Homework 3 of Introduction to Analysis (I), Honor Class

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1. (d_1) a. positive definite: $d(x,y)=(x-y)^2 \geq 0$ for any $x,y \in \mathbb{R}$ check

   b. $0 \iff$ equal:

   $(\Longrightarrow)$ $d(x,y)=0 \implies x-y=0 \implies x=y$

   $(\Longleftarrow)$ $d(x,y)=(x-y)^2=0^2=0$ check

   c. triangle inequality: $4=(2-0)^2=d(2,0)>1+1=d(2,1)+d(1,0)$ no

   Thus, d_1 is not a metric.

   (d_2) a. positive definite: since $|x-y|\geq 0$, then $\dfrac{|x-y|}{1+|x-y|}\geq 0$ check

   b. $0 \iff$ equal:

   $(\Longrightarrow)$ since $0=d(x,y)=\dfrac{0}{1+0} \implies x-y=0 \implies x=y$

   $(\Longleftarrow)$ since $x=y$, $d(x,y)=\dfrac{0}{1+0}=0$ check

   c. triangle inequality: If z is not in (x,y), the triangle inequality trivially holds.
   Then, WLOG, suppose $x<z<y$, then
   $$\frac{z-x}{1+z-x}+\frac{y-z}{1+y-z} > \frac{z-x}{1+(z-x)+(y-z)}+\frac{y-z}{1+(y-z)+(z-x)}$$
   $$=\frac{y-x}{1+(y-z)+(z-x)}=\frac{y-x}{1+y-x}$$

   Thus, d_2 is a metric.

2. We want to prove $s_{n+1}-s_n<\varepsilon$ for all $\varepsilon>0$ which means $x_n\to 0$.

   Since $\limsup_{n\to\infty}\dfrac{x_{n+1}}{x_n}<1$, there exists some $N\in\mathbb{N}$ s.t. $\dfrac{x_{n+1}}{x_n}<1$ for all $n>N$.
   Thus, $\{x_n\}_{n=N}^{\infty}$ is a monotone decreasing sequence bounded below by 0.
   For any $r<1$, $\dfrac{1}{r}>1$ and there exists some $k\in\mathbb{N}$ s.t. $(\dfrac{1}{r})^k>n$ for all $n\in\mathbb{N}$, then $r^k<\dfrac{1}{n}$.
   Therefore, $1-r<1 \implies (1-r)^k\to 0$ for $k\to\infty$
   Since $(0,1)$ is open, we can find some r s.t. $\dfrac{x_{n+1}}{x_n}<r<1$ for all $n>N$. Then, $\lim_{n\to\infty}x_n<\lim_{n\to\infty}x_1\times(1-r)^{n-1}=x_1\times 0=0$.
   Thus, $x_n\to 0$ when $n\to\infty$.

Then, for $\varepsilon > 0$ exists $N \in \mathbb{N}$ and $m > n > N$ with $x_n < \dfrac{\varepsilon}{m-n+1}$,

$|S_m - S_n| = S_m - S_n = x_n + x_{n+1} + \cdots x_{m-1} < (m-n+1)\dfrac{\varepsilon}{m-n+1} = \varepsilon$.

Thus, s_n is Cauchy $\implies s_n$ is convergence.

3. (a) WLOG, suppose $\{x_n\}$ is monotone increasing.

   Then, $\sigma_{n+1} - \sigma_n = (\dfrac{n}{n+1}S_n + \dfrac{x_{n+1}}{n+1}) - S_n = \dfrac{x_{n+1}}{n+1} - \dfrac{S_n}{n+1} > \dfrac{x_{n+1} - x_n}{n+1} > 0$ for all n. Therefore, σ_n is monotone.

   (b) we divided this question into $\{x_n\}$ diverges or bdd.

(diverges) WLOG, suppose $\limsup x_n = \infty$, then for any $M \in \mathbb{N}$, exists $N \in \mathbb{N}$ s.t. $x_n > M$ for $n \geq N$.
Thus, there exists $M_1, N_1 \in \mathbb{N}$ s.t. $x_n \geq M_1$ for all $n \geq N_1$ and $\sigma_{N_1} > \dfrac{M \times N_1}{N_1} = M$.
which means for any $M \in \mathbb{N}$, there exists some $N_1 \in \mathbb{N}$ s.t. $\sigma_n > M$ for all $n \geq N_1$.
That means, $\lim \sigma_n = \lim x_n = \infty$.
The case of diverging to $-\infty$ is the same.

(bounded) First, we how it works in a subsequence converging to a cluster point.
Let $\{x_{n_k}\}$ be a sequence converge to a and $\mu_k = \dfrac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}$, then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $|x_{n_k} - a| < \varepsilon$ for all $k > N$. Thus, there exists some $\varepsilon_1 < \varepsilon, N_1 \in \mathbb{N}$ s.t. $|\mu_k - a| \leq \dfrac{|x_{n_1} - a| + \cdots + |x_{n_k} - a|}{k} < \dfrac{N \times \mu_N + \varepsilon_1 \times (k-N)}{k} < \varepsilon$. Which means μ_k converge to a.
And since $\dfrac{n_1 a_1 + n_2 a_2 + \cdots + n_k a_k}{n_1 + \cdots + n_k} < \max\limits_{i \in [1,k]}(a_i)$ for all $n_i, k \in \mathbb{N}$ and $a_i \in \mathbb{R}$.
Thus, if there are cluster points of $\{x_n\}$ named as $a_1, a_2, \cdots a_k$, we can say that $\limsup \sigma_n \leq \max a_n = \limsup x_n$.

Then, $\limsup \sigma_n \leq \limsup x_n$ for all cases.