

Homework 10 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

April 30, 2024

1. (a) $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} 1 + 2h \sin\left(\frac{1}{h}\right) = 1.$
- (b) $\lim_{x \rightarrow 0} f(x) = \lim_{h \rightarrow \infty} \frac{1}{h} + 2 \frac{\sin(h)}{h^2} = 0.$ Thus, f is continuous on 0. And for any x close to 0, $|f(x+h) - f(x)| \leq |h| + 2|(x+h)^2| + 2|x^2| \rightarrow 0$ as $h, x \rightarrow 0$, f is continuous on a small interval I_1 .
- Since $f'(x) = 1 + 2(2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, there exists x in any interval contains 0 s.t. $f'(x) < 0$. Therefore, f is neither increasing nor decreasing and not one to one in any interval contains 0. Thus, f is not invertible near 0.
- (c) This is not contradict to inverse function theorem since f' is not continuous.

2.

$$\begin{aligned} \|f(x_1) - f(x_2) - (x_1 - x_2)\| &= \|x_1 + g(x_1) - (x_2 + g(x_2)) - (x_1 - x_2)\| \\ &= \|g(x_1) - g(x_2)\| \\ &\leq a\|x_1 - x_2\| \end{aligned}$$

And for any $x \in \mathbb{R}^n$, $\|Df(x)\| = \|I + Dg(x)\| \geq (1 - a)$, then $\|f(x) - f(y)\| = \|x - y\| - \|f(x) - f(y) - (x - y)\| \geq (1 - a)\|x - y\| > 0$ for $x, y \in \mathbb{R}^n$, $x \neq y$ and c on the line between x, y . Thus, $f(x) \neq f(y)$ implies that f is one to one.

Let $h_y(x) = y - g(x)$, then $\|h_y(x_1) - h_y(x_2)\| = \|g(x_1) - g(x_2)\| \leq a\|x_1 - x_2\|$. Thus, h_y is contraction mapping and exists unique fixed point x^* s.t. $x^* = h_y(x^*) = y - g(x^*)$. Then, for any y , exists x^* s.t. $y = x^* + g(x^*) = f(x^*)$. Therefore, f is surjective. Hence, f is bijective.

3. Since $\|f(x) - f(y)\| \geq C\|x - y\|$ and $C > 0$, f is one to one.