

Homework 8 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

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1. $f_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{x \cdot 0(x^2 - 0^2)/(x^2 + 0^2)}{x} = 0$ exists. Also, we can easily get that $f_y(0,0) = 0$. Then, $f_{xy}(0,0) = \frac{\partial f_x}{\partial y}(0,0) = \lim_{y \rightarrow 0} \frac{y(0^4 + 4 \cdot 0^2 y^2 - y^4)/(0^2 + y^2)^2}{y} = \lim_{y \rightarrow 0} \frac{-y^4}{y^4} = -1$. And, $f_{yx}(0,0) = 1$. Therefore, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

2. Since Df is continuous on S , for any $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\|f(x) - f(y)\| < \frac{\varepsilon}{\|b-a\|}$ if $\|x-y\| < \delta$. And since S is a closed line in R^p , we can find a sequence $\{x_k\}_{k=1}^n$ s.t. $D(x_k, \frac{\delta}{2}) \supseteq S$ and $\|x_{k+1} - a\| > \|x_k - a\|$. Let $x_0 = a$ and $x_{n+1} = b$ and $x_k = a + t_k(b-a)$. Then, by MCT, $Df(x_k) - Df(x_{k-1}) = Df(c_k)(x_k - x_{k-1})$ for c_k on the line between x_k, x_{k-1} . Thus,

$$\begin{aligned} |f(b) - f(a) - \int_0^1 Df(a+t(b-a))(b-a)dt| &\leq \sum_{k=1}^{n+1} \|f(x_k) - f(x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\ &= \sum_{k=1}^{n+1} \|Df(c_k)(x_k - x_{k-1}) - \int_{t_{k-1}}^{t_k} Df(a+t(b-a))(b-a)dt\| \\ &= \sum_{k=1}^{n+1} \|\int_{t_{k-1}}^{t_k} Df(c_k)(b-a) - Df(a+t(b-a))(b-a)dt\| \\ &= \sum_{k=1}^{n+1} \frac{\varepsilon(t_k - t_{k-1})}{\|(b-a)\|} \\ &= \varepsilon \end{aligned}$$

Therefore, $f(b) - f(a) = \int_0^1 Df(tb + (1-t)a)(b-a) dt$.

$$3. \lim_{u \rightarrow 0} \frac{\|g(x+u) - g(x) - Dg(x)(u)\|}{\|u\|} = \lim_{u \rightarrow 0} \frac{\|g(x) + g(u) + B(x,u) + B(u,x) - g(x) - Dg(x)(u)\|}{\|u\|} = 0.$$

4. For any $(x_0, y_0) \in \mathbb{R}^2$ and $\varepsilon > 0$, we can find small enough h, k s.t. $|f_{xy}(x_0, y_0) - \frac{f_x(x_0, y_0 + k) - f_x(x_0, y_0)}{k}| < \varepsilon$ and so as $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

Then,

$$\begin{aligned} \left| \frac{f_y(x_0 + h, y_0) - f_y(x_0, y_0)}{h} \right| &\leq \frac{1}{|h|} \cdot \left| \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k} - \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} \right| + 2 \frac{\varepsilon}{|h|} \\ &\leq \frac{2\varepsilon + 2|k|\varepsilon}{|h||k|} + \frac{1}{|k|} \left(\left| \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)}{h} - \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \right| \right) \\ &\leq \frac{2}{\varepsilon} + \frac{1}{|k|} \end{aligned}$$