Exercises(12)

November 28, 2023

1. (6 points) Given a function f defined and having a finite derivative f' in the half-open interval $0 < x \leq 1$ and such that $|f'(x)| < 1$. Define $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots$, and show that $\lim_{n\to\infty} a_n$ exists.

2. (7 points) Let $f : (a, b] \to \mathbb{R}$ be continuous and such that $f'(x)$ exists on (a, b) and $\lim_{x\to a^+} f'(x)$ exists. Prove that f is uniformly continuous on $(a, b]$.

3. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable on $(0, \infty)$.

   (a) (3 points) If $f'(x) \to b \in \mathbb{R}$ as $x \to \infty$, show that for any $h > 0$ we have
   $$\lim_{x\to\infty} \frac{f(x+h) - f(x)}{h} = b.$$

   (b) (3 points) If $f(x) \to a \in \mathbb{R}$ and $f'(x) \to b \in \mathbb{R}$ as $x \to \infty$, show that $b = 0$.

   (c) (3 points) Assume that $f'(x) \to b \in \mathbb{R}$ as $x \to \infty$. Don't use L'Hospital Rule, show that $f(x)/x \to b$ as $x \to \infty$.

4. Suppose that f is three times differentiable on an interval containing a. Show that

   (a) (4 points)
   $$\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a),$$

   (b) (4 points)
   $$\lim_{h\to 0} \frac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3} = f^{(3)}(a).$$