# Exercises(9)

November 7, 2023

1. (6 points) Let the series $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. Show that there is a rearrangement of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums tend to $\infty$.

2. (8 points) Given that $\sum_{n=1}^{\infty} a_n$ converges, where each $a_n > 0$. Prove that
$$\sum_{n=1}^{\infty} (a_n a_{n+1})^{1/2}$$
also converges. Show that the converse is also true if $\{a_n\}$ is monotonic.

3. Suppose that each $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Put
$$r_n = \sum_{m=n}^{\infty} a_m.$$
   (a) (5 points) Prove that
   $$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$
   if $m < n$, and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

   (b) (5 points) Prove that
   $$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$
   and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges.

4. (6 points) If each $a_n \geq 0$, show that $\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n$.