

Homework 6 of Introduction to Analysis (I), Honor Class

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1. First, we show that S is closed and bounded. There exists $D(2,2) \subseteq S \implies S$ is bounded. And for $x \in \mathbb{Q} \setminus S$, if $x \leq \sqrt{2}$, $D(x, d(x, \sqrt{2})) \subseteq \mathbb{Q} \setminus S$. If $x \geq \pi$, $D(x, \pi) \subseteq \mathbb{Q} \setminus S \implies \mathbb{Q} \setminus S$ is open. Then, S is closed.

$G_i = (\sqrt{2} - \frac{1}{n}, \pi - \frac{1}{n}) \cap \mathbb{Q}$, then $\bigcup_{i=1}^{\infty} G_i$ is an open cover of S but doesn't have finite subcover. Thus, S is not compact.

2. Assume $\bigcap_{n=1}^{\infty} V_n = \emptyset$. For \bar{V}_2 , V_1 is open and $\bar{V}_2 \subseteq V_1$, then V_1 is a open cover of \bar{V}_2 . Then, since \bar{V}_2 is compact, we can find finite $\{y_n \in V_1 \setminus \bar{V}_2\}$ s.t. $\bigcup D(y_i, \epsilon_i) \cup V_2$ is a open cover of \bar{V}_2 .

Using the same way,

3. Since $x, y \in K$, then assume there doesn't exists a set in G s.t. it contains both of x, y , then G doesn't contain one of x, y or both neither. Thus, G doesn't cover K , contradict to G is a open cover. Thus, there exists a set in G containing both x, y .

4. Since A is totally bounded, for all $\epsilon > 0$, exists $\{x_i\}_{i=1}^N$ is a finite set s.t. $A \subseteq \bigcup_{i=1}^N D(x_i, \epsilon)$.

And M is complete \implies every Cauchy sequence in M converge to a point in M .

Then, for any sequence $\{y_i\}$ in \bar{A} , if there is not any subsequence converge to \bar{A} , there exists $\epsilon' > 0$ s.t. $|y_m - y_n| > \epsilon'$ for all $m, n > N \in \mathbb{N}$. $\bigcup D(y_i, \epsilon')$ will be a infinite open cover of A , which contradicts to there has a finite N s.t. $\bigcup D(x_i, \epsilon')$ is cover of A .

Thus, every sequence in \bar{A} have subsequence converge to a point in $\bar{A} \implies \bar{A}$ is sequence compact.

Then, by BWT, \bar{A} is compact.