Exercises(14)

December 12, 2023

1. Let p and q be positive real numbers such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

Prove the following statements.

(a) (5 points) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p}+\frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

(b) (5 points) If $f \in R([a,b]), g \in R([a,b]), f \geq 0, g \geq 0$, and

$$\int_a^b f(x)^p dx = 1 = \int_a^b g(x)^q dx,$$

then

$$\int_a^b f(x)g(x)dx \leq 1.$$

(c) (3 points) If f and g are integrable in $[a,b]$, then

$$|\int_a^b f(x)g(x)dx| \leq (\int_a^b |f(x)|^p)^{1/p}(\int_a^b |g(x)|^q)^{1/q}.$$

2. Let f be defined on $[0,1]$ as follows: $f(0)=0$; if $2^{-n-1} < x \leq 2^{-n}$, then $f(x)=2^{-n}$, for $n=0,1,2,\ldots$.

(a) (5 points) Prove that $\int_0^1 f(x)dx$ exists.

(b) (5 points) Let $F(x)=\int_0^x f(t)dt$. Show that for $0 < x \leq 1$ we have

$$F(x) = xA(x) - \frac{1}{3}A(x)^2,$$

where $A(x) = 2^{-[-\ln x/\ln 2]}$ and where $[y]$ is the greatest integer in y.

3. (7 points) Let f be continuous on a closed interval $[a,b]$ and let

$$M = \sup_{x\in[a,b]} |f(x)|.$$

Prove that

$$\lim_{p\to\infty} (\int_a^b |f(x)|^p dx)^{1/p} = M.$$