Exercises(5)

March 19, 2024

1. Let A and B be two disjoint closed subsets of $\mathbb{R}^n$.

   (a) (7 points) For $x \in \mathbb{R}^n$, let $d(x, A) = \inf\{\|x - y\| : y \in A\}$ and $d(x, B) = \inf\{\|x - y\| : y \in B\}$. Show that $\phi(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$ is a continuous on $\mathbb{R}^n$ with

   $$\phi(x) = 0,\ x \in A;\ \ \phi(x) = 1,\ x \in B;\ \ 0 \leq \phi(x) \leq 1,\ x \in \mathbb{R}^n.$$

   (b) (5 points) Let a, b be two real numbers with $a < b$. Show that there exists a continuous function g defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ such that

   $$\phi(x) = a,\ x \in A;\ \ \phi(x) = b,\ x \in B;\ \ a \leq \phi(x) \leq b,\ x \in \mathbb{R}^n.$$

2. (5 points) Let $f : S \to S$ be a function from a complete metric space (S, d) into itself. Assume there is a real sequence $\{a_n\}$ which converges to 0 such that $d(f^n(x), f^n(y)) \leq a_n d(x, y)$ for all $n \geq 1$ and all x, y in S, where f^n is the nth iterate of f; this is,

   $$f^1(x) = f(x),\ f^{n+1}(x) = f(f^n(x)) \text{ for } n \geq 1.$$

   Prove that f has a unique fixed point.

3. (8 points) Let $\mathbb{M} = C([a, b], \mathbb{R})$. We define $T : \mathbb{M} \to \mathbb{M}$ by

   $$T(u)(t) = \int_a^t u(s)ds.$$

   Prove that T has the unique fixed point $u = 0$. (Hint:Use Exercise 2 and show that $T(T^{m-1})(u)(t) = T^m(u)(t) = \frac{1}{(m-1)!}\int_a^t (t-s)^{m-1}u(s)ds$.)

4. (5 points) If f is continuous on $[0, 1]$ and if

   $$\int_0^1 f(x)x^n dx = 0 \qquad (n = 0, 1, 2, \ldots),$$

   prove that $f(x) = 0$ on $[0, 1]$.