Homework 13 of Introduction to Analysis(II)

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May 21, 2024

1. (a) Since a rectangle R is compact, for all $\varepsilon > 0$, we can find finite open cubes C_i s.t. $\cup C_i \supseteq R$ and $\text{Vol}(\cup C_i) < \text{Vol}(R) + \varepsilon$. Therefore, for R_j, we can find $\text{Vol}(\cup C_{i,j}) < \text{Vol}(R_j) + \dfrac{\varepsilon}{2^j}$. And the total volume of $\cup_{i,j} C_{i,j} < \text{Vol}(\cup R_j) + \varepsilon < 2\varepsilon$. Thus, we can find a countable set of open cubes covers a measure zero set.

   (b) Since for any E_i is measure zero, we can find $\{R_{i,j}\}_j$ s.t. $\displaystyle\sum_j |R_{i,j}| < \frac{\varepsilon}{2^i}$. Then, let $R_{1,1} = R_1$, $R_{1,2} = R_2$, $R_{2,1} = R_3$, $R_{1,3} = R_4$, $R_{2,2} = R_5$ etc. And we can get $\{R_k\}_k$ contains $\displaystyle\bigcup_i E_i$ and $\displaystyle\sum_k |R_k| < \varepsilon(\frac{1}{2} + \frac{1}{4} + \cdots) = \varepsilon$. Therefore, union of countable measure zero sets is measure zero.

   (c) Since A is compact and (a), we can find finite open cubes which covers A and volume of their union is less than ε. Thus, take union of closure on each open cube and we can find a rectangle covers A with volume less than ε. Therefore, volume of A is 0.

2. For any grid g, $U(1,g) = L(1,g)$. Then, 1 is integrable. And for any $\varepsilon > 0$, there exists rectangles R_i s.t. $|\sum |R_i| - \text{Vol}(E)| < \varepsilon$. Thus, $\displaystyle\int_E 1\, dE \le U(1,g) = \sum 1 \cdot |R_i| < \text{Vol}(E) + \varepsilon$. Also, $\displaystyle\int_E 1\, dE \ge \text{Vol}(E) - \varepsilon$. Therefore, $\displaystyle\int_E 1\, dE = \text{Vol}(E)$.

3. (a) If x is rational, $G(x,t) = \displaystyle\int_0^t 1\, dy = t$. IF x is irrational, $G(x,t) = \displaystyle\int_0^t 2y\, dy = t^2$.
   Since $t \in [0,1]$, $t^2 \le t$. Then, (U)$\displaystyle\int_0^1 G(x,t)\, dx = t$ and (L)$\displaystyle\int_0^1 G(x,t)\, dx = t^2$.
   Therefore, $\displaystyle\int_0^1 (\int_0^1 f(x,y)\, dy)\, dx = \int_0^1 G(x,1)\, dx = \int_0^1 1\, dx = 1$.

   (b) As $y \to 0$, (U)$\displaystyle\int_0^1 f(x,y)\, dx = 1$ but (L)$\displaystyle\int_0^1 f(x,y)\, dx = 2y \to 0$. Thus, $\displaystyle\int_Q f(X)\, dx$ doesn't exists.