Homework 12 of Introduction to Analysis (I), Honor Class

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1. We claim that a_n is a Cauchy. Then, for $m > n \geq N$

$$\begin{aligned}|f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\ &< \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} \leq \frac{1}{N}\end{aligned}$$

   Thus, for any $\varepsilon > 0$, we take $N > \frac{1}{\varepsilon}$. Therefore, a_n is Cauchy $\Longrightarrow \lim_{n\to\infty} a_n$ exists.

2. Since $f'(x)$ exists on (a,b), we can find $s_1 = \sup\{f'(x) \mid x \in (a,b)\}$, $s_2 = \lim_{x\to a^+} f(x)$ and $s = \max\{s_1, s_2\}$.

   Then, for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{s}$, then $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$. Thus, f is uniform continuous.

3. (a) Claim that for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\frac{f(x+h)-f(x)}{h} - b| < \varepsilon$ for $x > N$ and all h.

   Suppose $b \geq 0$. Since $f'(x) \to b$ as $x \to \infty$, for any $\varepsilon > 0$, exists a $N_0 \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N_0$. Thus, $b - \varepsilon < f'(x) < b + \varepsilon$ for all $x > N_0$. By MVT, $f(x+h) = f(x) + h \cdot f'(c)$ for all h and some $c \in [x+h, x]$ or $c \in [x, x+h]$. Thus, $|f(x+h) - f(x)| < |h|(b+\varepsilon)$ for all h and $x > N_0$. Then, we can get the result by taking $N = N_0$.

   If $b < 0$, we can get the same result $|f(x+h) - f(x)| < |h||b - \varepsilon|$ by the silmilar way.

   Therefore, $\lim_{x\to\infty} \frac{f(x+h)-f(x)}{h} - b = 0$.

(b) Since $f(x) \to a$, for $\varepsilon, h > 0$, exsits $N \in \mathbb{N}$ s.t. $|f(x) - a| < \dfrac{\varepsilon \cdot h}{2}$.

Then, for $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} |\dfrac{f(x+h)-f(x)}{h}| \leq \lim_{x\to\infty} \dfrac{|f(x+h)-a|+|a-f(x)|}{h} < 2\dfrac{\varepsilon \cdot h}{2h} = \varepsilon$.

Thus, $f'(x) \to 0$ as $x \to \infty$

(c) For any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Then, for any $x_0 > N$

$$\begin{aligned}
\lim_{x\to\infty} |\frac{f(x)}{x} - b| &= \lim_{x\to\infty} |\frac{f(x_0)+f'(x_1)(x-x_0)}{x} - b| \\
&= \lim_{x\to\infty} |\frac{f(x_0)-f'(x_1)x_0}{x}| + |f'(x_1) - b| \text{ for } x_1 \in [x_0, x] \\
&< 0 + \varepsilon = \varepsilon
\end{aligned}$$

Thus, $\dfrac{f(x)}{x} \to b$ as $x \to \infty$.

4. (a) $\lim_{h\to 0} \dfrac{f(a+2h)-2f(a+h)+f(a)}{h^2} = \lim_{h\to 0} \dfrac{f'(a+2h)-f'(a+h)}{h} = \lim_{h\to 0} 2f''(a+2h) - f''(a+h)$

by L'Hospital Rule.

Since f is is three times differentiable, then f'' is continuous. Thus, we can find $\delta_1 > 0$ s.t. $|f''(a+h) - f''(a)| < \dfrac{\varepsilon}{3}$ for all $|h| < \delta_1$, and $\delta_2 > 0$ s.t. $|f''(a+2h) - f''(a)| < \dfrac{\varepsilon}{3}$ for all $|h| < \delta_2$.

Then, for $|h| < \min\{\delta_1, \delta_2\}$,

$$\begin{aligned}
|2f''(a+2h) - f''(a+h) - f''(a)| &= |f''(a+2h) - f''(a+h)| + |f''(a+2h) - f''(a)| \\
&< |f''(a+2h) - f''(a)| + |f''(a) - f(a+h)| + |f''(a+2h) - f''(a)| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

Thus, $\lim_{h\to 0} \dfrac{f(a+2h)-2f(a+h)+f(a)}{h^2} = f''(a)$.

(b)