Homework 3 of Introduction to Analysis (I), Honor Class

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1. (d_1) a. positive definite: $d(x,y) = (x-y)^2 \geq 0$ for any $x, y \in \mathbb{R}$ check
   b. $0 \iff$ equal:
   $(\implies)$ $d(x,y) = 0 \implies x - y = 0 \implies x = y$
   $(\impliedby)$ $d(x,y) = (x-y)^2 = 0^2 = 0$ check
   c. triangle inequality: $4 = (2-0)^2 = d(2,0) > 1+1 = d(2,1) + d(1,0)$ no
   Thus, d_1 is not a metric.

   (d_2) a. positive definite: since $|x-y| \geq 0$, then $\dfrac{|x-y|}{1+|x-y|} \geq 0$ check
   b. $0 \iff$ equal:
   $(\implies)$ since $0 = d(x,y) = \dfrac{0}{1+0} \implies x - y = 0 \implies x = y$
   $(\impliedby)$ since $x = y$, $d(x,y) = \dfrac{0}{1+0} = 0$ check
   c. triangle inequality: If z is not in (x,y), the triangle inequality trivially holds.
   Then, WLOG, suppose $x < z < y$, then
   $$\frac{z-x}{1+z-x} + \frac{y-z}{1+y-z} > \frac{z-x}{1+(z-x)+(y-z)} + \frac{y-z}{1+(y-z)+(z-x)}$$
   $$= \frac{y-x}{1+(y-z)+(z-x)} = \frac{y-x}{1+y-x}$$
   Thus, d_2 is a metric.

2. We want to prove $s_{n+1} - s_n < \varepsilon$ for all $\varepsilon > 0$ which means $x_n \to 0$.

   Since $\limsup\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} < 1$, there exists some $N \in \mathbb{N}$ s.t. $\dfrac{x_{n+1}}{x_n} < 1$ for all $n > N$.
   Thus, $\{x_n\}_{n=N}^{\infty}$ is a monotone decreasing sequence bounded below by 0.
   For any $r < 1$, $\dfrac{1}{r} > 1$ and there exists some $k \in \ltimes$ s.t. $(\dfrac{1}{r})^k > n$ for all $n \in \mathbb{N}$, then $r^k < \dfrac{1}{n}$.
   Therefore, $1 - r < 1 \implies (1-r)^k \to 0$ for $k \to \infty$
   Since $(0,1)$ is open, we can find some r s.t. $\dfrac{x_{n+1}}{x_n} < r < 1$ for all $n > N$. Then, $\lim x_n - x_{n+1} = \lim x_n(1 - \dfrac{x_{n+1}}{x_n}) < \lim x_N(1-r)^{n-N} = 0$.
   Thus, s_n is Cauchy $\implies$ s_n is convergence.

3. (a) WLOG, suppose $\{x_n\}$ is positive and monotone increasing.
   Then, $\sigma_{n+1} - \sigma_n = (\dfrac{n}{n+1}S_n + \dfrac{x_{n+1}}{n+1}) - S_n = \dfrac{1}{n+1}S_n + \dfrac{x_{n+1}}{n+1} > 0$. Therefore, σ_n is monotone increasing.

   (b)