Homework 7 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

February 19, 2024

1. First, we show that $\bigcap_{i=1}^{\infty} F_i$ is closed. For F_k, we let $V_k = M \setminus F_k$ which is a open set since F_k is compact(closed and bounded). Then, $M \setminus (\bigcap_{i=1}^{\infty} F_i) = \bigcup_{i=1}^{\infty} V_i = \lim_{i\to\infty} V_i$ is open. Thus, $\bigcap_{i=1}^{\infty} F_i$ is closed.

   Then, assume $\bigcap_{i=1}^{\infty} F_i$ is disconnected. Thus, we can find two non-empty open sets U, V in $\bigcap_{i=1}^{\infty} F_i$ s.t. $U \cap V = \emptyset$ and $\bigcap_{i=1}^{\infty} F_i = U \cup V$. Since $F_k \subseteq F_{k-1}$, $U, V \subseteq F_k$ for all k.

   And since F_k is connected, $U \cup V \subsetneq F_k$ for all k. Then, we let $S_i = F_i \setminus (U \cup V)$. Since U, V are open, $U \cup V$ is open $\implies$ S_i is closed in F_i.

   Since $S_i \subseteq F_i \subseteq F_1$ and F_1 is conpact, by Nested Interval Property, $\bigcap_{i=1}^{\infty} S_i \neq \emptyset \implies \bigcap_{i=1}^{\infty} F_i = U \cup V \cup \bigcap_{i=1}^{\infty} S_i$ contradict to $\bigcap_{i=1}^{\infty} F_i = U \cup V$.

   Thus, $\bigcap_{i=1}^{\infty} F_i$ is connected.

2. Assume there exists $a \in S$ and $r \geq 0$ s.t. $\{x \mid d(x,a) = r\} = \emptyset$. Then, $D(a,r) = \{a\}$ is open and $S \setminus \{a\}$ is open since for all $x \in S \setminus \{a\}, D(x, d(x,a)) \subseteq S \setminus \{a\}$. We can find two open sets $D(a,r), S \setminus \{a\}$ s.t. S is not connected.(contradiction)

   Thus, for every a in S and every $r > 0$, the set $\{x : d(x,a) = r\}$ is nonempty.

3. Assume B is disconnected, There are two non-empty open set $U, V \subseteq B$ s.t. $U \cup V = B$. Then, if $U \cap A$ and $V \cap A$ are relative open and nonempty in A and disjoint in $A \implies A$ is not connected.(contradiction)

   If one of U or V is empty in A, that means some of accumulation point is not connected to A. That is, A and some accumulation points of A, called $X = B \setminus A$ are relative open to B. But for every $x \in X$, $D(x,\varepsilon) \setminus \{x\} \cap A \neq \emptyset \implies D(x,\varepsilon) \nsubseteq X$. That means X is not open.(contradiction).

   Thus, B is connected.

4.

$(\implies)$ Assume A is not in the set, that is, A can be write as $\bigcup_{i \in E} I_i$ where I_i is the interval in the set and I_m, I_n is disconnected. Thus, A is disconnected.

therefore, A is connected $\implies A$ is in the set.

$(\impliedby)$ For any x, y in one of the four interval, exists $f : [0,1] \to [x,y]$ with $f(t) = tx + (1-t)y$ is a continuous path. Thus, A is path-connected $\implies A$ is connected.