

Homework 3 of Introduction to Analysis(II)

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March 12, 2024

1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t.

$$\frac{\sum_{i=1}^N a_i - a}{N'} < \frac{\varepsilon}{2}. \text{ Thus, for any } n > N',$$

$$\begin{aligned} \left| \frac{\sum_{i=1}^n a_i}{n} - a \right| &\leq \left| \frac{\sum_{i=1}^N a_i - a}{n} \right| + \left| \frac{\sum_{i=N+1}^n a_i - a}{n - N'} \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} b_n = a$.

2. Since $\lim_{n \rightarrow \infty} na_n = 0$, by 1., $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{na_n}{N} = 0$. Also, since $\lim_{n \rightarrow \infty} na_n = 0$, $\lim_{n \rightarrow \infty} a_n = 0$. Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $\left| \sum_{n=k+1}^{\infty} a_n \right| < \frac{\varepsilon}{3}$ for all $k > N$. Then, for $x \rightarrow 1^-$ s.t. $|f(x) - A| < \frac{\varepsilon}{3}$ and $\left| \sum_{n=1}^k a_n(1-x) \right| < \frac{\varepsilon}{3}$,

$$\begin{aligned} \left| \sum_{n=0}^N a_n - A \right| &= \left| \sum_{n=0}^N a_n(1-x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A) \right| \\ &\leq \left| \sum_{n=0}^N a_n(1-x) \right| + \left| \sum_{n=N+1}^{\infty} a_n x^n \right| + |f(x) - A| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3. Since $\lim_{x \rightarrow 1^-} \sum_{n=1}^{\infty} a_n x^n = A$, for any $\varepsilon > 0$, we can find a $x_1 \in (0, 1)$ s.t. $|\sum_{n=1}^{\infty} a_n x^n - A| < \frac{\varepsilon}{3}$ for all $x_1 < x < 1$. Then, there exists $N \in \mathbb{N}$ s.t. $|\sum_{n=N'+1}^{\infty} a_n x^n| < \frac{\varepsilon}{3}$ for all x and $N' > N$. And we have x_2 s.t. $|\sum_{n=1}^N a_n (1 - x^n)| < \frac{\varepsilon}{3}$ for all $x_2 < x < 1$.

Thus, for any $N' > N$ and $x > \max\{x_1, x_2\}$,

$$\begin{aligned} \left| \sum_{n=0}^N a_n - A \right| &\leq \left| \sum_{n=0}^{N'} a_n (1 - x^n) \right| + \left| \sum_{n=N'+1}^{\infty} a_n x^n \right| + \left| \sum_{n=1}^{\infty} a_n x^n - A \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus, $\sum_{n=1}^{\infty} a_n = A$.

4.

(\implies) Since $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=N}^{2N-1} a_n \sin(nx)| < \varepsilon$ for all x . Let $x = \frac{1}{2N}$, then $\sin(nx) = \sin(\frac{n}{2N}) \in [\sin(\frac{1}{2}), 1)$. Thus, $\sum_{n=N}^{2N-1} a_n \sin(nx) > \sum_{n=N}^{2N-1} a_n \sin(\frac{1}{2}) \geq \sum_{n=N}^{2N-1} a_{2n} \sin(\frac{1}{2}) = \frac{2n}{2} \sin(\frac{1}{2}) a_{2n}$. Therefore, $2na_{2n} < \frac{2}{\sin(\frac{1}{2})} \varepsilon$. Using the similar way, we can get $(2n-1)a_{2n-1} \rightarrow 0$ as $n \rightarrow \infty$ also. Hence, we get $na_n \rightarrow 0$ as $n \rightarrow \infty$.

(\impliedby) Since $na_n \rightarrow 0$ as $n \rightarrow \infty$, we can find an $N \in \mathbb{N}$ s.t. $na_n < \frac{\varepsilon}{\pi}$ for all $n > N$. And since $|\sin(nx)|$ is periodic function, we only need to check the series converges on $[0, \pi]$. Then, we want to proof $|\sum_{k=n}^{n+p} a_k \sin(kx)|$ uniformly converges for $n > N$ and $p \in \mathbb{N}$. We separate the interval into $[0, \frac{\pi}{n+p}]$, $[\frac{\pi}{n+p}, \frac{\pi}{n}]$, $[\frac{\pi}{n}, \pi]$ three interval.

First one,

$$\begin{aligned} \left| \sum_{k=n}^{n+p} a_k \sin(kx) \right| &\leq \sum_{k=n}^{n+p} a_k \cdot k \cdot x \\ &\leq \frac{k=n}{n+p} k \cdot k \cdot \frac{\pi}{n+p} \\ &\leq \end{aligned}$$