

Homework 3 of Introduction to Analysis (I), Honor Class

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1. (d_1) a. positive definite: $d(x, y) = (x - y)^2 \geq 0$ for any $x, y \in \mathbb{R}$ check
b. $0 \iff$ equal:
(\implies) $d(x, y) = 0 \implies x - y = 0 \implies x = y$
(\impliedby) $d(x, y) = (x - y)^2 = 0^2 = 0$ check
c. triangle inequality: $4 = (2 - 0)^2 = d(2, 0) > 1 + 1 = d(2, 1) + d(1, 0)$ no
Thus, d_1 is not a metric.
- (d_2) a. positive definite: since $|x - y| \geq 0$, then $\frac{|x - y|}{1 + |x - y|} \geq 0$ check
b. $0 \iff$ equal:
(\implies) since $0 = d(x, y) = \frac{0}{1 + 0} \implies x - y = 0 \implies x = y$
(\impliedby) since $x = y$, $d(x, y) = \frac{0}{1 + 0} = 0$ check
c. triangle inequality: If z is not in (x, y) , the triangle inequality trivially holds.
Then, WLOG, suppose $x < z < y$, then
$$\frac{z - x}{1 + z - x} + \frac{y - z}{1 + y - z} > \frac{z - x}{1 + (z - x) + (y - z)} + \frac{y - z}{1 + (y - z) + (z - x)}$$
$$= \frac{y - x}{1 + (y - z) + (z - x)} = \frac{y - x}{1 + y - x}$$

Thus, d_2 is a metric.
2. We want to prove $s_{n+1} - s_n < \varepsilon$ for all $\varepsilon > 0$ which means $x_n \rightarrow 0$.
Since $\limsup_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} < 1$, there exists some $N \in \mathbb{N}$ s.t. $\frac{x_{n+1}}{x_n} < 1$ for all $n > N$.
Thus, $\{x_n\}_{n=N}^{\infty}$ is a monotone decreasing sequence bounded below by 0.
For any $r < 1$, $\frac{1}{r} > 1$ and there exists some $k \in \mathbb{N}$ s.t. $(\frac{1}{r})^k > n$ for all $n \in \mathbb{N}$, then $r^k < \frac{1}{n}$.
Therefore, $1 - r < 1 \implies (1 - r)^k \rightarrow 0$ for $k \rightarrow \infty$
Since $(0, 1)$ is open, we can find some r s.t. $\frac{x_{n+1}}{x_n} < r < 1$ for all $n > N$. Then, $\lim_{n \rightarrow \infty} x_n - x_{n+1} =$
$$\lim_{n \rightarrow \infty} x_n \left(1 - \frac{x_{n+1}}{x_n}\right) < \lim_{n \rightarrow \infty} x_N (1 - r)^{n-N} = 0.$$

Thus, s_n is Cauchy $\implies s_n$ is convergence.

3. (a) WLOG, suppose $\{x_n\}$ is positive and monotone increasing.

Then, $\sigma_{n+1} - \sigma_n = \left(\frac{n}{n+1}S_n + \frac{x_{n+1}}{n+1}\right) - S_n = \frac{1}{n+1}S_n + \frac{x_{n+1}}{n+1} > 0$ for all n . Therefore, σ_n is monotone.

(b) Suppose $\{c_1, c_2, \dots, c_k\}$ are cluster points of $\{x_n\}$.

Then, $\limsup \sigma_n \leq \max\{c_i\} = \limsup x_n$.