Homework 2 of Introduction to Analysis(II)

AM15 黃琦翔 111652028

March 7, 2024

1. For all $\varepsilon > 0$, we can find a $N \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$ for all $n > N$. And we can find a $N' > N$ s.t. $\frac{\sum_{i=1}^{N} a_i - a}{N'} < \frac{\varepsilon}{2}$. Thus, for any $n > N'$,

$$
\begin{aligned}
|\frac{\sum_{i=1}^{n} a_i}{n} - a| &\leq |\frac{\sum_{i=1}^{N} a_i - a}{n}| + |\frac{\sum_{i=N+1}^{n} a_i - a}{n - N'}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

   Therefore, $\lim_{n\to\infty} b_n = a$.

2. Since $\lim_{n\to\infty} na_n = 0$, by 1., $\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{na_n}{N} = 0$ Also, since $\lim_{n\to\infty} na_n = 0$, $\lim_{n\to\infty} a_n = 0$. Thus, for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=0}^{k} a_n| < \frac{\varepsilon}{2}$ for all $k > N$. Then, for $x \to 1^-$ s.t. $|f(x) - A| < \frac{\varepsilon}{2}$,

$$
\begin{aligned}
\sum_{n=0}^{N} a_n - A &= \sum_{n=0}^{N} a_n(1 - x^n) - \sum_{n=N+1}^{\infty} a_n x^n + (f(x) - A) \\
&\leq \frac{\varepsilon}{2} - 0 + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$

   Therefore, $\sum_{n=0}^{\infty} a_n = A$.

3. Since $\lim_{x\to 1^-}\sum_{n=1}^{\infty} a_n x^n = A$, for any $\varepsilon > 0$, we can find a $x_1 \in (0,1)$ s.t. $|\sum_{n=1}^{\infty} a_n x^n - A| < \frac{\varepsilon}{3}$ for all $x_1 < x < 1$. Then, there exists $N \in \mathbb{N}$ s.t. $|\sum_{n=N'=1}^{\infty} a_n x^n| < \frac{\varepsilon}{3}$ for all x and $N' > N$. And we have x_2 s.t. $|\sum_{n=1}^{N} a_n(1-x^n)| < \frac{\varepsilon}{3}$ for all $x_2 < x < 1$.

Thus, for any $N' > N$ and $x > \max\{x_1, x_2\}$,

$$\begin{aligned}
|\sum_{n=0}^{N} a_n - A| &\leq |\sum_{n=0}^{N'} a_n(1-x^n)| + |\sum_{n=N'+1}^{\infty} a_n x^n| + |\sum_{n=1}^{\infty} a_n x^n - A| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} a_n = A$.

4.

$(\implies)$ Since $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly, there exists a $N \in \mathbb{N}$ s.t. $|\sum_{n=N+1}^{\infty} a_n \sin(nx)| < \varepsilon$ for all x. And since we can find x s.t. $\sin(nx) = 1$ for all x and all n, we can rewrite it as $\lim_{N\to\infty}\sum_{n=N+1}^{\infty} a_n| = \lim_{N\to\infty}\sum_{n=N+1}^{\infty} \frac{na_n}{n} = 0$. By 1., $\lim_{n\to\infty} na_n = 0$.

$(\impliedby)$ Since $na_n \to 0$, we have

$\left|\sum_{n=N+1}^{\infty} a_n\right| \to 0$ as $N \to \infty$. And since $|\sin(nx)| \leq 1$ for all x, $\left|\sum_{n=N+1}^{infty} a_n sin(nx)\right| \leq \sum_{n=N+1}^{\infty} a_n \to 0$.

By Cauchy Criterion, $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly.