

# Homework 4 of Introduction to Analysis(II)

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1. Since  $B_x$  are bounded on  $\mathbb{R}^n$ ,  $B_x$  is compact for all  $x \in A$ . Thus,  $B$  is pointwise compact.

And since  $\mathbb{R}^n$  is complete,  $B$  is complete, too. Thus,  $B$  is closed.

Then, by Arzela-Ascoli Theorem,  $B$  is compact. Therefore,  $B$  is sequentially compact.

2. First, we want to show that  $B$  is closed. For any sequence  $f_k \in B$  which converges to  $f$ , since  $f_k(0) = 0$  for all  $k$ , we can get  $f(0) = 0$ . Then, assume there exists an  $x_0, x_1 \in (0, 1)$  s.t.  $\frac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} = \alpha > 1$ . Take  $\varepsilon = \frac{\alpha - 1}{3}$ , there exists  $N \in \mathbb{N}$  s.t.  $|f(x) - f_k(x)| < \varepsilon$  for all  $x$  and  $k > N$ . Thus,  $|\frac{f_k(x_0) - f_k(x_1)}{x_0 - x_1}| = \frac{|f_k(x_0) - f(x_0) + f(x_0) - f(x_1) + f(x_1) - f_k(x_1)|}{|x_0 - x_1|} > \alpha - \frac{2\varepsilon}{|x_0 - x_1|} > 1$  which causes contradiction to  $|f'_k(x)| \leq 1$  for all  $x \in (0, 1)$ . Thus,  $\frac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} \leq 1 \implies f \in B$ . Hence,  $B$  is closed.

Then, since  $|f'(x)| \leq 1$  for all  $f \in B$  and  $x \in (0, 1)$ . For any  $\varepsilon > 0$ , we take  $\delta < \varepsilon$ , for any  $x, y \in [0, 1]$  s.t.  $|x - y| < \varepsilon$ ,  $|f(x) - f(y)| \leq 1 \cdot |x - y| = \delta < \varepsilon$ . Thus,  $B$  is equicontinuous.

Last, for any  $x \in [0, 1]$ , we want to proof  $B_x$  is compact. For  $x = 0$ ,  $B_0 = \{0\}$  obviously compact. For  $x > 0$ , since  $|f'| \leq 1$ , we can easily get  $B_x = [-x, x]$  by  $f(x) = ax$  for all  $|a| \leq 1$ . Thus,  $B$  is pointwise compact.

Therefore, by Arzela-Ascoli Theorem,  $B$  is compact.

3. For any  $\varepsilon > 0$ , we can find  $\delta > 0$  s.t.  $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$  for all  $f$  and  $x, y \in K$ . Since  $K$  is compact, we can find a increasing sequence  $\{x_k\}_{k=1}^n$  s.t.  $K \subset D(x_k, \frac{\delta}{2})$ . Then, we can find a  $N \in \mathbb{N}$

s.t.  $|f_k(x_1) - f(x_1)| < \varepsilon$  for all  $k > N$ . Thus, for any  $y \in K$ ,  $y \in x_l$  for some  $l$ ,

$$\begin{aligned} |f_k(y) - f(y)| &\leq |f_k(y) - f_k(x_l)| + |f_k(x_l) - f_k(x_{l-1})| + \cdots + |f_k(x_2) - f_k(x_1)| \\ &\quad + |f_k(x_1) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_l) - f(y)| \\ &< (2l + 1)\varepsilon \end{aligned}$$