Exercises(1)

February 20, 2024

1. For each $n \in \mathbb{N}$, let g_n be defined for $x \geq 0$ by the formula

$$g_n(x) = nx, \quad 0 \leq x \leq 1/n,$$
$$= \frac{1}{nx}, \quad 1/n < x.$$

   (a) (6 points) Show that $\lim_{n\to\infty} g_n(x) = 0$ for all $x > 0$.

   (b) (6 points) Show that the convergence is not uniform on the domain $x \geq 0$, but that it is uniform on a set $x \geq c$, where $c > 0$.

2. Let (M, d) be a metric space, E be a nonempty set, and $f_k, f : E \to M$ $(k = 1, 2, \ldots)$.

   (a) (6 points) Show that $f_k \to f$ uniformly on E if and only if $\sup\{d(f_k(x), f(x)) : x \in E\} \to 0$ as $k \to \infty$.

   (b) (6 points) Show that f_k does not converge uniformly to f on E (i.e. $f_k \nrightarrow f$ uniformly on E) if and only if there is a sequence $\{x_k\}_{k=1}^{\infty}$ in E such that $\overline{\lim}_{k\to\infty} d(f_k(x_k), f(x_k)) > 0$.

   (c) (6 points) Let $f_k(x) = (1/k)e^{-k^2x^2}$ if $x \in \mathbb{R}$, $k = 1, 2, \ldots$. Prove that $f_k \to 0$ uniformly on $\mathbb{R}$, that $f_k' \to 0$ pointwise on $\mathbb{R}$, but that convergence of $\{f_k'\}$ is not uniform on any interval containing the origin.