Homework 10 of Introduction to Analysis(II)

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1. (a) $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} 1 + 2h\sin(\frac{1}{h}) = 1.$

   (b) $\lim_{x\to 0} f(x) = \lim_{h\to\infty} \frac{1}{h} + 2\frac{\sin(h)}{h^2} = 0$. Thus, f is continuous on 0. And for any x close to 0, $|f(x+h)-f(x)| \leq |h| + 2|(x+h)^2| + 2|x^2| \to 0$ as $h, x \to 0$, f is continuous on a small interval I_1.

   Since $f'(x) = 1 + 2(2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, there exsits x in any interval contains 0 s.t. $f'(x) < 0$. Therefore, f is neither increasing nor decreasing and not one to one in any interval contains 0. Thus, f is not invertible near 0.

   (c) This is not contradict to inverse function theorem.

2.

$$
\begin{aligned}
\|f(x_1)-f(x_2)-(x_1-x_2)\| &= \|x_1+g(x_1)-(x_2+g(x_2))-(x_1-x_2)\| \\
&= \|g(x_1)-g(x_2)\| \\
&\leq a\|x_1-x_2\|
\end{aligned}
$$

And for any $x \in \mathbb{R}^n$, $[Df(x)] = I + [Dg(x)]$, then $\|Df(x)(y)\| = \|y + Dg(x)(y)\| \geq (1-a)\|y\|$. Thus, $Jf(x) \neq 0$ for any $x \in \mathbb{R}^n$, by inverse function theorem,

3. Since $\|f(x)-f(y)\| \geq C\|x-y\|$ and $C > 0$,