

Homework 8 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

April 13, 2024

1. $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{x \cdot 0(x^2 - 0^2)/(x^2 + 0^2)}{x} = 0$ exists. Also, we can easily get that $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y \rightarrow 0} \frac{y(0^4 + 4 \cdot 0^2 y^2 - y^4)/(0^2 + y^2)^2}{y} = \lim_{y \rightarrow 0} \frac{-y^4}{y^4} = -1$. And, $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

2. Since Df is continuous on S , for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x) - f(y)\| < \frac{\varepsilon}{\|b-a\|}$ if $\|x - y\| < \delta$. And since S is a closed line in R^p , we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k, \frac{\delta}{2}) \supseteq S$ and $\|x_k + 1 - a\| > \|x_k - a\|$. Let $x_0 = a$ and $x_{n+1} = b$. Then, by MCT, $Df(x_1) - Df(a) = Df(c_1)(x_1 - a)$ for c_1 on the line between a, x_1 and c_k on the line between x_k, x_{k+1} .

Thus,

$$\begin{aligned} \|f(b) - f(a) - \int_0^1 Df(tb + (1-t)a)(b-a) dt\| &\leq \sum_{k=1}^{n+1} \|f(x_k) - f(x_{k-1}) - \int_0^1 Df(tx_k + (1-t)x_{k-1})(b-a) dt\| \\ &= \sum_{k=1}^{n+1} \left\| \int_0^1 Df(c_k)(x_k - x_{k-1}) dt - \int_0^1 Df(tx_k + (1-t)x_{k-1})(b-a) dt \right\| \\ &< \frac{\varepsilon}{b-a} (b-a) \\ &= \varepsilon \end{aligned}$$

Therefore, $f(b) - f(a) = \int_0^1 Df(tb + (1-t)a)(b-a) dt$.

3. $\lim_{u \rightarrow 0} \frac{\|g(x+u) - g(x) - Dg(x)(u)\|}{\|u\|} = \lim_{u \rightarrow 0} \frac{\|g(x) + g(u) + B(x,u) + B(u,x) - g(x) - Dg(x)(u)\|}{\|u\|} = 0$.