Homework 8 of Introduction to Analysis(II)

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1. $f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0)}{x} = \lim_{x\to 0} \frac{x\cdot 0(x^2-0^2)/(x^2+0^2)}{x} = 0$ exists. Also, we can esaily get that $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y\to 0} \frac{y(0^4+4\cdot 0^2y^2-y^4)/(0^2+y^2)^2}{y} = \lim_{y\to 0} \frac{-y^4}{y^4} = -1$. And, $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_y(0,0)$.

2. Since Df is continuous on S, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x)-f(y)\| < \frac{\varepsilon}{\|b-a\|}$ if $\|x-y\| < \delta$. And since S is a closed line in R^p, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k, \frac{\delta}{2}) \supseteq S$ and $\|x_{k+1}-a\| > \|x_k - a\|$. Let $x_0 = a$ and $x_{n+1} = b$ and $x_k = a + t_k(b-a)$. Then, by MCT, $Df(x_k) - Df(x_{k-1}) = Df(c_k)(x_k - x_{k-1})$ for c_k on the line between x_k, x_{k-1}. Thus,

$$\begin{aligned}
|f(b)-f(a)-\int_0^1 Df(a+t(b-a))(b-a)dt| &\leq \sum_{k=1}^{n+1} \|f(x_k)-f(x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \|Df(c_k)(x_k - x_{k-1}) \\
&\quad - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)\ dt\| \\
&= \sum_{k=1}^{n+1} \| \int_{t_{k-1}}^{t_k} Df(c_k)(b-a) - Df(a+t(b-a))(b-a)dt\| \\
&= \sum_{k=1}^{n+1} \varepsilon(t_k - t_{k-1}) \\
&= \varepsilon
\end{aligned}$$

Therefore, $f(b) = f(a) = \int_0^1 Df(tb+(1-t)a)(b-a)\ dt$.

3. Since B is bilinear, $g(x+u) = B(x+u,x+u) = B(x,x+u) + B(u,x+u) = B(x,x) + B(x,u) + B(u,x) + B(u,u) = g(x) + g(u) + (B(u,x) + B(x,u))$.

   And since B is bilinear, $B(0,0) = 0$ and $g(0) = 0$.

   Then, for any $x, u \in \mathbb{R}^p$ and $\varepsilon > 0$, $Dg(x)(u) = (\lim_{h \to 0} \frac{\|g(x+h) - g(x) - Dg(x)(h)\|}{\|h\|})(u)$

4. For any $x_0, y_0 \in \mathbb{R}$, $f(x_0,y_0), f_x(x_0,y_0), f_y(x_0,y_0), f_{xy}(x_0,y_0)$ are all continuous. Then,

$$\begin{aligned}
\lim_{h \to 0} \frac{f_y(x_0+h,y_0) - f_y(x_0,y_0)}{h} &= \lim_{h,k \to 0} \frac{1}{h} \frac{f(x_0+h,y_0+k) - f(x_0+h,y_0)}{k} - \frac{f(x_0,y_0+k) - f(x_0,y_0)}{k} \\
&= \lim_{h,k \to 0} \frac{1}{k} \frac{f(x_0+h,y_0+k) - f(x_0,y_0+k)}{h} - \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h} \\
&= \lim_{k \to 0} \frac{f_x(x_0,y_0+k) - f_x(x_0,y_0)}{k} \\
&= f_{xy}(x_0,y_0)
\end{aligned}$$

   Thus, f_{yx} exists and $f_{yx} = f_{xy}$.