APPENDIX C

Effective Elastic Media:

Voigt–Reuss–Hill Average

and Backus Average

D.1 Voigt–Reuss–Hill average moduli estimate

The Voigt–Reuss–Hill average is simply the arithmetic average of the Voigt upper bound and the Reuss lower bound. This average is expressed as

$$M_{VRH} = \frac{M_V + M_R}{2} \tag{D-1}$$

where

$$M_V = \sum_{i=1}^{N} f_i M_i \tag{D-2}$$

$$\frac{1}{M_R} = \sum_{i=1}^{N} \frac{1}{f_i M_i} \tag{D-3}$$

The terms f_i and M_i are the volume fraction and the modulus of the ith component, respectively. Although M can be any modulus, it makes most sense for it to be the shear modulus or the bulk modulus.

The Voigt–Reuss–Hill average is useful when an estimate of the moduli is needed, not just the allowable range of values. An obvious extension would be to average, instead, the Hashin–Shtrikman upper and lower bounds.

This resembles, but is not exactly the same as the average of the algebraic and harmonic means of velocity used by Greenberg and Castagna (1992) in their empirical $V_P - V_S$ relation.

Uses of Voigt–Reuss–Hill average

The Voigt–Reuss–Hill average is used to estimate the effective elastic moduli of a rock in terms of its constituents and pore space.

Assumptions and limitations of Voigt–Reuss–Hill average

The following limitation and assumption apply to the Voigt–Reuss–Hill average the result is strictly heuristic. Hill (1952) showed that the Voigt and Reuss averages

- are upper and lower bounds, respectively. Several authors have shown that the average of these bounds can be a useful and sometimes accurate estimate of rock properties;

- the rock is isotropic.

D.2 Elastic constants in finely layered media: Backus average

A transversely isotropic medium with the symmetry axis in the x_3-direction has an elastic stiffness tensor that can be written in the condensed Voigt matrix form

$$\begin{bmatrix} a & b & f & 0 & 0 & 0 \\ b & a & f & 0 & 0 & 0 \\ f & f & c & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}, \quad m = \frac{1}{2}(a - b)$$

wherea, a, b, c, d and f are five independent elastic constants. Backus (1962) showed that in the long-wavelength limit a stratified medium composed of layers of transversely isotropic materials(each with its symmetry axis normal to the strata) is also effectively anisotropic, with effective stiffness as follows:

$$\begin{bmatrix} A & B & F & 0 & 0 & 0 \\ B & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}, \quad M = \frac{1}{2}(A - B)$$

where

$$A = \langle a - f^2 c^{-1} \rangle + \langle c^{-1} \rangle^{-1} \langle f c^{-1} \rangle^2$$

$$B = \langle b - f^2 c^{-1} \rangle + \langle c^{-1} \rangle^{-1} \langle f c^{-1} \rangle^2$$

$$C = \langle c^{-1} \rangle^{-1}$$

$$F = \langle c^{-1} \rangle^{-1} \langle f c^{-1} \rangle$$

$$M = \langle m \rangle$$

The brackets $\langle \cdot \rangle$ indicate averages of the enclosed properties weighted by their volumetric proportions. This is often called the Backus average.

If the individual layers are isotropic, the effective medium is still transversely isotropic, but the number of independent constants needed to describe each individual layer is reduced to 2:

$$a = c = \lambda + 2\mu, \qquad b = f = \lambda, \qquad d = m = \mu$$

giving for the effective medium

$$A = \langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \rangle + \langle \frac{1}{\lambda+2\mu} \rangle^{-1} \langle \frac{\lambda}{\lambda+2\mu} \rangle^2$$

$$B = \langle \frac{2\mu\lambda}{\lambda+2\mu} \rangle + \langle \frac{1}{\lambda+2\mu} \rangle^{-1} \langle \frac{\lambda}{\lambda+2\mu} \rangle^2$$

$$C = \langle \frac{1}{\lambda+2\mu} \rangle^{-1}$$

$$F = \langle \frac{1}{\lambda+2\mu} \rangle^{-1} \langle \frac{\lambda}{\lambda+2\mu} \rangle$$

$$D = \langle \frac{1}{\mu} \rangle^{-1}$$

$$M = \langle \mu \rangle$$

The P- and S-wave velocities in the effective anisotropic medium can be written as

$$V_{SH,h} = \sqrt{M/\rho}$$

$$V_{SH,v} = V_{SV,h} = V_{SV,v} = \sqrt{D/\rho} \tag{D-4}$$

$$V_{P,h} = \sqrt{A/\rho}$$

$$V_{P,v} = \sqrt{C/\rho} \tag{D-5}$$

wherer ρ is the average density $V_{P,v}$ is for the vertically propagating P-wave; $V_{P,h}$ is for the horizontally propagating P-wave; $V_{SH,h}$ is for the horizontally propagating, horizontally polarized S-wave; $V_{SH,h}$ is for the horizontally propagating, vertically polarized S-wave; and $V_{SV,v}$ and $V_{SH,v}$ are for the vertically propagating S-waves of any polarization (vertical is defined as being normal to the layering).

Uses of Backus average

The Backus average is used to model a finely stratified medium as a single homogeneous medium

Assumptions and limitations of Backus average

The following presuppositions and conditions apply to the Backus average:

- all materials are linearly elastic;

- there are no sources of intrinsic energy dissipation, such as friction or viscosity; and

- the layer thickness must be much smaller than the seismic wavelength. How small is still a subject of disagreement and research, but a rule of thumb is that the wavelength must be at least ten times the layer thickness.