# Homework 4 of Introduction to Analysis(II)

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1. Since $B_x$ are bounded on $\mathbb{R}^n$, $B_x$ is compact for all $x \in A$. Thus, $B$ is pointwise compact.

   And since $\mathbb{R}^n$ is complete, $B$ is complete, too. Thus, $B$ is closed.

   Then, by Arzela-Ascoli Theorem, $B$ is compact. Therefore, $B$ is sequentially compact.

2. First, we want to show that $B$ is closed. For any sequence $f_k \in B$ which converges to $f$, since $f_k(0) = 0$ for all $k$, we can get $f(0) = 0$. Then, assume there exists an $x_0, x_1 \in (0,1)$ s.t. $\dfrac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} = \alpha > 1$. Take $\varepsilon = \dfrac{\alpha - 1}{3}$, there exists $N \in \mathbb{N}$ s.t. $|f(x) - f_k(x)| < \varepsilon$ for all $x$ and $k > N$. Thus, $|\dfrac{f_k(x_0) - f_k(x_1)}{x_0 - x_1}| = \dfrac{|f_k(x_0) - f(x_0) + f(x_0) - f(x_1) + f(x_1) - f_k(x_1)|}{|x_0 - x_1|} > \alpha - \dfrac{2\varepsilon}{|x_0 - x_1|} > 1$ which causes contradiction to $|f_k'(x)| \leq 1$ for all $x \in (0,1)$. Thus, $\dfrac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} \leq 1 \implies f \in B$. Hence, $B$ is closed.

   Then, since $|f'(x)| \leq 1$ for all $f \in B$ and $x \in (0,1)$. For any $\varepsilon > 0$, we take $\delta < \varepsilon$, for any $x, y \in [0,1]$ s.t. $|x - y| < \varepsilon$, $|f(x) - f(y)| \leq 1 \cdot |x - y| = \delta < \varepsilon$. Thus, $B$ is equicontinuous.

   Last, for any $x \in [0,1]$, we want to proof $B_x$ is compact. For $x = 0$, $B_0 = \{0\}$ obviously compact. For $x > 0$, since $|f'| \leq 1$, we can easily get $B_x = [-x, x]$ by $f(x) = ax$ for all $|a| \leq 1$. Thus, $B$ is pointwise compact.

   Therefore, by Arzela-Ascoli Theorem, $B$ is compact.

3. For any $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ for all $f$ and $x, y \in K$. Since $K$ is compact, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $K \subset D(x_k, \dfrac{\delta}{2})$. Then, we can find a $N \in \mathbb{N}$ s.t.

$|f_k(x_1) - f(x_1)| < \varepsilon$ for all $k > N$. Thus, for any $y \in K$, $y \in x_l$ for some $l$,

$$
\begin{aligned}
|f_k(y) - f(y)| &\leq |f_k(y) - f_k(x_l)| + |f_k(x_l) - f_k(x_{l-1})| + \cdot + |f_k(x_2) - f_k(x_1)| \\
&+ |f_k(x_1) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_l) - f(y)| \\
&< (2l+1)\varepsilon
\end{aligned}
$$

4. Since $\mathfrak{F}$ is equicontinuous, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x - y\| < \delta \implies |f(x) - f(y)| < \varepsilon$.

   Then, assume there exists $x \in D$ and $x_k \in D$ s.t. $x_k$ converges to $x$ and $|f^*(x) - f^*(x_k)| = \alpha > \varepsilon$ for all $k$(that means $f^*$ is discontinuous).

   Since there exists $N \in \mathbb{N}$ s.t. $\|x_k - x\| < \delta$ for all $k > N$, there exists $f_1 \in \mathfrak{F}$ s.t. $f^*(x) - f_1(x) < \dfrac{\alpha - \varepsilon}{2}$.