Homework 12 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

November 30, 2023

1. We claim that a_n is a Cauchy. Then, for $m > n \geq N$

$$\begin{aligned}|f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\ &< \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} \\ &\leq \frac{1}{N}\end{aligned}$$

Thus, for any $\varepsilon > 0$, we take $N > \frac{1}{\varepsilon}$. Therefore, a_n is Cauchy $\Longrightarrow \lim_{n\to\infty} a_n$ exists.

2. Since $f'(x)$ exists on (a,b), we can find $s_1 = \sup\{f'(x) \mid x \in (a,b)\}$, $s_2 = \lim_{x\to a^+} f(x)$ and $s = \max\{s_1, s_2\}$.

Then, for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{s}$, then $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$. Thus, f is uniform continuous.

3. (a) We want to show that when $x \to \infty$, $f(x+h) - f(x) \to hb$ for all h.

Since $f'(x) \to b$ as $x \to \infty$, for any $\varepsilon > 0$, exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Thus, for $x > N$, $(b-\varepsilon)h < f(x+h) - f(x) < (b+\varepsilon)h$. Thus, $|\frac{f(x+h) - f(x)}{h} - b| < \varepsilon$ for $x > N$.

Therefore, $\lim_{x\to\infty} \frac{f(x+h) - f(x)}{h} - b = 0$.

(b) Since $f(x) \to a$, for $\varepsilon, h > 0$, exsits $N \in \mathbb{N}$ s.t. $|f(x) - a| < \frac{\varepsilon \cdot h}{2}$.

Then, for $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} |\frac{f(x+h) - f(x)}{h}| \leq \lim_{x\to\infty} \frac{|f(x+h) - a| + |a - f(x)|}{h} < 2\frac{\varepsilon \cdot h}{2h} = \varepsilon$.

Thus, $f'(x) \to 0$ as $x \to \infty$

(c) For any $\varepsilon > 0$, exists a $N \in \mathbb{N}$ s.t. $|f'(x_0) - b| < \varepsilon$ for all $x_0 > N$.

$$\begin{aligned}
\lim_{x\to\infty} |\frac{f(x)}{x} - b| &= \lim_{x\to\infty} |\frac{f(x_0) + f'(x_0)(x - x_0)}{x} - b| \\
&= \lim_{x\to\infty} |\frac{f(x_0)}{x} - \frac{f'(x_0)x_0}{x}| + |f'(x_0) - b| \\
&< 0 + \varepsilon \text{ for } x_0 > N
\end{aligned}$$

Thus, $\dfrac{f(x)}{x} \to b$ as $x \to \infty$.