

# Homework 5 of Introduction to Analysis (I), Honor Class

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1. First we want to show given  $\varepsilon > 0$ , we can find finite sequence  $x_1, x_2, \dots, x_n$  s.t.  $\cup D(x_i, \varepsilon) = X$ .

Proving this by contrapositive way.

Let  $y_1$  be a point in  $X$ , then  $D(y_1, \varepsilon)$  can not cover  $X$ . Then, we can find  $y_2 \in X - D(y_1, \varepsilon)$  s.t.  $D(y_1, \varepsilon) \cup D(y_2, \varepsilon)$  can not cover  $X$ .

Using the same way, if we can't find  $m \in \mathbb{N}$  s.t.  $\cup D(y_i, \varepsilon)$  does not cover  $X$ . Which means distance of  $y_i$  are greater than  $\varepsilon$ . This, is contradiction to the infinite set in  $X$  has a accumulation points.

Thus, we can find finite set  $\{x_{i,n}\}_{i=1}^{m_i}$  s.t.  $\{D(x_{i,n}, \frac{1}{n})\}_{i=0}^{m_n}$  cover  $X$ . Then,  $\{x_{i,n}\}_{1 \leq n \leq \infty, 1 \leq i \leq m_n}$  is countable.

Then, we want to show  $S = \cup D(x_{i,n}, \frac{1}{n})$  is dense in  $X$ .

For any  $x \in X - S$ , then for all  $\varepsilon > 0$ , exists  $\frac{1}{n_i} \leq \varepsilon$  s.t.  $D(x, \frac{1}{n_i}) \subseteq D(x, \varepsilon)$  and  $x \in D(x_{i,n}, \frac{1}{n_i})$  for some  $1 \leq i \leq m_i \implies x$  is a accumulation point of  $S$ . Then,  $\bar{S} = X \implies S$  is dense.

2. First, we let  $\{U\}$  be disjoint segments that covers the open set in  $\mathbb{R}$ . Let  $x \sim y$  if  $[x, y] \subseteq U$  with  $x \leq y$  or  $[y, x] \subseteq U$  with  $y \leq x$ . Then, if  $x, y \in U$  with  $x \neq y$ , since exists  $q \in [x, y] \cap \mathbb{Q}$ ,  $q \in U \cap \mathbb{Q}$ . Then, for every  $U$  contains more than one number, we call it  $U_x$ , for a  $x \in \mathbb{Q} \cap U$ , which  $\{U_x\}$  is countable.

3. Since  $A_1 \supseteq A_2 \supseteq \cdots A_n, A'_1 \supseteq A'_2 \supseteq \cdots A'_n$ .

$$\begin{aligned} \text{For } x \in A, \text{ since } \bigcap_{n=1}^{\infty} A_n = \emptyset, A &= \bigcap_{n=1}^{\infty} \bar{A}_n = (A_1 \cup A'_1) \cap \left( \bigcap_{n=2}^{\infty} \bar{A}_n \right) \\ &= (A'_1 \cap \left( \bigcap_{n=2}^{\infty} \bar{A}_n \right)) \cup (A \cap \left( \bigcap_{n=2}^{\infty} \bar{A}_n \right)) = \cdots = \left( \bigcap_{i=1}^{\infty} A'_i \right) \cup \left( \bigcap_{i=2}^{\infty} A'_i \right) \cup \cdots = \bigcap_{n=1}^{\infty} A'_n. \end{aligned}$$

Thus,  $x \in A'_1$ .

4. First, we want to proof  $\overline{M-A} \supseteq (\bar{A}-A)$ .

$$\text{Since } \overline{M-A} = (M-A)' \cup (M-A) \text{ and } \bar{A} \subseteq M \implies (\bar{A}-A) \subseteq (M-A) \subseteq \overline{M-A}.$$

$$(A \cap \overline{M-A}) \cup (\bar{A}-A) = (A \cup (\bar{A}-A)) \cap (\overline{M-A} \cup (\bar{A}-A)) = \bar{A} \cap (\overline{M-A}) = \partial A$$