Exercises(4)

October 3, 2023

1. (10 points) Show that if $A \subseteq \mathbb{R}^n$ is open, then A is the union of a countable collection of open balls.

2. Define a **limit point** of a set A in a matric space (M, d) be a point $x \in M$ such that $U \cap A \neq \emptyset$ for every neighborhood U of x.

   (a) (4 points) If x is an accumulation point of A, then show that x is a limit point of A. Is the converse true? (Justify your answer!)

   (b) (4 points) Prove: A set is closed if and only if it contains all of its limit points.

3. Let $A_1, A_2, \ldots$ be subsets of a metric space (M, d):

   (a) (4 points) If $x \in \bar{A}_1$, prove that $\inf\{d(x, y) : y \in A_1\} = 0$.

   (b) (4 points) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$.

   (c) (4 points) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supseteq \bigcup_{i=1}^{\infty} \bar{A}_i$. Show, by an example, that this inclusion can be proper.