

# Homework 8 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

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1.  $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{x \cdot 0(x^2 - 0^2)/(x^2 + 0^2)}{x} = 0$  exists. Also, we can easily get that  $f_y(0,0) = 0$ . Then,  $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y \rightarrow 0} \frac{y(0^4 + 4 \cdot 0^2 y^2 - y^4)/(0^2 + y^2)^2}{y} = \lim_{y \rightarrow 0} \frac{-y^4}{y^4} = -1$ . And,  $f_{yx}(0,0) = 1$ . Therefore,  $f_{xy}(0,0) \neq f_{yx}(0,0)$ .

2. Since  $Df$  is continuous on  $S$ , for any  $\varepsilon > 0$ , there exists  $\delta > 0$  s.t.  $\|f(x) - f(y)\| < \frac{\varepsilon}{\|b-a\|}$  if  $\|x - y\| < \delta$ . And since  $S$  is a closed line in  $R^p$ , we can find a sequence  $\{x_k\}_{k=1}^n$  s.t.  $D(x_k, \frac{\delta}{2}) \supseteq S$  and  $\|x_{k+1} - a\| > \|x_k - a\|$ . Let  $x_0 = a$  and  $x_{n+1} = b$  and  $x_k = a + t_k(b-a)$ . Then, by MVT,  $f(x_k) - f(x_{k-1}) = Df(c_k)(x_k - x_{k-1})$  for  $c_k$  on the line between  $x_k, x_{k-1}$ . Thus,

$$\begin{aligned} |f(b) - f(a) - \int_0^1 Df(a + t(b-a))(b-a) dt| &\leq \sum_{k=1}^{n+1} \|f(x_k) - f(x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a + t(b-a))(b-a) dt\| \\ &= \sum_{k=1}^{n+1} \|Df(c_k)(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a + t(b-a))(b-a) dt\| \\ &= \sum_{k=1}^{n+1} \|\int_{t_{k-1}}^{t_k} Df(c_k)(b-a) - Df(a + t(b-a))(b-a) dt\| \\ &= \sum_{k=1}^{n+1} \varepsilon(t_k - t_{k-1}) \\ &= \varepsilon \end{aligned}$$

Therefore,  $f(b) - f(a) = \int_0^1 Df(tb + (1-t)a)(b-a) dt$ .

3. Since  $B$  is bilinear,  $g(x+u) = B(x+u, x+u) = B(x, x+u) + B(u, x+u) = B(x, x) + B(x, u) + B(u, x) + B(u, u) = g(x) + g(u) + (B(u, x) + B(x, u))$ .

And since  $B$  is bilinear,  $B(0, 0) = 0$  and  $g(0) = 0$ .

For any  $x, u \in \mathbb{R}^p$ ,

$$\begin{aligned} \|B(x, u) + B(u, x) - Dg(x)(u)\| &= \|cB(x, u') + cB(u', x) - cDg(x)(u')\| \\ &\leq c(\|g(x) + g(u') + B(x, u') + B(u', x) - Dg(x)(u')\| + \|g(u')\|) \\ &\leq c(\varepsilon\|u'\| + \|g(u')\|) \\ &\leq \end{aligned}$$

4. For any  $x_0, y_0 \in \mathbb{R}$ ,  $f(x_0, y_0), f_x(x_0, y_0), f_y(x_0, y_0), f_{xy}(x_0, y_0)$  are all continuous. Then,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f_y(x_0 + h, y_0) - f_y(x_0, y_0)}{h} &= \lim_{h, k \rightarrow 0} \frac{1}{h} \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k} - \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} \\ &= \lim_{h, k \rightarrow 0} \frac{1}{k} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)}{h} - \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\ &= \lim_{k \rightarrow 0} \frac{f_x(x_0, y_0 + k) - f_x(x_0, y_0)}{k} \\ &= f_{xy}(x_0, y_0) \end{aligned}$$

Thus,  $f_{yx}$  exists and  $f_{yx} = f_{xy}$ .