Homework 11 of Introduction to Analysis(II)

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1. For $f \in \mathscr{C}^1(E, \mathbb{R}^n)$ with E is open subset of $\mathbb{R}^n$ and $Jf(x_0) \neq 0$ for some $x_0 \in E$ and $y_0 = f(x_0)$. Then, let $h(x,y) = f(x) - y \in \mathscr{C}^1(E \times \mathbb{R}^n \to \mathbb{R}^n)$ and $h(x_0,y_0) = f(x_0) - y_0 = y_0 - y_0 = 0$. We also can get $\dfrac{\partial h}{\partial x} = Df$. Then, by Implicit Function Theorem, there exists open sets $U \subseteq E \times \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$ with $(x_0,y_0) \in U$ and $y_0 \in W$, and unique $g \in \mathscr{C}^1(W, \mathbb{R}^n)$ with $g(y_0) = x_0$ s.t. $0 = h(g(y),y) = f(g(y)) - y \implies y = f(g(y))$ for all $y \in W$ and g is locally inverse function of f. And we can get $[Df^{-1}(y)] = [Dg(y)] = -[Df(g(y))]^{-1}[Dg(y)]_y = [Df(g(y))]^{-1} = [Df(f^{-1}(y))]^{-1}$ by Implicit Function Theorem, too.

2. Suppose $f \in \mathscr{C}^1(\mathbb{R}^2, \mathbb{R})$, $Df(x_0,y_0) \neq 0$ for some x_0, y_0 (or f is constant function and not one-to-one). Then, suppose $\dfrac{\partial f}{\partial x} \neq 0$ for neighborhood of (x_0,y_0), and let $h(x,y) = f(x,y) - f(x_0,y_0)$ with $\dfrac{\partial h}{\partial x} \neq 0$. by Implicit Function Theorem, there is a neighborhood $U \subseteq \mathbb{R}^2$ and $W \subseteq \mathbb{R}$ s.t. $(x_0,y_0) \in U$ and $y_0 \in W$ and a function $g : W \to \mathbb{R}^2$ s.t. $h(g(y),y) = f(g(y),y) - f(x_0,y_0) = 0$. Then, $f(g(y),y) = f(x_0,y_0)$ for $y \in W$ and f is not one-to-one. If $\dfrac{\partial f}{\partial x} = 0$, then $\dfrac{\partial f}{\partial y} \neq 0$ and use the same argument can get the same result.

3. (a) $\nabla f(x,y,z) = \begin{pmatrix} y_0 & x_0 & 0 \end{pmatrix}$.

   And $\nabla g_1(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix}$, $\nabla g_2(x,y,z) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$.