Exercises(8)

November 1, 2023

1. Suppose that $A = \{a_n\}$ is a sequence of strictly real numbers (this is, $a_n > 0$ for all n). The infinite product generated by A is the sequence $P = \{p_n\}$ defined by

   $$p_1 = a_1, p_2 = p_1 \times a_2, \ldots, p_n = p_{n-1} \times a_n (= a_1 \times a_2 \times \cdots \times a_n), \ldots.$$

   If the sequence P is convergent to a non-zero number, then we call $\lim_{n\to\infty} p_n$ the product of the infinite product generated by A. In this case we say that the infinite product is convergent and write

   $$\lim_{n\to\infty} p_n = \prod_{n=1}^{\infty} a_n.$$

   (a) (6 points) Prove that $\prod_{n=1}^{\infty} a_n$ is convergent if and only if $\sum_{n=1}^{\infty} \ln a_n$ is convergent.

   (b) (8 points) Infinite products often have terms of the form $a_n = 1 + u_n$. In keeping with our standing restriction, we suppose $u_n > -1$ for all n. If $u_n \geq 0$, show that a necessary and sufficient condition for the convergence of the infinite product is the convergence of the infinite series $\sum_{n=1}^{\infty} u_n$.

   (c) (6 points) Let $u_n > -1$. Show that if the infinite series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent, then the infinite product $\prod_{n=1}^{\infty}(1 + u_n)$ is convergent.

2. (10 points) Given a convergent series $\sum_{n=1}^{\infty} a_n$, where each $a_n \geq 0$. Prove that $\sum_{n=1}^{\infty} \sqrt{a_n} n^{-p}$ converges if $p > 1/2$. Give a counterexample for $p = 1/2$.