

Homework 6 of Introduction to Analysis (I), Honor Class

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1. First, we show that S is closed and bounded. There exists $D(2,2) \subseteq S \implies S$ is bounded. And for $x \in \mathbb{Q} \setminus S$, if $x \leq \sqrt{2}$, $D(x, d(x, \sqrt{2})) \subseteq \mathbb{Q} \setminus S$. If $x \geq \pi$, $D(x, \pi) \subseteq \mathbb{Q} \setminus S \implies \mathbb{Q} \setminus S$ is open. Then, S is closed.

$G_i = (\sqrt{2} - \frac{1}{n}, \pi - \frac{1}{n}) \cap \mathbb{Q}$, then $\bigcup_{i=1}^{\infty} G_i$ is an open cover of S but doesn't have finite subcover. Thus, S is not compact.

2. Assume $\bigcap_{n=1}^{\infty} V_n = \emptyset$.

And since $\text{int}(V_n) \subseteq V_n \subseteq \text{cl}(V_n) \subseteq V_{n-1}$, $V_n \subseteq V_{n-1}$. Then, $\bigcap_{n=1}^k V_n = V_k$.

Since $\text{cl}(V_n)$ is compact,

3. Since $x, y \in K$, then assume there doesn't exist a set in G s.t. it contains both of x, y , then G doesn't contain one of x, y or both neither. Thus, G doesn't cover K , contradict to G is a open cover. Thus, there exists a set in G containing both x, y .

4. Since A is totally bounded, for all $\varepsilon > 0$, exists $\{x_i\}_{i=1}^N$ is a finite set s.t. $A \subseteq \bigcup_{i=1}^N D(x_i, \varepsilon)$.

And M is complete \implies every Cauchy sequence in M converge to a point in M .

Then, for any sequence $\{y_i\}$ in $\text{cl}(A)$, if there is not any subsequence converge to $\text{cl}(A)$, there exists $\varepsilon' > 0$ s.t. $|y_m - y_n| > \varepsilon'$ for all $m, n > N \in \mathbb{N}$. $\bigcup D(y_i, \varepsilon')$ will be a infinite open cover of A , which contradicts to there has a finite N s.t. $\bigcup D(x_i, \varepsilon')$ is cover of A .

Thus, every sequence in $\text{cl}(A)$ have subsequence converge to a point in $\text{cl}(A) \implies \text{cl}(A)$ is sequence compact. Then, by BWT, $\text{cl}(A)$ is compact.