Homework 15 of Introduction to Analysis (I), Honor Class

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February 19, 2024

1. (a) Since f is continuous on Q is compact, $\exists \delta > 0$ s.t. for all $\alpha_1, \alpha_2 \in Q$, $\|\alpha_1 - \alpha_2\| < \delta$,

   $|f(\alpha_1) - f(\alpha_2)| < \dfrac{\varepsilon}{b-a}$.

   Thus, for $y_0 \in [c,d]$,

$$\begin{aligned}
|F(y) - F(y_0)| &= |\int_a^b f(x,y)dx - \int_a^b f(x,y_0)dx| \\
&\leq \int_a^b |f(x,y) - f(x,y_0)|dx \\
&< (b-a) \cdot \frac{\varepsilon}{b-a} = \varepsilon
\end{aligned}$$

   Thus, $\lim\limits_{y \to y_0} \int_a^b f(x,y)dx = \int_a^b f(x,y_0)dx = \int_a^b \lim\limits_{y \to y_0} f(x,y)dx$.

   (b) Since $\dfrac{\partial F}{\partial y}$ is continuous on Q, then $\dfrac{\partial F}{\partial y}$ exists for all $y \in (c,d)$

   $F'(y) = \lim\limits_{h \to 0} \dfrac{F(y+h) - F(y)}{h} = \int_a^b \lim\limits_{h \to 0} \dfrac{f(x,y+h) - f(x,y)}{h} dx = \int_a^b \dfrac{\partial f}{\partial y}(x,y)dx$.

2. (a) By 1(b), $g'(x) = \int_0^1 -2(t^2+1)x \dfrac{e^{-x^2(t^2+1)}}{t^2+1} dt = \int_0^1 -2xe^{-x^2(t^2+1)} dt$.

   By Fundamental Theorem of Calculus, $f'(x) = 2e^{-x^2}(\int_0^x e^{-t^2} dt)$

   Thus, take $u = xt$, $du = xdt$

$$\begin{aligned}
f'(x) + g'(x) &= 2e^{-x^2}(\int_0^x e^{-t^2} dt) + \int_0^1 -2xe^{-x^2(t^2)+1} dt \\
&= 2e^{-x^2}(\int_0^x e^{-t^2} dt) - 2e^{-x^2} \int_0^1 xe^{-x^2 t^2} dt \\
&= 2e^{-x^2}(\int_0^x e^{-t^2} dt - \int_0^x e^{-u^2} du) \\
&= 2e^{-x^2} \cdot 0 = 0
\end{aligned}$$

Since $f'(x)+g'(x)=0$ for all x, $f(x)+g(x)=f(0)+g(0)+\int_0^x f'(x)+g'(x)dx=g(0)=\int_0^1 \frac{1}{t^2+1}dt=\int_0^{\frac{\pi}{4}} du=\frac{\pi}{4}$.

(b) Since $g(x)\to 0$ as $x\to\infty$, $f(x)\to\frac{\pi}{4}$. Then, $\lim_{x\to\infty}\int_0^x e^{-t^2}dt=\lim_{x\to\infty}\sqrt{f(x)}=\sqrt{\frac{\pi}{4}}=\frac{\sqrt{\pi}}{2}$.

3. (a) If $L=\lim_{x\to\infty} f(x)>0$, we can find a $N\in\mathbb{N}$ s.t. $f(x)>\frac{L}{2}$ for all $x>N$.

Then, for $t>N$, $\int_0^t f(x)dx=\int_0^N f(x)dx+(t-N)\frac{L}{2}$. Thus, $\lim_{t\to\infty}\int_0^t f(x)dx=\infty$ doesn't exists(contradiction).

Using the same way, $L<0$ also causes contradiction. Therefore, $L=0$.

(b) First, for any $0<\varepsilon<1$, we can not find a $N\in\mathbb{N}$ s.t. $f(x)<\varepsilon$ for all $x>N$.

Then, let $n=[x]$, $\int_0^x f(t)dt\leq\sum_{i=1}^{n}1\cdot 2^{-i}$.

Since $\sum_{i=1}^{\infty}2^{-i}=1<\infty$, $\lim_{x\to\infty}\int_0^x f(t)dt\leq 1$, f is improperly integrable.