Homework 10 of Introduction to Analysis (I), Honor Class

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1. (a)

   ($\implies$) Since f is continuous, then for $x \in \bar{A}$, we can find a $\{x_i \mid x_i \in A\}$ converges to x. Thus, for all $\varepsilon > 0$, exists $\delta > 0$ s.t. $d(x_i, x) < \delta \implies \rho(f(x_i), f(x)) < \varepsilon$.

   Then, for $y \in f(\bar{A})$, $D(y, \varepsilon) \cap f(A) \neq \emptyset$ for all $\varepsilon > 0$. Thus, $y \in \overline{f(A)}$.

   ($\impliedby$) For V closed in T, then $f(\mathrm{cl}(f^{-1}(V))) \subseteq \mathrm{cl}(V) = V$. Applied f^{-1} on the both side, $\mathrm{cl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{cl}(V)) = f^{-1}(V)$. Thus, $f^{-1}(V)$ is closed if V is closed implies f is continuous.

   (b) Since $f(p) = g(p)$ for all $p \in S$ and $\bar{E} = S$. For any $x \in S$, $x \in \bar{E}$, then we can find a sequence $\{x_i\} \subseteq E$ converges to x.

   Since $f(x_i) = g(x_i)$ for all $i \in \mathbb{N}$, $|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x)| = |f(x) - f(x_i)| + |g(x) - g(x_i)| \to 0$ as $i \to \infty$. Then, $f(x) = g(x)$ for all $x \in S$.

2. Since B is compact, $f(B)$ is compact. Then, we want to show for an sequence y_k converges to $y \in f(B)$, we can get $f^{-1}(y_k)$ converges to $f^{-1}(y)$. Since f is one-to-one, we can find $x_k = f^{-1}(y_k) \in B$ and $x = f^{-1}(y)$. Then, assume x_k is not converges to x, $|x_k - x| > \delta$ for some $\delta > 0$ and $k \in \mathbb{N}$. Then, $f(D(x, \delta) \cap B) \not\subseteq D(f(x), \varepsilon) \cap f(B)$ since f is one-to-one. Thus, f is not continuous.(contradiction)

   Therefore, f^{-1} is continuous.

3. Since f is strictly increasing, f and f^{-1} is one-to-one.

   First, e want to check f^{-1} is strictly increasing. For $f(x_1) < f(x_2)$ with $x_1, x_2 \in \mathbb{R}$. Assume $x_1 > x_2$. Then, $f(x_1) > f(x_2)$.(contradiction) Thus, $f(x_1) < f(x_2) \implies f^{-1}(f(x_1)) = x_1 < x_2 = f^{-1}(f(x_2))$.

   Then, we want to check f^{-1} is continuous. For any decreasing sequence $y_k \to y$ for $y_k, y \in f(\mathbb{R})$. Since $y_k > y$ for all k, $f^{-1}(y_k) > f^{-1}(y)$ for all y. By MCT, $f^{-1}(y_k)$ converges. Assume $f^{-1}(y_k)$ converges to $x > f^{-1}(y)$. Then, we can find $y' = f(x) > y$. Therefore, we can find some $N \in \mathbb{N}$ s.t. $y_k < f(x)$ for all $k > N$. Then, $f^{-1}(y_k) > f(x)$ for all $k > N$. Thus, $f^{-1}(y_k)$ doesn't converge to x. Hence, $f^{-1}(y_k)$ converges to $f^{-1}(y)$.

   For increasing sequence, we can get the same result by similar way. Thus, by BWT, sequence converges to $y \in f(\mathbb{R})$, it converges to $f^{-1}(y)$.

   Therefore, f^{-1} is continuous and strictly increasing.