Homework 1 of Introduction to Analysis (I), Honor Class

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2023/9/19

1. Without loss of generality, we take a sequence which is increasing. Then, if $\{x_n\}$ is not Cauchy then there exists some $\varepsilon > 0$ s.t. for any $N \in \mathbb{N}$, $n \geq m \geq N \Rightarrow x_n - x_m \geq \varepsilon$. Then, we want to proof "not Cauchy implies not bounded"

   if $\{x_n\}$ is bounded above by y, then choose $N = 1$ and $n_2 - n_1 > \varepsilon$ implies $n_3 - n_2 > \varepsilon$ and $n_4 - n_3 > \varepsilon \Rightarrow n_k - n_1 > (k-1)\varepsilon$. By Archimedean property, $(k-1)\varepsilon > y$ for some k. Thus, y is not upper bound of $\{x_n\}$ contradiction.

   That is, bounded and monotone sequence is Cauchy. For any bounded convergence sequence, it is Cauchy and converges in $\mathbb{R}$. Therefore, for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|x - x_n| < \varepsilon$ for some $n > N$. We want to proof x is least upper bound of $\{x_n\}$.

   Since x_n is increasing, if there exists some $n \in \mathbb{N}$ s.t. $x_n > x$, then there exists some $\varepsilon > 0$ s.t. $|x_n - x| > \varepsilon$ contradiction to the convergence. Thus, $x \geq x_n$ for all n.
   For $x - \frac{1}{k}$, $k \in \mathbb{N}$, exists $0 < \varepsilon < \frac{1}{k} \Rightarrow$ there exists some $n \in \mathbb{N}$ s.t. $x - x_n < \varepsilon < \frac{1}{k}$. That is, $x_n > x - \frac{1}{k}$ for all $k \in \mathbb{N}$.

   Therefore, x is the least upper bound of the sequence. Thus, for any Cauchy sequence converges in $\mathbb{R}$, $\mathbb{R}$ has least upper bound property.

2. (a) If we can find infinite many points labeled by $n_1, n_2, \cdots$ s.t. $\{x_{n_k}\}_{k=1}^{\infty}$ is decreasing, thus $\{x_n\}_{n=1}^{\infty}$ have a decreasing subsequence.
   If not, then we only can find finite points s.t. $\{x_{n_k}\}_{k=1}^{l}$ is decreasing. Thus, we name the last element of the decreasing sequence as N. Thus, we can say that $n'_1 = N + 1$, since n'_1 is not in decreasing sequence, implies that $\exists n'_2 > n'_1 \Rightarrow x_{n'_2} \geq x_{n'_1}$. And doing the same way, we can find $n'_1 < n'_2 < n'_3 < \cdots$ s.t. $\{x_{n'_i}\}_{i=1}^{\infty}$ is a increasing subsequence.
   Then, every sequence in $\mathbb{R}$ either has an increasing subsequence or a decreasing subsequence

   (b) Since every sequence have either increasing or decreasing subsequence and $\{x_n\}$ is bounded, then by monotone convergence theorem, the subsequence converges.

3.

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$
$$x_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

Thus, $x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+2)}$. And $x_{n+k+1} - x_{n+k} = \frac{1}{(2(n+k)+1)(2(n+k)+2)} < \frac{1}{(2n+1)(2n+2)}$.

Thus, for any $\varepsilon > 0$, there exists $n, m \in \mathbb{N}, m > n$ s.t. $|x_n - x_{n+1}| = \frac{1}{(2n+1)(2n+1)} \leq \frac{\varepsilon}{m-n}$. Then, $|x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \leq (m-n)\frac{\varepsilon}{m-n} = \varepsilon$.

Then, $\{x_n\}$ is Cauchy $\Rightarrow$ converges.

4. If $\liminf\limits_{n\to\infty} x_n = \infty$, then assume $\{x_n\}$ is bounded above by y. Then, $\limsup\{x_n\} < y < \liminf x_n = \infty$ contradiction to $\inf S \leq \sup S$ for all set S.

   Thus, $\{x_n\}$ is not bounded above$\Rightarrow$ diverges to ∞.