Homework 14 of Introduction to Analysis(II)

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1. Let $A_M = \{x \in E \mid f_M \text{ is discontinuous at } x\}$ and $A = \{x \in E \mid f \text{ is discontinuous at } x\}$. For any $M \in \mathbb{N}$, if x is a point that f_M is discontinuous at x, That means $f(x) \leq M$ and f is discontinuous at x. That is, $A_M \subseteq A$ for all M implies that $\cup A_m \subseteq A$. And for any $x \in A$, there exists a $N \in \mathbb{N}$ s.t. $f(x) < N$. Then, $x \in A_N$. Therefore, $A = \cup A_M$.

2. First, want to proof E contains finite union of intervals. For all $x \in E$ and f is continuous on neighborhood of x, we can find finite union of open intervals cover it. Thus, we can find finite union of intervals $\cup I_n$ such that $f(x) \geq \dfrac{\alpha}{2}$ and f is continuous on some open interval U contains x for all $x \in \cup I_n$.

   And for $f(x) \geq \dfrac{\alpha}{2}$ but $f(x)$ is not continuous. If $x \in \mathrm{cl}(I_n)$ for some n, then just add into the interval. Else $x \notin \mathrm{cl}(I_n)$ for all n, just drop it since the length is zero. And we have the length $L = \cup |I_n|$.

   Suppose $M_0 = \sup f(x)$ and $\alpha_0 = \int_0^1 f(x)\, dx$, If $L < \dfrac{\alpha}{4M}$,

3. Let $A = \{x \in E \mid f(x) \neq 0\}$. If A is empty, then A is measure zero.

   Suppose A is non-empty. Then, for a large enough $N \in \mathbb{N}$, $A_N = \{x \in E \mid f(x) > \dfrac{1}{N}\}$. Using the same argument of 1. , we can have $A_N \to A$ as $N \to \infty$. Since $\int_E f(x)\, dx = 0$, $\int_{A_N} f(x)\, dx = 0$. Thus, for any $\varepsilon > 0$, there exists rectangles such that $\dfrac{1}{N} \sum |R_i| \leq (L) \int_{A_N} f(x)\, dx \leq \dfrac{\varepsilon}{N}$. Therefore, $\sum |R_i| < N \cdot \dfrac{\varepsilon}{N} = \varepsilon$ and A_N is measure zero.

   By the theorem that countable set of measure zero is also measure zero, we can have A is measure zero.