

Homework 4 of Introduction to Analysis(II)

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1. Since $B \subset C(A, \mathbb{R}^n)$, B_x is totally bounded. From lemma mentioned in class, since A is compact and B_x is totally bounded, B is totally bounded. And since B is equicontinuous, for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. $d(x, y) < \delta \implies \|f(x) - f(y)\| < \frac{\varepsilon}{2}$ for all $f \in B$ and $x, y \in A$. Then, there exists $\{x_i \in A\}$ s.t. $A \subseteq \cup D(x_i, \delta)$ and $\{y_{i,j} \mid f(x_i) = y_{i,j} \text{ for some } f \in B\}$ s.t. $B_{x_i} \subseteq \cup D(y_{i,j}, \frac{\varepsilon}{2})$. Thus, for any sequence $f_k \in B$, $f_k(x_i)$ will have subsequence fall into a $D(y_{i,j}, \frac{\varepsilon}{2})$ for k is large. We collect all of the $y_{i,j}$ for all i , and we can find a function f s.t. $f(x_i) = y_{i,j}$ in B . Then, for the subsequence and any $x \in A$, $\|f_k(x) - f(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, every sequence in B has uniformly convergent subsequence.

2. First, we want to show that B is closed. For any sequence $f_k \in B$ which converges to f , since $f_k(0) = 0$ for all k , we can get $f(0) = 0$. Then, assume there exists an $x_0, x_1 \in (0, 1)$ s.t. $\frac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} = \alpha > 1$. Take $\varepsilon = \frac{\alpha - 1}{3}$, there exists $N \in \mathbb{N}$ s.t. $|f(x) - f_k(x)| < \varepsilon$ for all x and $k > N$. Thus, $|\frac{f_k(x_0) - f_k(x_1)}{x_0 - x_1}| = \frac{|f_k(x_0) - f(x_0) + f(x_0) - f(x_1) + f(x_1) - f_k(x_1)|}{|x_0 - x_1|} > \alpha - \frac{2\varepsilon}{|x_0 - x_1|} > 1$ which causes contradiction to $|f'_k(x)| \leq 1$ for all $x \in (0, 1)$. Thus, $\frac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} \leq 1 \implies f \in B$. Hence, B is closed.

Then, since $|f'(x)| \leq 1$ for all $f \in B$ and $x \in (0, 1)$. For any $\varepsilon > 0$, we take $\delta < \varepsilon$, for any $x, y \in [0, 1]$ s.t. $|x - y| < \varepsilon$, $|f(x) - f(y)| \leq 1 \cdot |x - y| = \delta < \varepsilon$. Thus, B is equicontinuous.

Last, for any $x \in [0, 1]$, we want to proof B_x is compact. For $x = 0$, $B_0 = \{0\}$ obviously compact. For $x > 0$, since $|f'| \leq 1$, we can easily get $B_x = [-x, x]$ by $f(x) = ax$ for all $|a| \leq 1$. Thus, B is pointwise compact.

Therefore, by Arzela-Ascoli Theorem, B is compact.

3. For any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x - y\| < \delta \implies \|f_k(x) - f_k(y)\| < \frac{\varepsilon}{3}$. And since K is compact, we can find finite $p \in \mathbb{N}$ and $\{x_k\}_{k=1}^p$ s.t. $K \subseteq \bigcup_{k=1}^p D(x_k, \delta)$. Also, for all x_k , we can find a $N_{x_k} \in \mathbb{N}$ s.t. $|f_n(x_k) - f_m(x_k)| < \frac{\varepsilon}{3}$ for all $n, m > N_{x_k}$.

Then, we take $N = \max_{x \in [1, p]} \{N_{x_k}\}$, and for any $x \in K$, we can find a $k \in [1, p]$ s.t. $x \in D(x_k, \delta)$. Thus,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus, by Cauchy Criterion, $f_k \rightarrow f$ uniformly.

4. Since \mathfrak{F} is equicontinuous, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x - y\| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ for all $f \in \mathfrak{F}$. For all $x, y \in D$ and $\|x - y\| < \delta$, we want to show $|f^*(x) - f^*(y)| < \varepsilon$. First, if $f^*(x) > f^*(y)$, we can find a $f \in \mathfrak{F}$ s.t. $f^*(x) - f(x) < \frac{\varepsilon}{2}$. Then, $f^*(y) > f^*(x) - \frac{\varepsilon}{2} - |f(x) - f(y)| > f^*(x) - \varepsilon$. For the other case, just change x, y and get the same result. Thus, f^* is continuous.