

# Homework 11 of Introduction to Analysis(II)

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1. For  $f \in \mathcal{C}^1(E, \mathbb{R}^n)$  with  $E$  is open subset of  $\mathbb{R}^n$  and  $Jf(x_0) \neq 0$  for some  $x_0 \in E$  and  $y_0 = f(x_0)$ . Then, let  $h(x, y) = f(x) - y \in \mathcal{C}^1(E \times \mathbb{R}^n \rightarrow \mathbb{R}^n)$  and  $h(x_0, y_0) = f(x_0) - y_0 = y_0 - y_0 = 0$ . We also can get  $\frac{\partial h}{\partial x} = Df$ . Then, by Implicit Function Theorem, there exists open set  $W \subseteq \mathbb{R}^n$  with  $y_0 \in W$ , and unique  $g \in \mathcal{C}^1(W, \mathbb{R}^n)$  with  $g(y_0) = x_0$  s.t.  $0 = h(g(y), y) = f(g(y)) - y \implies y = f(g(y))$  for all  $y \in W$  and  $g$  is locally inverse function of  $f$ . And we can get  $[Df^{-1}(y)] = [Dg(y)] = -[Df(g(y))]^{-1}[Dg(y)]_y = [Df(g(y))]^{-1} = [Df(f^{-1}(y))]^{-1}$  by Implicit Function Theorem, too.
2. Suppose  $f \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ ,  $Df(x_0, y_0) \neq 0$  for some  $x_0, y_0$  (or  $f$  is constant function and not one-to-one). Then, suppose  $\frac{\partial f}{\partial x} \neq 0$  for neighborhood of  $(x_0, y_0)$ , and let  $h(x, y) = f(x, y) - f(x_0, y_0)$  with  $\frac{\partial h}{\partial x} \neq 0$ . by Implicit Function Theorem, there is a neighborhood  $U \subseteq \mathbb{R}^2$  and  $W \subseteq \mathbb{R}$  s.t.  $(x_0, y_0) \in U$  and  $y_0 \in W$  and a function  $g : W \rightarrow \mathbb{R}^2$  s.t.  $h(g(y), y) = f(g(y), y) - f(x_0, y_0) = 0$ . Then,  $f(g(y), y) = f(x_0, y_0)$  for  $y \in W$  and  $f$  is not one-to-one. If  $\frac{\partial f}{\partial x} = 0$ , then  $\frac{\partial f}{\partial y} \neq 0$  and use the same argument can get the same result.
3. (a)  $\nabla f(x, y, z) = (y, x, 0)$  And  $\nabla g_1(x, y, z) = (2x, 2y, 2z)$ ,  $\nabla g_2(x, y, z) = (1, 1, 1)$ .

Thus, we have

$$\begin{cases} y &= 2ax + b \\ x &= 2ay + b \\ 0 &= 2az + b \\ 1 &= x^2 + y^2 + z^2 \\ 0 &= x + y + z \end{cases}$$

Then,  $a = \frac{-1}{2}$ ,  $b = 0$ ,  $(x, y, z) = (\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}, 0)$ .

(b)  $\nabla f(x, y, z, w) = (3, 1, 0, 1)$ ,  $\nabla g_1(x, y, z, w) = (6x, 1, 12z^2, 0)$  and  $\nabla g_2(x, y, z, w) = (-3x^2, 0, 12z^3, 1)$ .

Then, we have

$$\begin{cases} 3 &= 6cx - 3dx^2 \\ 1 &= c \\ 0 &= 12cz^2 + 12dz^3 \\ 1 &= d \\ 1 &= 3x^2 + y + 4z^3 \\ 0 &= -x^3 + 3z^4 + w \end{cases}$$

Thus,  $c = 1, d = 1$  and  $(x, y, z, w) = (1, 2, -1, -2)$  or  $(1, -2, 0, 1)$ .

4. We show that for any  $\varepsilon > 0$ , we can not find  $\phi : D(0, \varepsilon) \rightarrow \mathbb{R}$  s.t.  $F(\phi(x)) = (\phi(x))^2 - x$  for all  $x \in D(0, \varepsilon)$ .