Exercises(15)

December 19, 2023

1. Let f be continuous at each point (x, y) of a rectangle
$$Q = \{(x, y) : a \le x \le b, c \le y \le d\}.$$
Let F be the function defined on $[c, d]$ by the equation
$$F(y) = \int_a^b f(x, y)dx.$$
   (a) Prove that F is continuous on $[c, d]$. In other words, if $y_0 \in [c, d]$, we have
$$\lim_{y \to y_0} \int_a^b f(x, y)dx = \int_a^b \lim_{y \to y_0} f(x, y)dx = \int_a^b f(x, y_0)dx.$$
   (b) If the partial derivative $\frac{\partial f}{\partial y}$ is continuous on Q, show that for each $y \in (c, d)$,
$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)dx.$$
   Note: The partial derivative $\frac{\partial f}{\partial y}$ is defined by
$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

2. Define
$$f(x) = \left(\int_0^x e^{-t^2} \right)^2, \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2 + 1} dt.$$
   (a) Show that $g'(x) + f'(x) = 0$ for all x and deduce that $g(x) + f(x) = \pi/4$.
   (b) Use (a) to prove that
$$\lim_{x \to \infty} \int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

3. (a) Suppose that f is improperly integrable on $[0, \infty)$. Prove that if $L = \lim_{x \to \infty} f(x)$ exists, then $L = 0$.
   (b) Let
$$f(x) = \begin{cases} 1 & n \le x < n + 2^{-n}, n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$
   Prove that f is improperly integrable on $[0, \infty)$ but $\lim_{x \to \infty} f(x)$ does not exist.