Homework 5 of Introduction to Analysis(II)

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March 20, 2024

1. (a) If $x \in A$, $d(x,A) = \|x-x\| = 0$ and $d(x,B) = k > 0$, $\phi(x) = \dfrac{0}{0+k} = 0$. If $x \in B$, $d(x,A) = l > 0$ and $d(x,B) = 0$, $\phi(x) = \dfrac{l}{l+0} = 1$. If $x \notin A \wedge x \notin B$, $d(x,A) = l$ and $d(x,B) = k$, $\phi(x) = \dfrac{l}{l+k} < 1$ and is positive. Thus, $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^n$.

   (b) Let $\phi(x) = (b-a)\dfrac{d(x,A)}{d(x,A)+d(x,B)} + a$. From (a), we can get $\phi(x \in A) = (b-a)\cdot 0 + a = a$, $\phi(x \in B) = (b-a)\cdot 1 + a = b$, and $a \leq phi(x) \leq b$ for all $x \in A$.

2. We let $x_n = f^n(x)$, $y_n = f^n(y)$. Since $d(x_n, y_n) = d(f(x_{n-1}), f(y_{n-1})) \leq a_1 d(x_{n_1}, y_{n-1})$ for all x, y and x_n, y_n in S. And a_n converges to 0, we can get an $a < 1$ s.t. $a_n \leq a^n$. Thus, by Contraction Mapping Principle, since $\sup\{\dfrac{d(f(x), f(y))}{d(x,y)}\} \leq a < 1$, f has unique fixed point.

3.