

# Homework 13 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

December 12, 2023

1. First, we want to check there are at most  $\frac{n(n+1)}{2}$  points for  $x \in [0, 1]$  for which  $f(x) > \frac{1}{n}$ .

For  $f(x) > \frac{1}{n}$ , that means  $x$  is a rational number  $\frac{p}{q}$  and  $q < n$ . Thus, we have  $x = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \dots$ . The number of  $x$  is less than  $\frac{n(n+1)}{2}$ . Thus, we suppose  $a_n$  is the number of  $x$  s.t.  $f(x) > \frac{1}{n}$

For any  $n \in \mathbb{N}$ , we can find a  $\delta > 0$  and partition  $P_n = \bigcup_{i=1}^n [x_i - \delta, x_i + \delta]$  s.t.  $x_i$  is the only point  $x \in [x_i - \delta, x_i + \delta]$  that  $f(x) > \frac{1}{n}$  and  $P'_n = [0, 1] \setminus P_n$ . Thus,  $|U(f, P_n) - L(f, P_n)| + |U(f, P'_n) - L(f, P'_n)| = U(f, P_n) + U(f, P'_n) < a_n \cdot (2\delta \cdot 1) + (1 - a_n \cdot 2\delta) \frac{1}{n} < \frac{1}{n} + a_n \cdot 2\delta$ . Thus, we can find large enough  $n$  and a tiny  $\delta$  s.t.  $f$  is integrable.

2. Since  $f$  is bounded, we can find  $M \in \mathbb{R}$  s.t.  $|f(x)| < M$  for all  $x \in [a, b]$ . Suppose there are  $n$  points of discontinuity of  $f$  on  $[a, b]$ . Thus, for any  $\varepsilon > 0$ , we take  $\delta = \frac{\varepsilon}{8nM}$ . Then, suppose the set of points of discontinuity is  $\{y_i \mid i \in \mathbb{N}, i \leq n\}$ . Take partition  $P = \{x_1 = y_1 - \varepsilon, x_2 = y_1 + \varepsilon, x_3 = y_2 - \varepsilon, \dots\}$  is finite. Then, for  $f$  is continuous on  $[a, x_1], [x_2, x_3], [x_4, x_5], \dots$ , we can find a partition  $P_1$  s.t.  $|U(f, P_1) - L(f, P_1)| < \frac{\varepsilon}{2}$  we only need to check other intervals  $P_2 = [x_1, x_2] \cup [x_3, x_4] \cup \dots \cup [x_{2n-1}, x_{2n}]$  are Riemann integrable.  $|U(f, P_2) - L(f, P_2)| \leq 2U(f, P_2) \leq 2 \cdot n \cdot (2\delta) \cdot M = 4nM \cdot \frac{\varepsilon}{8nM} = \frac{\varepsilon}{2}$ . Thus,  $|U(f, P) - L(f, P)| \leq |U(f, P_1) - L(f, P_1)| + |U(f, P_2) - L(f, P_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ , which implies that  $f \in R([a, b])$ .

$$3. f^{(n)}(x) = \left(\prod_{k=0}^{n-1} m-k\right)(1+x)^{m-n}. R_n = x^n \frac{f^{(n)}(\theta_n x)}{n!} = \left(\prod_{k=1}^n \frac{m-k+1}{k}\right)(1+\theta_n x)^{m-n} x^n.$$

For  $m > 0$ , we can find  $N \in \mathbb{N}$  s.t.  $N > m$  and  $N-1 \leq m$ . Since  $x > 0$ ,  $1+\theta_n x > 1$ ,  $(1+\theta_n x)^{m-n} < 1$  for  $n > N$ . And since  $n > N > M$ ,  $\prod_{k=1}^n \frac{m-k+1}{k} = \frac{m}{N} \cdot \frac{m-1}{N-1} \cdots \frac{m-N+1}{1} \cdot \frac{m-N}{N+1} \cdots < 1$ . Thus,  $R_n < x^n$  for  $m > 0$  and  $n > m$ .

For  $m < 0$ , since  $|\frac{m-k}{k}| > |\frac{m-k-1}{k+1}| > \cdots > 1$ , there exists  $N \in \mathbb{N}$  s.t.  $|\frac{m-n}{n}| < \frac{1}{x}$  for all  $n > N$ .

Thus,  $R_n < \prod_{k=1}^n \left(\frac{m-k+1}{k} x\right) = A \cdot \prod_{k=N}^n \frac{m-k+1}{k} x \rightarrow 0$  as  $n \rightarrow \infty$  since  $\frac{m-k+1}{k} x < l < 1$  for all  $n \geq N$ .