Homework 5 of Introduction to Analysis (I), Honor Class

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1. First we want to show given $\varepsilon > 0$, we can find finite sequence $x_1, x_2, \cdots x_n$ s.t. $\cup D(x_i, \varepsilon) = X$.

    Let y_1 be a point in X, then $D(y_1, \varepsilon)$ can not cover X. Then, we can find $y_2 \in X - D(y_1, \varepsilon)$ s.t. $D(y_1, \varepsilon) \cup D(y_2, \varepsilon)$ can not cover X.

    Using the same way, we can find $m \in \mathbb{N}, m < \infty$ s.t. $\bigcap_{i=1}^{m} D(y_i, \varepsilon) = X$. If we can't, in the other word, we found a infinite sequence y_n s.t. the distance of two element in them is greater than ε, which is contradict to the infinite sequence in X has a accumulation point.

   Thus, we can find finite set $\{x_{i,n}\}_{i=1}^{m_i}$ s.t. $\bigcup_{i=1}^{m} D(x_{i,n}, \frac{1}{n})$ cover X. Then, $\{x_{i,n}\}_{1 \le n \le \infty,\ 1 \le i \le m_n}$ is countable.

   Then, we want to show $S = \cup D(x_{i,n}, \frac{1}{n})$ is dense in X.

   For any $x \in X - S$, then for all $\varepsilon > 0$, exists $\frac{1}{n_i} \le \varepsilon$ s.t. $D(x, \frac{1}{n_i}) \subseteq D(x, \varepsilon)$ and $x \in D(x_{i,n}, \frac{1}{n_i})$ for some $1 \le i \le m_i \implies x$ is a accumulation point of S. Then, $\bar{S} = X \implies S$ is dense.

2. First, we let $\{U\}$ be disjoint segments that covers the open set in $\mathbb{R}$. Then, if $x, y \in U$ with $x \neq y$, since exists $q \in [x, y] \cap \mathbb{Q}$, $q \in U \cap \mathbb{Q}$. Then, for every U contains more than one number, we call it U_x, for a $x \in \mathbb{Q} \cap U$, which $\{U_x\}$ is countable.

3. Since $A_1 \supseteq A_2 \supseteq \cdots A_n$, $A'_1 \supseteq A'_2 \supseteq \cdots A'_n$.

   For $x \in A$, since $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$, $A = \cap_{n=1}^{\infty} \bar{A}_n = (A_1 \cup A'_1) \cap (\bigcap\limits_{n=2}^{\infty} \bar{A}_n)$
   $= (A'_1 \cap (\bigcap\limits_{n=2}^{\infty} \bar{A}_n)) \cup (A \cap (\bigcap\limits_{n=2}^{\infty} \bar{A}_n)) = \cdots = (\bigcap\limits_{i=1}^{\infty} A'_i) \cup (\bigcap\limits_{i=2}^{\infty} A'_i) \cup \cdots = \bigcap\limits_{n=1}^{\infty} A'_n$.
   Thus, $x \in A'_1$.

4. First, we want to proof $\overline{M-A} \supseteq (\overline{A} - A)$.

   Since $\overline{M-A} = (M-A)' \cup (M-A)$ and $\overline{A} \subseteq M \implies (\overline{A} - A) \subseteq (M-A) \subseteq \overline{M-A}$.

   $(A \cap \overline{M-A}) \cup (\overline{A} - A) = (A \cup (\overline{A} - A)) \cap (\overline{M-A} \cup (\overline{A} - A)) = \overline{A} \cap (\overline{M-A}) = \partial A$