

Homework 2 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

March 1, 2024

1. Suppose $f_k(x) = \sum_{n=1}^k \frac{x}{n^\alpha(1+nx^2)}$ and $E_l = [-L, L]$ for $L \in \mathbb{N}$. Then, we want to proof that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|f_k(x) - f_l(x)| < \varepsilon$ for all $k, l > N$ and all $x \in I_L$.

First, suppose that $l > k > N$, then

$$\begin{aligned} |f_k(x) - f_l(x)| &= \sum_{n=k}^l \frac{x}{n^\alpha(1+nx^2)} \\ &\leq \sum_{n=k}^l \frac{L}{n^\alpha(1+nL^2)} \\ &\leq \frac{L}{n^{\alpha+1}L^2} \\ &= \frac{1}{n^{\alpha+1}L} \\ &= \end{aligned}$$

2. Since $f_k \rightarrow f$ uniformly and f_k are continuous, f is continuous. Then, for any $\varepsilon > 0$, we have $\delta > 0$ s.t. if $|y - y'| < \delta$ then $|f(y) - f(y')| < \frac{\varepsilon}{2}$ for all $y, y' \in \mathbb{R}$. Since $x_k \rightarrow x$, there exists $N_1 \in \mathbb{N}$ s.t. $|x_k - x| < \delta$ for all $k > N_1$. Also we have $N_2 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$ for all $k > N_2$.
- Then, take $N = \max\{N_1, N_2\}$, we can get $|f_k(x_k) - f(x)| \leq |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $k > N$. Thus, $\lim_{k \rightarrow \infty} f_k(x_k) = f(x)$.

3. Since f_n are continuous and converges uniformly, f is continuous(also integrable on $[0, 1]$). And since f is continuous, we can find the maximum of $|f|$ on $[0, 1]$ which is called as M . For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ s.t. $|f_k(x) - f(x)| < \frac{\varepsilon}{4}$ for all $k > N_1$. And there exists $N_2 \in \mathbb{N}$ s.t. $N_2 > \frac{M}{\varepsilon}$. Then, take

$$n > N = \max N_1, N_2$$

$$\begin{aligned} \left| \int_0^{1-\frac{1}{n}} f_n(x) dx - \int_0^1 f(x) dx \right| &\leq \left| \int_0^{1-\frac{1}{n}} f_n(x) - f(x) dx \right| + \left| \int_{1-\frac{1}{n}}^1 f(x) \right| \\ &\leq \int_0^1 |f_n(x) - f(x)| dx + \frac{1}{n} \cdot M \\ &\leq 1 \cdot 2 \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \int_0^{1-\frac{1}{n}} f_n(x) dx = \int_0^1 f(x) dx.$$

4.