Homework 3 of Introduction to Analysis (I), Honor Class

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1. (d_1) a. positive definite: $d(x,y) = (x-y)^2 \geq 0$ for any $x, y \in \mathbb{R}$ check

   b. $0 \iff$ equal:

   $(\implies)$ $d(x,y) = 0 \implies x - y = 0 \implies x = y$

   $(\impliedby)$ $d(x,y) = (x-y)^2 = 0^2 = 0$ check

   c. triangle inequality: $4 = (2-0)^2 = d(2,0) > 1 + 1 = d(2,1) + d(1,0)$ no

   Thus, d_1 is not a metric.

   (d_2) a. positive definite: since $|x-y| \geq 0$, then $\dfrac{|x-y|}{1+|x-y|} \geq 0$ check

   b. $0 \iff$ equal:

   $(\implies)$ since $0 = d(x,y) = \dfrac{0}{1+0} \implies x - y = 0 \implies x = y$

   $(\impliedby)$ since $x = y$, $d(x,y) = \dfrac{0}{1+0} = 0$ check

   c. triangle inequality: If z is not in (x,y), the triangle inequality trivially holds.
   Then, WLOG, suppose $x < z < y$, then

   $$\frac{z-x}{1+z-x} + \frac{y-z}{1+y-z} > \frac{z-x}{1+(z-x)+(y-z)} + \frac{y-z}{1+(y-z)+(z-x)}$$
   $$= \frac{y-x}{1+(y-z)+(z-x)} = \frac{y-x}{1+y-x}$$

   Thus, d_2 is a metric.

2. We want to prove $s_{n+1} - s_n < \varepsilon$ for all $\varepsilon > 0$, which is equivalent to $x_n \to 0$.

   Since $\limsup\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} < 1$, there exists some $N \in \mathbb{N}$ s.t. $\dfrac{x_{n+1}}{x_n} < 1$ for all $n > N$.

   Thus, $\{x_n\}_{n=N}^{\infty}$ is a monotone decreasing sequence bounded below by 0.

   For any $r < 1$, $\dfrac{1}{r} > 1$ and there exists some $k \in \mathbb{N}$ s.t. $(\dfrac{1}{r})^k > n$ for all $n \in \mathbb{N}$, then $r^k < \dfrac{1}{n}$.

   Since $(0,1)$ is open, we can find some r s.t. $\dfrac{x_{n+1}}{x_n} < r < 1$ for all $n > N$. Then, $\lim\limits_{n\to\infty} x_n < \lim\limits_{n\to\infty} x_1 \times r^{n-1} = x_1 \times 0 = 0$.

   Thus, $x_n \to 0$ when $n \to \infty$.

Then, for $\varepsilon > 0$ exists $N \in \mathbb{N}$ and $m > n > N$ with $x_n < \dfrac{\varepsilon}{m-n+1}$,

$|S_m - S_n| = S_m - S_n = x_n + x_{n+1} + \cdots x_{m-1} < (m-n+1)\dfrac{\varepsilon}{m-n+1} = \varepsilon.$

Thus, s_n is Cauchy $\implies$ s_n is convergence.

3. (a) WLOG, suppose $\{x_n\}$ is monotone increasing.

   Then, $\sigma_{n+1} - \sigma_n = (\dfrac{n}{n+1}S_n + \dfrac{x_{n+1}}{n+1}) - S_n = \dfrac{x_{n+1}}{n+1} - \dfrac{S_n}{n+1} > \dfrac{x_{n+1} - x_n}{n+1} > 0$ for all n. Therefore, σ_n is monotone.

   (b) we divided this question into $\{x_n\}$ diverges or bdd.

(diverges) WLOG, suppose $\limsup x_n = \infty$, then for any $M \in \mathbb{N}$, exists $N \in \mathbb{N}$ s.t. $x_n > M$ for $n \geq N$.
Thus, there exists $M_1, N_1 \in \mathbb{N}$ s.t. $x_n \geq M_1$ for all $n \geq N_1$ and $\sigma_{N_1} > \dfrac{M \times N_1}{N_1} = M$.
which means for any $M \in \mathbb{N}$, there exists some $N_1 \in \mathbb{N}$ s.t. $\sigma_n > M$ for all $n \geq N_1$.
That means, $\lim \sigma_n = \lim x_n = \infty$.
The case of diverging to $-\infty$ is the same.

(bounded) Since $\{x_n\}$ is bdd, let $\beta = \limsup x_n = \sup\{\beta_i\}$ where β_i is cluster point of x_n for all i.
Thus, for $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ s.t. $x_n < \beta + \varepsilon$ for all $n > N$.
Then, since the density of $\mathbb{R}$, there exists $\beta < \delta < \beta + \varepsilon$. Thus, there exists some $M \in \mathbb{N}$ s.t. $x_n \leq \delta$ for all $n > M$.
Therefore, $\sigma_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n} < \dfrac{x_1 + x_2 + \cdots + x_M}{n} + \dfrac{\delta(n-M)}{n}$
$= \delta \cdot \dfrac{1}{n}(x_1 + x_2 + \cdots + x_M - M)$. By Archimedean properity, there exists some $N' \in \mathbb{N}$ s.t. $\dfrac{1}{n}(x_1 + x_2 + \cdots + x_M - M) < \beta + \varepsilon - \delta$ for $n > N'$. Thus, $\sigma < \beta + \varepsilon$ for all $\varepsilon > 0$ and some $N' \in \mathbb{N}$, which is equivalent to $\limsup \sigma_n \leq \beta = \limsup x_n$.

Then, $\limsup \sigma_n \leq \limsup x_n$ for all cases.