Homework 7 of Introduction to Analysis (I), Honor Class

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1. First, we show that $\bigcap_{i=1}^{\infty} F_i$ is closed. For F_k, we let $V_k = M \setminus F_k$ which is a open set. Then, $M \setminus (\bigcap_{i=1}^{\infty} F_i) = \bigcup_{i=1}^{\infty} V_i = \lim_{i \to \infty} V_i$ is open. Thus, $\bigcap_{i=1}^{\infty} F_i$ is closed.

   Then, assume $\bigcap_{i=1}^{\infty} F_i$ is disconnected. Thus, we can find two non-empty open sets U, V in $\bigcap_{i=1}^{\infty} F_i$ s.t. $U \cap V = \emptyset$ and $\bigcap_{i=1}^{\infty} F_i = U \cup V$. Since $F_k \subseteq F_{k-1}$, $U, V \subseteq F_k$ for all k.

   And since F_k is connected, $U \cup V \subsetneq F_k$ for all k. Then, there exists some $x_{k,i} \in F_k \setminus U \cup V$.

2. Assume there exists $a \in S$ and $r \geq 0$ s.t. $\{x \mid d(x,a) = r\} = \emptyset$. Then, $D(a,r) = \{a\}$ is open and $S \setminus \{a\}$ is open since for all $x \in S \setminus \{a\}, D(x, d(x,a)) \subseteq S \setminus \{a\}$. We can find two open sets $D(a,r), S \setminus \{a\}$ s.t. S is not connected.(contradiction)

   Thus, for every a in S and every $r > 0$, the set $\{x : d(x,a) = r\}$ is nonempty.

3.

4.

($\Longrightarrow$) Assume A is not in the set, that is, A can be write as $\bigcup_{i \in E} I_i$ where I_i is the interval in the set and I_m, I_n is disconnected. Thus, A is disconnected.

therefore, A is connected $\Longrightarrow$ A is in the set.

($\Longleftarrow$) For any x, y in one of the four interval, exists $f : [0,1] \to [x,y]$ with $f(t) = tx + (1-t)y$ is a continuous path. Thus, A is path-connected $\Longrightarrow$ A is connected.