

# Homework 14 of Introduction to Analysis(II)

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1. Let  $A_M = \{x \in E \mid f_M \text{ is discontinuous at } x\}$  and  $A = \{x \in E \mid f \text{ is discontinuous at } x\}$ . For any  $M \in \mathbb{N}$ , if  $x$  is a point that  $f_M$  is discontinuous at  $x$ , That means  $f(x) \leq M$  and  $f$  is discontinuous at  $x$ . That is,  $A_M \subseteq A$  for all  $M$  implies that  $\cup A_m \subseteq A$ . And for any  $x \in A$ , there exists a  $N \in \mathbb{N}$  s.t.  $f(x) < N$ . Then,  $x \in A_N$ . Therefore,  $A = \cup A_M$ .

2. First, want to proof  $E$  contains finite union of intervals. Since  $f$  is integrable, we can find some  $x \in E$  and  $U \subseteq E$  for  $U$  is neighborhood of  $x$ . That is, we can find union of open intervals  $\cup_N I_n$  in  $E$  with  $|I_1| \geq |I_2| \geq \dots$ , and we let  $L = \sum_{n=1}^N |I_n|$ .

Suppose  $M_0 = \sup f(x)$  and  $\alpha_0 = \int_0^1 f(x) dx$ , We want to show that there exists finite  $N_0$  s.t.  $\sum_{i=1}^{N_0} |I_n| \geq \frac{\alpha_0}{4M_0}$ .

3. Let  $A = \{x \in E \mid f(x) \neq 0\}$ . If  $A$  is empty, then  $A$  is measure zero.

Suppose  $A$  is non-empty. Then, for a large enough  $N \in \mathbb{N}$ ,  $A_N = \{x \in E \mid f(x) > \frac{1}{N}\}$ . Using the same argument of 1. , we can have  $A_N \rightarrow A$  as  $N \rightarrow \infty$ . Since  $\int_E f(x) dx = 0$ ,  $\int_{A_N} f(x) dx = 0$ . Thus, for any  $\varepsilon > 0$ , there exists rectangles such that  $\frac{1}{N} \sum |R_i| \leq (L) \int_{A_N} f(x) dx \leq \frac{\varepsilon}{N}$ . Therefore,  $\sum |R_i| < N \cdot \frac{\varepsilon}{N} = \varepsilon$  and  $A_N$  is measure zero.

By the theorem that countable set of measure zero is also measure zero, we can have  $A$  is measure zero.