

Homework 15 of Introduction to Analysis (I), Honor Class

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1. (a) For any $y \in [c, d]$, we can find a partition $P_y =$

$$(b) F'(y) = \lim_{h \rightarrow 0} \frac{F(y+h) - F(y)}{h} = \int_a^b \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} dx = \int_a^b \frac{\partial f}{\partial y}(x, y) dx.$$

$$2. (a) g'(x) = \int_0^1 -2(t^2 + 1)x \frac{e^{-x^2(t^2+1)}}{t^2 + 1} dt = \int_0^1 -2xe^{-x^2(t^2+1)} dt.$$

$$f'(x) = 2e^{-x^2} \left(\int_0^x e^{-t^2} dt \right)$$

$$\text{Since } f'(x) + g'(x) = 0 \text{ for all } x, f(x) + g(x) = f(0) + g(0) = g(0) = \int_0^1 \frac{1}{t^2 + 1} dt = \int_0^{\frac{\pi}{4}} du = \frac{\pi}{4}.$$

$$(b) \text{ Since } g(x) \rightarrow 0 \text{ as } x \rightarrow \infty, f(x) \rightarrow \frac{\pi}{4}. \text{ Then, } \lim_{x \rightarrow \infty} \int_0^x e^{-t^2} dt = \lim_{x \rightarrow \infty} \sqrt{f(x)} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$

3. (a) If $L = \lim_{x \rightarrow \infty} f(x) > 0$, we can find a $N \in \mathbb{N}$ s.t. $f(x) > \frac{L}{2}$ for all $x > N$.

Then, for $t > N$, $\int_0^t f(x) dx = \int_0^N f(x) dx + (t - N) \frac{L}{2}$. Thus, $\lim_{t \rightarrow \infty} \int_0^t f(x) dx = \infty$ doesn't exist (contradiction).

Using the same way, $L < 0$ also causes contradiction. Therefore, $L = 0$.

- (b) First, for any $0 < \varepsilon < 1$, we can not find a $N \in \mathbb{N}$ s.t. $f(x) < \varepsilon$ for all $x > N$.

$$\text{Then, let } n = [x], \int_0^x f(t) dt \leq \sum_{i=1}^n 1 \cdot 2^{-i}.$$

Since $\sum_{i=1}^{\infty} 2^{-i} = 1 < \infty$, $\lim_{x \rightarrow \infty} \int_0^x f(t) dt \leq 1$, f is improperly integrable.