UVA CS 4774: Machine Learning

S3 : Lecture 15: Probability Review

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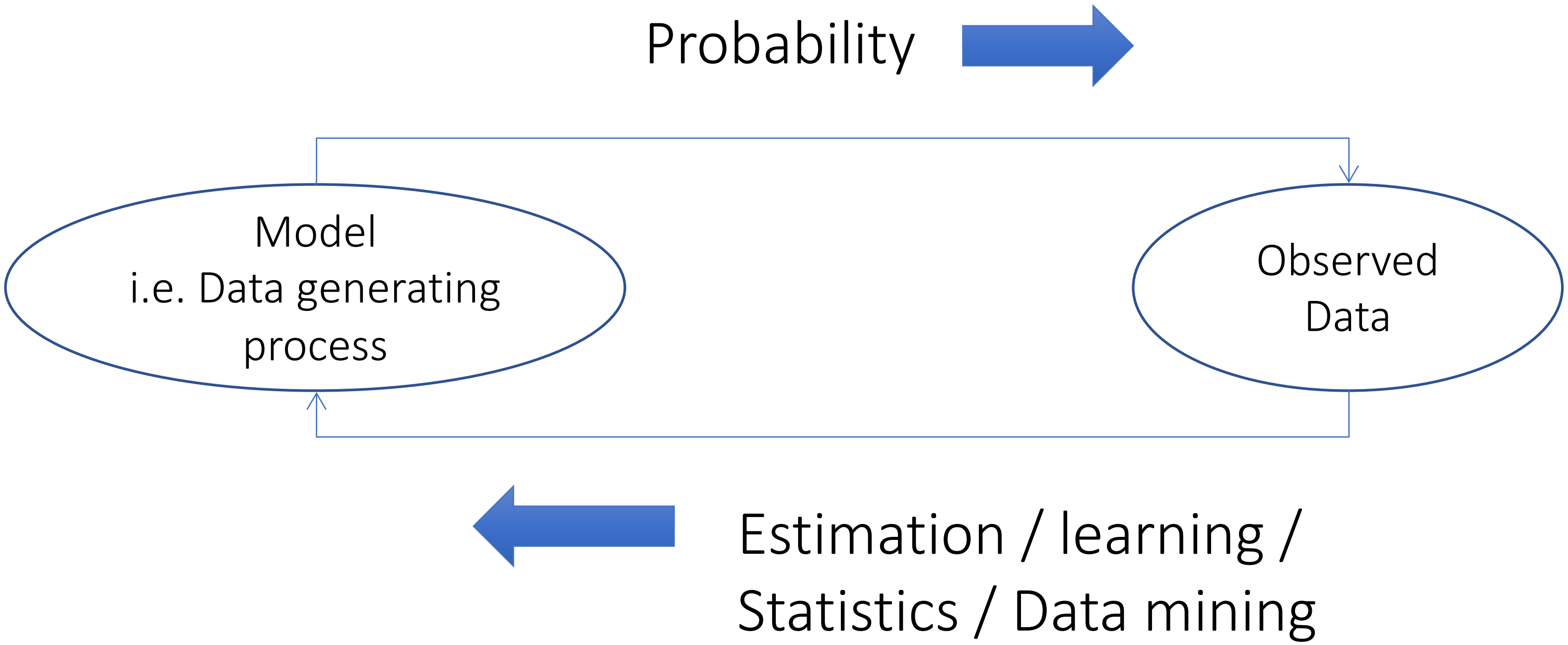
Today : Probability Review

- The big picture
- Events and Event spaces
- Random variables
- Joint probability, Marginalization, conditioning, chain rule, Bayes Rule, law of total probability, etc.
- Structural properties, e.g., Independence, conditional independence
- Maximum Likelihood Estimation

The Big Picture



Probability

- Counting
- Basics of probability
- Conditional probability
- Random variables
- Discrete and continuous distributions
- Expectation and variance
- Tail bounds and central limit theorem
- ……

Statistics

- Maximum likelihood estimation
- Bayesian estimation
- Hypothesis testing
- Linear regression
- [Machine learning]
- ……

Probability as frequency

- Consider the following questions:
  - 1. What is the probability that when I flip a coin it is "heads"?
  - 2. why ?
  - 3. What is the probability of Blue Ridge Mountains to have an erupting volcano in the near future ?

We can count → ~1/2

→ could not count

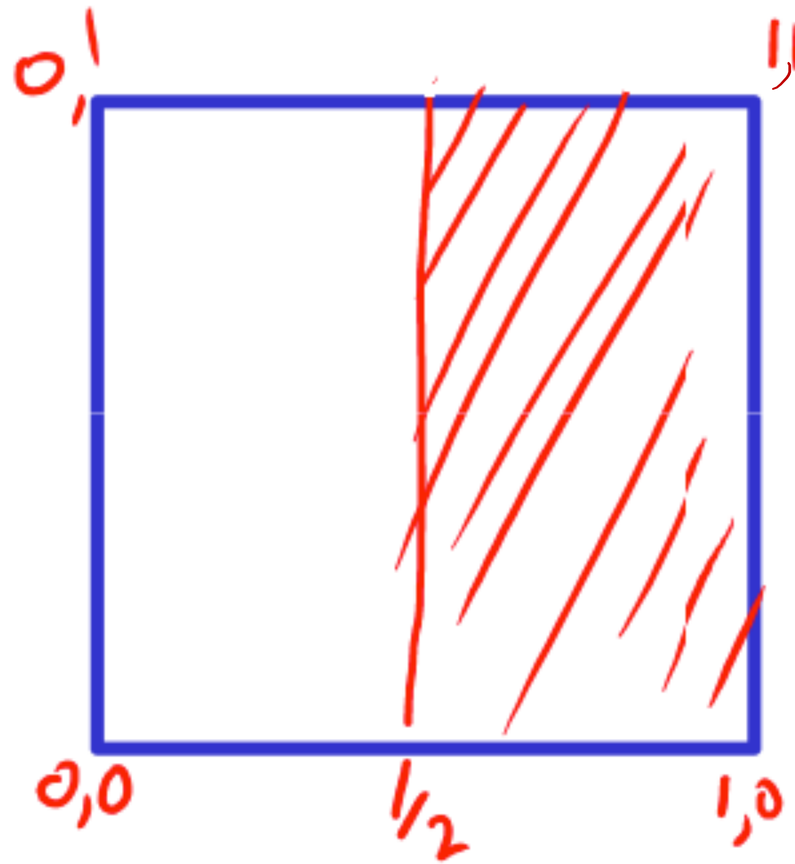
Message: The frequentist view is very useful, but it seems that we can also use domain knowledge to come up with probabilities.

Adapt from Prof. Nando de Freitas's review slides

Probability as a measure of uncertainty

- Imagine we are throwing darts at a wa
size 1x1 and that all darts are guarant
to fall within this 1x1 wall.

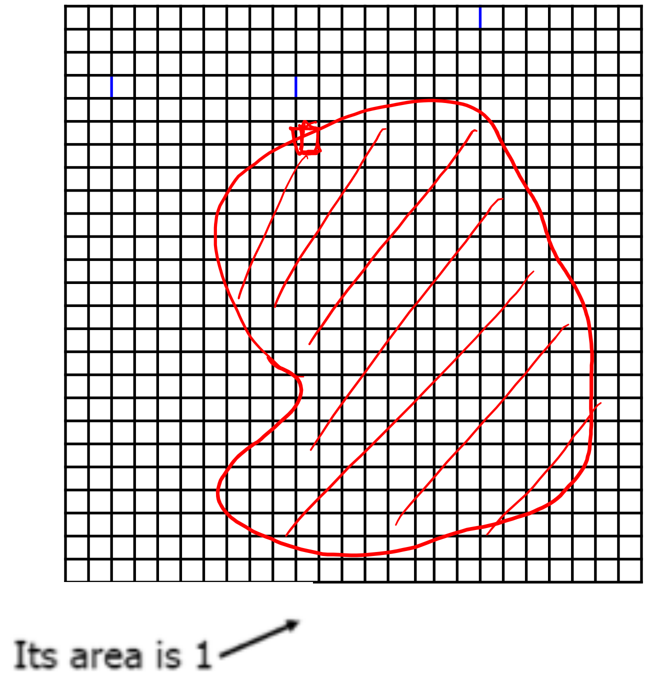
- What is the probability that a dart wil
the shaded area?



Adapt from Prof. Nando de Freitas's review slides

Probability as a measure of uncertainty

- Probability is a measure of certainty of an event taking place.
- i.e. in the example, we were measuring the chances of hitting the shaded area.



$$prob = \frac{\#\,RedBoxes}{\#\,Boxes}$$

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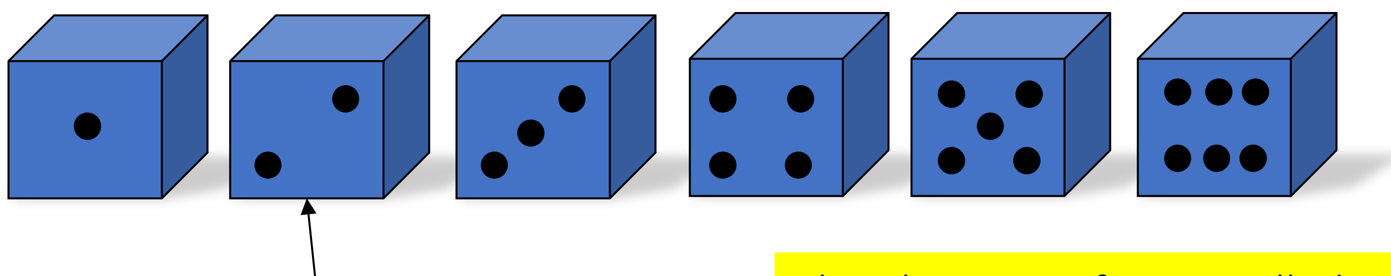
Probability

***Probability** is the formal study of the laws of chance. Probability allows us to **manage uncertainty**.*

*The **sample space** is the set of all **outcomes**. For example, for a die we have 6 outcomes:*

$$O_{die} = \{1,2,3,4,5,6\}$$

O:



Elementary Event “Throw 2”

The elements of O are called elementary events.

Probability

- *Probability allows us to measure many* ***events****.*
- ***The events are subsets of the sample space O****. For example, for a die we may consider the following events: e.g.,*

GREATER = {5, 6}

EVEN = {2, 4, 6}

- ***Assign probabilities to these events: e.g.,***

$$P(\text{EVEN}) = 1/2$$

Adapt from Prof. Nando de Freitas's review slides

Sample space and Events

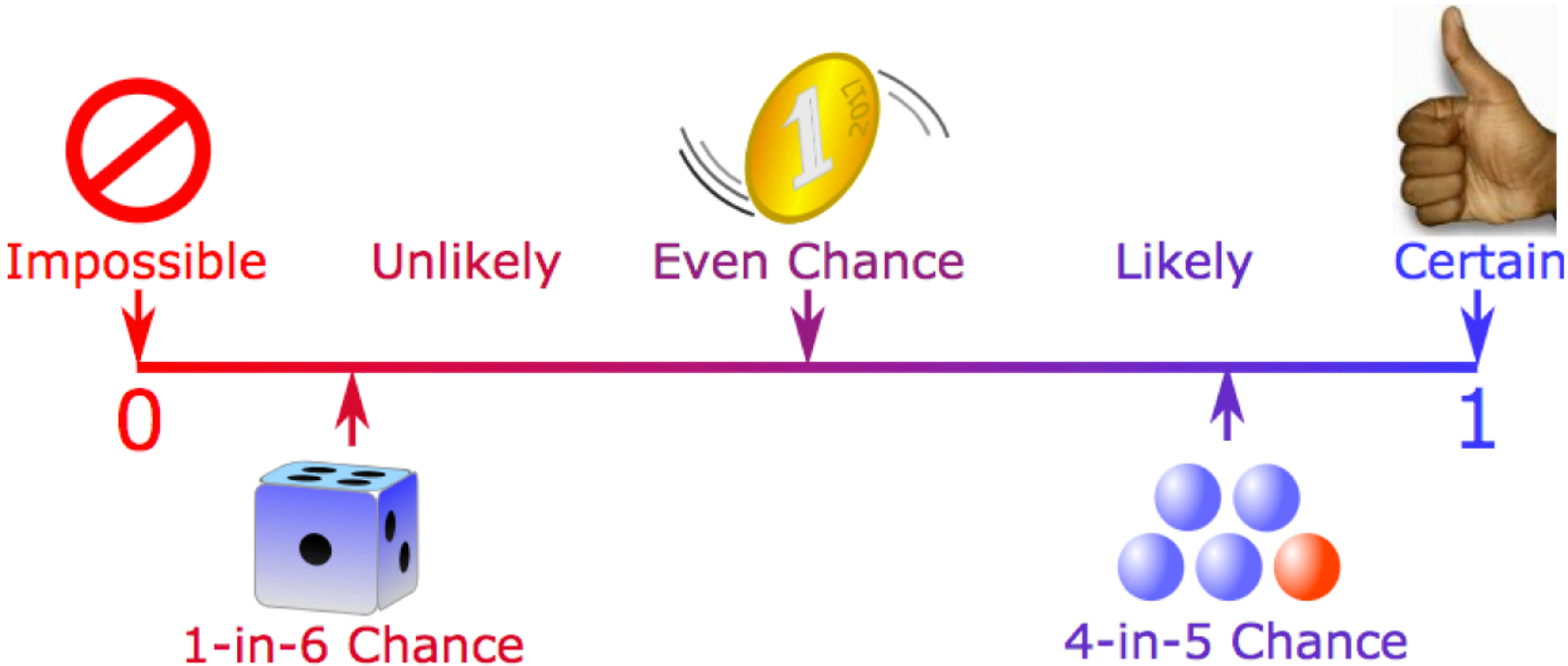
- O : Sample Space,
  - result of an experiment / set of all outcomes
  - If you toss a coin twice O = {HH,HT,TH,TT}
- Event: a subset of O
  - First toss is head = {HH,HT}
- S: event space, a set of events:
  - Contains the empty event and O

Axioms for Probability

Sample Space

Event Space

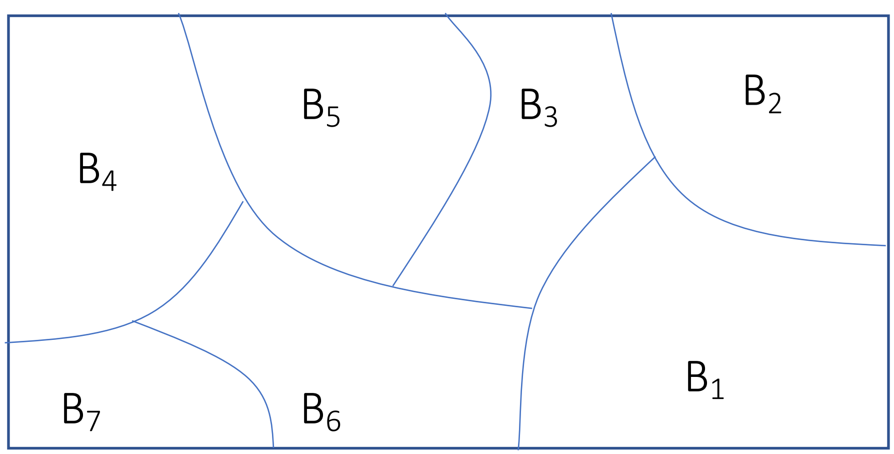
- Defined over (O,S) s.t.
  - 1 >= P(a) >= 0 for all a in S
  - P(O) = 1
  - If A, B are disjoint, then
    - P(A U B) = p(A) + p(B)



Probability is always between 0 and 1

Axioms for Probability

- $P(O) = \sum P(B_i)$ $= 1$

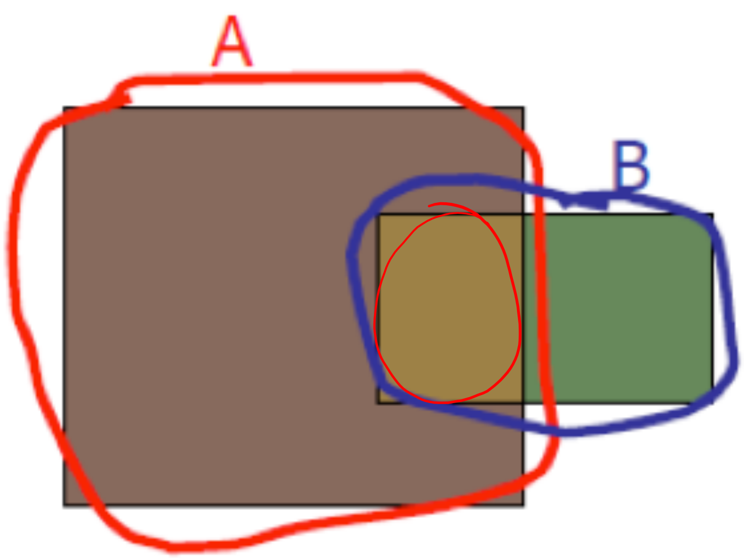


OR operation for Probability

- We can deduce other axioms from the above ones
  - Ex: P(A U B) for non-disjoint events

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

P(Union of A set and B set)

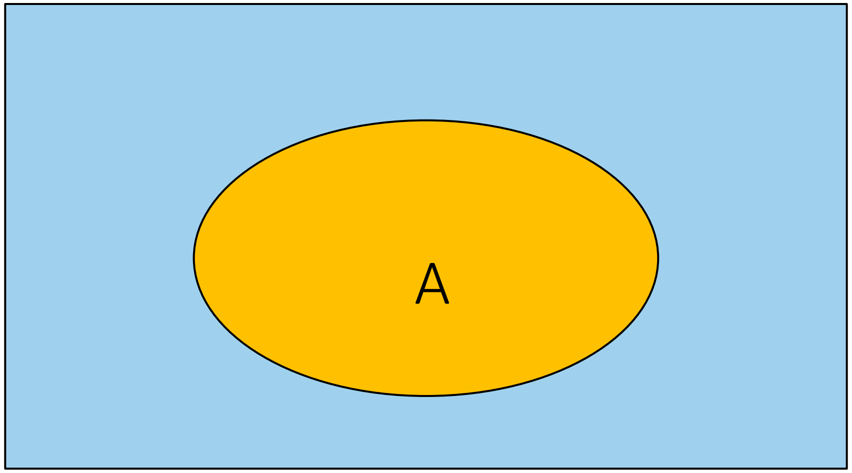


NOT operation for Probability

- $0 <= P(A) <= 1$,
- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

From these we can prove:

$$P(\text{not } A) = P(\sim A) = 1 - P(A)$$



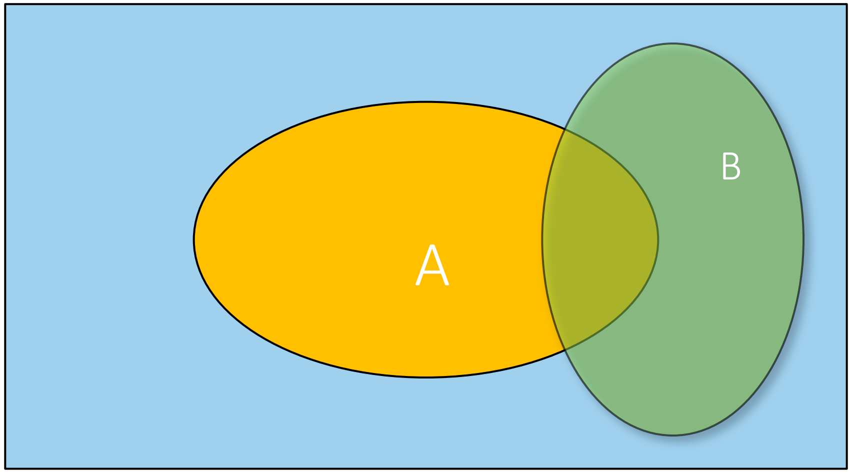
Law of Total Probability

- $0 <= P(A) <= 1$,
- P(A or B) = P(A) + P(B) - P(A and B)

From these we can prove:

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$

P(Intersection of A and B)



Law of Total Probability

- $0 <= P(A) <= 1$,
- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

From these we can prove:

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$

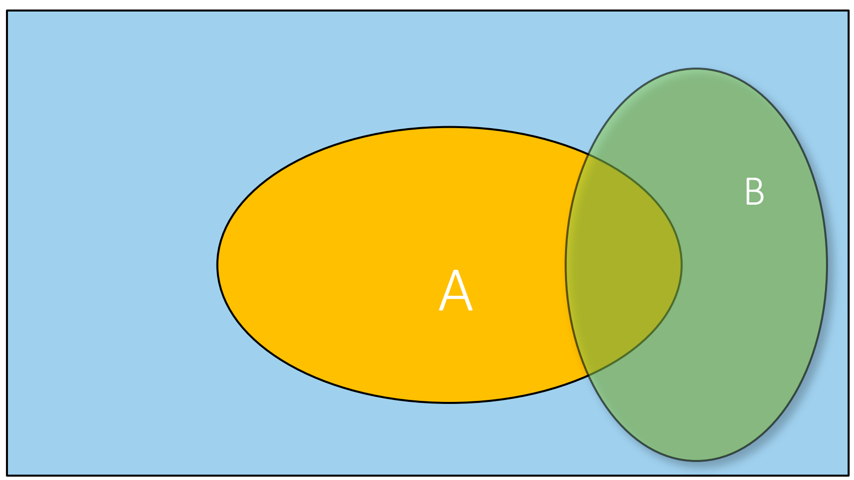
$P(A)$

$= P(A \wedge \Omega)$

$= P(A \cap (B \cup \neg B))$

$= P((A \cap B) \cup (A \cap \neg B))$

$= P(A \cap B) + P(A \cap \neg B)$



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- Maximum Likelihood Estimation

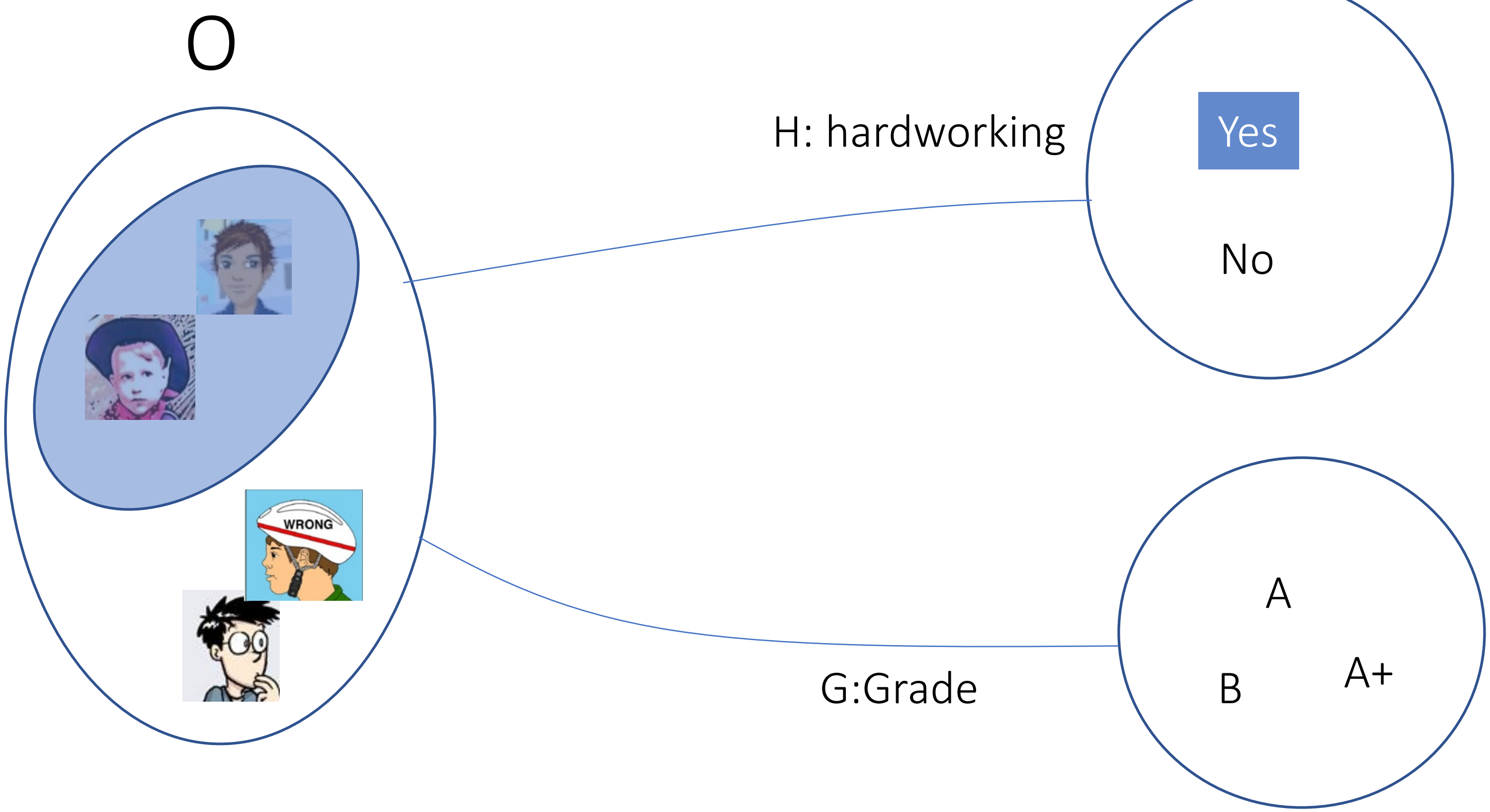
From Events to Random Variable (RV)

- Concise way of specifying attributes of outcomes
- Modeling students (Grade and Intelligence):
  - O = all possible students (sample space)
  - What are events (subset of sample space)
    - Grade_A = all students with grade A
    - Grade_B = all students with grade B
    - HardWorking_Yes = … who works hard
  - Very cumbersome

  - Need "functions" that maps from O to an attribute space T.
  - P(H = YES) = P({student ϵ O : H(student) = YES})

Random Variables (RV)

P(H=Yes)



P(H = Yes) = P({all students who is working hard on the course})

- "functions" that maps from O to an attribute space T.

Notations

- P(A) is shorthand for P(A=true)
- P(~A) is shorthand for P(A=false)
- Same notation applies to other binary RVs: P(Gender=M), P(Gender=F)
- Same notation applies to *multivalued* RVs: P(Major=history), P(Age=19), P(Q=c)
- Note: upper case letters/names for *variables*, lower case letters/names for *values*

Discrete Random Variables

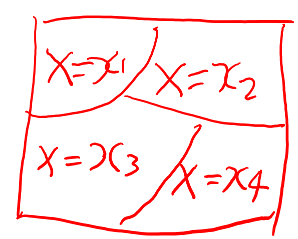
- Random variables (RVs) which may take on only a countable number of distinct values

- X is a RV with arity k if it can take on exactly one value out of $\{x_1, \ldots, x_k\}$

Probability of Discrete RV

- Probability mass function (pmf): $P(X = x_i)$
- Easy facts about pmf
  - $\Sigma_i P(X = x_i) = 1$
  - $P(X = x_i \cap X = x_j) = 0$ if $i \neq j$
  - $P(X = x_i \cup X = x_j) = P(X = x_i) + P(X = x_j)$ if $i \neq j$
  - $P(X = x_1 \cup X = x_2 \cup \ldots \cup X = x_k) = 1$

$$\sum_{i=1}^{4} P(X = x_i) = 1$$



e.g. Coin Flips

- You flip a coin
  - Head with probability p, e.g. =0.5

- You flip a coin for *k*, e.g., =100 times
  - How many heads would you expect

e.g. Coin Flips cont.

- You flip a coin
  - Head with probability *p*
  - Binary random variable
  - Bernoulli trial with success probability *p*
- You flip *a* coin for *k* times
  - How many heads would you expect
  - Number of heads X is a discrete random variable
  - Binomial distribution with parameters *k* and *p*

Binary = {H, T}

P(#Heads)

Integer {1, 2, ..., k}

Discrete Random Variables

- Random variables (RVs) which may take on only a **countable** number of **distinct** values
  - E.g. the total number of heads X you get if you flip 100 coins

- X is a RV with arity *k* if it can take on exactly one value out of
  - E.g. the possible values that X can take on are 0, 1, 2,…, 100

$$\left\{x_1,\ldots,x_k\right\}$$

e.g., two Common Distributions

- Uniform
  - X takes values 1, 2, …, *N*
  - E.g. picking balls of different colors from a box

$$X \sim U[1,\ldots,N]$$

$$P(X=i)=1/N$$

- Binomial
  - X takes values 0, 1, …, *k*
  - E.g. coin flips k times

$$X \sim Bin(k,p)$$

$$P(X=i)=\binom{k}{i}p^i(1-p)^{k-i}$$

$\Downarrow$

i heads out k

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- Structural properties
  - Independence, conditional independence

If hard to directly estimate from data, most likely we can estimate

- 1. Joint probability
  - Use Chain Rule

$$P(A, B) = P(B)\,P(A|B)$$

- 2. Marginal probability
  - Use the total law of probability

$$P(B) = P(B, A) + P(B, \sim A)$$
$$\parallel$$
$$P(B, A \cup \sim A) \,/\!/$$

- 3. Conditional probability
  - Use the Bayes Rule

$$P(A|B)$$
$$P(B|A) = \frac{P(A, B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

(1). To calculate Joint Probability: Use Chain Rule

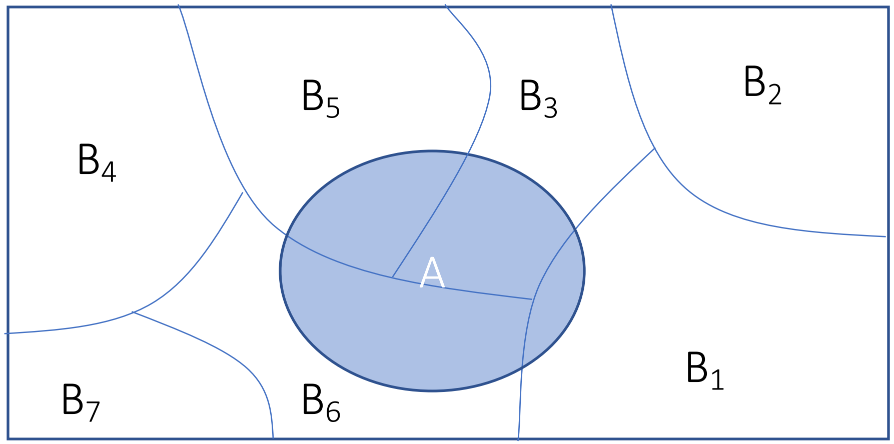
- Two ways to use chain rules on joint probability

joint conditional

$$P(A,B) = p(B|A)p(A)$$ marginal

$$P(A,B) = p(A|B)p(B)$$

(2). To calculate Marginal Probability:
Use Rule of total probability (e.g. event version)



$P(A) =$
$P(A \wedge B) +$
$P(A \wedge \sim B)$

$$\Rightarrow p(A) = \sum P\overbrace{(B_i) P(A | B_i)}^{P(B_i \wedge A)}$$

WHY ???

$P(A) = P(A \cap S)$
$= P(A \cap (B_1 \cup B_2 \cdots \cup B_k))$
$= \sum P(A \cap B_i)$

(2). To calculate Marginal Probability: Use Rule of total probability (e.g. RV version)

- Given two discrete RVs X and Y, which take values in:

$$\{x_1, \ldots, x_k\} \qquad \{y_1, \ldots, y_m\}$$

$$\mathrm{P}(\mathrm{X} = x_i) = \sum_j \mathrm{P}(\mathrm{X} = x_i \cap \mathrm{Y} = y_j)$$

$$= \sum_j \mathrm{P}(\mathrm{X} = x_i | \mathrm{Y} = y_j) \mathrm{P}(\mathrm{Y} = y_j)$$

$$P(A) = P(A \wedge B) + P(A \wedge \sim B)$$

(3). To calculate Conditional Probability: Use Bayes Rule (e.g. RV version)

$$P\left(X = x \middle| Y = y\right) = \frac{P\left(X = x \cap Y = y\right)}{P\left(Y = y\right)}$$

One Example

Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the ***set*** *{**r,r,r,b**}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$P(B_1 = r, B_2 = r) =$$

$$P(B_2 = r)$$

$$P(B_1 = r | B_2 = r)$$

One Example: Joint

Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the **set** *{r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$P(B_1 = r, B_2 = r) \; =$$

Adapt from Prof. Nando de Freitas's review slides

One Example: Joint

*Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the **set** {r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$P(B_1 = r, B_2 = r) = P(B_1 = r)\underbrace{P(B_2 = r \mid B_1 = r)}_{\frac{2}{3}}$$

$$P(B_1 = r) = \frac{3}{4}$$

$$P(B_1 = b) = \frac{1}{4}$$

Adapt from Prof. Nando de Freitas's review slides

One Example: Joint

*Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the **set** {r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$\mathbf{P(B_1 = r, B_2 = r)} = P(B_1 = r)\,P(B_2 = r \mid B_1 = r)$$

$$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

Adapt from Prof. Nando de Freitas's review slides

One Example: Marginal

What is the probability that the 2nd ball drawn from the* set *{r,r,r,b} will be red?

Using marginalization, $P(B_2 = r)$ $=$ $P(B_2 = r, B_1 = r)$

$+ P(B_2 = r, B_1 = b)$

One Example: Marginal

What is the probability that the 2nd ball drawn from the set {r,r,r,b} will be red?

Using marginalization, $\boldsymbol{P(B_2 = r)}$ $= P(B_2 = r \wedge B_1 = r)$

$+ P(B_2 = r \wedge B_1 = b)$

$= P(B_1 = r)P(B_2 = r \mid B_1 = r) + P(B_1 = b)P(B_2 = r \mid B_1 = b)$

$= \frac{3}{4} \times \frac{2}{3} + \frac{1}{4} \times 1$

One Example

Assume we have a dark box with 3 red balls and 1 blue ball. That is, we have the **set** *{r,r,r,b}. What is the probability of drawing 2 red balls in the first 2 tries?*

$$P(B_1 = r, B_2 = r) = \underbrace{P(B_1=r)}_{\frac{3}{4}} P(B_2=r \mid B_1=r) = \frac{1}{2}$$

$$\frac{3}{4} \times \frac{2}{3}$$

$$P(B_2 = r) = P(B_1=r, B_2=r) + P(B_1=b, B_2=r)$$

$$P(B_1 = r | B_2 = r) = \frac{P(B_1=r, B_2=r)}{P(B_2=r)}$$

One Example: Conditional

$\begin{cases} \text{Chain Rule} \\ \text{total law Prob} \end{cases}$

$$P(B_1=r \mid B_2=r)$$

$$= \frac{P(B_2=r \mid B_1=r)\, P(B_1=r)}{P(B_2=r)} \Rightarrow \text{last last page}$$

$\Downarrow$ Last

$$= \frac{P(B_2=r \mid B_1=r)\, P(B_1=r)}{P(B_2=r, B_1=r) + P(B_2=r, B_1=b)}$$

Bayes Rule

$$P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

$$\frac{P(Y=yes)\,P(X|Y=yes)}{P(X)}$$

if $P(Y=yes|X) > P(Y=No|X)$

$$\frac{P(Y=No)\,P(X|Y=No)}{P(X)}$$

$\Rightarrow \hat{Y}=yes$

This is Bayes Rule

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London*, **53:370-418**

More General Forms of Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\sim A)P(\sim A)}$$

$$\frac{P(B_2=r, B_1=r)}{P(B_2=r, B_1=r) + P(B_2=r, B_1=b)}$$

$$P(A|B \wedge X) = \frac{P(B|A \wedge X)P(A \wedge X)}{P(B \wedge X)}$$

$$P(A = a_1 | B) = \frac{P(B | A = a_1)P(A = a_1)}{\sum_i P(B | A = a_i)P(A = a_i)}$$

E.g.: Use both Bayes Rule and Marginal

- X and Y are discrete RVs…

$$P(X = xi | Y = yj) = \frac{P(X = xi \cap Y = yj)}{P(Y = yj)}$$

$$\{x_1, \ldots, x_k\} \Downarrow$$

$$P\left(X = x_i \middle| Y = y_j\right) = \frac{P\left(Y = y_j \middle| X = x_i\right) P\left(X = x_i\right)}{\sum_k P\left(Y = y_j \middle| X = x_k\right) P\left(X = x_k\right)}$$

Simplify Notation: Conditional Probability

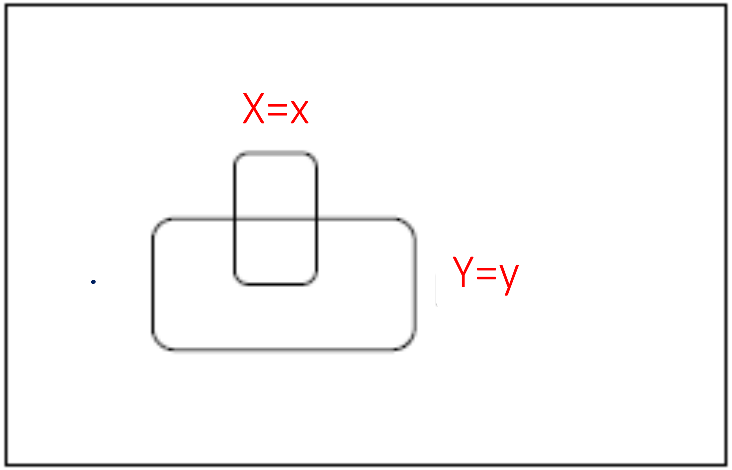
events

$$P(X = x | Y = y) = \frac{P(X = x \cap Y = y)}{P(Y = y)}$$

But we will always write it this way:

$$P(x \mid y) = \frac{p(x, y)}{p(y)}$$

P(X=x true) -> P(X=x) -> P(x)



$P(x) \leftarrow P(X=x) \leftarrow P(X=x \text{ true})$

value RV event

Simplify Notation:

An Example of estimating conditional

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(rain \mid wet) = \frac{P(rain)P(wet \mid rain)}{P(wet)}$$

W= G=

Simplify Notation:

An Example of estimating conditional

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

W= G=

$$P(rain \mid wet) = \frac{P(rain)P(wet \mid rain)}{P(wet)}$$

$P(W=S \mid wet)$

$$P(x \mid y) = \frac{P(x)P(y \mid x)}{P(y)} = \frac{p(x,y)}{p(y)}$$

Simplify Notation: Conditional

- Bayes Rule

$$P(x \mid y) = \frac{P(x)P(y \mid x)}{P(y)}$$

- You can condition on more variables

$$P(x \mid y, z) = \frac{P(x \mid z)P(y \mid x, z)}{P(y \mid z)}$$

Simplify Notation: Marginal

- We know p(X, Y), what is P(Y=y) or P(X=x)?
- We can use the law of total probability

$$p(x) = \sum_{y} P(x, y)$$

$$= \sum_{y} P(y) P(x \mid y)$$

$$\{y_1, \ldots, y_m\}$$

all possible Y values

$$p(x) = \sum_{y,z} P(x, y, z)$$

$$= \sum_{z,y} P(y, z) P(x \mid y, z)$$

$$\sum_{y} \sum_{z} p(y, z) = 1$$

Simplify Notation:
An Example

- We know that P(rain) = 0.5
  - If we also know that the grass is wet, then how this affects our belief about whether it rains or not?

$$P(rain \mid wet) = \frac{\overset{0.5}{P(rain)}\,\overset{1}{P(wet \mid rain)}}{P(wet)}$$

$p(wet, rain)$

$P(wet, rain) + P(wet, sunny)$

$P(rain)\,p(wet \mid rain) + P(sunny)\,P(wet \mid sunny)$

Weather   Grass

$\{rain, sunny\}$ $\{wet, dry\}$

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Independent RVs

- Definition: X and Y are independent *iff*

$$\mathrm{P}\left(\mathrm{X}=x \cap \mathrm{Y}=y\right)=\mathrm{P}\left(\mathrm{X}=x\right)\mathrm{P}\left(\mathrm{Y}=y\right)$$

More on Independence

$$P\left(X = x \cap Y = y\right) = P\left(X = x\right)P\left(Y = y\right)$$

$$P\left(X = x \middle| Y = y\right) = P\left(X = x\right)$$

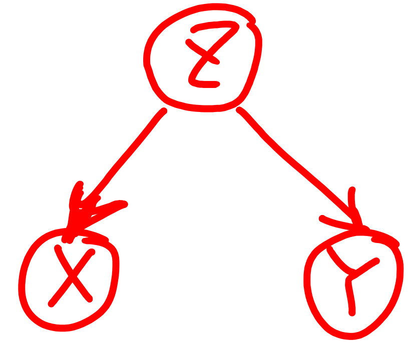
$$P\left(Y = y \middle| X = x\right) = P\left(Y = y\right)$$

- E.g. no matter how many heads you get, your friend will not be affected, and vice versa

More on Independence

- X is independent of Y means that knowing Y does not change our belief about X.
- The following forms are equivalent:
  - $P(X=x, Y=y) = P(X=x)\, P(Y=y)$
  - $P(X=x \mid Y=y) = P(X=x)$
  - The above should hold for all x_i, y_j
  - It is symmetric and written as $X \perp Y$

Conditionally Independent RVs

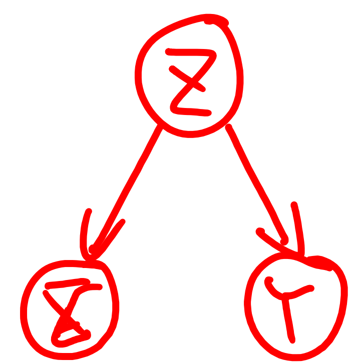


- Intuition: X and Y are conditionally independent given Z means that once Z is **known**, the value of X does not add any **additional** information about Y
- Definition: X and Y are conditionally independent given Z *iff*

$$\mathrm{P}\left(\mathrm{X}=x \cap \mathrm{Y}=y \middle| \mathrm{Z}=z\right)=\mathrm{P}\left(\mathrm{X}=x \middle| \mathrm{Z}=z\right)\mathrm{P}\left(\mathrm{Y}=y \middle| \mathrm{Z}=z\right)$$

If holding for all x_i, y_j, z_k

$$X \perp Y \mid Z$$



More on Conditional Independence

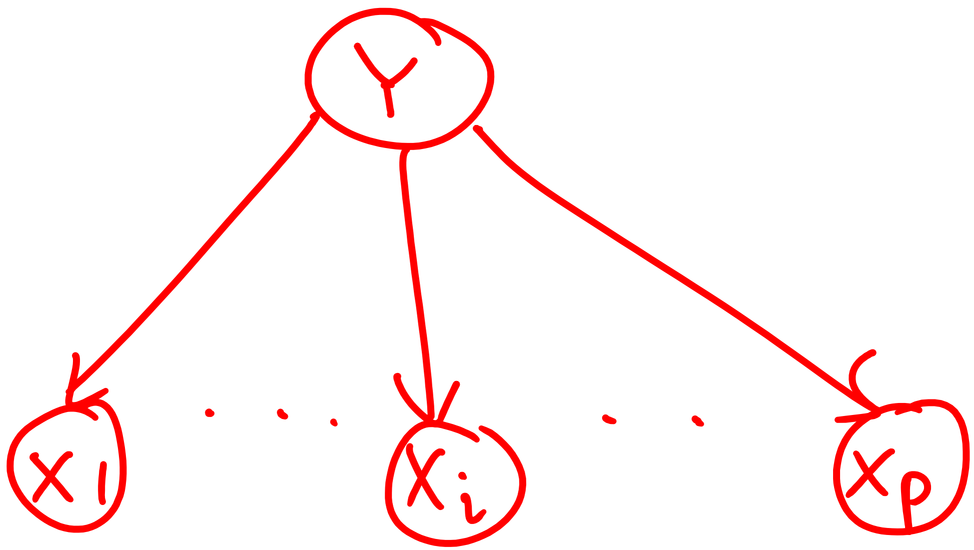
$$\mathrm{P}\left(\mathrm{X}=x\cap\mathrm{Y}=y\middle|\mathrm{Z}=z\right)=\mathrm{P}\left(\mathrm{X}=x\middle|\mathrm{Z}=z\right)\mathrm{P}\left(\mathrm{Y}=y\middle|\mathrm{Z}=z\right)$$

$$\mathrm{P}\left(\mathrm{X}=x\middle|\mathrm{Y}=y,\mathrm{Z}=z\right)=\mathrm{P}\left(\mathrm{X}=x\middle|\mathrm{Z}=z\right)$$

$$\mathrm{P}\left(\mathrm{Y}=y\middle|\mathrm{X}=x,\mathrm{Z}=z\right)=\mathrm{P}\left(\mathrm{Y}=y\middle|\mathrm{Z}=z\right)$$

independence and conditional independence

- Independence does not imply conditional independence.
- Conditional independence does not imply independence.



Today Recap: Probability Review

- The big picture
- Events and Event spaces
- Random variables
- Joint probability, Marginalization, conditioning, chain rule, Bayes Rule, law of total probability, etc.
- Structural properties, e.g., Independence, conditional independence
- Maximum Likelihood Estimation (next class)

References

- Prof. Andrew Moore’s review tutorial
- Prof. Nando de Freitas’s review slides
- Prof. Carlos Guestrin recitation slides