

# Homework 3 of Introduction to Analysis (I), Honor Class

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1.  $(d_1)$  a. positive definite:  $d(x, y) = (x - y)^2 \geq 0$  for any  $x, y \in \mathbb{R}$  check  
b.  $0 \iff$  equal:  
 $(\implies)$   $d(x, y) = 0 \implies x - y = 0 \implies x = y$   
 $(\impliedby)$   $d(x, y) = (x - y)^2 = 0^2 = 0$  check  
c. triangle inequality:  $4 = (2 - 0)^2 = d(2, 0) > 1 + 1 = d(2, 1) + d(1, 0)$  no  
Thus,  $d_1$  is not a metric.
- $(d_2)$  a. positive definite: since  $|x - y| \geq 0$ , then  $\frac{|x - y|}{1 + |x - y|} \geq 0$  check  
b.  $0 \iff$  equal:  
 $(\implies)$  since  $0 = d(x, y) = \frac{0}{1 + 0} \implies x - y = 0 \implies x = y$   
 $(\impliedby)$  since  $x = y$ ,  $d(x, y) = \frac{0}{1 + 0} = 0$  check  
c. triangle inequality: If  $z$  is not in  $(x, y)$ , the triangle inequality trivially holds.  
Then, WLOG, suppose  $x < z < y$ , then  
$$\frac{z - x}{1 + z - x} + \frac{y - z}{1 + y - z} > \frac{z - x}{1 + (z - x) + (y - z)} + \frac{y - z}{1 + (y - z) + (z - x)}$$
$$= \frac{y - x}{1 + (y - z) + (z - x)} = \frac{y - x}{1 + y - x}$$
  
Thus,  $d_2$  is a metric.
2. We want to prove  $s_{n+1} - s_n < \varepsilon$  for all  $\varepsilon > 0$ , which is equivalent to  $x_n \rightarrow 0$ .
- Since  $\limsup_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} < 1$ , there exists some  $N \in \mathbb{N}$  s.t.  $\frac{x_{n+1}}{x_n} < 1$  for all  $n > N$ .  
Thus,  $\{x_n\}_{n=N}^{\infty}$  is a monotone decreasing sequence bounded below by 0.  
For any  $r < 1$ ,  $\frac{1}{r} > 1$  and there exists some  $k \in \mathbb{N}$  s.t.  $(\frac{1}{r})^k > n$  for all  $n \in \mathbb{N}$ , then  $r^k < \frac{1}{n}$ .  
Since  $(0, 1)$  is open, we can find some  $r$  s.t.  $\frac{x_{n+1}}{x_n} < r < 1$  for all  $n > N$ . Then,  $\lim_{n \rightarrow \infty} x_n < r$   
$$\lim_{n \rightarrow \infty} x_1 \times r^{n-1} = x_1 \times 0 = 0.$$
  
Thus,  $x_n \rightarrow 0$  when  $n \rightarrow \infty$ .

Then, for  $\varepsilon > 0$  exists  $N \in \mathbb{N}$  and  $m > n > N$  with  $x_n < \frac{\varepsilon}{m-n+1}$ ,

$$|S_m - S_n| = S_m - S_n = x_n + x_{n+1} + \cdots + x_{m-1} < (m-n+1) \frac{\varepsilon}{m-n+1} = \varepsilon.$$

Thus,  $s_n$  is Cauchy  $\implies s_n$  is convergence.

3. (a) WLOG, suppose  $\{x_n\}$  is monotone increasing.

Then,  $\sigma_{n+1} - \sigma_n = \left(\frac{n}{n+1}S_n + \frac{x_{n+1}}{n+1}\right) - S_n = \frac{x_{n+1}}{n+1} - \frac{S_n}{n+1} > \frac{x_{n+1} - x_n}{n+1} > 0$  for all  $n$ . Therefore,  $\sigma_n$  is monotone.

(b) we divided this question into  $\{x_n\}$  diverges or bdd.

(diverges) WLOG, suppose  $\limsup x_n = \infty$ , then for any  $M \in \mathbb{N}$ , exists  $N \in \mathbb{N}$  s.t.  $x_n > M$  for  $n \geq N$ .

Thus, there exists  $M_1, N_1 \in \mathbb{N}$  s.t.  $x_n \geq M_1$  for all  $n \geq N_1$  and  $\sigma_{N_1} > \frac{M \times N_1}{N_1} = M$ .

which means for any  $M \in \mathbb{N}$ , there exists some  $N_1 \in \mathbb{N}$  s.t.  $\sigma_n > M$  for all  $n \geq N_1$ .

That means,  $\lim \sigma_n = \lim x_n = \infty$ .

The case of diverging to  $-\infty$  is the same.

(bounded) First, we show it works in a subsequence converging to a cluster point.

Let  $\{x_{n_k}\}$  be a sequence converge to  $a$  and  $\mu_k = \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}$ , then for  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  s.t.  $|x_{n_k} - a| < \varepsilon$  for all  $k > N$ . Thus, there exists some  $\varepsilon_1 < \varepsilon, N_1 \in \mathbb{N}$  s.t.  $|\mu_k - a| \leq \frac{|x_{n_1} - a| + \cdots + |x_{n_k} - a|}{k} < \frac{N \times \mu_N + \varepsilon_1 \times (k - N)}{k} < \varepsilon$ . Which means  $\mu_k$  converge to  $a$ .

And since  $\frac{n_1 a_1 + n_2 a_2 + \cdots + n_k a_k}{n_1 + \cdots + n_k} < \max_{i \in [1, k]} (a_i)$  for all  $n_i, k \in \mathbb{N}$  and  $a_i \in \mathbb{R}$ .

Thus, if there are cluster points of  $\{x_n\}$  named as  $a_1, a_2, \cdots, a_k$ , we can say that  $\limsup \sigma_n \leq \max a_n = \limsup x_n$ .

Then,  $\limsup \sigma_n \leq \limsup x_n$  for all cases.