Exercises(9)

April 16, 2024

1. Let $\Omega \subseteq \mathbb{R}^2$ be open, let $f \in C^2(\Omega, \mathbb{R})$, let $c \in \Omega$ be a critical point of f, and let
$$\Delta(x) = f_{xx}(x) f_{yy}(x) - (f_{xy}(x))^2$$
for $x \in \Omega$. Suppose that for some $\delta > 0$, $\Delta(x) \geq 0$ for all $\|x - c\| < \delta$.

   (a) (5 points) If $f_{xx}(x) > 0$ for all x such that $\|x - c\| < \delta$, show that c is a local minimum point of f.

   (b) (5 points) If $f_{xx}(x) < 0$ for all x such that $\|x - c\| < \delta$, show that c is a local maximum point of f.

2. (10 points) Suppose that V is open in $\mathbb{R}^2$, that $x_0 \in V$, and that $f \in C^2(V, \mathbb{R})$. Suppose that $f_x(x_0) = f_y(x_0) = f_{xx}(x_0) = f_{yy}(x_0) = 0$. Prove that x_0 is a saddle point if $f_{xy}(x_0) \neq 0$.

3. (10 points) Let $r \in \mathbb{N}$, E be open in $\mathbb{R}^n$, and suppose that $f \in C^r(E, \mathbb{R})$. Let $x_0, x \in E$ and suppose that the segment joining x and x_0 lies in E. By Tayolr's formula, we have
$$f(x) = \sum_{k=0}^{r-1} \frac{1}{k!} D^k f(x_0)(x - x_0, \ldots, x - x_0) + R_{r-1}(x_0, x - x_0),$$
where $R_{r-1}(x_0, h) = \frac{1}{r!} D^r f(c)(h, \ldots, h)$ and the point c is on that segment joining x and x_0. Show that
$$\frac{R_{r-1}(x_0, h)}{\|h\|^{r-1}} \quad \text{as } h \to 0.$$