COMP5313/COMP4313 - Large Scale Networks

Week 2b&3a: Structural Balance

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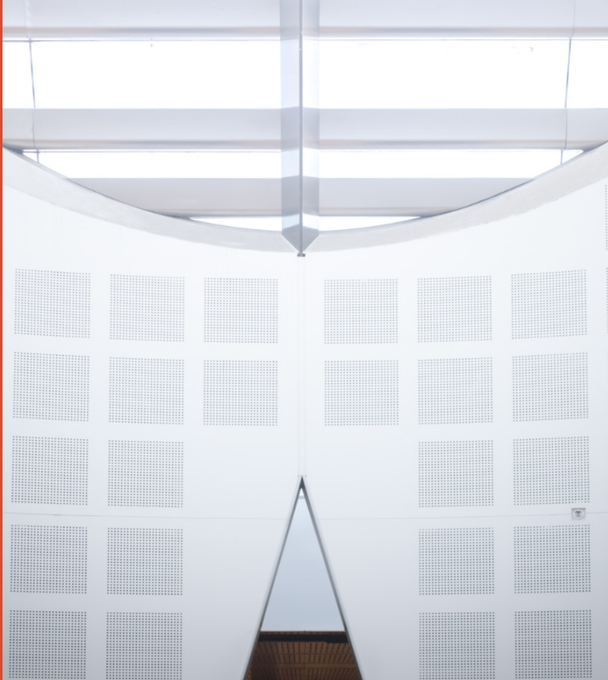


Introduction

- We have looked at the strength (i.e., strong ties and weak ties) of links in a network.
- Now we will talk about the positive and negative relationships that affect the structures
  - Signed graph and structural balance
  - Example applications of signed graph

Outline

Structural Balance

Generalization

Example Applications of Signed Graphs

Structural Balance

- The theory behind structural balance comes from social psychology in 1940s. [1]
- Take two connected persons in isolation
  - Label the edge + if they are friends
  - Label the edge - if they are enemies
  - A signed graph is a graph in which each edge has a positive or a negative sign
- Take three connected persons, certain configurations are more plausible than others

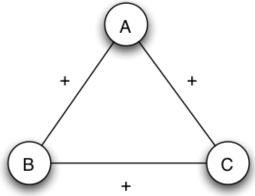
[1] F. Heider, "Attitudes and cognitive organization," *The Journal of psychology*, vol. 21, no. 1, pp. 107–112, 1946.

Structural Balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

Scenario 1: Three 'plus' edges

- This is a natural situation
- It corresponds to three people who are mutually friends



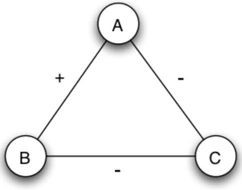
A, B and C are mutual friends: balanced

Structural Balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

Scenario 2: One 'plus' edge and two 'minus' edges

- This is also a natural situation
- Two of the three are friends and they have a mutual enemy



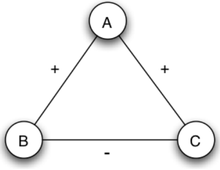
A and B are friends and have C as a mutual enemy: balanced

Structural Balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

Scenario 3: Two ‘plus’ edges and one ‘minus’ edges

- Creates some instability
- A is friends with B and C who do not get along with each other



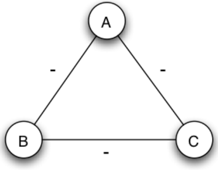
A is friends with B and C but they are not friends: not balanced

Structural Balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

Scenario 4: Three ‘minus’ edges

- Creates some instability
- There are forces motivating two people to team up to against the third



A, B and C are mutual enemies: not balanced

Structural Balance

- Conclusions
  - We refer to triangles with one or three '+' as balanced since they are free from instability
  - We refer to triangles with zero or two '+' as unbalanced since they are unstable
- Unbalanced triangles are sources of stress so that people strive to minimize them in their personal relationships
  - Unbalanced triangles will thus be less abundant in real social settings than balanced triangles

Structural Balance

- How to generalize structural balance to any complete graph?
- A labeled complete graph is balanced if every one of its triangles is balanced

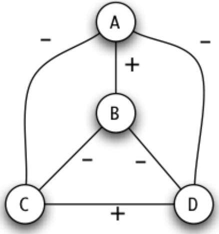
Structural balance property (SBP):

For every set of three nodes, if we consider the three edges connecting them, either all three of these edges are labeled $+$ or exactly one of them is labeled $+$.

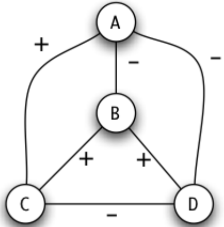
Structural Balance

Examples:

- The labeled four-node complete graph on the left is balanced because each set of 3 nodes satisfies the structural balance property



Balanced



Not balanced

- The one on the right is unbalanced because triangle A, B, C and triangle B, C, D violate the structural balance property

Structural Balance

- At a high level, how does a balance network look like?
- One way to be balanced, is if everyone likes each other
  - All triangles have thus three '$+$' labels
- A slightly more complicated representation would be:
  - Consider two groups X and Y
  - Everyone in X likes each other
  - Everyone in Y likes each other
  - And everyone in X is the enemy of everyone in Y

Structural Balance

- This leads to two basic ways of achieving structural balance:
  - Everyone likes each other
  - The world consists of two groups of mutual friends with complete antagonism between the groups

The Balance Theorem: [2]

If a labeled complete graph is balanced, then either all pairs of nodes are friends, or else the nodes can be divided into two groups, X and Y, such that each pair of people in X likes each other, each pair of people in Y likes each other and everyone in X is the enemy of everyone in Y

- The balance theorem takes a local property (structural balance property) and implies a global property:
  - either everyone gets along
  - or the world is divided into two enemy groups

[2] F. Harary *et al.*, "On the notion of balance of a signed graph.," *The Michigan Mathematical Journal*, vol. 2, no. 2, pp. 143–146, 1953.

Structural Balance

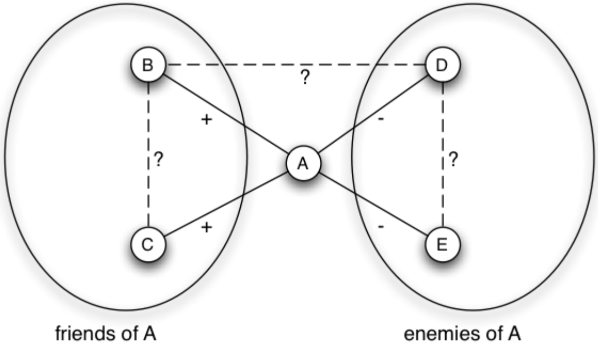
Proof

- Suppose we have a balanced arbitrary labeled complete graph
- If the graph has only + labels, then we are done. Assume this isn't the case
  - Let A be a node of a group X and let Y be another group
    - Every other node is either a friend of A or an enemy of A (due to completeness)
    - Let X be A and all its friends and Y be the rest
  - We need to show 3 properties:
    1. Every two nodes in X are friends
    2. Every two nodes in Y are friends
    3. Every node in X is an enemy of every node in Y
  - We now show that our definition of X and Y satisfies these 3 properties

Structural Balance

Proof (contd.)

- A schematic illustration of our analysis of balanced networks. (There may be other nodes not illustrated here.)



Structural Balance

Proof (contd.)

- We now show that our definition of X and Y satisfies these 3 properties
  1. A is friends with every other node in X
     $\Longrightarrow$ B and C in X are friends as well, otherwise triangle A, B, C would violate SBP
  2. A is enemy with every node in Y
     $\Longrightarrow$ D and E in Y are friends, otherwise triangle A, D, E would violate SBP
  3. A is friend with any B in X and enemy with any D in Y
     $\Longrightarrow$ B and D are enemies, otherwise triangle A, B, D would violate SBP
- This concludes the proof

Outline

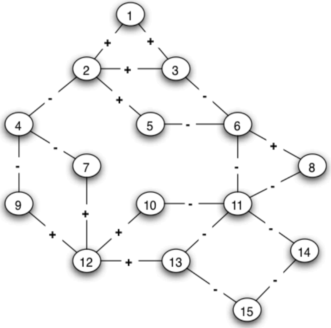
Structural Balance

Generalization

Example Applications of Signed Graphs

Generalization

- So far our definitions are restrictive:
  - They only apply to complete graphs
    However, persons may not have an opinion on others



- Can we generalize the balance theorem to incomplete graphs?

Generalization

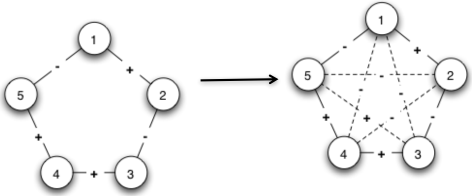
- Let's consider a social network that is not necessarily complete
  - Two nodes may be linked by a positive edge
  - Two nodes may be linked by a negative edge
  - Two node may not be linked to each other

- We can relax the former definition in two ways:
  1. We consider that the given graph misses some edges: The network is balanced if we can complete it with some edges that lead to a complete graph that is balanced under the former definition
  2. We consider that the given graph should be divisible into two sets: The network is balanced if it is possible to divide the nodes into two sets, so that any edge within one set is positive, any edge across sets is negative

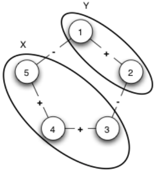
Generalization

Examples

1. A graph can be completed into a complete graph that satisfies the former property



2. A graph can be divided into two sets with positive intra-set and negative inter-set edges



Generalization

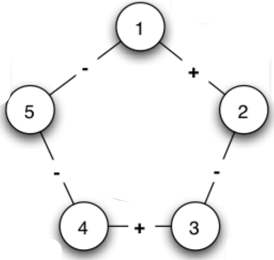
These two definitions (by completing edges or dividing nodes) are equivalent

- Definition (1) implies definition (2)
  - If a signed graph is balanced under the definition (1) then after filling in all the missing edges appropriately we obtain a signed complete graph where we can apply the Balance Theorem
  - This approach divides the network into two sets, X and Y, that satisfy the properties of the definition (2)

- Definition (2) implies definition (1)
  - If a signed graph is balanced under definition (2) then after finding a division of the nodes into X and Y, we can fill positive edges inside X and inside Y and fill in negative edges between X and Y and check that all triangles will be balanced, satisfying definition (1)

Generalization

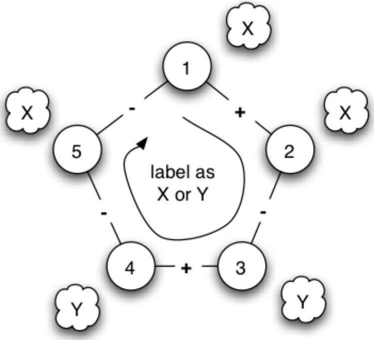
- Is this graph balanced?



Generalization

No it is not balanced

- Try going through each edge clock-wise
- Place endpoints in the same set if you cross a + edge
- Place them in different sets if you cross a - edge
- You cannot do that for all edges without changing your initial decision



Getting back to node 1 induced crossing an odd number of negative edges

Generalization

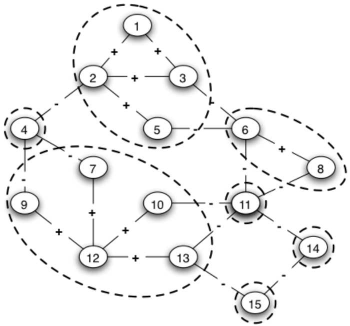
Claim

A signed graph is balanced if and only if it contains no cycle with an odd number of negative edges.

- The proof proceeds by
  - Either finding a balanced division in sets X and Y in which all edges are positive and across which all edges are negative
  - Or finding a cycle with an odd number of negative edges
- Find the supernodes representing blobs of positively connected nodes so that supernodes are connected through negative edges

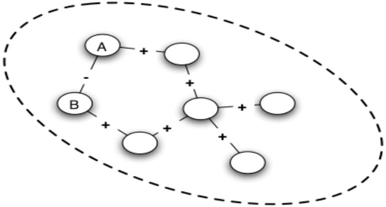
Generalization

- To determine whether a signed graph is balanced, we use a two-step approach.
  1. the first step considers only positive edges, to find the supernodes.



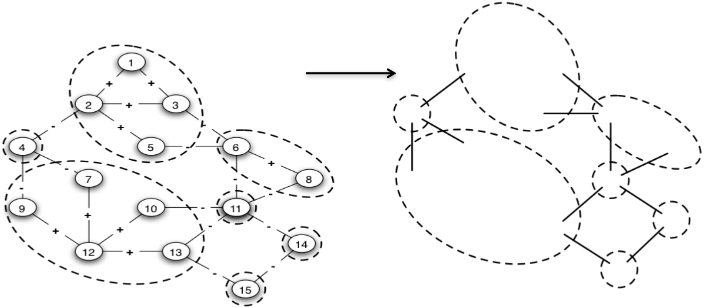
Generalization

- If any supernode contains a negative edge between some pair of nodes A and B, then the graph contains a cycle with an odd number of negative edges and thus is not balanced:
  - take a path of positive edges from B to A and
  - take the negative edge between A and B



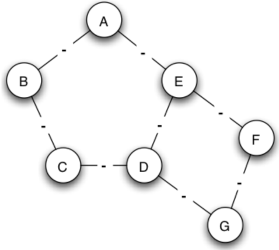
Generalization

- To determine whether a signed graph is balanced, we use a two-step approach.
  1. the first step considers only positive edges, to find the supernodes.
  2. the second step considers a simpler graph, with only negative edges
     - nodes are the supernodes of the previous graph
     - there is an edge between two supernodes if there is an edge in the previous graph whose two end-points are from the two supernodes, respectively.



Generalization

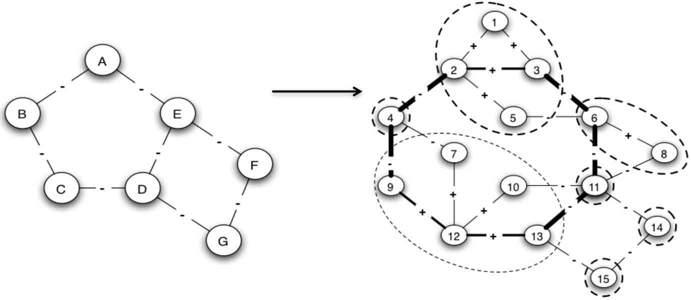
- A more standard drawing of the previous graph where we visualize the negative edges between supernodes



- From now on, there are two options:
  - Either we label each node in this reduced graph as X or Y so that each edge connects X to Y
  - Or we find a cycle in the reduced graph with an odd number of edges

Generalization

- Once we have found a cycle of an odd number of negative edges in the reduced graph, we can determine a cycle of an odd number of negative edges in the original graph by listing the nodes connected within a supernode with positive edges and that connect this cycle



Generalization

- This version of finding an "odd" cycle where the underlying graph has only negative edges is known as the problem of determining whether a graph is bipartite
  - Whether its nodes can be divided into two groups (e.g., X and Y) so that each edge goes from one group to the other

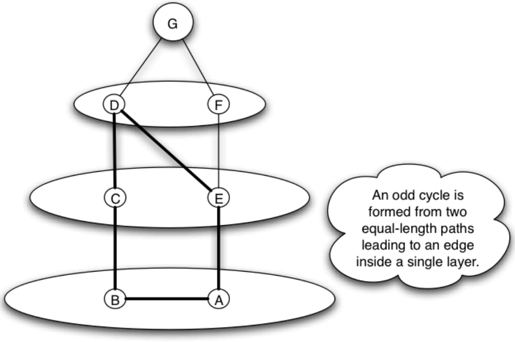
- If we can find whether the graph is bipartite, then we know whether there are no odd cycles?

- How to determine whether a graph is bipartite using breadth-first search (BFS) ?

Generalization

- We start a BFS from any node in the graph (e.g. G), producing layers
- Because edges cannot jump over a layer of the breadth-first search, then
  - Edges connect nodes in adjacent layers or nodes in the same layer

- **Case 1:** Balanced division
  - even-numbered layers as part of X
  - odd-numbered layers as part of Y
- **Case 2:** Cycle
  - two connected nodes (A and B) in the same layer have an immediate common ancestor (D)
  - the length of paths from D to A and from D to B are of same size k
    $\Longrightarrow$ This creates a cycle of size $2k+1$: an odd number



Outline

Applications

Let's consider two types of applications of structural balance

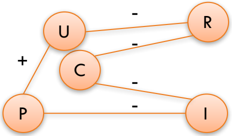
- **International relations** can be represented as a network of countries whose relations are a combination of alliances and animosities

- **Online rating Web sites** offer individuals the possibility to express positive or negative opinions about each other

International Relations

- International relations is a setting in which it is natural to assume that a collection of nodes all have opinions (positive or negative) about one another
  - Nodes are nations
  - Edge labeled + indicate alliance
  - Edge labeled - indicate animosity

- Structural balance sometimes explains behaviors of nations during crises [3]
  - Conflict over Bangladesh's separation from Pakistan in 1972
  - US's support to Pakistan is not surprising considering that:
    - USSR (R) was China's enemy
    - China was India's enemy
    - India had bad relations with Pakistan



[3] M. Moore, "An international application of heider's balance theory," *European Journal of Social Psychology*, vol. 8, no. 3, pp. 401–405, 1978.

Online Ratings

- Slashdot is a news website on science and technology
  - Users can designate each other as a “friend” or a “foe”
- Epinions is an online product rating site
  - Users evaluate products
  - Users express trust or distrust of other users

Online Ratings

- Epinion analysis revealed differences between online ratings and friend-enemy dichotomy of structural balance theory. [4]
  - Users of Epinion form a directed graph
  - If A trusts B and B trusts C, then A should trust C
  - If A distrusts B and B distrusts C, then should A trust C?
    - If distrust was like enmity, then yes
    - If someone distrust someone else because she is more knowledgeable, then we should expect the opposite

[4] R. Guha, R. Kumar, P. Raghavan, and A. Tomkins, "Propagation of trust and distrust," in *Proceedings of the 13th international conference on World Wide Web*, pp. 403–412, ACM, 2004.

Conclusion

- A signed graph represents the positive and negative relations in a network
- The Balance Theorem illustrates how local relations impact globally the network
- Determining whether an incomplete network is balanced can be achieved through a BFS on the supernodes of the network