Homework 10 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

April 27, 2024

1. (a) $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} 1 + 2h\sin(\frac{1}{h}) = 1.$

   (b) $\lim_{x \to 0} f(x) = \lim_{h \to \infty} \frac{1}{h} + 2\frac{\sin(h)}{h^2} = 0$. Thus, f is continuous on 0. And for any x close to 0, $|f(x+h) - f(x)| \leq |h| + 2|(x+h)^2| + 2|x^2| \to 0$ as $h, x \to 0$, f is continuous on a small interval I_1.

   Since $f'(x) = 1 + 2(2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, there exsits x in any interval contains 0 s.t. $f'(x) < 0$. Therefore, f is neither increasing nor decreasing and not one to one in any interval contains 0. Thus, f is not invertible near 0.

   (c) This is not contradict to inverse function theorem.

2.

$$
\begin{aligned}
\|f(x_1) - f(x_2) - (x_1 - x_2)\| &= \|x_1 + g(x_1) - (x_2 + g(x_2)) - (x_1 - x_2)\| \\
&= \|g(x_1) - g(x_2)\| \\
&\leq a\|x_1 - x_2\|
\end{aligned}
$$

And we can get g is a contraction mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, we can find an unique fixed point x^* s.t. $g(x^*) = x^*$ and $f(x^*) = 2x^*$.

Then, for any $x_1, x_2 \in \mathbb{R}^n$ and $x_1 \neq x_2$, $\|f(x_1) - f(x_2)\| \geq \|f(x_1) - f(x_2) - (x_1 - x_2)\| + \|x_1 - x_2\| \geq (1-a)\|x_1 - x_2\| > 0$. Thus, f is one-to-one.

3.