# Homework 12 of Introduction to Analysis (I), Honor Class

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1. We claim that $a_n$ is a Cauchy. Then, for $m > n \geq N$

$$\begin{aligned}
|f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\
&< \frac{1}{n} - \frac{1}{m} \\
&< \frac{1}{n} \\
&\leq \frac{1}{N}
\end{aligned}$$

Thus, for any $\varepsilon > 0$, we take $N > \frac{1}{\varepsilon}$. Therefore, $a_n$ is Cauchy $\Longrightarrow \lim_{n\to\infty} a_n$ exists.

2. Since $f'(x)$ exists on $(a,b)$, we can find $s_1 = \sup\{f'(x) \mid x \in (a,b)\}$, $s_2 = \lim_{x\to a^+} f(x)$ and $s = \max\{s_1, s_2\}$.

Then, for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{s}$, then $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$. Thus, $f$ is uniform continuous.

3. (a) We want to show that when $x \to \infty$, $f(x+h) - f(x) \to hb$ for all $h$.

Since $f'(x) \to b$ as $x \to \infty$, for any $\varepsilon > 0$, exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Thus, for $x > N$, $(b-\varepsilon)h < f(x+h) - f(x) < (b+\varepsilon)h$. Thus, $|\frac{f(x+h)-f(x)}{h} - b| < \varepsilon$ for $x > N$. Therefore, $\lim_{x\to\infty} \frac{f(x+h)-f(x)}{h} - b = 0$.

(b) Since $f(x) \to a$, we assume $f'(x)$