Homework 1 of Introduction to Analysis(II)

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February 25, 2024

1. (a) For any $x > 0$, and for any $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $\varepsilon x > \frac{1}{N}$. Also, we can get $x > \frac{1}{\varepsilon N} > \frac{1}{N}$. Thus, $|g_k(x) - 0| = \frac{1}{kx} < \varepsilon$ for all $k > N$. And for $x = 0$, whatever k we take, $g_k(0) = n \cdot 0 = 0$. Therefore, for any $x > 0$, $\lim_{n\to\infty} g_n(x) = 0$.

   (b) Assume for any $0 < \varepsilon < 1$, exists $N \in \mathbb{N}$, we have $|g_k(x) - 0| < \varepsilon$ for all $x \geq 0$ with any $k \geq N$. Then, for g_N, we can find $x = \frac{1}{N}$ s.t. $g_N(x) = Nx = 1 > \varepsilon$(contradiction). Thus, $g_n(x)$ is not uniform convergence on $x \geq 0$.

   For $x \geq c > 0$ and any $\varepsilon > 0$, there exists $N_c \in \mathbb{N}$ s.t. $N_c \cdot c > \frac{1}{\varepsilon}$. Therefore, $|g_n(x) - 0| = \frac{1}{nx} < \frac{1}{nc} < \varepsilon$ for all $n > N_c$. Thus, $g_n(x)$ is uniform convergence on $x \geq c > 0$.

2. (a)

   ($\Longrightarrow$) Since $f_k \to f$ uniformly on E, for any $\varepsilon > 0$, exsits $N \in \mathbb{N}$ s.t. $d(f_k(x), f(x)) < \varepsilon$ for all $x \in E$ and $k > N$. Thus, we can get for all $\varepsilon > 0$, exists $N \in \mathbb{N}$ s.t. $\sup\{d(f_k(x), f(x)) \mid x \in E\} < \varepsilon$ for $k > N$. That means $\sup\{d(f_k(x), f(x)) \mid x \in E\} \to 0$ as $k \to \infty$.

   ($\Longleftarrow$) Since $\sup\{d(f_k(x), f(x)) \mid x \in E\} \to 0$ as $k \to \infty$, for any $\varepsilon > 0$, exists $N \in \mathbb{N}$ s.t. $\sup\{d(f_k(x), f(x)) \mid x \in E\} < \varepsilon$ for $k > N$. That means $d(f_k(x), f(x)) < \varepsilon$ for all x and $k > N$. Therefore, $f_k \to f$ uniformly.

(b)

($\Longrightarrow$) First, for any $\varepsilon > 0$ and a $x_0 \in E$, we take$x_k \in \{x \mid f_k(x) - f(x) > \varepsilon, x_0\}$ for any k. Sicne $f_k \nrightarrow f$ uniformly, we can find some $k > N$ for any $N \in \mathbb{N}$ s.t. $f_k(x) - f(x) > \varepsilon$. Thus, the sequence $\{x_k\}$ satisfies that $\limsup_{k\to\infty} d(f_k(x_k), f(x_k)) \geq \varepsilon > 0$.

($\Longleftarrow$) Since there exists a sequence $\{x_k\}$ in E s.t. $\limsup_{k\to\infty} d(f_k(x_k), f(x_k)) = \varepsilon > 0$, $\sup\{d(f_k(x), f(x)) \mid x \in E\} \nrightarrow 0$ as $k \to \infty$. Thus, by (a), we can get $f_k \nrightarrow f$.

(c) Let $f_k(x) = \frac{1}{k}e^{-k^2x^2}$ and $f(x) = 0$. First, we get $f_k'(x) = -2kxe^{-k^2x^2}$ and $f(x) > 0$ for all x. And it is positive for $x < 0$ and is negative for $x > 0$. Thus, the maximun of f_k occurs at $x = 0$.

Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Therefore, $|f_k(x) - f(x)| < \frac{1}{k} < \frac{1}{N} < \varepsilon$ for all $k > N$ and all $x \in \mathbb{N}$. Thus, $f_k \to f$ uniformly on $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$, and for any $\varepsilon > 0$, exists $N \in \mathbb{N}$ s.t. $2Nxe^{-N^2x^2} < \varepsilon$ since

$$\lim_{n\to\infty} ne^{-n^2x^2} = \lim_{n\to\infty} \frac{n}{e^{n^2x^2}} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{1}{2nx^2e^{n^2x^2}} = 0.$$

Thus, $|f_k'(x) - f'(x)| = |2kxe^{-k^2x^2} - 0| < |2Nxe^{-N^2x^2}| < \varepsilon$ for all $k > N$. Therefore, $f_k' \to f'$ pointwisely.

But for any interval contains 0, we can find $N \in \mathbb{N}$ s.t. $(\frac{-1}{N}, 0]$ or $[0, \frac{1}{N})$ lies in the interval. Suppose $[0, \frac{1}{N})$ lies in the interval. Then, let $f_k''(x) = 2ke^{-2k^2x^2}(2k^2x^2 - 1) = 0$, we can get $x = \frac{1}{\sqrt{2}k}$. Then, for $\varepsilon = \frac{1}{2}$, we can find $x = \frac{1}{\sqrt{2}k} \in (\frac{-1}{N}, \frac{1}{N})$ for all $k > \sqrt{2}N$. Thus, $|f_k'(x)| = 2\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} = \sqrt{\frac{2}{e}} > \frac{1}{2}$.

For the other case, we take $x = \frac{-1}{\sqrt{2}k}$ and the argument also right. Therefore, $f_k'(x) \nrightarrow f$ uniformly on any interval contains 0.