

Homework 4 of Introduction to Analysis(II)

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March 15, 2024

1. Since B_x are bounded on \mathbb{R}^n , B_x is compact for all $x \in A$. Thus, B is pointwise compact.

And since \mathbb{R}^n is complete, B is complete, too. Thus, B is closed.

Then, by Arzela-Ascoli Theorem, B is compact. Therefore, B is sequentially compact.

2. First, we want to show that B is closed. For any sequence $f_k \in B$ which converges to f , since $f_k(0) = 0$

for all k , we can get $f(0) = 0$. Then, assume there exists an $x_0, x_1 \in (0, 1)$ s.t. $\frac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} = \alpha > 1$.

Take $\varepsilon = \frac{\alpha - 1}{3}$, there exists $N \in \mathbb{N}$ s.t. $|f(x) - f_k(x)| < \varepsilon$ for all x and $k > N$. Thus, $|\frac{f_k(x_0) - f_k(x_1)}{x_0 - x_1}| = \frac{|f_k(x_0) - f(x_0) + f(x_0) - f(x_1) + f(x_1) - f_k(x_1)|}{|x_0 - x_1|} > \alpha - \frac{2\varepsilon}{|x_0 - x_1|} > 1$ which causes contradiction to $|f'_k(x)| \leq 1$ for all $x \in (0, 1)$. Thus, $\frac{|f(x_0) - f(x_1)|}{|x_0 - x_1|} \leq 1 \implies f \in B$. Hence, B is closed.

Then, since $|f'(x)| \leq 1$ for all $f \in B$ and $x \in (0, 1)$. For any $\varepsilon > 0$, we take $\delta < \varepsilon$, for any $x, y \in [0, 1]$ s.t. $|x - y| < \varepsilon$, $|f(x) - f(y)| \leq 1 \cdot |x - y| = \delta < \varepsilon$. Thus, B is equicontinuous.

Last, for any $x \in [0, 1]$, we want to proof B_x is compact. For $x = 0$, $B_0 = \{0\}$ obviously compact. For $x > 0$, since $|f'| \leq 1$, we can easily get $B_x = [-x, x]$ by $f(x) = ax$ for all $|a| \leq 1$. Thus, B is pointwise compact.

Therefore, by Arzela-Ascoli Theorem, B is compact.

3. For any $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ for all f and $x, y \in K$. Since K is compact, we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $K \subset D(x_k, \frac{\delta}{2})$. Then, we can find a $N \in \mathbb{N}$ s.t.

$|f_k(x_1) - f(x_1)| < \varepsilon$ for all $k > N$. Thus, for any $y \in K$, $y \in x_l$ for some l ,

$$\begin{aligned} |f_k(y) - f(y)| &\leq |f_k(y) - f_k(x_l)| + |f_k(x_l) - f_k(x_{l-1})| + \cdots + |f_k(x_2) - f_k(x_1)| \\ &\quad + |f_k(x_1) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_l) - f(y)| \\ &< (2l + 1)\varepsilon \end{aligned}$$

4. Since \mathfrak{F} is equicontinuous, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|x - y\| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$ for all $f \in \mathfrak{F}$. For all $x, y \in D$ and $\|x - y\| < \delta$, we want to show $|f^*(x) - f^*(y)| < \varepsilon$. First, if $f^*(x) > f^*(y)$, we can find a $f \in \mathfrak{F}$ s.t. $f^*(x) - f(x) < \frac{\varepsilon}{2}$. Then, $f^*(y) > f^*(x) - \frac{\varepsilon}{2} - |f(x) - f(y)| > f^*(x) - \varepsilon$. For the other case, just change x, y and get the same result. Thus, f^* is continuous.