

Homework 5 of Introduction to Analysis(II)

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1. (a) If $x \in A$, $d(x, A) = \|x - x\| = 0$ and $d(x, B) = k > 0$, $\phi(x) = \frac{0}{0+k} = 0$. If $x \in B$, $d(x, A) = l > 0$ and $d(x, B) = 0$, $\phi(x) = \frac{l}{l+0} = 1$. If $x \notin A \wedge x \notin B$, $d(x, A) = l$ and $d(x, B) = k$, $\phi(x) = \frac{l}{l+k} < 1$ and is positive. Thus, $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^n$.

- (b) Let $\phi(x) = (b-a) \frac{d(x, A)}{d(x, A) + d(x, B)} + a$. From (a), we can get $\phi(x \in A) = (b-a) \cdot 0 + a = a$, $\phi(x \in B) = (b-a) \cdot 1 + a = b$, and $a \leq \phi(x) \leq b$ for all $x \in A$.

2. If f has more than one fixed point, there exists $x, y \in S$ s.t. $d(f^n(x), f^n(y)) = d(x, y)$ for all $n \in \mathbb{N}$ (contradiction to $a_n \rightarrow 0$). Then, we want to show that f has fixed point. For any $x_0 \in S$, we let $x_k = f^k(x)$ and we can find a $N \in \mathbb{N}$ s.t. $a_n < \varepsilon$ for all $n > N$. Then, $d(x_{n+k}, x_n) \leq a_n d(x_k, x_0) < \varepsilon \cdot d(x_k, x_0)$ for any k and $n > N$. Thus, $x_n \rightarrow x^* \in S$ by S is complete, and x^* is a fixed point of f .

3. Since $T(u)(t) = \int_a^t u(s) ds$,

$$\begin{aligned} T^2(u)(t) &= T(T(u))(t) \\ &= \int_a^t T(u)(\tau) d\tau \\ &= \int_a^t \int_a^\tau u(s) ds d\tau \\ &= \int_a^t \int_a^s u(s) d\tau ds \\ &= \int_a^t (t-s)u(s)ds \end{aligned}$$