Exercises(8)

April 9, 2024

1. (6 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   Show that the second partial derivatives f_{xy} and f_{yx} exist at $(0, 0)$ but are not equal.

2. (8 points) Let $A \subseteq \mathbb{R}^p$ be open and let $f : A \to \mathbb{R}^q$. Suppose that A contains the points a, b and the line segment S joining the points, and that f has continuous partial derivatives on S. Show that

$$f(b) - f(a) = \int_0^1 Df(a + t(b - a))(b - a)dt.$$

3. (8 points) Let $B : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ be bilinear (this is, B is a map that is linear in each variable separately) and let $g(x) = B(x, x)$ for all $x \in \mathbb{R}^p$. Show that if $x, u \in \mathbb{R}^p$, then $Dg(x)(u) = B(x, u) + B(u, x) = Dg(u)(x)$.

4. (8 points) Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$. Show that if $\partial^2 f/\partial x\partial y$ exists and is continuous, then $\partial^2 f/\partial y\partial x$ exists and $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.