Homework 13 of Introduction to Analysis (I), Honor Class

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1. First, we want to check there are at most $\dfrac{n(n+1)}{2}$ points for $x \in [0,1]$ for which $f(x) > \dfrac{1}{n}$.

   For $f(x) > \dfrac{1}{n}$, that means x is a rational number $\dfrac{p}{q}$ and $q < n$. Thus, we have $x = \dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{2}{3}, \cdots$. The number of x is less than $\dfrac{n(n+1)}{2}$.

   Then, we want to proof for any $\varepsilon > 0$, we have a partition P s.t. $|U(f,P) - L(f,p)| < \varepsilon$.

   Then, for any $\varepsilon > 0$, we take P is partition of $[0,1]$ which norms are $\dfrac{2}{n(n+1)}$. Since

   $$\begin{aligned} |U(f,P) - L(f,P)| &= U(f,P) \\ &< n \cdot \frac{2}{n(n+1)} \\ &= \frac{2}{n+1} \end{aligned}$$

   , for any $\varepsilon > 0$, we can find n s.t. $U(f,P) < \varepsilon$. Thus, f is integrable.

2. Since f is bounded, we can find $M \in \mathbb{R}$ s.t. $|f(x)| < M$ for all $x \in [a,b]$. Suppose there are n points of discontinuity of f on $[a,b]$.

   Thus, for any $\varepsilon > 0$, we take $\delta = \dfrac{\varepsilon}{4nM}$. Then, suppose the set of points of discontinuity is $\{y_i \mid i \in \mathbb{N}, i \leq n\}$. Take partition $P = \{x_1 = y_1 - \varepsilon,\ x_2 = y_1 + \varepsilon,\ x_3 = y_2 - \varepsilon,\ \cdots\}$ is finite. Then, for f is continuous on $[a,x_1],\ [x_2,x_3],\ [x_4,x_5],\ \cdots$, we only need to check other intervals P' are Reimann integrable. $|U(f,P') - L(f,P')| \leq 2U(f,P') \leq 2 \cdot n \cdot (2\delta) \cdot M = 4nM \cdot \dfrac{\varepsilon}{4nM} = \varepsilon$. Thus, $f \in R([a,b])$.

3. $f^{(n)}(x) = (\prod_{k=0}^{n-1} m - k)(1+x)^{m-n}$