

Homework 12 of Introduction to Analysis(II)

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1. (a) Since $\partial \text{int}(E) = \text{cl}(\text{int}(E)) \setminus \text{int}(\text{int}(E)) = \text{cl}(E) \setminus \text{int}(E) = \partial E$ and $\partial \text{cl}(E) = \partial E$, we can get

$$\text{Vol}(\partial \text{int}(E)) = \text{Vol}(\partial \text{cl}(E)) = \text{Vol}(\partial E) = 0 \text{ and } \text{int}(E), \text{cl}(E) \text{ are Jordan regions.}$$

- (b) Since $\text{cl}(E) = \text{int}(E) \cup \partial E$ and $\text{int}(E) \subseteq E \subseteq \text{cl}(E)$,

$$\text{Vol}(\text{cl}(E)) \leq \text{Vol}(\text{int}(E)) + \text{Vol}(\partial E) = \text{Vol}(\text{int}(E)) \leq \text{Vol}(E) \leq \text{Vol}(\text{cl}(E)).$$

$$\text{Therefore, } \text{Vol}(\text{cl}(E)) = \text{Vol}(\text{int}(E)) = E.$$

(c)

(\implies) From (b), we know $\text{Vol}(\text{int}(E)) = \text{Vol}(E) > 0$, then we can find a set of rectangles R_n s.t.

$$\sum |R_n| > 0 \text{ and } \cup R_n \subseteq \text{int}(E). \text{ Therefore, } \text{int}(E) \neq \emptyset.$$

(\impliedby) Since $\text{int}(E)$ is non-empty, for any $x_0 \in \text{int}(E)$, there exists $\varepsilon > 0$ s.t. $D(x_0, \varepsilon) \subseteq \text{int}(E)$. Then,

we can find a small rectangle R with each length is $\frac{\varepsilon}{2}$ and R is contained in $D(x_0, \varepsilon)$. Thus,

$$\text{Vol}(\text{int}(E)) > \left(\frac{\varepsilon}{2}\right)^2 > 0.$$

- (d) Since f is continuous, for any $x_0 \in [a, b]$, we can find a sequence $x_k \rightarrow x_0$ s.t. $f(x_k) \rightarrow f(x_0)$. That

is, $A = \{(x, f(x)) \mid x \in [a, b]\}$ is closed. And since $\partial A \subseteq A$, $\text{Vol}(\partial A) \leq \text{Vol}(A)$.

Since f is continuous on $[a, b]$ is compact, for all $\varepsilon > 0$, we can find $\delta > 0$ s.t. $|x - y| < \delta$ implies

That $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Then, we can find a finite increasing sequence $\{x_i \mid x_i \in [a, b]\}_{i=1}^N$ s.t.

$[a, b] \subseteq D(x_i, \frac{\delta}{2})$. Therefore, for any $y = f(x)$, $y \in D(f(x_i), \frac{\varepsilon}{b-a})$ for some i .

Then, take $u_0 = a$, $u_i \in D(x_i, \frac{\delta}{2}) \cap D(x_{i+1}, \frac{\delta}{2})$, $u_N = b$, and we can get $[a, b] = \cup [u_i, u_{i+1}]$. Thus,

$A \subseteq \bigcup_{i=0}^N [u_i, u_{i+1}] \times D(\xi_i, \frac{\varepsilon}{b-a})$ for some $\xi_i \in \{f(x) \mid x \in [u_i, u_{i+1}]\}$ with the sum of the rectangles is $2 \frac{\varepsilon}{b-a} \cdot (b-a) = 2\varepsilon$. Hence, $\text{Vol}(A) = 0$ and $\text{Vol}(\partial A) = 0$.

(e) Yes. Since f is integrable, for any $\varepsilon > 0$, we can find a partition P s.t. $|U(f, P) - L(f, P)| < \varepsilon$.

That is, $\sum_{i=0}^N (\sup_{x \in [x_i, x_{i+1}]} \{f(x)\} - \inf_{x \in [x_i, x_{i+1}]} \{f(x)\}) \cdot (x_{i+1} - x_i) < \varepsilon$, and each one of the summation is a rectangle that contains all $(x, f(x))$ in the interval. Thus, $\text{Vol}(A) = 0$ and $\text{Vol}(\partial A) = 0$.

2. (a) For $x \in \partial(E_1 \cap E_2)$, $x \in \text{cl}(E_1 \cap E_2) \cap \text{cl}(M \setminus E_1 \cap E_2)$. Then, if $E_1 \cap E_2$ is not Jordan region,

$$\text{Vol}(E_1 \cap E_2) = a > 0.$$

(b) Since E_1, E_2 are non-overlapping, $E_1 \cap E_2 \subseteq \partial E_1 \cup \partial E_2$. And since E_1, E_2 are Jordan regions,

$$\text{Vol}(E_1) = \text{Vol}(\text{int}(E_1)) \text{ and } \text{Vol}(E_2) = \text{Vol}(\text{int}(E_2)). \text{ Also we have } \text{int}(E_1) \cup \text{int}(E_2) \subseteq E_1 \cup E_2$$

$$\text{and } \text{int}(E_1) \cap \text{int}(E_2) = \emptyset.$$

$$\begin{aligned} \text{Vol}(\text{int}(E_1)) + \text{Vol}(\text{int}(E_2)) &= \text{Vol}(\text{int}(E_1) \cup \text{int}(E_2)) \\ &\leq \text{Vol}(\text{int}(E_1) \cup \text{int}(E_2)) + \text{Vol}(E_1 \cap E_2) \\ &= \text{Vol}(E_1 \cup E_2) \\ &\leq \text{Vol}(\text{int}(E_1)) + \text{Vol}(\text{int}(E_2)) + \text{Vol}(\partial E_1) + \text{Vol}(\partial E_2) \\ &= \text{Vol}(\text{int}(E_1)) + \text{Vol}(\text{int}(E_2)) \\ &= \text{Vol}(E_1) + \text{Vol}(E_2) \end{aligned}$$

$$\text{Thus, } \text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2).$$

(c) Let $E_1 \setminus E_2 = S_1$, $E_2 = S_2$. We have S_1 and S_2 are Jordan regions and non-overlapping. Thus,

$$\text{Vol}(E_1) = \text{Vol}(S_1 \cup S_2) = \text{Vol}(S_1) + \text{Vol}(S_2) \text{ and } \text{Vol}(E_1 \setminus E_2) = \text{Vol}(E_1) - \text{Vol}(E_2).$$

(d) Using (b), (c) and the fact $E_1 \cup E_2 = (E_1 \setminus E_2 \cap E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_2 \cap E_1)$, we can get $\text{Vol}(E_1 \cup$

$$E_2) = (\text{Vol}(E_1) - \text{Vol}(E_1 \cap E_2)) + \text{Vol}(E_1 \cap E_2) + (\text{Vol}(E_2) - \text{Vol}(E_1 \cap E_2)) = \text{Vol}(E_1) + \text{Vol}(E_2) -$$

$$\text{Vol}(E_1 \cap E_2).$$