Homework 5 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

October 15, 2023

1. First we want to show given $\varepsilon > 0$, we can find finite sequence $x_1, x_2, \cdots x_n$ s.t. $\cup D(x_i, \varepsilon) = X$.

   Proving this by contrapositive way.

   Let y_1 be a point in X, then $D(y_1, \varepsilon)$ can not cover X. Then, we can find $y_2 \in X - D(y_1, \varepsilon)$ s.t. $D(y_1, \varepsilon) \cup D(y_2, \varepsilon)$ can not cover X.

   Using the same way, if we cann't find $m \in \mathbb{N}$ s.t. $\cup D(y_i, \varepsilon)$ does not cover X. Which means distance of y_i are greateer than ε. This, is contradiction to the infinite set in X has a accumulation points.

   Thus, we can find finite set $\{x_{i,n}\}_{i=1}^{m_i}$ s.t. $\{D(x_{i,n}, \frac{1}{n})\}_{i=0}^{m_n}$ cover X. Then, $\{x_{i,n}\}_{1 \leq n \leq \infty,\ 1 \leq i \leq m_n}$ is countable.

   Then, we want to show $S = \cup D(x_{i,n}, \frac{1}{n})$ is dense in X.

   For any $x \in X - S$, then for all $\varepsilon > 0$, exists $\frac{1}{n_i} \leq \varepsilon$ s.t. $D(x, \frac{1}{n_i}) \subseteq D(x, \varepsilon)$ and $x \in D(x_{i,n}, \frac{1}{n_i})$ for some $1 \leq i \leq m_i \implies x$ is a accumulation point of S. Then, $\bar{S} = X \implies S$ is dense.

2. Since we can find collection of open sets covers open set $U \subseteq \mathbb{R}$. For $\{U'\}$ be disjoint non-empty open sets covers U. Since U' is open, U' contains rational numbers. Then, for every U', we call it U'_x, for a $x \in \mathbb{Q} \cap U'$. Thus, we can label the open sets by $\mathbb{Q}$ which is countable.

3. Since $A_1 \supseteq A_2 \supseteq \cdots A_n, A_1' \supseteq A_2' \supseteq \cdots A_n'$.

   For $x \in A$, since $\cap_{n=1}^{\infty} A_n = \emptyset, A = \cap_{n=1}^{\infty} \bar{A}_n = (A_1 \cup A_1') \cap (\cap_{n=2}^{\infty} \bar{A}_n)$

   $= (A_1' \cap (\cap_{n=2}^{\infty} \bar{A}_n)) \cup (A \cap (\cap_{n=2}^{\infty} \bar{A}_n)) = (\cap_{i=1}^{\infty} A_i') \cup (\cap_{i=2}^{\infty} A_i') \cup \cdots = \cap_{n=1}^{\infty} A_n'$.

   Thus, $x \in A_1'$.

4. First, we want to proof $\overline{M-A} \supseteq (\overline{A} - A)$. Since $\overline{M-A} = (M-A)' \cup (M-A)$ and $\overline{A} \subseteq M \implies (\overline{A} - A) \subseteq (M-A) \subseteq \overline{M-A}$.

   $(A \cap \overline{M-A}) \cup (\overline{A} - A) = (A \cup (\overline{A} - A)) \cap (\overline{M-A} \cup (\overline{A} - A)) = \overline{A} \cap (\overline{M-A}) = \partial A$