Homework 10 of Introduction to Analysis(II)

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1. (a) $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} 1+2h\sin(\frac{1}{h}) = 1.$

   (b) $\lim_{x\to 0} f(x) = \lim_{h\to\infty} \frac{1}{h} + 2\frac{\sin(h)}{h^2} = 0$. Thus, f is continuous on 0. And for any x close to 0, $|f(x+h)-f(x)| \leq |h|+2|(x+h)^2|+2|x^2| \to 0$ as $h,x\to 0$, f is continuous on a small interval I_1.

   Since $f'(x) = 1+2(2x\sin(\frac{1}{x})-\cos(\frac{1}{x}))$, there exsits x in any interval contains 0 s.t. $f'(x) < 0$. Therefore, f is neither increasing nor decreasing and not one to one in any interval contains 0. Thus, f is not invertible near 0.

   (c) This is not contradict to inverse function theorem since f' is not continuous.

2.

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\begin{aligned}
\|f(x_1)-f(x_2)-(x_1-x_2)\| &= \|x_1+g(x_1)-(x_2+g(x_2))-(x_1-x_2)\| \\
&= \|g(x_1)-g(x_2)\| \\
&\leq a\|x_1-x_2\|
\end{aligned}
$$

And for any $x\in\mathbb{R}^n$, $\|Df(x)\| = \|I+Dg(x)\| \geq (1-a)$, then $\|f(x)-f(y)\| = \|x-y\| - \|f(x)-f(y)-(x-y)\| \geq (1-a)\|x-y\| > 0$ for $x,y\in\mathbb{R}^n$, $x\neq y$ and c on the line between x,y. Thus, $f(x)\neq f(y)$ implies that f is one to one.

Let $h_y(x) = y-g(x)$, then $\|h_y(x_1)-h_y(x_2)\| = \|g(x_1)-g(x_2)\| \leq a\|x_1-x_2\|$. Thus, h_y is contraction mapping and exsits unique fixed point x^* s.t. $x^* = h_y(x^*) = y-g(x^*)$.

Then, for any y, exsits x^* s.t. $y = x^* + g(x^*) = f(x^*)$. Therefore, f is surjective. Hence, f is bijective.

3. Since $\|f(x) - f(y)\| \geq C\|x - y\|$ and $C > 0$, if $x \neq y$, $\|f(x) - f(y)\| \geq C\|x - y\| > 0$. Then, f is injective. And since f is continuous, $f(\mathbb{R}^n)$ is closed since $\mathbb{R}^n$ is closed.

   Now proof $f(\mathbb{R}^n)$ is open. For any x_0 and $\varepsilon \in (0, C)$, since f is differentiable, there exsits h s.t. $\|f(x_0 + h) - f(x_0) - Df(x_0)(h)\| < \varepsilon\|h\|$. Then, since $\|f(x_0 + h) - f(x_0)\| - \|Df(x_0)(h)\| \leq \|f(x_0 + h) - f(x_0) - Df(x_0)(h)\|$, $C\|h\| - \|Df(x_0)(h)\| < \varepsilon\|h\| < C\|h\|$. Thus, $\|Df(x_0)(h)\| > 0$. Therefore, by inverse function theorem, there exsits a neighborhood U near x_0 s.t. $f(U)$ is open. Then, $f(\mathbb{R}^n)$ is open.

   Therefore, f is bijective and then f is invertible. And by inverse function theorem, f^{-1} is differentiable and continuous on $\mathbb{R}^n$.