Homework 15 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

December 24, 2023

1. (a) For any $y \in [c,d]$, we can find a partition $P_y =$

   (b) $F'(y) = \lim_{h\to 0} \dfrac{F(y+h)-F(y)}{h} = \displaystyle\int_a^b \lim_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h} dx = \int_a^b \dfrac{\partial f}{\partial y}(x,y)dx.$

2. (a) $g'(x) = \displaystyle\int_0^1 -2(t^2+1)x\frac{e^{-x^2(t^2+1)}}{t^2+1}dt = \int_0^1 -2xe^{-x^2(t^2+1)}dt.$

   $f'(x) = 2e^{-x^2}(\displaystyle\int_0^x e^{-t^2}dt)$

   Since $f'(x)+g'(x) = 0$ for all x, $f(x)+g(x) = f(0)+g(0) = g(0) = \displaystyle\int_0^1 \frac{1}{t^2+1}dt = \int_0^{\frac{\pi}{4}} du = \frac{\pi}{4}.$

   (b) Since $g(x) \to 0$ as $x \to \infty$, $f(x) \to \dfrac{\pi}{4}$. Then, $\displaystyle\lim_{x\to\infty}\int_0^x e^{-t^2}dt = \lim_{x\to\infty}\sqrt{f(x)} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$

3. (a) If $L = \lim_{x\to\infty} f(x) > 0$, we can find a $N \in \mathbb{N}$ s.t. $f(x) > \dfrac{L}{2}$.

   Then, for $t > N$, $\displaystyle\int_0^t f(x)dx = \int_0^N f(x)dx + (t-N)\frac{L}{2}$. Thus, $\displaystyle\lim_{t\to\infty}\int_0^t f(x)dx = \infty$ doesn't exists(contradiction).

   Using the same way, $L < 0$ also causes contradiction. Therefore, $L = 0$.

   (b) First, for any $0 < \varepsilon < 1$, we can not find a $N \in \mathbb{N}$ s.t. $f(x) < \varepsilon$ for all $x > N$.

   Then, let $n = [x]$, $\displaystyle\int_0^x f(t)dt < \sum_{i=1}^n 1\cdot 2^{-i}$.

   Since $\displaystyle\sum_{i=1}^\infty 2^{-i} = 1 < \infty$, $\displaystyle\lim_{x\to\infty}\int_0^x f(t)dt < 1$, f is improperly integrable.