Homework 13 of Introduction to Analysis (I), Honor Class

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1. (a) Since $f(x) = \ln(x)$ is concave down, $\ln(uv) = \ln(u) + \ln(v) = \frac{1}{p}\ln(u^p) + \frac{1}{q}\ln(v^q) \leq \ln(\frac{u^p}{p} + \frac{v^q}{q})$.

   And since $\ln(x)$ is strictly increasing, $uv \leq \frac{u^p}{p} + \frac{v^q}{q}$.

   If $u^p = v^q$, $\frac{u^p}{p} + \frac{v^q}{q} = (\frac{1}{p} + \frac{1}{q})u^p = u^p = u^{p(\frac{1}{p}+\frac{1}{q})} = u \cdot u^{\frac{p}{q}} = uv$.

   (b) $\int_a^b \frac{(f(x))^p}{p} dx = \frac{1}{p}$ and $\int_a^b \frac{(g(x))^q}{q} = \frac{1}{q}$. Then, $\int_a^b \frac{(f(x))^p}{p} + \frac{(g(x))^q}{q} dx = \frac{1}{p} + \frac{1}{q} = 1$.

   And since $f(x)g(x) \leq \frac{(f(x)^p)}{p} + \frac{(g(x))^q}{q}$ for all $x \in [a,b]$ and f, g are Reimann integral, $\int_a^b f(x)g(x)dx$ exists and $\int_a^b f(x)g(x)dx \leq \int_a^b \frac{(f(x))^p}{p} + \frac{(g(x))^q}{q} dx = 1$.

   (c) Take $F = \int_a^b |f(x)|^p dx, \int_a^b |g(x)|^q dx$.

   Then,

$$
\begin{aligned}
\frac{|\int_a^b f(x)g(x)dx|}{F^{\frac{1}{p}}G^{\frac{1}{q}}} &\leq \frac{\int_a^b |f(x)||g(x)|dx}{F^{\frac{1}{p}}G^{\frac{1}{q}}} \\
&= \int_a^b (\frac{|f(x)|^p}{F})^{\frac{1}{p}} (\frac{|g(x)|^p}{G})^{\frac{1}{q}} dx \\
&\leq \int_a^b \frac{1}{p}(\frac{|f(x)|^p}{F}) + \frac{1}{q}(\frac{|g(x)|^q}{G}) dx \\
&= \frac{1}{p}(\frac{\int_a^b |f(x)|^p dx}{F})^{\frac{1}{p}} + \frac{1}{q}(\frac{\int_a^b |g(x)|^q dx}{G})^{\frac{1}{q}} \\
&= \frac{1}{p} = \frac{1}{q} = 1
\end{aligned}
$$

Thus, $|\int_a^b f(x)g(x)dx| \leq (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}(\int_a^b |g(x)|^q dx)^{\frac{1}{q}}$.

2. (a) Take $\delta = \frac{1}{2^{n+1}}$ and $P = \{0, \frac{1}{2^n}, \frac{1}{2^n}, \frac{1}{2^n}+\delta, \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}+\delta, \cdots, \frac{1}{2}+\delta, 1\}$.

Then,

$$\begin{aligned}
U(f,P)-L(f,P) &= \delta\cdots(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}-\frac{1}{2^n})+\frac{1}{2^n}(\frac{1}{2^n}-0) \\
&= \frac{1}{2^{n+1}}\cdot\frac{1}{2^n}+\frac{1}{2^{2n}} \\
&< \frac{1}{2^{2n-1}} \to 0 \text{ as } n\to\infty
\end{aligned}$$

(b) Let $n = \left[-\frac{\ln x}{\ln 2}\right]$, then $A(x) = 2^{-n}$.

$$\begin{aligned}
\int_0^x f(t)dt &= \int_0^1 f(t)dt - \int_x^1 f(t)dt \\
&= \sum_{k=n}^{\infty}\frac{1}{2^k}\cdot\frac{1}{2^{k+1}} - (\frac{1}{2^n}-x)\cdot 2^{-n} \\
&= x2^{-n}+\frac{1}{2^{2n+1}}\cdot\frac{4}{3}-\frac{1}{2^{2n}} \\
&= x2^{-n}-\frac{1}{3}2^{-2n} \\
&= xA(x)-\frac{1}{3}(A(x))^2
\end{aligned}$$

3. Since f is continuous on $[a,b]$, f is uniform continuous and exists $c\in[a,b]$ s.t. $f(c)=M$. And let $I_n = (\int_a^b (|f(x)|)^n dx)^{\frac{1}{n}}$.

Then, for all $\varepsilon>0$, there exists $\delta>0$ s.t. if $x\in(c-\delta,c+\delta)\bigcap[a,b]$, $|f(x)-M|<\varepsilon \implies f(x)\geq M-\varepsilon$.

Thus, $I_n \geq (\frac{1}{2\delta})^{\frac{1}{n}}(M-\varepsilon)$. And since $f(x)\leq M$ for all x, $I_n \leq (\frac{1}{b-a})^{\frac{1}{n}}M$.

Then, we can get for all $\varepsilon>0$ and some $\delta>0$, $(\frac{1}{2\delta})^{\frac{1}{n}}(M-\varepsilon) \leq I_n \leq (\frac{1}{b-a})^{\frac{1}{n}}M$. And since $r^{\frac{1}{n}}\to 1$ as $n\to\infty$ for all $r>0$, $\lim_{n\to\infty}(\frac{1}{2\delta})^{\frac{1}{n}}(M-\varepsilon)=M$ and $\lim_{n\to\infty}(\frac{1}{b-a})^{\frac{1}{n}}M=M$.

By squeeze theorem, $\lim_{n\to\infty} I_n = M$.