Homework 3 of Introduction to Analysis(II)

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1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \dfrac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t. $\dfrac{\sum_{i=1}^{N} a_i - a}{N'} < \dfrac{\varepsilon}{2}$. Thus, for any $n > N'$,

$$\begin{aligned}
|\frac{\sum_{i=1}^{n} a_i}{n} - a| &\leq |\frac{\sum_{i=1}^{N} a_i - a}{n}| + |\frac{\sum_{i=N+1}^{n} a_i - a}{n - N'}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} b_n = a$.

2. Since $\lim_{n\to\infty} na_n = 0$, by 1., $\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \dfrac{na_n}{N} = 0$ Also, since $\lim_{n\to\infty} na_n = 0$, $\lim_{n\to\infty} a_n = 0$. Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=k+1}^{\infty} a_n| < \dfrac{\varepsilon}{3}$ for all $k > N$. Then, for $x \to 1^-$ s.t. $|f(x) - A| < \dfrac{\varepsilon}{3}$ and $|\sum_{n=1}^{k} a_n(1-x)| < \dfrac{\varepsilon}{3}$,

$$\begin{aligned}
|\sum_{n=0}^{N} a_n - A| &= |\sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A)| \\
&\leq |\sum_{n=0}^{N} a_n(1-x)| + |\sum_{n=N+1}^{\infty} a_n x^n| + |f(x) - A| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3. Since $\lim_{x\to 1^-}\sum_{n=1}^{\infty} a_n x^n = A$, for any $\varepsilon > 0$, we can find a $x_1 \in (0,1)$ s.t. $|\sum_{n=1}^{\infty} a_n x^n - A| < \frac{\varepsilon}{3}$ for all $x_1 < x < 1$. Then, there exists $N \in \mathbb{N}$ s.t. $|\sum_{n=N'+1}^{\infty} a_n x^n| < \frac{\varepsilon}{3}$ for all x and $N' > N$. And we have x_2 s.t. $|\sum_{n=1}^{N} a_n(1-x^n)| < \frac{\varepsilon}{3}$ for all $x_2 < x < 1$.

   Thus, for any $N' > N$ and $x > \max\{x_1, x_2\}$,

$$
\begin{aligned}
|\sum_{n=0}^{N} a_n - A| &\leq |\sum_{n=0}^{N'} a_n(1-x^n)| + |\sum_{n=N'+1}^{\infty} a_n x^n| + |\sum_{n=1}^{\infty} a_n x^n - A| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}
$$

   Thus, $\sum_{n=1}^{\infty} a_n = A$.

4.

($\Longrightarrow$) Since $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=N}^{2N-1} a_n \sin(nx)| < \varepsilon$ for all x. Let $x = \frac{1}{2N}$, then $\sin(nx) = \sin(\frac{n}{2N}) \in [\sin(\frac{1}{2}), 1)$. Thus, $\sum_{n=N}^{2N-1} a_n \sin(nx) > \sum_{n=N}^{2N-1} a_n \sin(\frac{1}{2}) \geq \sum_{n=N}^{2N-1} a_{2n} \sin(\frac{1}{2}) = \frac{2n}{2}\sin(\frac{1}{2}) a_{2n}$. Therefore, $2na_{2n} < \frac{2}{\sin(\frac{1}{2})}\varepsilon$. Using the similar way, we can get $(2n-1)a_{2n-1} \to 0$ as $n \to \infty$ also. Hence, we get $na_n \to 0$ as $n \to \infty$.

($\Longleftarrow$) Since $na_n \to 0$ as $n \to \infty$, we can find an $N \in \mathbb{N}$ s.t. $na_n < \frac{\varepsilon}{\pi}$ for all $n > N$. And since $|\sin(nx)|$ is periodic function, we only need to check the series converges on $[0, \pi]$. Then, we want to proof $|\sum_{k=n}^{n+p} a_k \sin(kx)|$ uniformly converges for $n > N$ and $p \in N$. We seperate the interval into $[0, \frac{\pi}{n+p}]$, $[\frac{\pi}{n+p}, \frac{\pi}{n}]$, $[\frac{\pi}{n}, \pi]$ three interval. First one,

$$
\begin{aligned}
|\sum_{k=n}^{n+p} a_k \sin(kx)| &\leq \sum_{k=n}^{n+p} a_k \cdot k \cdot x \\
&\leq \frac{k=n}{n+p} k \cdot k \cdot \frac{\pi}{n+p} \\
&\leq (p+1)\frac{\varepsilon}{\pi} \cdot \frac{\pi}{n+p} \\
&< \varepsilon.
\end{aligned}
$$

And the second interval,