_0 (or f is constant function and not one-to-one). Then, suppose $\frac{\partial f}{\partial x} \neq 0$ for neighborhood of (x_0, y_0) , and let $h(x, y) = f(x, y) - f(x_0, y_0)$ with $\frac{\partial h}{\partial x} \neq 0$. by Implicit Function Theorem, there is a neighborhood $U \subseteq \mathbb{R}^2$ and $W \subseteq \mathbb{R}$ s.t. $(x_0, y_0) \in U$ and $y_0 \in W$ and a function $g : W \rightarrow \mathbb{R}^2$ s.t. $h(g(y), y) = f(g(y), y) - f(x_0, y_0) = 0$. Then, $f(g(y), y) = f(x_0, y_0)$ for $y \in W$ and f is not one-to-one. If $\frac{\partial f}{\partial x} = 0$, then $\frac{\partial f}{\partial y} \neq 0$ and use the same argument can get the same result.
3. (a) $\nabla f(x, y, z) = (y, x, 0)$ And $\nabla g_1(x, y, z) = (2x, 2y, 2z)$, $\nabla g_2(x, y, z) = (1, 1, 1)$.

Thus, we have

$$\begin{cases} y &= 2ax + b \\ x &= 2ay + b \\ 0 &= 2az + b \\ 1 &= x^2 + y^2 + z^2 \\ 0 &= x + y + z \end{cases}$$

Then, $a = \frac{-1}{2}$, $b = 0$, $(x, y, z) = (\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}, 0)$.

(b) $\nabla f(x, y, z, w) = (3, 1, 0, 1)$, $\nabla g_1(x, y, z, w) = (6x, 1, 12z^2, 0)$ and $\nabla g_2(x, y, z, w) = (-3x^2, 0, 12z^3, 1)$.

Then, we have

$$\begin{cases} 3 &= 6cx - 3dx^2 \\ 1 &= c \\ 0 &= 12cz^2 + 12dz^3 \\ 1 &= d \\ 1 &= 3x^2 + y + 4z^3 \\ 0 &= -x^3 + 3z^4 + w \end{cases}$$

Thus, $c = 1, d = 1$ and $(x, y, z, w) = (1, 2, -1, -2)$ or $(1, -2, 0, 1)$.