Homework 11 of Introduction to Analysis (I), Honor Class

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1. (a) Let $g(x_0) = \lim\limits_{x \to x_0} f(x)$ for all $x_0 \in \bar{A}$. Since f is uniform continuous, $g(x) = f(x)$ for all $x \in A$ and g is continuous. Then, we want to show g is uniform continuous.

   Assume g is not uniform continuous, exists a $\varepsilon > 0$, for all $\delta > 0$ s.t. $|g(x) - g(x_0)| > \varepsilon$ for all $x \in D(x, \delta) \cap A$. Since f is uniform continuous, for $x_1, x_2 \in D(x, \delta) \cap A \setminus \{x\}$, we can find a $\varepsilon' > 0$ s.t. $|g(x_1) - g(x_2)| = |f(x_1) - f(x_2)| < \varepsilon$ for $|x_1 - x_2| < 2\delta$. Which implies $|g(x_1) - g(x_0)| < \varepsilon$, or $g(x)$ is not continuous.

   Thus, g is uniform continuous.

   (b) Since A is bounded, $\bar{A}$ is compact. Then, $f(A) \subseteq f(\bar{A})$ is bounded since $f(\bar{A})$ is compact.

2. (a) For any $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, $f_A(x_0) - \varepsilon < f_A(x) < f_A(x_0) + \varepsilon$ for $x \in D(x_0, \varepsilon)$ tirvially. Then, $|f_A(x) - f_A(x_0)| < \delta = \varepsilon$ for $\|x - x_0\| < \varepsilon$. Thus, f_A is uniform continuous.

   (b) For $x \in \bar{A}$, $\inf\{\|x - y\| \mid y \in A\} = 0$. Thus, $f_A(x) = 0$ for all $x \in \bar{A}$

   And since $x \notin \bar{A}$, $\inf\{\|x - y\| \mid y \in A\} > 0$, $\bar{A} = \{x \mid f_A(x) = 0\}$

3. Since f is one-to-one, then $f(x) = f(y)$ implies $x = y$. Assume f is not monotonic but continuous, exists a x s.t. $x < x'$ for all $x' \in D(x, \varepsilon)$ or $x > x'$ for all $x' \in D(x, \varepsilon)$. Then, since (a, b) is path-connected, by IVT, we can find $y, y' \in D(x, \varepsilon)$ s.t. $f(y) = f(y')$ with $y \neq y'$. That is, f is not one-to-one(contradiction).

   Thus, f is one-to-one and monotonic.