Exercises(10)

April 23, 2024

1. Let
$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
   (a) (4 points) Find $f'(0)$?

   (b) (4 points) Is f locally invertible near 0? Justify your answer. (Hint: If f is continuous and one-to-one in an interval I, then f is strictly monotonic in I. (By IVT.))

   (c) (4 points) Does this this result contradict the inverse function theorem? Why?

2. (8 points) Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and satisfy $\|[Dg(x)]\| \leq a < 1$ for all $x \in \mathbb{R}^n$. If $f(x) = x + g(x)$ for $x \in \mathbb{R}^n$, show that f satisfies
$$\|f(x_1) - f(x_2) - (x_1 - x_2)\| \leq a\|x_1 - x_2\|$$
for all $x_1, x_2 \in \mathbb{R}^n$ and that f is a bijection (one-to-one and onto) of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

3. (10 points) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Assume that f satisfies
$$\|f(x) - f(y)\| \geq C\|x - y\| \text{ for all } x, y \in \mathbb{R}^n,$$
where $C > 0$ is a constant. Prove that $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ exists, and it is continuous. (Hint: Show that $f(\mathbb{R}^n)$ is open and closed.) Must f^{-1} be differentiable? (Justify your answer!)