Homework 10 of Introduction to Analysis (I), Honor Class

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1. (a)

   ($\Longrightarrow$) Since f is continuous, then for $x \in \bar{A}$, we can find a $\{x_i \mid x_i \in A\}$ converges to x. Thus, for all $\varepsilon > 0$, exists $\delta > 0$ s.t. $d(x_i, x) < \delta \implies \rho(f(x_i), f(x)) < \varepsilon$.

   Then, for $y \in f(\bar{A})$, $D(y, \varepsilon) \cap f(A) \neq \emptyset$ for all $\varepsilon > 0$. Thus, $y \in \overline{f(A)}$.

   ($\Longleftarrow$) For V closed in T, then $f(\mathrm{cl}(f^{-1}(V))) \subseteq \mathrm{cl}(V) = V$. Applied f^{-1} on the both side, $\mathrm{cl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{cl}(V)) = f^{-1}(V)$. Thus, $f^{-1}(V)$ is closed if V is closed implies f is continuous.

   (b) Since $f(p) = g(p)$ for all $p \in S$ and $\bar{E} = S$. For any $x \in S$, $x \in \bar{E}$, then we can find a sequence $\{x_i\} \subseteq E$ converges to x.

   Since $f(x_i) = g(x_i)$ for all $i \in \mathbb{N}$, $|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x)| = |f(x) - f(x_i)| + |g(x) - g(x_i)| \to 0$ as $i \to \infty$. Then, $f(x) = g(x)$ for all $x \in S$.

2. Since B is compact in $\mathbb{R}^n$, B is closed and bounded. And since f is 1-to-1 and continuous function, $f^{-1} : f(B) \to f$ is also 1-to-1 and onto.