

Homework 3 of Introduction to Analysis (I), Honor Class

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1. (d_1) a. positive definite: $d(x, y) = (x - y)^2 \geq 0$ for any $x, y \in \mathbb{R}$ check
b. $0 \iff$ equal:
 (\implies) $d(x, y) = 0 \implies x - y = 0 \implies x = y$
 (\impliedby) $d(x, y) = (x - y)^2 = 0^2 = 0$ check
c. triangle inequality: $4 = (2 - 0)^2 = d(2, 0) > 1 + 1 = d(2, 1) + d(1, 0)$ no
Thus, d_1 is not a metric.
- (d_2) a. positive definite: since $|x - y| \geq 0$, then $\frac{|x - y|}{1 + |x - y|} \geq 0$ check
b. $0 \iff$ equal:
 (\implies) since $0 = d(x, y) = \frac{0}{1 + 0} \implies x - y = 0 \implies x = y$
 (\impliedby) since $x = y$, $d(x, y) = \frac{0}{1 + 0} = 0$ check
c. triangle inequality: If z is not in (x, y) , the triangle inequality trivially holds.
Then, WLOG, suppose $x < z < y$, then
$$\frac{z - x}{1 + z - x} + \frac{y - z}{1 + y - z} > \frac{z - x}{1 + (z - x) + (y - z)} + \frac{y - z}{1 + (y - z) + (z - x)}$$
$$= \frac{y - x}{1 + (y - z) + (z - x)} = \frac{y - x}{1 + y - x}$$

Thus, d_2 is a metric.
2. We want to prove $s_{n+1} - s_n < \varepsilon$ for all $\varepsilon > 0$, which is equivalent to $x_n \rightarrow 0$.
Since $\limsup_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} < 1$, there exists some $N \in \mathbb{N}$ s.t. $\frac{x_{n+1}}{x_n} < 1$ for all $n > N$.
Thus, $\{x_n\}_{n=N}^{\infty}$ is a monotone decreasing sequence bounded below by 0.
For any $r < 1$, $\frac{1}{r} > 1$ and there exists some $k \in \mathbb{N}$ s.t. $(\frac{1}{r})^k > n$ for all $n \in \mathbb{N}$, then $r^k < \frac{1}{n}$.
Since $(0, 1)$ is open, we can find some r s.t. $\frac{x_{n+1}}{x_n} < r < 1$ for all $n > N$. Then, $\lim_{n \rightarrow \infty} x_n < r$.
$$\lim_{n \rightarrow \infty} x_1 \times r^{n-1} = x_1 \times 0 = 0.$$

Thus, $x_n \rightarrow 0$ when $n \rightarrow \infty$.

Then, for $\varepsilon > 0$ exists $N \in \mathbb{N}$ and $m > n > N$ with $x_n < \frac{\varepsilon}{m-n+1}$,

$$|S_m - S_n| = S_m - S_n = x_n + x_{n+1} + \cdots + x_{m-1} < (m-n+1) \frac{\varepsilon}{m-n+1} = \varepsilon.$$

Thus, s_n is Cauchy $\implies s_n$ is convergence.

3. (a) WLOG, suppose $\{x_n\}$ is monotone increasing.

Then, $\sigma_{n+1} - \sigma_n = \left(\frac{n}{n+1}S_n + \frac{x_{n+1}}{n+1}\right) - S_n = \frac{x_{n+1}}{n+1} - \frac{S_n}{n+1} > \frac{x_{n+1} - x_n}{n+1} > 0$ for all n . Therefore, σ_n is monotone.

(b) we divided this question into $\{x_n\}$ diverges or bdd.

(diverges) WLOG, suppose $\limsup x_n = \infty$, then for any $M \in \mathbb{N}$, exists $N \in \mathbb{N}$ s.t. $x_n > M$ for $n \geq N$.

Thus, there exists $M_1, N_1 \in \mathbb{N}$ s.t. $x_n \geq M_1$ for all $n \geq N_1$ and $\sigma_{N_1} > \frac{M \times N_1}{N_1} = M$.

which means for any $M \in \mathbb{N}$, there exists some $N_1 \in \mathbb{N}$ s.t. $\sigma_n > M$ for all $n \geq N_1$.

That means, $\lim \sigma_n = \lim x_n = \infty$.

The case of diverging to $-\infty$ is the same.

(bounded) Since $\{x_n\}$ is bdd, let $\beta = \limsup x_n = \sup\{\beta_i\}$ where β_i is cluster point of x_n for all i .

Thus, for $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ s.t. $x_n < \beta + \varepsilon$ for all $n > N$.

Then, since the density of \mathbb{R} , there exists $\beta < \delta < \beta + \varepsilon$. Thus, there exists some $M \in \mathbb{N}$ s.t. $x_n \leq \delta$ for all $n > M$.

$$\text{Therefore, } \sigma_n = \frac{x_1 + x_2 + \cdots + x_n}{n} < \frac{x_1 + x_2 + \cdots + x_M}{n} + \frac{\delta(n-M)}{n}$$

$= \delta \cdot \frac{1}{n}(x_1 + x_2 + \cdots + x_M - M)$. By Archimedean property, there exists some $N' \in \mathbb{N}$ s.t.

$\frac{1}{n}(x_1 + x_2 + \cdots + x_M - M) < \beta + \varepsilon - \delta$ for $n > N'$. Thus, $\sigma < \beta + \varepsilon$ for all $\varepsilon > 0$ and some $N' \in \mathbb{N}$, which is equivalent to $\limsup \sigma_n \leq \beta = \limsup x_n$.

Then, $\limsup \sigma_n \leq \limsup x_n$ for all cases.