Exercises(15)

May 28, 2024

1. (a) Suppose that E is a Jordan region in $\mathbb{R}^n$ and that $f_k : E \to \mathbb{R}$ are integrable on E for $k \in \mathbb{N}$. If $f_k \to f$ uniformly on E as $k \to \infty$, prove that f is integrable on E and
$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx.$$
   (b) Prove that
$$\lim_{k\to\infty} \int_E \cos(x/k)e^{y/k} dV$$
exists, and find its value for any Jordan region E in $\mathbb{R}^2$.

2. Suppose that V is open in $\mathbb{R}^n$ and that $f : V \to \mathbb{R}$ is continuous. Prove that if
$$\int_E f dV = 0$$
for all nonempty Jordan regions $E \subseteq V$, then $f = 0$ on V.

3. Suppose that E is a Jordan region and that $f : E \to \mathbb{R}$ is integrable.
   (a) If $f(E) \subseteq H$, for some compact set H, and $\phi : H \to \mathbb{R}$ is continuous, prove that $\phi \circ f$ is integrable on E.
   (b) Show that part (a) is false if ϕ has even one point of discontinuity.

4. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y) = \int_a^b f(x, y)dx$.
   (a) Prove that $F(y) = \int_a^b f(x, y)dx = \int_a^b (\int_c^y D_2 f(x, t)dt + f(x, c))dx$.
   (b) Use (a) to prove that $F'(y) = \int_a^b D_2 f(x, y)dx$.