Homework 12 of Introduction to Analysis(II)

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1. (a) Since $\partial\,\mathrm{int}(E) = \mathrm{cl}(\mathrm{int}(E)) \setminus \mathrm{int}(\mathrm{int}(E)) = \mathrm{cl}(E) \setminus \mathrm{int}(E) = \partial E$ and $\partial\,\mathrm{cl}(E) = \partial E$, we can get $\mathrm{Vol}(\partial\,\mathrm{int}(E)) = \mathrm{Vol}(\partial\,\mathrm{cl}(E)) = \mathrm{Vol}(\partial E) = 0$ and $\mathrm{int}(E), \mathrm{cl}(E)$ are Jordan regions.

   (b) Since $\mathrm{cl}(E) = \mathrm{int}(E) \cup \partial E$ and $\mathrm{int}(E) \subseteq E \subseteq \mathrm{cl}(E)$,

   $\mathrm{Vol}(\mathrm{cl}(E)) \leq \mathrm{Vol}(\mathrm{int}(E)) + \mathrm{Vol}(\partial E) = \mathrm{Vol}(\mathrm{int}(E)) \leq \mathrm{Vol}(E) \leq \mathrm{Vol}(\mathrm{cl}(E))$.

   Therefore, $\mathrm{Vol}(\mathrm{cl}(E)) = \mathrm{Vol}(\mathrm{int}(E)) = E$.

   (c)

   ($\Longrightarrow$) From (b), we know $\mathrm{Vol}(\mathrm{int}(E)) = \mathrm{Vol}(E) > 0$, then we can find a set of rectangles R_n s.t. $\sum |R_n| > 0$ and $\cup R_n \subseteq \mathrm{int}(E)$. Therefore, $\mathrm{int}(E) \neq \emptyset$.

   ($\Longleftarrow$) Since $\mathrm{int}(E)$ is non-empty, for any $x_0 \in \mathrm{int}(E)$, there exists $\varepsilon > 0$ s.t. $D(x_0, \varepsilon) \subseteq \mathrm{int}(E)$. Then, we can find a small rectangle R with each length is $\dfrac{\varepsilon}{2}$ and R is contained in $D(x_0, \varepsilon)$. Thus, $\mathrm{Vol}(\mathrm{int}(E)) > \left(\dfrac{\varepsilon}{2}\right)^2 > 0$.

   (d) Since f is continuous, for any $x_0 \in [a,b]$, we can find a sequence $x_k \to x_0$ s.t. $f(x_k) \to f(x_0)$. That is, $A = \{(x, f(x)) \mid x \in [a,b]\}$ is closed. And since $\partial A \subseteq A$, $\mathrm{Vol}(\partial A) \leq \mathrm{Vol}(A)$.

   Since f is continuous on $[a,b]$ is compact, for all $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|x - y| < \delta$ implies That $|f(x) - f(y)| < \dfrac{\varepsilon}{b-a}$. Then, we can find a finite increasing sequence $\{x_i \mid x_i \in [a,b]\}_{i=1}^{N}$ s.t. $[a,b] \subseteq D(x_i, \dfrac{\delta}{2})$. Therefore, for any $y = f(x)$, $y \in D(f(x_i), \dfrac{\varepsilon}{b-a})$ for some i.

   Then, take $u_0 = a$, $u_i \in D(x_i, \dfrac{\delta}{2}) \cap D(x_{i+1}, \dfrac{\delta}{2})$, $u_N = b$, and we can get $[a,b] = \cup [u_i, u_{i+1}]$. Thus,

$A \subseteq \bigcup_{i=0}^{N} [u_i, u_{i+1}] \times D(\xi_i, \frac{\varepsilon}{b-a})$ for some $\xi_i \in \{f(x) \mid x \in [u_i, u_{i+1}]\}$ with the sum of the rectangles is $2\frac{\varepsilon}{b-a} \cdot (b-a) = 2\varepsilon$. Hence, $\mathrm{Vol}(A) = 0$ and $\mathrm{Vol}(\partial A) = 0$.

(e) Yes. Since f is integrable, for any $\varepsilon > 0$, we can find an partition P s.t. $|U(f,P) - L(f,P)| < \varepsilon$. That is, $\sum_{i=0}^{N} (\sup_{x \in [x_i, x_{i+1}]} \{f(x)\} - \inf_{x \in [x_i, x_{i+1}]} \{f(x)\}) \cdot (x_{i+1} - x_i) < \varepsilon$, and each one of the summation is a rectangle that contains all $(x, f(x))$ in the intevral. Thus, $\mathrm{Vol}(A) = 0$ and $\mathrm{Vol}(\partial A) = 0$.

2. (a) $\partial(E_1 \cap E_2) = \mathrm{cl}(E_1 \cap E_2) \cap \mathrm{cl}(M \setminus E_1 \cap E_2) \subseteq (\mathrm{cl}(E_1) \cap \mathrm{cl}(M \setminus E_1)) \cup (\mathrm{cl}(E_2) \cap \mathrm{cl}(M \setminus E_2)) = \partial E_1 \cup \partial E_2$. Then, $\mathrm{Vol}(\partial(E_1 \cap E_2)) \leq \mathrm{Vol}(\partial E_1) + \mathrm{Vol}(\partial E_2) = 0$.

   Also $\partial(E_1 \setminus E_2) \subseteq \partial E_1 \cup \partial E_2$. Therefore, $E_1 \cap E_2$ and $E_1 \setminus E_2$ are Jordan regions.

   (b) Since E_1, E_2 are non-overlapping, $E_1 \cap E_2 \subseteq \partial E_1 \cup \partial E_2$. And since E_1, E_2 are Jordan regions, $\mathrm{Vol}(E_1) = \mathrm{Vol}(\mathrm{int}(E_1))$ and $\mathrm{Vol}(E_2) = \mathrm{Vol}(\mathrm{int}(E_2))$. Also we have $\mathrm{int}(E_1) \cup \mathrm{int}(E_2) \subseteq E_1 \cup E_2$ and $\mathrm{int}(E_1) \cap \mathrm{int}(E_2) = \emptyset$.

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\begin{aligned}
\mathrm{Vol}(\mathrm{int}(E_1)) + \mathrm{Vol}(\mathrm{int}(E_1)) &= \mathrm{Vol}(\mathrm{int}(E_1) \cup \mathrm{int}(E_2)) \\
&\leq \mathrm{Vol}(\mathrm{int}(E_1) \cup \mathrm{int}(E_2)) + \mathrm{Vol}(E_1 \cap E_2) \\
&= \mathrm{Vol}(E_1 \cup E_2) \\
&\leq \mathrm{Vol}(\mathrm{int}(E_1)) + \mathrm{Vol}(\mathrm{int}(E_1)) + \mathrm{Vol}(\partial E_1) + \mathrm{Vol}(\partial E_2) \\
&= \mathrm{Vol}(\mathrm{int}(E_1)) + \mathrm{Vol}(\mathrm{int}(E_2)) \\
&= \mathrm{Vol}(E_1) + \mathrm{Vol}(E_2)
\end{aligned}
$$

   Thus, $\mathrm{Vol}(E_1 \cup E_2) = \mathrm{Vol}(E_1) + \mathrm{Vol}(E_2)$.

   (c) Let $E_1 \setminus E_1 = S_1$, $E_2 = S_2$. We have S_1 and S_2 are Jordan regions and non-overlapping. Thus, $\mathrm{Vol}(E_1) = \mathrm{Vol}(S_1 \cup S_2) = \mathrm{Vol}(S_1) + \mathrm{Vol}(S_1)$ and $\mathrm{Vol}(E_1 \setminus E_2) = \mathrm{Vol}(E_1) - \mathrm{Vol}(E_2)$.

   (d) Using (b), (c) and the fact $E_1 \cup E_2 = (E_1 \setminus E_1 \cap E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_2 \cap E_1)$, we can get $\mathrm{Vol}(E_1 \cup E_2) = (\mathrm{Vol}(E_1) - \mathrm{Vol}(E_1 \cap E_2)) + \mathrm{Vol}(E_1 \cap E_2) + (\mathrm{Vol}(E_2) - \mathrm{Vol}(E_1 \cap E_2)) = \mathrm{Vol}(E_1) + \mathrm{Vol}(E_2) - \mathrm{Vol}(E_1 \cap E_2)$.