Homework 13 of Introduction to Analysis (I), Honor Class

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1. (a) Since $f(x)=\ln(x)$ is concave down, $\ln(uv)=\ln(u)+\ln(v)=\frac{1}{p}\ln(u^p)+\frac{1}{q}\ln(v^q)\leq\ln(\frac{u^p}{p}+\frac{v^q}{q})$.
   And since $\ln(x)$ is strictly increasing, $uv\leq\frac{u^p}{p}+\frac{v^q}{q}$.
   If $u^p=v^q$, $\frac{u^p}{p}+\frac{v^q}{q}=(\frac{1}{p}+\frac{1}{q})u^p=u^p=u^{p(\frac{1}{p}+\frac{1}{q})}=u\cdot u^{\frac{p}{q}}=uv$.

   (b) $\int_a^b\frac{(f(x))^p}{p}dx=\frac{1}{p}$ and $\int_a^b\frac{(g(x))^q}{q}=\frac{1}{q}$. Then, $\int_a^b\frac{(f(x))^p}{p}+\frac{(g(x))^q}{q}dx=\frac{1}{p}+\frac{1}{q}=1$.
   And since $f(x)g(x)\leq\frac{(f(x)^p)}{p}+\frac{(g(x))^q}{q}$ for all $x\in[a,b]$ and f,g are Reimann integral, $\int_a^b f(x)g(x)dx$ exists and $\int_a^b f(x)g(x)dx\leq\int_a^b\frac{(f(x))^p}{p}+\frac{(g(x))^q}{q}dx=1$.

   (c) Take $F=\int_a^b|f(x)|^pdx, \int_a^b|g(x)|^qdx$.

   Then,

$$
\begin{aligned}
\frac{|\int_a^b f(x)g(x)dx|}{F^{\frac{1}{p}}G^{\frac{1}{q}}} &\leq \frac{\int_a^b|f(x)||g(x)|dx}{F^{\frac{1}{p}}G^{\frac{1}{q}}}\\
&=\int_a^b(\frac{|f(x)|^p}{F})^{\frac{1}{p}}(\frac{|g(x)|^p}{G})^{\frac{1}{q}}dx\\
&\leq\int_a^b\frac{1}{p}(\frac{|f(x)|^p}{F})+\frac{1}{q}(\frac{|g(x)|^q}{G})dx\\
&=\frac{1}{p}(\frac{\int_a^b|f(x)|^pdx}{F})^{\frac{1}{p}}+\frac{1}{q}(\frac{\int_a^b|g(x)|^qdx}{G})^{\frac{1}{q}}\\
&=\frac{1}{p}=\frac{1}{q}=1
\end{aligned}
$$

Thus, $|\int_a^b f(x)g(x)dx| \le (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}(\int_a^b |g(x)|^q dx)^{\frac{1}{q}}$.

2. (a) Take $\delta = \frac{1}{2^{n+1}}$ and $P = \{0, \frac{1}{2^n}, \frac{1}{2^n}, \frac{1}{2^n} + \delta, \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}} + \delta, \cdots, \frac{1}{2} + \delta, 1\}$.

Then,

$$\begin{aligned} U(f,P) - L(f,P) &= \delta \cdots (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} - \frac{1}{2^n}) + \frac{1}{2^n}(\frac{1}{2^n} - 0) \\ &= \frac{1}{2^{n+1}} \cdot \frac{1}{2^n} + \frac{1}{2^{2n}} \\ &< \frac{1}{2^{2n-1}} \to 0 \text{ as } n \to \infty \end{aligned}$$

(b) $F(x) = xA(x) - \frac{1}{3}A(x)^2$, then $F'(x) = (A(x) + xA'(x)) - \frac{2}{3}A(x)A'(x)$.

$A'(x) =$

3.