Homework 6 of Introduction to Analysis(II)

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March 28, 2024

1. Let $\phi(x) = \arctan(x)$, then $\phi \circ f(x) = \phi(f(x))$ is bdd by $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then, by Tietze's Extension Theorem, there exists $g \in C(\mathbb{R}^n, \mathbb{R})$ s.t. $g(x) = \phi(f(x))$ for all $x \in D$ and $\sup|g(x)| = \sup|\phi(f(x))| \leq \frac{\pi}{2}$.

   Thus, there exists $h(x) = \tan(g(x))$ in $C(\mathbb{R}^n, \mathbb{R})$ and $h(x) = f(x)$ for all $x \in D$ by $\phi(x)$ is invertible.

2. Since $\frac{\partial f_i}{\partial x}$ exists for all x and all i, we get the Jacobian matrix

$$[J(x)] = \begin{bmatrix} \frac{\partial f_1}{\partial x}(x) \\ \frac{\partial f_2}{\partial x}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x}(x) \end{bmatrix}.$$

   Then, we want to show that $J = Df$. For any x and any i,

$$\begin{aligned} \lim_{h \to 0} \frac{|f_i(x+h) - f_i(x) - J_i(x)(h)|}{|h|} &= \lim_{h \to 0} \frac{|f_i(x+h) - f_i(x) - \lim_{k \to 0} \frac{|f_i(x+k) - f_i(x)|}{|k|}(h)|}{|h|} \\ &= \lim_{h \to 0} \frac{|f_i(x+h) - f_i(x) - f_i(x+h) + f_i(x)|}{|h|} \\ &= 0. \end{aligned}$$

   Thus, $J(x) = Df(x)$ for all x. Therefore, Df exists.

3.