Homework 13 of Introduction to Analysis(II)

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1. (a) For any $\varepsilon > 0$, we can find an open cube $D(x_0, \delta)$ that $R_j \subseteq D(x_0, \delta)$ and

   (b) Since for any E_i is measure zero, we can find $\{R_{i,j}\}_j$ s.t. $\sum_j |R_{i,j}| < \dfrac{\varepsilon}{2^i}$. Then, let $R_{1,1} = R_1$, $R_{1,2} = R_2$, $R_{2,1} = R_3$, $R_{1,3} = R_4$, $R_{2,2} = R_5$ etc. And we can get $\{R_k\}_k$ contains $\bigcup_i E_i$ and $\sum_k |R_k| < \varepsilon(\dfrac{1}{2} + \dfrac{1}{4} + \cdots) = \varepsilon$. Therefore, union of countable measure zero sets is measure zero.

   (c) Since A is compact and (a), we can find finite open cubes which covers A and volume of their union is less than ε. Thus, take union of closure on each open cube and we can find a retangle covers A with volume less than ε. Therefore, volume of A is 0.

2. For any grid g, $U(1,g) = L(1,g)$. Then, 1 is integrable. And for any $\varepsilon > 0$, there exists rectangles R_i s.t. $|\sum |R_i| - \text{Vol}(E)| < \varepsilon$. Thus, $\int_E 1\, dE \le U(1,g) = \sum 1 \cdot |R_i| < \text{Vol}(E) + \varepsilon$. Also, $\int_E 1\, dE \ge \text{Vol}(E) - \varepsilon$. Therefore, $\int_E 1\, dE = \text{Vol}(E)$.

3. (a) If x is rational, $G(x,t) = \int_0^t 1\, dy = t$. IF x is irrational, $G(x,t) = \int_0^t 2y\, dy = t^2$.
   Since $t \in [0,1]$, $t^2 \le t$. Then, (U)$\int_0^1 G(x,t)\, dx = t$ and (L)$\int_0^1 G(x,t)\, dx = t^2$.
   Therefore, $\int_0^1 (\int_0^1 f(x,y)\, dy)\, dx = \int_0^1 G(x,1)\, dx = \int_0^1 1\, dx = 1$.

   (b) As $y \to 0$, (U)$\int_0^1 f(x,y)\, dx = 1$ but (L)$\int_0^1 f(x,y)\, dx = 2y \to 0$. Thus, $\int_Q f(X)\, dx$ doesn't exists.