# Homework 12 of Introduction to Analysis (I), Honor Class

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December 5, 2023

1. We claim that $a_n$ is a Cauchy. Then, for $m > n \geq N$

$$\begin{aligned}|f(\frac{1}{n}) - f(\frac{1}{m})| &< |(\frac{1}{n} - \frac{1}{m}) \cdot f'(c)| \text{ for some } c \in [\frac{1}{m}, \frac{1}{n}] \\ &< \frac{1}{n} - \frac{1}{m} \\ &< \frac{1}{n} \leq \frac{1}{N}\end{aligned}$$

   Thus, for any $\varepsilon > 0$, we take $N > \frac{1}{\varepsilon}$. Therefore, $a_n$ is Cauchy $\Longrightarrow \lim_{n\to\infty} a_n$ exists.

2. Since $f'(x)$ exists on $(a,b)$, we can find $s_1 = \sup\{f'(x) \mid x \in (a,b)\}$, $s_2 = \lim_{x\to a^+} f(x)$ and $s = \max\{s_1, s_2\}$.

   Then, for $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{s}$, then $|f(x) - f(x_0)| < |x - x_0|s < \delta s = \varepsilon$. Thus, $f$ is uniform continuous.

3. (a) Claim that for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|\frac{f(x+h) - f(x)}{h} - b| < \varepsilon$ for $x > N$ and all $h$.

   Suppose $b \geq 0$. Since $f'(x) \to b$ as $x \to \infty$, for any $\varepsilon > 0$, exists a $N_0 \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N_0$. Thus, $b - \varepsilon < f'(x) < b + \varepsilon$ for all $x > N_0$. By MVT, $f(x+h) = f(x) + h \cdot f'(c)$ for all $h$ and some $c \in [x+h, x]$ or $c \in [x, x+h]$. Thus, $|f(x+h) - f(x)| < |h|(b+\varepsilon)$ for all $h$ and $x > N_0$. Then, we can get the result by taking $N = N_0$.

   If $b < 0$, we can get the same result $|f(x+h) - f(x)| < |h||b - \varepsilon|$ by the silmilar way.

   Therefore, $\lim_{x\to\infty} \frac{f(x+h) - f(x)}{h} - b = 0$.

(b) Since $f(x) \to a$, for $\varepsilon, h > 0$, exsits $N \in \mathbb{N}$ s.t. $|f(x) - a| < \frac{\varepsilon \cdot h}{2}$.

Then, for $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} |\frac{f(x+h)-f(x)}{h}| \leq \lim_{x\to\infty} \frac{|f(x+h)-a|+|a-f(x)|}{h} < 2\frac{\varepsilon \cdot h}{2h} = \varepsilon$.

Thus, $f'(x) \to 0$ as $x \to \infty$

(c) For any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|f'(x) - b| < \varepsilon$ for all $x > N$. Then, for any $x_0 > N$

$$
\begin{aligned}
\lim_{x\to\infty} |\frac{f(x)}{x} - b| &= \lim_{x\to\infty} |\frac{f(x_0) + f'(x_1)(x - x_0)}{x} - b| \\
&= \lim_{x\to\infty} |\frac{f(x_0) - f'(x_1)x_0}{x}| + |f'(x_1) - b| \text{ for } x_1 \in [x_0, x] \\
&< 0 + \varepsilon = \varepsilon
\end{aligned}
$$

Thus, $\frac{f(x)}{x} \to b$ as $x \to \infty$.

4. (a)

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} &= \lim_{h\to 0} \frac{f'(a+2h) - f'(a+h)}{h} \\
&= 2\lim_{h\to 0} \frac{f'(a+2h) - f'(a)}{2h} - \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{h} \\
&= 2f''(a) - f''(a) = f''(a)
\end{aligned}
$$

Thus, $\lim_{h\to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2} = f''(a)$.

(b)

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)}{h^3} &= \lim_{h\to 0} \frac{3f'(a+3h) - 6f'(a+2h) + 3f(a+h)}{3h^2} \\
&= \lim_{h\to 0} \frac{f'(a+3h) - 2f'(a+2h) + f'(a+h)}{h^2} \\
&= \lim_{h\to 0} \frac{f'(a+2h) - 2f'(a+h) + f'(a)}{h^2} \\
&= f^{(3)}(a)
\end{aligned}
$$