Homework 4 of Introduction to Analysis (I), Honor Class

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1. Let $B = A \cap \mathbb{Q}^n$. Then, for any $p \in B$, we can find a $\varepsilon \in \mathbb{Q} > 0$ s.t. $D(p,\varepsilon) \subseteq B$. Since $\mathbb{Q}$ is countable, the number of open balls contains any points in B is isomorphic to $\mathbb{Q}^n \oplus \mathbb{Q}$, which is also countable. Thus, we want to proof the union of the collection of open balls contains any points in A.

   For any $x \in A$, since A is open, exists $\varepsilon > 0$ s.t. $D(x,\varepsilon) \subseteq A$. Taking $p \in D(x,\varepsilon) \cap \mathbb{Q}^n$ with $d(p,x) < \dfrac{\varepsilon}{2}$. Then, exists $\varepsilon' < \dfrac{\varepsilon}{2}$, $\varepsilon' \in \mathbb{Q}$ s.t. $D(p,\varepsilon')$ contains $x \implies$ every points in A is contained in a open ball in B. Therefore, any $A \subseteq \mathbb{R}^n$, A is union of a countable collection of open balls.

2. (a) i. x is an accumulation point, for all $\varepsilon > 0$, $D(x,\varepsilon)/\{x\} \cap A \neq \emptyset \implies D(x,\varepsilon) \cap A \neq \emptyset$. Thus, x is a limit point.

   ii. Let $M = \mathbb{R}, d(x,y) = |x-y|$, $A = \{n \mid n \in \mathbb{N}\}$, then 1 is a limit point of A since $B(0,\varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

   But for $0 < \varepsilon < 1$, $B(1,\varepsilon)/\{1\} \cap A = \emptyset$. Thus, limit point is not accumulation all the time.

   (b) Assume there exists a limit point x of A is not in A. Then, $x \in A^c$. We want to show that A is not close.

   For A^c, since all neighborhood U of x has intersection with A, then $U \nsubseteq A^c$. Thus, A^c is not open.

   Therefore, if A is close, A contains all its limit points.

3. (a) Since $x \in A_1'$, for all $\varepsilon > 0$, $(D(x,\varepsilon)/\{x\}) \cap A_1 \neq \emptyset$. Then, let $y \in (D(x,\varepsilon)/\{x\}) \cap A_1$, $d(x,y) < \varepsilon$ for all $\varepsilon > 0 \implies \inf\{d(x,y) \mid y \in A_1\} = 0$.

   (b) $(\subseteq)$ For $x \in B_n'$, for any $\varepsilon > 0$, $(D(x,\varepsilon)/\{x\}) \cap B_n = (D(x,\varepsilon)/\{x\}) \cap (\cup A_i)$ $= \cup((D(x,\varepsilon)/\{x\}) \cap A_i) \neq \emptyset$. Thus, $(D(x,\varepsilon)/\{x\}) \cap A_i \neq \emptyset$ for some $i \implies x \in \cup A_i'$.

   $(\supseteq)$ If $x \in A_i'$, for any $\varepsilon > 0$, $(D(x,\varepsilon)/\{x\}) \cap A_i \neq \emptyset \implies \emptyset \neq (D(x,\varepsilon)/\{x\}) \cap (\cup A_i)$ $= (D(x,\varepsilon)/\{x\}) \cap B_n$.

   Thus, $B_n' = \cup A_i'$.

4. False.

   Let $A_i = (\frac{1}{i}, 1)$. Then, $B = (0,1)$. And for 0, $0 \in B'$ but not in $\cup A_i'$.