Homework 3 of Introduction to Analysis(II)

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1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \dfrac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t.

$$\dfrac{\sum\limits_{i=1}^{N} a_i - a}{N'} < \dfrac{\varepsilon}{2}.$$ Thus, for any $n > N'$,

$$|\frac{\sum_{i=1}^{n} a_i}{n} - a| \leq |\frac{\sum_{i=1}^{N} a_i - a}{n}| + |\frac{\sum_{i=N+1}^{n} a_i - a}{n - N'}|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

Therefore, $\lim\limits_{n \to \infty} b_n = a$.

2. Since $\lim\limits_{n \to \infty} n a_n = 0$, by 1., $\lim\limits_{N \to \infty} \sum\limits_{n=0}^{N} a_n = \lim\limits_{N \to \infty} \sum\limits_{n=0}^{N} \dfrac{n a_n}{N} = 0$ Also, since $\lim\limits_{n \to \infty} n a_n = 0$, $\lim\limits_{n \to \infty} a_n = 0$. Thus, for

any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\sum\limits_{n=k+1}^{\infty} a_n| < \dfrac{\varepsilon}{3}$ for all $k > N$. Then, for $x \to 1^-$ s.t. $|f(x) - A| < \dfrac{\varepsilon}{3}$

and $|\sum\limits_{n=1}^{k} a_n(1-x)| < \dfrac{\varepsilon}{3}$,

$$|\sum_{n=0}^{N} a_n - A| = |\sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A)|$$
$$\leq |\sum_{n=0}^{N} a_n(1 - x)| + |\sum_{n=N+1}^{\infty} a_n x^n| + |f(x) - A|$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Therefore, $\sum\limits_{n=0}^{\infty} a_n = A$.

3. Since $\displaystyle\lim_{x\to 1^-}\sum_{n=1}^{\infty} a_n x^n = A$, for any $\varepsilon > 0$, we can find a $x_1 \in (0,1)$ s.t. $\left|\sum_{n=1}^{\infty} a_n x^n - A\right| < \dfrac{\varepsilon}{3}$ for all $x_1 <$

   $x < 1$. Then, there exists $N \in \mathbb{N}$ s.t. $\left|\sum_{n=N'+1}^{\infty} a_n x^n\right| < \dfrac{\varepsilon}{3}$ for all x and $N' > N$. And we have x_2 s.t.

   $\left|\sum_{n=1}^{N} a_n(1-x^n)\right| < \dfrac{\varepsilon}{3}$ for all $x_2 < x < 1$.

   Thus, for any $N' > N$ and $x > \max\{x_1, x_2\}$,

   $$\left|\sum_{n=0}^{N} a_n - A\right| \le \left|\sum_{n=0}^{N'} a_n(1-x^n)\right| + \left|\sum_{n=N'+1}^{\infty} a_n x^n\right| + \left|\sum_{n=1}^{\infty} a_n x^n - A\right|$$

   $$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

   $$= \varepsilon$$

   Thus, $\displaystyle\sum_{n=1}^{\infty} a_n = A$.

4.

   ($\Longrightarrow$) Since $\displaystyle\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $\left|\sum_{n=N}^{2N-1} a_n \sin(nx)\right| < \varepsilon$ for all

   x. Let $x = \dfrac{1}{2N}$, then $\sin(nx) = \sin(\dfrac{n}{2N}) \in [\sin(\tfrac{1}{2}), 1)$. Thus, $\displaystyle\sum_{n=N}^{2N-1} a_n \sin(nx) > \sum_{n=N}^{2N-1} a_n \sin(\dfrac{1}{2}) \ge$

   $\displaystyle\sum_{n=N}^{2N-1} a_{2n} \sin(\dfrac{1}{2}) = \dfrac{2n}{2} \sin(\dfrac{1}{2}) a_{2n}$. Therefore, $2na_{2n} < \dfrac{2}{\sin(\tfrac{1}{2})} \varepsilon$. Using the similar way, we can get

   $(2n-1)a_{2n-1} \to 0$ as $n \to \infty$ also. Hence, we get $na_n \to 0$ as $n \to \infty$.

   ($\Longleftarrow$) Since $na_n \to 0$ as $n \to \infty$, we can find an $N \in \mathbb{N}$ s.t. $na_n < \dfrac{\varepsilon}{\pi}$ for all $n > N$. And since $|\sin(nx)|$

   is periodic function, we only need to check the series converges on $[0, \pi]$. Then, we want to

   proof $\left|\sum_{k=n}^{n+p} a_k \sin(kx)\right|$ uniformly converges for $n > N$ and $p \in \mathbb{N}$. We seperate the interval into

   $[0, \dfrac{\pi}{n+p}]$, $[\dfrac{\pi}{n+p}, \dfrac{\pi}{n}]$, $[\dfrac{\pi}{n}, \pi]$ three interval. First one,

   $$\left|\sum_{k=n}^{n+p} a_k \sin(kx)\right| \le \sum_{k=n}^{n+p} a_k \cdot k \cdot x$$

   $$\le \frac{k=n}{n+p} k \cdot k \cdot \frac{\pi}{n+p}$$

   $$\le (p+1)\frac{\varepsilon}{\pi} \cdot \frac{\pi}{n+p}$$

   $$< \varepsilon.$$

And the second interval, let $m = [\frac{\pi}{x}]$. We want to show that for any $n \in \mathbb{N}$ and $x \in [\frac{\pi}{n+p}, \pi]$,

$$\sum_{k=1}^{n} a_k \sin(n_k) \leq \frac{1}{\sin(\frac{x}{2})}. \text{ Since } 2\sin(\frac{x}{2})\sin(kx) = \cos((k-\frac{1}{2})x) - \cos((k+\frac{1}{2})x),$$

$$|\sum_{k=1}^{n} \sin(kx)| = \frac{1}{2\sin(\frac{x}{2})}|\sum_{k=1}^{n} \cos((k-\frac{1}{2})x) - \cos((k+\frac{1}{2})x)|$$

$$= \frac{1}{2\sin(\frac{x}{2})}|\cos(\frac{1}{2}x) - \cos((n-\frac{1}{2})x)|$$

$$\leq \frac{1}{\sin(\frac{x}{2})}.$$

Then, we want to show $|\sum_{k=m+1}^{n+p} a_k \sin(kx)| \leq \frac{2a_{m+1}}{\sin(\frac{x}{2})}$. Using Abel's formula,

$$|\sum_{k=m+1}^{n+p} a_k \sin(kx)| \leq a_{n+p}|\sum_{k=m+1}^{n+p} \sin(kx)| + \sum_{k=m+1}^{n+p-1} |\sum_{j=m+1}^{k} \sin(jx)|(a_k - a_{k-1})$$

$$\leq a_{n+p}\frac{2}{\sin(\frac{x}{2})} + \frac{2}{\sin(\frac{x}{2})}\sum_{k=m+1}^{n+p-1} (a_k - a_{k-1})$$

$$= \frac{2}{\sin(\frac{x}{2})}(a_{n+p} + a_{m+1} - a_{n+p})$$

$$= \frac{2a_{m+1}}{\sin(\frac{x}{2})}$$

Then,

$$|\sum_{k=n}^{n+p} a_k \sin(kx)| \leq \sum_{k=n}^{m} a_k \sin(kx) + |\sum_{k=m+1}^{n+p} a_k \sin(kx)|$$

$$\leq$$