Homework 6 of Introduction to Analysis(II)

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March 28, 2024

1. Let $\phi(x) = \arctan(x)$, then $\phi \circ f(x) = \phi(f(x))$ is bdd by $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then, by Tietze's Extension Theorem, there exists $g \in C(\mathbb{R}^n, \mathbb{R})$ s.t. $g(x) = \phi(f(x))$ for all $x \in D$ and $\sup|g(x)| = \sup|\phi(f(x))| \leq \frac{\pi}{2}$.

   Thus, there exists $h(x) = \tan(g(x))$ in $C(\mathbb{R}^n, \mathbb{R})$ and $h(x) = f(x)$ for all $x \in D$ by $\phi(x)$ is invertible.

2. Since $\frac{\partial f_i}{\partial x}$ exists for all x and all i, we get the Jacobian matrix

$$[J(x)] = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x) \\ \frac{\partial f_2}{\partial x}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x}(x) \end{bmatrix}.$$

   Then, we want to show that $J = Df$. Since $h \to 0$, $\frac{f_i}{\partial x}(x) * h = f_i(x+h) - f_i(x)$, then for any x,
   $\lim_{h\to 0} \frac{\|f(x+h) - f(x) - [J(x)]h\|}{|h|} = \lim_{h\to 0} \frac{\|(f_1(x+h) - f(x) - [J(x)]_i h,\ \cdots,\ f_m(x+h) - f_m(x) - [J(x)]_m h)\|}{|h|}$
   $= \frac{\|(0, 0, \cdots, 0)\|}{|h|} = 0$. Therefore, $Df = J$ exists.

3. Since the partial derivative are all bdd, we can find a M s.t. M is an upper bound of all partial derivative.

Then, for all $\varepsilon > 0$, take $\delta = \dfrac{\varepsilon}{n \cdot M}$. Thus, for any $\|x - y\| < \delta$,

$$
\begin{aligned}
\|f(x) - f(y)\| &= \|f((x_1, x_2, \cdots, x_n)) - f((y_1, y_2, \cdots, y_n))\| \\
&\leq \|f(x_1, x_2, \cdots, x_n) - f(y_1, x_2, \cdots, y_n)\| \\
&+ \|f(y_1, x_2, \cdots, y_n) - f(y_1, y_2, \cdots, y_n)\| \\
&+ \cdots \\
&+ \|f(y_1, y_2, \cdots, x_n) - f(y_1, y_2, \cdots, y_n)\| \\
&= \sum_{i=1}^{n} |x_i - y_i| \cdot \frac{\partial f}{\partial x_i} \\
&\leq n \cdot \delta \cdot M = \varepsilon
\end{aligned}
$$

Therefore, f is continuous.