NAME →

ISyE 6644 — Fall 2015 — Test #1 Solutions

(revised 10/5/16)

You have 85 minutes. You get one cheat sheet. Put your succinct answers below. All questions are 3 points, unless indicated. You get 1 point for writing your name correctly.

1. Let X be the number of tickets that a fan will buy to attend a Panic at the `DISCo` concert (yes, this is the name of an actual band). It turns out that $X = 1$ with probability 0.5, $X = 2$ w.p. 0.3, and $X = 3$ w.p. 0.2. Show how to generate X with one Arena expression.

   Solution: `DISC(0.5,1,0.8,2,1.0,3)`. $\Box$

2. In Arena, what is the value of the expression `((2 == 2) + (2 == 1)) == 1`?

   Solution: $(2 == 2) + (2 == 1) = (1 + 0) == 1 = 1 == 1 = 1$. $\Box$

3. TRUE or FALSE? In Arena, you can use a `DECIDE` block to move a customer to a choice of more than 2 possible destinations.

   Solution: True. $\Box$

4. TRUE or FALSE? In Arena, you can use an `ASSIGN` block to change an entity's picture.

   Solution: True. $\Box$

5. In Arena, there are multiple places (templates) to find a `SEIZE` block. Name at least two.

   Solution: Basic Process template (inside the `Process` block), Advanced Process template, and Blocks template. $\Box$

6. Suppose I toss two dice and I tell you that the sum is at least 10. What's the probability that the sum is exactly 12?

   Solution: Let A be the event that the sum is exactly 12, and B be the event that the sum is at least 10. Then
$$\Pr(A|B) \;=\; \frac{\Pr(A\cap B)}{\Pr(B)} \;=\; \frac{\Pr(12)}{\Pr(10,\,11,\,12)} \;=\; \frac{\frac{1}{36}}{\frac{3}{36}+\frac{2}{36}+\frac{1}{36}} \;=\; 1/6. \quad \Box$$

7. YES or NO? Again suppose I toss two dice. Are the events "sum is less than 3" and "sum is greater than 10" independent?

   Solution: No. In fact, they're highly dependent, since information about one event gives us info about the other. $\Box$

8. Suppose X has p.d.f. $f(x) = 4x^3$, $0 < x < 1$. Find $\Pr(X < 0.5)$.

   Solution: Note that the c.d.f. (which we'll use a little later) is
$$F(x) \;=\; \int_0^x f(t)\,dt \;=\; x^4, \quad 0 < x < 1.$$
   Thus, $\Pr(X < 0.5) = 1/16$. $\Box$

9. Again suppose X has p.d.f. $f(x) = 4x^3$, $0 < x < 1$. Find $\mathsf{E}[3X - 2]$.

   Solution: $\mathsf{E}[X] = \int_0^1 x f(x)\,dx = 4/5$. Thus, $\mathsf{E}[3X-2] = 3\mathsf{E}[X] - 2 = 2/5$. $\Box$

10. Yet again suppose X has p.d.f. $f(x) = 4x^3$, $0 < x < 1$. Write the LOTUS expression for (but do not evaluate) $\mathsf{E}[\ell\text{n}(1+X)]$.

    Solution: $\mathsf{E}[\ell\text{n}(1+X)] = \int_{\mathbb{R}} \ell\text{n}(1+x) f(x)\,dx = 4\int_0^1 \ell\text{n}(1+x)x^3\,dx$. $\Box$

11. Still yet again suppose X has p.d.f. $f(x) = 4x^3$, $0 < x < 1$. Find the p.d.f. of $Y = \ell\text{n}(1+X)$.

Solution: Denote the c.d.f. of Y by $G(y)$. Then

$$G(y) \;=\; \Pr(Y \le y) \;=\; \Pr(\ell\text{n}(1{+}X) \le y) \;=\; \Pr(X \le e^y{-}1) \;=\; F(e^y{-}1) \;=\; (e^y{-}1)^4,$$

where the c.d.f. $F(x)$ was given in Question 8. Then the p.d.f. of Y is

$$g(y) \;=\; \frac{d}{dy}G(y) \;=\; 4e^y(e^y-1)^3, \quad 0 < y < \ell\text{n}(2),$$

where the limits are determined after plugging $0 < x < 1$ into $y = \ell\text{n}(1+x)$. $\square$

12. Yet still yet again suppose X has p.d.f. $f(x) = 4x^3$, $0 < x < 1$. Find the p.d.f. of X^4.

    Solution: Note that, from Question 8, the c.d.f. of X is $F(x) = x^4$. Thus, by the Inverse Transform Theorem, we have that $X^4 \sim \text{Unif}(0,1)$. So the p.d.f. of $Z = X^4$ is $h(z) = 1$, $0 < z < 1$. $\square$

13. If X_1 and X_2 are i.i.d. Unif$(-2,2)$, what's the distribution of $X_1 + X_2$?

    Solution: In the following manipulations, we'll assume that all random variables are independent. Then

$$\begin{aligned} X_1 + X_2 &\sim [-2 + \text{Unif(0,4)}] + [-2 + \text{Unif(0,4)}] \\ &\sim -4 + 4\text{Unif(0,1)} + 4\text{Unif(0,1)} \\ &\sim -4 + 4\text{Tria(0,1,2)} \\ &\sim -4 + \text{Tria(0,4,8)} \\ &\sim \text{Tria}(-4,0,4). \quad \square \end{aligned}$$

14. I'm at DragonCon, but I've lost my n-sided die! Show me how to generate a realization of such a toss using only one Unif(0,1) random number.

    Solution: $\lceil nU \rceil$. $\square$

15. Joey's Lightbulb Company sells bulbs that have Exp(2/year) lifetimes. Suppose one of his lightbulbs has already lasted two years. What's the probability that it'll survive another year?

Solution: Let X denote the lifetime of a bulb. By the memoryless property,

$$\Pr(X > 3 \,|\, X > 2) \;=\; \Pr(X > 1) \;=\; e^{-\lambda x} \;=\; e^{-2(1)} \;=\; 0.1353. \quad \Box$$

16. How would Joey generate one of his exponential lifetimes in Arena? Be precise!

    Solution: `EXPO(0.5)`. $\Box$

17. In Arena, `LN(`x`)` denotes $\ell\text{n}(x)$ and `UNIF(`a,b`)` generates a Unif(a,b) r.v. Using these facts, give Joey a second way to generate his exponential lifetimes in Arena.

    Solution: `-0.5 * LN(UNIF(0,1))`. (Other solutions possible.) $\Box$

18. Suppose we replace Joey's lightbulbs each time they fail. What is the probability that we'll have exactly 4 lightbulbs fail by the end of Year 1?

    Solution: Let Y denote the number of bulbs that fail in one year. Then $Y \sim \text{Pois}(2)$, so that $\Pr(Y = 4) = e^{-2}2^4/4! = 0.0902$. $\Box$

19. Tommy's Lightbulb Company sells bulbs that have Exp(1/year) lifetimes. Now I have a choice — I can buy *one* of Tommy's bulbs or *two* of Joey's. Which selection will have the higher probability of yielding a total lifetime of at least a year? One of Tommy's or two of Joey's?

    Solution: Let $T \sim \text{Exp}(\lambda = 1)$ denote the life of Tommy's one bulb, and let $J \sim \text{Erlang}_{k=2}(\lambda = 2)$ denote the total life of Joey's two bulbs. Then

    $$\Pr(T > 1) \;=\; e^{-\lambda x} \;=\; e^{-1} \;=\; 0.3678,$$

    while

    $$\Pr(J > 1) \;=\; \sum_{i=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^i}{i!} \;=\; e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!}\right] \;=\; 0.4060.$$

    So Joey wins! Go Joey! $\Box$

20. I've just bought 100 lightbulbs from Tommy. What's the (approximate) probability that the average bulb will last less than a year?

Solution: Let W denote the average lifetime of the $n = 100$ bulbs. By the Central Limit Theorem,

$$W \approx \text{Nor}\left(\mu, \frac{\sigma^2}{n}\right) \sim \text{Nor}\left(\frac{1}{\lambda}, \frac{1}{n\lambda^2}\right) \sim \text{Nor}(1, 0.01),$$

where $\mu = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$ are the mean and variance of the exponential bulbs, and $\lambda = 1$. In any case, by symmetry of the normal distribution, $\Pr(W < 1) \approx 0.5$. $\Box$.

21. Suppose U_1 and U_2 are i.i.d. Unif(0,1), and let $X = \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2)$. Find $\Pr(X \le 1)$?

Solution: By Box–Muller, $X \sim \text{Nor}(0,1)$. So by tables, $\Pr(X \le 1) = 0.8413$. $\Box$

22. If $\Phi(x)$ is the Nor(0,1) c.d.f. and $X \sim \text{Nor}(0,1)$, what is $\Pr(\Phi(X) \le 1)$?

Solution: By Inverse Transform, $\Phi(X) \sim \text{Unif}(0,1)$. So $\Pr(\Phi(X) \le 1) = 1$. $\Box$

23. Suppose U_1 and U_2 are i.i.d. Unif(0,1). What's the distribution of $-3\ell\text{n}[U_1(1-U_2)]$?

Solution:

$$-3\ell\text{n}[U_1(1-U_2)] = -3\ell\text{n}(U_1) - 3\ell\text{n}(1-U_2) \sim \text{Exp}(1/3) + \text{Exp}(1/3) \sim \text{Erlang}_2(1/3). \quad \Box$$

24. If X, Y, and Z are i.i.d. Exp(1), find $\Pr(\min(X,Y,Z) < 1)$.

Solution: Let $W = \min(X,Y,Z)$. Then the c.d.f. of W is

$$\begin{aligned} \Pr(W \le w) &= 1 - \Pr(W > w) \\ &= 1 - \Pr(X > w \text{ and } Y > w \text{ and } Z > w) \\ &= 1 - [\Pr(X > w)]^3 \quad (\text{since } X, Y, Z \text{ are i.i.d.}) \\ &= 1 - [e^{-w}]^3 \quad (\text{since } X \sim \text{Exp}(1)) \\ &= 1 - e^{-3w}. \end{aligned}$$

Thus, $\Pr(W < 1) = 1 - e^{-3} = 0.9502$.

This answer actually makes a lot of sense, since it is well known that if X, Y, Z are i.i.d. Exp(1), then $\min(X, Y, Z) \sim \text{Exp}(3)$. $\Box$

25. TRUE or FALSE? $\mathsf{Cov}(X, Y) = 0$ implies X and Y are independent.

Solution: False. $\Box$

26. TRUE or FALSE? Covariance is always between -1 and 1.

Solution: False. (The correlation is always between -1 and 1, but not necessarily the covariance.) $\Box$

27. Suppose we pick 1000 random points in a unit $(1 \times 1 \times 1)$ cube and that 543 of those points land inside the sphere with radius 1/2 that is inscribed in the cube. Use this result to estimate π.

Solution: From Homework 2, we use the estimator $\hat{\pi}_n = 6\hat{p}_n = \frac{6(543)}{1000} = 3.258$. $\Box$

28. (6 points) Consider the integral $I = \int_1^2 \ell\text{n}(x)\, dx$.

    (a) Evaluate I via Monte Carlo integration. Use the following $n = 5$ Unif$(0, 1)$ random numbers to come up with the usual estimator $\hat{I}_n$ for I:

    0.85 0.53 0.98 0.12 0.60

    (b) What is the exact value of I? Hint: Maybe try integration by parts.

    (c) What's the expected value of $\hat{I}_n$?

Solution:

(a) We have

$$\begin{aligned}\hat{I}_n &= \frac{b-a}{n}\sum_{i=1}^{n} f(a+(b-a)U_i) \\ &= \frac{1}{5}\sum_{i=1}^{5} f(1+U_i) \\ &= \frac{1}{5}\sum_{i=1}^{5} \ell\text{n}(1+U_i) \\ &= \frac{1}{5}\,\ell\text{n}\left[\prod_{i=1}^{5}(1+U_i)\right] \\ &= 0.461.\end{aligned}$$

(b) Parts yields $I = x\ell\text{n}(x)|_1^2 - \int_1^2 dx = 2\ell\text{n}(2) - 1 = 0.386.$ $\square$

(c) Since $\hat{I}_n$ is unbiased, $\mathsf{E}[\hat{I}_n] = I = 0.386.$ $\square$

29. (6 points) Suppose that exactly 5 customers arrive at the First National Bank during its first hour of operation. The interarrival times (in minutes) are as follows.

Bill	Xavier	Yolanda	Andy	Arnold
0	3.1	4.4	3.2	21.8

Each customer takes exactly 10 minutes to get served. Customers are served *alphabetically.*

(a) How many customers do not have to wait before they are served?

(b) There is a penalty to pay for customers having to wait. Namely, if a customer has to wait t units, we lose \$$(2+t^2)$. Find the total penalty we incur for customer waits.

(c) What's the average number of customers in the system over the first 30 minutes?

Solution:

cust	arrl time	start serv	serv time	depart	wait
Bill	0.0	0.0	10	10.0	0
Xavier	3.1	10.0	10	20.0	6.9
Yolanda	7.5	30.0	10	40.0	22.5
Andy	10.7	20.0	10	30.0	9.3
Arnold	32.5	40.0	10	50.0	7.5

(a) 1 (customer 1). $\square$

(b) $0 + (2 + 6.9^2) + (2 + 22.5^2) + (2 + 9.3^2) + (2 + 7.5^2) = \$704.60.$ $\square$

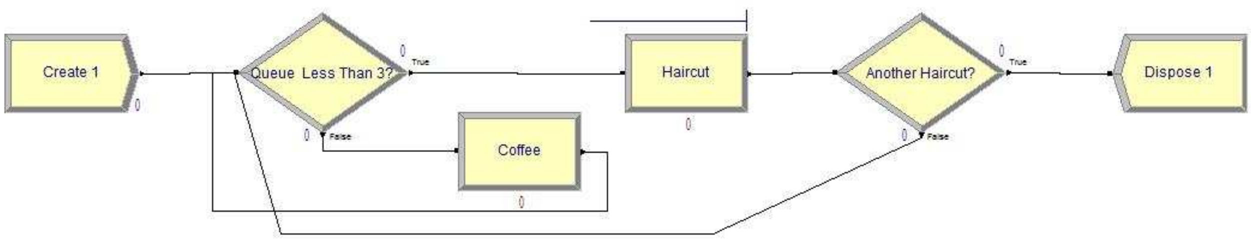
(c) Let's list out the total time during [0,30] that various customers are in the system:

cust	interval	length
Bill	$[0.0, 10.0]$	10.0
Xavier	$[3.1, 20.0]$	16.9
Yolanda	$[7.5, 30.0]$	22.5
Andy	$[10.7, 30.0]$	19.3

So the total customer time in the system during the first 30 minutes is 58.7, so that $\bar{L} = 68.7/30 = 2.29$. $\square$

30. (6 points) Customers come into a barber shop according to a Poisson process. If the line in front of the barber is ≥ 3 people, the customer will instead go get a coffee around the corner and come back in about 30 minutes to try again. Otherwise, he gets a haircut. 10% of all customers are dissatisfied with their haircut and go back for another (perhaps even multiple times). For this model, draw a rough Arena flowchart with a high-level explanation of what's going on.

Solution: The figure below should be self-explanatory. The only interesting item is the n-way by condition `DECIDE` block — if the expression `NQ(Haircut.Queue) < 3` is true, get a haircut.



Create 1
Queue Less Than 3?
True
False
Coffee
Haircut
Another Haircut?
True
False
Dispose 1