Homework 9 of Introduction to Analysis (I), Honor Class

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1. Since $\sum a_n$ is conditionally converges, then $\sum |a_n|$ diverges.

   We take $b_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \le 0 \end{cases}$ and $c_n = \begin{cases} 0 & \text{if } a_n > 0 \\ -a_n & \text{if } a_n \ge 0 \end{cases}$ Then, $\sum a_n = \sum b_n - c_n$.

   We want to show that $\sum b_n = \infty$. First, we assume $\sum c_n$ converges. Then, $\sum b_n = \sum (a_n + c_n) = \sum a_n + \sum c_n$ both converges. Then, $\sum |a_n| = \sum |b_n + c_n| = \sum b_n + \sum c_n$ both converges(contradiction).

   Thus, c_n diverges, which implies b_n diverges, too.

2. Since $\sum \sqrt{a_n a_{n+1}} \le \sum \dfrac{a_n + a_{n+1}}{2} = \sum a_n - \dfrac{a_1}{2}$. By comparison test, $\sum \sqrt{a_n a_{n+1}}$ converges.

   Since a_n is monotoneic, a_n are all greater than 0 or all lower than 0. Then, we suppose a_n are greater than 0 and monotone decreasing. $\sum \sqrt{a_n a_{n+1}} \ge \sum a_{n+1} = \sum a_n - a_1$. Then, by comparison test, $\sum a_n$ converges.

3. (a)

$$\begin{aligned} \frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdot + \frac{a_n}{r_n} &> \frac{a_m + a_{m+1} + \cdot a_m}{a_m} \\ &= \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m} \end{aligned}$$

   Then, we want to show $\sum \dfrac{a_n}{r_n}$ is not Cauchy. For all $N \in \mathbb{N}$ and $n > m \ge N$, $\displaystyle\sum_{k=m}^{n} \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m} > \frac{a_{m+1}}{r_m}$

(b)

$$
\begin{aligned}
2(\sqrt{r_n}-\sqrt{r_{n+1}}) &= 2\frac{(\sqrt{r_n})(\sqrt{r_n}-\sqrt{r_n+1})}{\sqrt{r_n}} \\
&= 2\frac{(\sqrt{r_n})^2-(\sqrt{r_n}\sqrt{r_{n+1}})}{\sqrt{r_n}} \\
&\geq 2\frac{r_n-r_{n+1}}{\sqrt{r_n}} \\
&= 2\frac{a_n}{r_n} \\
&> \frac{a_n}{r_n}
\end{aligned}
$$

We want to proof $\sum_{n=m}^{\infty}\frac{a_n}{\sqrt{r_n}} \to \infty$ as $m \to \infty$. $\sum_{n=m}^{\infty}\frac{a_n}{\sqrt{r_n}} < \sum_{n=m}^{\infty} 2(\sqrt{r_n}-\sqrt{r_{n+1}}) < 2\sqrt{r_m}$. And since $\sum a_n$ converges, $r_n \to 0$ as $n \to \infty$. Thus, by comparison test, $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

4. We want to show $\sum a_n - \lim_{n\to 1^-}\sum a_n x^n = \lim_{n\to i^-}\sum a_n(1-x^n) = 0$. Since $\sum a_n$ converges, $a_n \to 0$ as $n \to \infty$. Then, let $x = 1-\varepsilon$, then $x \to 1- \implies \varepsilon \to 0$. $\sum a_n(1-x^n) = \sum a_n \cdot \varepsilon^n < \varepsilon \cdot \sum a_n \to 0$ as $\varepsilon \to 0$. Thus, $\lim_{x\to 1^-}\sum a_n x^n = \sum a_n$.