Derives the double pendulum E.O.M

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Here is the derivation of double pendulum E.O.M(take the forces into consideration):

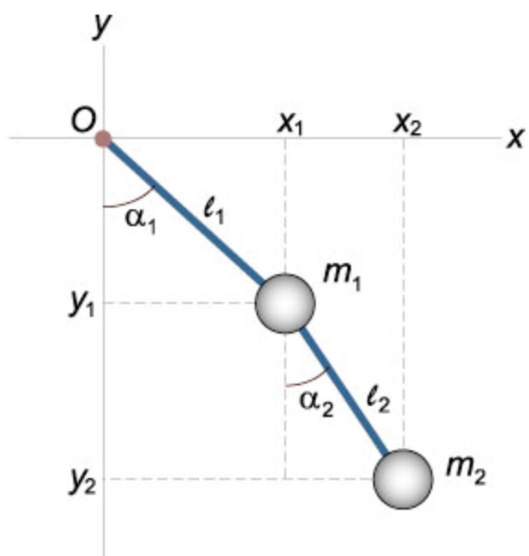

Figure 1: Two Joints

Location:

$$x_1 = l_1 sin\alpha_1, x_2 = l_1 sin\alpha_1 + l_2 sin\alpha_2, y_1 = -l_1 cos\alpha_1, y_2 = -l_1 cos\alpha_1 - l_2 cos\alpha_2 \tag{1}$$

Lagrangian of pendulum bob positions

$$T = \frac{m_1(\dot{x_1}^2 + \dot{y_1}^2)}{2} + \frac{m_2(\dot{x_2}^2 + \dot{y_2}^2)}{2}, V = m_1 g y_1 + m_2 g y_2, L = T - V \tag{2}$$

Calculate $\dot{x}, \dot{y}$

$$\dot{x_1} = l_1 cos\alpha_1 \cdot \alpha_1, \dot{x_2} = l_1 cos\alpha_1 \cdot \dot{\alpha_1} + l_2 cos\alpha_2, \dot{y_1} = l_1 sin\alpha_1 \cdot \alpha_1, \dot{y_2} = l_1 sin\alpha_1 \cdot \alpha_1 + l_2 sin\alpha_2 \cdot \alpha_2 \tag{3}$$

Take $\dot{x}, \dot{y}$ into L = T - V

$$L = (\frac{m_1}{2} + \frac{m_2}{2}) l_1^2 \dot{\alpha_1}^2 + \frac{m_2}{2} l_2^2 \dot{\alpha_2}^2 + m_2 l_1 l_2 \dot{\alpha_1}\dot{\alpha_2} cos(\alpha_1 - \alpha_2) + (m_1 + m_2) g l_1 cos\alpha_1 + m_2 g l_2 cos\alpha_2 \tag{4}$$

Lagrangian equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha_i}} - \frac{\partial L}{\partial \alpha_i} = \tau_i, i = 1, 2 \tag{5}$$

$$\frac{\partial L}{\partial \dot{\alpha_1}} = (m_1+m_2)l_1^2\dot{\alpha_1} + m_2l_1l_2\dot{\alpha_2}cos(\alpha_1-\alpha_2), \frac{\partial L}{\partial \dot{\alpha_2}} = m_2l_2^2\dot{\alpha_2} + m_2l_1l_2\dot{\alpha_1}cos(\alpha_1-\alpha_2) \tag{6}$$

$$\frac{\partial L}{\partial \alpha_1} = -m_2l_1l_2\dot{\alpha_1}\dot{\alpha_2}sin(\alpha_1-\alpha_2) - (m_1+m_2)gl_1sin\alpha_1, \frac{\partial L}{\partial \alpha_2} = m_2l_1l_2\dot{\alpha_1}\dot{\alpha_2}sin(\alpha_1-\alpha_2) - m_2gl_2sin\alpha_2 \tag{7}$$

Lagrangian EOM for i=1

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha_1}} - \frac{\partial L}{\partial \alpha_1} = \tau_1 \tag{8}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha_1}} = (m_1+m_2)l_1^2\ddot{\alpha_1} + m_2l_1l_2(\ddot{\alpha_2}cos(\alpha_1-\alpha_2) - \dot{\alpha_2}sin(\alpha_1-\alpha_2)(\dot{\alpha_1}-\dot{\alpha_2})) \tag{9}$$

$$\frac{\partial L}{\partial \alpha_1} = -m_2l_1l_2\dot{\alpha_1}\dot{\alpha_2}sin(\alpha_1-\alpha_2) - (m_1+m_2)gl_1sin\alpha_1, \tag{10}$$

$$(m_1+m_2)l_1^2\ddot{\alpha_1} + m_2l_1l_2\ddot{\alpha_2}cos(\alpha_1-\alpha_2) + m_2l_1l_2\dot{\alpha_2}^2sin(\alpha_1-\alpha_2) + (m_1+m_2)l_1gsin\alpha_1 = \tau_1 \tag{11}$$

$$(m_1+m_2)l_1\ddot{\alpha_1} + m_2l_2\ddot{\alpha_2}cos(\alpha_1-\alpha_2) + m_2l_2\dot{\alpha_2}^2sin(\alpha_1-\alpha_2) + (m_1+m_2)gsin\alpha_1 = \frac{\tau_1}{l_1} \tag{12}$$

Lagrangian EOM for i=2

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha_2}} - \frac{\partial L}{\partial \alpha_2} = \tau_2 \tag{13}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha_2}} = m_2l_2^2\ddot{\alpha_2} + m_2l_1l_2(\ddot{\alpha_1}cos(\alpha_1-\alpha_2) - \dot{\alpha_1}sin(\alpha_1-\alpha_2)(\dot{\alpha_1}-\dot{\alpha_2})) \tag{14}$$

$$\frac{\partial L}{\partial \alpha_2} = m_2l_1l_2\dot{\alpha_1}\dot{\alpha_2}sin(\alpha_1-\alpha_2) - m_2gl_2sin\alpha_2 \tag{15}$$

$$m_2l_2\ddot{\alpha_2} + m_2l_1\ddot{\alpha_1}cos(\alpha_1-\alpha_2) - m_2l_1\dot{\alpha_1}^2sin(\alpha_1-\alpha_2) + m_2gsin(\alpha_2) = \tau_2 \tag{16}$$

$$l_2\ddot{\alpha_2} + l_1\ddot{\alpha_1}cos(\alpha_1-\alpha_2) - l_1\dot{\alpha_1}^2sin(\alpha_1-\alpha_2) + gsin\alpha_2 = \frac{\tau_2}{m_2} \tag{17}$$

First, calculate $\ddot{\alpha_1}$, use $(12) - (17) * m_2cos(\alpha_1-\alpha_2)$

$$\begin{aligned}(m_1+m_2)l_1\ddot{\alpha_1} + m_2l_2\ddot{\alpha_2}cos(\alpha_1-\alpha_2) + m_2l_2\dot{\alpha_2}^2sin(\alpha_1-\alpha_2) + (m_1+m_2)gsin\alpha_1 - (m_2l_2\ddot{\alpha_2}cos(\alpha_1-\alpha_2)\\ +m_2l_1\ddot{\alpha_1}cos(\alpha_1-\alpha_2)cos(\alpha_1-\alpha_2) - m_2l_1\dot{\alpha_1}^2sin(\alpha_1-\alpha_2)cos(\alpha_1-\alpha_2) + m_2gsin\alpha_2cos(\alpha_1-\alpha_2))\\ = \frac{\tau_1}{l_1} - m_2cos(\alpha_1-\alpha_2)\frac{\tau_2}{m_2}\end{aligned} \tag{18}$$

$$\begin{aligned}\ddot{\alpha_1}((m_1+m_2)l_1 - m_2l_1cos(\alpha_1-\alpha_2)cos(\alpha_1-\alpha_2)) = \frac{\tau_1}{l_1} - cos(\alpha_1-\alpha_2)\tau_2\\ -m_2l_1\dot{\alpha_1}^2sin(\alpha_1-\alpha_2)cos(\alpha_1-\alpha_2) + m_2gsin\alpha_2cos(\alpha_1-\alpha_2) - m_2l_2\dot{\alpha_2}^2sin(\alpha_1-\alpha_2) - (m_1+m_2)gsin\alpha_1\end{aligned} \tag{19}$$

$$\begin{aligned}\ddot{\alpha_1} = (\frac{\tau_1}{l_1} - cos(\alpha_1-\alpha_2)\tau_2 - m_2l_1\dot{\alpha_1}^2sin(\alpha_1-\alpha_2)cos(\alpha_1-\alpha_2) + m_2gsin\alpha_2cos(\alpha_1-\alpha_2)\\ -m_2l_2\dot{\alpha_2}^2sin(\alpha_1-\alpha_2) - (m_1+m_2)gsin\alpha_1)/((m_1+m_2)l_1 - m_2l_1cos(\alpha_1-\alpha_2)cos(\alpha_1-\alpha_2))\end{aligned} \tag{20}$$

Second, use$\ddot{\alpha_1}$ to calculate $\ddot{\alpha_2}$ from (17)

$$\ddot{\alpha_2} = \frac{\frac{\tau_2}{m_2} - (l_1\ddot{\alpha_1}cos(\alpha_1-\alpha_2) - l_1\dot{\alpha_1}^2sin(\alpha_1-\alpha_2) + gsin\alpha_2)}{l_2} \tag{21}$$

As a result, $\ddot{\alpha_1}$,$\ddot{\alpha_2}$ are as follows:

$$\ddot{\alpha_1} = (\frac{\tau_1}{l_1} - cos(\alpha_1 - \alpha_2)\tau_2 - m_2 l_1 \dot{\alpha_1}^2 sin(\alpha_1 - \alpha_2) cos(\alpha_1 - \alpha_2) + m_2 g sin\alpha_2 cos(\alpha_1 - \alpha_2)$$
$$-m_2 l_2 \dot{\alpha_2}^2 sin(\alpha_1 - \alpha_2) - (m_1 + m_2) g sin\alpha_1)/((m_1 + m_2) l_1 - m_2 l_1 cos(\alpha_1 - \alpha_2) cos(\alpha_1 - \alpha_2)) \tag{22}$$

$$\ddot{\alpha_2} = \frac{\frac{\tau_2}{m_2} - (l_1 \ddot{\alpha_1} cos(\alpha_1 - \alpha_2) - l_1 \dot{\alpha_1}^2 sin(\alpha_1 - \alpha_2) + g sin\alpha_2)}{l_2} \tag{23}$$

In the old JavaScript document, the acceleration is give as follows, I think a l_1 is miss there, I also add the force, which is also not in the old JavaScript document.

```
var a1 =
    ( -m2 * Math.cos(x1-x2) * l1* v1*v1 * Math.sin(x1-x2)
    + m2 * Math.cos(x1-x2) * g * Math.sin(x2)
    - m2 * l2 * v2*v2 * Math.sin(x1-x2)
    - (m1+m2) * g * Math.sin(x1) )
    / (l1 * (m1+m2)
    - m2 * Math.cos(x1-x2) * Math.cos(x1-x2) );
```

Figure 2: old version