

Homework 12 of Introduction to Analysis (I), Honor Class

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1. We claim that a_n is a Cauchy. Then, for $m > n \geq N$

$$\begin{aligned} |f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\ &< \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} \\ &\leq \frac{1}{N} \end{aligned}$$

Thus, for any $\varepsilon > 0$, we take $N > \frac{1}{\varepsilon}$. Therefore, a_n is Cauchy $\implies \lim_{n \rightarrow \infty} a_n$ exists.

2. Since $f'(x)$ exists on (a, b) , we can find $s_1 = \sup\{f'(x) \mid x \in (a, b)\}$, $s_2 = \lim_{x \rightarrow a^+} f(x)$ and $s = \max\{s_1, s_2\}$.

Then, for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{s}$, then $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$. Thus, f is uniform continuous.

3. (a) We want to show that when $x \rightarrow \infty$, $f(x+h) - f(x) \rightarrow hb$ for all h .

Since $f'(x) \rightarrow b$ as $x \rightarrow \infty$, for any $\varepsilon > 0$, exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Thus,

for $x > N$, $(b - \varepsilon)h < f(x+h) - f(x) < (b + \varepsilon)h$. Thus, $|\frac{f(x+h) - f(x)}{h} - b| < \varepsilon$ for $x > N$.

Therefore, $\lim_{x \rightarrow \infty} \frac{f(x+h) - f(x)}{h} - b = 0$.

- (b) Since $f(x) \rightarrow a$, for $\varepsilon, h > 0$, exists $N \in \mathbb{N}$ s.t. $|f(x) - a| < \frac{\varepsilon \cdot h}{2}$.

Then, for $\lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} |\frac{f(x+h) - f(x)}{h}| \leq \lim_{x \rightarrow \infty} \frac{|f(x+h) - a| + |a - f(x)|}{h} < 2 \frac{\varepsilon \cdot h}{2h} = \varepsilon$.

Thus, $f'(x) \rightarrow 0$ as $x \rightarrow \infty$

(c) For any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Then, for any $x_0 > N$

$$\begin{aligned}\lim_{x \rightarrow \infty} \left| \frac{f(x)}{x} - b \right| &= \lim_{x \rightarrow \infty} \left| \frac{f(x_0) + f'(x_1)(x - x_0)}{x} - b \right| \\ &= \lim_{x \rightarrow \infty} \left| \frac{f(x_0) - f'(x_1)x_0}{x} \right| + |f'(x_1) - b| \text{ for } x_1 \in [x_0, x] \\ &< 0 + \varepsilon = \varepsilon\end{aligned}$$

Thus, $\frac{f(x)}{x} \rightarrow b$ as $x \rightarrow \infty$.

$$4. \quad (a) \quad \lim_{h \rightarrow 0} \left(\frac{f(a+2h) - f(a+h)}{h^2} - \frac{f(a+h) - f(a)}{h^2} \right) = \lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a)}{h} = f''(a).$$