

Homework 14 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

February 19, 2024

1. (a) Since $f(x) = \ln(x)$ is concave down, $\ln(uv) = \ln(u) + \ln(v) = \frac{1}{p} \ln(u^p) + \frac{1}{q} \ln(v^q) \leq \ln\left(\frac{u^p}{p} + \frac{v^q}{q}\right)$.

And since $\ln(x)$ is strictly increasing, $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$.

If $u^p = v^q$, $\frac{u^p}{p} + \frac{v^q}{q} = \left(\frac{1}{p} + \frac{1}{q}\right)u^p = u^p = u^{p\left(\frac{1}{p} + \frac{1}{q}\right)} = u \cdot u^{\frac{p}{q}} = uv$.

- (b) $\int_a^b \frac{(f(x))^p}{p} dx = \frac{1}{p}$ and $\int_a^b \frac{(g(x))^q}{q} dx = \frac{1}{q}$. Then, $\int_a^b \frac{(f(x))^p}{p} + \frac{(g(x))^q}{q} dx = \frac{1}{p} + \frac{1}{q} = 1$.

And since $f(x)g(x) \leq \frac{(f(x))^p}{p} + \frac{(g(x))^q}{q}$ for all $x \in [a, b]$ and f, g are Riemann integral, $\int_a^b f(x)g(x)dx$

exists and $\int_a^b f(x)g(x)dx \leq \int_a^b \frac{(f(x))^p}{p} + \frac{(g(x))^q}{q} dx = 1$.

- (c) Take $F = \int_a^b |f(x)|^p dx$, $G = \int_a^b |g(x)|^q dx$.

Then,

$$\begin{aligned} \frac{\left| \int_a^b f(x)g(x)dx \right|}{F^{\frac{1}{p}} G^{\frac{1}{q}}} &\leq \frac{\int_a^b |f(x)||g(x)|dx}{F^{\frac{1}{p}} G^{\frac{1}{q}}} \\ &= \int_a^b \left(\frac{|f(x)|^p}{F} \right)^{\frac{1}{p}} \left(\frac{|g(x)|^q}{G} \right)^{\frac{1}{q}} dx \\ &\leq \int_a^b \frac{1}{p} \left(\frac{|f(x)|^p}{F} \right) + \frac{1}{q} \left(\frac{|g(x)|^q}{G} \right) dx \\ &= \frac{1}{p} \left(\frac{\int_a^b |f(x)|^p dx}{F} \right)^{\frac{1}{p}} + \frac{1}{q} \left(\frac{\int_a^b |g(x)|^q dx}{G} \right)^{\frac{1}{q}} \\ &= \frac{1}{p} = \frac{1}{q} = 1 \end{aligned}$$

Thus, $|\int_a^b f(x)g(x)dx| \leq (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} (\int_a^b |g(x)|^q dx)^{\frac{1}{q}}.$

2. (a) Take $\delta = \frac{1}{2^{n+1}}$ and $P = \{0, \frac{1}{2^n}, \frac{1}{2^n}, \frac{1}{2^n} + \delta, \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}} + \delta, \dots, \frac{1}{2} + \delta, 1\}.$

Then,

$$\begin{aligned} U(f, P) - L(f, P) &= \delta \cdots (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} - \frac{1}{2^n}) + \frac{1}{2^n} (\frac{1}{2^n} - 0) \\ &= \frac{1}{2^{n+1}} \cdot \frac{1}{2^n} + \frac{1}{2^{2n}} \\ &< \frac{1}{2^{2n-1}} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

- (b) Let $n = \left\lceil -\frac{\ln x}{\ln 2} \right\rceil$, then $A(x) = 2^{-n}.$

$$\begin{aligned} \int_0^x f(t)dt &= \int_0^1 f(t)dt - \int_x^1 f(t)dt \\ &= \sum_{k=n}^{\infty} \frac{1}{2^k} \cdot \frac{1}{2^{k+1}} - (\frac{1}{2^n} - x) \cdot 2^{-n} \\ &= x2^{-n} + \frac{1}{2^{2n+1}} \cdot \frac{4}{3} - \frac{1}{2^{2n}} \\ &= x2^{-n} - \frac{1}{3}2^{-2n} \\ &= xA(x) - \frac{1}{3}(A(x))^2 \end{aligned}$$

3. Since f is continuous on $[a, b]$, f is uniform continuous and exists $c \in [a, b]$ s.t. $f(c) = M$. And let

$$I_n = (\int_a^b (|f(x)|)^n dx)^{\frac{1}{n}}.$$

Then, for all $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $x \in (c - \delta, c + \delta) \cap [a, b]$, $|f(x) - M| < \varepsilon \implies f(x) \geq M - \varepsilon.$

Thus, $I_n \geq (\frac{1}{2\delta})^{\frac{1}{n}}(M - \varepsilon).$ And since $f(x) \leq M$ for all x , $I_n \leq (\frac{1}{b-a})^{\frac{1}{n}}M.$

Then, we can get for all $\varepsilon > 0$ and some $\delta > 0$, $(\frac{1}{2\delta})^{\frac{1}{n}}(M - \varepsilon) \leq I_n \leq (\frac{1}{b-a})^{\frac{1}{n}}M.$ And since $r^{\frac{1}{n}} \rightarrow 1$ as $n \rightarrow \infty$ for all $r > 0$, $\lim_{n \rightarrow \infty} (\frac{1}{2\delta})^{\frac{1}{n}}(M - \varepsilon) = M$ and $\lim_{n \rightarrow \infty} (\frac{1}{b-a})^{\frac{1}{n}}M = M.$

By squeeze theorem, $\lim_{n \rightarrow \infty} I_n = M.$