Homework 8 of Introduction to Analysis(II)

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1. $f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0)}{x} = \lim_{x\to 0} \frac{x\cdot 0(x^2-0^2)/(x^2+0^2)}{x} = 0$ exists. Also, we can esaily get that $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y\to 0} \frac{y(0^4+4\cdot 0^2y^2-y^4)/(0^2+y^2)^2}{y} = \lim_{y\to 0} \frac{-y^4}{y^4} = -1$. And, $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_y(0,0)$.

2. Since Df is continuous on S, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x)-f(y)\| < \frac{\varepsilon}{\|b-a\|}$ if $\|x-y\| < \delta$. And since S is a closed line in R^p, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k,\frac{\delta}{2}) \supseteq S$ and $\|x_k+1-a\| > \|x_k-a\|$. Let $x_0 = a$ and $x_{n+1} = b$. Then, by MCT, $Df(x_1)-Df(a) = Df(c_1)(x_1-a)$ for c_1 on the line between a, x_1 and c_k on the line between x_k, x_k-1.

   Thus,

$$
\begin{aligned}
|f(b)-f(a)-\int_0^1 Df(a+t(b-a))(b-a)\ dt| &\leq \sum_{k=1}^{n+1} \|f(x_k)-f(x_{k-1})-\int_0^1 Df(x_k+t(x_k-x_{k-1}))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \|\int_0^1 Df(c_k)(x_k-x_{k-1})\ dt \\
&- \int_0^1 Df(tx_k+(1-t)x_{k-1})(b-a)\ dt\| \\
&< \frac{\varepsilon}{b-a}(b-a) \\
&= \varepsilon
\end{aligned}
$$

   Therefore, $f(b) = f(a) = \int_0^1 Df(tb+(1-t)a)(b-a)\ dt$.

3. $\lim_{u\to 0} \frac{\|g(x+u)-g(x)-Dg(x)(u)\|}{\|u\|} = \lim_{u\to 0} \frac{\|g(x)+g(u)+B(x,u)+B(u,x)-g(x)-Dg(x)(u)\|}{\|u\|} = 0.$

4. Since $\dfrac{\partial^2 f}{\partial x \partial y}$ exists, for any $(x_0, y_0) \in \mathbb{R}^2$ and $h, k \in \mathbb{R}$,