Homework 12 of Introduction to Analysis (I), Honor Class

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1. We claim that a_n is a Cauchy. Then, for $m > n \geq N$

$$\begin{aligned} |f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\ &< \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} \leq \frac{1}{N} \end{aligned}$$

   Thus, for any $\varepsilon > 0$, we take $N > \frac{1}{\varepsilon}$. Therefore, a_n is Cauchy $\Longrightarrow \lim_{n\to\infty} a_n$ exists.

2. Since $f'(x)$ exists on (a,b), we can find $s_1 = \sup\{f'(x) \mid x \in (a,b)\}$, $s_2 = \lim_{x\to a^+} f(x)$ and $s = \max\{s_1, s_2\}$.

   Then, for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{s}$, then $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$. Thus, f is uniform continuous.

3. (a) Since f is differentiable, f is continuous. Thus, for any $h > 0$, there exists a $c \in (x, x+h)$ s.t. $f'(c)h = f(x+h) - f(x)$ by MCT. For $h < 0$, there also exists $c \in (x+h, x)$ s.t. $f'(c)h = f(x+h) - f(x)$.

   Thus, $\lim_{x\to\infty} \frac{f(x+h) - f(x)}{h} = \lim_{x\to\infty} f'(c) = b$.

   (b) Since $f(x) \to a$, for $\varepsilon, h > 0$, exsits $N \in \mathbb{N}$ s.t. $|f(x) - a| < \frac{\varepsilon \cdot h}{2}$.

   Then, for $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} |\frac{f(x+h) - f(x)}{h}| \leq \lim_{x\to\infty} \frac{|f(x+h) - a| + |a - f(x)|}{h} < 2\frac{\varepsilon \cdot h}{2h} = \varepsilon$.

   Thus, $f'(x) \to 0$ as $x \to \infty$

(c) For any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Then, for any $x_0 > N$

$$\begin{aligned}\lim_{x\to\infty} \left|\frac{f(x)}{x} - b\right| &= \lim_{x\to\infty} \left|\frac{f(x_0) + f'(x_1)(x - x_0)}{x} - b\right| \\ &= \lim_{x\to\infty} \left|\frac{f(x_0) - f'(x_1)x_0}{x}\right| + |f'(x_1) - b| \text{ for } x_1 \in [x_0, x] \\ &< 0 + \varepsilon = \varepsilon\end{aligned}$$

Thus, $\dfrac{f(x)}{x} \to b$ as $x \to \infty$.

4. (a)

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} &= \lim_{h\to 0} \frac{f'(a+2h) - f'(a+h)}{h} \\ &= 2\lim_{h\to 0} \frac{f'(a+2h) - f'(a)}{2h} - \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h} \\ &= 2f''(a) - f''(a) = f''(a)\end{aligned}$$

(b)

$$\begin{aligned}\lim_{h\to 0} \frac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3} &= \lim_{h\to 0} \frac{3f'(a+3h) - 6f'(a+2h) + 3f(a+h)}{3h^2} \\ &= \lim_{h\to 0} \frac{f'(a+3h) - 2f'(a+2h) + f'(a+h)}{h^2} \\ &= \lim_{h\to 0} \frac{3f''(a+3h) - 4f''(a+2h) + f''(a+h)}{2h} \\ &= \lim_{h\to 0} \frac{3f''(a+3h) - 3f''(a)}{2h} - \frac{4f''(a+2h) - 4f''(a)}{2h} \\ &\quad + \frac{f''(a+h) - f''(a)}{2h} \\ &= \left(\frac{9}{2} - 4 + \frac{1}{2}\right) f^{(3)}(a) \\ &= f^{(3)}(a)\end{aligned}$$