Homework 6 of Introduction to Analysis (I), Honor Class

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1. First, we show that S is closed and bounded. There exists $D(2,2) \subseteq S \implies S$ is bounded. And for $x \in \mathbb{Q} \setminus S$, if $x \leq \sqrt{2}$, $D(x, d(x, \sqrt{2})) \subseteq \mathbb{Q} \setminus S$. If $x \geq \pi$, $D(x, \pi) \subseteq \mathbb{Q} \setminus S \implies \mathbb{Q} \setminus S$ is open. Then, S is closed.

   $G_i = (\sqrt{2} - \frac{1}{n}, \pi - \frac{1}{n}) \cap \mathbb{Q}$, then $\bigcap_{i=1}^{\infty} G_i$ is an open cover of S but doesn't have finite subcover. Thus, S is not compact.

2. Assume $\bigcap_{n=1}^{\infty} V_n = \emptyset$. For $\bar{V}_2$, V_1 is open and $\bar{V}_2 \subseteq V_1$, then V_1 is a open cover of $\bar{V}_2$. Then, since $\bar{V}_2$ is compact, we can find finite $\{y_n \in V_1 \setminus \bar{V}_2\}$ s.t. $\cup D(y_i, \varepsilon_i) \cup V_2$ is a open cover of $\bar{V}_2$.

   Using the same way,

3. Since $x, y \in K$, then assume there doesn't exists a set in G s.t. it contains both of x, y, then G doesn't contain one of x, y or both neither. Thus, G doesn't cover K , contradict to G is a open cover. Thus, there exists a set in G containing both x, y.

4. Since A is totally bounded, for all $\varepsilon > 0$, exists $\{x_i\}_{i=1}^{N}$ is a finite set s.t. $A \subseteq \bigcup_{i=1}^{N} D(x_i, \varepsilon)$.

   And M is conplete $\implies$ every Cauchy sequence in M converge to a point in M.

   Then, for any sequence $\{y_i\}$ in $\bar{A}$, if there is not any subsequence converge to $\bar{A}$, there exists $\varepsilon' > 0$ s.t. $|y_m - y_n| > \varepsilon'$ for all $m, n > N \in \mathbb{N}$. $\cup D(y_i, \varepsilon')$ will be a infinite open cover of A, which contradicts to there has a finite N s.t. $\cup D(x_i, \varepsilon')$ is cover of A.

Thus, every sequence in $\bar{A}$ have subsequence converge to a point in $\bar{A} \implies \bar{A}$ is sequence compact.

Then, by BWT, $\bar{A}$ is compact.