ISyE 6644 — Spring 2018 — Practice Test #2

Hi Again, Gentle Class!

Our Test 2 will cover material up to and including some material on multivariate normal and nonhomogeneous Poisson random variate generation (but nothing after that) — i.e., towards the end of Module 7 (or, equivalently, the first part of HW 10). You'll be responsible for all of the material from Test 1, but I'll emphasize mostly the new stuff since Test 1.

Practice Test 2 below is actually an amalgamation of Test 2 and part of Test 3 originally given in my "live" 6644 class during the Fall 2017 semester. Your Test 2 won't be as long, and I'll be more generous with time and cheat sheets. ☺

Also, all of our test questions will be multiple choice, in spite of what you see below.

Dave

1. Consider a Poisson process with rate $\lambda = 2$. Find the probability that the time between the 3rd and 5th arrivals is at least 1.

   Solution: The times between Poisson arrivals are i.i.d. Exp(2). Thus, the sum of the 2 interarrival times between the 3rd and 5th arrivals is $X \sim \text{Erlang}_2(2)$. We want

   $$\mathsf{P}(X \geq 1) = \sum_{i=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} = \sum_{i=0}^{1} \frac{e^{-2(1)}(2(1))^i}{i!} = 3e^{-2} = 0.4060. \quad \square$$

2. Consider the following joint p.m.f.

$f(x,y)$	$X = 1$	$X = 2$
$Y = 0$	0.2	0.5
$Y = 1$	0.0	0.3

   Find $\mathsf{E}[XY]$.

Solution:

$$
\begin{aligned}
\mathsf{E}[XY] &= \sum_x \sum_y xy f(x,y) \\
&= (1)(0)(0.2) + (1)(1)(0.0) + (2)(0)(0.5) + (2)(1)(0.3) \\
&= 0.6. \quad \square
\end{aligned}
$$

3. TRUE or FALSE? (Almost) every simulation language utilizes a future events list to process events in order by time.

   Solution: TRUE. $\square$

4. TRUE or FALSE? In Arena, variables are global.

   Solution: TRUE. $\square$

5. TRUE or FALSE? In Arena, you can use the `SEIZE - DELAY - RELEASE` module series instead of a single `PROCESS` module.

   Solution: TRUE. $\square$

6. TRUE or FALSE? In Arena, you can find a `SEIZE` (i) inside a `PROCESS` module, (ii) in the Advanced Process template, and (iii) in the Primitive Blocks template.

   Solution: TRUE. $\square$

7. In Arena, what kind of module would you use to give a new value to an attribute?

   (a) `RECORD`

   (b) `SET`

   (c) `ASSIGN`

   (d) `TALLY`

Solution: (c) `ASSIGN`. □

8. TRUE or FALSE? In Arena, certain resources can be in more than one resource set.

Solution: TRUE. □

9. TRUE or FALSE? An Arena `DECIDE` module can route customers probabilistically or conditionally to multiple locations.

Solution: TRUE. □

10. In Arena, where would you find the `Expression` spreadsheet?

    (a) `Basic Process` panel
    (b) `Advanced Process` panel
    (c) `Basic Transfer` panel
    (d) `Advanced Transfer` panel

    Solution: (b) `Advanced Process`. □

11. TRUE or FALSE? In a particular Arena `PROCESS` module, it is possible to initiate a `SEIZE-DELAY` pair *without* a `RELEASE`.

    Solution: TRUE, though you'd have to do a `RELEASE` in a later module. □

12. What is the name of the Arena module that can be used to split one customer into two or more clones?

    Solution: `Separate` or `Clone`. □

13. TRUE or FALSE? In Arena, you can assign different customers completely different sequences of visitation stations.

Solution: TRUE. □

14. What is the variance of the random variable generated by the Arena expression `2*DISC(0.5,-1,1,1) + (1==1)`?

Solution: Let $X = \text{DISC}(0.5, -1, 1, 1)$. Then $\mathsf{P}(X = -1) = \mathsf{P}(X = 1) = 0.5$, so that $\mathsf{E}[X] = 0$ and $\mathsf{E}[X^2] = 1$. Thus, $\mathsf{Var}(X) = 1$, and we have

$$\mathsf{Var}(2 * \text{DISC}(0.5, -1, 1, 1) + (1 == 1)) = \mathsf{Var}(2X + 1) = 4\mathsf{Var}(X) = 4. \quad \square$$

15. YES or NO? Is it *ever* OK to use the midsquare random number generator?

Solution: NO. □

16. What does "LCG" mean?

Solution: Linear Congruential Generator. □

17. What's the period of the generator $X_i = (3X_{i-1} + 2) \bmod (5)$ if we use the seed $X_0 = 3$?

Solution: Note that $X_0 = 3$, $X_1 = 1$, $X_2 = 0$, $X_3 = 2$, $X_4 = 3$, so the period is 4. □

(Interestingly, $X_0 = 4$ would have caused an immediate degenerate cycle.)

18. TRUE or FALSE? The generator $X_i = (32X_{i-1} + 124)\bmod(2048)$ has full period.

Solution: FALSE. (You get all evens.) □

19. Which uniform generator was recommended in class, at least as a "desert island" generator?

   (a) $X_i = 16807X_{i-1} \bmod (2^{31})$

(b) $X_i = 16807 X_{i-1} \bmod (2^{31} - 1)$

(c) $X_i = 16807 (X_{i-1} - 1) \bmod (2^{31})$

(d) $X_i = 16807 (X_{i-1} - 1) \bmod (2^{31} - 1)$

Solution: (b). □

20. Suppose that a Tausworthe generator gave you the series of bits 1010101. If you use all 7 bits, what Unif(0,1) random number would that translate to?

Solution: Using the usual base-2 notation, we have

$$\frac{1010101_2}{2^7} = \frac{64 + 16 + 4 + 1}{128} = \frac{85}{128} = 0.6641. \quad □$$

21. Suppose we observe 1000 numbers to obtain the following data.

interval	$[0.00, 0.25)$	$[0.25, 0.50)$	$[0.50, 0.75)$	$[0.75, 1.0]$
number observed	240	255	243	262

Conduct a χ^2 goodness-of-fit test to see if these numbers are approximately Unif(0,1). Use level of significance $\alpha = 0.05$. Here are some table entries that you may need: $\chi^2_{0.05,3} = 7.81$, $\chi^2_{0.05,4} = 9.49$, and $\chi^2_{0.05,5} = 11.1$. ACCEPT or REJECT?

Solution: The expected number of observations per equal-probability cell is $E_i = n/k = 1000/4 = 250$. This gives us the following augmented table.

interval i	$[0.00, 0.25)$	$[0.25, 0.50)$	$[0.50, 0.75)$	$[0.75, 1.0]$
number observed O_i	240	255	243	262
expected number E_i	250	250	250	250

Thus, the χ^2 goodness-of-fit statistic is

$$\chi^2_0 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \frac{100 + 25 + 49 + 144}{250} = 1.272.$$

Meanwhile, the appropriate quantile $\chi^2_{\alpha, k-1} = \chi^2_{0.05,3} = 7.81$. Since $\chi^2_0 < \chi^2_{\alpha, k-1}$, we ACCEPT H_0. In other words, we are willing to assume that the numbers are approximately Unif(0,1). □

22. Consider the following $n = 30$ PRNs. (Read from left to right, and then down.)

$$0.79 \quad 0.68 \quad 0.46 \quad 0.69 \quad 0.90 \quad 0.93 \quad 0.99 \quad 0.86 \quad 0.33 \quad 0.22$$
$$0.60 \quad 0.18 \quad 0.59 \quad 0.38 \quad 0.69 \quad 0.76 \quad 0.91 \quad 0.62 \quad 0.22 \quad 0.19$$
$$0.11 \quad 0.45 \quad 0.72 \quad 0.88 \quad 0.65 \quad 0.55 \quad 0.31 \quad 0.27 \quad 0.46 \quad 0.89$$

Let's conduct a runs *up and down* test to test H_0 : the U_i's are independent with level $\alpha = 0.05$. ACCEPT or REJECT?

Solution: Let's associate $+$ and $-$ for up and down, respectively. Then the 30 PRNs translate to

$$-\; -\quad +\,+\,+\,+\quad -\,-\,-\quad +\; -\quad +\; -\quad +\,+\,+\quad -\,-\,-\,-\quad +\,+\,+\quad -\,-\,-\,-\quad +\,+$$

so that we have $A = 12$ runs up and down.

Recall that

$$A \sim \text{Nor}\left(\frac{2n-1}{3}, \frac{16n-29}{90}\right) \sim \text{Nor}\,(19.67,\, 5.01)\,.$$

so that the test stat is

$$Z_0 \;=\; \frac{A - \mathsf{E}[A]}{\sqrt{\mathsf{Var}(A)}} \;=\; \frac{12 - 19.67}{\sqrt{5.01}} \;=\; -3.43.$$

Meanwhile, the appropriate quantile is $z_{\alpha/2} = 1.96$.

Since $Z_0 > z_{\alpha/2}$, we REJECT H_0. In other words, these fellas probably ain't indep. $\square$

23. Suppose that we have two PRNs, $U_1 = 0.7$ and $U_2 = 0.4$. Use these to generate a realization from an $\text{Erlang}_{k=2}(\lambda = 3)$ distribution.

Solution:

$$-\frac{1}{\lambda}\ell\text{n}\left(\prod_{i=1}^{k} U_i\right) \;=\; -\frac{1}{3}\ell\text{n}(0.28) \;=\; 0.424. \quad \square$$

Other answers are possible, e.g.,

$$-\frac{1}{\lambda}\ell\text{n}\left(\prod_{i=1}^{k} (1 - U_i)\right) \;=\; -\frac{1}{3}\ell\text{n}(0.18) \;=\; 0.572. \quad \square$$

24. If X is a Nor(0,1) random variate, and $\Phi(x)$ is the Nor(0,1) c.d.f., what is the distribution of $\Phi(X)$?

   Solution: By the Inverse Transform Theorem, $\Phi(X) \sim \text{Unif}(0,1)$.   □

25. If U is a Unif(0,1) random variate, and $\Phi(x)$ is the Nor(0,1) c.d.f., what is the distribution of $2\Phi^{-1}(U) + 3$?

   Solution: By the Inverse Transform Theorem,
   $$2\Phi^{-1}(U) + 3 \sim 2\,\text{Nor}(0,1) + 3 \sim \text{Nor}(3,4).$$ □

26. If U and V are i.i.d. Unif(0,1), what's the distribution of $-3\ell n(U^2 V^2)$?

   Solution: Note that
   $$\begin{aligned}
   -3\ell n(U^2 V^2) &= -6\ell n(U) - 6\ell n(V) \\
   &\sim \text{Exp}(1/6) + \text{Exp}(1/6) \\
   &\sim \text{Erlang}_2(1/6).
   \end{aligned}$$ □

27. Suppose the random variable X has the following c.d.f.
   $$F(x) = \begin{cases} \frac{1}{2}e^{3x} & \text{if } x \leq 0 \\ 1 - \frac{1}{2}e^{-5x} & \text{if } x > 0 \end{cases}.$$
   Give an inverse transform method for generating realizations of X.

   Solution: Two cases:

   For $X \leq 0$ (i.e., $0 \leq U \leq 1/2$), set $F(X) = \frac{1}{2}e^{3X} = U$, so that $X = \frac{1}{3}\ell n(2U)$.

   Similarly, for $X > 0$ (i.e., $1/2 < U \leq 1$), set $F(X) = 1 - \frac{1}{2}e^{-5X} = U$, so that $X = -\frac{1}{5}\ell n(2(1-U))$.

   Putting all of this together, we have
   $$X = \begin{cases} \frac{1}{3}\ell n(2U) & \text{if } U \leq 1/2 \\ -\frac{1}{5}\ell n(2(1-U)) & \text{if } U > 1/2 \end{cases}.$$ □

28. Consider your inverse transform solution to Problem 27. Use the Unif(0,1) random number 0.6 to generate a realization of X.

Solution: Have to use the second case above, i.e.,

$$X = -\frac{1}{5}\ell n(2(1-U)) = -0.2\ell n(0.8) = 0.0446. \quad \square$$

29. ~~If U_1 and U_2 are i.i.d. Unif(0,1) with $U_1 = 0.75$ and $U_2 = 0.75$, use Box-Muller to generate two i.i.d. Nor(0,1) realizations. [Ha! I made U_1 and U_2 the same value so that the problem will be easier for me to grade! But note that you should still get two different Z_1 and Z_2 values.]~~

Solution: We have

$$Z_1 = \sqrt{-2\ell n(U_1)}\cos(2\pi U_2) = \sqrt{-2\ell n(0.75)}\cos(1.5\pi) = 0$$
$$Z_2 = \sqrt{-2\ell n(U_1)}\sin(2\pi U_2) = \sqrt{-2\ell n(0.75)}\sin(1.5\pi) = -0.759. \quad \square$$

30. If Z_1, Z_2, and Z_3 are i.i.d. Nor(0,1) random variables, find the value of c such that $P(Z_1^2 + Z_2^2 + Z_3^2 < c) = 0.99$.

Solution:

$$0.99 = P(Z_1^2 + Z_2^2 + Z_3^2 < c) = P(\chi^2(1) + \chi^2(1) + \chi^2(1) < c) = P(\chi^2(3) < c),$$

which implies that $c = \chi^2_{0.01,3} = 11.34.$ $\quad \square$

31. If U_1 and U_2 are PRNs, what is the distribution of $-4(U_1 + U_2) - 2$?

Solution: By class notes,

$$-4(U_1+U_2)-2 \sim -4\text{Tria}(0,1,2)-2 \sim \text{Tria}(-8,-4,0)-2 \sim \text{Tria}(-10,-6,-2). \quad \square$$

32. If a random variable X has the beta distribution, then its p.d.f. is of the form $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, for parameters α and $\beta > 0$, and where $\Gamma(\cdot)$ is the gamma function. How might you generate such a random variate? Pick the best answer.

(a) Ask a UGA student.

(b) Convolution (add up appropriate random variates).

(c) Inversion.

(d) Acceptance-Rejection.

(e) Composition.

Solution: (a) is a sick joke; (b) isn't right because there aren't any sums involved; (c) is wrong because you can't invert the resulting c.d.f. in closed-form; (e) won't work in any natural way (except in a couple of special cases). In fact, from a similar example given in the class notes, we know that the correct answer is...

(d) acceptance-rejection. □

33. Consider a bivariate normal random variable (X, Y), for which $\mathsf{E}[X] = 3$, $\mathsf{Var}(X) = 4$, $\mathsf{E}[Y] = -2$, $\mathsf{Var}(Y) = 9$, and $\mathsf{Cov}(X, Y) = -2$. Find the Cholesky matrix associated with (X, Y), i.e., the lower-triangular matrix C such that $\Sigma = CC'$, where Σ is the variance-covariance matrix.

Solution: By class notes, we saw that for $k = 2$, we obtain

$$C = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} & \sqrt{\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}} \end{pmatrix} = \begin{pmatrix} \sqrt{4} & 0 \\ \frac{-2}{\sqrt{4}} & \sqrt{9 - \frac{(-2)^2}{4}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 2\sqrt{2} \end{pmatrix}. \quad \square$$

34. Suppose $Z_1 = 1$ and $Z_2 = 2$ are i.i.d. Nor(0,1) realizations. Using the notation from and your answer to Question 33, generate (X, Y).

Solution: By class notes, we have

$$\begin{aligned} X &= \mathsf{E}[X] + c_{11}Z_1 = 3 + 2(1) = 5 \\ Y &= \mathsf{E}[Y] + c_{21}Z_1 + c_{22}Z_2 = -2 - 1(1) + 2\sqrt{2}(2) = 2.657. \end{aligned}$$

35. ~~Consider a nonhomogeneous Poisson arrival process with rate function $\lambda(t) = t/2$ for $t \geq 0$. Find the probability that there will be exactly 3 arrivals before time $t = 2$.~~

Solution: The distribution of the number of arrivals by time 2 is

$$N(2) \ \sim \ \text{Pois}\left(\int_0^2 \lambda(t)\,dt\right) \ \sim \ \text{Pois}\left(\int_0^2 (t/2)\,dt\right) \ \sim \ \text{Pois}(1).$$

Thus,

$$\mathsf{P}(N(2) = 3) \ = \ \mathsf{P}(\text{Pois}(1) = 3) \ = \ \frac{e^{-1}(1)^3}{3!} \ = \ 0.0613. \quad \square$$

36. What is thinning good for? Circle the best choice.

(a) It's used to generate independent exponential interarrivals when the rate changes over time.

(b) It's a method to generate i.i.d. normal observations.

(c) It's a technique for generating multivariate normal observations.

(d) It's used to produce Brownian motion paths.

Solution: (a) Thinning is used to generate the independent exponential interarrivals arising from a nonhomogeneous Poisson process. $\square$

37. The Zombies or Justin Bieber?

Solution: Gimme a break. www.youtube.com/watch?v=FmuswTEGF-U $\heartsuit$