Homework 5 of Introduction to Analysis(II)

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1. (a) If $x \in A$, $d(x,A) = \|x-x\| = 0$ and $d(x,B) = k > 0$, $\phi(x) = \dfrac{0}{0+k} = 0$. If $x \in B$, $d(x,A) = l > 0$ and $d(x,B) = 0$, $\phi(x) = \dfrac{l}{l+0} = 1$. If $x \notin A \wedge x \notin B$, $d(x,A) = l$ and $d(x,B) = k$, $\phi(x) = \dfrac{l}{l+k} < 1$ and is positive.

   And for $x, y \in A$ and $\varepsilon > 0$, if $\|x-y\| < \delta = \varepsilon$, $|\phi(x) - \phi(y)| < |\phi(x) - (\phi(x) + \delta)| = \varepsilon$. Thus, $0 \leq \phi(x) \leq 1$ is continuous for all $x \in \mathbb{R}^n$.

   (b) Let $\phi(x) = (b-a)\dfrac{d(x,A)}{d(x,A)+d(x,B)} + a$. From (a), we can get continuous funtion $\phi(x \in A) = (b-a) \cdot 0 + a = a$, $\phi(x \in B) = (b-a) \cdot 1 + a = b$, and $a \leq phi(x) \leq b$ for all $x \in A$.

2. If f has more than one fixed point, there exists $x, y \in S$ s.t. $d(f^n(x), f^n(y)) = d(x,y)$ for all $n \in \mathbb{N}$(contradiction to $a_n \to 0$). Then, we want to show that f has fixed point. For any $x_0 \in S$, we let $x_k = f^k(x)$ and we can find a $N \in \mathbb{N}$ s.t. $a_n < \varepsilon$ for all $n > N$. Then, $d(x_{n+k}, x_n) \leq a_n d(x_k, x_0) < \varepsilon \cdot d(x_k, x_0)$ for any k and $n > N$. Thus, $x_n \to x^* \in S$ by S is complete, and x^* is a fixed point of f.

3. Since $T(u)(t) = \int_a^t u(s)\, ds$, assume $T^m(u)(t) = \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} u(s)\, ds$. Then,

$$\begin{aligned} T^{m+1}(u)(t) &= T(T^m(u))(t) \\ &= \int_a^t T^m(u)(s)\, ds \\ &= \int_a^t \frac{1}{(m-1)!} \int_a^s (s-\tau)^{m-1} u(\tau)\, d\tau\, ds \\ &= \int_a^t \frac{1}{(m-1)!} \int_a^\tau (s-\tau)^{m-1} u(\tau)\, ds\, d\tau \\ &= \int_a^t \frac{1}{m!} (t-\tau)^m u(\tau)\, d\tau \end{aligned}$$

Thus, by M.I., $T^m(u) = \frac{1}{(m-1)!} \int_a^t (t-s)^m u(s) ds \leq \int_a^t \frac{(t-s)^m}{(m-1)!} M\, ds$ for M is upper bound of u.

Thus, T has unique fixed point by 2., and $T(0) = 0$ trivially.

4. We want to proof $T(f)(x) = \int_0^x f(t) t\, dt$ has unique fixed point.

Since f is on $[0,1]$, $T(|f|)(x) \leq |f(x)|$ for all f and $x \in [0,1]$. Thus, T has unique fixed point and $T(0) = 0$. Therefore, $\int_0^1 f(x) x^n dx = 0$ for all x implies that $f(x) = 0$.