Numberical Results Summary

Qiyuan Pang

June 14, 2021

According to the preprint, we solve $Kx = b$ by solving

$$\hat{K}^*\hat{K}x = \hat{K}^*b,$$

preconditioned with and without $\hat{G} = (\hat{K}^*\hat{K})^{-1}$. The following quantities are used in the rest of the section to evaluate the performance of the preconditioner:

- N: problem size;
- e_a: the relative error set for the butterfly approximation $\hat{K}$ of K;
- ϵ: the fixed tolerance set in HIF;
- r: the fixed maximum rank set in HIF;
- e_s: the relative error of the approximation $\hat{G}\hat{K}^*$ of K^{-1}, defined as $\|\hat{G}\hat{K}^*b - x\|/\|x\|$ where x is a random vector and $b = Kx$;
- n_i: the number of iterations used in PCG until covergence;
- e: the relative error of the solution returned by PCG.

Among all experiments below, the stopping criteria set for PCG is tolerance $1e-8$.

Examples (1D). We begin with an example of 1D discrete FIO of the form

$$u(x) = \int_{\mathbb{R}} a(x)e^{2\pi i\Phi(x,\xi)}\hat{f}(\xi)d\xi$$

with uniform amplitude $a(x, \xi) = 1$.

There are five 1D kernels to test here, as follows:

$$\Phi(x, \xi) = x \cdot \xi + c(x)|\xi|, c(x) = (2 + \sin(2\pi x))/8, \tag{1}$$

$$\Phi(x, \xi) = x \cdot \xi + c(x)\xi, c(x) = (2 + 0.2\sin(2\pi x))/16, \tag{2}$$

$$\Phi(x, \xi) = x \cdot \xi + (2 + 0.2\sin(2\pi x))(2 + 0.5\cos(2\pi\xi) + \xi^2), \tag{3}$$

$$\Phi(x, \xi) = (x + \sin(2\pi x))(\xi + \cos(2\pi\xi)), \tag{4}$$

$$\Phi(x,\xi) = x\cdot\xi + c(x)|\xi|, c(x) = (2+\sin(2\pi x))/32, \tag{5}$$

Discretizing x and ξ on $[0,1)$ and $[-N/2, N/2)$ with N points,

$$x_i = (i-1)/N, \xi_j = j-1-N/2.$$

leads to the discrete system $u = Kf$.

Table **??** and **??** summarize the results for 1D kernel (1). Table **??** and **??** summarize the results for 1D kernel (2). Table **??** and **??** summarize the results for 1D kernel (3). Table **??** and **??** summarize the results for 1D kernel (4). Table **??** and **??** summarize the results for 1D kernel (5).

Examples (2D). Then, we consider some 2D analogs of the 1D examples,

$$u(x) = \sum_{\xi\in\Omega} e^{2\pi\Phi(x,\xi)}\hat{f}(\xi), x\in X,$$

and

$$X = \{x = (\frac{n_1}{n}, \frac{n_2}{n}), 0 \le n_1, n_2 < n, with n_1, n_2 \in \mathbb{Z}\},$$
$$\Omega = \{\xi = (n_1, n_2), -\frac{n}{2} \le n_1, n_2 < \frac{n}{2}, with n_1, n_2 \in \mathbb{Z}\},$$

with n being the number of points in each dimension and $N = n^2$.

We consider four kernels as follows:

$$\Phi(x,\xi) = x\cdot\xi + \sqrt{c_1^2(x)\xi_1^2 + c_2^2(x)\xi_2^2}, c_1(x) = (2+\sin(2\pi x_1)\sin(2\pi x_2))/32, c_2(x) = (2+\cos(2\pi x_1)\cos(2\pi x_2))/32, \tag{6}$$

$$\Phi(x,\xi) = x\cdot\xi + \sqrt{c_1^2(x)\xi_1^2 + c_2^2(x)\xi_2^2}, c_1(x) = (2+\sin(2\pi x_1)\sin(2\pi x_2))/16, c_2(x) = (2+\cos(2\pi x_1)\cos(2\pi x_2))/16, \tag{7}$$

$$\Phi(x,\xi) = x\cdot\xi + \sqrt{c_1^2(x)\xi_1^2 + c_2^2(x)\xi_2^2}, c_1(x) = (2+\sin(2\pi x_1)\sin(2\pi x_2))/8, c_2(x) = (2+\cos(2\pi x_1)\cos(2\pi x_2))/8, \tag{8}$$

$$\Phi(x,\xi) = x\cdot\xi + \sqrt{c_1^2(x)\xi_1^2 + c_2^2(x)\xi_2^2}, c_1(x) = (2+\sin(2\pi x_1)\sin(2\pi x_2))/4, c_2(x) = (2+\cos(2\pi x_1)\cos(2\pi x_2))/4, \tag{9}$$

Table **??** and **??** summarize the results for 2D kernel (6). Table **??** and **??** summarize the results for 2D kernel (7). Table **??** and **??** summarize the results for 2D kernel (8). Table **??** and **??** summarize the results for 2D kernel (9).

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
256	1e-7	1e-6	2.1336e-07	2	2.4560e-13	25	9.9473e-09
		1e-3	3.3399e-04	3	2.4695e-10	25	9.9473e-09
625	1e-7	1e-6	3.6876e-07	2	2.9753e-13	26	9.2094e-09
		1e-3	2.3161e-04	3	1.0952e-10	26	9.2094e-09
1296	1e-7	1e-6	4.0501e-07	2	5.4640e-13	27	5.1549e-09
		1e-3	2.4908e-04	3	1.0500e-10	27	5.1549e-09
2401	1e-7	1e-6	2.0698e-07	2	2.4927e-13	27	5.5965e-09
		1e-3	2.7872e-04	3	2.5319e-10	27	5.5965e-09
4096	1e-7	1e-6	3.7231e-07	2	1.1721e-12	27	5.4880e-09
		1e-3	4.4070e-04	3	3.7392e-10	27	5.4880e-09
6561	1e-7	1e-6	3.4479e-07	2	8.3411e-13	27	5.3598e-09
		1e-3	2.1708e-04	3	2.3232e-10	27	5.3598e-09

Table 1: Numerical results for 1D uniform amplitude FIO (1) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
256	1e-7	1e-6	5.5818e+00	50	5.8919e-05	48	1.5806e-02
		1e-3	2.9826e+00	29	5.0284e-03	48	1.5806e-02
625	1e-7	1e-6	1.4768e+00	0	1.0000e+00	2	1.4901e-01
		1e-3	7.6218e+00	0	1.0000e+00	2	1.4901e-01
1296	1e-7	1e-6	3.4273e+00	0	1.0000e+00	2	1.9735e-01
		1e-3	1.3683e+00	0	1.0000e+00	2	1.9735e-01
2401	1e-7	1e-6	7.8132e+00	1	9.5723e-01	2	9.6268e-02
		1e-3	3.1408e+01	1	9.3398e-01	2	9.6268e-02
4096	1e-7	1e-6	2.2074e+01	0	1.0000e+00	1	3.0659e-01
		1e-3	4.0021e+00	0	1.0000e+00	1	3.0659e-01
6561	1e-7	1e-6	2.4005e+01	0	1.0000e+00	1	3.1116e-01
		1e-3	2.5085e+00	0	1.0000e+00	1	3.1116e-01

Table 2: Numerical results for 1D uniform amplitude FIO (2) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
256	1e-7	1e-6	3.2323e-11	1	2.3212e-11	46	4.1806e-01
		1e-3	3.2323e-11	1	2.3212e-11	46	4.1806e-01
625	1e-7	1e-6	4.7458e-11	1	5.2122e-11	7	4.3607e-01
		1e-3	4.7458e-11	1	5.2122e-11	7	4.3607e-01
1296	1e-7	1e-6	2.2310e+01	1	5.1605e-10	32	4.6936e-01
		1e-3	2.2310e+01	1	5.1605e-10	32	4.6936e-01
2401	1e-7	1e-6	1.2082e-04	1	1.8969e-10	8	4.8654e-01
		1e-3	1.2082e-04	1	1.8969e-10	8	4.8654e-01
4096	1e-7	1e-6	1.7371e+01	1	3.0852e-09	0	1.0000e+00
		1e-3	1.7371e+01	1	3.0852e-09	0	1.0000e+00
6561	1e-7	1e-6	2.9124e+01	2	6.6032e-09	0	1.0000e+00
		1e-3	2.9124e+01	2	6.6032e-09	0	1.0000e+00

Table 3: Numerical results for 1D uniform amplitude FIO (3) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
256	1e-7	1e-6	1.7666e-08	2	5.8438e-16	5	2.4401e-09
		1e-3	1.5120e+01	0	1.0000e+00	3	4.5023e-01
625	1e-7	1e-6	4.5527e-08	2	4.6174e-15	5	2.3783e-09
		1e-3	2.0976e+01	0	1.0000e+00	1	4.1921e-01
1296	1e-7	1e-6	3.7013e-08	2	4.5873e-15	5	2.1702e-09
		1e-3	1.4169e+02	0	1.0000e+00	1	3.9825e-01
2401	1e-7	1e-6	9.9584e+01	0	1.0000e+00	1	5.7565e-01
		1e-3	9.9584e+01	0	1.0000e+00	1	5.7565e-01
4096	1e-7	1e-6	1.6244e+03	0	1.0000e+00	1	4.3098e-01
		1e-3	1.6244e+03	0	1.0000e+00	1	4.3098e-01
6561	1e-7	1e-6	1.4026e+04	0	1.0000e+00	0	1.0000e+00
		1e-3	1.4026e+04	0	1.0000e+00	0	1.0000e+00

Table 4: Numerical results for 1D uniform amplitude FIO (4) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
256	1e-7	1e-6	1.7666e-08	2	5.8438e-16	5	2.4401e-09
		1e-3	1.7696e-06	2	6.1726e-12	5	2.4401e-09
625	1e-7	1e-6	4.5527e-08	2	4.6174e-15	5	2.3783e-09
		1e-3	2.0556e-06	2	6.6328e-12	5	2.3783e-09
1296	1e-7	1e-6	3.7013e-08	2	4.5873e-15	5	2.1702e-09
		1e-3	2.9471e-06	2	8.2380e-12	5	2.1702e-09
2401	1e-7	1e-6	5.6659e-08	2	9.6044e-15	5	2.2438e-09
		1e-3	2.1969e-06	2	6.7013e-12	5	2.2438e-09
4096	1e-7	1e-6	2.8581e-07	2	7.7674e-13	5	2.2087e-09
		1e-3	1.8748e-06	2	6.2216e-12	5	2.2087e-09
6561	1e-7	1e-6	2.8826e-07	2	7.4096e-13	5	2.1898e-09
		1e-3	2.3606e-06	2	7.0394e-12	5	2.1898e-09

Table 5: Numerical results for 1D uniform amplitude FIO (5) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
256	1e-7	$15log_2N$	1.7940e-08	1	9.8290e-16	25	9.9473e-09
		$8log_2N$	2.1437e-05	2	2.8929e-09	25	9.9473e-09
625	1e-7	$15log_2N$	9.0664e-05	3	4.2247e-11	26	9.2094e-09
		$8log_2N$	1.9254e-03	4	7.9901e-09	26	9.2094e-09
1296	1e-7	$15log_2N$	1.8609e-04	3	1.5774e-09	27	5.1549e-09
		$8log_2N$	1.2149e-03	5	1.2136e-10	27	5.1549e-09
2401	1e-7	$15log_2N$	5.9234e-04	3	6.2116e-09	27	5.5965e-09
		$8log_2N$	6.1468e-03	6	2.1640e-10	27	5.5965e-09
4096	1e-7	$15log_2N$	3.6670e-03	6	2.9362e-10	27	5.4880e-09
		$8log_2N$	7.3599e-03	5	7.0574e-09	27	5.4880e-09
6561	1e-7	$15log_2N$	5.3460e-03	5	5.9872e-09	27	5.3598e-09
		$8log_2N$	9.6651e-03	6	1.5906e-09	27	5.3598e-09

Table 6: Numerical results for 1D uniform amplitude FIO (1) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
256	1e-7	$15log_2N$	4.4706e-02	50	1.8611e-08	48	1.5806e-02
		$8log_2N$	2.6151e+04	14	4.6737e-03	48	1.5806e-02
625	1e-7	$15log_2N$	3.6370e+07	0	1.0000e+00	2	1.4901e-01
		$8log_2N$	6.4132e+06	0	1.0000e+00	2	1.4901e-01
1296	1e-7	$15log_2N$	7.8196e+02	0	1.0000e+00	2	1.9735e-01
		$8log_2N$	5.8627e+01	0	1.0000e+00	2	1.9735e-01
2401	1e-7	$15log_2N$	7.8292e+00	0	1.0000e+00	2	9.6268e-02
		$8log_2N$	1.0408e+01	0	1.0000e+00	2	9.6268e-02
4096	1e-7	$15log_2N$	8.2260e+00	0	1.0000e+00	1	3.0659e-01
		$8log_2N$	9.7050e+00	0	1.0000e+00	1	3.0659e-01
6561	1e-7	$15log_2N$	3.4527e+01	0	1.0000e+00	1	3.1116e-01
		$8log_2N$	1.0948e+01	2	9.6515e-01	1	3.1116e-01

Table 7: Numerical results for 1D uniform amplitude FIO (2) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
256	1e-7	$15log_2N$	3.2323e-11	1	2.3212e-11	46	4.1806e-01
		$8log_2N$	4.9004e+00	35	5.7483e-09	46	4.1806e-01
625	1e-7	$15log_2N$	2.2940e+01	44	4.3912e-01	7	4.3607e-01
		$8log_2N$	2.2687e+01	32	4.4483e-01	7	4.3607e-01
1296	1e-7	$15log_2N$	1.3739e+01	50	4.7349e-01	32	4.6936e-01
		$8log_2N$	6.9500e+00	34	4.9653e-01	32	4.6936e-01
2401	1e-7	$15log_2N$	2.7433e+00	9	5.0112e-01	8	4.8654e-01
		$8log_2N$	1.3079e+00	7	4.8838e-01	8	4.8654e-01
4096	1e-7	$15log_2N$	4.4946e+00	0	1.0000e+00	0	1.0000e+00
		$8log_2N$	6.4175e+00	0	1.0000e+00	0	1.0000e+00
6561	1e-7	$15log_2N$	5.6588e+00	0	1.0000e+00	0	1.0000e+00
		$8log_2N$	1.2717e+00	0	1.0000e+00	0	1.0000e+00

Table 8: Numerical results for 1D uniform amplitude FIO (3) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
256	1e-7	$15log_2N$	1.5120e+01	0	1.0000e+00	3	4.5023e-01
		$8log_2N$	2.4166e+01	0	1.0000e+00	3	4.5023e-01
625	1e-7	$15log_2N$	2.0845e+01	10	7.5184e-01	1	4.1921e-01
		$8log_2N$	1.6137e+01	13	7.3399e-01	1	4.1921e-01
1296	1e-7	$15log_2N$	2.8380e+01	12	6.3824e-01	1	3.9825e-01
		$8log_2N$	1.4015e+01	8	6.0355e-01	1	3.9825e-01
2401	1e-7	$15log_2N$	3.8884e+01	12	6.8831e-01	1	5.7565e-01
		$8log_2N$	2.4676e+00	5	6.3549e-01	1	5.7565e-01
4096	1e-7	$15log_2N$	1.9453e+01	3	5.4779e-01	1	4.3098e-01
		$8log_2N$	3.7073e+00	2	5.0636e-01	1	4.3098e-01
6561	1e-7	$15log_2N$	1.1592e+01	0	1.0000e+00	0	1.0000e+00
		$8log_2N$	6.2269e+00	0	1.0000e+00	0	1.0000e+00

Table 9: Numerical results for 1D uniform amplitude FIO (4) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
256	1e-7	$15log_2N$	1.9513e-08	1	8.2233e-16	5	2.4401e-09
		$8log_2N$	1.9337e-08	1	1.8995e-09	5	2.4401e-09
625	1e-7	$15log_2N$	5.8074e-07	2	2.3352e-12	5	2.3783e-09
		$8log_2N$	5.2226e-07	2	1.9176e-12	5	2.3783e-09
1296	1e-7	$15log_2N$	8.7199e-07	2	3.6109e-12	5	2.1702e-09
		$8log_2N$	7.4380e-07	2	4.9661e-12	5	2.1702e-09
2401	1e-7	$15log_2N$	6.1061e-07	2	2.8134e-12	5	2.2438e-09
		$8log_2N$	5.4379e-07	2	1.8375e-12	5	2.2438e-09
4096	1e-7	$15log_2N$	6.2313e-07	2	4.3019e-12	5	2.2087e-09
		$8log_2N$	2.1256e-06	2	4.5048e-11	5	2.2087e-09
6561	1e-7	$15log_2N$	1.0210e-06	2	1.7526e-11	5	2.1898e-09
		$8log_2N$	1.6611e-06	2	6.7902e-11	5	2.1898e-09

Table 10: Numerical results for 1D uniform amplitude FIO (5) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
16^2	1e-7	1e-3	5.0303e-05	2	8.1092e-09	10	9.3295e-09
25^2	1e-7	1e-3	6.4488e-05	3	1.5497e-12	11	4.7878e-09
36^2	1e-7	1e-3	6.9689e-05	3	1.8378e-12	11	8.1950e-09
49^2	1e-7	1e-3	5.2556e-04	3	2.4241e-10	12	2.0339e-09
64^2	1e-7	1e-3	6.7135e-05	3	2.2070e-11	12	2.2750e-09
81^2	1e-7	1e-3	6.0597e-05	3	3.4154e-12	12	2.4716e-09

Table 11: Numerical results for 2D uniform amplitude FIO (6) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
16^2	1e-7	1e-3	5.8652e-05	3	1.4229e-12	17	3.2067e-09
25^2	1e-7	1e-3	1.1249e-04	3	8.1081e-12	17	9.8648e-09
36^2	1e-7	1e-3	1.5834e-04	3	3.2388e-11	19	3.6075e-09
49^2	1e-7	1e-3	5.2357e-04	3	6.1662e-10	19	6.9227e-09
64^2	1e-7	1e-3	1.6066e-04	3	5.6871e-11	20	4.7774e-09
81^2	1e-7	1e-3	1.5370e-04	3	6.5701e-11	20	7.3115e-09

Table 12: Numerical results for 2D uniform amplitude FIO (7) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
16^2	1e-7	1e-3	1.8957e-12	1	1.2056e-15	29	4.9053e-09
25^2	1e-7	1e-3	2.8679e-04	3	2.8057e-10	35	7.9026e-09
36^2	1e-7	1e-3	4.4268e-04	3	1.8384e-09	40	8.3131e-09
49^2	1e-7	1e-3	6.1393e-03	4	3.5965e-11	47	6.0100e-09
64^2	1e-7	1e-3	7.5916e-04	3	3.6429e-09	50	1.3345e-08
81^2	1e-7	1e-3	9.4295e-04	3	8.6226e-09	50	8.9494e-08

Table 13: Numerical results for 2D uniform amplitude FIO (8) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	ϵ	e_s	n_i	e	n_i	e
16^2	1e-7	1e-3	1.9113e-11	1	3.4568e-13	50	3.8826e-01
25^2	1e-7	1e-3	2.3893e-10	1	6.1817e-11	5	7.8271e-01
36^2	1e-7	1e-3	2.5031e+02	35	5.6451e-08	7	9.2996e-01
49^2	1e-7	1e-3	2.1935e+03	0	1.0000e+00	5	9.5320e-01
64^2	1e-7	1e-3	5.2873e+03	0	1.0000e+00	5	9.5751e-01
81^2	1e-7	1e-3	4.3305e+03	0	1.0000e+00	0	1.0000e+00

Table 14: Numerical results for 2D uniform amplitude FIO (9) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
16^2	1e-7	$8log_2N$	3.0037e-05	2	4.6138e-09	10	9.3295e-09
25^2	1e-7	$8log_2N$	3.9225e-04	3	2.2359e-09	11	4.7878e-09
36^2	1e-7	$8log_2N$	1.5552e-03	4	2.8455e-10	11	8.1950e-09
49^2	1e-7	$8log_2N$	3.6387e-03	5	1.1175e-09	12	2.0339e-09
64^2	1e-7	$8log_2N$	6.1327e-03	5	6.9027e-09	12	2.2750e-09
81^2	1e-7	$8log_2N$	9.3773e-03	6	3.4127e-09	12	2.4716e-09

Table 15: Numerical results for 2D uniform amplitude FIO (6) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
16^2	1e-7	$8log_2N$	2.5844e-04	3	5.9544e-10	17	3.2067e-09
25^2	1e-7	$8log_2N$	2.8387e-03	4	1.7453e-09	17	9.8648e-09
36^2	1e-7	$8log_2N$	6.7332e-03	5	1.8601e-09	19	3.6075e-09
49^2	1e-7	$8log_2N$	9.6055e-03	6	5.8901e-09	19	6.9227e-09
64^2	1e-7	$8log_2N$	1.5655e-02	7	7.8324e-10	20	4.7774e-09
81^2	1e-7	$8log_2N$	2.0175e-02	8	1.3675e-09	20	7.3115e-09

Table 16: Numerical results for 2D uniform amplitude FIO (7) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
16^2	1e-7	$8log_2N$	1.0579e-02	5	1.7571e-09	29	4.9053e-09
25^2	1e-7	$8log_2N$	3.1286e-02	7	5.1513e-10	35	7.9026e-09
36^2	1e-7	$8log_2N$	5.8317e-02	9	7.9130e-09	40	8.3131e-09
49^2	1e-7	$8log_2N$	6.6517e-02	10	2.9707e-09	47	6.0100e-09
64^2	1e-7	$8log_2N$	7.2656e-02	11	4.3443e-09	50	1.3345e-08
81^2	1e-7	$8log_2N$	8.1396e-02	13	3.6063e-09	50	8.9494e-08

Table 17: Numerical results for 2D uniform amplitude FIO (8) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.

	$\hat{K} \approx K$	$\hat{G}\hat{K}^* \approx K^{-1}$				$\hat{K}^* \approx \hat{K}^*$	
N	e_a	r	e_s	n_i	e	n_i	e
16^2	1e-7	$8log_2N$	5.4117e+01	27	3.1304e-09	50	3.8826e-01
25^2	1e-7	$8log_2N$	5.8317e+00	0	1.0000e+00	5	7.8271e-01
36^2	1e-7	$8log_2N$	3.2884e+02	0	1.0000e+00	7	9.2996e-01
49^2	1e-7	$8log_2N$	1.8200e+01	0	1.0000e+00	5	9.5320e-01
64^2	1e-7	$8log_2N$	1.2028e+02	0	1.0000e+00	5	9.5751e-01
81^2	1e-7	$8log_2N$	8.5404e+01	30	9.3999e-01	0	1.0000e+00

Table 18: Numerical results for 2D uniform amplitude FIO (9) using the approximate inverse $\hat{G}\hat{K}^*$ and the adjoint FIO matrix $\hat{K}^*$ as preconditioners for PCG with tolerance $1e-8$.