

# Homework 12 of Introduction to Analysis(II)

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1. (a) Since  $\partial \text{int}(E) = \text{cl}(\text{int}(E)) \setminus \text{int}(\text{int}(E)) = \text{cl}(E) \setminus \text{int}(E) = \partial E$  and  $\partial \text{cl}(E) = \partial E$ , we can get  $\text{Vol}(\partial \text{int}(E)) = \text{Vol}(\partial \text{cl}(E)) = \text{Vol}(\partial E) = 0$  and  $\text{int}(E), \text{cl}(E)$  are Jordan regions.

- (b) Since  $\text{cl}(E) = \text{int}(E) \cup \partial E$  and  $\text{int}(E) \subseteq E \subseteq \text{cl}(E)$ ,

$$\text{Vol}(\text{cl}(E)) \leq \text{Vol}(\text{int}(E)) + \text{Vol}(\partial E) = \text{Vol}(\text{int}(E)) \leq \text{Vol}(E) \leq \text{Vol}(\text{cl}(E)).$$

Therefore,  $\text{Vol}(\text{cl}(E)) = \text{Vol}(\text{int}(E)) = E$ .

(c)

( $\implies$ ) From (b), we know  $\text{Vol}(\text{int}(E)) = \text{Vol}(E) > 0$ , then for any  $\varepsilon > 0$ , we can find a set of rectangles  $R_n$  s.t.  $\text{Vol}(E) > \sum |R_n| > \text{Vol}(E) - \varepsilon$  and  $\cup R_n \subseteq \text{int}(E)$ . Therefore,  $\text{int}(E) \neq \emptyset$ .

( $\impliedby$ ) Since  $\text{int}(E)$  is non-empty, for any  $x_0 \in \text{int}(E)$ , there exists  $\varepsilon > 0$  s.t.  $D(x_0, \varepsilon) \subseteq \text{int}(E)$ . Then, we can find a small rectangle  $R$  with each length is  $\frac{\varepsilon}{2}$  and  $R$  is contained in  $D(x_0, \varepsilon)$ . Thus,  $\text{Vol}(\text{int}(E)) > \left(\frac{\varepsilon}{2}\right)^2 > 0$ .

- (d) Since  $f$  is continuous, for any  $x_0 \in [a, b]$ , we can find a sequence  $x_k \rightarrow x_0$  s.t.  $f(x_k) \rightarrow f(x_0)$ . That is,  $A = \{(x, f(x)) \mid x \in [a, b]\}$  is closed. And since  $\partial A \subseteq A$ ,  $\text{Vol}(\partial A) \leq \text{Vol}(A)$ .

Since  $f$  is continuous on  $[a, b]$  is compact, for all  $\varepsilon > 0$ , we can find  $\delta > 0$  s.t.  $|x - y| < \delta$  implies

That  $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ . Then, we can find a finite increasing sequence  $\{x_i \mid x_i \in [a, b]\}_{i=1}^N$  s.t.

$[a, b] \subseteq D(x_i, \frac{\delta}{2})$ . Therefore, for any  $y = f(x)$ ,  $y \in D(f(x_i), \frac{\varepsilon}{b-a})$  for some  $i$ .

Then, take  $u_0 = a$ ,  $u_i \in D(x_i, \frac{\delta}{2}) \cap D(x_{i+1}, \frac{\delta}{2})$ ,  $u_N = b$ , and we can get  $[a, b] = \cup [u_i, u_{i+1}]$ . Thus,

$A \subseteq \bigcup_{i=0}^N [u_i, u_{i+1}] \times D(\xi_i, \frac{\varepsilon}{b-a})$  for some  $\xi_i \in \{f(x) \mid x \in [u_i, u_{i+1}]\}$  with the sum of the rectangles is  $2 \frac{\varepsilon}{b-a} \cdot (b-a) = 2\varepsilon$ . Hence,  $\text{Vol}(A) = 0$  and  $\text{Vol}(\partial A) = 0$ .

(e) Yes. Since  $f$  is integrable, for any  $\varepsilon > 0$ , we can find a partition  $P$  s.t.  $|U(f, P) - L(f, P)| < \varepsilon$ .

That is,  $\sum_{i=0}^N (\sup_{x \in [x_i, x_{i+1}]} \{f(x)\} - \inf_{x \in [x_i, x_{i+1}]} \{f(x)\}) \cdot (x_{i+1} - x_i) < \varepsilon$ , and each one of the summation is a rectangle that contains all  $(x, f(x))$  in the interval. Thus,  $\text{Vol}(A) = 0$  and  $\text{Vol}(\partial A) = 0$ .

2. (a)  $\partial(E_1 \cap E_2) = \text{cl}(E_1 \cap E_2) \cap \text{cl}(M \setminus E_1 \cap E_2) \subseteq (\text{cl}(E_1) \cap \text{cl}(M \setminus E_1)) \cup (\text{cl}(E_2) \cap \text{cl}(M \setminus E_2)) = \partial E_1 \cup \partial E_2$ . Then,  $\text{Vol}(\partial(E_1 \cap E_2)) \leq \text{Vol}(\partial E_1) + \text{Vol}(\partial E_2) = 0$ .

Also  $\partial(E_1 \setminus E_2) \subseteq \partial E_1 \cup \partial E_2$ . Therefore,  $E_1 \cap E_2$  and  $E_1 \setminus E_2$  are Jordan regions.

(b) Since  $E_1, E_2$  are non-overlapping,  $E_1 \cap E_2 \subseteq \partial E_1 \cup \partial E_2$ . And since  $E_1, E_2$  are Jordan regions,  $\text{Vol}(E_1) = \text{Vol}(\text{int}(E_1))$  and  $\text{Vol}(E_2) = \text{Vol}(\text{int}(E_2))$ . Also we have  $\text{int}(E_1) \cup \text{int}(E_2) \subseteq E_1 \cup E_2$  and  $\text{int}(E_1) \cap \text{int}(E_2) = \emptyset$ .

$$\begin{aligned}
\text{Vol}(\text{int}(E_1)) + \text{Vol}(\text{int}(E_2)) &= \text{Vol}(\text{int}(E_1) \cup \text{int}(E_2)) \\
&\leq \text{Vol}(\text{int}(E_1) \cup \text{int}(E_2)) + \text{Vol}(E_1 \cap E_2) \\
&= \text{Vol}(E_1 \cup E_2) \\
&\leq \text{Vol}(\text{int}(E_1)) + \text{Vol}(\text{int}(E_2)) + \text{Vol}(\partial E_1) + \text{Vol}(\partial E_2) \\
&= \text{Vol}(\text{int}(E_1)) + \text{Vol}(\text{int}(E_2)) \\
&= \text{Vol}(E_1) + \text{Vol}(E_2)
\end{aligned}$$

Thus,  $\text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2)$ .

(c) Let  $E_1 \setminus E_2 = S_1$ ,  $E_2 = S_2$ . We have  $S_1$  and  $S_2$  are Jordan regions and non-overlapping. Thus,

$$\text{Vol}(E_1) = \text{Vol}(S_1 \cup S_2) = \text{Vol}(S_1) + \text{Vol}(S_2) \text{ and } \text{Vol}(E_1 \setminus E_2) = \text{Vol}(E_1) - \text{Vol}(E_2).$$

(d) Using (b), (c) and the fact  $E_1 \cup E_2 = (E_1 \setminus E_2 \cap E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_2 \cap E_1)$ , we can get  $\text{Vol}(E_1 \cup E_2) = (\text{Vol}(E_1) - \text{Vol}(E_1 \cap E_2)) + \text{Vol}(E_1 \cap E_2) + (\text{Vol}(E_2) - \text{Vol}(E_1 \cap E_2)) = \text{Vol}(E_1) + \text{Vol}(E_2) - \text{Vol}(E_1 \cap E_2)$ .