Exercises(2)

February 27, 2024

1. (8 points) Prove that $\sum_{n=1}^{\infty} \frac{x}{n^\alpha(1+nx^2)}$ converges uniformly on every finite interval in $\mathbb{R}$ if $\alpha > 1/2$. Is the convergence uniform on $\mathbb{R}$? (Justify your answer.)

2. (6 points) Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous function. Prove if that f_k converges uniformly to a function f on $\mathbb{R}$, then
$$\lim_{k\to\infty} f_k(x_k) = f(x)$$
for every sequence of points x_k with $x_k \to x$ in $\mathbb{R}$.

3. (8 points) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real value continuous functions defined on $[0,1]$ and assume that $f_n \to f$ uniformly on $[0,1]$. Prove or disprove
$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)dx = \int_0^1 f(x)dx.$$

4. (8 points) Suppose g and $f_n (n = 1, 2, 3, \ldots)$ are defined on $(0,\infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0,\infty)$, and
$$\int_0^{\infty} g(x) < \infty.$$
Prove that
$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)dx = \int_0^{\infty} f(x)dx.$$