Homework 1 of Introduction to Analysis (I), Honor Class

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1. Let $R = \{f(x) \mid x \in X\}$ and $r = \sup R$. Then, for any β which is upper bound of R, $r \leq \beta$. Thus, $a+\beta$ is an upper bound of $R' = \{a + f(x) \mid x \in X\}$.

   Assume there exists $\beta' = \sup R'$ s.t. $\beta' < a + r$. Since $\beta' \geq a + f(x)$, $\forall x \in X \Rightarrow \beta' - a \geq f(x)$, $\forall x \in X$. Therefore, $\beta' - a$ is an upper bound of R and it should greater or equal to r. This causes contradiction to the assumption.

   Thus, for every upper bound f R', it is greater or equal to $a + r$.

   That is $\sup\{a + f(x) \mid x \in X\} = a + \{f(x) \mid x \in X\}$

2. let a be greastest lower bound of $\{f(x) \mid x \in X\}$, b be greastest lower bound of $\{f(x) \mid x \in X\}$. That is $a \leq f(x) \forall x \in X$ and $b \leq g(x) \forall x \in X$. Then, $a+b$ is an lower bound of $\{f(x)+g(x) \mid x \in X\}$. Therefore $a+b \leq \inf\{f(x)+g(x) \mid x \in X\}$. In other world, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x)+g(x) \mid x \in X\}$

   Using the same way we can prove $\sup\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

   Then, we want to show $\inf S \leq \sup S$ for all set S:

   > For a upper bound of S "β" and lower bound of S "α". For any element $x \in S$, $\alpha \leq x \leq \beta$.
   > Then, $\inf S \leq x \leq \sup S$ for all $x \in S$

   Then, $\inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\} \leq \inf\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$

   Ex. Let $f(x) = \sin(x),\ g(x) = \cos(x),\ X = (0, 2\pi)$, then $-2 < -\sqrt{2} < 2$.

3. Using the steps from Rudin's book.

(a) For any $n \in \mathbb{N}$, $b^n - 1 \geq n(b-1)$

Proof.

- $n = 1$, $b - 1 \geq 1(b-1)$ trivially true
- Assume $n = k$, $b^k - 1 \geq k(b-1)$
- $n = k+1$, $b^{k+1} - 1 = (b-1)(b^k + b^{k-1} + \cdots + 1) \geq (b-1)(k+1)$ since $b \geq 1$

Then, by math induction, $b^n - 1 \geq n(b-1)$ for all $n \in \mathbb{N}$

(b) Then, $b - 1 \geq n(b^{1/n} - 1)$

Proof.$(b^{1/n})^n - 1 \geq n(b^{1/n} - 1)$

(c) If $t > 1$ and $n > \dfrac{b-1}{t-1}$, then $b^{1/n} < t$

Proof. $n > \dfrac{b-1}{t-1} \geq \dfrac{n(b^{1/n}-1)}{t-1}$, then $t - 1 > b^{1/n} - 1 \Rightarrow b^{1/n} < t$ for $t > 1$

(d) If w s.t. $b^w < y$, then exists some $n \in \mathbb{N}$ s.t. $b^{w+1/n} < y$

Proof. $b^{w+1/n} = b^w * b^{1/n}$, then from (c), we can know that there exists some $n \in \mathbb{N}$ s.t. $b^{1/n} < \dfrac{y}{b^w}$ since $\dfrac{y}{b^w} > 1$. Thus, $b^{w+1/n} < y$ if $b^w < y$ with some $n \in \mathbb{N}$

(e) If $b^w > y$, then $b^{w-1/n} > y$

Proof. $b^{w-1/n} = \dfrac{b^w}{b^1/n}$. By (c), there exists some $n \in \mathbb{N}$ s.t. $b^{1/n} \leq \dfrac{b^w}{y}$. Thus, $b^{w-1/n} > y$ if $b^w > y$ for some $n \in \mathbb{N}$

(f) $A = \{w \mid b^w < y\}$, $x = supA$ satisfies $b^x = y$

By Bernoulli's inequality, $b^n = (1+c)^n \geq 1 + nc$ by let $c = b - 1 > 0$, then by Archimedean property, there exists some $n \in \mathbb{N}$ s.t. $b^n > 1 + nc > y$. Thus, A is bounded above. And since $b^{-n} = \dfrac{1}{b^n}$, there exists some $n \in \mathbb{N}$ s.t. $b^n \geq \dfrac{1}{y} \Rightarrow b^{-n} \leq y$. Thus, $A \neq \emptyset$.

Thus, by the completement of real number, there exists $x = \sup A$.

If $b^x < y$, from (d), $b^{x+1/n} < y \Rightarrow x + 1/n \in A$. Thus, x is not upper bound of A. If $b^x > y$, form (f), $b^{x-1/n} > y$. Then, x is not the least upper bound of A.

So, b^x must to be equal to y.

(g) Uniqueness

Proof. Assume there exists x, x' s.t. $b^x = y = b^{x'}$.

Then, assume $x \neq x'$, since $b > 1$ and $x' - x \neq 0 \Rightarrow b^{x'-x} \neq 1$, then $b^x - b^{x'} = b^x(1 - b^{x'-x}) \neq 0$

Thus, if $b^x = b^{x'} \Rightarrow x = x' \Rightarrow x$ is unique.