Homework 8 of Introduction to Analysis (I), Honor Class

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1. (a) Suppose $\prod_{n=1}^{\infty} a_n = a \in \mathbb{R}^+$, $\exp(\sum_{n=1}^{\infty} \ln(a_n)) = \prod_{n=1}^{\infty} a_n = a$. Thus, $\sum_{n=1}^{\infty} ln(a_n) = \ln(a) \in (-\infty, \infty)$
   Therefore, $\prod_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} \ln(a_n)$ converges.

   (b) Since $u_n \geq 0$ and converges, $\sum_{n=1}^{\infty} u_n$ converges if u_n converges to 0. By limit comparison test $\lim_{n\to\infty} \frac{\ln(1+u_n)}{u_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, u_n converges $\iff \prod_{n=1}^{\infty}(1+u_n)$ converges.

   (c) Since $\sum_{n=1}^{\infty} u_n$ is absolutely convergent, by (b), $\prod_{n=1}^{\infty}(1+|u_n|)$ is converges. For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$, for all $a, b > N$ s.t. $|\sum_{n=1}^{a} \ln(1+|u_n|) - \sum_{n=1}^{b} \ln(1+|u_n|)| < \varepsilon$. Then, $|\sum_{n=1}^{a} \ln(1+u_n) - \sum_{n=1}^{b} \ln(1+u_n)| < \varepsilon$. Which implies $\sum_{n=1}^{\infty} \ln(1+u_n)$ converges. Thus, $\prod_{n=1}^{\infty}(1+u_n)$ converges.

2. $\frac{\sqrt{a_n}}{n^p} = \sqrt{\frac{a_n}{n^{2p}}} \leq \frac{a_n + n^{-2p}}{2}$ by AM-GM Inequality. Then, if $p \leq \frac{1}{2}$, then $\frac{1}{n^{2p}} \geq \frac{1}{n}$ diverges. Therefore, if $p > \frac{1}{2}$, $\sum_{n=1}^{\infty} \sqrt{a_n} \cdot n^{-p} \leq \sum_{n=1}^{\infty} (\frac{a_n + n^{-2p}}{2})$ converges by p-test.

   Counter example: $a_n = \frac{1}{n(\ln(n))^2}$, then $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}} = \sum_{n=1}^{\infty} \frac{1}{n(\ln(n))}$ diverges by integral-test.