

# Homework 4 of Introduction to Analysis (I), Honor Class

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1. Let  $B = A \cap \mathbb{Q}^n$ . Then, for any  $p \in B$ , we can find a  $\varepsilon \in \mathbb{Q} > 0$  s.t.  $D(p, \varepsilon) \subseteq B$ . Since  $\mathbb{Q}$  is countable, the number of open balls contains any points in  $B$  is isomorphic to  $\mathbb{Q}^n \oplus \mathbb{Q}$ , which is also countable. Thus, we want to proof the union of the collection of open balls contains any points in  $A$ .

For any  $x \in A$ , since  $A$  is open, exists  $\varepsilon > 0$  s.t.  $D(x, \varepsilon) \subseteq A$ . Taking  $p \in D(x, \varepsilon) \cap \mathbb{Q}^n$  with  $d(p, x) < \frac{\varepsilon}{2}$ . Then,  $D(p, \frac{\varepsilon}{2})$  contains  $x \implies$  every points in  $A$  is contained in a open ball in  $B$ . Therefore, any  $A \subseteq \mathbb{R}^n$ ,  $A$  is union of a countable collection of open balls.

2. (a) i.  $x$  is an accumulation point, for all  $\varepsilon > 0$ ,  $D(x, \varepsilon) \setminus \{x\} \cap A \neq \emptyset \implies D(x, \varepsilon) \cap A \neq \emptyset$ . Thus,  $x$  is a limit point.
- ii. Let  $M = \mathbb{R}$ ,  $d(x, y) = |x - y|$ ,  $A = \{n \mid n \in \mathbb{N}\}$ , then 1 is a limit point of  $A$  since  $B(0, \varepsilon) \cap A \neq \emptyset$  for all  $\varepsilon > 0$ .
- But for  $0 < \varepsilon < 1$ ,  $B(1, \varepsilon) \setminus \{1\} \cap A = \emptyset$ . Thus, limit point is not accumulation all the time.

- (b) Assume there exists a limit point  $x$  of  $A$  is not in  $A$ . Then,  $x \in A^c$ . We want to show that  $A$  is not close.

For  $A^c$ , since all neighborhood  $U$  of  $x$  has intersection with  $A$ , then  $U \not\subseteq A^c$ . Thus,  $A^c$  is not open.

Therefore, if  $A$  is close,  $A$  contains all its limit points.

3. (a) Since  $x \in A'_1$ , for all  $\varepsilon > 0$ ,  $(D(x, \varepsilon)/\{x\}) \cap A_1 \neq \emptyset$ . Then, let  $y \in (D(x, \varepsilon)/\{x\}) \cap A_1$ ,  $d(x, y) < \varepsilon$  for all  $\varepsilon > 0 \implies \inf\{d(x, y) \mid y \in A_1\} = 0$ .

(b) ( $\subseteq$ ) For  $x \in B'_n$ , for any  $\varepsilon > 0$ ,  $(D(x, \varepsilon)/\{x\}) \cap B_n = (D(x, \varepsilon)/\{x\}) \cap (\cup A_i)$   
 $= \cup((D(x, \varepsilon)/\{x\}) \cap A_i) \neq \emptyset$ . Thus,  $(D(x, \varepsilon)/\{x\}) \cap A_i \neq \emptyset$  for some  $i \implies x \in \cup A'_i$ .

( $\supseteq$ ) If  $x \in A'_i$ , for any  $\varepsilon > 0$ ,  $(D(x, \varepsilon)/\{x\}) \cap A_i \neq \emptyset \implies \emptyset \neq (D(x, \varepsilon)/\{x\}) \cap (\cup A_i)$   
 $= (D(x, \varepsilon)/\{x\}) \cap B_n$ .

Thus,  $B'_n = \cup A'_i$ .

4. False.

Let  $A_i = (\frac{1}{i}, 1)$ . Then,  $B = (0, 1)$ . And for 0,  $0 \in B'$  but not in  $\cup A'_i$ .