

# Homework 12 of Introduction to Analysis (I), Honor Class

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1. We claim that  $a_n$  is a Cauchy. Then, for  $m > n \geq N$

$$\begin{aligned} |f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\ &< \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} \\ &\leq \frac{1}{N} \end{aligned}$$

Thus, for any  $\varepsilon > 0$ , we take  $N > \frac{1}{\varepsilon}$ . Therefore,  $a_n$  is Cauchy  $\implies \lim_{n \rightarrow \infty} a_n$  exists.

2. Since  $f'(x)$  exists on  $(a, b)$ , we can find  $s_1 = \sup\{f'(x) \mid x \in (a, b)\}$ ,  $s_2 = \lim_{x \rightarrow a^+} f(x)$  and  $s = \max\{s_1, s_2\}$ .

Then, for  $\varepsilon > 0$ , we take  $\delta = \frac{\varepsilon}{s}$ , then  $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$ . Thus,  $f$  is uniform continuous.

3. (a) We want to show that when  $x \rightarrow \infty$ ,  $f(x+h) - f(x) \rightarrow hb$  for all  $h$ .

Since  $f'(x) \rightarrow b$  as  $x \rightarrow \infty$ , for any  $\varepsilon > 0$ , exists a  $N \in \mathbb{N}$  s.t.  $|f'(x) - b| < \varepsilon$  for all  $x > N$ . Thus,

for  $x > N$ ,  $(b - \varepsilon)h < f(x+h) - f(x) < (b + \varepsilon)h$ . Thus,  $|\frac{f(x+h) - f(x)}{h} - b| < \varepsilon$  for  $x > N$ .

Therefore,  $\lim_{x \rightarrow \infty} \frac{f(x+h) - f(x)}{h} - b = 0$ .

- (b) Since  $f(x) \rightarrow a$ , for  $\varepsilon, h > 0$ , exists  $N \in \mathbb{N}$  s.t.  $|f(x) - a| < \frac{\varepsilon \cdot h}{2}$ .

Then, for  $\lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} |\frac{f(x+h) - f(x)}{h}| \leq \lim_{x \rightarrow \infty} \frac{|f(x+h) - a| + |a - f(x)|}{h} < 2 \frac{\varepsilon \cdot h}{2h} = \varepsilon$ .

Thus,  $f'(x) \rightarrow 0$  as  $x \rightarrow \infty$

(c) For any  $\varepsilon > 0$ , there exists a  $N \in \mathbb{N}$  s.t.  $|f'(x) - b| < \varepsilon$  for all  $x > N$ .

$$\begin{aligned}\lim_{x \rightarrow \infty} \left| \frac{f(x)}{x} - b \right| &= \lim_{x \rightarrow \infty} \left| \frac{f(x_0) + f'(x_1)(x - x_0)}{x} - b \right| \\ &= \lim_{x \rightarrow \infty} \left| \frac{f(x_0) - f'(x_1)x_0}{x} \right| + |f'(x_1) - b| \\ &< 0 + \varepsilon \text{ for } x_1 > x_0 > N\end{aligned}$$

Thus,  $\frac{f(x)}{x} \rightarrow b$  as  $x \rightarrow \infty$ .

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