Homework 1 of Introduction to Analysis(II)

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1. (a) For any $x > 0$, and for any $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $\varepsilon x > \frac{1}{N}$. Also, we can get $x > \frac{1}{\varepsilon N} > \frac{1}{N}$. Thus, $|g_k(x) - 0| = \frac{1}{kx} < \varepsilon$ for all $k > N$. And for $x = 0$, whatever k we take, $g_k(0) = n \cdot 0 = 0$. Therefore, for any $x > 0$, $\lim_{n\to\infty} g_n(x) = 0$.

   (b) Assume for any $0 < \varepsilon < 1$, exists $N \in \mathbb{N}$, we have $|g_k(x) - 0| < \varepsilon$ for all $x \geq 0$ with any $k \geq N$. Then, for g_N, we can find $x = \frac{1}{N}$ s.t. $g_N(x) = Nx = 1 > \varepsilon$(contradiction). Thus, $g_n(x)$ is not uniform convergence on $x \geq 0$.

   For $x \geq c > 0$ and any $\varepsilon > 0$, $|g_n(x) - 0| = \frac{1}{nx} < \frac{1}{nc} < \varepsilon$ for some $n > N_c \in \mathbb{N}$. Thus, $g_n(x)$ is uniform convergence on $x \geq c > 0$.

2. (a)

   ($\Longrightarrow$) Since $f_k \to f$ uniformly on E, for any $\varepsilon > 0$, exsits $N \in \mathbb{N}$ s.t. $d(f_k(x), f(x)) < \varepsilon$ for all $x \in E$ and $k > N$. Thus, we can get for all $\varepsilon > 0$, exists $N \in \mathbb{N}$ s.t. $\sup\{d(f_k(x), f(x)) \mid x \in E\} < \varepsilon$ for $k > N$. That means $\sup\{d(f_k(x), f(x)) \mid x \in E\} \to 0$ as $k \to \infty$.