

# Homework 6 of Introduction to Analysis (I), Honor Class

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1. First, we show that  $S$  is closed and bounded. There exists  $D(2,2) \subseteq S \implies S$  is bounded. And for  $x \in \mathbb{Q} \setminus S$ , if  $x \leq \sqrt{2}$ ,  $D(x, d(x, \sqrt{2})) \subseteq \mathbb{Q} \setminus S$ . If  $x \geq \pi$ ,  $D(x, \pi) \subseteq \mathbb{Q} \setminus S \implies \mathbb{Q} \setminus S$  is open. Then,  $S$  is closed.

$G_i = (\sqrt{2} - \frac{1}{n}, \pi - \frac{1}{n}) \cap \mathbb{Q}$ , then  $\bigcup_{i=1}^{\infty} G_i$  is an open cover of  $S$  but doesn't have finite subcover. Thus,  $S$  is not compact.

2. Assume  $\bigcap_{n=1}^{\infty} V_n = \emptyset$ . And since  $\text{int}(V_n) \subseteq V_n \subseteq \text{cl}(V_n) \subseteq V_{n-1}$ ,  $V_n \subseteq V_{n-1}$ . Then,  $\bigcap_{n=1}^k V_n = V_k$ .

Take open set  $C_i = (\text{cl } V_i)^c$ . Since  $\bigcap V_i = \emptyset$ ,  $\bigcup_{i=1}^{\infty} C_i$  is an open cover of  $\text{cl}(V_1)$ .

Since  $\text{cl}(V_1)$  is compact, there exists  $n' \in \mathbb{N}$  s.t.  $\bigcup_{i=1}^{n'} C_i$  is finite subcover of  $\text{cl}(V_1)$ . That implies  $V_{n'} = \emptyset$ , contradict to  $V_i$  are non-empty for all  $i$ .

Thus,  $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$ .

3. Since  $x, y \in K$ , then assume there doesn't exist a set in  $G$  s.t. it contains both of  $x, y$ , then  $G$  doesn't contain one of  $x, y$  or both neither. Thus,  $G$  doesn't cover  $K$ , contradict to  $G$  is a open cover. Thus, there exists a set in  $G$  containing both  $x, y$ .

4. Since  $A$  is totally bounded, for all  $\varepsilon > 0$ , exists  $\{x_i\}_{i=1}^N$  is a finite set s.t.  $A \subseteq \bigcup_{i=1}^N D(x_i, \varepsilon)$ .

And  $M$  is complete  $\implies$  every Cauchy sequence in  $M$  converge to a point in  $M$ .

Then, for any sequence  $\{y_i\}$  in  $\text{cl}(A)$ , if there is not any subsequence converge to  $\text{cl}(A)$ , there exists  $\varepsilon' > 0$  s.t.  $|y_m - y_n| > \varepsilon'$  for all  $m, n > N \in \mathbb{N}$ .  $\cup D(y_i, \varepsilon')$  will be a infinite open cover of  $A$ , which contradicts to there has a finite  $N$  s.t.  $\cup D(x_i, \varepsilon')$  is cover of  $A$ .

Thus, every sequence in  $\text{cl}(A)$  have subsequence converge to a point in  $\text{cl}(A) \implies \text{cl}(A)$  is sequence compact. Then, by BWT,  $\text{cl}(A)$  is compact.