# Homework 10 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

April 28, 2024

1. (a) $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} 1 + 2h\sin(\frac{1}{h}) = 1.$

   (b) $\lim_{x\to 0} f(x) = \lim_{h\to\infty} \frac{1}{h} + 2\frac{\sin(h)}{h^2} = 0$. Thus, $f$ is continuous on 0. And for any $x$ close to 0, $|f(x+h)-f(x)| \le |h| + 2|(x+h)^2| + 2|x^2| \to 0$ as $h, x \to 0$, $f$ is continuous on a small interval $I_1$.

   Since $f'(x) = 1 + 2(2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, there exsits $x$ in any interval contains 0 s.t. $f'(x) < 0$. Therefore, $f$ is neither increasing nor decreasing and not one to one in any interval contains 0. Thus, $f$ is not invertible near 0.

   (c) This is not contradict to inverse function theorem since $f'$ is not continuous.

2. 

$$\begin{aligned}\|f(x_1)-f(x_2)-(x_1-x_2)\| &= \|x_1+g(x_1)-(x_2+g(x_2))-(x_1-x_2)\| \\ &= \|g(x_1)-g(x_2)\| \\ &\le a\|x_1-x_2\|\end{aligned}$$

And for any $x \in \mathbb{R}^n$, $\|Df(x)\| = \|I + Dg(x)\| \ge (1-a)$, then $\|f(x)-f(y)\| = \|Df(c)\|\|x-y\| \ge (1-a)\|x-y\| > 0$ for $x, y \in \mathbb{R}^n$, $x \ne y$ and $c$ on the line between $x, y$. Thus, $f(x) \ne f(y)$ implies that $f$ is one to one.

3. Since $\|f(x)-f(y)\| \ge C\|x-y\|$ and $C > 0$,