Homework 9 of Introduction to Analysis (I), Honor Class

AM15 黃琦翔 111652028

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1. Let $\{a_k^+\}$ be the positive elements in $\{a_k\}$, and $\{a_k^-\}$ be the negative ones.

   By Riemann's Theorem, if $\sum a_k$ is c.c., for any $i \in \mathbb{N}$, we can find smallest $p_i, q_i \in \mathbb{N}$ s.t. $i+1 < \sum_{k=1}^{p_{i+1}} a_k^+ + \sum_{k=1}^{q_i} a_k^-$ and $i+1 < \sum_{k=1}^{p_{i+1}} a_k^+ + \sum_{k=1}^{q_{i+1}} a_k^-$.

   Then, $i < \sum_{k=1}^{p_i} a_k^+ + \sum_{k=1}^{q_i} a_k^-$ for all $i \in \mathbb{N}$. Thus, we can find a rearrangement which partial sum diverges to infinity.

2. Since $\sum \sqrt{a_n a_{n+1}} \le \sum \frac{a_n + a_{n+1}}{2} = \sum a_n - \frac{a_1}{2}$. By comparison test, $\sum \sqrt{a_n a_{n+1}}$ converges.

   Since a_n is monotoneic, a_n are all greater than 0 or all lower than 0. Then, we suppose a_n are greater than 0 and monotone decreasing. $\sum \sqrt{a_n a_{n+1}} \ge \sum a_{n+1} = \sum a_n - a_1$. Then, by comparison test, $\sum a_n$ converges.

3. (a)

$$\begin{aligned}\frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdot + \frac{a_n}{r_n} &> \frac{a_m + a_{m+1} + \cdot a_m}{a_m} \\ &= \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m}\end{aligned}$$

Then, we want to show $\sum \frac{a_n}{r_n}$ is not Cauchy. For any $m \in \mathbb{N}$, we can find a $n \in \mathbb{N}$, $n > m$ s.t. $r_n = \sum_{k=n}^{\infty} a_k > \frac{1}{2} \sum_{k=m}^{\infty} a_k = \frac{1}{2} r_m$ since $\sum a_k$ converges.

Thus, for $N \in \mathbb{N}$ and $m > N$, we can find a $n > m$ s.t. $\sum_{k=m}^{n} \frac{a_k}{r_k} > 1 - \frac{r_n}{r_m} > 1 - \frac{1}{2} = \frac{1}{2}$. Therefore, $\sum \frac{a_k}{r_k}$ is not Cauchy implies it diverges.

(b)

$$
\begin{aligned}
2(\sqrt{r_n} - \sqrt{r_{n+1}}) &= 2\frac{(\sqrt{r_n})(\sqrt{r_n} - \sqrt{r_n + 1})}{\sqrt{r_n}} \\
&= 2\frac{(\sqrt{r_n})^2 - (\sqrt{r_n}\sqrt{r_{n+1}})}{\sqrt{r_n}} \\
&\geq 2\frac{r_n - r_{n+1}}{\sqrt{r_n}} \\
&= 2\frac{a_n}{r_n} \\
&> \frac{a_n}{r_n}
\end{aligned}
$$

We want to proof $\sum_{n=m}^{\infty} \frac{a_n}{\sqrt{r_n}} \to \infty$ as $m \to \infty$. $\sum_{n=m}^{\infty} \frac{a_n}{\sqrt{r_n}} < \sum_{n=m}^{\infty} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) < 2\sqrt{r_m}$. And since $\sum a_n$ converges, $r_n \to 0$ as $n \to \infty$. Thus, by comparison test, $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

4. We want to show $\sum a_n - \lim_{n \to 1^-} \sum a_n x^n = \lim_{n \to i^-} \sum a_n (1 - x^n) = 0$. Since $\sum a_n$ converges, $a_n \to 0$ as $n \to \infty$.

   Then, let $x = 1 - \varepsilon$, then $x \to 1- \implies \varepsilon \to 0$. $\sum a_n (1 - x^n) = \sum a_n \cdot \varepsilon^n < \varepsilon \cdot \sum a_n \to 0$ as $\varepsilon \to 0$. Thus, $\lim_{x \to 1^-} \sum a_n x^n = \sum a_n$.