Homework 5 of Introduction to Analysis(II)

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1. (a) If $x \in A$, $d(x,A) = \|x-x\| = 0$ and $d(x,B) = k > 0$, $\phi(x) = \dfrac{0}{0+k} = 0$. If $x \in B$, $d(x,A) = l > 0$ and $d(x,B) = 0$, $\phi(x) = \dfrac{l}{l+0} = 1$. If $x \notin A \wedge x \notin B$, $d(x,A) = l$ and $d(x,B) = k$, $\phi(x) = \dfrac{l}{l+k} < 1$ and is positive.

   And for $x, y \in A$ and $\varepsilon > 0$, if $\|x-y\| < \delta = \varepsilon$, $|d(x,A) - d(x,Y)| < \delta = \varepsilon$. Thus, $d(x,A)$ is continuous. Using the same way, $d(x,B)$ is also continuous. Since $\phi(x) = \dfrac{d(x,A)}{d(x,A)+d(x,B)}$ is a continuous function divided by a continuous function which is greater than 0, $\phi(x)$ is also continuous.

   (b) Let $\phi(x) = (b-a)\dfrac{d(x,A)}{d(x,A)+d(x,B)} + a$. From (a), we can get continuous function $\phi(x \in A) = (b-a)\cdot 0 + a = a$, $\phi(x \in B) = (b-a)\cdot 1 + a = b$, and $a \leq \phi(x) \leq b$ for all $x \in A$.

2. If f has more than one fixed point, there exists $x, y \in S$ s.t. $d(f^n(x), f^n(y)) = d(x,y)$ for all $n \in \mathbb{N}$(contradiction to $a_n \to 0$). Then, we want to show that f has fixed point. Since $a_n \to 0$, we can find an $N \in \mathbb{N}$ s.t. $d(f^N(x), f^N(y)) \leq ad(x,y)$ for $a < 1$. Thus, f^N is a contraction mapping. Take x^* be fixed point of f^N, that is, $f^N(x^*) = x^*$. Then, since $f^{N+1}(x^*) = f(f^N(x^*)) = f(x^*)$ and $f^{N+1} = f^N(f(x^*))$, $f(x^*) = f^N(f(x^*))$ is fixed point of f^N. Since f^N has unique fixed point, $f(x^*) = x^*$. Thus, f has unique fixed point.

3. Since $T(u)(t) = \displaystyle\int_a^t u(s)\, ds$, $(T^m(u)(t))' = \left(T(T^{m-1}(u))(t)\right)' = \left(\displaystyle\int_a^t T^{m-1}(u)(s)ds\right)' = T^{m-1}(u)(t)$

by the FTC. And since $T^m(u)(a) = 0$ for all $m \in \mathbb{N}$, we can get

$$\begin{aligned}
T^m(u)(t) &= T(T^{m-1}(u))(t) \\
&= \int_a^t T^{m-1}(u)(s)\, ds \\
&= -\int_a^t T^{m-1}(u)(s)\, d(t-s) \\
&= -(t-s)T^{m-1}(u)(s) \mid_a^t + \int_a^t (t-s)\, d(T^{m-1}(u)(s)) \\
&= \int_a^t (t-s)T^{m-2}(u)(s) \\
&= \int_a^t T^{m-2}(u)(s)\, d(\frac{-(t-s)^2}{2}) \\
&= \int_a^t \frac{(t-s)^2}{2!} T^{m-3}(u)(s)\, ds \\
&\vdots \\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!}\, d(T(u)(s)) \\
&= \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} u(s)\, ds
\end{aligned}$$

Thus, $T^m(u) = \dfrac{1}{(m-1)!}\displaystyle\int_a^t (t-s)^{m-1}u(s)ds \leq \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} M\, ds \leq \int_a^t \frac{M \cdot (t-a)^{m-1}}{(m-1)!}\, ds$ with M is upper bound of u. Then, taking $a_n = \dfrac{(b-a)^n}{(n-1)!}$, $d(T^n(u), T^n(v)) \leq a_n d(u,v)$ and $a_n \to 0$. Therefore, by 2., T has unique fixed point, and $T(0) = 0$ trivially.

4. Since there exists P_n is polynomial on $[0,1]$ with degree n and $P_n \to f$,

$$\begin{aligned}
\left|\int_0^1 (f(x))^2 dx\right| &\leq \left|\int_0^1 (f(x))^2 - f(x)P_n(x) + f(x)P_n(x)\, dx\right| \\
&\leq \max|f(x)| \cdot \int_0^1 f(x)P_n(x)\, dx \\
&= 0
\end{aligned}$$

Then, since $(f(x))^2 \geq 0$ for all x, we get $f(x) = 0$ for all $x \in [0,1]$.