

Homework 9 of Introduction to Analysis (I), Honor Class

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1. Since $\sum a_n$ is conditionally converges, then $\sum |a_n|$ diverges.

$$\text{We take } b_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} \text{ and } c_n = \begin{cases} 0 & \text{if } a_n > 0 \\ -a_n & \text{if } a_n \leq 0 \end{cases} \text{ Then, } \sum a_n = \sum b_n - \sum c_n.$$

We want to show that $\sum b_n = \infty$. First, we assume $\sum c_n$ converges. Then, $\sum b_n = \sum (a_n + c_n) = \sum a_n + \sum c_n$ both converges. Then, $\sum |a_n| = \sum |b_n + c_n| = \sum b_n + \sum c_n$ both converges(contradiction).

Thus, c_n diverges, which implies b_n diverges, too.

2. Since $\sum \sqrt{a_n a_{n+1}} \leq \sum \frac{a_n + a_{n+1}}{2} = \sum a_n - \frac{a_1}{2}$. By comparison test, $\sum \sqrt{a_n a_{n+1}}$ converges.

Since a_n is monotoneic, a_n are all greater than 0 or all lower than 0. Then, we suppose a_n are greater than 0 and monotone decreasing. $\sum \sqrt{a_n a_{n+1}} \geq \sum a_{n+1} = \sum a_n - a_1$. Then, by comparison test, $\sum a_n$ converges.

3. (a)

$$\begin{aligned} \frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdots + \frac{a_n}{r_n} &> \frac{a_m + a_{m+1} + \cdots + a_n}{a_m} \\ &= \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m} \end{aligned}$$

Then, we want to show $\sum \frac{a_n}{r_n}$ is not Cauchy. For any $m \in \mathbb{N}$, we can find a $n \in \mathbb{N}$, $n > m$ s.t.

$$r_n = \sum_{k=n}^{\infty} a_k > \frac{1}{2} \sum_{k=m}^{\infty} a_k = \frac{1}{2} r_m \text{ since } \sum a_k \text{ converges.}$$

Thus, for $N \in \mathbb{N}$ and $m > N$, we can find a $n > m$ s.t. $\sum_{k=m}^n \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m} > 1 - \frac{1}{2} = \frac{1}{2}$. Therefore, $\sum \frac{a_k}{r_k}$ is not Cauchy implies it diverges.

(b)

$$\begin{aligned} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) &= 2 \frac{(\sqrt{r_n})(\sqrt{r_n} - \sqrt{r_{n+1}})}{\sqrt{r_n}} \\ &= 2 \frac{(\sqrt{r_n})^2 - (\sqrt{r_n}\sqrt{r_{n+1}})}{\sqrt{r_n}} \\ &\geq 2 \frac{r_n - r_{n+1}}{\sqrt{r_n}} \\ &= 2 \frac{a_n}{r_n} \\ &> \frac{a_n}{r_n} \end{aligned}$$

We want to proof $\sum_{n=m}^{\infty} \frac{a_n}{\sqrt{r_n}} \rightarrow \infty$ as $m \rightarrow \infty$. $\sum_{n=m}^{\infty} \frac{a_n}{\sqrt{r_n}} < \sum_{n=m}^{\infty} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) < 2\sqrt{r_m}$. And since $\sum a_n$ converges, $r_n \rightarrow 0$ as $n \rightarrow \infty$. Thus, by comparison test, $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

4. We want to show $\sum a_n - \lim_{n \rightarrow 1^-} \sum a_n x^n = \lim_{n \rightarrow i^-} \sum a_n (1 - x^n) = 0$. Since $\sum a_n$ converges, $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Then, let $x = 1 - \varepsilon$, then $x \rightarrow 1^- \implies \varepsilon \rightarrow 0$. $\sum a_n (1 - x^n) = \sum a_n \cdot \varepsilon^n < \varepsilon \cdot \sum a_n \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus,

$$\lim_{x \rightarrow 1^-} \sum a_n x^n = \sum a_n.$$