3.

Lemma

Given $b, t \in \mathbb{R}$, and $b>1, t>1$. Then $\exists\, n \in \mathbb{N}$, s.t. $b^{\frac{1}{n}} < t$.

pf: $\forall\, n \in \mathbb{N}$, $b'>1$

$$b'^{n} - 1 = (b'-1)(b'^{n-1} + b'^{n-2} + \cdots + 1)$$

$$= b'^{n-1}(b'-1) + b'^{n-2}(b'-1) + \cdots + (b'-1)$$

$$(\because b'>1) \geq (b'-1) + (b'-1) + \cdots + (b'-1) = n(b'-1).$$

Since $b>1$, we have $b^{\frac{1}{n}} > 1$ (If $0 \leq b^{\frac{1}{n}} \leq 1$, $b = (b^{\frac{1}{n}})^{n} \leq 1 \rightarrow \times$)

Take $b' = b^{\frac{1}{n}}$. Hence we have $b-1 \geq n(b^{\frac{1}{n}} - 1)$

Now let $m \in \mathbb{N}$, s.t. $m > \frac{b-1}{t-1}$.

Then $b-1 \geq m(b^{\frac{1}{m}} - 1) > \frac{b-1}{t-1}(b^{\frac{1}{m}} - 1)$

$$\Rightarrow 1 > \frac{1}{t-1}(b^{\frac{1}{m}} - 1) \Rightarrow t-1 > b^{\frac{1}{m}} - 1 \Rightarrow t > b^{\frac{1}{m}}$$

pf of the problem.

Fix $b>1$, $y>0$. Let $A = \{ w \in \mathbb{R} \mid b^{w} < y \}$.

① (non-empty)

(i) If $y \geq 1$, then $b^{-1} < 1 \leq y$. $\Rightarrow -1 \in A$.

(ii) If $0<y<1$, then $\frac{1}{y} > 1$, and $b>1$, by Lemma, $\exists\, n \in \mathbb{N}$ s.t.

$$\left(\frac{1}{y}\right)^{\frac{1}{n}} < b \Rightarrow \frac{1}{y} < b^{n} \quad y > b^{-n}. \Rightarrow -n \in A.$$

$\therefore$ If $y>0, b>1$, $A \neq \emptyset$.

② (bdd-above)

(i) If $0<y\leq 1$, then $b^1>1\geq y \Rightarrow 1$ is an upper bound of A.

(ii) If $y>1$, then by Lemma $\exists\, n\in\mathbb{N}$, s.t. $b>y^{\frac{1}{n}} \Rightarrow b^n>y \Rightarrow n$ is an upper bound of A.

$\therefore$ By the least-upper-bound property, $\sup A = x$ exists.

③ ($b^x=y$)

i) Suppose $b^x>y$, then $\frac{b^x}{y}>1 \Rightarrow \exists\, n\in\mathbb{N}$, s.t. $\frac{b^x}{y}>b^{\frac{1}{n}} \Rightarrow b^{x-\frac{1}{n}}>y$.

$\exists\, \bar{x}$, where $x-\frac{1}{n}<\bar{x}<x$ s.t. $b^{\bar{x}}<y<b^{x-\frac{1}{n}}$ $\times$

(ii) Suppose $b^x<y$, then $\frac{y}{b^x}>1 \Rightarrow \exists\, n\in\mathbb{N}$ s.t. $\frac{y}{b^x}>b^{\frac{1}{n}} \Rightarrow y>b^{x+\frac{1}{n}}$.

$x+\frac{1}{n}\in A \Rightarrow \sup A\neq x$ $\times$

$\therefore b^x=y$

④ (Uniqueness)

Suppose $\exists$ distinct $x_1,x_2\in\mathbb{R}$ s.t. $b^{x_1}=y$, $b^{x_2}=y$, and let $x_1 \gneqq x_2$.

Then $0=b^{x_1}-b^{x_2}=b^{x_2}(b^{x_1-x_2}-1)$

$\Rightarrow b^{x_1-x_2}-1=0 \Rightarrow b^{x_1-x_2}=1 \Rightarrow x_1-x_2=0 \Rightarrow x_1=x_2$ $\times$.

$\therefore$ The solution for $b^x=y$ is unique.