Exercises(14)

May 21, 2024

1. (10 points) Assume that $f \geq 0$ on $E \subseteq \mathbb{R}^n$. For $M > 0$, let

$$f_M(x) = \begin{cases} f(x) & \text{if } f(x) \leq M, \\ M & \text{if } f(x) > M. \end{cases}$$

   Prove that

$$\{x \in E : f \text{ is discontinuous at } x\} = \bigcup_{M \in \mathbb{N}} \{x \in E : f_M \text{ is discontinuous at } x\}.$$

2. (10 points) Suppose that f is Riemann integrable, $f : [0,1] \to \mathbb{R}$, $|f(x)| \leq M$ for all $x \in [0,1]$, and $\int_0^1 f(x)dx \geq \alpha > 0$. Show that $E = \{x \in [0,1] : f(x) \geq \alpha/2\}$ contains a finite union of intervals of total length $\ell \geq \alpha/(4M)$. (Hint: let P be a partition of $[0,1]$ such that $0 \leq \int_0^1 f(x)dx - L(f,P) \leq \alpha/4$.)

3. (10 points) Suppose that E is a Jordan region in $\mathbb{R}^n$, and that $f : E \to [0,\infty)$ is integrable. If $\int_E f(x)dx = 0$, show that $\{x \in E : f(x) \neq 0\}$ is measure zero.