Homework 1 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

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1. Let $R = \{f(x) \mid x \in X\}$ and $r = \sup R$. Then, for any β which is upper bound of R, $r \leq \beta$. Thus, $a+\beta$ is an upper bound of $R' = \{a + f(x) \mid x \in X\}$.

   Assume there exists $\beta' = \sup R'$ s.t. $\beta' < a+r$. Since $\beta' \geq a + f(x)$, $\forall x \in X \Rightarrow \beta' - a \geq f(x)$, $\forall x \in X$. Therefore, $\beta' - a$ is an upper bound of R and it should greater or equal to r. This causes contradiction to the assumption.

   Thus, for every upper bound f R', it is greater or equal to $a+r$.

   That is $\sup\{a + f(x) \mid x \in X\} = a + \{f(x) \mid x \in X\}$

2. let a be greastest lower bound of $\{f(x) \mid x \in X\}$, b be greastest lower bound of $\{f(x) \mid x \in X\}$. That is $a \leq f(x) \forall x \in X$ and $b \leq g(x) \forall x \in X$. Then, $a+b$ is an lower bound of $\{f(x)+g(x) \mid x \in X\}$. Therefore $a+b \leq \inf\{f(x)+g(x) \mid x \in X\}$. In other world, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x)+g(x) \mid x \in X\}$

   Using the same way we can prove $\sup\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

   Then, we want to show $\inf S \leq \sup S$ for all set S:

   > For a upper bound of S "β" and lower bound of S "α". For any element $x \in S$, $\alpha \leq x \leq \beta$.
   > Then, $\inf S \leq x \leq \sup S$ for all $x \in S$

   Then, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

   Ex. Let $f(x) = \sin(x),\ g(x) = \cos(x),\ X = (0, 2\pi)$, then $-2 < -\sqrt{2} < 2$.

3. Let $A = \{x \mid x \in \mathbb{R},\ b^x < y\}$.

   By Bernoulli's Inequality(In Bartle's book page.35) and let $b = 1 + c$ with $c \in \mathbb{R},\ c > 0$, then $(1+c)^n \geq 1 + nc$. Then, we can find some $n \in \mathbb{N}$ s.t. $c > \dfrac{y}{n}$. Therefore, $b^n = (1+c)^n \geq 1 + nc > y$ and n is upper bound of A. Thus, by completeness of $\mathbb{R}$, there exists $\beta \in \mathbb{R}$ be $\sup A$.

   If $b^\beta < y$, then assume $\beta + \dfrac{1}{n} \in A$. Thus $b^{\beta + \frac{1}{n}} = b^\beta \cdot b^{\frac{1}{n}} > b^\beta \cdot 1 = b^\beta$ for all $n \in \mathbb{N}$. Then, we want to proof $b^{\beta + \frac{1}{n}} < y$, thus we check the inequality $b^{\beta + \frac{1}{n}} - b^\beta < y - b^\beta$ for some $n \in \mathbb{N}$.

   > **Proof.** $b^{\beta + \frac{1}{n}} - b^\beta = b^\beta (b^{\frac{1}{n}} - 1)$. Then, we want to proof $b^{\frac{1}{n}} - 1 \to 0$ when $n \to \infty$. Since $b^{\frac{1}{n}} \leq \dfrac{1}{n}$

   $\beta \in A$ by the definition of A(contradiction). Therefore, $b^\beta \geq y$.

   If $b^\beta > y$, we can find a real number r' s.t. $b^\beta > r' > y$. Therefore, we can find $x \in A$ s.t $b^x \leq r'$. Thus, β is not least upper bound of A.

   These two line implies the supremun of A, β, is the unique real number s.t. $b^\beta = y$ since the supremun is unique.