# Homework 1 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

2023/9/12

1. Let $R = \{f(x) \mid x \in X\}$ and $r = \sup R$. Then, for any $\beta$ which is upper bound of $R$, $r \leq \beta$. Thus, $a + \beta$ is an upper bound of $R' = \{a + f(x) \mid x \in X\}$.

   Assume there exists $\beta' = \sup R'$ s.t. $\beta' < a + r$. Since $\beta' \geq a + f(x)$, $\forall x \in X \Rightarrow \beta' - a \geq f(x)$, $\forall x \in X$. Therefore, $\beta' - a$ is an upper bound of $R$ and it should greater or equal to $r$. This causes contradiction to the assumption.

   Thus, for every upper bound f $R'$, it is greater or equal to $a + r$.

   That is $\sup\{a + f(x) \mid x \in X\} = a + \{f(x) \mid x \in X\}$

2. let $a$ be greastest lower bound of $\{f(x) \mid x \in X\}$, $b$ be greastest lower bound of $\{f(x) \mid x \in X\}$. That is $a \leq f(x) \forall x \in X$ and $b \leq g(x) \forall x \in X$. Then, $a + b$ is an lower bound of $\{f(x) + g(x) \mid x \in X\}$. Therefore $a + b \leq \inf\{f(x) + g(x) \mid x \in X\}$. In other world, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}$

   Using the same way we can prove $\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

   Then, we want to show $\inf S \leq \sup S$ for all set $S$:

   > For a upper bound of $S$ "$\beta$" and lower bound of $S$ "$\alpha$". For any element $x \in S$, $\alpha \leq x \leq \beta$.
   > Then, $\inf S \leq x \leq \sup S$ for all $x \in S$

   Then, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

   Ex. Let $f(x) = \sin(x),\ g(x) = \cos(x),\ X = (0, 2\pi)$, then $-2 < -\sqrt{2} < 2$.

3. **Proof Bernoulli's Inequality and its Generalization of exponent**

Statement. For $x > 0$

$(1+x)^n \geq 1+nx, \ \forall n \in \mathbb{N}$
$(1+x)^r \leq 1+rx, \ \forall 0 \leq r \leq 1$

**Proof.** For $n \in \mathbb{N}$

(1) $(1+x)^1 \geq 1+x$

(2) $(1+x)^2 = 1+2x+x^2 \geq 1+2x$

Assume $n = k$ is true

(k+1) $(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1+kx+x+kx^2 \geq 1+(k+1)x$ is true

Then, the first line is proof by math induction.
Since $((1+x)^r)' = (r-1)(1+x)^{r-1} \leq r = (1+rx)'$, for all $0 \leq r \leq 1$ and $(1+x)^r = 1+rx$ when $x = 0$, $(1+x)^r \leq 1+rx$, for all $x > 0$ and $0 \leq r \leq 1$.

Let $A = \{x \mid x \in \mathbb{R}, \ b^x < y\}$.
Let $b = 1+c$ with $c \in \mathbb{R}, \ c > 0$, then $(1+c)^n \geq 1+nc$ with all $n \in N$. Then, we can find some $n \in \mathbb{N}$ s.t. $c > \frac{y}{n}$. Therefore, $b^n = (1+c)^n \geq 1+nc > y$ and $n$ is upper bound of $A$. Thus, by completeness of $\mathbb{R}$, there exists $\beta \in \mathbb{R}$ be $\sup A$.

If $b^\beta < y$, then assume $\beta + \frac{1}{n} \in A$. Thus $b^{\beta+\frac{1}{n}} = b^\beta \cdot b^{\frac{1}{n}} > b^\beta \cdot 1 = b^\beta$ for all $n \in \mathbb{N}$. Then, we want to proof $b^{\beta+\frac{1}{n}} < y$, thus we check the inequality $b^{\beta+\frac{1}{n}} < y$ for some $n \in \mathbb{N}$.

**Proof.** $b^{\beta+\frac{1}{n}} = b^\beta b^{\frac{1}{n}}$. Since $b^{\frac{1}{n}} = (1+c)^{\frac{1}{n}} \leq 1+\frac{1}{n}c$ and $\frac{y}{b^\beta}$ is a positive real number which is greater than 1, by Archimedeam Property there exists some $n \in \mathbb{N}$ s.t. $n > \frac{c \cdot b^\beta}{y - b^\beta} > 0$. In the other world $1+\frac{c}{n} < \frac{y}{b^\beta}$. Therefore, $b^{\beta+\frac{1}{n}} < y$.

Thus, $b^{\beta+\frac{1}{n}} \in A \Rightarrow b^\beta$ is not a upper bound of $A$. Therefore, $b^\beta$ should greater or equal to $y$.

If $b^\beta > y$, assume exists $n \in \mathbb{N}$ s.t. $b^{\beta-\frac{1}{n}} > y$. Then, $b^{\beta-\frac{1}{n}} = \frac{b^\beta}{b^{\frac{1}{n}}} \geq \frac{b^\beta}{1+\frac{1}{n}c}$. And since $\frac{b^\beta}{y} - 1$ is a positive real number by $b^\beta > y$, there exists some $n \in \ltimes$ s.t. $\frac{c}{n} > \frac{b^\beta}{y} - 1$ by Archimedeam Property. Therefore, $b^\beta$ is not the least upper bound of $A$(contradiction).

These two line implies the supremun of $A$, $\beta$, is the unique real number s.t. $b^\beta = y$ since the supremun is unique.