

Homework 1 of Introduction to Analysis (I), Honor Class

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2023/9/12

1. Let $R = \{f(x) \mid x \in X\}$ and $r = \sup R$. Then, for any β which is upper bound of R , $r \leq \beta$. Thus, $a + \beta$ is an upper bound of $R' = \{a + f(x) \mid x \in X\}$.

Assume there exists $\beta' = \sup R'$ s.t. $\beta' < a + r$. Since $\beta' \geq a + f(x)$, $\forall x \in X \Rightarrow \beta' - a \geq f(x)$, $\forall x \in X$. Therefore, $\beta' - a$ is an upper bound of R and it should be greater or equal to r . This causes contradiction to the assumption.

Thus, for every upper bound of R' , it is greater or equal to $a + r$.

That is $\sup\{a + f(x) \mid x \in X\} = a + \sup\{f(x) \mid x \in X\}$

2. let a be greatest lower bound of $\{f(x) \mid x \in X\}$, b be greatest lower bound of $\{g(x) \mid x \in X\}$. That is $a \leq f(x) \forall x \in X$ and $b \leq g(x) \forall x \in X$. Then, $a + b$ is a lower bound of $\{f(x) + g(x) \mid x \in X\}$. Therefore $a + b \leq \inf\{f(x) + g(x) \mid x \in X\}$. In other words, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\}$

Using the same way we can prove $\sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

Then, we want to show $\inf S \leq \sup S$ for all set S :

For an upper bound of S " β " and lower bound of S " α ". For any element $x \in S$, $\alpha \leq x \leq \beta$.

Then, $\inf S \leq x \leq \sup S$ for all $x \in S$

Then, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) + g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

Ex. Let $f(x) = \sin(x)$, $g(x) = \cos(x)$, $X = (0, 2\pi)$, then $-2 < -\sqrt{2} < 2$.

3. Let $A = \{x \mid x \in \mathbb{R}, b^x < y\}$.

By Bernoulli's Inequality (In Bartle's book page.35) and let $b = 1 + c$ with $c \in \mathbb{R}, c > 0$, then $(1 + c)^n \geq 1 + nc$ with all $n \in \mathbb{N}$. Then, we can find some $n \in \mathbb{N}$ s.t. $c > \frac{y}{n}$. Therefore, $b^n = (1 + c)^n \geq 1 + nc > y$ and n is upper bound of A . Thus, by completeness of \mathbb{R} , there exists $\beta \in \mathbb{R}$ be $\sup A$.

If $b^\beta < y$, then assume $\beta + \frac{1}{n} \in A$. Thus $b^{\beta + \frac{1}{n}} = b^\beta \cdot b^{\frac{1}{n}} > b^\beta \cdot 1 = b^\beta$ for all $n \in \mathbb{N}$. Then, we want to proof $b^{\beta + \frac{1}{n}} < y$, thus we check the inequality $b^{\beta + \frac{1}{n}} < y$ for some $n \in \mathbb{N}$.

Proof. $b^{\beta + \frac{1}{n}} = b^\beta b^{\frac{1}{n}}$. Since $b^{\frac{1}{n}} = (1 + c)^{\frac{1}{n}} \leq 1 + \frac{1}{n}c$ and $\frac{y}{b^\beta}$ is a positive real number which is greater than 1, by Archimedean Property there exists some $n \in \mathbb{N}$ s.t. $n > \frac{c \cdot b^\beta}{y - b^\beta} > 0$. In the other world $1 + \frac{c}{n} < \frac{y}{b^\beta}$. Therefore, $b^{\beta + \frac{1}{n}} < y$.

Thus, $b^{\beta + \frac{1}{n}} \in A \Rightarrow b^\beta$ is not a upper bound of A . Therefore, b^β should greater or equal to y .

If $b^\beta > y$, assume exists $n \in \mathbb{N}$ s.t. $b^{\beta - \frac{1}{n}} > y$. Then, $b^{\beta - \frac{1}{n}} = \frac{b^\beta}{b^{\frac{1}{n}}} \geq \frac{b^\beta}{1 + \frac{1}{n}c}$. And since $\frac{b^\beta}{y} - 1$ is a positive real number by $b^\beta > y$, there exists some $n \in \mathbb{N}$ s.t. $\frac{c}{n} > \frac{b^\beta}{y} - 1$ by Archimedean Property. Therefore, b^β is not the least upper bound of A (contradiction).

These two line implies the supremum of A , β , is the unique real number s.t. $b^\beta = y$ since the supremum is unique.