Homework 5 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

March 21, 2024

1. (a) If $x \in A$, $d(x,A) = \|x - x\| = 0$ and $d(x,B) = k > 0$, $\phi(x) = \dfrac{0}{0+k} = 0$. If $x \in B$, $d(x,A) = l > 0$ and $d(x,B) = 0$, $\phi(x) = \dfrac{l}{l+0} = 1$. If $x \notin A \land x \notin B$, $d(x,A) = l$ and $d(x,B) = k$, $\phi(x) = \dfrac{l}{l+k} < 1$ and is positive. Thus, $0 \le \phi(x) \le 1$ for all $x \in \mathbb{R}^n$.

   (b) Let $\phi(x) = (b-a)\dfrac{d(x,A)}{d(x,A)+d(x,B)} + a$. From (a), we can get $\phi(x \in A) = (b-a)\cdot 0 + a = a$, $\phi(x \in B) = (b-a)\cdot 1 + a = b$, and $a \le phi(x) \le b$ for all $x \in A$.

2. If f has more than one fixed point, there exists $x, y \in S$ s.t. $d(f^n(x), f^n(y)) = d(x,y)$ for all $n \in \mathbb{N}$(contradiction to $a_n \to 0$). Then, we want to show that f has fixed point. For any $x_0 \in S$, we let $x_k = f^k(x)$ and we can find a $N \in \mathbb{N}$ s.t. $a_n < \varepsilon$ for all $n > N$. Then, $d(x_{n+k}, x_n) \le a_n d(x_k, x_0) < \varepsilon \cdot d(x_k, x_0)$ for any k and $n > N$. Thus, $x_n \to x^* \in S$ by S is complete, and x^* is a fixed point of f.

3. Since $T(u)(t) = \int_a^t u(s)\,ds$, assume $T^m(u)(t) = \int_a^t \dfrac{(t-s)^{m-1}}{(m-1)!}u(s)\,ds$. Then,

$$\begin{aligned} T^{m+1}(u)(t) &= T(T^m(u))(t) \\ &= \int_a^t T^m(u)(s)\,ds \\ &= \int_a^t \frac{1}{(m-1)!}\int_a^s (s-\tau)^{m-1}u(\tau)\,d\tau\,ds \\ &= \int_a^t \frac{1}{(m-1)!} \end{aligned}$$

   Thus, $T^m(u) \le \dfrac{1}{(m-1)!}u$

4.