Homework 2 of Introduction to Analysis(II)

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1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t. $\frac{\sum_{i=1}^{N} a_i - a}{N'} < \frac{\varepsilon}{2}$. Thus, for any $n > N'$,

$$\begin{aligned} |\frac{\sum_{i=1}^{n} a_i}{n} - a| &\leq |\frac{\sum_{i=1}^{N} a_i - a}{n}| + |\frac{\sum_{i=N+1}^{n} a_i - a}{n - N'}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

   Therefore, $\lim_{n\to\infty} b_n = a$.

2. Since $\lim_{n\to\infty} na_n = 0$, by 1., $\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{na_n}{N} = 0$ Also, since $\lim_{n\to\infty} na_n = 0$, $\lim_{n\to\infty} a_n = 0$.

   Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=0}^{k} a_n| < \frac{\varepsilon}{2}$ for all $k > N$.

   Then, for $x \to 1^-$ s.t. $|f(x) - A| < \frac{\varepsilon}{2}$,

$$\begin{aligned} \sum_{n=0}^{N} a_n - A &= \sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A) \\ &\leq \frac{\varepsilon}{2} - 0 + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

   Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3.