Homework 6 of Introduction to Analysis (I), Honor Class

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1. First, we show that S is closed and bounded. There exists $D(2,2) \subseteq S \implies S$ is bounded. And for $x \in \mathbb{Q} \setminus S$, if $x \leq \sqrt{2}$, $D(x, d(x, \sqrt{2})) \subseteq \mathbb{Q} \setminus S$. If $x \geq \pi$, $D(x, \pi) \subseteq \mathbb{Q} \setminus S \implies \mathbb{Q} \setminus S$ is open. Then, S is closed.

   $G_i = (\sqrt{2} - \frac{1}{n}, \pi - \frac{1}{n}) \cap \mathbb{Q}$, then $\bigcup_{i=1}^{\infty} G_i$ is an open cover of S but doesn't have finite subcover. Thus, S is not compact.

2. Assume $\bigcap_{n=1}^{\infty} V_n = \emptyset$. And since $\text{int}(V_n) \subseteq V_n \subseteq \text{cl}(V_n) \subseteq V_{n-1}$, $V_n \subseteq V_{n-1}$. Then, $\bigcap_{n=1}^{k} V_n = V_k$.

   Take open set $C_i \supseteq (V_i)^c$. Since $\cap V_i = \emptyset$, $\bigcup_{i=1}^{\infty} C_i$ is an open cover of V_n for all n. That is, $\bigcup_{i=1}^{\infty} C_i$ is a open cover of $\text{cl}(V_n)$ for all n.

   Since $\text{cl}(V_1)$ is compact, there exists $n' \in \mathbb{N}$ s.t. $\bigcup_{i=1}^{n'} C_i$ is finite subcover of $\text{cl}(V_1)$. That implies $V_n = \emptyset$, contradict to V_i are non-empty for all i.

   Thus, $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$.

3. Since $x, y \in K$, then assume there doesn't exists a set in G s.t. it contains both of x, y, then G doesn't contain one of x, y or both neither. Thus, G doesn't cover K , contradict to G is a open cover. Thus, there exists a set in G containing both x, y.

4. Since A is totally bounded, for all $\varepsilon > 0$, exists $\{x_i\}_{i=1}^N$ is a finite set s.t. $A \subseteq \bigcup_{i=1}^{N} D(x_i, \varepsilon)$.

   And M is conplete $\implies$ every Cauchy sequence in M converge to a point in M.

   Then, for any sequence $\{y_i\}$ in $\mathrm{cl}(A)$, if there is not any subsequence converge to $\mathrm{cl}(A)$, there exists $\varepsilon' > 0$ s.t. $|y_m - y_n| > \varepsilon'$ for all $m, n > N \in \mathbb{N}$. $\cup D(y_i, \varepsilon')$ will be a infinite open cover of A, which contradicts to there has a finite N s.t. $\cup D(x_i, \varepsilon')$ is cover of A.

   Thus, every sequence in $\mathrm{cl}(A)$ have subsequence converge to a point in $\mathrm{cl}(A)$ $\implies$ $\mathrm{cl}(A)$ is sequence compact. Then, by BWT, $\mathrm{cl}(A)$ is compact.