

Homework 1 of Introduction to Analysis (I), Honor Class

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1. Let $R = \{f(x) \mid x \in X\}$ and $r = \sup R$. Then, for any β which is upper bound of R , $r \leq \beta$. Thus, $a + \beta$ is an upper bound of $R' = \{a + f(x) \mid x \in X\}$.

Assume there exists $\beta' = \sup R'$ s.t. $\beta' < a + r$. Since $\beta' \geq a + f(x)$, $\forall x \in X \Rightarrow \beta' - a \geq f(x)$, $\forall x \in X$. Therefore, $\beta' - a$ is an upper bound of R and it should be greater or equal to r . This causes contradiction to the assumption.

Thus, for every upper bound of R' , it is greater or equal to $a + r$.

That is $\sup\{a + f(x) \mid x \in X\} = a + \sup\{f(x) \mid x \in X\}$

2. let a be greatest lower bound of $\{f(x) \mid x \in X\}$, b be greatest lower bound of $\{g(x) \mid x \in X\}$. That is $a \leq f(x) \forall x \in X$ and $b \leq g(x) \forall x \in X$. Then, $a + b$ is a lower bound of $\{f(x) + g(x) \mid x \in X\}$. Therefore $a + b \leq \inf\{f(x) + g(x) \mid x \in X\}$. In other words, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}$

Using the same way we can prove $\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

Then, we want to show $\inf S \leq \sup S$ for all set S :

For an upper bound of S " β " and lower bound of S " α ". For any element $x \in S$, $\alpha \leq x \leq \beta$.

Then, $\inf S \leq x \leq \sup S$ for all $x \in S$

Then, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

Ex. Let $f(x) = \sin(x)$, $g(x) = \cos(x)$, $X = (0, 2\pi)$, then $-2 < -\sqrt{2} < 2$.

3. Using the steps from Rudin's book.

(a) For any $n \in \mathbb{N}$, $b^n - 1 \geq n(b - 1)$

Proof.

- $n = 1$, $b - 1 \geq 1(b - 1)$ trivially true
- Assume $n = k$, $b^k - 1 \geq k(b - 1)$
- $n = k + 1$, $b^{k+1} - 1 = (b - 1)(b^k + b^{k-1} + \dots + 1) \geq (b - 1)(k + 1)$ since $b \geq 1$

Then, by math induction, $b^n - 1 \geq n(b - 1)$ for all $n \in \mathbb{N}$

(b) Then, $b - 1 \geq n(b^{1/n} - 1)$

Proof. $(b^{1/n})^n - 1 \geq n(b^{1/n} - 1)$

(c) If $t > 1$ and $n > \frac{b-1}{t-1}$, then $b^{1/n} < t$

Proof. $n > \frac{b-1}{t-1} \geq \frac{n(b^{1/n} - 1)}{t - 1}$, then $t - 1 > b^{1/n} - 1 \Rightarrow b^{1/n} < t$ for $t > 1$

(d) If w s.t. $b^w < y$, then exists some $n \in \mathbb{N}$ s.t. $b^{w+1/n} < y$

Proof. $b^{w+1/n} = b^w * b^{1/n}$, then from (c), we can know that there exists some $n \in \mathbb{N}$ s.t. $b^{1/n} < \frac{y}{b^w}$ since $\frac{y}{b^w} > 1$. Thus, $b^{w+1/n} < y$ if $b^w < y$ with some $n \in \mathbb{N}$

(e) If $b^w > y$, then $b^{w-1/n} > y$

Proof. $b^{w-1/n} = \frac{b^w}{b^{1/n}}$. By (c), there exists some $n \in \mathbb{N}$ s.t. $b^{1/n} \leq \frac{b^w}{y}$. Thus, $b^{w-1/n} > y$ if $b^w > y$ for some $n \in \mathbb{N}$

(f) $A = \{w \mid b^w < y\}$, $x = \sup A$ satisfies $b^x = y$

By Bernoulli's inequality, $b^n = (1 + c)^n \geq 1 + nc$ by let $c = b - 1 > 0$, then by Archimedean property, there exists some $n \in \mathbb{N}$ s.t. $b^n > 1 + nc > y$. Thus, A is bounded above. And since $b^{-n} = \frac{1}{b^n}$, there exists some $n \in \mathbb{N}$ s.t. $b^n \geq \frac{1}{y} \Rightarrow b^{-n} \leq y$. Thus, $A \neq \emptyset$.

Thus, by the complement of real number, there exists $x = \sup A$.

If $b^x < y$, from (d), $b^{x+1/n} < y \Rightarrow x + 1/n \in A$. Thus, x is not upper bound of A . If $b^x > y$, from (e), $b^{x-1/n} > y$. Then, x is not the least upper bound of A .

So, b^x must to be equal to y .

(g) x is unique because it is supremum of A .