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ISyE 6644 — Fall 2017 — Test #1 Solutions

This test is 75 minutes. You're allowed one cheat sheet. Good luck!

1. Toss two dice and observe their sum. What is the expected number of tosses until you observe a sum of 11?

   Solution: $X \sim \text{Geom}(1/18)$, so $\mathsf{E}[X] = 18$. $\Box$

2. Consider a Poisson process with rate $\lambda = 2$. What is the distribution of the time between the 5rd and 6th arrivals?

   Solution: Exp(2). $\Box$

3. YES or NO? If X and Y are independent *exponential* random variables, are they necessarily uncorrelated?

   Solution: Yes. $\Box$

4. TRUE or FALSE? A Poisson process has stationary and independent increments.

   Solution: True. $\Box$

5. Suppose X has p.d.f. $f(x) = 4x^3$, $0 < x < 1$. Find $\mathsf{E}[\frac{2}{X} - 3]$.

   Solution:
$$\mathsf{E}\Big[\frac{1}{X}\Big] \;=\; \int_0^1 \frac{1}{x}\, 4x^3\, dx \;=\; \int_0^1 4x^2\, dx \;=\; 4/3.$$
   This implies that
$$\mathsf{E}\Big[\frac{2}{X} - 3\Big] \;=\; 2\mathsf{E}\Big[\frac{1}{X}\Big] - 3 \;=\; -1/3. \quad \Box$$

6. Suppose that X has p.d.f. $f(x) = 3x^2$, $0 \le x \le 1$. What's the p.d.f. of the random variable e^X?

Solution: Let $Y = e^X$. The c.d.f. of Y is

$$G(y) \;=\; \mathsf{P}(Y \le y) \;=\; \mathsf{P}(e^X \le y) \;=\; \mathsf{P}(X \le \ell\mathrm{n}(y)) \;=\; \int_0^{\ell\mathrm{n}(y)} 3x^2\,dx \;=\; [\ell\mathrm{n}(y)]^3.$$

This implies that the p.d.f. of Y is

$$g(y) \;=\; \frac{d}{dy}G(y) \;=\; \frac{3[\ell\mathrm{n}(y)]^2}{y}, \quad 1 \le y \le e. \quad \Box$$

7. Suppose that X and Y have joint p.d.f. $f(x,y) = 8xy$ for $0 < y < x < 1$.

   (a) Find $\mathsf{E}[\mathrm{X}]$.

   (b) Find $\mathsf{Cov}(X,Y)$.

Solution: We have

$$f_X(x) \;=\; \int_{\mathbb{R}} f(x,y)\,dy \;=\; \int_0^x 8xy\,dy \;=\; 4x^3, \quad 0 < x < 1,$$

so that

$$\mathsf{E}[X] \;=\; \int_{\mathbb{R}} x f_X(x)\,dx \;=\; \int_0^1 4x^4\,dx \;=\; 4/5. \quad \Box$$

Similarly,

$$f_Y(y) \;=\; \int_{\mathbb{R}} f(x,y)\,dx \;=\; \int_y^1 8xy\,dx \;=\; 4(y - y^3), \quad 0 < y < 1,$$

$$\mathsf{E}[Y] \;=\; \int_{\mathbb{R}} y f_Y(y)\,dy \;=\; \int_0^1 4(y^2 - y^4)\,dy \;=\; 8/15,$$

and

$$\mathsf{E}[XY] \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} xyf(x,y)\,dy\,dx \;=\; \int_0^1\int_0^x 8x^2y^2\,dy\,dx \;=\; 4/9.$$

So after all of this, $\mathsf{Cov}(X,Y) = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] = 4/225. \quad \Box$

8. Let X and Y be i.i.d. Exponential with rate 1. Find $\mathsf{P}(X+Y \geq 2)$.

Solution: $X+Y \sim \text{Erlang}_2(1)$. So

$$\mathsf{P}(X+Y \geq 2) \;=\; \sum_{i=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^i}{i!} \;=\; 3e^{-2}. \quad \Box$$

9. Suppose X is a normal random variable with mean 1 and variance 9. What is the probability that $X \geq 4$?

Solution:

$$\mathsf{P}(X \geq 4) \;=\; \mathsf{P}\left(Z \geq \frac{4-1}{\sqrt{9}}\right) \;=\; \mathsf{P}(Z \geq 1) \;=\; 1-\Phi(1) \;=\; 0.1587. \quad \Box$$

10. If X and Y are i.i.d. Nor(0,1) random variables, find $\mathsf{P}(Y < 2X+1)$.

Solution: $Y - 2X \sim \text{Nor}(0,5)$. So $\mathsf{P}(Y < 2X+1) = \Phi(1/\sqrt{5}) \approx 0.672$. $\Box$

11. If X and Y are both Normal(4,10) with $\mathsf{Cov}(X,Y) = 6$, find $\mathsf{Var}(X+Y)$.

Solution:

$$\mathsf{Var}(X+Y) \;=\; \mathsf{Var}(X) + \mathsf{Var}(Y) + 2\mathsf{Cov}(X,Y) \;=\; 32. \quad \Box$$

12. BONUS: What beloved American actress had her birthday on Valentine's Day?

Solution: Florence Henderson. $\Box$

13. Suppose that the Atlanta Hawks play i.i.d. games, each of which has win probability 0.6. Let X be the number of games until the Hawks achieve their first win. Find the smallest x such that $\mathsf{P}(X \leq x) \geq 0.9$.

Solution: The number of games until the first win is $X \sim \text{Geom}(0.6)$, so that $\mathsf{P}(X = x) = q^{x-1}p = (0.4)^{x-1}(0.6)$. It can also be shown that the c.d.f. is

$$F(x) \;=\; \mathsf{P}(X \le x) \;=\; 1 - q^x \;=\; 1 - (0.4)^x$$

(though you don't need to know this if you do a trial-and-error argument to find the smallest x). Now, $F(x) = 1 - (0.4)^x \ge 0.9$ iff $0.1 \ge (0.4)^x$, which is achieved by $x = 3$. $\square$

14. Suppose $X_1, \ldots, X_n$ are i.i.d. from some distribution. What tells us that the sample mean of the X_i's is approximately normal for large enough n?

    Solution: CLT. $\square$

15. Suppose that $X_1, \ldots, X_{100}$ are i.i.d. with values 1 and -1, each with probability 0.5. (This is a simple random walk.) Find the approximate probability that the sum $\sum_{i=1}^{100} X_i$ will be at least 10.

    Solution: Note that $\mathsf{E}[X_i] = 0$ and $\mathsf{Var}(X_i) = \mathsf{E}[X_i^2] - (\mathsf{E}[X_i])^2 = 1$. Then the Central Limit Theorem implies that $\sum_{i=1}^{100} X_i \approx \text{Nor}(0, 100)$, and so

$$\mathsf{P}\Big(\sum_{i=1}^{100} X_i > 10\Big) \;\approx\; \mathsf{P}\left(Z > \frac{10-0}{\sqrt{100}}\right) \;=\; \mathsf{P}(Z > 1) \;=\; 0.1587. \quad \square$$

16. Consider the linear congruential generator $X_{i+1} = (3X_i + 1)\text{mod}(8)$.

    (a) Using $X_0 = 1$, calculate the first pseudo-random number U_1.

    Solution: We immediately have $X_1 = 4$, so that $U_1 = 0.5$. $\square$

    (b) Using $X_0 = 1$, calculate the pseudo-random number U_{801}.

    Solution: If $X_0 = 1$, then we get $X_1 = 4$, $X_2 = 5$, $X_3 = 0$, and $X_4 = 1$, so that the thing repeats every 4 tries. Thus, $X_{801} = X_1 = 4$, so that $U_1 = 0.5$. $\square$

17. If $U \sim \text{Unif}(0, 1)$, what's the distribution of $-3\ell\text{n}(U)$?

    Solution: Exp(1/3). $\Box$

18. If U_1 and U_2 are i.i.d. Unif(0,1), what's the distribution of $-3\ell\text{n}(U_1U_2)$?

    Solution: Note that $-3\ell\text{n}(U_1U_2) = -3\ell\text{n}(U_1) - 3\ell\text{n}(U_2)$. Thus, $\text{Erlang}_2(1/3)$ (or gamma). $\Box$

19. If U_1 and U_2 are i.i.d. Unif(0,1), what's the distribution of $2+\sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2)$?

    Solution: Nor(2,1). $\Box$

20. Suppose that you want to estimate the integral

$$I = \int_0^1 [1 + \cos(\pi x)]dx.$$

The following numbers are a Unif(0,1) sample:

0.419	0.109	0.732	0.893

Use the Monte Carlo method from class to approximate the integral via the estimator $\bar{I}_4$.

Solution:

$$\begin{aligned}
\bar{I}_4 &= \frac{b-a}{n}\sum_{i=1}^{n} g(a + (b-a)U_i) \\
&= \frac{1-0}{4}\sum_{i=1}^{4} g(U_i) \\
&= \frac{1}{4}\sum_{i=1}^{4}\int_0^1 [1 + \cos(\pi U_i)] \\
&= 0.896. \quad \Box
\end{aligned}$$

21. Consider the differential equation $f'(x) = (x-2)f(x)$ with $f(0) = 1$. Use Euler's method with increment $h = 0.01$ to find the approximate value of $f(0.02)$.

Solution. You can actually get the true answer using separation of variables, and it turns out to be $f(x) = \exp\{\frac{x^2}{2} - 2x\}$.

But our job is to use Euler to come up with an iterative approximation, so here goes. As usual, we start with

$$f(x+h) \;=\; f(x) + hf'(x) \;=\; f(x) + h(x-2)f(x) \;=\; f(x)[1 + h(x-2)],$$

from which we obtain the following table.

x	Euler approx	true $f(x)$
0.00	1.0000	1.0000
0.01	0.9800	0.9802
0.02	**0.9605**	0.9610

Thus, the desired Euler approximation for $f(0.2)$ is 0.9605. $\square$

22. Joey works at a chocolate store. Starting at time 0, we have the following 4 customer interarrival times (in minutes):

$$8 \quad 2 \quad 5 \quad 2$$

Customers are served in LIFO fashion. The 4 customers order the following numbers of chocolate products, respectively:

$$6 \quad 2 \quad 3 \quad 1$$

Suppose it takes Joey 3 minutes to prepare each chocolate product. Further suppose that he charges \$2/chocolate. Unfortunately, the customers are unruly and each customer causes \$0.50 in damage for every minute the customer has to wait in line.

(a) When does the first customer leave?

(b) What is the average number of customers in the system during the first 20 minutes?

(c) How much money will Joey make or lose with the above 4 customers?

Solution: Consider the following table.

cust	intrarrl	arrl time	serv start	serve time	depart	wait	sys time
1	8	8	8	18	26	0	18
2	2	10	38	6	44	28	34
3	5	15	29	9	38	14	23
4	2	17	26	3	29	9	12

(a) The first customer leaves at time 26. $\square$

(b) Let X_i denote the amount of time Customer i spends in the system during the time interval [0,20]. In particular, $X_1 = 20 - 8 = 12$, $X_2 = 20 - 10 = 10$, $X_3 = 5$, and $X_4 = 3$. The average number of customers in the system during the first 20 minutes is

$$\frac{\text{total customer time}}{20} = \frac{12 + 10 + 5 + 3}{20} = 1.5. \quad \square$$

(c) Joey makes $2(6 + 2 + 3 + 1) - 0.5(0 + 28 + 14 + 9) = -\$1.50$. $\square$