Exercises(3)

September 26, 2023

1. (8 points) For $x, y \in \mathbb{R}$, define

$$d_1(x,y) = (x-y)^2;\ d_2(x,y) = \frac{|x-y|}{1+|x-y|}.$$

   Determine, for each of these, whether it is a metric or not.(Justify your answer!)

2. (7 points) Let $x_n > 0$, $n = 1, 2, \ldots$ and let $s_n = x_1 + \cdots + x_n$. If $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$, show that $\{s_n\}_{n=1}^{\infty}$ converges.

3. Definition. If $X = \{x_n\}_{n=1}^{\infty}$ is a sequence of elements in $\mathbb{R}$, then the sequence $\{\sigma_n\}_{n=1}^{\infty}$ defined by

$$\sigma_1 = x_1, \sigma_2 = \frac{x_1 + x_2}{2}, \ldots, \sigma_n = \frac{x_1 + \cdots + x_n}{n}, \ldots,$$

   is called the sequence of arithmetic means of X.

   (a) (7 points) If $\{x_n\}_{n=1}^{\infty}$ is a monotone sequence in $\mathbb{R}$, show that the sequence of arithmetic means is monotone.

   (b) (8 points) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ and $\{\sigma_n\}_{n=1}^{\infty}$ is the sequence of arithmetic means, show that

$$\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} x_n.$$