Data Analytics for Business

Regression Diagnostics

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Visual Exploration Before Doing Regression

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Lessons in this Module

A. Visual Exploration Before Doing Regression
B. Anscombe's Quartet
C. Assumptions of Linear Models
D. Common Problems and Fixes in Fitting Linear Regression, Part 1
E. Common Problems and Fixes in Fitting Linear Regression, Part 2

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Some Useful Graphs Before Regression

- Univariate graphs
- Two-dimensional graphs
- 3d plots
- Correlation Matrix plots

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The First 10 Records in Housing

Housing Dataset in the *Ecdat* package in R

price	lotsize	bedrooms	bathrms	stories	driveway	recroom	fullbase	gashw	airco	garagepl	prefarea
42000	5850	3	1	2	yes	no	yes	no	no	1	no
38500	4000	2	1	1	yes	no	no	no	no	0	no
49500	3060	3	1	1	yes	no	no	no	no	0	no
60500	6650	3	1	2	yes	yes	no	no	no	0	no
61000	6360	2	1	1	yes	no	no	no	no	0	no
66000	4160	3	1	1	yes	yes	yes	no	yes	0	no
66000	3880	3	2	2	yes	no	yes	no	no	2	no
69000	4160	3	1	3	yes	no	no	no	no	0	no
83800	4800	3	1	1	yes	yes	yes	no	no	0	no
88500	5500	3	2	4	yes	yes	no	no	yes	1	no

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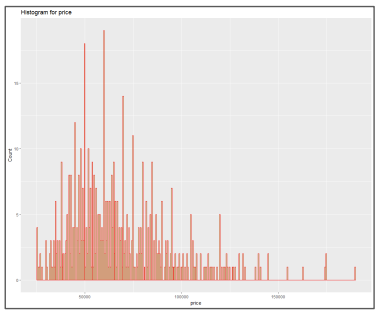
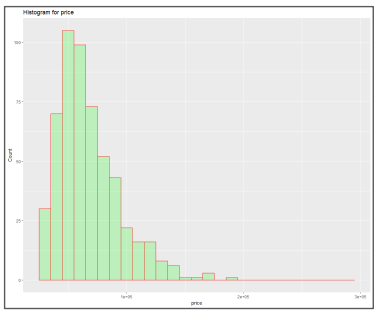
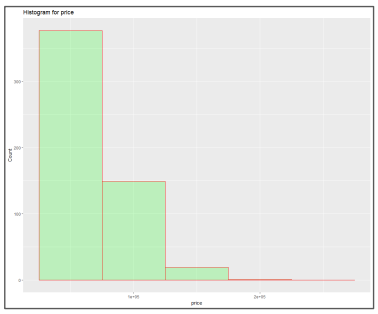
Univariate - Histograms



Bin size = $500



Bin size = $10,000



Bin size = $50,000

- Histograms of house prices with different bin sizes
- You can get an idea if the data is normally distributed or is skewed
- You need to be careful about the bin size (the form depends on bin size)

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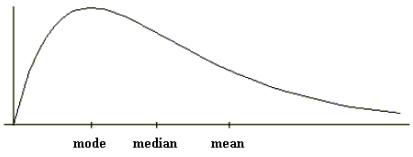
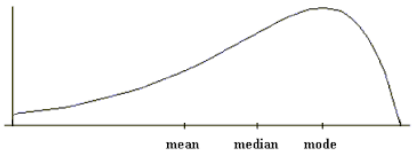
If a Variable is Skewed



Right skewed



Left skewed

- You may need to use Log(variable) in your model (add 1 to the variable in case 0 values are present)
- Usually, we use natural logs since they are easier to interpret
- Logarithm of a large number is a small number, i.e., taking log is one way to do a variance reducing transformation

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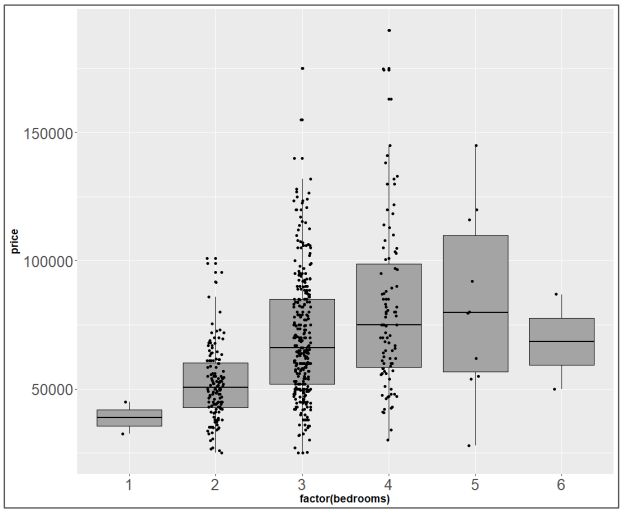
Univariate - Boxplots

- The central box is drawn between the 25th and 75th percentile of the data
- The line in the box represents the median value
- Can be used to detect skewness (e.g., boxplot for houses with 3 bedrooms which is right skewed)
- Can be used to show outliers (e.g. boxplot for 4 bedrooms which has some very high priced houses)



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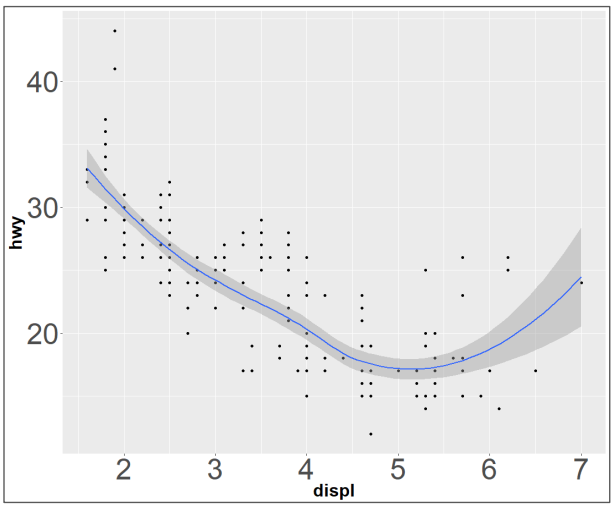
Bivariate Data - Scatterplots

- The scatterplot is probably the most helpful type of graph for displaying two variables
- It is helpful (sometimes) to add a function, like the loess smoothing function in this plot
- Scatterplots with two discrete variables are difficult to use



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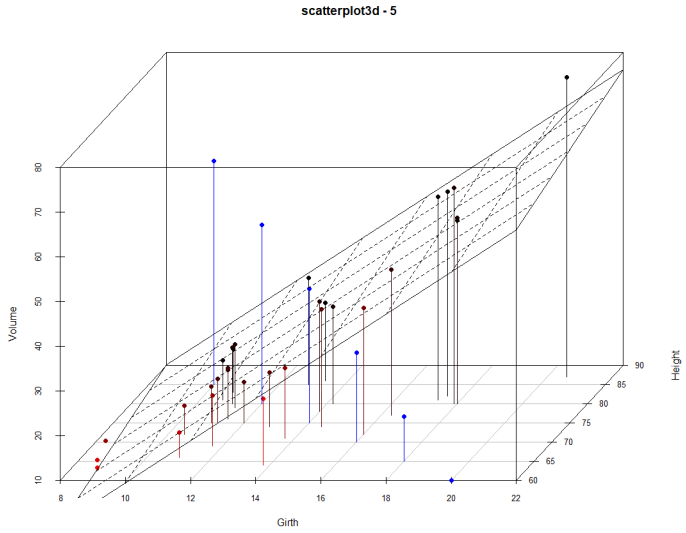
Scatterplot3d



Example 5 in scatterplot3.pdf with linear model, lm(Volume ~ Girth + Height)

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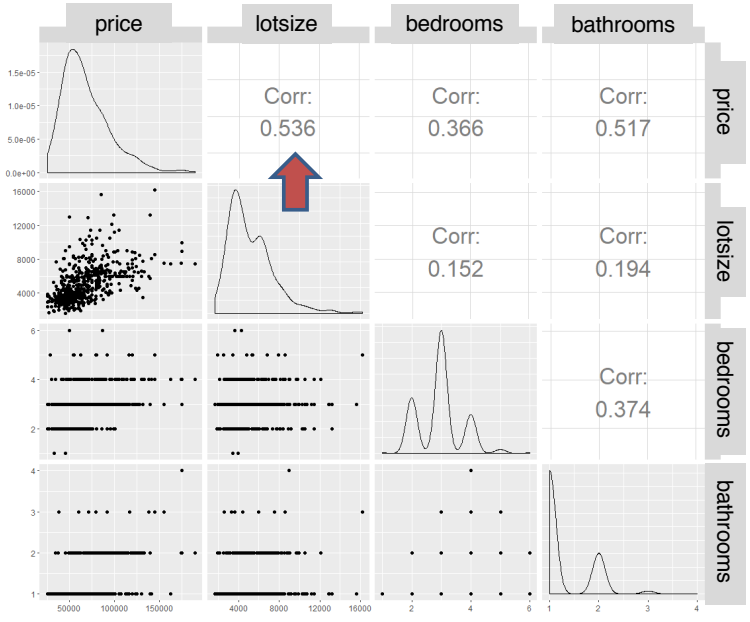
Correlation Matrix



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Anscombe's Quartet

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Anscombe's Quartet (1973).

Francis Anscombe's main idea in this paper is that graphs are very valuable for checking modeling assumptions.

He wanted to counter the impression among statisticians that:

1. "numerical calculations are exact, but graphs are rough;
2. for any particular kind of statistical data there is just one set of calculations constituting a correct statistical analysis;
3. performing intricate calculations is virtuous, whereas actually looking at the data is cheating."

Anscombe, F. J. (1973). "Graphs in Statistical Analysis," American Statistician. 27 (1): 17–21.

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Anscombe's Quartet: Four Regressions

$$y_1 = a_0 + a_1*x_1 \; ; \; y_2 = b_0 + b_1*x_2 \; ; \; y_3 = c_0 + c_1*x_3 \; ; \; y_4 = d_0 + d_1*x_4$$

x1	x2	x3	x4	y1	y2	y3	y4
10	10	10	8	8.04	9.14	7.46	6.58
8	8	8	8	6.95	8.14	6.77	5.76
13	13	13	8	7.58	8.74	12.74	7.71
9	9	9	8	8.81	8.77	7.11	8.84
11	11	11	8	8.33	9.26	7.81	8.47
14	14	14	8	9.96	8.1	8.84	7.04
6	6	6	8	7.24	6.13	6.08	5.25
4	4	4	19	4.26	3.1	5.39	12.5
12	12	12	8	10.84	9.13	8.15	5.56
7	7	7	8	4.82	7.26	6.42	7.91
5	5	5	8	5.68	4.74	5.73	6.89

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```
lm(formula = y1 ~ x1, data = anscombe)

Residuals:
     Min       1Q   Median       3Q      Max
-1.92127 -0.45577 -0.04136  0.70941  1.83882

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   3.0001     1.1247   2.667  0.02573 *
x1            0.5001     0.1179   4.241  0.00217 **
---
Signif. codes:  '**' 0.01 '*' 0.05

Residual standard error: 1.237 on 9 degrees of freedom
Multiple R-squared:  0.6665,
Adjusted R-squared:  0.6295
F-statistic: 17.99 on 1 and 9 DF,  p-value: 0.00217
```

```
lm(formula = y2 ~ x2, data = anscombe)

Residuals:
    Min      1Q  Median      3Q     Max
-1.9009 -0.7609  0.1291  0.9491  1.2691

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    3.001      1.125   2.667  0.02576 *
x2             0.500      0.118   4.239  0.00218 **
---
Signif. codes:  '**' 0.01 '*' 0.05

Residual standard error: 1.237 on 9 degrees of freedom
Multiple R-squared:  0.6662,
Adjusted R-squared:  0.6292
F-statistic: 17.97 on 1 and 9 DF,  p-value: 0.002179
```

```
lm(formula = y3 ~ x3, data = anscombe)

Residuals:
    Min      1Q  Median      3Q     Max
-1.1586 -0.6146 -0.2303  0.1540  3.2411

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   3.0025     1.1245   2.670  0.02562 *
x3            0.4997     0.1179   4.239  0.00218 **
---
Signif. codes:  '**' 0.01 '*' 0.05

Residual standard error: 1.236 on 9 degrees of freedom
Multiple R-squared:  0.6663,
Adjusted R-squared:  0.6292
F-statistic: 17.97 on 1 and 9 DF,  p-value: 0.002176
```

```
lm(formula = y4 ~ x4, data = anscombe)

Residuals:
   Min     1Q Median     3Q    Max
-1.751 -0.831  0.000  0.809  1.839

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   3.0017     1.1239   2.671  0.02559 *
x4            0.4999     0.1178   4.243  0.00216 **
---
Signif. codes:  '**' 0.01 '*' 0.05

Residual standard error: 1.236 on 9 degrees of freedom
Multiple R-squared:  0.6667,
Adjusted R-squared:  0.6297
F-statistic:    18 on 1 and 9 DF,  p-value: 0.002165
```

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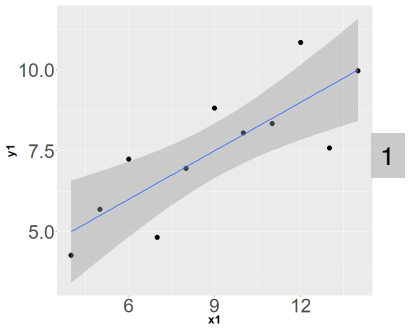
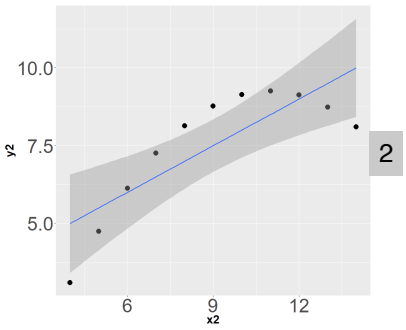
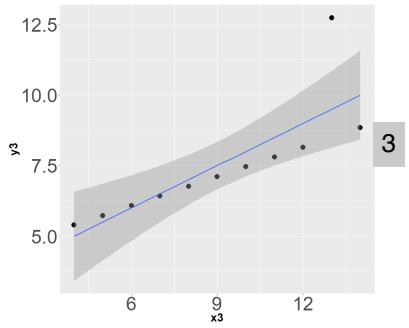
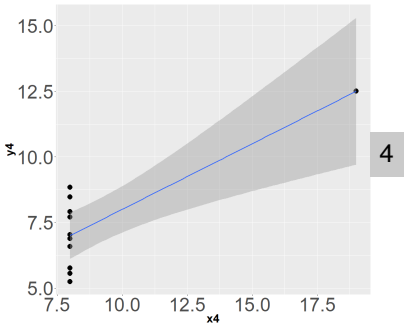






The regression lines are identical. But are there any problems?



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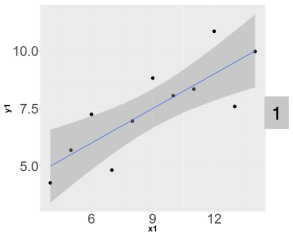
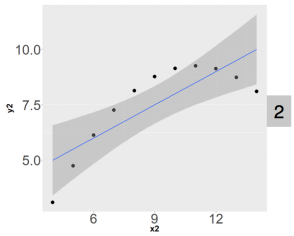
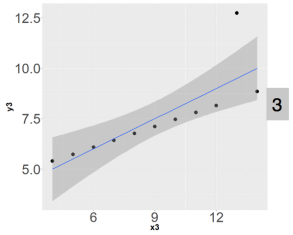
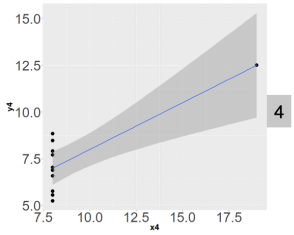






The regression lines are identical. But are there any problems?



- Regression 1 looks fine.
- Regression 2 requires a nonlinear transformation.
- Regression 3 has a single large (vertical) outlier that has a large residual which is influential on the fitted line.
- Regression 4 has a large (horizontal) outlier (also called a leverage point) which is influential on the fitted line.

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Quiz (True/False)

- Anscombe's main idea in his 1973 paper was that graphs are NOT valuable for checking modeling assumptions.
  Answer: **FALSE**

- For the four models Anscombe fitted for their respective data sets, the parameter estimates are exactly the same and the statistics are identical.
  Answer: **TRUE**

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Assumptions of Linear Models

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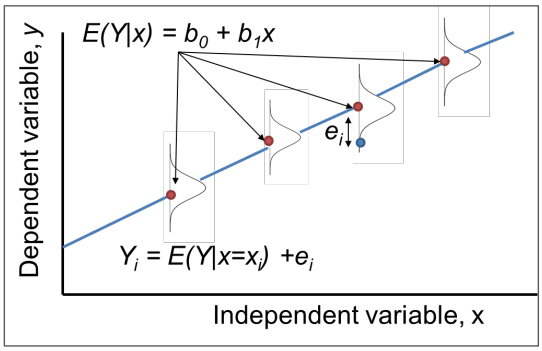
Assumptions of Linear Regression



- **Linearity assumption**: E(y) = $b_0 + b_1x$, i.e., the expected value of Y at each value of X approximates to a straight line
- **Assumption about errors**: The error terms e_i are independently and identically distributed (iid) normal random variables, each with mean zero and constant variance σ^2 (homoscedasticity)
- **Assumptions about predictors**: In multiple regression, the predictor variables are assumed to be linearly independent of one another

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Common Problems and Fixes in Fitting Linear Regression, Part 1

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Most Common Problems in Fitting Linear Regression

1. Non-linearity of the response-predictor relationships
2. Correlation of error terms
3. Non-constant variance of error terms
4. Outliers
5. High-leverage points
6. Collinearity

James, Gareth, et al. *An introduction to statistical learning: with applications in R* (Section 3.3.3). Springer, 2017.

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1. Is the Relationship Nonlinear?

- Check the scatter plots of **Y** vs. each **X** variable. Linear?
- Another plot to use is the residuals plot vs. fitted values plot (especially useful in multiple regression)
  - We want to see no patterns

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1. Is the Relationship Nonlinear?

- If there are concerns, then
  - Can you model non-linear relationship with higher order terms (e.g., square)?
  - Use variance reducing transformation (such as log) that will give a better linear fit
  - Are there outliers or certain sections of the observations that seem to drive the non-linearity?
  - Is there any important variable that you left out from your model (e.g., age or gender)?
  - Or, maybe, was there systematic bias when collecting data, hence redesign data collection
- Checking residuals helps discover useful insights about your model and data

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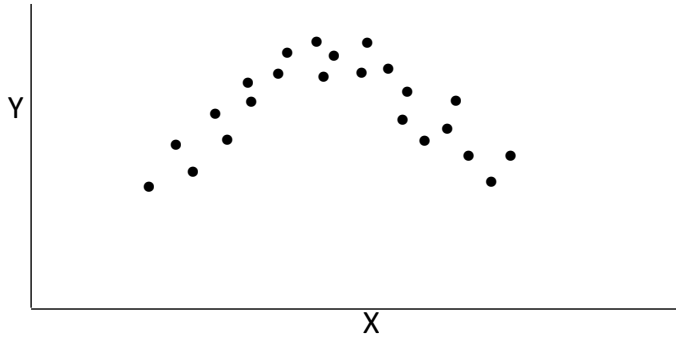
Scatterplot of the Response Variable Versus the Explanatory Variable

- This is useful to do before fitting a model.
- You can plot Y vs. X to identify any patterns, For example, the scatter plot below suggests using X^2 rather than X as a predictor.



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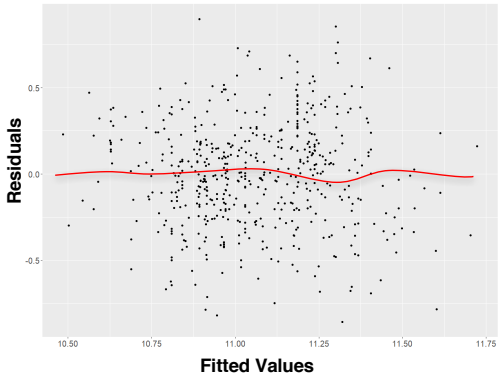
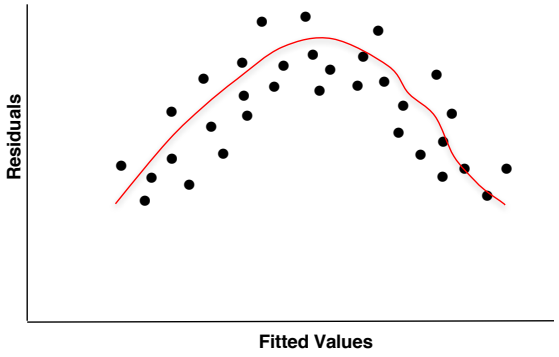
Residuals vs. Fitted Values Plot

Case 1



Case 2



This plot should be examined after a model is fitted.

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2. Correlation of Error Terms

- An important assumption is that error terms $e_1, e_2, \ldots, e_n$ are uncorrelated. If they aren't, then we have autocorrelation.
- So knowing the value of e_i should not have any influence on the magnitude or size of e_{i+1}.
- This property is used to estimate the standard errors of the parameters of the model.
- If there is correlation in the error terms:
  - The estimated standard errors will underestimate the true standard errors.
  - Confidence and prediction intervals will be narrower than they should be and p values will be lower than they should be.
  - We may have sense of confidence in the model that is not warranted.
- The Durbin-Watson test is used to detect autocorrelations in a linear model

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3. Heteroskedasticity (Non-constant Error Variance)

- The assumption is that the spread of the responses around the straight line is the same at all levels of the explanatory variable (i.e., we have constant variance or homoscedasticity).
- You may have non-constant error present (e.g., the errors increase in size with the fitted values). You can detect this with the residuals vs. fitted values plot.
- If non-constant error is present, then Hypotheses tests and Confidence Intervals can be misleading.
- If there is Heteroscedasticity, then transformation of the Y variable may be called for.
- Example: ln(Y), or 1/Y, etc.

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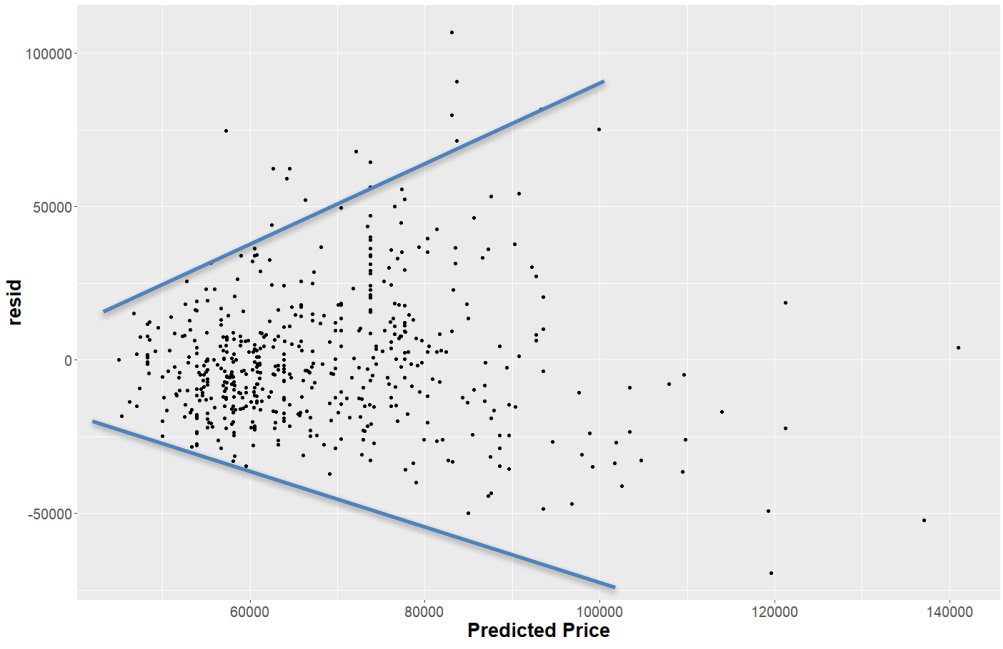

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Quiz (True/False)

- If the scatterplot of Y vs. X shows a nonlinear pattern, then we should not change our linear regression model.
  Answer: **FALSE**
- Autocorrelation is the correlation between each of the e_i variables.
  Answer: **TRUE**
- Heteroskedasticity means having constant Error Variance.
  Answer: **FALSE**

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Common Problems and Fixes in Fitting Linear Regression, Part 2

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4. Outliers

- An outlier is a point that has a y_i value that is far from its predicted value, $\hat{y}_i$
- One way to visualize outliers is to plot residuals (or, better yet, standardized residuals) against predicted values of *y.*
- Outliers could occur because of incorrect data recording or because the phenomenon could very well be non-linear.
- Do not assume that an outlier observation should be removed. It may signal a model deficiency (for e.g., a missing predictor).

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5. High Leverage Points

- In Anscombe's quartet, Regression 4 has a large (horizontal) outlier (also called a leverage point), which is influential on the fitted line.
- In simple regression, look for observations that have a predictor value outside the normal range of observations.
- A point has high leverage if its deletion (by itself or with 2 or 3 other points) causes noticeable changes in the model.
- With many predictors in a model, one could have an observation that is within the range of each predictor's value but still be unusual.

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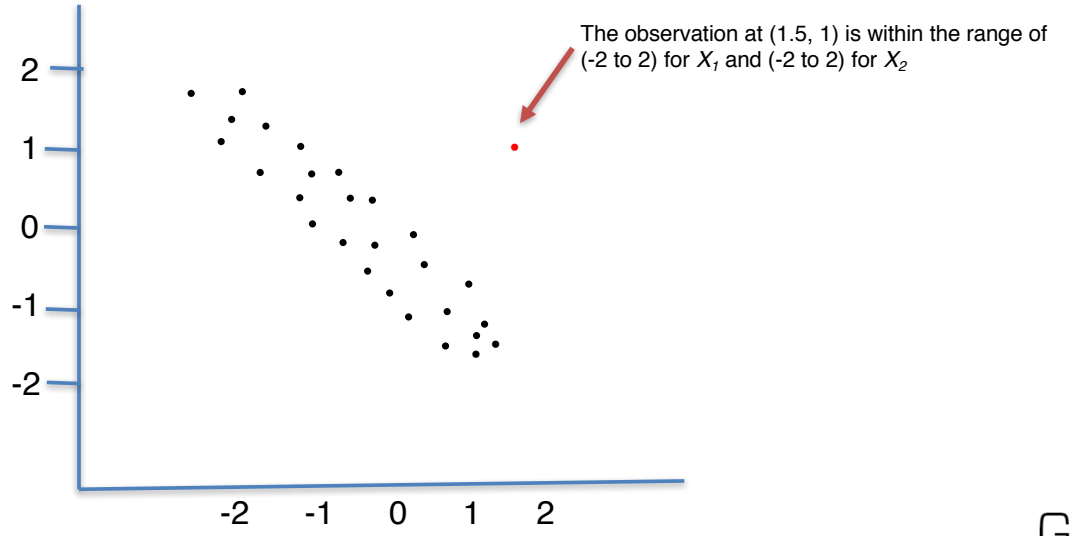
. High Leverage Points - Example



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Cook's Distance

- One statistic to identify influential points is the ***Cook's Distance*** C_i that measures the difference between the regression coefficients obtained
  (a) from the full data and
  (b) from deleting observation *i*
- A rule of thumb is to identify points with $C_i > 1$ as highly influential.

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plot(model) in R

- The plot function in R, when applied to a linear model, provides four plots that are:
  - Residual vs. Fitted (check if residuals have non-linear patterns)
  - Normal Q-Q (check if residuals are normally distributed)
  - Scale-Location (check if √ (standardized residuals) are spread equally along the range of fitted values)
  - Residuals vs Leverage (to find influential points, if any, with $C_i > 1$)

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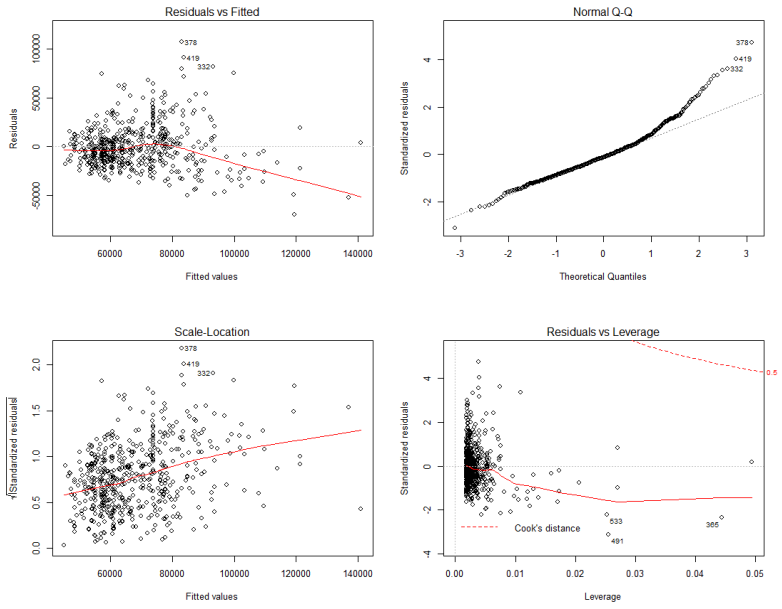
a.lm <- lm(formula = price ~ lotsize , data = Housing)
plot(a.lm)



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6. (Multi)Collinearity

- Run these two linear regression models in R and then compare their respective coefficient of cylinders:
  - Reg1 <- lm(formula = mpg ~ cylinders, data = Auto)
  - Reg2 <- lm(formula = mpg ~ cylinders + displacement + weight, data = Auto)

- Reg1
  cylinders -3.5581 0.1457 -24.43 <2e-16 ***

- Reg2
  cylinders -0.2678 0.4131 -0.648 0.517

- In Reg2, cylinder's coefficient is no longer statistically significant!

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What Could Be the Reason?

- Such a change in a parameter estimate could indicate the presence of multicollinearity in Reg2.
- Multicollinearity: two or more of the explanatory variables are more or less linearly related.
- To detect multicollinearity, one approach is to use **Variance Inflation Factors (VIF).**
- Regress predictor variable X_j against all other predictor (X) variables. Name the resulting R^2 as R_j^2
- Define $VIF_j = 1/(1-R_j^2)$, $j = 1,2,\ldots, p$
- If X_j has a strong linear relationship to other X variables, then R_j^2 is close to 1, and VIF_j will be large.
- Values of VIF > 5 signify presence of multicollinearity (rule of thumb).

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VIF

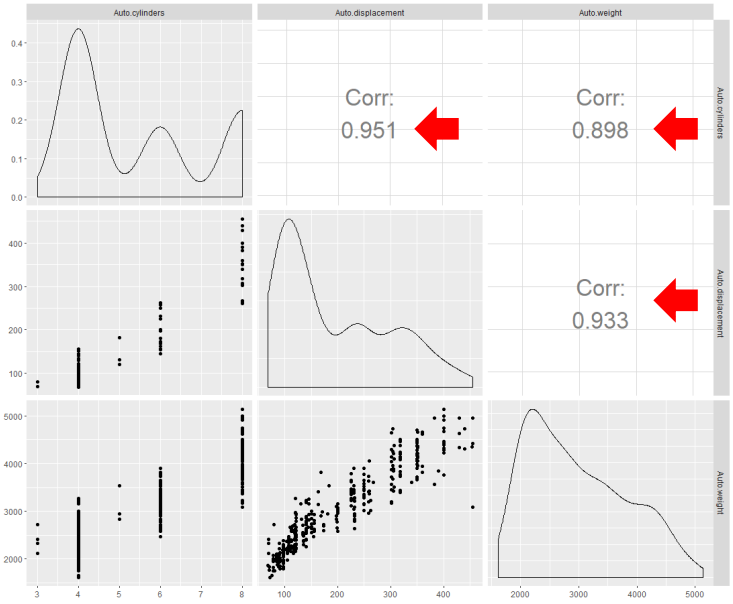
- We use the vif function on the predictors of the Reg2 model and obtain

```
vif(Reg2)
     cylinders      displacement      weight
     10.515508      15.786455         7.788716
```

- These high VIF values indicate Multicollinearity problems

GTx

Correlation Matrix



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Consequences of Multicollinearity

- Multicollinearity: the explanatory variables are highly correlated
  - If $VIF_j = 1/(1-R_j^2) > 5$, multicollinearity present…
- Consequences of multicollinearity
  - OLS estimated parameters may have large variances and covariances, thus making precise estimation difficult
  - The confidence intervals of the estimated parameters tend to be bigger, hence we may not be able to reject H_0 (the null hypothesis, $b_i = 0$)
  - Regression coefficients have the wrong sign, or
  - Regression coefficients are not significantly different from 0 although R^2 is high
  - Adding an explanatory variable changes other variables' coefficients

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Consequences of Multicollinearity

- Solution?
  - Pick one variable if two measure the same "thing"
  - Use Principal Components Analysis or Factor Analysis to create more useful variable(s)

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Recap of this Module

A. Visual Exploration Before Doing Regression
B. Anscombe's Quartet
C. Assumptions of Linear Models
D. Common Problems and Fixes in Fitting Linear Regression, Part 1
E. Common Problems and Fixes in Fitting Linear Regression, Part 2

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