Exercises(4)

March 12, 2024

1. (8 points) Let (M, d) be a metric spaces and $A \subseteq M$ be compact. Assume that $B \subseteq C(A, \mathbb{R}^n)$ is equicontinuous and pointwise bounded. Show that every sequence in B has a uniformly convergent subsequence.

2. (6 points) Let $B = \{f \in C([0,1], \mathbb{R}) : f'$ is continuous on $(0,1), f(0) = 0,$ and $|f'(x)| \leq 1\}$. Show that the closure of B is compact.

3. (8 points) Let $f_k : K \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a sequence of equicontinuous functions on a compact set K converging pointwise. Prove that the convergence is uniform.

4. (8 points) Let $\mathfrak{F}$ be a bounded and equicontinuous collection of functions on $D \subseteq \mathbb{R}^n$ to $\mathbb{R}$ and let f^* be defined on $D \to \mathbb{R}$ by

   $$f^*(x) = \sup\{f(x) : f \in \mathfrak{F}\}.$$

   Show that f^* is continuous on $D \to \mathbb{R}$.