CHAPTER 6 MODEL SPECIFICATION

6.1 Properties of the Sample Autocorrelation Function

6.2 The Partial and Extended Autocorrelation Functions

6.3 Specification of Some Simulated Time Series

6.4 Nonstationarity

6.5 Other Specification Methods

6.6 Specification of Some Actual Time Series

6.7 Summary

CHAPTER 6 MODEL SPECIFICATION

statistical inference for the ARIMA models

1. How to ***choose*** appropriate values for *p*, *d*, and *q* for a given series (Ch 6);

2. How to ***estimate*** the parameters of a specific ARIMA(*p*,*d*,*q*) model (Ch 7);

3. How to ***check*** on the appropriateness of the fitted model and improve it if needed (Ch 8).

model-building strategy

6.1 Properties of the Sample Autocorrelation Function

For General MA(*q*) Process

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

Can we use the "cut off" characteristic of autocorrelation function to determine q for MA process? Since theta are unknown, how to compute rho? ***Sample Autocorrelation Function***?

6.1 Properties of the Sample Autocorrelation Function

Definition of the sample autocorrelation function:

For the observed series $Y_1, Y_2, \dots, Y_n$, we have

$$r_k = \frac{\sum_{t=k+1}^{n} (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n} (Y_t - \bar{Y})^2} \qquad \text{for } k = 1, 2, \dots$$

Why do we need to investigation the ***sampling properties*** of sample autocorrelation function?

6.1 Properties of the Sample Autocorrelation Function

large-sample results of sampling properties

We suppose that $Y_t = \mu + \sum_{j=0}^{\infty} \psi_j e_{t-j}$

With $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $\sum_{j=0}^{\infty} j\psi_j^2 < \infty$

6.1 Properties of the Sample Autocorrelation Function

For any fixed *m*, the joint distribution of

$$\sqrt{n}(r_1 - \rho_1), \sqrt{n}(r_2 - \rho_2), \ldots, \sqrt{n}(r_m - \rho_m)$$

approaches, as $n \to \infty$, a joint normal distribution with zero means, variances c_{jj}, and covariances c_{ij}, where

$$c_{ij} = \sum_{k=-\infty}^{\infty} (\rho_{k+i}\rho_{k+j} + \rho_{k-i}\rho_{k+j} - 2\rho_i\rho_k\rho_{k+j} - 2\rho_j\rho_k\rho_{k+i} + 2\rho_i\rho_j\rho_k^2)$$

For large *n*, we would say that r_k is approximately normally distributed with mean ρ_k and variance c_{kk}/n

6.1 Properties of the Sample Autocorrelation Function

general MA(q) process:

$$c_{kk} = 1 + 2\sum_{j=1}^{q} \rho_j^2 \quad \text{for } k > q$$

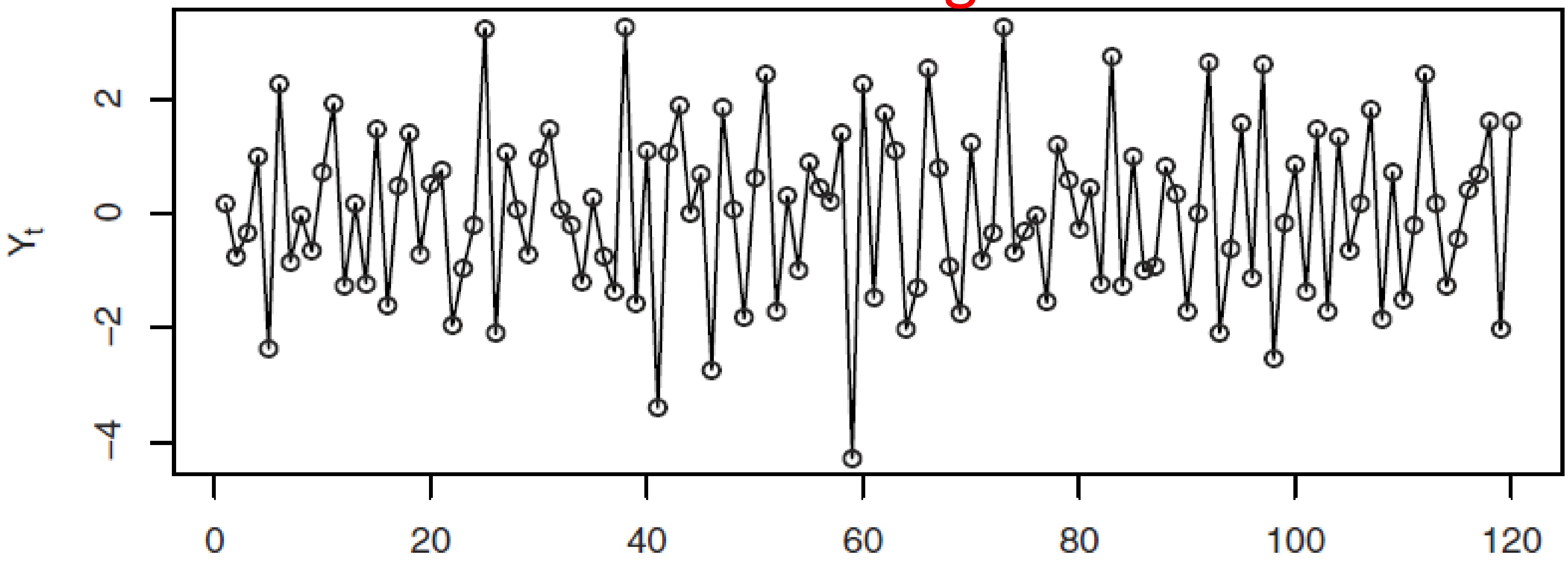
$$Var(r_k) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{q} \rho_j^2\right] \quad \text{for } k > q$$

A test of the ***hypothesis*** that the series is MA(q) could be carried out by comparing r_k to ***plus and minus two standard errors***. We would reject the null hypothesis if and only if r_k lies outside these bounds.

6.3 Specification of Some Simulated Time Series

Exhibit 4.5 Time Plot of an MA(1) Process with $\theta = +0.9$

theoretical autocorrelation at lag 1 is –0.4972

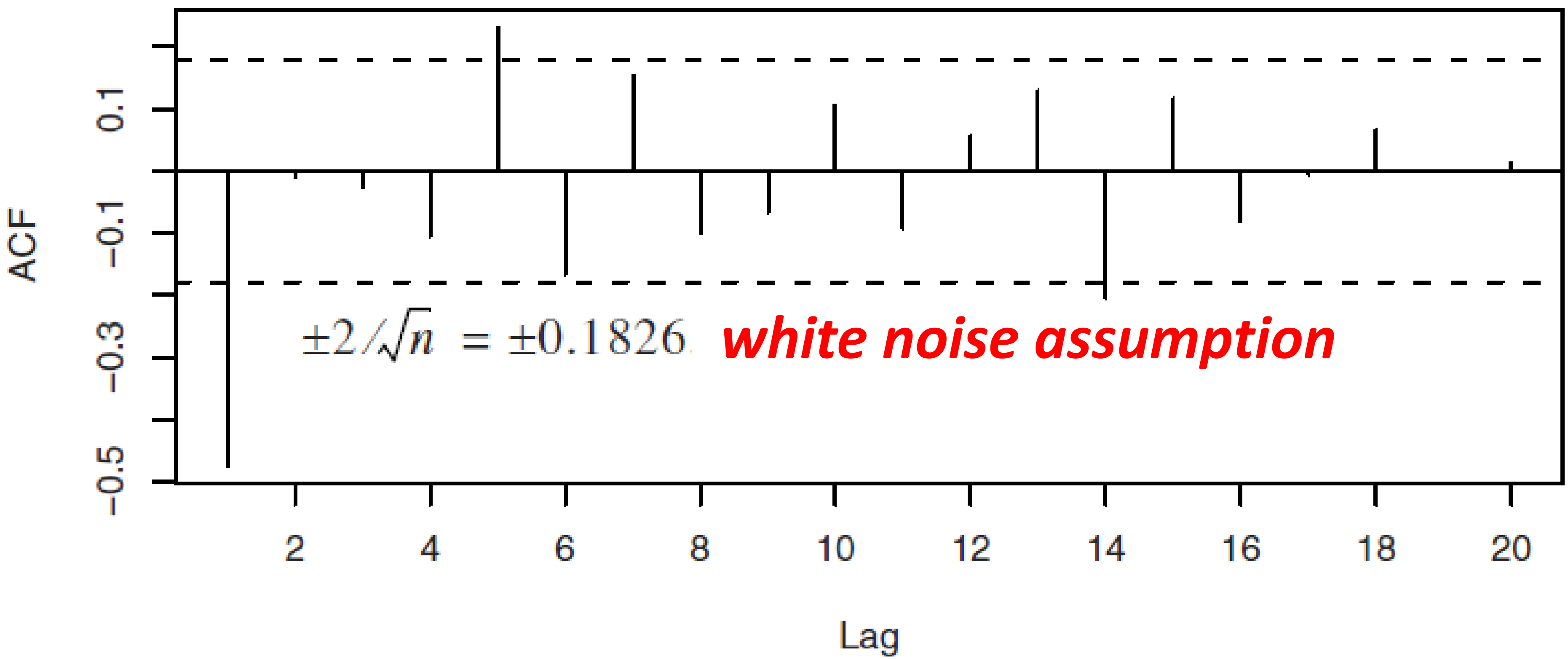


sample autocorrelation at lag 1 is –0.474

```
> win.graph(width=4.875,height=3,pointsize=8)
> data(ma1.1.s)
> plot(ma1.1.s,ylab=expression(Y[t]),type='o')
```

6.3 Specification of Some Simulated Time Series

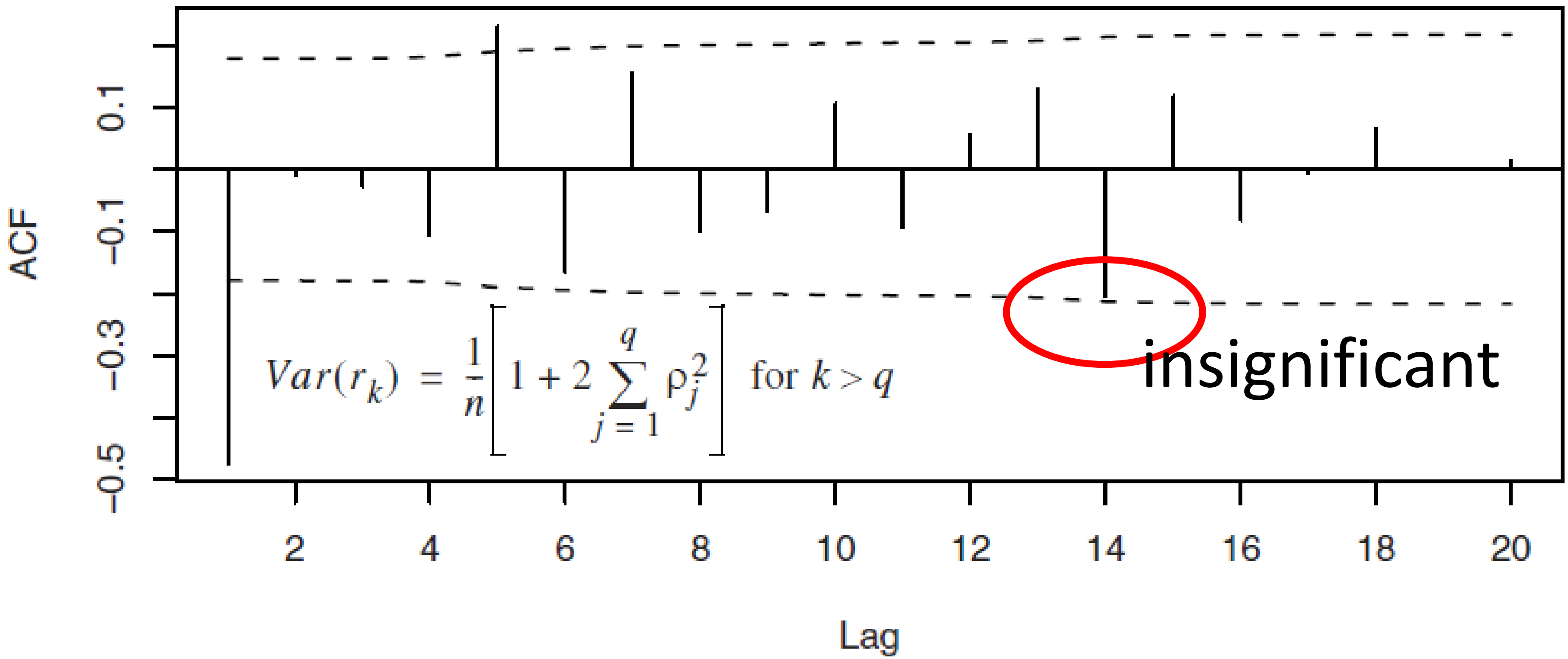
Exhibit 6.5 Sample Autocorrelation of an MA(1) Process with $\theta = 0.9$




```
> data(ma1.1.s)
> win.graph(width=4.875,height=3,pointsize=8)
> acf(ma1.1.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

Exhibit 6.6 **Alternative Bounds for the Sample ACF for the MA(1) Process**

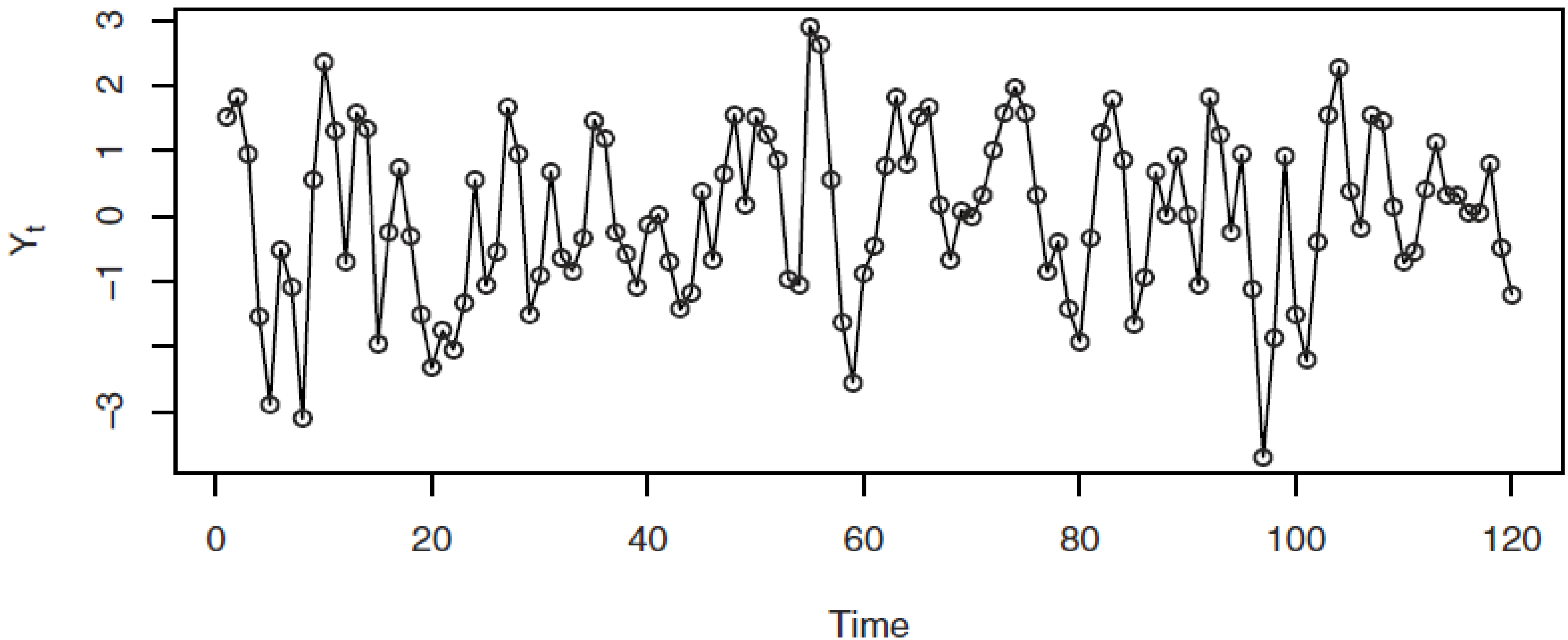


$$Var(r_k) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{q}\rho_j^2\right] \quad \text{for } k > q$$

```
> acf(ma1.1.s,ci.type='ma',xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

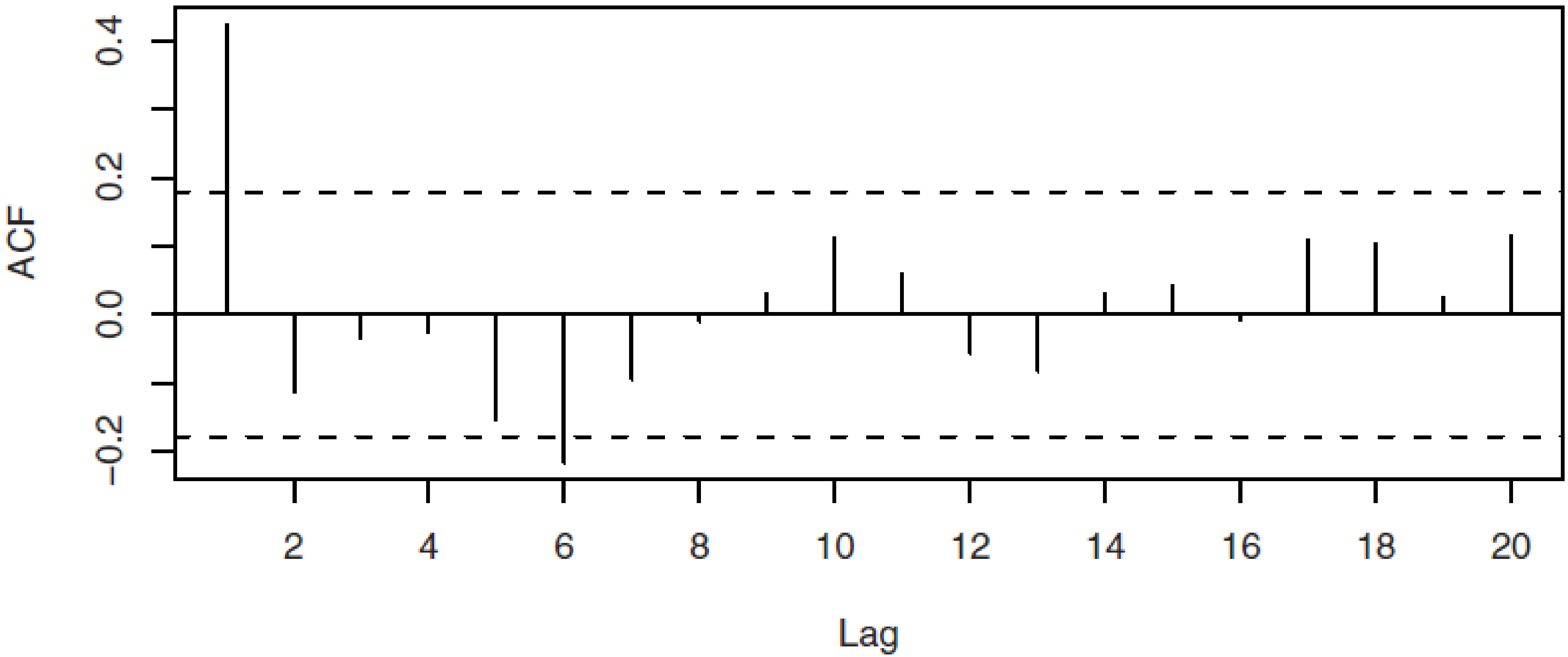
Exhibit 4.2 Time Plot of an MA(1) Process with $\theta = -0.9$




```
> win.graph(width=4.875,height=3,pointsize=8)
> data(ma1.2.s); plot(ma1.2.s,ylab=expression(Y[t]),type='o')
```

6.3 Specification of Some Simulated Time Series

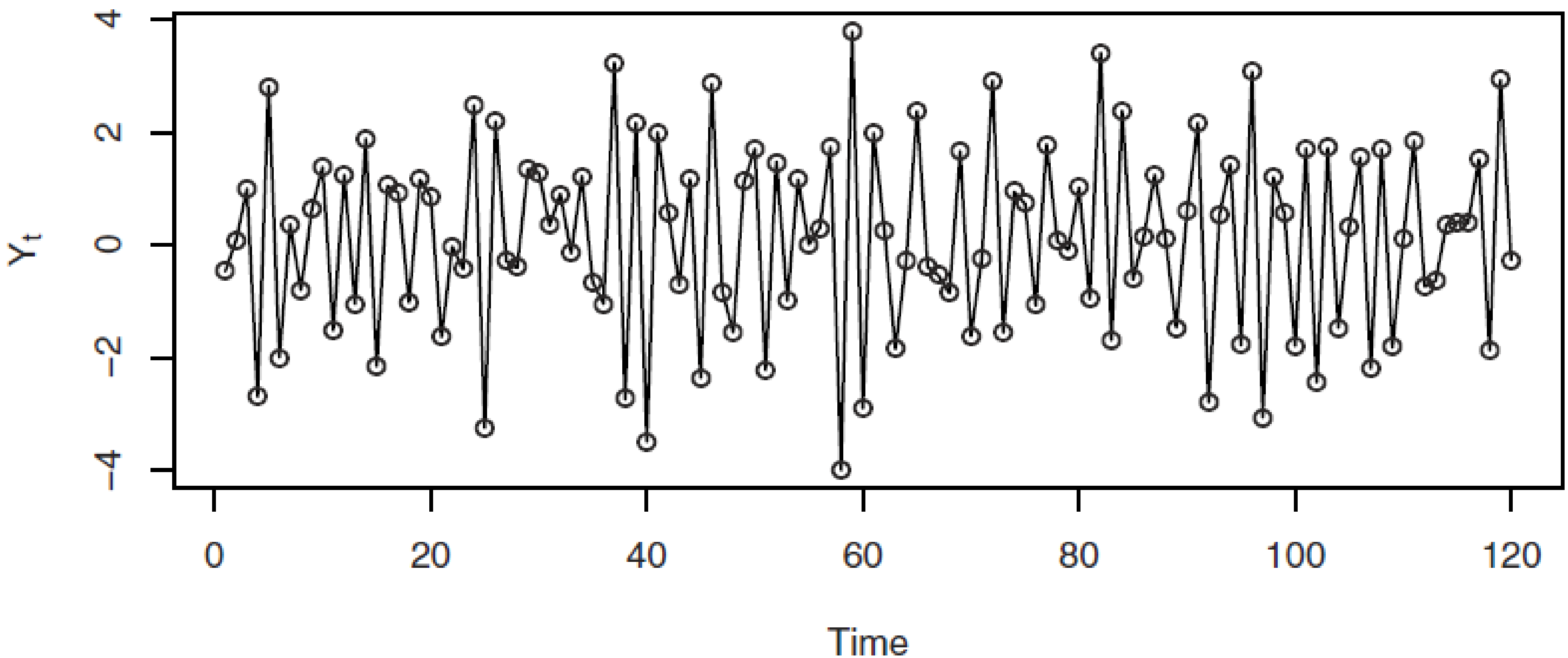
Exhibit 6.7 Sample Autocorrelation for an MA(1) Process with $\theta = -0.9$




```
> data(ma1.2.s); acf(ma1.2.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

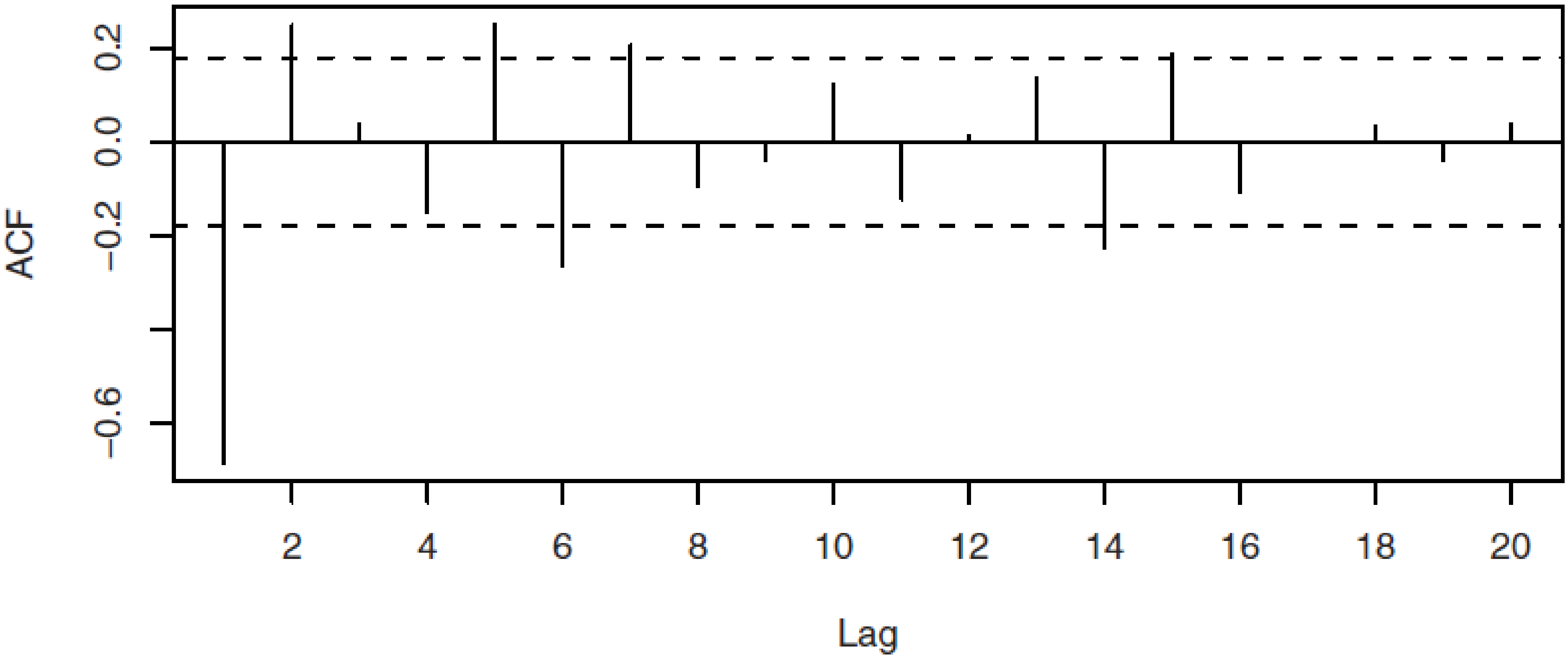
Exhibit 4.8 **Time Plot of an MA(2) Process with $\theta_1 = 1$ and $\theta_2 = -0.6$**




```
> win.graph(width=4.875, height=3,pointsize=8)
> data(ma2.s); plot(ma2.s,ylab=expression(Y[t]),type='o')
```

6.3 Specification of Some Simulated Time Series

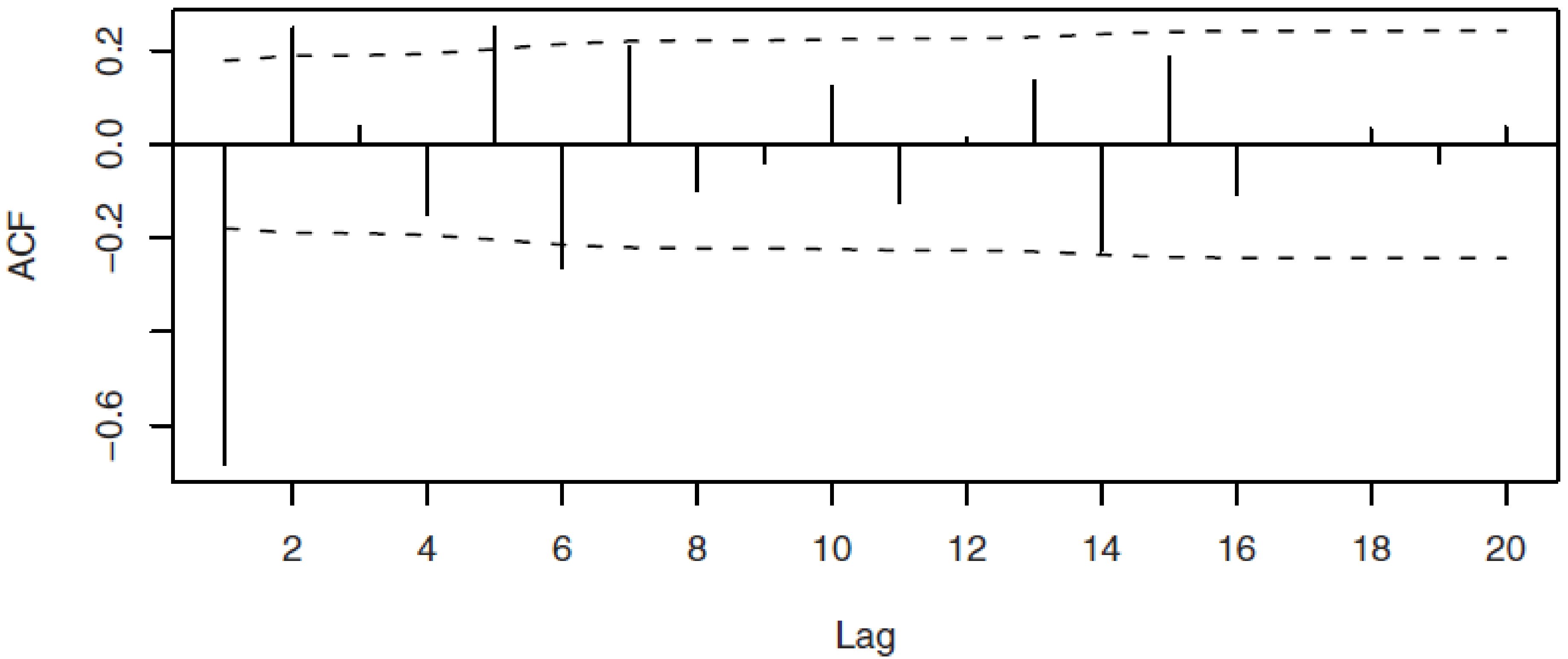
Exhibit 6.8 Sample ACF for an MA(2) Process with $\theta_1 = 1$ and $\theta_2 = -0.6$




```
> data(ma2.s); acf(ma2.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

Exhibit 6.9 **Alternative Bounds for the Sample ACF for the MA(2) Process**

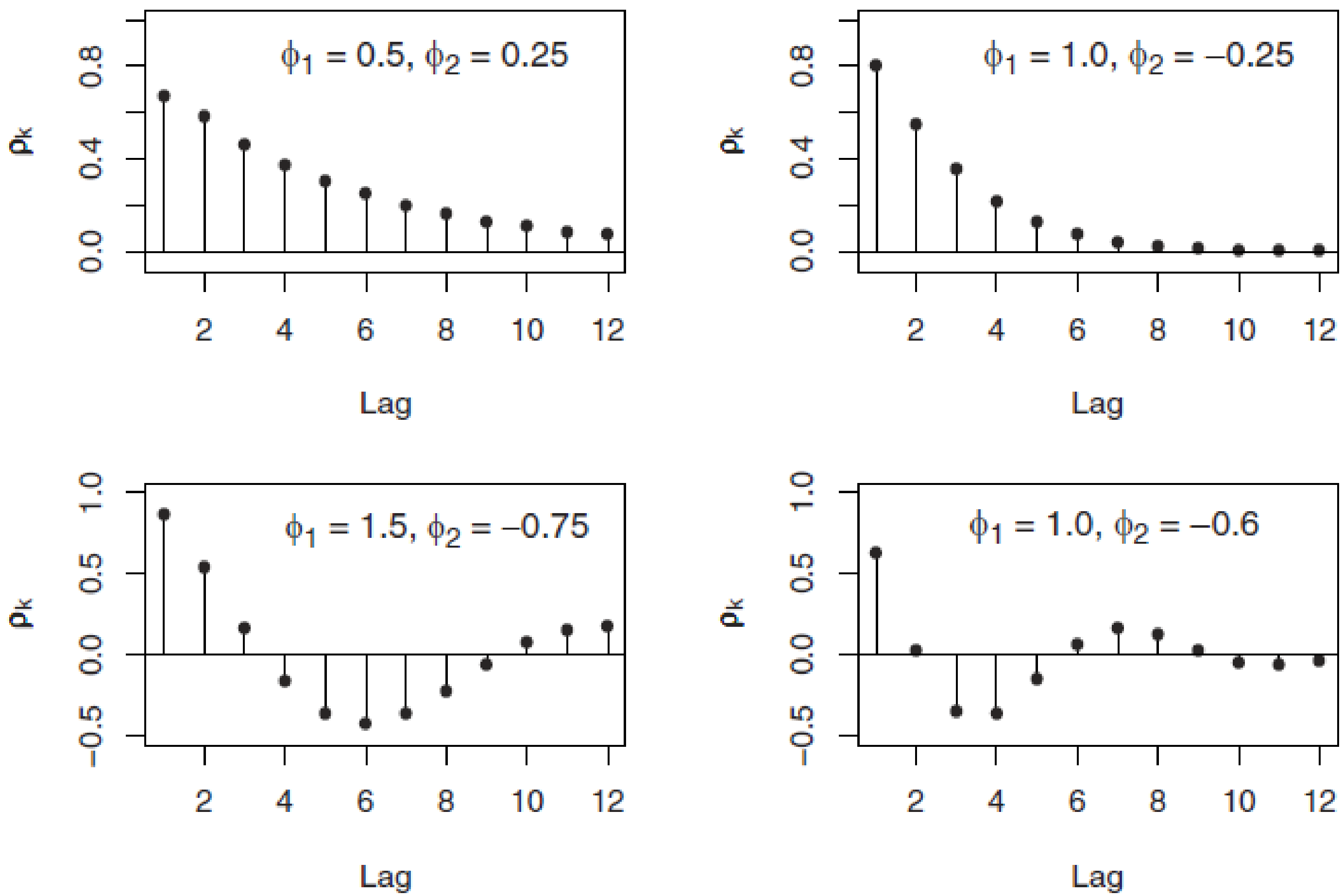



```
> acf(ma2.s,ci.type='ma',xaxp=c(0,20,10))
```

6.2 The Partial and Extended Autocorrelation Functions

The autocorrelations of an AR(p) model do not become ***zero*** after a certain number of lags—***they die off rather than cut off.***

Exhibit 4.18 Autocorrelation Functions for Several AR(2) Models



6.2 The Partial and Extended Autocorrelation Functions

Autocorrelation Functions does not work well.

partial autocorrelation (ϕ_{kk})***:***

correlation between Y_t and Y_{t-k} *after removing the effect of the intervening variables* $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots, Y_{t-k+1}$.

If $\{Y_t\}$ is a normally distributed time series

$$\phi_{kk} = Corr(Y_t, Y_{t-k} | Y_{t-1}, Y_{t-2}, \ldots, Y_{t-k+1})$$

6.2 The Partial and Extended Autocorrelation Functions

Consider predicting Y_t based on a linear function of the intervening variables $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-k+1}$, say, $\beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_{k-1} Y_{t-k+1}$, with the β's chosen to minimize the mean square error of prediction.

stationarity

the best "predictor" of Y_{t-k} based on the same $Y_{t-1}, Y_{t-2}, \ldots, Y_{t-k+1}$ will be $\beta_1 Y_{t-k+1} + \beta_2 Y_{t-k+2} + \cdots + \beta_{k-1} Y_{t-1}$.

$$\phi_{kk} = Corr(Y_t - \beta_1 Y_{t-1} - \beta_2 Y_{t-2} - \cdots - \beta_{k-1} Y_{t-2},$$
$$Y_{t-k} - \beta_1 Y_{t-k+1} - \beta_2 Y_{t-k+2} - \cdots - \beta_{k-1} Y_{t-1})$$

6.2 The Partial and Extended Autocorrelation Functions

By convention, we take $\Phi_{11} = \rho_1$

As an example, consider ϕ_{22}.

$$Cov(Y_t - \rho_1 Y_{t-1}, Y_{t-2} - \rho_1 Y_{t-1}) = \gamma_0(\rho_2 - \rho_1^2 - \rho_1^2 + \rho_1^2) = \gamma_0(\rho_2 - \rho_1^2)$$

$$\begin{aligned} Var(Y_t - \rho_1 Y_{t-1}) &= Var(Y_{t-2} - \rho_1 Y_{t-1}) \\ &= \gamma_0(1 + \rho_1^2 - 2\rho_1^2) \\ &= \gamma_0(1 - \rho_1^2) \end{aligned}$$

For any stationary process

$$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

6.2 The Partial and Extended Autocorrelation Functions

Consider now an AR(1) model. Recall that $\rho_k = \phi^k$ so that

$$\phi_{22} = \frac{\phi^2 - \phi^2}{1 - \phi^2} = 0$$

Consider a general AR(p) process.

$k>p$

$$Cov(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p},$$
$$Y_{t-k} - h(Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_{t-1}))$$

$$= Cov(e_t, Y_{t-k} - h(Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_{t-1}))$$

$= 0$ since e_t is independent of $Y_{t-k}, Y_{t-k+1}, Y_{t-k+2}, \ldots, Y_{t-1}$

$$\phi_{kk} = 0 \text{ for } k > p$$

6.2 The Partial and Extended Autocorrelation Functions

For an MA(1) model

$$\phi_{22} = \frac{-\theta^2}{1+\theta^2+\theta^4} \qquad \phi_{kk} = -\frac{\theta^k(1-\theta^2)}{1-\theta^{2(k+1)}} \quad \text{for } k \geq 1$$

The partial autocorrelation of an MA(1) model **never equals zero** but **essentially decays to zero exponentially fast as the lag increases**—rather like the autocorrelation function of the AR(1) process. More generally, the partial autocorrelation of an MA(*q*) model behaves very much like the *autocorrelation* of an AR(*q*) model.

6.2 The Partial and Extended Autocorrelation Functions

A general method for finding the partial autocorrelation function for any ***stationary process*** with autocorrelation functic ρ_k

For a given lag *k*, it can be shown that the ϕ_{kk} satisfy the Yule-Walker equations:

$$\rho_j = \phi_{k1}\rho_{j-1} + \phi_{k2}\rho_{j-2} + \phi_{k3}\rho_{j-3} + \cdots + \phi_{kk}\rho_{j-k} \quad \text{for } j = 1, 2, \ldots, k$$

$$r_{yx1.X} = \beta_{x1}\sqrt{\frac{\operatorname{var}(e_{x1\leftarrow X})}{\operatorname{var}(e_{y\leftarrow X})}},$$

6.2 The Partial and Extended Autocorrelation Functions

A general method for finding the partial autocorrelation function for any ***stationary process*** with autocorrelation functic ρ_k

For a given lag *k*, it can be shown that the ϕ_{kk} satisfy the Yule-Walker equations:

$$\rho_j = \phi_{k1}\rho_{j-1} + \phi_{k2}\rho_{j-2} + \phi_{k3}\rho_{j-3} + \cdots + \phi_{kk}\rho_{j-k} \quad \text{for } j = 1, 2, \ldots, k$$

$$\boxed{\left.\begin{array}{rrrrr}
\phi_{k1} + & \rho_1\phi_{k2} + & \rho_2\phi_{k3} & + \cdots + \rho_{k-1}\phi_{kk} = & \rho_1 \\
\rho_1\phi_{k1} + & \phi_{k2} + & \rho_1\phi_{k3} & + \cdots + \rho_{k-2}\phi_{kk} = & \rho_2 \\
\vdots & & & & \\
\rho_{k-1}\phi_{k1} + & \rho_{k-2}\phi_{k2} + & \rho_{k-3}\phi_{k3} & + \cdots + \phantom{\rho_{k-1}}\phi_{kk} = & \rho_k
\end{array}\right\}}$$

6.2 The Partial and Extended Autocorrelation Functions

A general method for finding the partial autocorrelation function for any ***stationary process*** with autocorrelation functicρ_k

For a given lag *k*, it can be shown that the ϕ_{kk} satisfy the Yule-Walker equations:

$$\rho_j = \phi_{k1}\rho_{j-1} + \phi_{k2}\rho_{j-2} + \phi_{k3}\rho_{j-3} + \cdots + \phi_{kk}\rho_{j-k} \quad \text{for } j = 1, 2, \ldots, k$$

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix}$$

6.2.1 The Sample Partial Autocorrelation Function

sample partial autocorrelation function (sample PACF): denote by $\hat{\phi}_{kk}$

recursive equations

$$\phi_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1} \phi_{k-1,j}\, \rho_{k-j}}{1 - \sum_{j=1}^{k-1} \phi_{k-1,j}\, \rho_j}$$

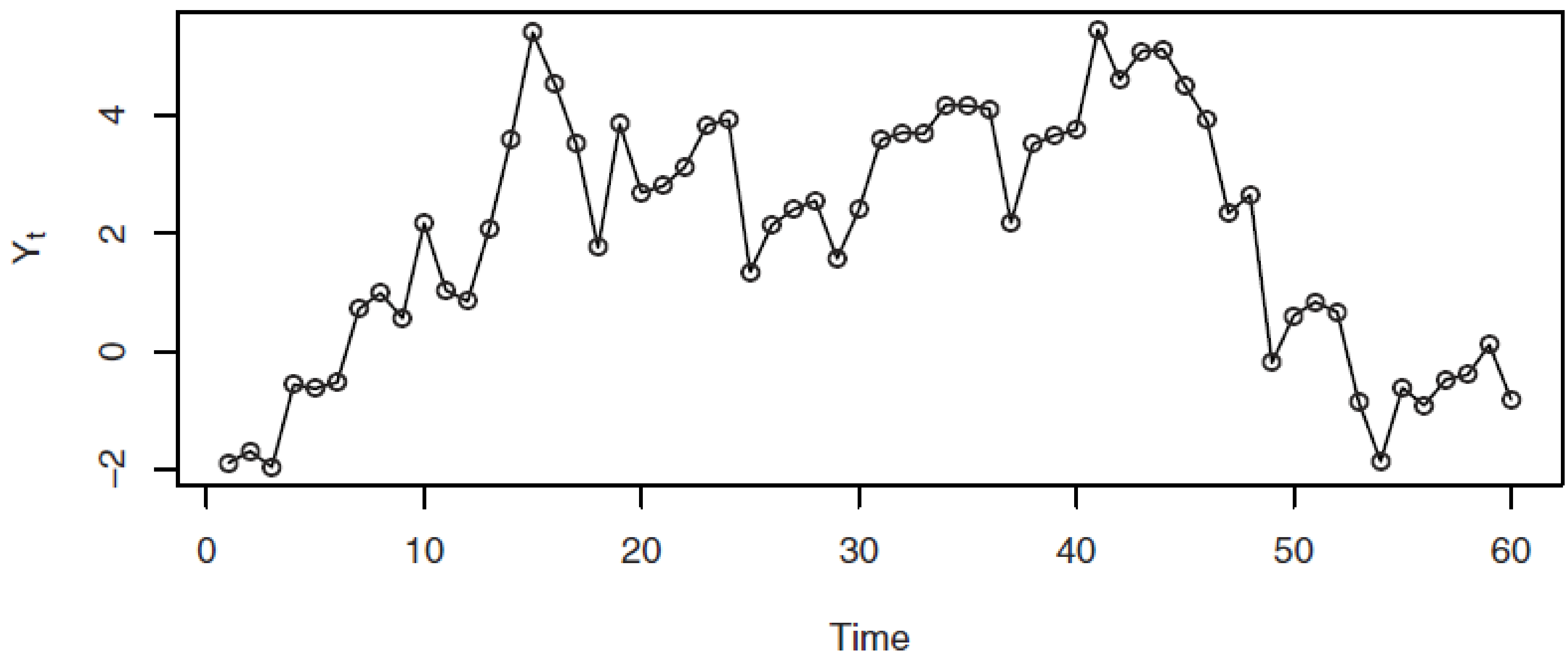
where $\phi_{k,j} = \phi_{k-1,j} - \phi_{kk}\phi_{k-1,k-j}$ for $j = 1, 2, \ldots, k-1$

6.2.1 The Sample Partial Autocorrelation Function

To assess the possible magnitude of the sample partial autocorrelations, Quenoulle (1949) has shown that, **under the hypothesis that an AR(*p*) model is correct**, the sample partial autocorrelations ***at lags greater than p are approximately normally distributed with zero means and variances 1/n.*** Thus, for $k > p$, $\pm 2/\sqrt{n}$ can be used as critical limits on to test the null hypothesis that an AR(p) model is correct.

6.3 Specification of Some Simulated Time Series

Exhibit 4.13 Time Plot of an AR(1) Series with $\phi = 0.9$

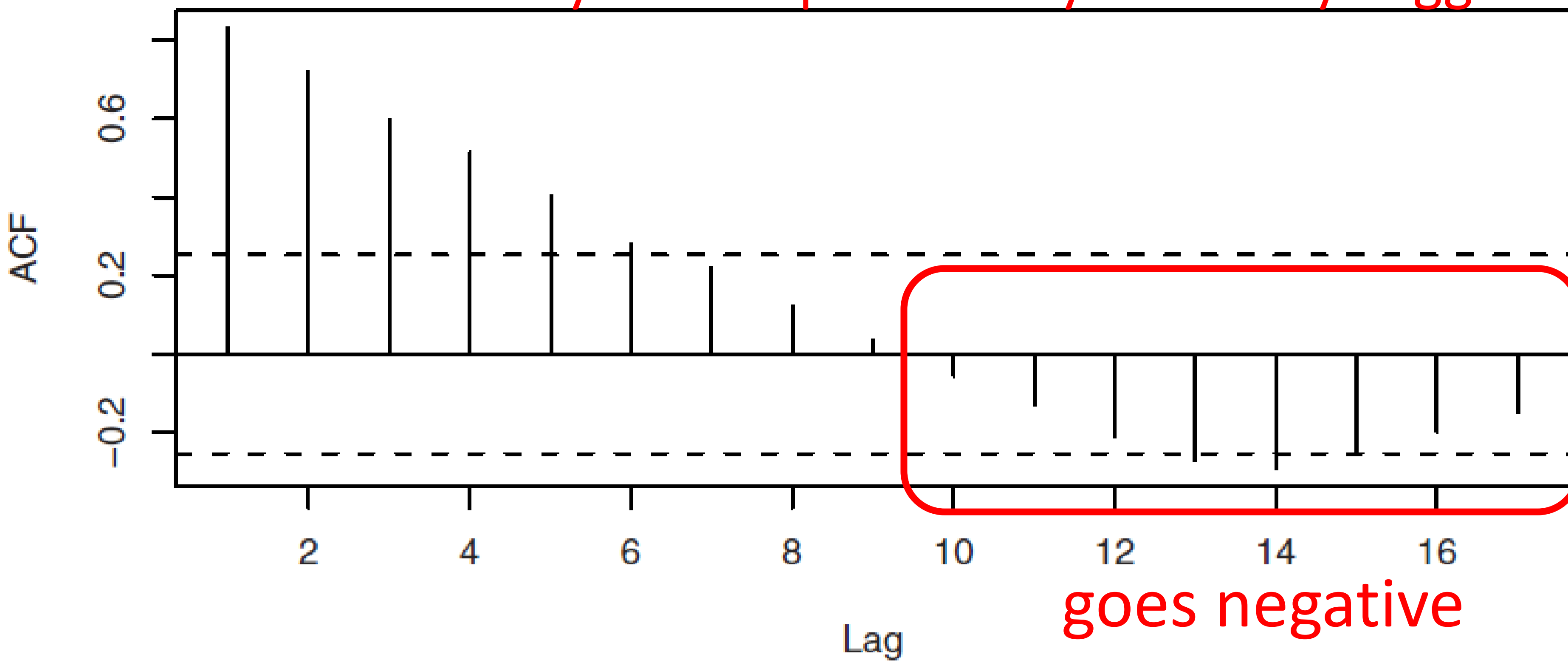



```
> win.graph(width=4.875, height=3,pointsize=8)
> data(ar1.s); plot(ar1.s,ylab=expression(Y[t]),type='o')
```

6.3 Specification of Some Simulated Time Series

Exhibit 6.10 Sample ACF for an AR(1) Process with $\phi = 0.9$

decreases more linearly than exponentially as theory suggests

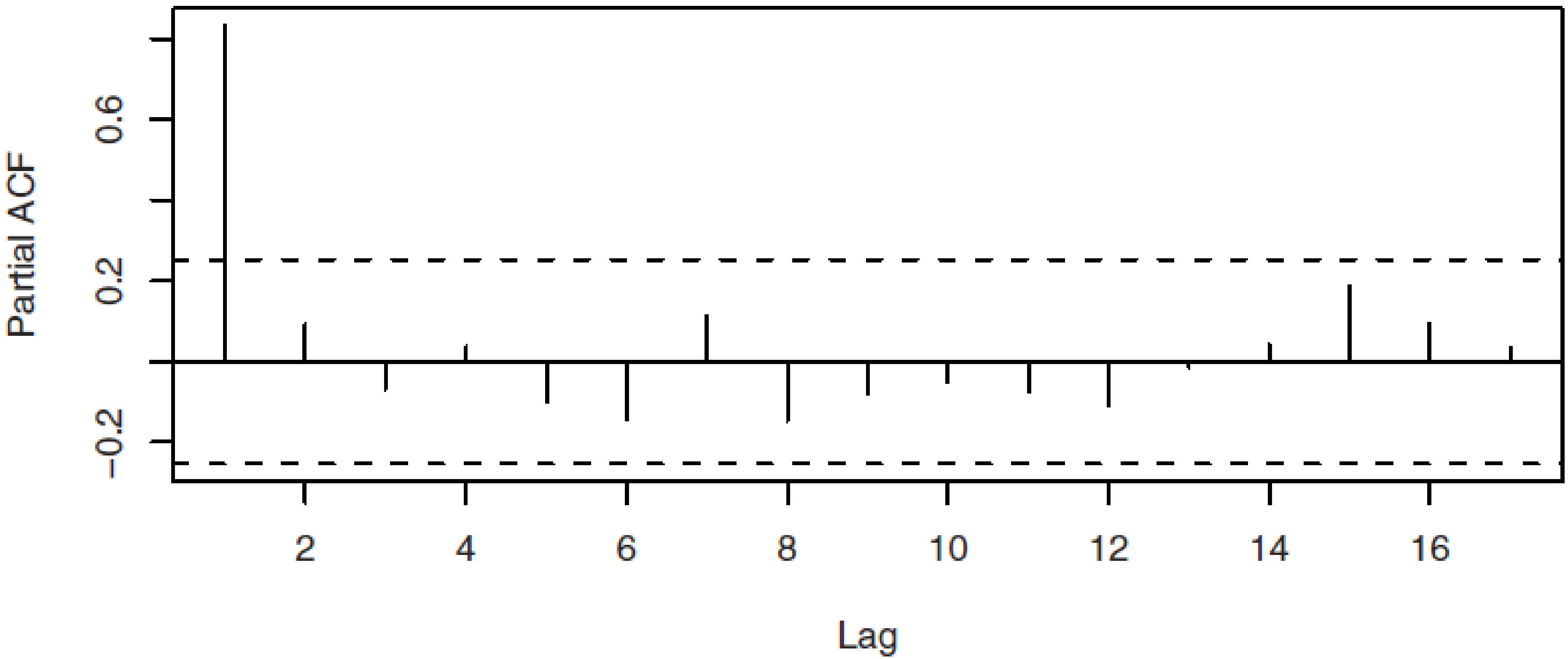


goes negative

```
> data(ar1.s); acf(ar1.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

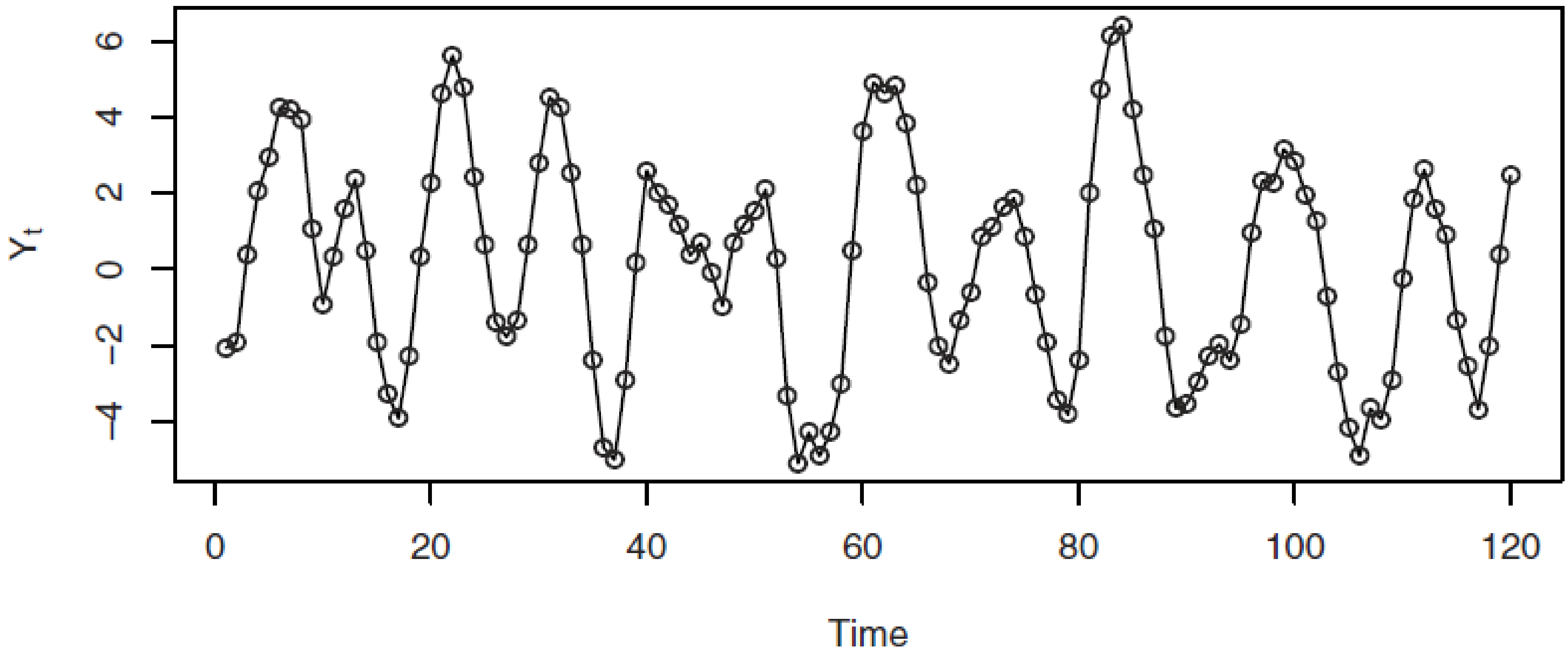
Exhibit 6.11 Sample Partial ACF for an AR(1) Process with $\phi = 0.9$




```
> pacf(ar1.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

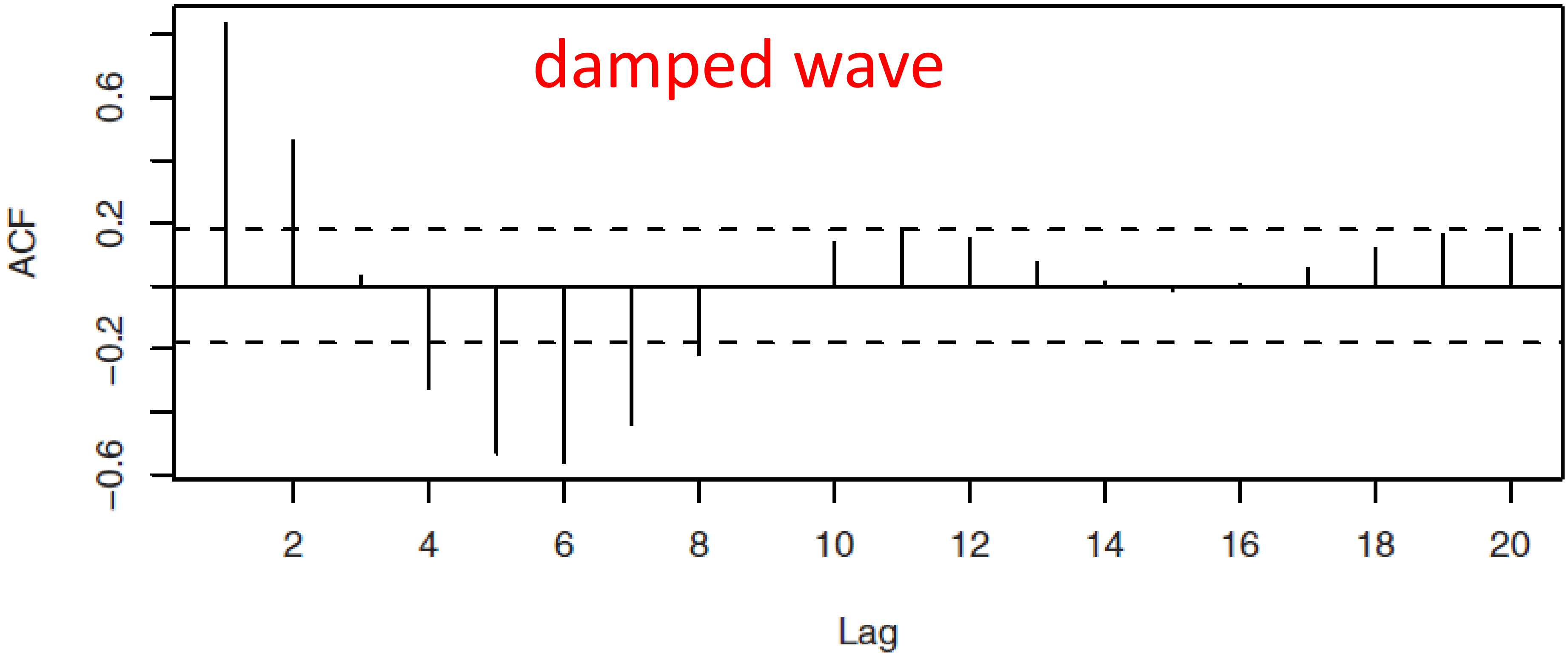
Exhibit 4.19 Time Plot of an AR(2) Series with $\phi_1 = 1.5$ and $\phi_2 = -0.75$




```
> win.graph(width=4.875,height=3,pointsize=8)
> data(ar2.s); plot(ar2.s,ylab=expression(Y[t]),type='o')
```

6.3 Specification of Some Simulated Time Series

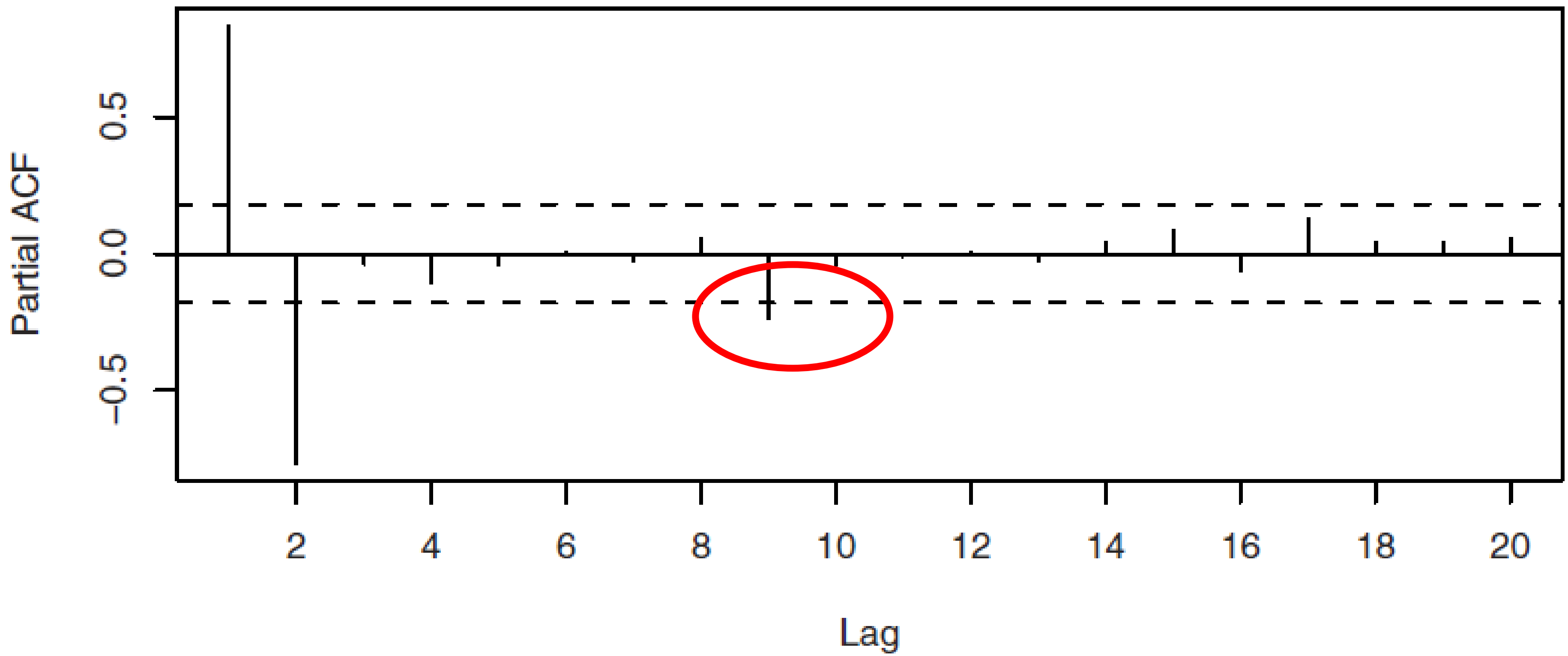
Exhibit 6.12 Sample ACF for an AR(2) Process with $\phi_1 = 1.5$ and $\phi_2 = -0.75$




```
> acf(ar2.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

Exhibit 6.13 Sample PACF for an AR(2) Process with $\phi_1 = 1.5$ and $\phi_2 = -0.75$




```
> pacf(ar2.s,xaxp=c(0,20,10))
```

6.2.2 Mixed Models and the Extended Autocorrelation Function

Exhibit 6.3 General Behavior of the ACF and PACF for ARMA Models

	AR(p)	MA(q)	ARMA(p,q), p>0, and q>0
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

The sample ACF and PACF provide effective tools for identifying pure AR(p) or MA(q) models. However, for a mixed ARMA model, its theoretical ACF and PACF have ***infinitely many nonzero values***, making it difficult to identify mixed models from the sample ACF and PACF.

6.2.2 Mixed Models and the *Extended Autocorrelation Function*

The extended autocorrelation (EACF) method uses the fact that if ***the AR part of a mixed ARMA model is known, "filtering out" the autoregression from the observed time series results in a pure MA process*** that enjoys the cutoff property in its ACF. The AR coefficients may be estimated by ***a finite sequence of regressions***

6.2.2 Mixed Models and the Extended Autocorrelation Function

We illustrate the procedure for the case where the true model is an ARMA(1,1) model:

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

Setp 1 linear regression of Y_t on Y_{t-1} inconsistent

Setp 2 linear regression

Y_t on Y_{t-1} and on the lag 1 of the residuals

Define $W_t = Y_t - \tilde{\phi} Y_{t-1}$, which is then approximately an MA(1) process.

6.2.2 Mixed Models and the Extended Autocorrelation Function

For an ARMA(1,2) model, a third regression that regresses Y_t on ***its lag 1, the lag 1 of the residuals from the second regression, and the lag 2 of the residuals from the first regression*** leads to the coefficient of Y_{t-1} being a consistent estimator of ϕ. Similarly, the AR coefficients of an ARMA(p,q) model can be consistently estimated via ***a sequence of q regressions.***

Consistent estimates of autoregressive parameters and extended sample autocorrelation function for stationary and nonstationary ARMA models. JASA, 1984.

ARMA(p, q) process

$$Z_t = \sum_{l=1}^{p} \Phi_l Z_{t-l} - \sum_{j=1}^{q} \theta_j a_{t-j} + a_t.$$

$$Z_t = \sum_{l=1}^{k} \Phi_{l(k)}{}^{(j)} Z_{t-l} + \sum_{i=1}^{j} \beta_{i(k)}{}^{(j)} \hat{e}_{k,t-i}{}^{(j-i)} + e_{k,t}{}^{(j)},$$

$t = k + j + 1, \ldots, n;\ \ j = 0, \ldots;$

where

$$\hat{e}_{k,t}{}^{(i)} = Z_t - \sum_{l=1}^{k} \hat{\Phi}_{l(k)}{}^{(i)} Z_{t-l} - \sum_{h=1}^{i} \hat{\beta}_{h(k)}{}^{(i)} \hat{e}_{k,t-h}{}^{(i-h)}$$

Table 1. The ESACF Table

	MA					
AR	0	1	2	3	·	·
0	$r_{1(0)}$	$r_{2(0)}$	$r_{3(0)}$	$r_{4(0)}$	·	·
1	$r_{1(1)}$	$r_{2(1)}$	$r_{3(1)}$	$r_{4(1)}$	·	·
2	$r_{1(2)}$	$r_{2(2)}$	$r_{3(2)}$	$r_{4(2)}$	·	·
3	$r_{1(3)}$	$r_{2(3)}$	$r_{3(3)}$	$r_{4(3)}$	·	·
·	·	·	·	·	·	·
·	·	·	·	·	·	·

3. In general, for any nonnegative integer k we define the kth ESACF of Z_t as

$$r_{j(k)} = r_j(W_{k,t}{}^{(j)}) \tag{3.4}$$

Table 3. The ESACF of Series C

a. The ESACF Table

AR	MA 0	1	2	3	4	5	6	7	8
0	.98	.94	.90	.85	.80	.75	.69	.64	.58
1	.81	.66	.55	.48	.43	.38	.34	.28	.25
2	−.04	−.03	−.12	−.06	.02	−.01	.07	−.04	−.12
3	−.50	.01	−.07	−.11	−.01	.00	.03	−.03	−.10
4	−.25	−.27	−.05	−.11	−.01	.03	.00	−.02	−.09
5	−.48	.28	−.29	−.07	.04	−.05	−.00	−.01	−.08

b. The Indicator Symbols

AR	MA 0	1	2	3	4	5	6	7	8
0	X	X	X	X	X	X	X	X	X
1	X	X	X	X	X	X	X	X	X
2	0	0	0	0	0	0	0	0	0
3	X	0	0	0	0	0	0	0	0
4	X	X	0	0	0	0	0	0	0
5	X	X	X	0	0	0	0	0	0

6.2.2 Mixed Models and the Extended Autocorrelation Function

As the AR and MA orders (p, q) are **unknown**, an iterative procedure is required. Let

$$W_{t,k,j} = Y_t - \tilde{\phi}_1 Y_{t-1} - \cdots - \tilde{\phi}_k Y_{t-k}$$

be the autoregressive residuals defined with the AR coefficients estimated iteratively ***assuming the AR order is k and the MA order is j***. The sample autocorrelations of $W_{t,k,j}$ are referred to as the **extended sample autocorrelations**.

6.2.2 Mixed Models and the Extended Autocorrelation Function

For $k = p$ and $j \geq q$, $W_{t,\,k,\,j}$ is approximately an MA(q) model, so that its theoretical autocorrelations of **lag *q* + 1 or higher** are equal to **zero**.

For $k > p$, an ***overfitting*** problem occurs, and this ***increases*** the MA order for the W process by the minimum of ***k* – *p*** and ***j* – *q***.

How about k < p?

6.2.2 Mixed Models and the Extended Autocorrelation Function

Tsay and Tiao (1984) suggested summarizing the information in the sample EACF by a table with the element in the ***k*th row** and ***j*th column** equal to the symbol X if the **lag *j* + 1 sample correlation** of $W_{t,k,j}$ is significantly different from 0 and 0 otherwise. In such a table, an MA(*p*,*q*) process will have a theoretical pattern of ***a triangle of zeroes***, with the ***upper left-hand vertex*** corresponding to the ARMA orders.

6.2.2 Mixed Models and the Extended Autocorrelation Function

Exhibit 6.4 Theoretical Extended ACF (EACF) for an ARMA(1,1) Model

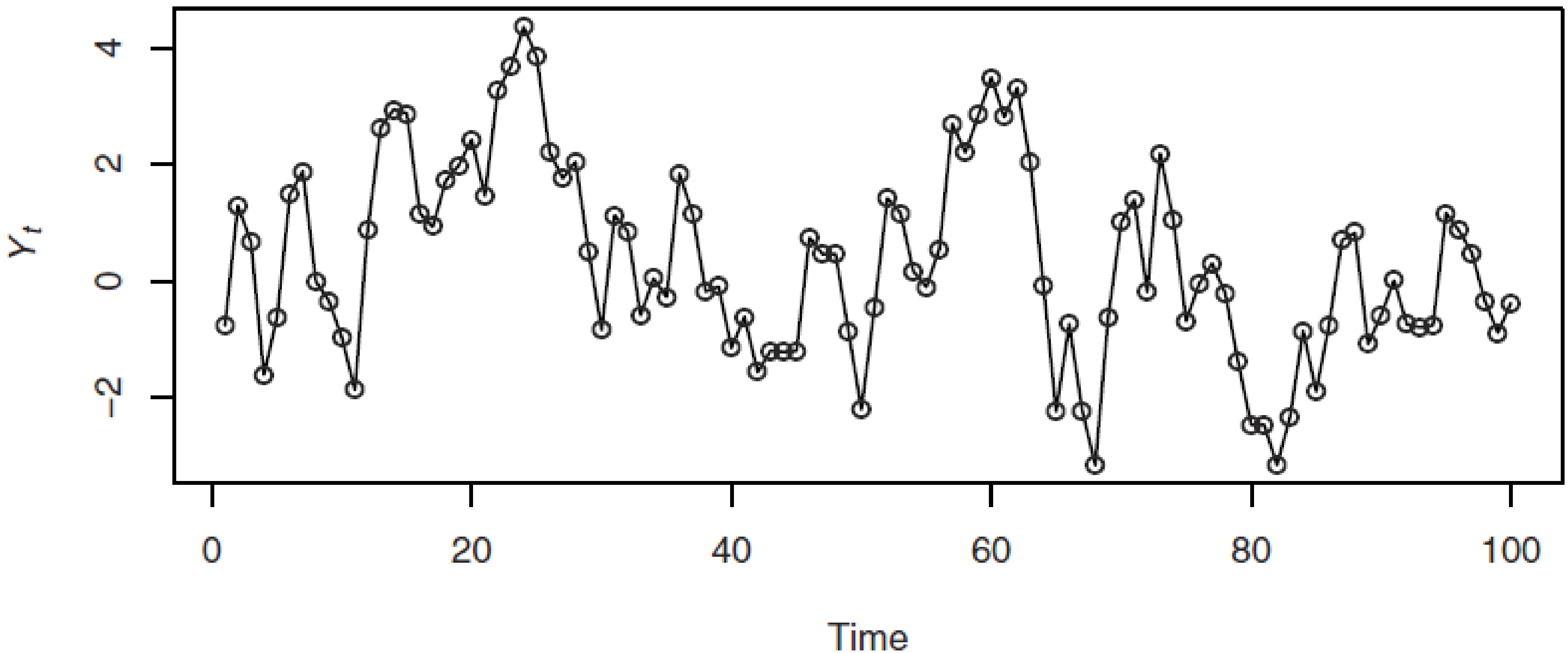
k=p, j>= q

k-p>j-q，阶增加min(k-p,j-q)

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	x	x	x	x	x	x	x	x	x	x	x	x	x
1	x	0*	0	0	0	0	0	0	0	0	0	0	0	0
2	x	x	0	0	0	0	0	0	0	0	0	0	0	0
3	x	x	x	0	0	0	0	0	0	0	0	0	0	0
4	x	x	x	x	0	0	0	0	0	0	0	0	0	0
5	x	x	x	x	x	0	0	0	0	0	0	0	0	0
6	x	x	x	x	x	x	0	0	0	0	0	0	0	0
7	x	x	x	x	x	x	x	0	0	0	0	0	0	0

6.3 Specification of Some Simulated Time Series

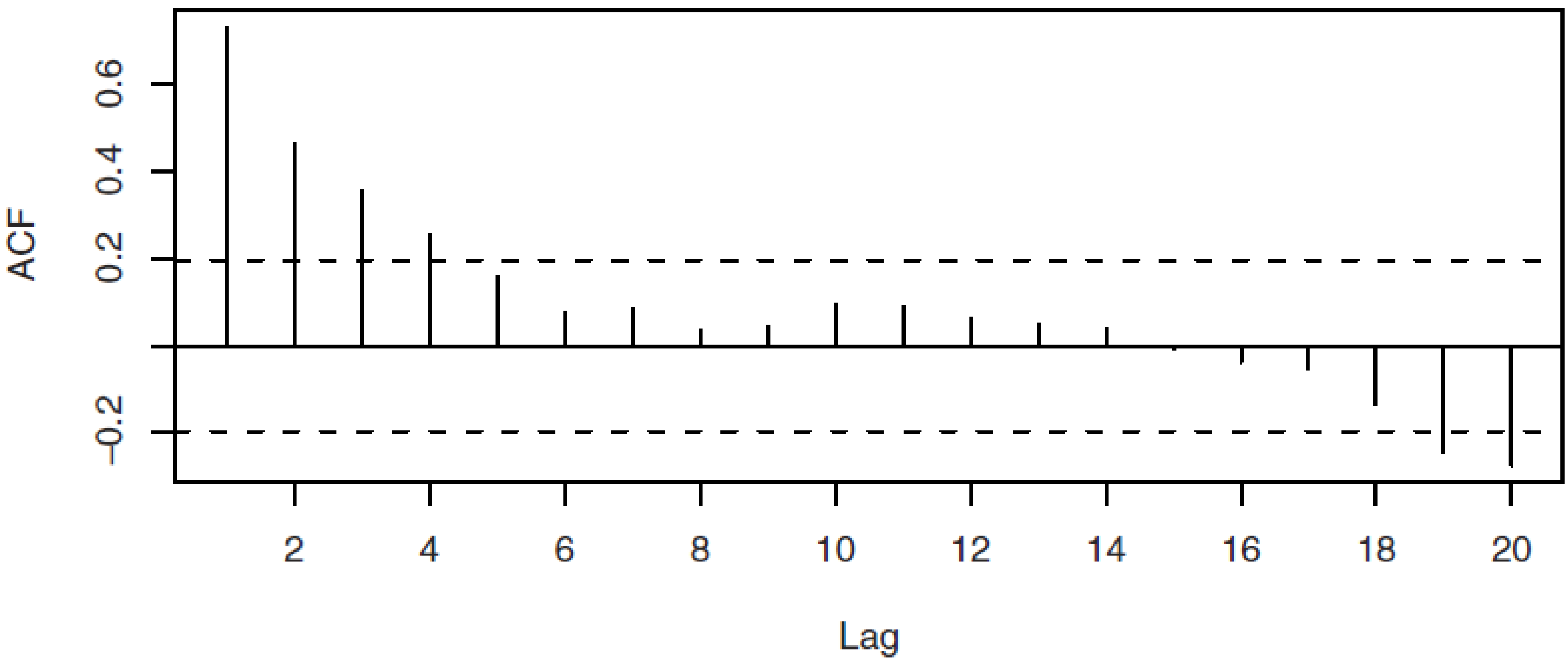
Exhibit 6.14 Simulated ARMA(1,1) Series with $\phi = 0.6$ and $\theta = -0.3$.




```
> data(arma11.s)
> plot(arma11.s, type='o',ylab=expression(Y[t]))
```

6.3 Specification of Some Simulated Time Series

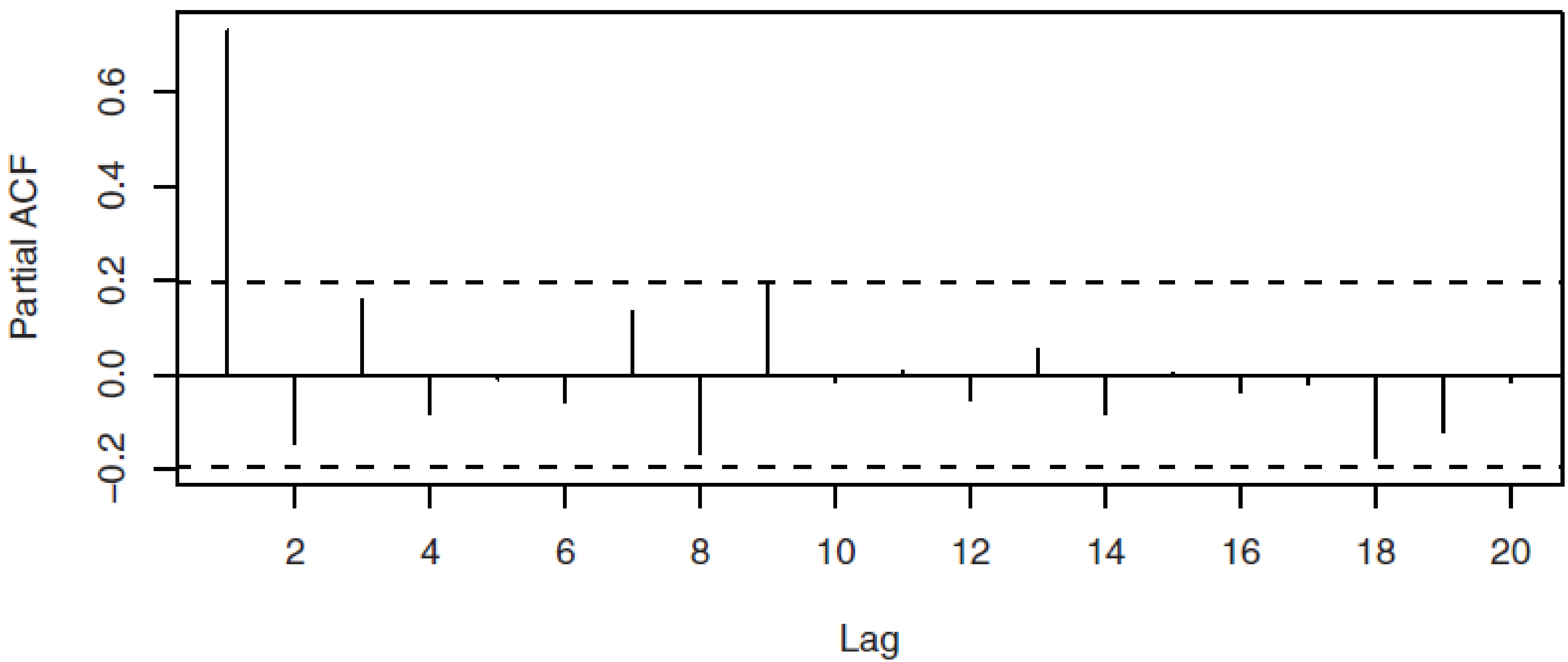
Exhibit 6.15 Sample ACF for Simulated ARMA(1,1) Series




```
> acf(arma11.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

Exhibit 6.16 Sample PACF for Simulated ARMA(1,1) Series




```
> pacf(arma11.s,xaxp=c(0,20,10))
```

6.3 Specification of Some Simulated Time Series

Exhibit 6.17 Sample EACF for Simulated ARMA(1,1) Series

AR / MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	x	x	x	o	o	o	o	o	o	o	o	o	o
1	x	o	o	o	o	o	o	o	o	o	o	o	o	o
2	x	o	o	o	o	o	o	o	o	o	o	o	o	o
3	x	x	o	o	o	o	o	o	o	o	o	o	o	o
4	x	o	x	o	o	o	o	o	o	o	o	o	o	o
5	x	o	o	o	o	o	o	o	o	o	o	o	o	o
6	x	o	o	o	x	o	o	o	o	o	o	o	o	o
7	x	o	o	o	x	o	o	o	o	o	o	o	o	o

```
> eacf(arma11.s)
```

q = 1 and with p = 1 or 2?

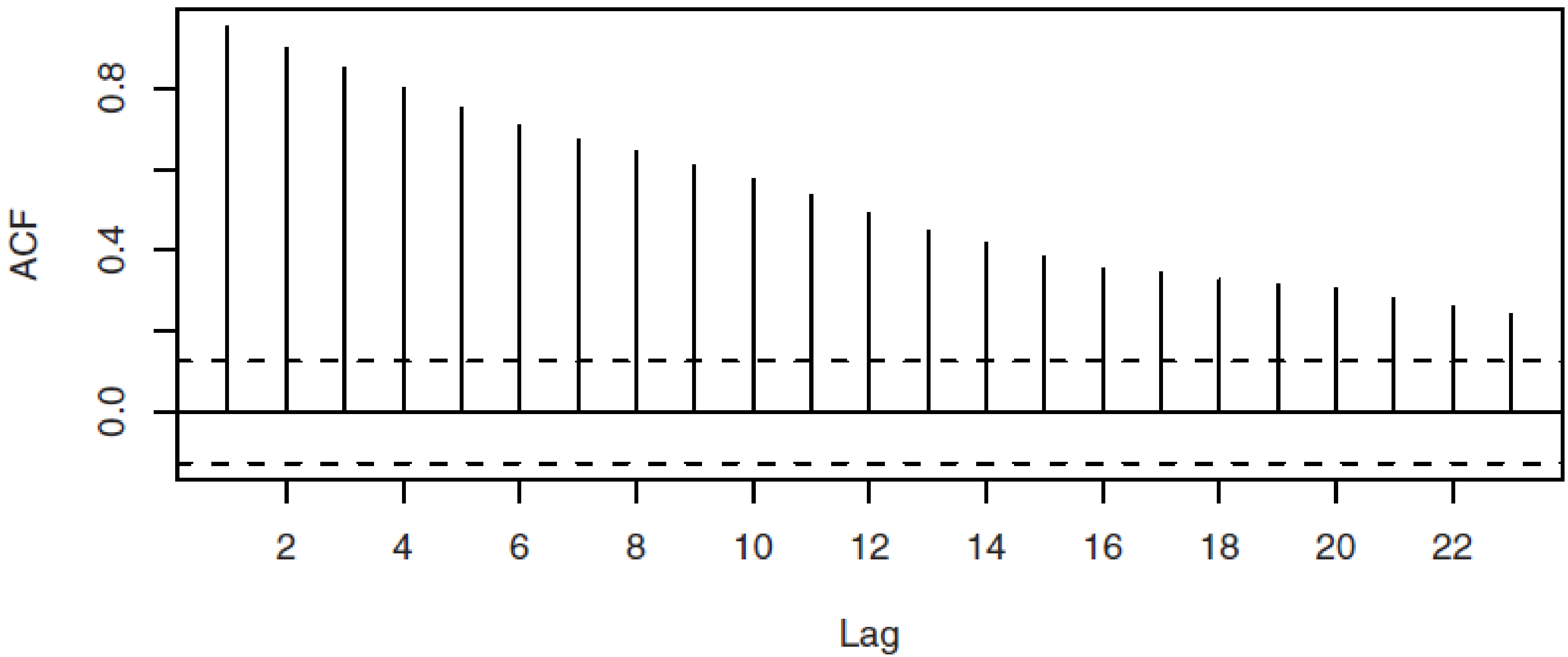
6.4 Nonstationarity

1. The **nonstationarity** will frequently be apparent in the ***time series plot*** of the series. E.g., Exhibits 5.1, 5.5, and 5.8.

2. The ***sample ACF*** computed for nonstationary series will also usually indicate the ***nonstationarity***. For nonstationary series, the sample ACF typically ***fails to die out*** rapidly as the lags increase. The values of r_k need not be large even for low lags, but often they are.

6.4 Nonstationarity

Exhibit 6.18 Sample ACF for the Oil Price Time Series

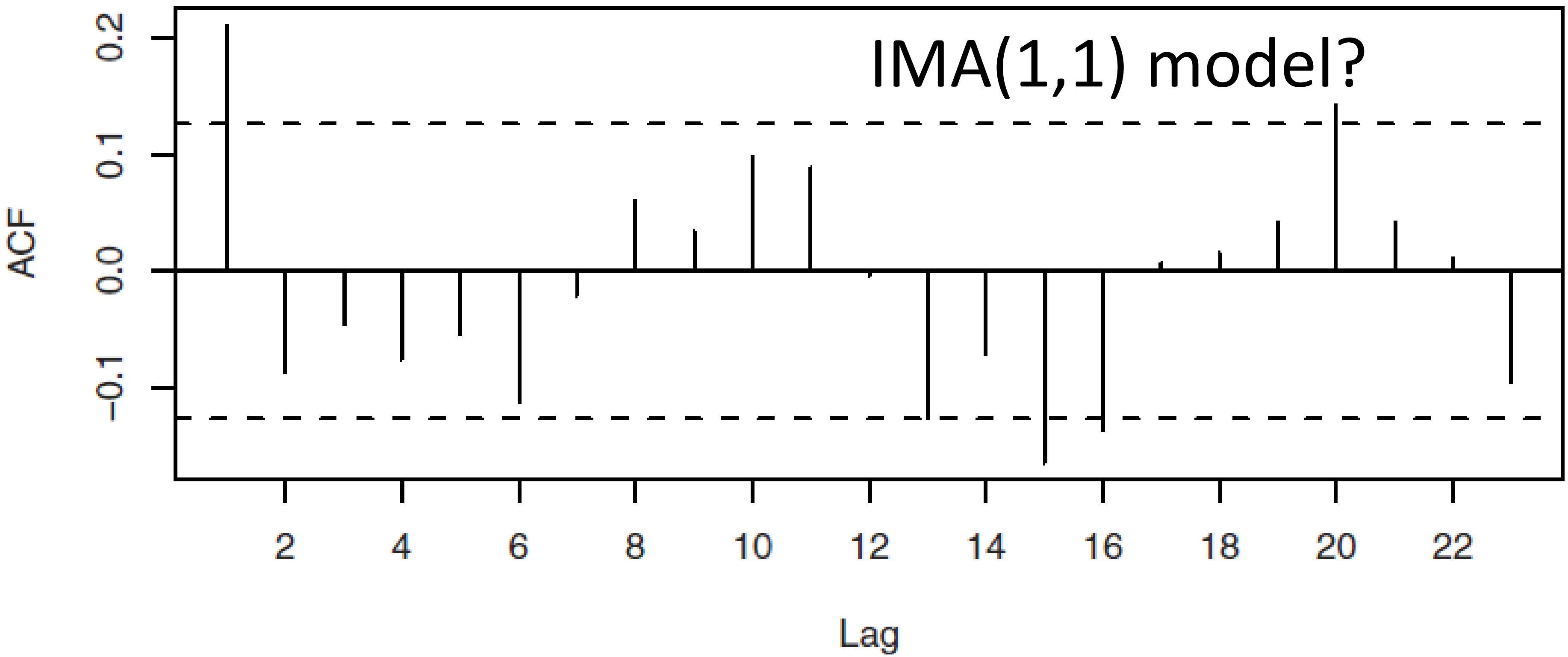



```
> data(oil.price)
> acf(as.vector(oil.price),xaxp=c(0,24,12))
```

All values shown are "***significantly far from zero***," and the only pattern is perhaps a linear decrease with increasing lag.

6.4 Nonstationarity

Exhibit 6.19 Sample ACF for the Difference of the Log Oil Price Series




```
> acf(diff(as.vector(log(oil.price))),xaxp=c(0,24,12))
```

The pattern emerges much more clearly—after differencing, a moving average model of order 1 seems appropriate.

6.4 Nonstationarity

If the first difference of a series and its sample ACF do not appear to support a stationary ARMA model, then we take ***another difference*** and again compute the sample ACF and PACF to look for characteristics of a stationary ARMA process. Usually ***one or at most two*** differences, perhaps combined with a logarithm or other ***transformation***, will accomplish this reduction to stationarity.

6.4.1 Overdifferencing

Overdifferencing will introduce unnecessary correlations into a series and will **complicate** the modeling process.

$$Y_t = Y_{t-1} + e_t$$ random walk

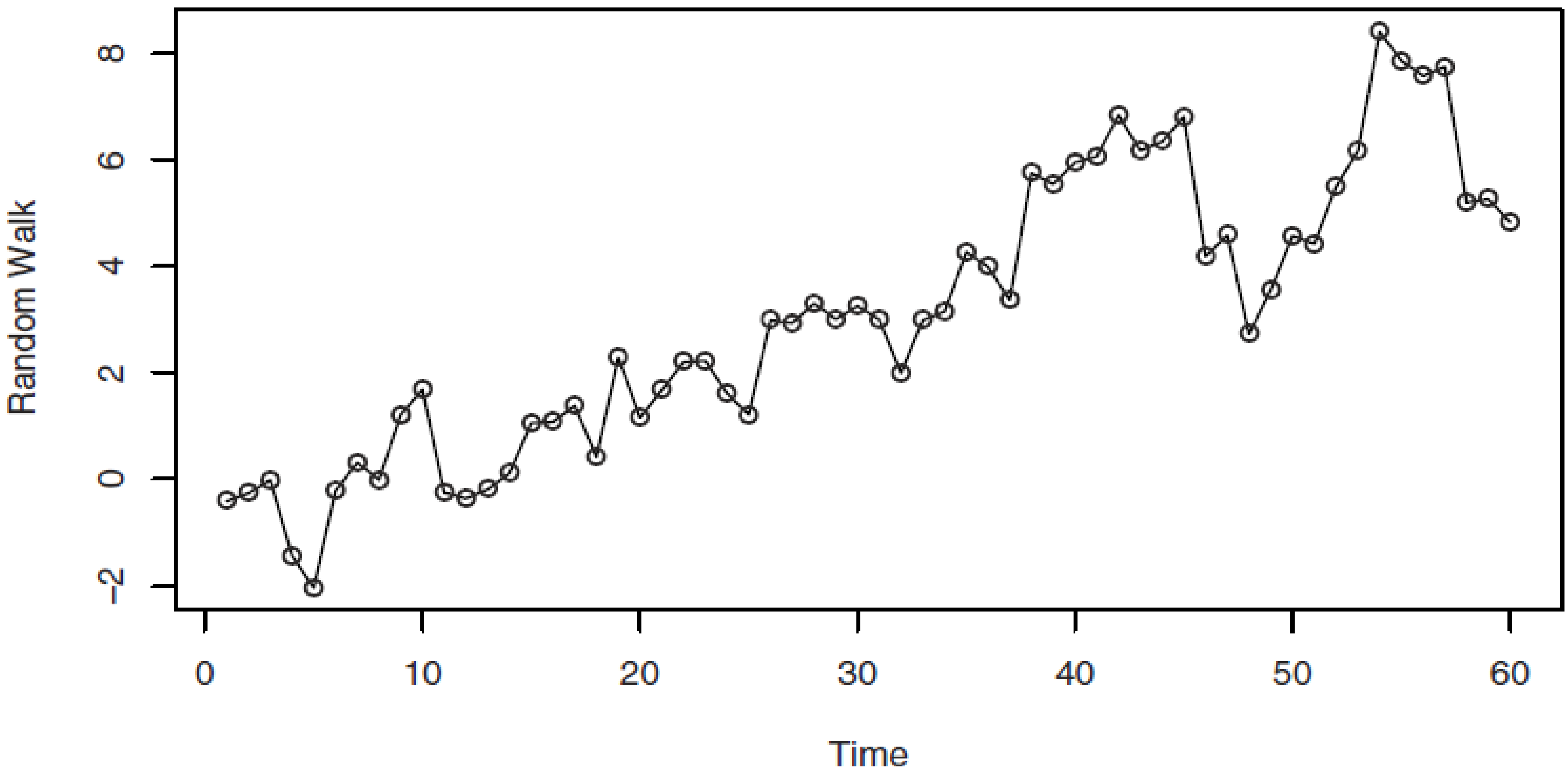
$$\nabla Y_t = Y_t - Y_{t-1} = e_t$$ white noise model

$$\nabla^2 Y_t = e_t - e_{t-1}$$ MA(1) model

Specifying an IMA(2,1) model would not be appropriate. The random walk model, which can be thought of as IMA(1,1) with θ = 0, is the correct model.

6.4.1 Overdifferencing

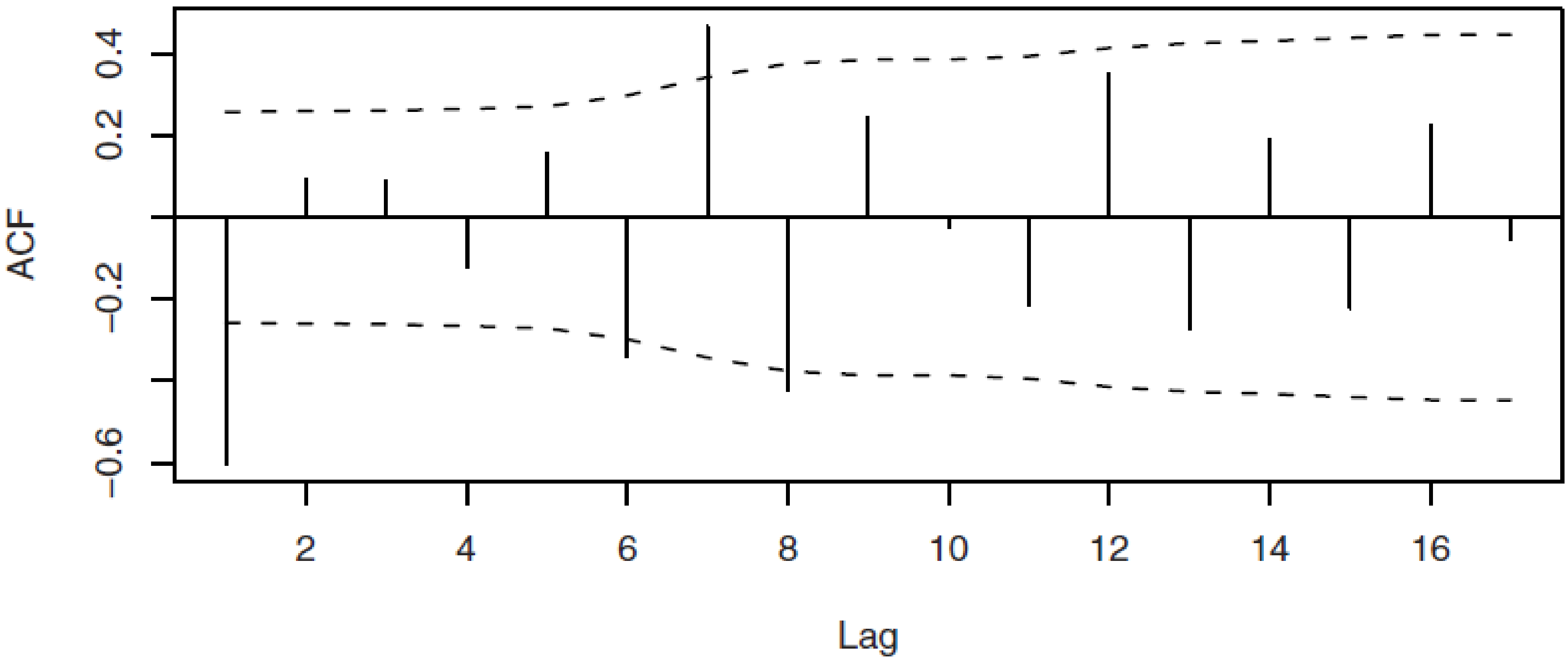
Exhibit 2.1 Time Series Plot of a Random Walk




```
> win.graph(width=4.875, height=2.5,pointsize=8)
> data(rwalk) # rwalk contains a simulated random walk
> plot(rwalk,type='o',ylab='Random Walk')
```

6.4.1 Overdifferencing

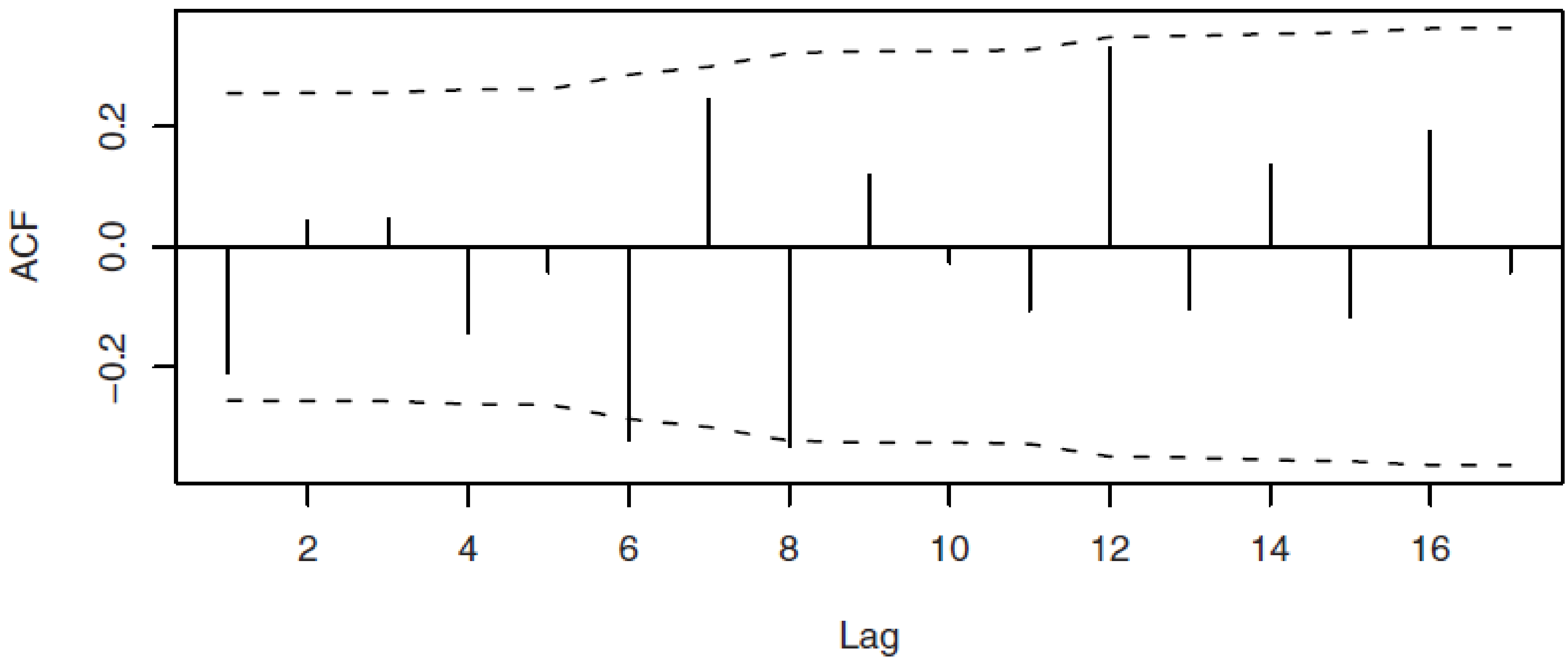
Exhibit 6.20 Sample ACF of Overdifferenced Random Walk




```
> data(rwalk)
> acf(diff(rwalk,difference=2),ci.type='ma', xaxp=c(0,18,9))
```

6.4.1 Overdifferencing

Exhibit 6.21 Sample ACF of Correctly Differenced Random Walk




```
> acf(diff(rwalk),ci.type='ma',xaxp=c(0,18,9))
```

To avoid overdifferencing, look carefully at each difference and keep the principle of parsimony in mind—*models should be simple, but not too simple*.

6.4.2 The Dickey-Fuller Unit-Root Test

While the ***approximate linear decay*** of the sample ACF is often taken as a symptom that the underlying time series is ***nonstationary*** and requires differencing, it is also useful to ***quantify*** the evidence of nonstationarity in the data-generating mechanism. ***Hypothesis testing***?

Consider the model:

$$Y_t = \alpha Y_{t-1} + X_t \text{ for } t = 1, 2, \ldots$$

where $\{X_t\}$ is a stationary process.

The test for differencing amounts to ***testing for a unit root*** in the AR characteristic polynomial of $\{Y_t\}$.

6.5 Other Specification Methods

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2\log(\text{maximum likelihood}) + 2k$$

where $k = p + q + 1$ if the model contains an intercept or constant term and $k = p + q$ otherwise.

corrected AIC (AICc):

$$\text{AIC}_c = \text{AIC} + \frac{2(k+1)(k+2)}{n-k-2}$$

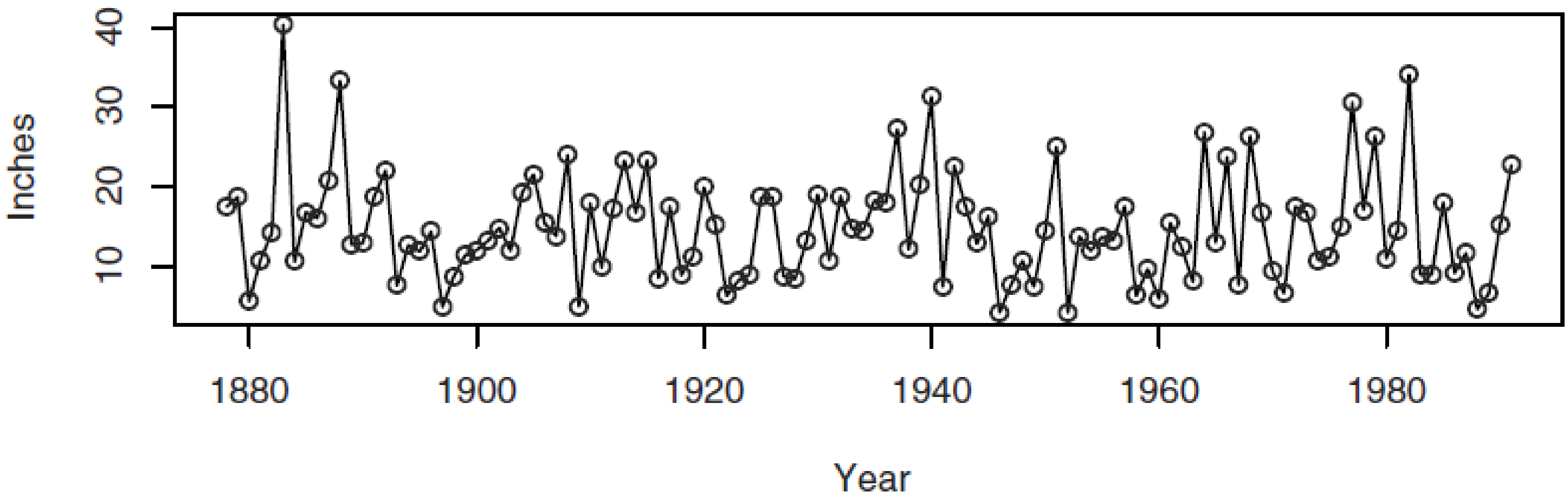
Bayesian Information Criterion (BIC):

$$\text{BIC} = -2\log(\text{maximum likelihood}) + k\log(n)$$

Best Subset ARMA Selection

6.6.1 The Los Angeles Annual Rainfall Series

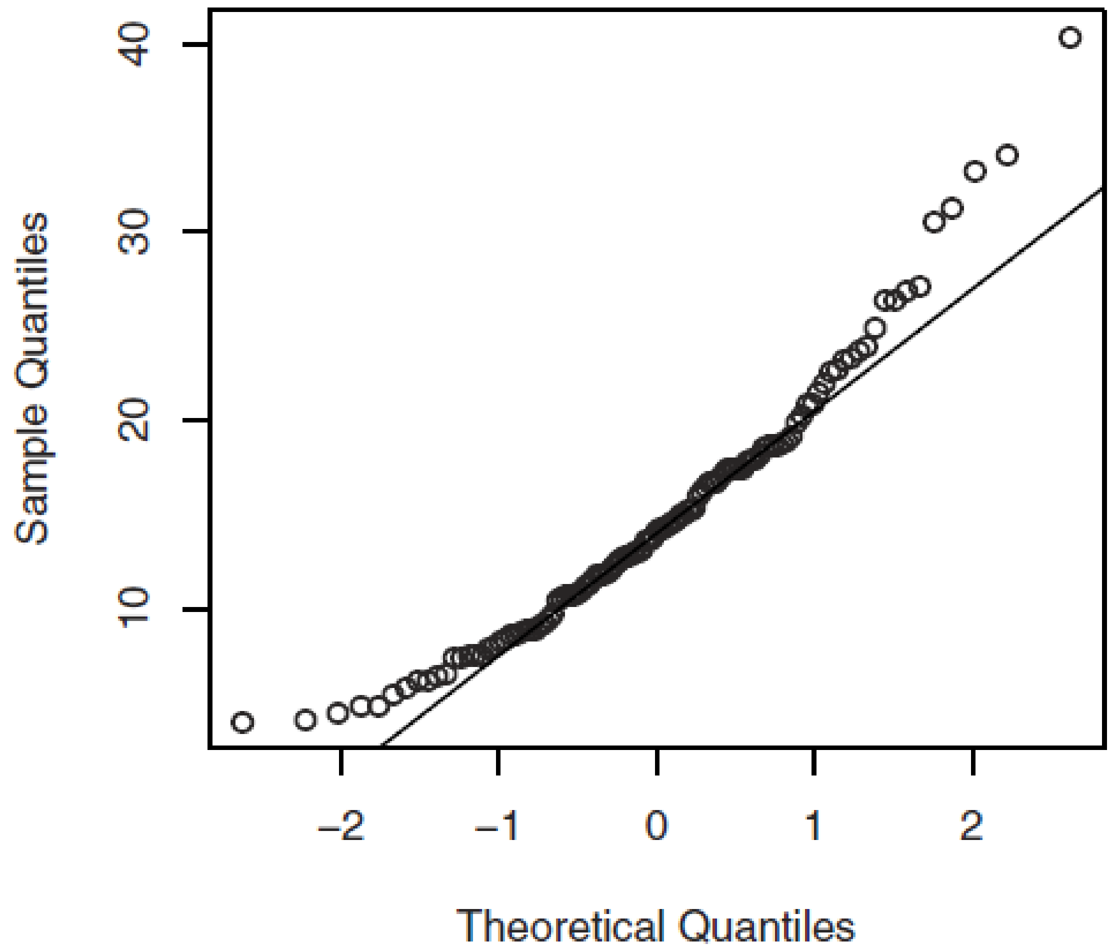
Exhibit 1.1 Time Series Plot of Los Angeles Annual Rainfall




```
> library(TSA)
> win.graph(width=4.875, height=2.5,pointsize=8)
> data(larain); plot(larain,ylab='Inches',xlab='Year',type='o')
```

6.6.1 The Los Angeles Annual Rainfall Series

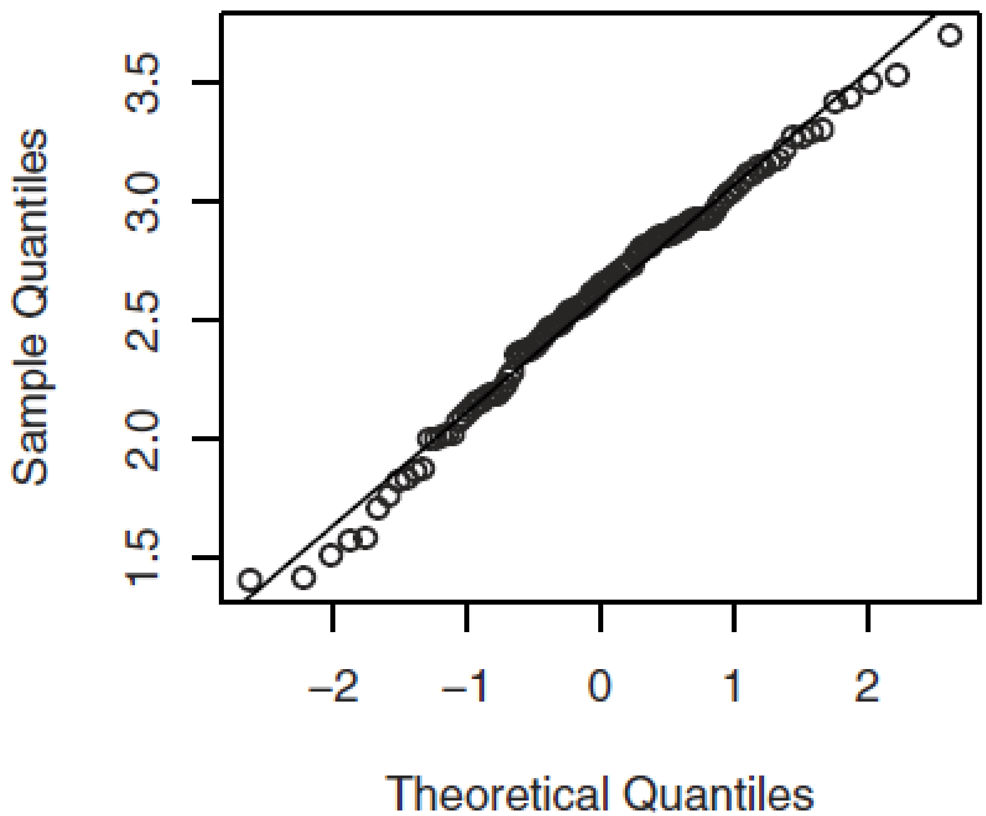
Exhibit 3.17 Quantile-Quantile Plot of Los Angeles Annual Rainfall Series




```
> win.graph(width=2.5,height=2.5,pointsize=8)
> qqnorm(larain); qqline(larain)
```

6.6.1 The Los Angeles Annual Rainfall Series

Exhibit 6.23 QQ Normal Plot of the Logarithms of LA Annual Rainfall

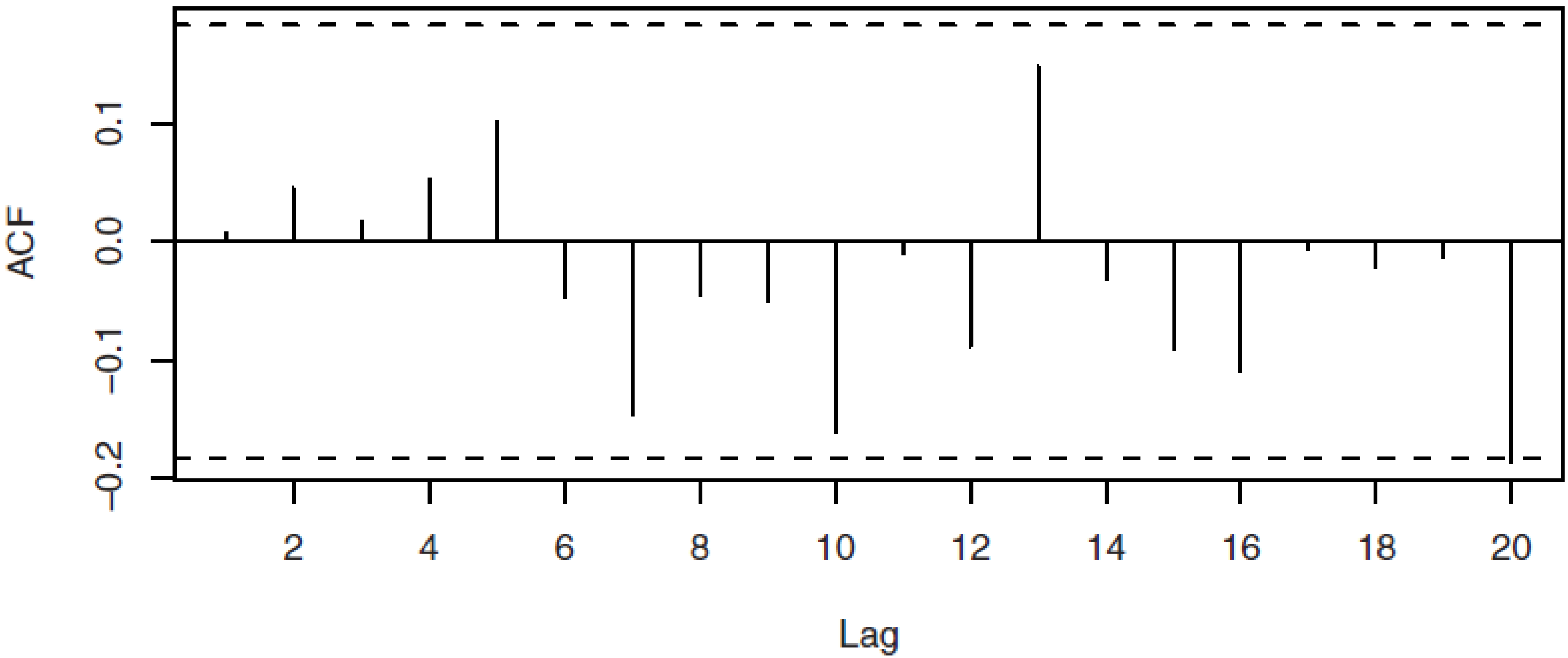



```
> data(larain); win.graph(width=2.5,height=2.5,pointsize=8)
> qqnorm(log(larain)); qqline(log(larain))
```

Taking logarithms improves the normality.

6.6.1 The Los Angeles Annual Rainfall Series

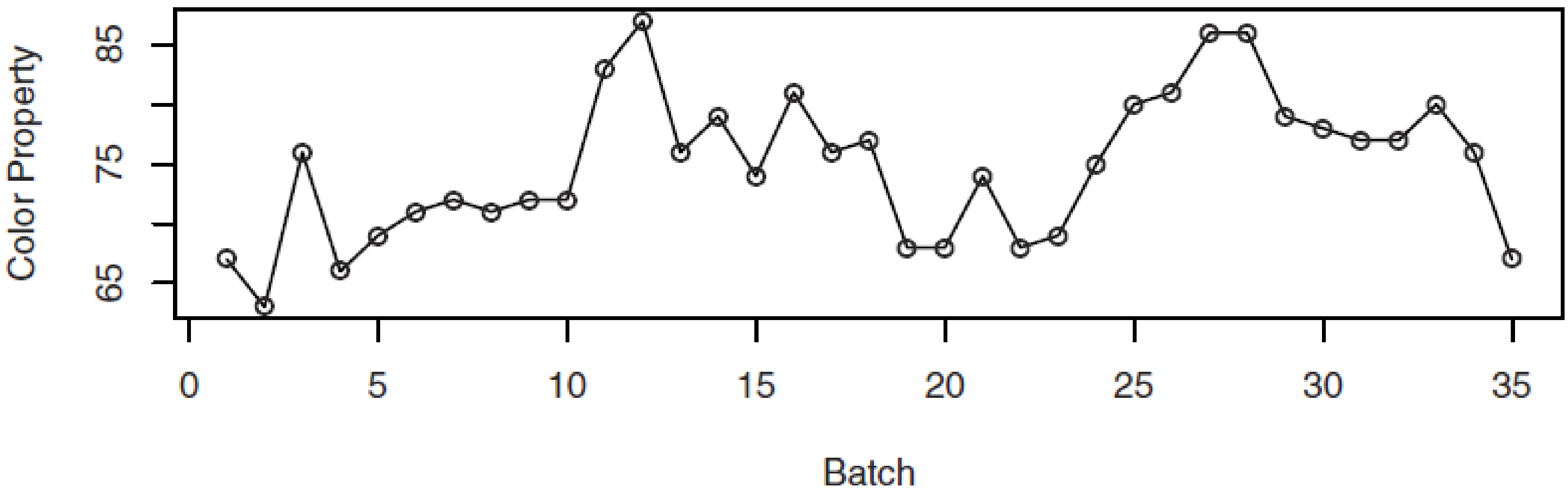
Exhibit 6.24 Sample ACF of the Logarithms of LA Annual Rainfall




```
> win.graph(width=4.875,height=3,pointsize=8)
> acf(log(larain),xaxp=c(0,20,10))
```

6.6.2 The Chemical Process Color Property Series

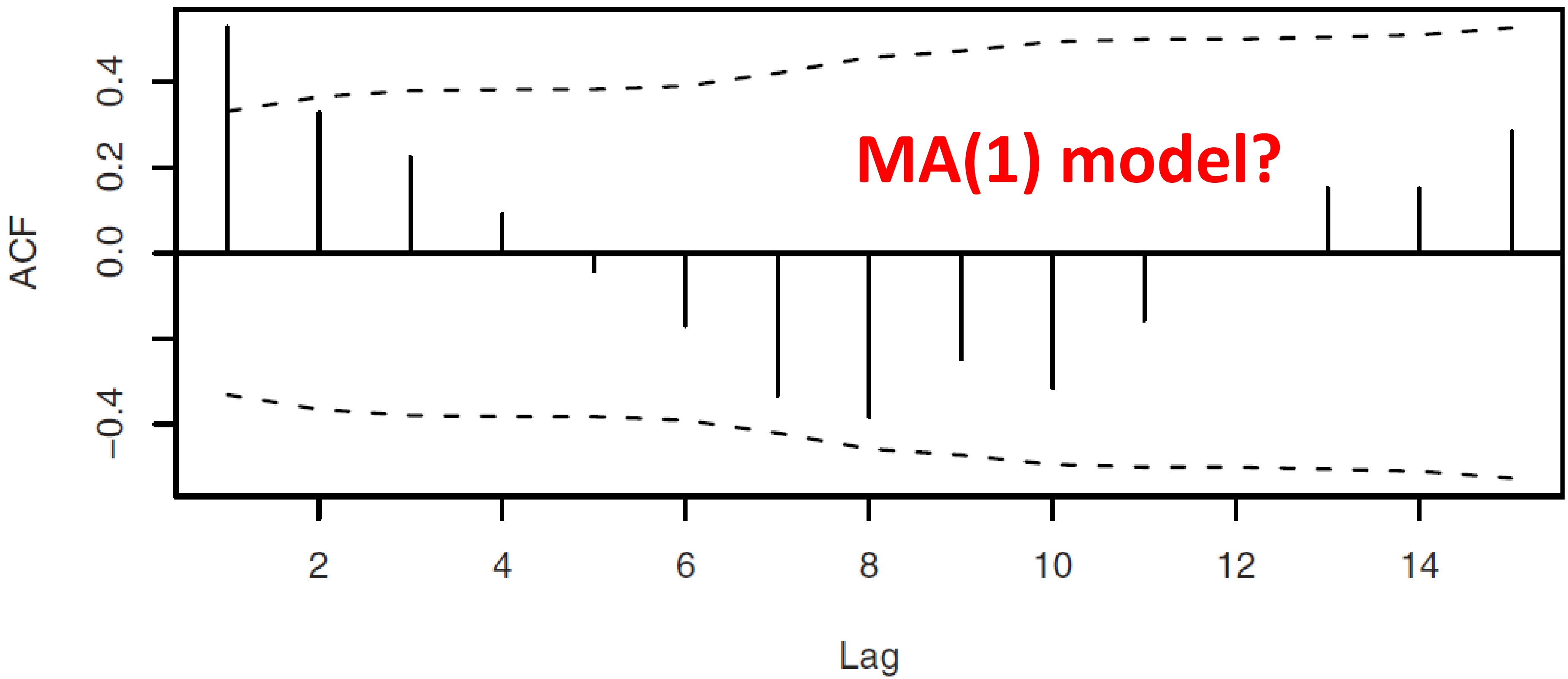
Exhibit 1.3 Time Series Plot of Color Property from a Chemical Process




```
> win.graph(width=4.875, height=2.5,pointsize=8)
> data(color)
> plot(color,ylab='Color Property',xlab='Batch',type='o')
```

6.6.2 The Chemical Process Color Property Series

Exhibit 6.25 Sample ACF for the Color Property Series

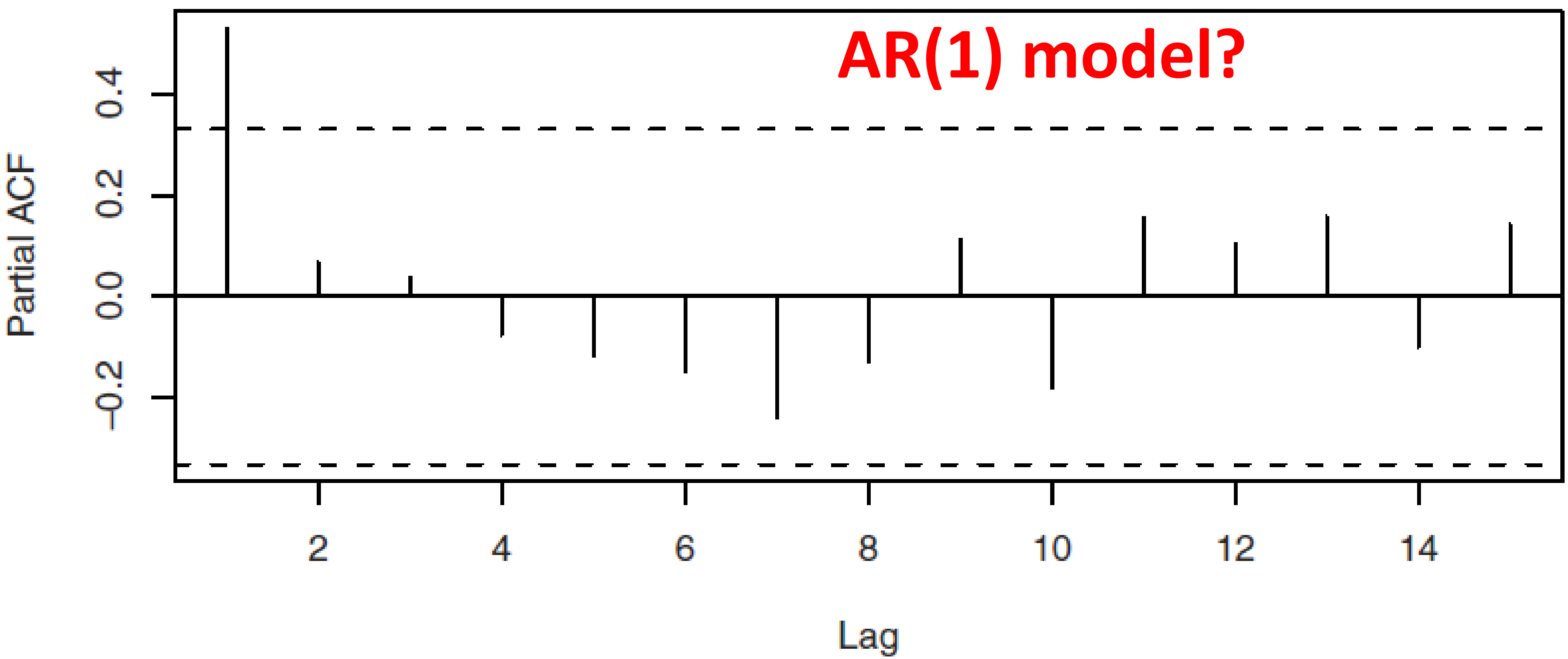



```
> data(color); acf(color,ci.type='ma')
```

The ***damped sine wave*** appearance of the plot encourages us to look further at the sample ***partial autocorrelation***.

6.6.2 The Chemical Process Color Property Series

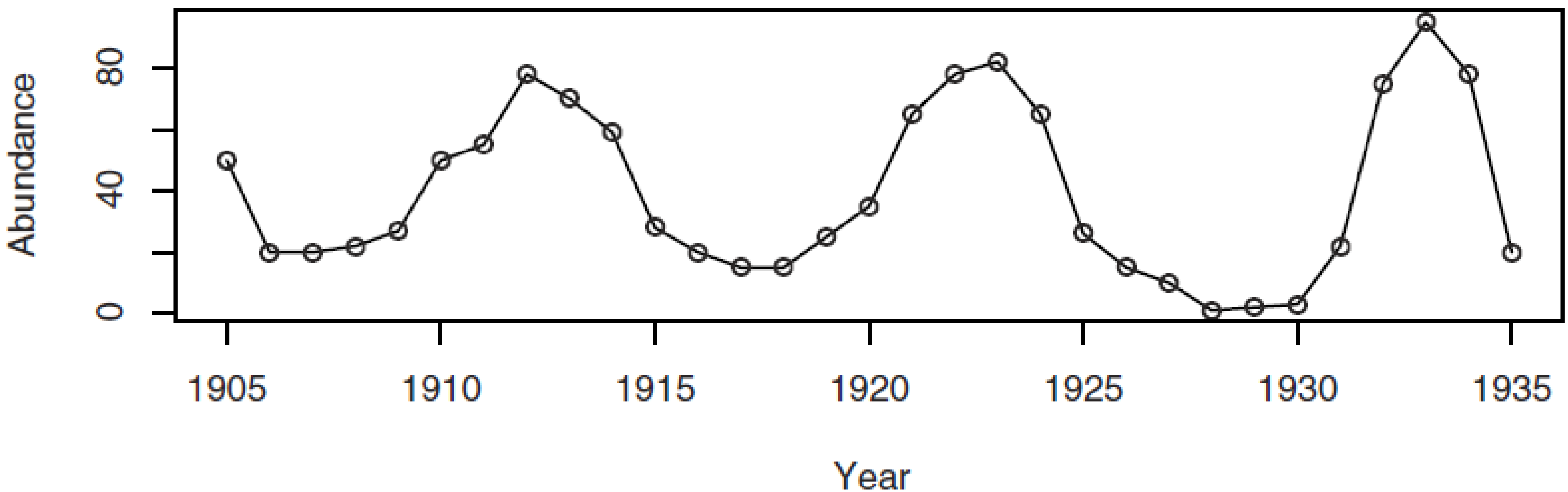
Exhibit 6.26 Sample Partial ACF for the Color Property Series




```
> pacf(color)
```

6.6.3 The Annual Abundance of Canadian Hare Series

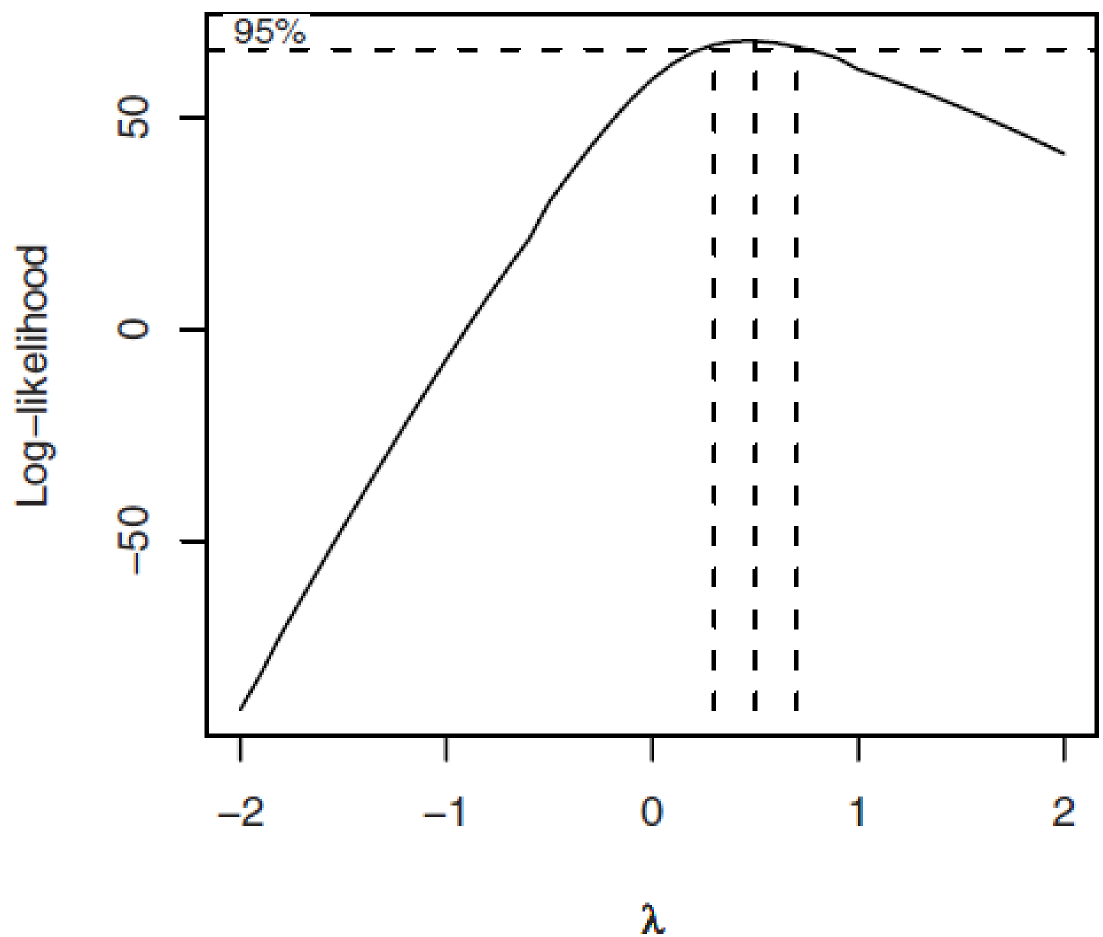
Exhibit 1.5 Abundance of Canadian Hare




```
> win.graph(width=4.875, height=2.5,pointsize=8)
> data(hare); plot(hare,ylab='Abundance',xlab='Year',type='o')
```

6.6.3 The Annual Abundance of Canadian Hare Series

Exhibit 6.27 Box-Cox Power Transformation Results for Hare Abundance

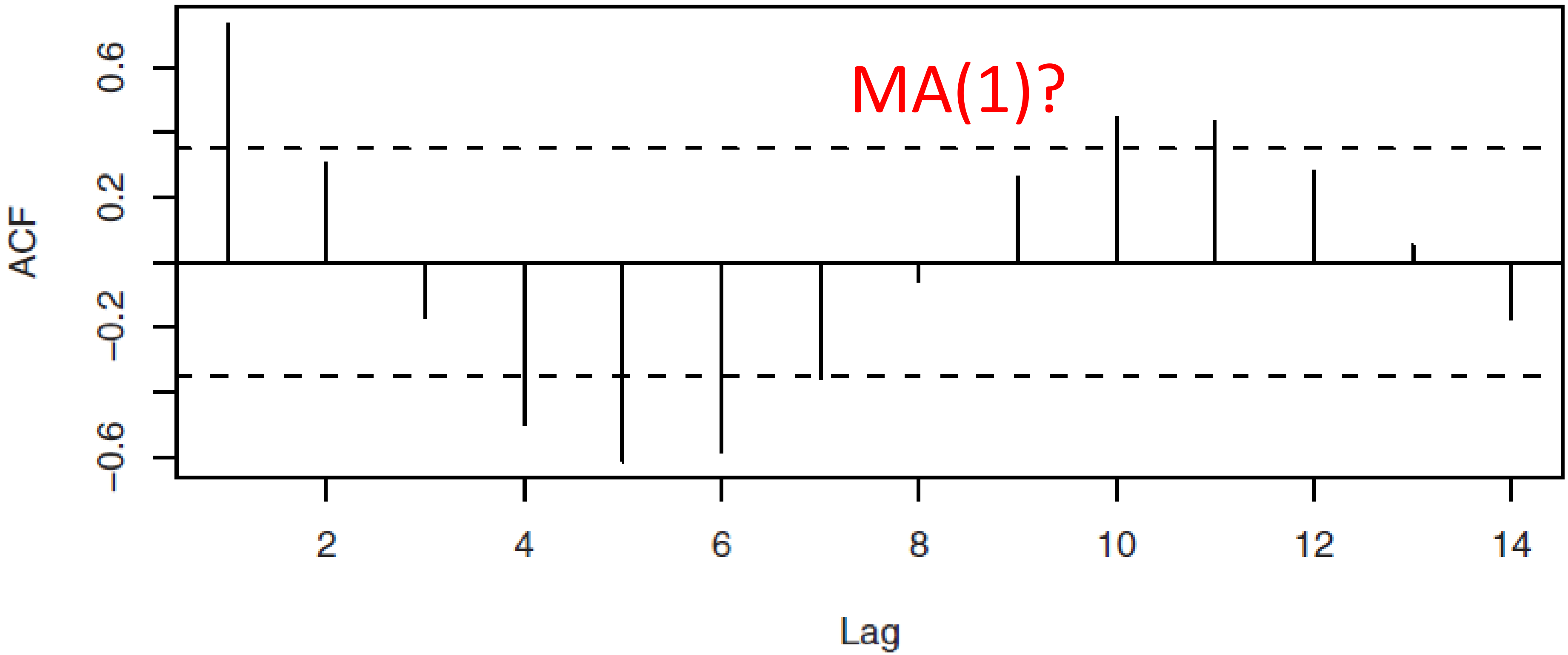


$\lambda = 0.5$

```
> win.graph(width=3,height=3,pointsize=8)
> data(hare); BoxCox.ar(hare)
```

6.6.3 The Annual Abundance of Canadian Hare Series

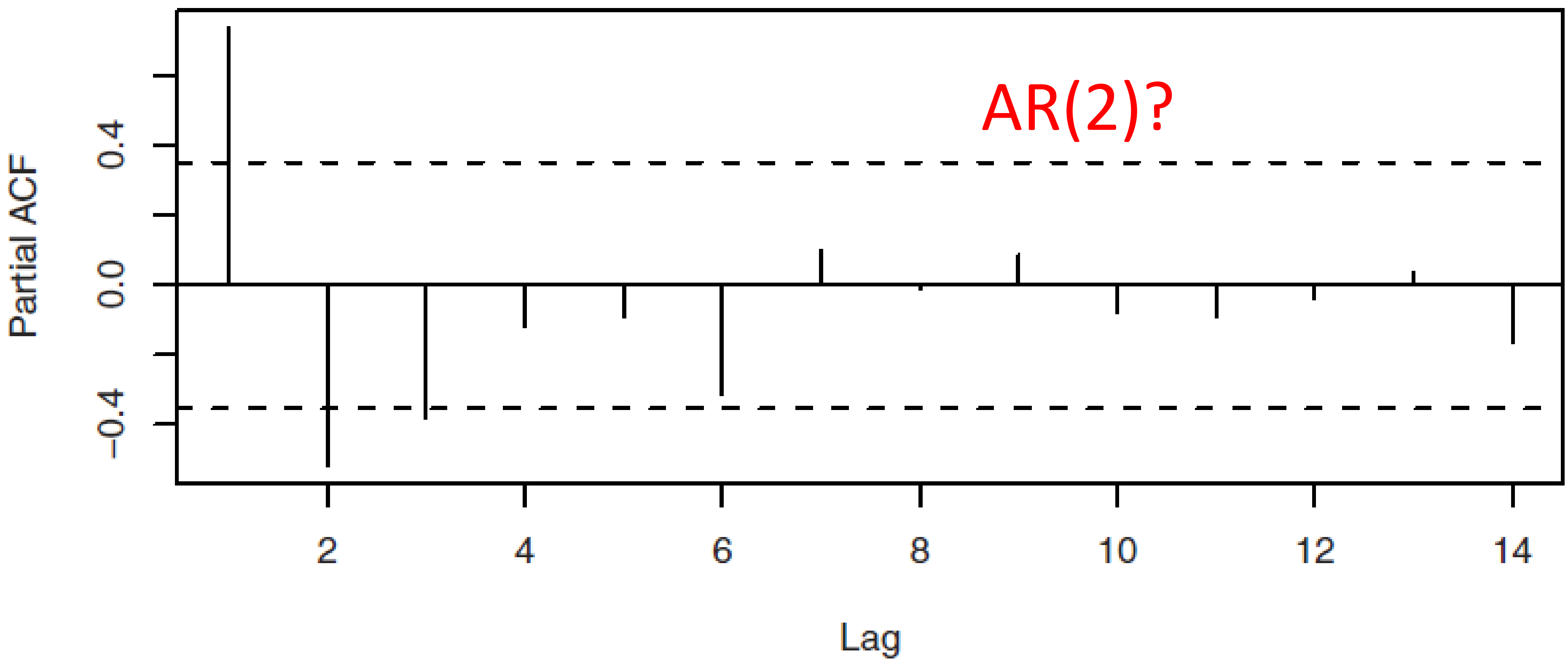
Exhibit 6.28 Sample ACF for Square Root of Hare Abundance




```
> acf(hare^.5)
```

6.6.3 The Annual Abundance of Canadian Hare Series

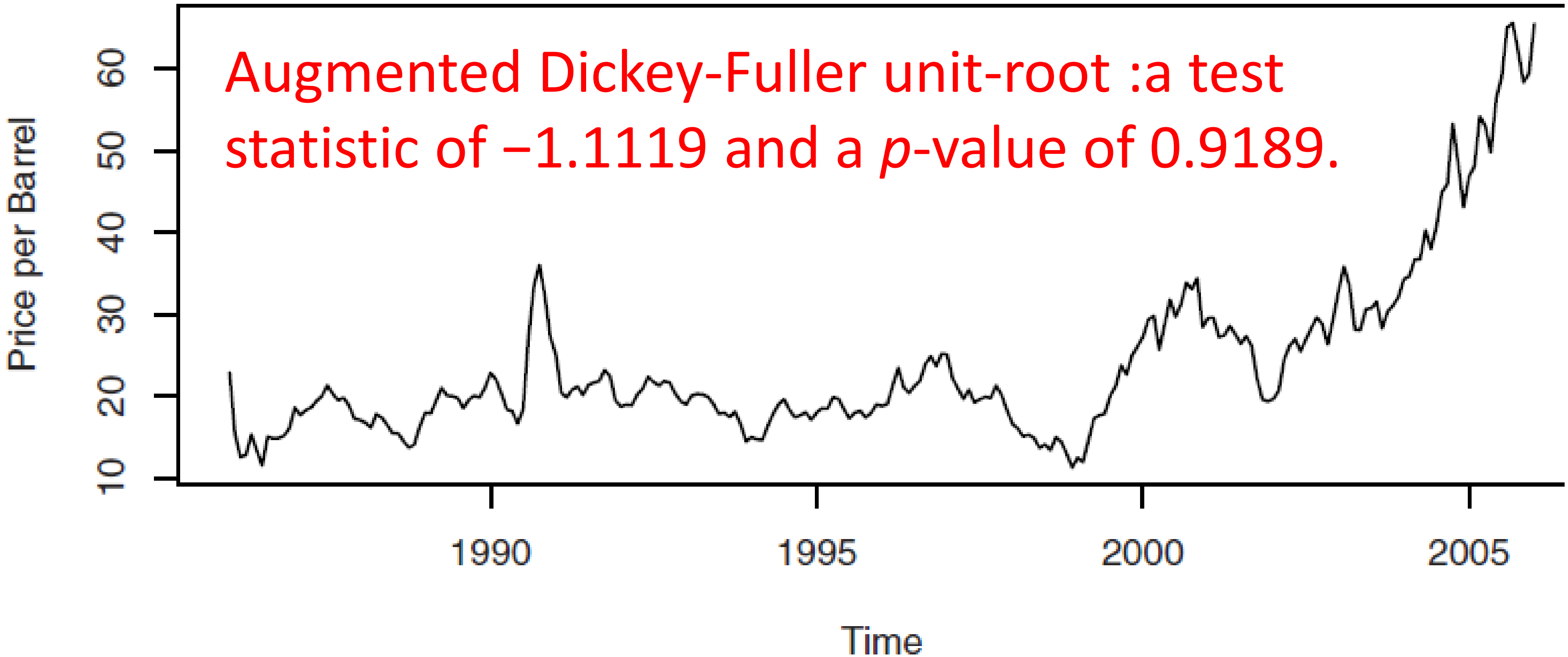
Exhibit 6.29 Sample Partial ACF for Square Root of Hare Abundance




```
> pacf(hare^.5)
```

6.6.4 The Oil Price Series

Exhibit 5.1 Monthly Price of Oil: January 1986–January 2006




```
> win.graph(width=4.875,height=3,pointsize=8)
> data(oil.price)
> plot(oil.price, ylab='Price per Barrel',type='l')
```

6.6.4 The Oil Price Series

Exhibit 6.30 Extended ACF for Difference of Logarithms of Oil Price Series

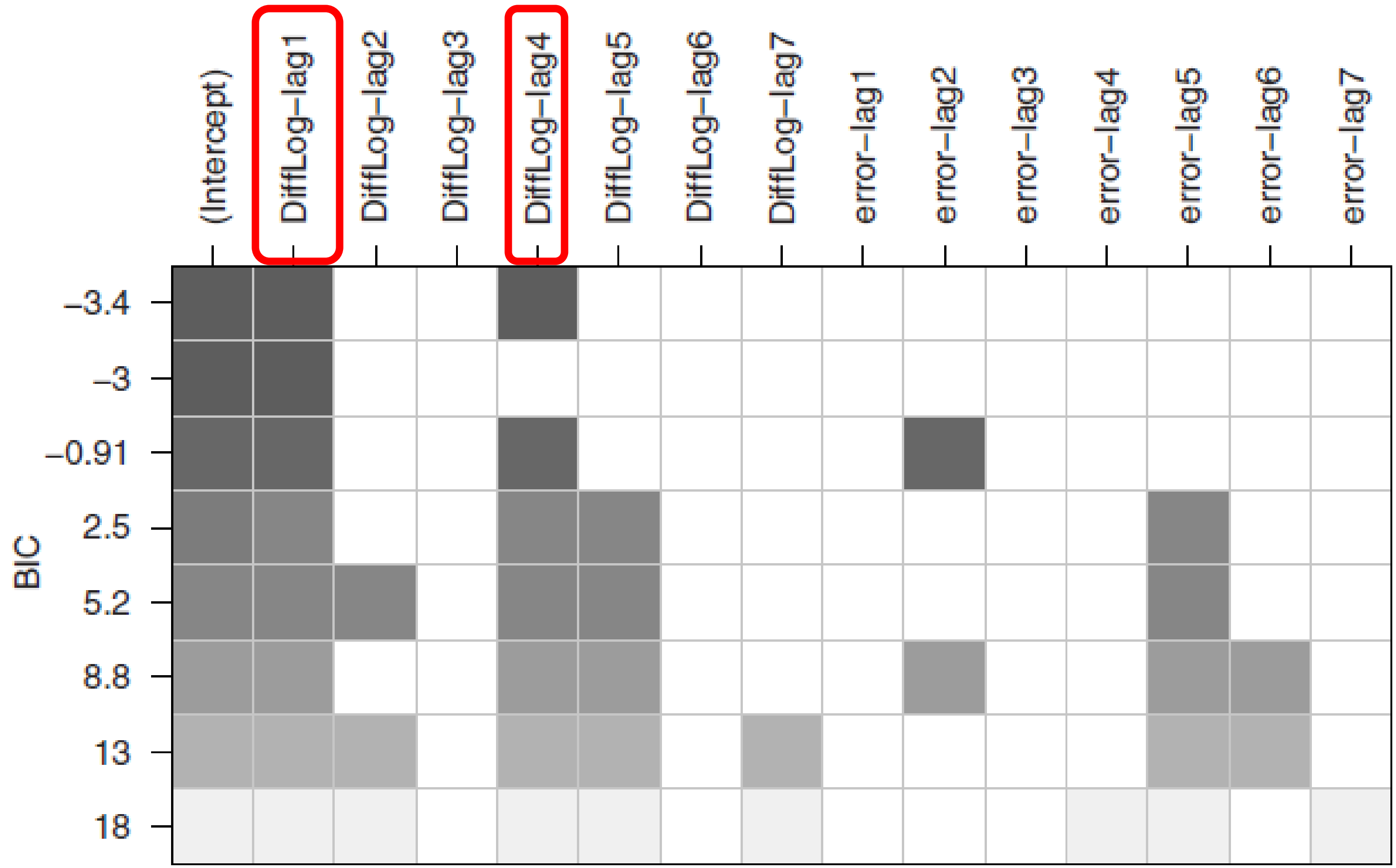
AR / MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	o	o	o	o	o	o	o	o	o	o	o	o	o
1	x	x	o	o	o	o	o	o	o	o	x	o	o	o
2	o	x	o	o	o	o	o	o	o	o	o	o	o	o
3	o	x	o	o	o	o	o	o	o	o	o	o	o	o
4	o	x	x	o	o	o	o	o	o	o	o	o	o	o
5	o	x	o	x	o	o	o	o	o	o	o	o	o	o
6	o	x	o	x	o	o	o	o	o	o	o	o	o	o
7	x	x	o	x	o	o	o	o	o	o	o	o	o	o

```
> eacf(diff(log(oil.price)))
```

suggests an ARMA model with $p = 0$ and $q = 1$?

6.6.4 The Oil Price Series

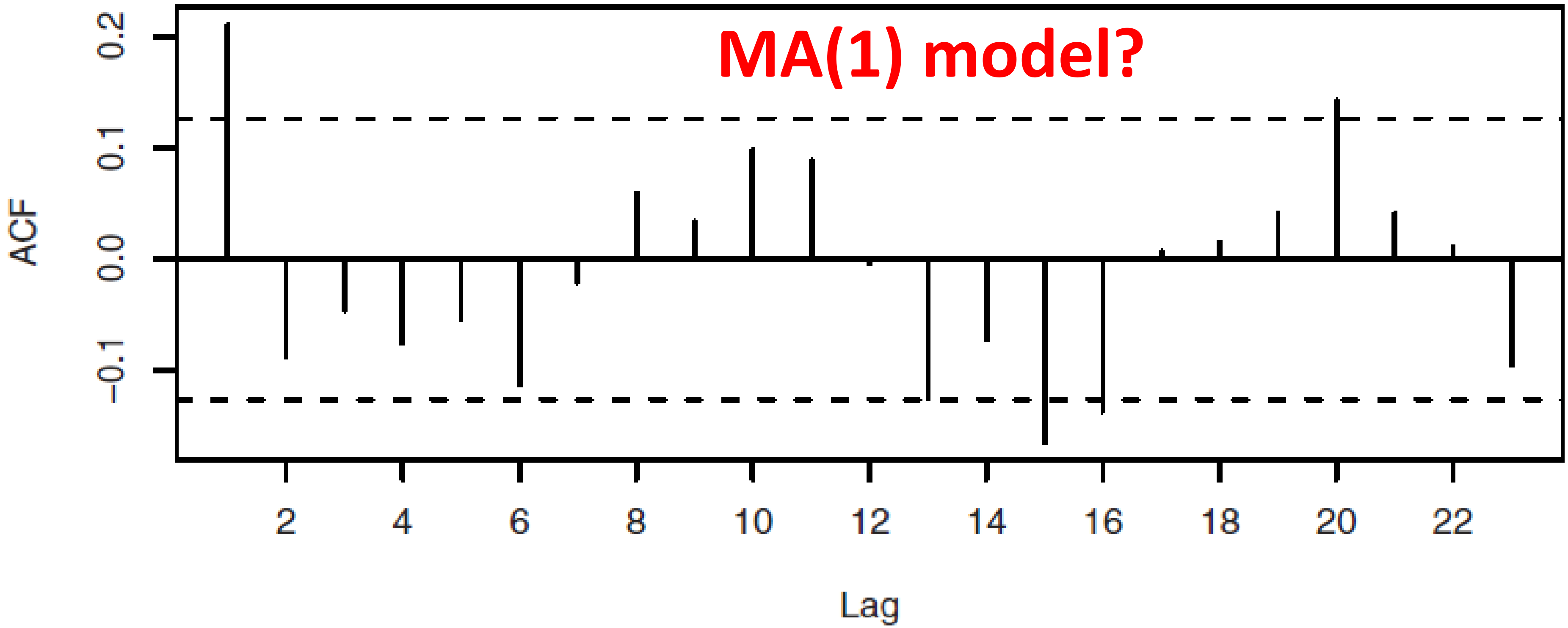
Exhibit 6.31 Best Subset ARMA Model for Difference of Log(Oil)




```
> res=armasubsets(y=diff(log(oil.price)),nar=7,nma=7,
   y.name='test', ar.method='ols')
> plot(res)
```

6.6.4 The Oil Price Series

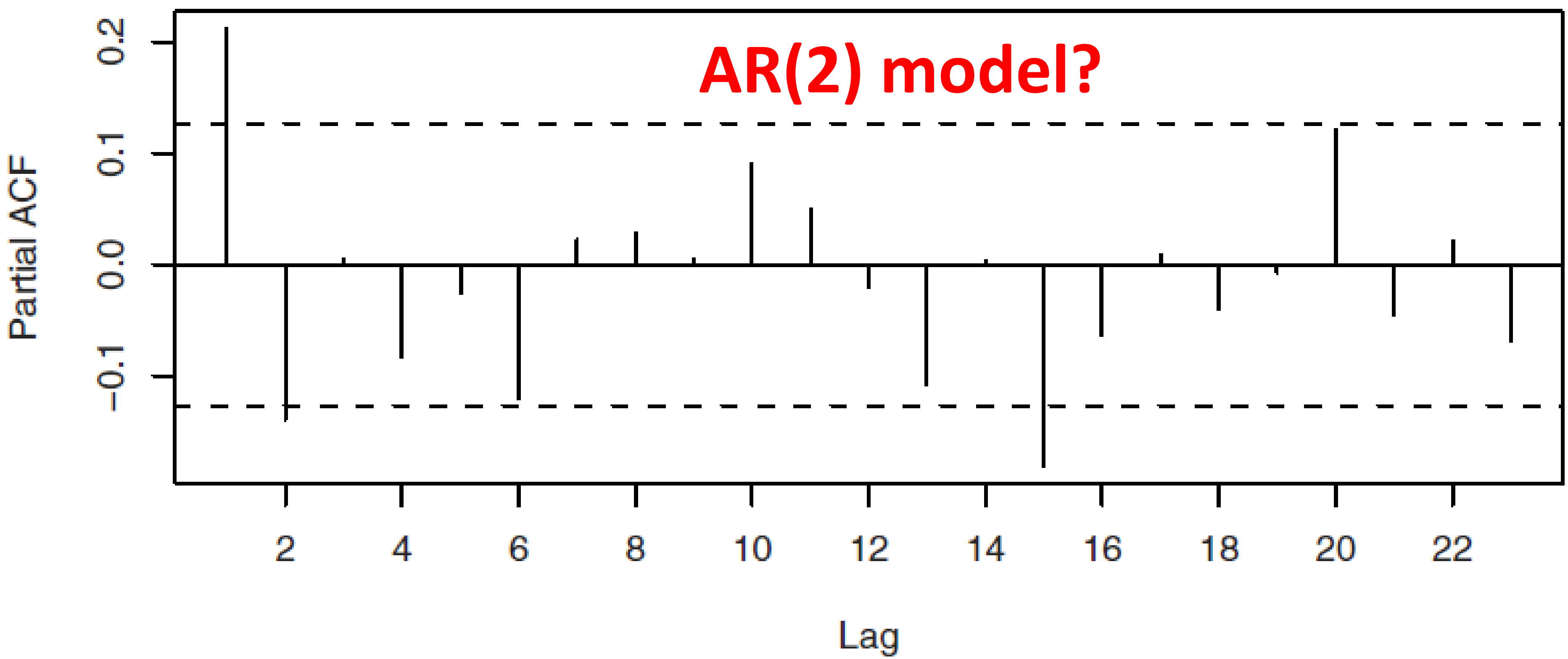
Exhibit 6.32 Sample ACF of Difference of Logged Oil Prices




```
> acf(as.vector(diff(log(oil.price))),xaxp=c(0,22,11))
```

6.6.4 The Oil Price Series

Exhibit 6.33 Sample PACF of Difference of Logged Oil Prices




```
> pacf(as.vector(diff(log(oil.price))),xaxp=c(0,22,11))
```

6.7 Summary

We considered the problem of ***specifying*** reasonable models for observed times series. In particular, we investigated tools for ***choosing the orders*** (p, d, and q) for ARIMA(p,d,q) models. Three tools, ***the sample autocorrelation function***, ***the sample partial autocorrelation function***, ***and the sample extended autocorrelation function***, were introduced and studied to help with this difficult task. The Dickey-Fuller unit-root test was also introduced to help ***distinguish between stationary and nonstationary series***. These ideas were all illustrated with both simulated and actual time series.

作业

1. 根据（6.1.1）式**写出计算样本自相关函数的R程序**，并用该程序完成6.24题

$$r_k = \frac{\sum_{t=k+1}^{n} (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n} (Y_t - \bar{Y})^2} \qquad \text{for } k = 1, 2, \ldots$$

2. 根据（6.2.8）式**写出计算样本偏自相关函数的R程序**，并用该程序完成6.25，6.27，6.28，6.33题

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix}$$