NAME →

ISyE 6644 — Fall 2016 — Test #1 Solutions

This test is 85 minutes. You're allowed one cheat sheet. Good luck!

1. Suppose X has p.d.f. $f(x) = 3x^2$, $0 < x < 1$. Find $\mathsf{E}[3X + 2]$.

   Solution: $\mathsf{E}[X] = \int_0^1 x\, 3x^2\, dx = 3/4$. Thus, $\mathsf{E}[3X + 2] = 17/4$. $\Box$

2. Suppose X has p.d.f. $f(x) = 3x^2$, $0 < x < 1$. Find $\mathsf{E}[\frac{1}{X}]$.

   Solution: $\mathsf{E}[\frac{1}{X}] = \int_0^1 \frac{1}{x}\, 3x^2\, dx = 1.5$. $\Box$

3. TRUE or FALSE? For any positive random variable X with mean μ, we have $\mathsf{Var}(\mu^2 - X) \geq 0$.

   Solution: TRUE. $\mathsf{Var}(\mu^2 - X) = \mathsf{Var}(X) \geq 0$. $\Box$

4. The number of arrivals to a parking lot over time can best be represented by

   (a) a Brownian motion process.
   (b) an independent, identically distributed process
   (c) a Poisson process
   (d) a nonhomogeneous Poisson process
   (e) a Yellowian motion process

   Solution (d) a nonhomogeneous Poisson process. $\Box$

5. Suppose that the Atlanta Braves play i.i.d. games, each of which has win probability 0.4. Find the probability that their first win will occur on game 4.

   Solution: The number of games until the first win is $X \sim \text{Geom}(0.4)$, so that $\Pr(X = 4) = q^{x-1}p = (0.6)^3(0.4) = 0.0864$. $\Box$

6. Suppose that machine breakdowns occur according to a Poisson process at the rate of 2/week. What's the probability that there will be exactly 5 breakdowns during the next 3 weeks?

   Solution: Let X denote the number of breaks during the next 3 weeks. Then $X \sim \text{Pois}(6)$, so that

   $$\Pr(X = 5) \;=\; \frac{e^{-6}6^5}{5!} \;=\; 0.1606. \quad \Box$$

7. Again suppose that machine breakdowns occur according to a Poisson process at the rate of 2/week. What is the distribution of the time between the 3rd and 6th breakdowns? (Name the distribution with any relevant parameters.)

   Solution: Add up 3 Exp(2)'s, so that the answer is $\text{Erlang}_3(2)$. $\Box$

8. Suppose that X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. What's the distribution of the random variable $\sqrt{X}$?

   Solution: Let $Y = \sqrt{X}$. We have

   $$\Pr(Y \le y) \;=\; \Pr(\sqrt{X} \le y) \;=\; \Pr(X \le y^2) \;=\; \int_0^{y^2} 2x\,dx \;=\; y^4.$$

   This implies that $f_Y(y) = 4y^3$, $0 \le y \le 1$. $\Box$

9. Suppose that U_1 and U_2 are i.i.d. Unif(0,1). Name the distribution of $-4\,\ell\text{n}(U_1U_2)$. (Include any relevant parameters.)

   Solution: We have

   $$-4\,\ell\text{n}(U_1U_2) \;=\; -4\,\ell\text{n}(U_1) - 4\,\ell\text{n}(U_2) \;\sim\; \text{Exp}(1/4) + \text{Exp}(1/4) \;\sim\; \text{Erlang}_2(1/4). \quad \Box$$

10. Suppose X is a normal random variable with mean 1 and variance 4. What is the probability that $X \le 3$?

Solution:

$$\Pr(X \le 3) \;=\; \Pr\left(Z \le \frac{3-1}{\sqrt{4}}\right) \;=\; \Pr(Z \le 1) \;=\; \Phi(1) \;=\; 0.8413. \quad \Box$$

11. Suppose the joint p.d.f. of X and Y is $f(x,y) = 2x/9$, $0 < x < y < 3$. Find $\mathsf{E}[Y]$.

Solution: We have

$$f_Y(y) \;=\; \int_{-\infty}^{\infty} f(x,y)\,dx \;=\; \int_0^y \frac{2x}{9}\,dx \;=\; \frac{y^2}{9},\ 0 \le y \le 3,$$

so that

$$\mathsf{E}[Y] \;=\; \int_{-\infty}^{\infty} y f_Y(y)\,dy \;=\; \int_0^3 \frac{y^3}{9}\,dy \;=\; 9/4. \quad \Box$$

12. Suppose X and Y are discrete random variables with the following joint p.m.f.

$f(x,y)$	$X=0$	$X=1$
$Y=0$	0.5	0.1
$Y=1$	0.1	0.3

Find $\mathsf{Cov}(X,Y)$.

Solution: Let's expand the table in the usual way.

$f(x,y)$	$X=0$	$X=1$	$f_Y(y)$
$Y=0$	0.5	0.1	0.6
$Y=1$	0.1	0.3	0.4
$f_X(x)$	0.6	0.4	1

Then we immediately see that both X and Y are Bern(0.4). But let's see if they're correlated. To this end,

$$\mathsf{E}[XY] \;=\; \sum_x \sum_y xyf(x,y) = (0)(0)0.5 + (1)(0)0.1 + (1)(0)0.1 + (1)(1)0.3 \;=\; 0.3.$$

Thus,

$$\mathsf{Cov}(X,Y) \;=\; \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y] \;=\; 0.3 - (0.4)(0.4) \;=\; 0.14. \quad \Box$$

13. Consider the random variables X and Y, both of which are Nor$(0,3)$, but with $\mathsf{Cov}(X,Y) = 1$. Find $\mathsf{Var}(X-Y)$.

Solution:

$$\mathsf{Var}(X) + \mathsf{Var}(Y) - 2\mathsf{Cov}(X,Y) = 4. \quad \Box$$

14. Suppose that the joint p.d.f. of X and Y is $f(x,y) = c\,e^{-(x+y)}/(x^2+y^2)$, for $2 < x < 3$, $1 < y < 2$, and some appropriate value of c. Are X and Y independent?

Solution: No. You can't factor the joint p.d.f. in the form $a(x)b(y)$. $\Box$

15. Suppose that the times between buses are i.i.d. Exp(4/hr). Further suppose that I showed up at the bus stop at some random time and have already been waiting for an hour. What's the probability that the next bus will come within the next 15 minutes?

Solution: Let $X \sim$ Exp(4) be the time that the next bus arrives. We want the probability that the next bus will show up by time 75 minutes (i.e., 5/4 hours), given that we've already been waiting an hour. By the memoryless property,

$$\Pr(X \le 5/4 \,|\, X \ge 1) \;=\; \Pr(X \le 1/4) \;=\; 1 - e^{-\lambda x} \;=\; 1 - e^{-1} \;=\; 0.632. \quad \Box$$

16. What distribution do you get if you add up three i.i.d. Nor(3,6) random variables?

Solution: Nor$(9,18)$. $\Box$

17. Suppose that $X_1, \ldots, X_{400}$ are i.i.d. with values 1 and -1, each with probability 0.5. (This is a simple random walk.) Find the approximate probability that the sample mean $\bar{X}_{400}$ will be between -0.1 and 0.1.

Solution: Note that $\mathsf{E}[X_i] = 0$ and $\mathsf{Var}(X_i) = \mathsf{E}[X_i^2] - (\mathsf{E}[X_i])^2 = 1$. Then the Central Limit Theorem implies that the sample mean $\bar{X} \approx$ Nor$(0, 1/400)$, and so

$$\begin{aligned}\Pr(-0.1 < \bar{X} < 0.1) &\approx \Pr\left(\frac{-0.1-0}{\sqrt{1/400}} < Z < \frac{0.1-0}{\sqrt{1/400}}\right) \\ &= \Pr(-2 < Z < 2) = 0.954. \quad \Box\end{aligned}$$

18. Joey likes to play Dungeons and Dragons. If $U \sim \text{Unif}(0,1)$, give Joey a simple algorithm to generate a 10-sided die toss. (If your algorithm isn't simple enough, a Justin Bieber monster will kill him — so Joey is really counting on you to get this question correct.)

    Solution: If $U \sim \text{Unif}(0,1)$, then you can use $\lceil 10U \rceil$. $\Box$

19. How would you implement Joey's 10-sided die toss from Question 18 in Arena?

    Solution: Many possible answers, but a really simple one is AINT(UNIF(1,11)). $\Box$

20. TRUE or FALSE? If you add up two i.i.d. Unif(0,1) random variables, you get a triangular distribution.

    Solution: TRUE. $\Box$

21. Suppose that I pick 40000 points randomly in a unit cube, into which I've inscribed a sphere with radius 1/2. (This sphere has volume $\frac{4}{3}\pi r^3 = \pi/6$.) Further suppose that 21067 of those random points fall inside the inscribed sphere. Using this information, give me an estimate of π.

    Solution: Let $p = \pi/6$ denote the probability that a random point will be in the sphere. Then an estimator for p is the sample proportion $\hat{p}_n = 0.52668$, and so the desired estimate is $\hat{\pi}_n = 6\hat{p}_n = 3.1601$. $\Box$

22. If you were to write a simulation of a queueing system in Python, would you likely be using the (i) event-scheduling or (ii) process-interaction approach?

    Solution: (i) E-S. $\Box$

23. TRUE or FALSE? It is possible for an arrival event to cause multiple event deletions in the future events list.

Solution: TRUE. $\square$

24. What is the variance of the Arena expression NORM(-1,3) - EXPO(5)?

    Solution: $9 + 25 = 34$. $\square$

25. In Arena, what does NR(Barber) do?

    Solution: It indicates the number of servers in the resource Barber who are currently working. $\square$

26. In what Arena template can you find a DELAY?

    (a) Basic Process
    (b) Advanced Process
    (c) Blocks
    (d) all of the above

    Solution: (d) all of the above. $\square$

27. TRUE or FALSE? You can set the value of an Arena entity attribute to the value of the variable TNOW.

    Solution: TRUE. $\square$

28. TRUE or FALSE? Arena's Schedule spreadsheet can define both resource *and* customer arrival schedules.

    Solution: TRUE. $\square$

29. Suppose that you want to evaluate the integral

$$I = \int_0^2 e^{-x^2/2}\,dx$$

via Monte Carlo. The following numbers are a Unif(0,1) sample:

0.91	0.11	0.73	0.32

Use the MC method from class to approximate the integral via the estimator $\bar{I}_4$.

Solution:

$$\begin{aligned} \bar{I}_4 &= \frac{b-a}{n}\sum_{i=1}^{n} f(a+(b-a)U_i) \\ &= \frac{2}{4}\sum_{i=1}^{4} f(2U_i) \\ &= \frac{1}{2}\sum_{i=1}^{4} \exp\left[\frac{-(2U_i)^2}{2}\right] \\ &= \frac{1}{2}\sum_{i=1}^{4} e^{-2U_i^2} \\ &= 1.163. \quad \Box \end{aligned}$$

30. What is the expected value of the estimator $\bar{I}_4$?

Solution: Let $\Phi(\cdot)$ denote the standard normal c.d.f. Since $\bar{I}_4$ is unbiased for I, we have

$$\mathsf{E}[\bar{I}_4] = I = \sqrt{2\pi}\,[\Phi(2)-\Phi(0)] = \sqrt{2\pi}\,(0.4772) = 1.196. \quad \Box$$

31. Consider the differential equation $f'(x) = (x+2)f(x)$ with $f(0) = 2$. Use Euler's method with increment $h = 0.01$ to find the approximate value of $f(0.02)$.

Solution. You can actually get the true answer using separation of variables, and it turns out to be $f(x) = 2\exp\{\frac{x^2}{2} + 2x\}$.

But our job is to use Euler to come up with an iterative approximation, so here goes. As usual, we start with

$$f(x+h) = f(x) + hf'(x) = f(x) + h(x+2)f(x) = f(x)[1+h(x+2)],$$

from which we obtain the following table.

x	Euler approx	true $f(x)$
0.00	2.0000	2.0000
0.01	2.0400	2.0406
0.02	**2.0810**	2.0820

Thus, the desired Euler approximation for $f(0.2)$ is 2.0810. $\Box$

32. Joey works at an ice cream shop. Starting at time 0, four customer interarrival times are as follows (in minutes):

$$3 \qquad 2^{\mathbf{P}} \qquad 5 \qquad 2^{\mathbf{P}},$$

where a "**P**" indicates a *high-priority* customer. Customers are served in FIFO fashion within the regular and high-priority classes. However, high-priority customers *preempt* any non-high-priority customer currently in service; that is, when a high-priority guy shows up,

- If a regular guy is currently getting served, he is temporarily kicked out of service and has to go behind the high-priority guy.
- The high-priority guy gets served immediately.
- When the high-priority guy is done, then the regular guy gets served again, picking up where he left off.

In any case, the 4 customers order the following numbers of ice cream products, respectively:

$$6 \qquad 2 \qquad 3 \qquad 1$$

Suppose it takes Joey 2 minutes to prepare each ice cream product. Further suppose that he charges \$5/ice cream. Sadly, the customers are unruly and each customer causes \$1 in damage for every minute the customer has to wait in line.

When does customer #1 leave?

Solution: Consider the following beautiful table.

cust	arrl time	serv start / restart	remaining serve time	kicked out / departed	wait time	sys time
1	3	3	12	5	0	–
1	–	9	10	12	4	–
1	–	14	7	21	2	18
$2^{\mathbf{P}}$	5	5	4	9	0	4
3	10	21	6	27	11	17
$4^{\mathbf{P}}$	12	12	2	14	0	2

Thus, cust #1 leaves at time 21. $\square$

33. Continuing the ice cream question, how much money will Joey make or lose with the above 4 customers?

    Solution: Profit = 5(number of ice creams sold) − 1(total wait times) = $60 - 17 =$ \$43. $\square$