Homework 13 of Introduction to Analysis (I), Honor Class

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1. First, we want to check there are at most $\dfrac{n(n+1)}{2}$ points for $x \in [0,1]$ for which $f(x) > \dfrac{1}{n}$.

   For $f(x) > \dfrac{1}{n}$, that means x is a rational number $\dfrac{p}{q}$ and $q < n$. Thus, we have $x = \dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{2}{3}, \cdots$. The number of x is less than $\dfrac{n(n+1)}{2}$. Thus, we suppose a_n is the number of x s.t. $f(x) > \dfrac{1}{n}$

   For any $n \in \mathbb{N}$, we can find a $\delta > 0$ and partition $P_n = \bigcup\limits^{n} [x_i - \delta, x_i + \delta]$ s.t. x_i is the only point $x \in [x_i - \delta, x_i + \delta]$ that $f(x) > \dfrac{1}{n}$ and $P_n' = [0,1] \setminus P_n$. Thus, $|U(f,P_n) - L(f,P_n)| + |U(f,P_n') - L(f,P_n')| = U(f,p) + U(f,P_n') < a_n \cdot (2\delta \cdot 1) + (1 - a_n \cdot 2\delta)\dfrac{1}{n} < \dfrac{1}{n} + a_n \cdot 2\delta$. Thus, we can find large enough n and a tiny δ s.t. f is integrable.

2. Since f is bounded, we can find $M \in \mathbb{R}$ s.t. $|f(x)| < M$ for all $x \in [a,b]$. Suppose there are n points of discontinuity of f on $[a,b]$. Thus, for any $\varepsilon > 0$, we take $\delta = \dfrac{\varepsilon}{8nM}$. Then, suppose the set of points of discontinuity is $\{y_i \mid i \in \mathbb{N}, i \leq n\}$. Take partition $P = \{x_1 = y_1 - \varepsilon,\ x_2 = y_1 + \varepsilon,\ x_3 = y_2 - \varepsilon,\ \cdots\}$ is finite. Then, for f is continuous on $[a,x_1],\ [x_2,x_3],\ [x_4,x_5],\ \cdots$, we can find a partition P_1 s.t. $|U(f,P_1) - L(f,P_2)| < \dfrac{\varepsilon}{2}$ we only need to check other intervals $P_2 = [x_1,x_2] \bigcup [x_3,x_4] \bigcup \cdots \bigcup [x_{2n-1}, x_{2n}]$ are Reimann integrable. $|U(f,P_2) - L(f,P_2)| \leq 2U(f,P_2) \leq 2 \cdot n \cdot (2\delta) \cdot M = 4nM \cdot \dfrac{\varepsilon}{8nM} = \dfrac{\varepsilon}{2}$. Thus, $|U(f,P) - L(f,P)| \leq |U(f,P_1) - L(f,P_1)| + |U(f,P_2), L(f,P_2)| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$, which implies that $f \in R([a,b])$.

3. $f^{(n)}(x) = (\prod_{k=0}^{n-1} m-k)(1+x)^{m-n}$. $R_n = x^n \frac{f^{(n)}(\theta_n x)}{n!} = (\prod_{k=1}^{n} \frac{m-k+1}{k})(1+\theta_n x)^{m-n} x^n$.

For $m > 0$, we can find $N \in \mathbb{N}$ s.t. $N > m$ and $N - 1 \leq m$. Since $x > 0$, $1 + \theta_n x > 1$, $(1+\theta_n x)^{m-n} < 1$ for $n > N$. And since $n > N > m$, $\prod_{k=1}^{n} \frac{m-k+1}{k} = \frac{m}{N} \cdot \frac{m-1}{N-1} \cdots \frac{m-N+1}{1} \cdot \frac{m-N}{N+1} \cdots < 1$. Thus, $R_n < x^n$ for $m > 0$ and $n > m$.

For $m < 0$, since $|\frac{m-k}{k}| > |\frac{m-k-1}{k+1}| > \cdots > 1$, there exists $N \in \mathbb{N}$ s.t. $|\frac{m-n}{n}| < \frac{1}{x}$ for all $n > N$. Therefore, we can find $l < 1$ s.t. $|\frac{m-n}{n} x| < l$ for all $n > N$. Thus, $|R_n| < |\prod_{k=1}^{n} (\frac{m-k+1}{k} x)| = |A| \cdot |\prod_{k=N}^{n} \frac{m-k+1}{k} x| < |A| \cdot l^{n-N} \to 0$ as $n \to \infty$.

If $x < 0$, $(1 + \theta_n x) < 1 \implies (1+\theta_n x)^{m-n} > 1$ for all $n > m$. Thus, we can't use the same argument to prove.