

# Homework 1 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

2023/9/12

1. Let  $R = \{f(x) \mid x \in X\}$  and  $r = \sup R$ . Then, for any  $\beta$  which is upper bound of  $R$ ,  $r \leq \beta$ . Thus,  $a + \beta$  is an upper bound of  $R' = \{a + f(x) \mid x \in X\}$ .

Assume there exists  $\beta' = \sup R'$  s.t.  $\beta' < a + r$ . Since  $\beta' \geq a + f(x)$ ,  $\forall x \in X \Rightarrow \beta' - a \geq f(x)$ ,  $\forall x \in X$ . Therefore,  $\beta' - a$  is an upper bound of  $R$  and it should be greater or equal to  $r$ . This causes contradiction to the assumption.

Thus, for every upper bound of  $R'$ , it is greater or equal to  $a + r$ .

That is  $\sup\{a + f(x) \mid x \in X\} = a + \sup\{f(x) \mid x \in X\}$

2. let  $a$  be greatest lower bound of  $\{f(x) \mid x \in X\}$ ,  $b$  be greatest lower bound of  $\{g(x) \mid x \in X\}$ . That is  $a \leq f(x) \forall x \in X$  and  $b \leq g(x) \forall x \in X$ . Then,  $a + b$  is a lower bound of  $\{f(x) + g(x) \mid x \in X\}$ . Therefore  $a + b \leq \inf\{f(x) + g(x) \mid x \in X\}$ . In other words,  $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}$

Using the same way we can prove  $\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

Then, we want to show  $\inf S \leq \sup S$  for all set  $S$ :

For an upper bound of  $S$  " $\beta$ " and lower bound of  $S$  " $\alpha$ ". For any element  $x \in S$ ,  $\alpha \leq x \leq \beta$ .

Then,  $\inf S \leq x \leq \sup S$  for all  $x \in S$

Then,  $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

Ex. Let  $f(x) = \sin(x)$ ,  $g(x) = \cos(x)$ ,  $X = (0, 2\pi)$ , then  $-2 < -\sqrt{2} < 2$ .

3. Let  $A = \{x \mid x \in \mathbb{R}, b^x < y\}$ .

By Bernoulli's Inequality (In Bartle's book page.35) and let  $b = 1 + c$  with  $c \in \mathbb{R}$ ,  $c > 0$ , then  $(1 + c)^n \geq 1 + nc$ . Then, we can find some  $n \in \mathbb{N}$  s.t.  $c > \frac{y}{n}$ . Therefore,  $b^n = (1 + c)^n \geq 1 + nc > y$  and  $n$  is upper bound of  $A$ . Thus, by completeness of  $\mathbb{R}$ , there exists  $\beta \in \mathbb{R}$  be  $\sup A$ .

If  $b^\beta < y$ , then assume  $\beta + \frac{1}{n} \in A$ . Thus  $b^{\beta + \frac{1}{n}} = b^\beta \cdot b^{\frac{1}{n}} > b^\beta \cdot 1 = b^\beta$  for all  $n \in \mathbb{N}$ . Then, we want to proof  $b^{\beta + \frac{1}{n}} < y$ , thus we check the inequality  $b^{\beta + \frac{1}{n}} < y$  for some  $n \in \mathbb{N}$ .

**Proof.**  $b^{\beta + \frac{1}{n}} = b^\beta b^{\frac{1}{n}}$ . Then, we want to proof  $b^{\frac{1}{n}} - 1 \rightarrow 0$  when  $n \rightarrow \infty$ .

Since  $b^{\frac{1}{n}} = (1 + c)^{\frac{1}{n}} \leq 1 + \frac{1}{n}c$  and  $\frac{y}{b^\beta}$  is a positive real number which is greater than 1, there exists some  $n \in \mathbb{N}$  s.t.  $n > \frac{c \cdot b^\beta}{y - b^\beta} > 0$ . In the other world  $1 + \frac{c}{n} < \frac{y}{b^\beta}$ . Therefore,  $b^{\beta + \frac{1}{n}} < y$ .

Thus,  $b^{\beta + \frac{1}{n}} \in A \Rightarrow b^\beta$  is not a upper bound of  $A$ . Therefore,  $b^\beta$  should greater or equal to  $y$ .

If  $b^\beta > y$ , we can find a real number  $r'$  s.t.  $b^\beta > r' > y$ . Therefore, we can find  $x \in A$  s.t.  $b^x \leq r'$ . Thus,  $\beta$  is not least upper bound of  $A$ .

These two line implies the supremum of  $A$ ,  $\beta$ , is the unique real number s.t.  $b^\beta = y$  since the supremum is unique.