Homework 4 of Introduction to Analysis (I), Honor Class

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1. Since $A \subseteq \mathbb{R}^n$ is open, then for any point $x \in A$, exists some $\varepsilon > 0$ s.t. $B(x, \varepsilon) \subset A$.

2. (a) i. x is an accumulation point, for all $\varepsilon > 0$, $D(x, \varepsilon)/\{x\} \cap A \neq \emptyset \implies D(x, \varepsilon) \cap A \neq \emptyset$. Thus, x is a limit point.

   ii. Let $M = \mathbb{R}, d(x, y) = |x - y|$, $A = \{n \mid n \in \mathbb{N}\}$, then 1 is a limit point of A since $B(0, \varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

   But for $0 < \varepsilon < 1$, $B(1, \varepsilon)/\{1\} \cap A = \emptyset$. Thus, limit point is not accumulation all the time.

   (b) Assume there exists a limit point x of A is not in A. Then, $x \in A^c$. We want to show that A is not close.

   For A^c, since all neighborhood U of x has intersection with A, then $U \nsubseteq A^c$. Thus, A^c is not open.

   Therefore, if A is close, A contains all its limit points.

3. (a) For $x \in \bar{A}_1 = A_1 \cup A_1'$, then we only need to show that the accumulation point x of $A_1 \implies \inf\{d(x,y) \mid y \in A_1\} = 0$.

Assume $\inf\{d(x,y) \mid y \in A_1\} = \alpha > 0$, then $B(x,\varepsilon) \cap A_1 = \emptyset$ for $0 < \varepsilon < \alpha$. Implies that x is not an accumulation point of A_1.

Thus, if $x \in \bar{A}_1$, $\inf\{d(x,y) \mid y \in A_1\} = 0$.

(b) ($\subseteq$) For $x \in B_n$, $D(x,\varepsilon) \cap B_n \neq \emptyset$ for all $\varepsilon > 0$.

$D(x,\varepsilon) \cap (\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} (D(x,\varepsilon) \cap A_i) \neq \emptyset \implies D(x,\varepsilon) \cap A_i \neq \emptyset$ for some $i \in [1,n]$. Thus, $x \in \bar{A}_i$ for any $\varepsilon > 0$ and some $i \in [1,n] \implies x \in \cup_{i=1}^{n} \bar{A}_i$.

($\supseteq$) For $x \in \cup_{i=1}^{n} \bar{A}_i$ and $\varepsilon > 0$, $D(x,\varepsilon) cap \bar{A}_i \neq \emptyset$ for some i. This implies that $D(x,\varepsilon) \cap (\cup_{i=1}^{n} A_i) = D(x,\varepsilon) \cap B_n \neq \emptyset \implies x \in \bar{B}_n$

(c) Using the same way in (b), we can know that $U_{i=1}^{\infty} A_i \subseteq B$. But the union of close sets may not be close for some time, then the inclusion can be proper. Like $A_i = (\frac{1}{I}, 1)$, $\bar{B} = [0,1]$ and $\cup A_i = (0,1]$.