Homework 3 of Introduction to Analysis (I), Honor Class

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1. (d_1) a. positive definite: $d(x,y) = (x-y)^2 \geq 0$ for any $x, y \in \mathbb{R}$ check

   b. $0 \iff$ equal:

   $(\implies)$ $d(x,y) = 0 \implies x - y = 0 \implies x = y$

   $(\impliedby)$ $d(x,y) = (x-y)^2 = 0^2 = 0$ check

   c. triangle inequality: $4 = (2-0)^2 = d(2,0) > 1 + 1 = d(2,1) + d(1,0)$ no

   Thus, d_1 is not a metric.

   (d_2) a. positive definite: since $|x-y| \geq 0$, then $\dfrac{|x-y|}{1+|x-y|} \geq 0$ check

   b. $0 \iff$ equal:

   $(\implies)$ since $0 = d(x,y) = \dfrac{0}{1+0} \implies x - y = 0 \implies x = y$

   $(\impliedby)$ since $x = y$, $d(x,y) = \dfrac{0}{1+0} = 0$ check

   c. triangle inequality: If z is not in (x, y), the triangle inequality trivially holds.
   Then, WLOG, suppose $x < z < y$, then

   $$\frac{z-x}{1+z-x} + \frac{y-z}{1+y-z} > \frac{z-x}{1+(z-x)+(y-z)} + \frac{y-z}{1+(y-z)+(z-x)}$$
   $$= \frac{y-x}{1+(y-z)+(z-x)} = \frac{y-x}{1+y-x}$$

   Thus, d_2 is a metric.

2. We want to prove $s_{n+1} - s_n < \varepsilon$ for all $\varepsilon > 0$ which means $x_n \to 0$.

   Since $\limsup\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} < 1$, there exists some $N \in \mathbb{N}$ s.t. $\dfrac{x_{n+1}}{x_n} < 1$ for all $n > N$.
   Thus, $\{x_n\}_{n=N}^{\infty}$ is a monotone decreasing sequence bounded below by 0.
   For any $r < 1$, $\dfrac{1}{r} > 1$ and there exists some $k \in \mathbb{N}$ s.t. $(\dfrac{1}{r})^k > n$ for all $n \in \mathbb{N}$, then $r^k < \dfrac{1}{n}$.
   Therefore, $1 - r < 1 \implies (1-r)^k \to 0$ for $k \to \infty$
   Since $(0,1)$ is open, we can find some r s.t. $\dfrac{x_{n+1}}{x_n} < r < 1$ for all $n > N$. Then, $\lim\limits_{n\to\infty} x_n - x_{n+1} = \lim\limits_{n\to\infty} x_n(1 - \dfrac{x_{n+1}}{x_n}) < \lim x_N(1-r)^{n-N} = 0$.
   Thus, s_n is Cauchy $\implies$ s_n is convergence.

3. (a) WLOG, suppose $\{x_n\}$ is monotone increasing.

   Then, $\sigma_{n+1} - \sigma_n = (\frac{n}{n+1}S_n + \frac{x_{n+1}}{n+1}) - S_n = \frac{x_{n+1}}{n+1} - \frac{S_n}{n+1} > \frac{x_{n+1} - x_n}{n+1} > 0$ for all n. Therefore, σ_n is monotone.

   (b) we divided this question into $\{x_n\}$ diverges or bdd.

(diverges) WLOG, suppose $\limsup x_n = \infty$, then for any $M \in \mathbb{N}$, exists $N \in \mathbb{N}$ s.t. $x_n > M$ for $n \geq N$.
Thus, there exists $M_1, N_1 \in \mathbb{N}$ s.t. $x_n \geq M_1$ for all $n \geq N_1$ and $\sigma_{N_1} > \frac{M \times N_1}{N_1} = M$.
which means for any $M \in \mathbb{N}$, there exists some $N_1 \in \mathbb{N}$ s.t. $\sigma_n > M$ for all $n \geq N_1$.
That means, $\lim \sigma_n = \lim x_n = \infty$.
The case of diverging to $-\infty$ is the same.

(bounded) First, we how it works in a subsequence converging to a cluster point.
Let $\{x_{n_k}\}$ be a sequence converge to a and $\mu_k = \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}$, then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_{n_k} - a| < \varepsilon$ for all $k > N$. Thus, there exists some $\varepsilon_1 < \varepsilon, N_1 \in \mathbb{N}$ s.t. $|\mu_k - a| \leq \frac{|x_{n_1} - a| + \cdots + |x_{n_k} - a|}{k} < \frac{N \times \mu_N + \varepsilon_1 \times (k - N)}{k} < \varepsilon$. Which means μ_k converge to a.
And since $\frac{n_1 a_1 + n_2 a_2 + \cdots + n_k a_k}{n_1 + \cdots + n_k} < \max_{i \in [1,k]} (a_i)$ for all $n_i, k \in \mathbb{N}$ and $a_i \in \mathbb{R}$.
Thus, if there are cluster points of $\{x_n\}$ named as $a_1, a_2, \cdots a_k$, we can say that $\limsup \sigma_n \leq \max a_n = \limsup x_n$.

Then, $\limsup \sigma_n \leq \limsup x_n$ for all cases.