Exercises(13)

December 5, 2023

1. (10 points) Let

$$f(x) = \begin{cases} 1/q & \text{if } x \text{ is a rational number } \frac{p}{q}, \text{ in lowest terms, with } q > 0 \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Prove that f is integrable on $[0, 1]$. Hint: Verify that there are at most $\frac{n(n+1)}{2}$ points x in $[0, 1]$ for which $f(x) > \frac{1}{n}$.

2. (10 points) Suppose f is bounded on $[a, b]$ and f has only finitely many points of discontinuity on $[a, b]$. Show that $f \in R([a, b])$.

3. (10 points) If $f(x) = (1 + x)^m$, where $m \in \mathbb{Q}$, $|x| < 1$, the usual differentiation formulas from calculus and Taylor's Theorem lead to the expression

$$(1 + x)^m = 1 + \binom{m}{1} x + \binom{m}{2} x^2 + \cdots + \binom{m}{n-1} x^{n-1} + R_n,$$

   where R_n can be given in Lagrange's form by $R_n = x^n f^{(n)}(\theta_n x)/n!$, where $0 < \theta_n < 1$. Show that if $0 \leq x < 1$, then $\lim_{n\to\infty} R_n = 0$. Show that if $-1 < x < 0$, then we cannot use the same argument to show that $\lim_{n\to\infty} R_n = 0$.