

# Homework 15 of Introduction to Analysis (I), Honor Class

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1. (a) Since  $f$  is continuous on  $Q$  is compact,  $\exists \delta > 0$  s.t. for all  $\alpha_1, \alpha_2 \in Q$ ,  $\|\alpha_1 - \alpha_2\| < \delta$ ,

$$|f(\alpha_1) - f(\alpha_2)| < \frac{\varepsilon}{b-a}.$$

Thus, for  $y_0 \in [c, d]$ ,

$$\begin{aligned} |F(y) - F(y_0)| &= \left| \int_a^b f(x, y) dx - \int_a^b f(x, y_0) dx \right| \\ &\leq \int_a^b |f(x, y) - f(x, y_0)| dx \\ &< (b-a) \cdot \frac{\varepsilon}{b-a} = \varepsilon \end{aligned}$$

$$\text{Thus, } \lim_{y \rightarrow y_0} \int_a^b f(x, y) dx = \int_a^b f(x, y_0) dx = \int_a^b \lim_{y \rightarrow y_0} f(x, y) dx.$$

- (b) Since  $\frac{\partial F}{\partial y}$  is continuous on  $Q$ , then  $\frac{\partial F}{\partial y}$  exists for all  $y \in (c, d)$

$$F'(y) = \lim_{h \rightarrow 0} \frac{F(y+h) - F(y)}{h} = \int_a^b \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx.$$

2. (a) By 1(b),  $g'(x) = \int_0^1 -2(t^2 + 1)x \frac{e^{-x^2(t^2+1)}}{t^2+1} dt = \int_0^1 -2xe^{-x^2(t^2+1)} dt.$

$$\text{By Fundamental Theorem of Calculus, } f'(x) = 2e^{-x^2} \left( \int_0^x e^{-t^2} dt \right)$$

Thus, take  $u = xt$ ,  $du = xdt$

$$\begin{aligned} f'(x) + g'(x) &= 2e^{-x^2} \left( \int_0^x e^{-t^2} dt \right) + \int_0^1 -2xe^{-x^2(t^2+1)} dt \\ &= 2e^{-x^2} \left( \int_0^x e^{-t^2} dt \right) - 2e^{-x^2} \int_0^1 xe^{-x^2 t^2} dt \\ &= 2e^{-x^2} \left( \int_0^x e^{-t^2} dt - \int_0^x e^{-u^2} du \right) \\ &= 2e^{-x^2} \cdot 0 = 0 \end{aligned}$$

Since  $f'(x) + g'(x) = 0$  for all  $x$ ,  $f(x) + g(x) = f(0) + g(0) + \int_0^x f'(x) + g'(x) dx = g(0) = \int_0^1 \frac{1}{t^2 + 1} dt = \int_0^{\frac{\pi}{4}} du = \frac{\pi}{4}$ .

(b) Since  $g(x) \rightarrow 0$  as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \frac{\pi}{4}$ . Then,  $\lim_{x \rightarrow \infty} \int_0^x e^{-t^2} dt = \lim_{x \rightarrow \infty} \sqrt{f(x)} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$ .

3. (a) If  $L = \lim_{x \rightarrow \infty} f(x) > 0$ , we can find a  $N \in \mathbb{N}$  s.t.  $f(x) > \frac{L}{2}$  for all  $x > N$ .

Then, for  $t > N$ ,  $\int_0^t f(x) dx = \int_0^N f(x) dx + (t - N) \frac{L}{2}$ . Thus,  $\lim_{t \rightarrow \infty} \int_0^t f(x) dx = \infty$  doesn't exist (contradiction).

Using the same way,  $L < 0$  also causes contradiction. Therefore,  $L = 0$ .

(b) First, for any  $0 < \varepsilon < 1$ , we can not find a  $N \in \mathbb{N}$  s.t.  $f(x) < \varepsilon$  for all  $x > N$ .

Then, let  $n = [x]$ ,  $\int_0^x f(t) dt \leq \sum_{i=1}^n 1 \cdot 2^{-i}$ .

Since  $\sum_{i=1}^{\infty} 2^{-i} = 1 < \infty$ ,  $\lim_{x \rightarrow \infty} \int_0^x f(t) dt \leq 1$ ,  $f$  is improperly integrable.