CSE 583 Solutions to Exercises of Lecture 2

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In this solution manual, we always follow the sign convention of continuum mechanics.

Exercise 1.1

If we take the derivative w.r.t. $\boldsymbol{\epsilon}$ for Eq. (3) in the lecture notes, we get

$$\frac{\partial W(\boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T)}{\partial \boldsymbol{\epsilon}} = \frac{\partial W(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}} = \boldsymbol{\sigma}(\boldsymbol{\epsilon})\,. \tag{1}$$

Therefore, we only need to deal with the LHS of the above equation. By using the index notation and the chain rule, we know that

$$\frac{\partial W}{\partial \epsilon_{ij}} = \frac{\partial W}{\partial \tilde{\epsilon}_{kl}} \frac{\partial \tilde{\epsilon}_{kl}}{\partial \epsilon_{ij}}\,, \tag{2}$$

where $\tilde{\epsilon}_{kl} = (\boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T)_{kl} = Q_{km}\epsilon_{mn}Q_{ln}$. According to the definition of stress, $\partial W/\partial \tilde{\epsilon}_{kl}$ is indeed the kl component of $\boldsymbol{\sigma}(\boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T)$. For convenience, let us denote $\boldsymbol{\sigma}(\boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T)$ as $\tilde{\boldsymbol{\sigma}}$, therefore, we have $\partial W/\partial \tilde{\epsilon}_{kl} = \tilde{\sigma}_{kl}$.

The next step is to calculate the second term on the RHS of Eq. (2) by using the index notation. We can write

$$\frac{\partial \tilde{\epsilon}_{kl}}{\partial \epsilon_{ij}} = \frac{\partial (Q_{km}\epsilon_{mn}Q_{ln})}{\partial \epsilon_{ij}}\,. \tag{3}$$

From the lecture notes of MIT, the derivative of ϵ_{mn} to ϵ_{ij} is given as

$$\frac{\partial \epsilon_{mn}}{\partial \epsilon_{ij}} = \frac{1}{2}\left(\delta_{mi}\delta_{nj} + \delta_{mj}\delta_{ni}\right)\,. \tag{4}$$

Note if you ignore the symmetry property of $\boldsymbol{\epsilon}$, Eq. (4) should be changed to $\delta_{mi}\delta_{nj}$. Now let us plug Eq. (4) into Eq. (3), we get

$$\frac{\partial \tilde{\epsilon}_{kl}}{\partial \epsilon_{ij}} = \frac{1}{2}(Q_{km}\delta_{mi}\delta_{nj}Q_{ln} + Q_{km}\delta_{mj}\delta_{ni}Q_{ln}) = \frac{1}{2}(Q_{ki}Q_{lj} + Q_{kj}Q_{li}) \tag{5}$$

Thus, Equation (2) will be simplified as (note $\tilde{\sigma}_{kl} = \tilde{\sigma}_{lk}$)

$$\frac{\partial W}{\partial \epsilon_{ij}} = \frac{1}{2}\left(Q_{ki}\tilde{\sigma}_{kl}Q_{lj} + Q_{li}\tilde{\sigma}_{lk}Q_{kj}\right) \tag{6}$$

The RHS is just the component of the following tensorial expression:

$$\frac{\partial W}{\partial \boldsymbol{\epsilon}} = \frac{1}{2}(\boldsymbol{Q}^T\tilde{\boldsymbol{\sigma}}\boldsymbol{Q} + \boldsymbol{Q}^T\tilde{\boldsymbol{\sigma}}\boldsymbol{Q}) = \boldsymbol{Q}^T\tilde{\boldsymbol{\sigma}}\boldsymbol{Q} \tag{7}$$

If we go back and look at the first equation, we will get the following conclusion

$$\boldsymbol{Q}^T\tilde{\boldsymbol{\sigma}}\boldsymbol{Q} = \boldsymbol{Q}^T\boldsymbol{\sigma}(\boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T)\boldsymbol{Q} = \boldsymbol{\sigma}(\boldsymbol{\epsilon})\,. \tag{8}$$

Since $\boldsymbol{Q}$ is an orthogonal tensor ($\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \mathbf{1}$), we can left-multiply $\boldsymbol{Q}$ and right-multiply $\boldsymbol{Q}^T$ to obtain

$$\boldsymbol{\sigma}(\boldsymbol{Q}\boldsymbol{\epsilon}\boldsymbol{Q}^T) = \boldsymbol{Q}\boldsymbol{\sigma}(\boldsymbol{\epsilon})\boldsymbol{Q}^T \tag{9}$$

In fact, in this derivation, the symmetry property in Eq. (4) is not required.

Exercise 1.2

(1) By using the chain rule

$$\frac{\partial W}{\partial \boldsymbol{\epsilon}} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{\epsilon}} + \frac{\partial W}{\partial \tilde{I}_2}\frac{\partial \tilde{I}_2}{\partial \boldsymbol{\epsilon}} + \frac{\partial W}{\partial I_4}\frac{\partial I_4}{\partial \boldsymbol{\epsilon}} + \frac{\partial W}{\partial I_5}\frac{\partial I_5}{\partial \boldsymbol{\epsilon}}\,, \tag{10}$$

For the specific form of W, we have

$$\begin{aligned}
\frac{\partial W}{\partial I_1} &= \lambda I_1 + \alpha I_4\,,\\
\frac{\partial W}{\partial \tilde{I}_2} &= \mu_T\,,\\
\frac{\partial W}{\partial I_4} &= \alpha I_1 + \beta I_4\,,\\
\frac{\partial W}{\partial I_5} &= 2(\mu_L - \mu_T)\,.
\end{aligned} \tag{11}$$

Thus,

$$\boldsymbol{\sigma} = (\lambda I_1 + \alpha I_4)\mathbf{1} + 2\mu_T\boldsymbol{\epsilon} + (\alpha I_1 + \beta I_4)\boldsymbol{M} + 2(\mu_L - \mu_T)(\boldsymbol{\epsilon}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{\epsilon})\,. \tag{12}$$

(2) In order to relate these five constants with another set of five constants used in Eq. (8) of the lecture notes, we need to think about the vector/matrix form of the stress-strain relation. To obtain the first column of $[\mathbb{C}^e]$ (matrix form), we only need to set

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ 2\epsilon_{12}\\ 2\epsilon_{13}\\ 2\epsilon_{23}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix} \tag{13}$$

and calculate the final stress vector (6 by 1), which is exactly the first column of $[\mathbb{C}^e]$. Same for other columns.

As an example, I will show the calculation process of the first column. Since $\mathbf{a} = \boldsymbol{e}_z$, the structural tensor $\boldsymbol{M}$ is

$$\boldsymbol{M} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Next, we calculate the value of I_1, $\tilde{I}_2$, and I_4, which are $I_1 = 1$, $\tilde{I}_2 = 1$, $I_4 = 0$. Finally, we can plug these results into Eq. (12) to get

$$\begin{aligned} \boldsymbol{\sigma} = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu_T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ + 2(\mu_L - \mu_T) \underbrace{\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)}_{\text{zero matrix}} \\ = \begin{bmatrix} \lambda + 2\mu_T & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda + \alpha \end{bmatrix}. \end{aligned} \tag{15}$$

In vector form, this is given as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu_T \\ \lambda \\ \lambda + \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{16}$$

which is just the first column of $[\mathbb{C}^e]$. We establish the following relation:

$$C_{1122} + 2C_{1212} = \lambda + 2\mu_T \quad \Longrightarrow \quad C_{1212} = \mu_T\,, \tag{17}$$

$$C_{1122} = \lambda\,, \tag{18}$$

$$C_{1133} = \lambda + \alpha\,. \tag{19}$$

Similarly, if we want to find the third column, we set

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

and calculate the invariants as $I_1 = \tilde{I}_2 = I_4 = 1$. We can then plug these results into Eq. (12) to get

$$
\begin{aligned}
\boldsymbol{\sigma} &= (\lambda+\alpha)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + 2\mu_T\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + (\alpha+\beta)\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \\
&+ 2(\mu_L-\mu_T)\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right) \\
&= (\lambda+\alpha)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + (2\mu_T+\alpha+\beta+4\mu_L-4\mu_T)\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \\
&= \begin{bmatrix}\lambda+\alpha&0&0\\0&\lambda+\alpha&0\\0&0&\lambda+2\alpha+\beta+4\mu_L-2\mu_T\end{bmatrix}.
\end{aligned}
\tag{21}
$$

In vector form, this is given as

$$
\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{13}\\\sigma_{23}\end{bmatrix} = \begin{bmatrix}\lambda+\alpha\\\lambda+\alpha\\\lambda+2\alpha+\beta+4\mu_L-2\mu_T\\0\\0\\0\end{bmatrix}, \tag{22}
$$

which is just the third column of $[\mathbb{C}^e]$. We establish the following relation:

$$
C_{1133} = \lambda+\alpha\,, \tag{23}
$$

$$
C_{3333} = \lambda+2\alpha+\beta+4\mu_L-2\mu_T\,. \tag{24}
$$

Exercise 1.3

1. By using the following relation

$$
\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{13}\\\sigma_{23}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu&\lambda&\lambda&&&\\\lambda&\lambda+2\mu&\lambda&&\boldsymbol{0}&\\\lambda&\lambda&\lambda+2\mu&&&\\&&&\mu&&\\&\boldsymbol{0}&&&\mu&\\&&&&&\mu\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\\epsilon_{22}\\\epsilon_{33}\\2\epsilon_{12}\\2\epsilon_{13}\\2\epsilon_{23}\end{bmatrix}, \tag{25}
$$

when there is only one non-zero component ϵ_{11}, we have

$$
\begin{aligned}
\sigma_{11} &= (\lambda+2\mu)\epsilon_{11}\,,\\
\sigma_{22} &= \lambda\epsilon_{11}\,,\\
\sigma_{33} &= \lambda\epsilon_{11}\,.
\end{aligned}
\tag{26}
$$

Therefore, we can present M and K_0 as

$$
M = \lambda+2\mu\,, \tag{27}
$$

$$K_0 = \frac{\lambda}{\lambda + 2\mu}\,. \tag{28}$$

2. For the plane strain problem, we still start from Eq. (25), and we obtain

$$\begin{aligned} \sigma_{11} &= (\lambda + 2\mu)\epsilon_{11} + \lambda\epsilon_{22}\,, \\ \sigma_{22} &= \lambda\epsilon_{11} + (\lambda + 2\mu)\epsilon_{22}\,, \\ \sigma_{33} &= \lambda(\epsilon_{11} + \epsilon_{22})\,. \end{aligned} \tag{29}$$

As a result

$$\frac{\sigma_{33}}{\sigma_{11} + \sigma_{22}} = \frac{\lambda(\epsilon_{11} + \epsilon_{22})}{2(\lambda + \mu)(\epsilon_{11} + \epsilon_{22})} = \frac{\lambda}{2(\lambda + \mu)}\,. \tag{30}$$

According to Eq. (14) in the lecture notes, this is exactly the Poisson's ratio ν.

3. For the plane stress problem, we will use the inverse form of Eq. (25), *i.e.*, the compliance matrix, which is given as

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} = \begin{bmatrix} \frac{\lambda+\mu}{\mu\,(3\lambda+2\mu)} & \frac{-\lambda}{2\mu\,(3\lambda+2\mu)} & \frac{-\lambda}{2\mu\,(3\lambda+2\mu)} & & \boldsymbol{0} & \\ \frac{-\lambda}{2\mu\,(3\lambda+2\mu)} & \frac{\lambda+\mu}{\mu\,(3\lambda+2\mu)} & \frac{-\lambda}{2\mu\,(3\lambda+2\mu)} & & & \\ \frac{-\lambda}{2\mu\,(3\lambda+2\mu)} & \frac{-\lambda}{2\mu\,(3\lambda+2\mu)} & \frac{\lambda+\mu}{\mu\,(3\lambda+2\mu)} & & & \\ & & & \mu^{-1} & & \\ & \boldsymbol{0} & & & \mu^{-1} & \\ & & & & & \mu^{-1} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}. \tag{31}$$

By using the expression for Young's modulus E and Poisson's ratio ν, *i.e.*, Eq. (14) in the lecture notes, above eqaution could be written in the equivalent form

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & & \boldsymbol{0} & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & \mu^{-1} & & \\ & \boldsymbol{0} & & & \mu^{-1} & \\ & & & & & \mu^{-1} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}. \tag{32}$$

Under the plane stress condition, we have

$$\begin{aligned} \epsilon_{11} &= \frac{\sigma_{11}}{E} - \frac{\nu}{E}\sigma_{22}\,, \\ \epsilon_{22} &= -\frac{\nu}{E}\sigma_{11} + \frac{\sigma_{22}}{E}\,, \\ \epsilon_{33} &= -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})\,. \end{aligned} \tag{33}$$

As a result

$$\frac{\epsilon_{33}}{\epsilon_{11} + \epsilon_{22}} = \frac{-\frac{\nu}{E}(\sigma_{11} + \sigma_{22})}{\frac{1-\nu}{E}(\sigma_{11} + \sigma_{22})} = -\frac{\nu}{1-\nu}\,. \tag{34}$$

Exercise 1.4

According to the hint, we will use the change in the volumetric strain $\Delta\epsilon_v$ to update the void ratio $\bar{e}$. Then the question is how to calculate $\Delta\epsilon_v$?

If we recall the physical meaning of bulk modulus, the $\Delta\epsilon_v$ is simply $\Delta p'/K_n$, where K_n represents the value of K from the previous step. Therefore, the algorithm is summarized as follows.

1. We are given the previous step result: $\bar{e}_n$, p'_n (negative number), and K_n.

2. For a new value of p'_{n+1} (negative number), calculate the $\Delta\epsilon_v$ as $(p'_{n+1} - p'_n)/K_n$, and $\Delta\bar{e}$ will thus be $\Delta\epsilon_v(1+\bar{e}_0)$.

3. Update the void ratio as
$$\bar{e}_{n+1} = \bar{e}_n + (1+\bar{e}_0)\frac{p'_{n+1} - p'_n}{K_n}$$

4. Update the bulk modulus K_{n+1} using p'_{n+1} (and $\bar{e}_{n+1}$ if necessary, depends on the formula for K).

5. Enter the next loop and back to Step 1.

Exercise 2.1

By using the relation $E_T = EH'/(H'+E)$, we can calculate $E_T = 20$ MPa, which is the slope $\Delta\sigma/\Delta\epsilon$ during the stage of plastic deformation. The basic idea is to find the initial yield point, then plot the stress-strain curve using E_T during the plastic deformation until reaching $\sigma = \pm 2$ MPa, and update α and σ_Y accordingly. The detailed calculation process is given as follows.

1. When $\beta = 0$, it is purely kinematic hardening, and $\sigma_Y \equiv \sigma_{Y0} = 1$ MPa. During the extension stage: the bar will firstly yield at $\sigma = 1$ MPa (Point B), the corresponding strain $\epsilon = \sigma/E = 1/100$, then it will enter the hardening stage with a slope of 20 MPa until $\sigma = 2$ MPa (Point C), the current total strain is $\epsilon = 1/100 + (2-1)/20 = 0.06$, elastic strain is $\epsilon^e = \sigma/E = 2/100 = 0.02$, so the plastic strain is $\epsilon^p = \epsilon - \epsilon^e = 0.06 - 0.02 = 0.04$.

   By using the information of $\epsilon^p = 0.04$, we can update α as $\alpha = \alpha_0 + H'\epsilon^p = 0 + 25 \times 0.04 = 1$ MPa. Therefore, during the unloading and compression stage, the yield function becomes $|\sigma - 1| - 1 \leq 0$, and it will yield at $\sigma = 0$ MPa (Point D), with zero elastic strain and plastic strain unchanged (still 0.04), so the total strain is $0 + 0.04 = 0.04$. Then it will enter the plastic deformation stage (in compression) until $\sigma = -2$ MPa (Point E). The current total strain is $\epsilon = 0.04 - \Big(0 - (-2)\Big)/20 = -0.06$, elastic strain is $\epsilon^e = \sigma/E = -2/100 = -0.02$, and plastic strain is $\epsilon^p = \epsilon - \epsilon^e = -0.06 - (-0.02) = -0.04$.

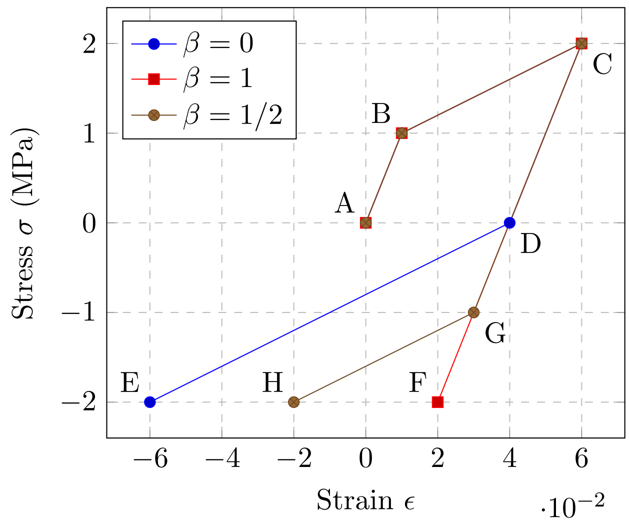
   From Point D to Point E, the change in plastic strain is $-0.04 - 0.04 = -0.08$, then α is updated as $\alpha = 1 - 25 \times 0.08 = -1$ MPa, the yield function is $|\sigma + 1| - 1 \leq 0$. You can easily check that Point E ($\sigma = -2$ MPa) satisfies the yield function.

2. When $\beta = 1$, it is purely isotropic hardening, and $\alpha \equiv 0$ MPa. During the extension stage, the behavior is exactly the same as that for kinematic hardening.

By using the information of $\epsilon^p = 0.04$, we can update σ_Y as $\sigma_Y = \sigma_{Y0} + H' |\epsilon^p| = 1 + 25 \times 0.04 = 2$ MPa. Now the yield function becomes $|\sigma| - 2 \leq 0$. Therefore, the unloading and compression stage will be purely elastic, with a residual plastic strain of 0.04. When $\sigma = -2$ MPa, the total strain is $0.04 + \sigma/E = 0.04 + (-2/100) = 0.02$ (Point F).

3. When $\beta = 1/2$, it is combined hardening, but the analysis is the same as before. During the extension stage, the behavior is exactly the same until Point C.

   By using the information of $\epsilon^p = 0.04$, we can update σ_Y as $\sigma_Y = \sigma_{Y0} + \beta H' |\epsilon^p| = 1 + 0.5 \times 25 \times 0.04 = 1.5$ MPa, and update α as $\alpha = \alpha_0 + (1-\beta) H' \epsilon^p = 0 + 0.5 \times 25 \times 0.04 = 0.5$ MPa. The yield function is $|\sigma - 0.5| - 1.5 \leq 0$. On reverse loading the yield point is encountered at Point G with $\sigma = -1$ MPa, and at Point H, σ_Y increases further to 2 MPa while α goes back to zero. From Point G to Point H, the change in plastic strain is -0.04.



Exercise 3.1

The answer is given on Page 78-79 of Borja, 2013, which is not repeated here.

Exercise 4.1 (An equivalent way to obtain D-P parameters based on M-C parameters [1])

Let us assume the initial confining stress is $p_0 < 0$. For the triaxial drained compression test, we simultaneously solve the following two equations at the critical state

$$\begin{cases} q &= -Mp' \\ q &= -3(p' - p_0) \end{cases} \tag{35}$$

[1]Since CSL can be treated as the D-P yield surface with no interception. If we consider a non-zero interception a, then a may be represented by c and ϕ as well. The procedures are basically the same. You may notice that our derived results of M_c and M_e are quite similar to $(4.76)_2$ of Borja, 2013. $(4.76)_1$ is obtained based on the vertex of the M-C yield surface (a pyramid): $\sigma_1 = \sigma_2 = \sigma_3 = c\cot\phi$ or $\sigma_3'' = \sqrt{3}/3 \times (\sigma_1 + \sigma_2 + \sigma_3) = \sqrt{3}c\cot\phi$, $\sigma_1'' = \sigma_2'' = 0$.

to get

$$q = \frac{3Mp_0}{M-3} = |\sigma_{\max} - \sigma_{\min}| \ . \tag{36}$$

Since $\sigma_{\min} = p_0$, we know $\sigma_{\max} = p_0 - q$ (on the horizontal axis, $\sigma_{\max} < \sigma_{\min}$ since compression is assumed to be negative). From the Mohr-Coulomb failure criterion when $c = 0$ (ϕ_{cs} represents the critical state friction angle, because the CSL may be considered as a limiting failure envelope for frictional materials)

$$(\sigma_{\max} - \sigma_{\min}) = (\sigma_{\max} + \sigma_{\min}) \sin\phi_{cs} - \underbrace{2c\cos\phi_{cs}}_{0} \tag{37}$$

while

$$\begin{cases} \sigma_{\max} - \sigma_{\min} & = -q \\ \sigma_{\max} + \sigma_{\min} & = 2p_0 - q \end{cases} , \tag{38}$$

which leads to the final conclusion by combining Eqs. (36)(37)(38) (add the subscript "c" to represent compression)

$$M_c = \frac{6\sin\phi_{cs}}{3 - \sin\phi_{cs}} \Longleftrightarrow \sin\phi_{cs} = \frac{3M_c}{6 + M_c} \ . \tag{39}$$

While for the <u>triaxial drained extension test</u>, we simultaneously solve the following two equations at the critical state (note the change of slope from -3 to 3)

$$\begin{cases} q & = -Mp' \\ q & = 3(p' - p_0) \end{cases} \tag{40}$$

to get

$$q = \frac{-3Mp_0}{3+M} = |\sigma_{\max} - \sigma_{\min}| \ . \tag{41}$$

Now we have $\sigma_{\max} = p_0$ and $\sigma_{\min} = p_0 + q$ (on the horizontal axis, $\sigma_{\max} < \sigma_{\min}$ since compression is assumed to be negative). From the Mohr-Coulomb failure criterion when $c = 0$ (ϕ_{cs} represents the critical state friction angle, because the CSL may be considered as a limiting failure envelope for frictional materials)

$$(\sigma_{\max} - \sigma_{\min}) = (\sigma_{\max} + \sigma_{\min}) \sin\phi_{cs} - \underbrace{2c\cos\phi_{cs}}_{0} \tag{42}$$

while

$$\begin{cases} \sigma_{\max} - \sigma_{\min} & = -q \\ \sigma_{\max} + \sigma_{\min} & = 2p_0 + q \end{cases} , \tag{43}$$

which leads to the final conclusion in the same manner as the previous case (add the subscript "e" to represent extension)

$$M_e = \frac{6\sin\phi_{cs}}{3 + \sin\phi_{cs}} \Longleftrightarrow \sin\phi_{cs} = \frac{3M_e}{6 - M_e} \ . \tag{44}$$

Finally, for the <u>drained shear test</u> (a numerical approach by prescribing shear strain while normal stress is fixed), the stress path is a vertical line from ($p' = p_0, \ q = 0$). Therefore, at the critical state, we have

$$q = -Mp_0 = \sqrt{3}\,|\tau| \ . \tag{45}$$

Now we have $\sigma_{\max} = p_0 - |\tau|$ and $\sigma_{\min} = p_0 + |\tau|$. In other words

$$\begin{cases} \sigma_{\max} - \sigma_{\min} & = -2\,|\tau| = \dfrac{2Mp_0}{\sqrt{3}} \\ \sigma_{\max} + \sigma_{\min} & = 2p_0 \end{cases}, \tag{46}$$

which leads to the final conclusion (add the subscript "s" to represent shear)

$$M_s = \sqrt{3}\sin\phi_{cs} \Longleftrightarrow \sin\phi_{cs} = \frac{M_s}{\sqrt{3}}\,. \tag{47}$$

Triaxial undrained or drained compression test (MCC model)

The typical results are shown here. It is your own duty to understand these results by using the knowledge that you have learned both from this course and previous similar courses such as CSE 578 and CSE 6012. If you have questions, you can email me or ask me during my office hour.

Note in literature, the ICL might be also known as the normal consolidation line or the virgin consolidation line.

Specific volume, void ratio, and volumetric strain

- From the following relation (based on conserved incompressible solid volume/mass), we know that $\Delta\epsilon = \Delta e/(1+e_0)$ where $\Delta e = e - e_0$ and $\Delta\epsilon = -\Delta H/H_0$ (compression has a **negative** sign, $\Delta\epsilon$ in fact represents the volumetric deformation, thus a subscript "v" could be added)

$$\frac{H_0}{1+e_0} = \frac{H_0 - \Delta H}{1+e} \tag{48}$$

- Since the specific volume $v = 1+e$, thus $\Delta v = \Delta e$ and $\Delta\epsilon = \Delta v/v_0$. If we assume $\Delta v = v - v_0$, we would have the equivalent relation (v_0 is not required to be the original value at $t = 0$, it could be the value determined from the last time step)

$$v = v_0\,(1 + \Delta\epsilon_v) \tag{49}$$

- In the modified Cam-Clay model, the bulk modulus $K = vp'/\kappa$, and the previous equation can be used to update K. Also, the hardening law also requires the new specific volume v

$$\dot{p}_c = \frac{v}{\lambda - \kappa} p_c \dot{\epsilon}_v^p \tag{50}$$

- For additional information, see http://docs.itascacg.com/flac3d700/common/models/camclay/doc/modelcamclay.html?node525.

References

Borja, R. I. (2013). "Plasticity: Modeling & Computation". In: *Springer*, pp. 1–260. URL: https://link.springer.com/book/10.1007/978-3-642-38547-6.

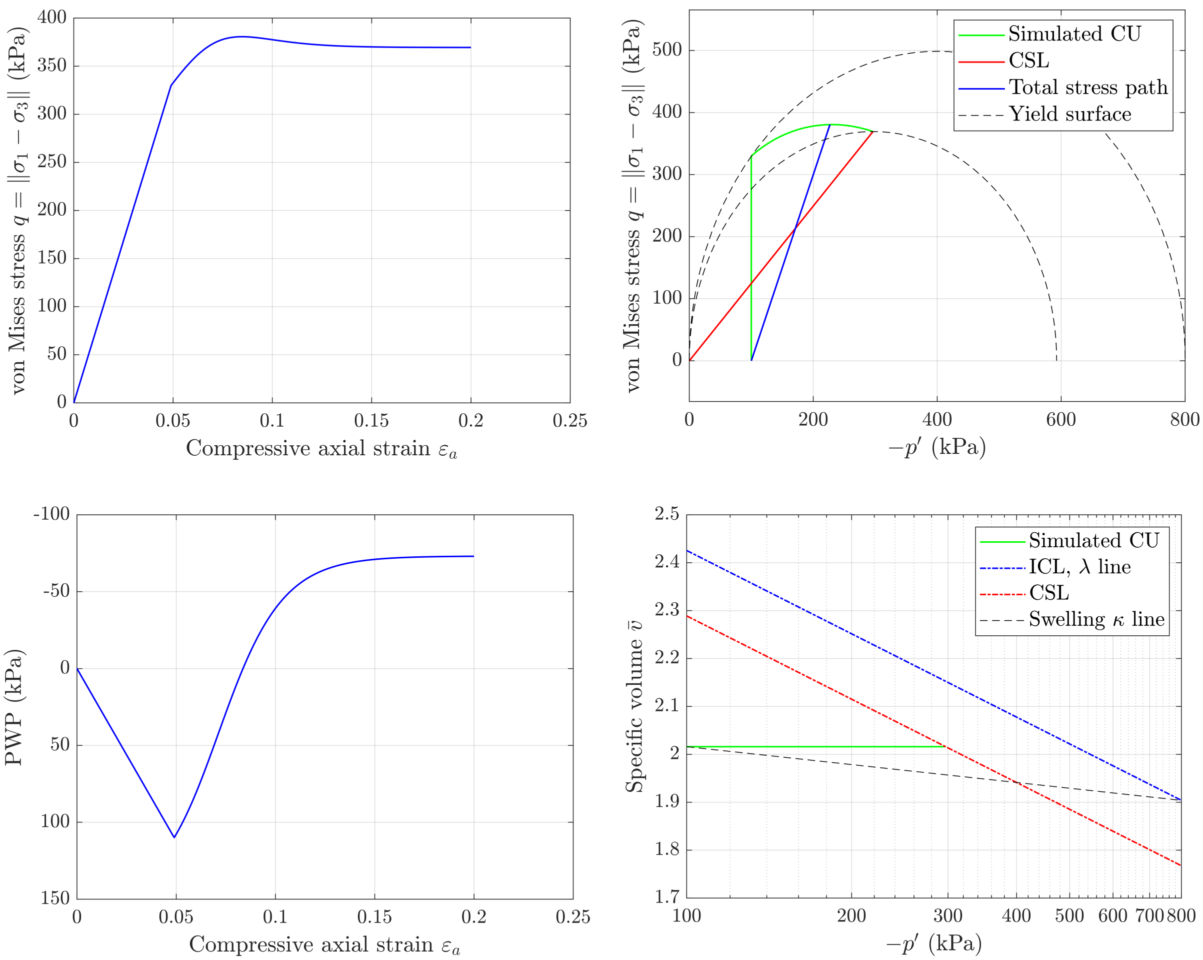


Figure 1: Numerical simulation results of a triaxial **undrained** compression test for soil with **OCR = 8**. The material parameters are given as follows: $M = 1.2468$, $\lambda = 0.2508$, $\kappa = 0.05387$, $\bar{v}$ at $p' = -1$ kPa of the ICL is 3.5804, Poisson's ratio $\nu = 0.25$, initial preconsolidation pressure is 800 kPa ($p_c|_{t=0} = -800$ kPa), initial confining pressure is 100 kPa ($p'|_{t=0} = -100$ kPa).

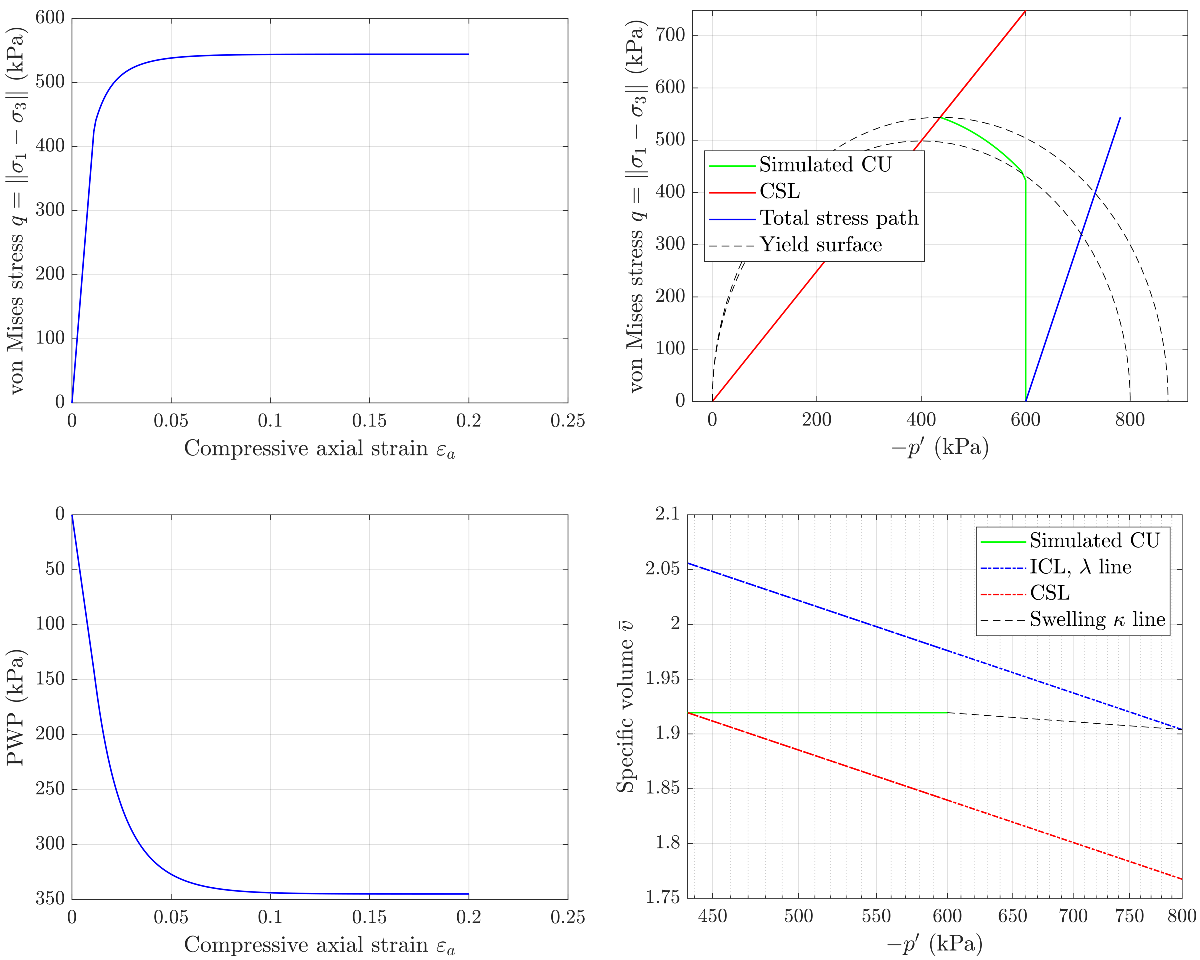


Figure 2: Numerical simulation results of a triaxial **undrained** compression test for soil with **OCR = 4/3**. The initial confining pressure is 600 kPa ($p'|_{t=0} = -600$ kPa). By default, the remaining parameters will be the same as those in Figure 1.

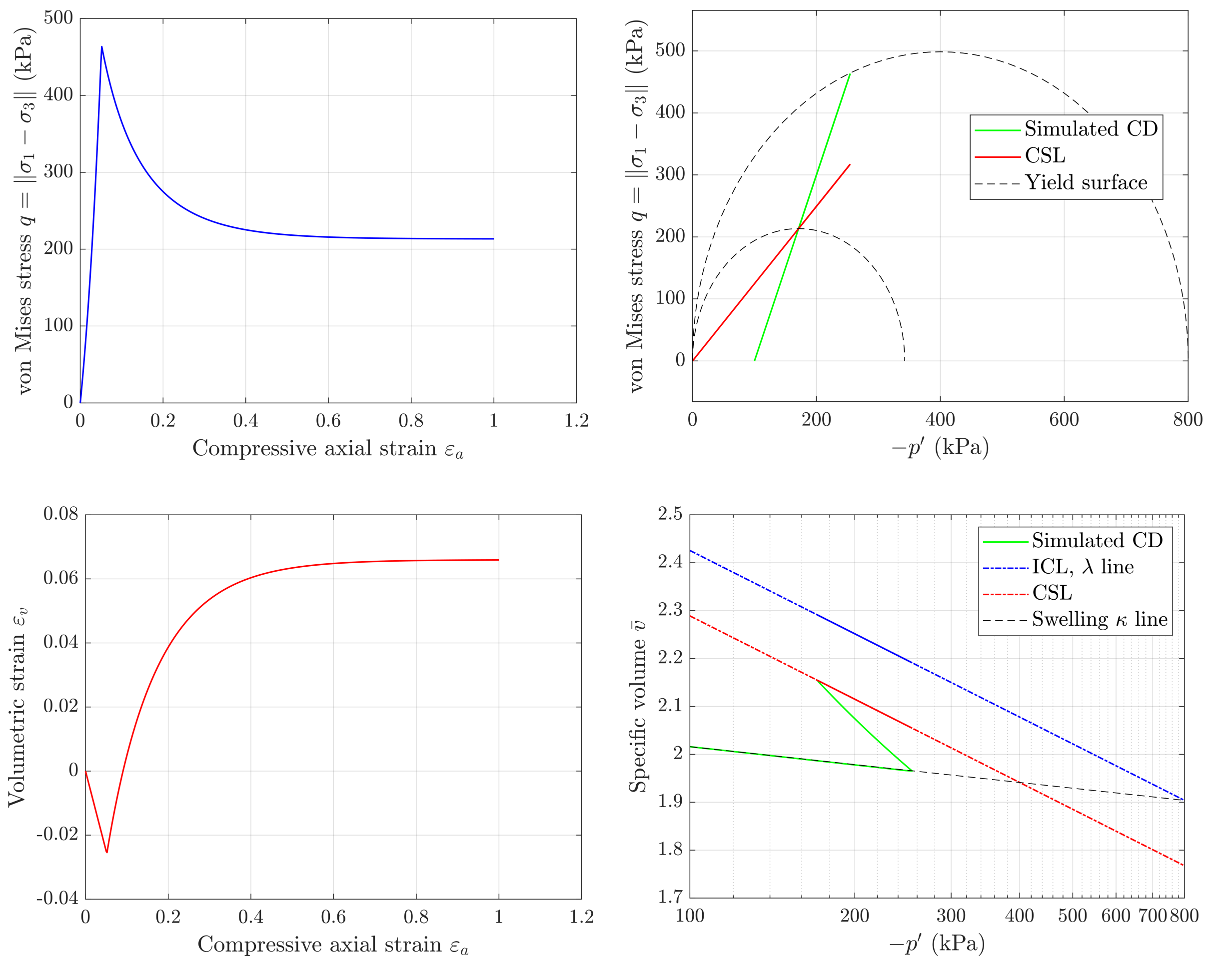


Figure 3: Numerical simulation results of a triaxial **drained** compression test for soil with **OCR = 8**. The initial confining pressure is 100 kPa ($p'|_{t=0} = -100$ kPa). By default, the remaining parameters will be the same as those in Figure 1.

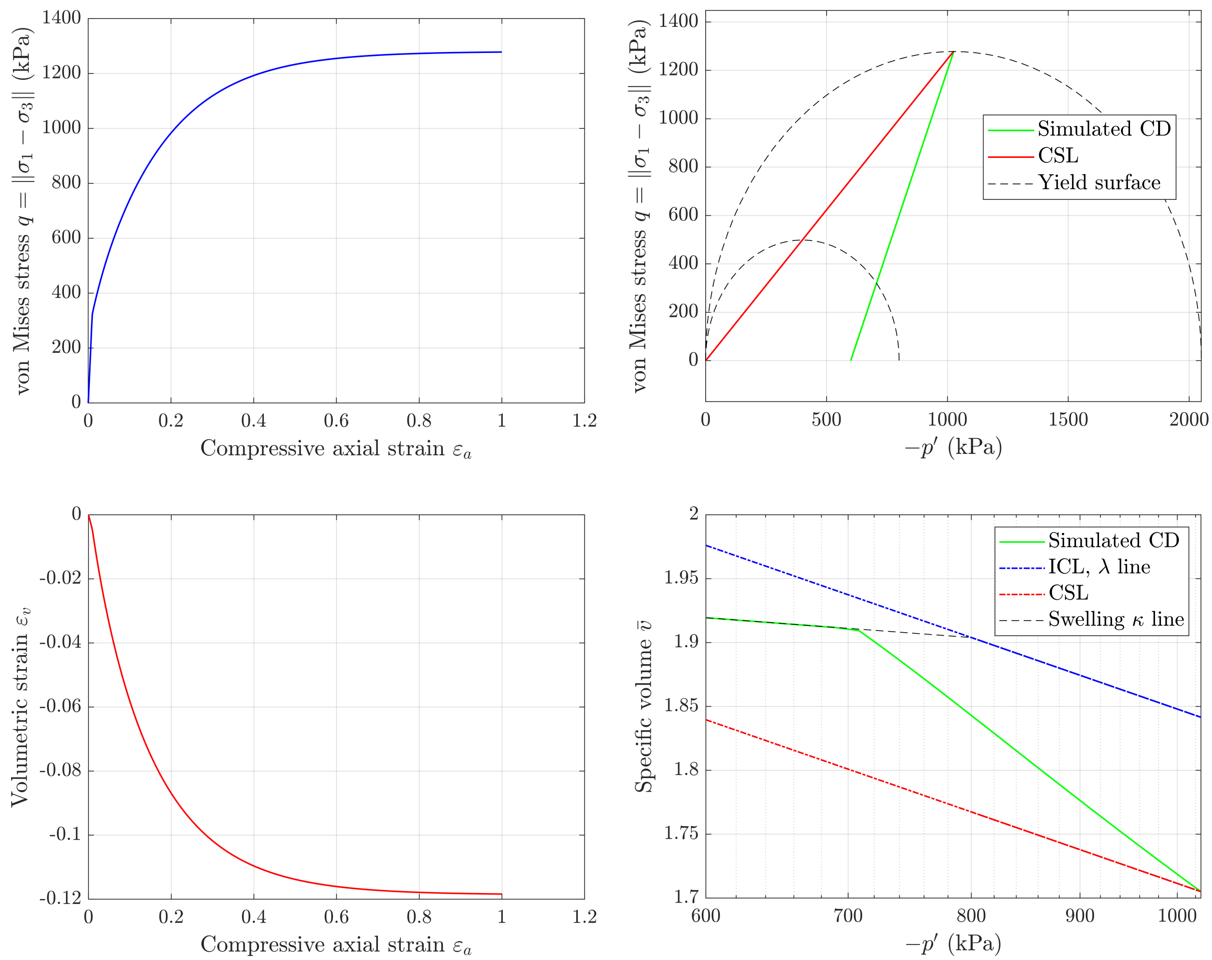
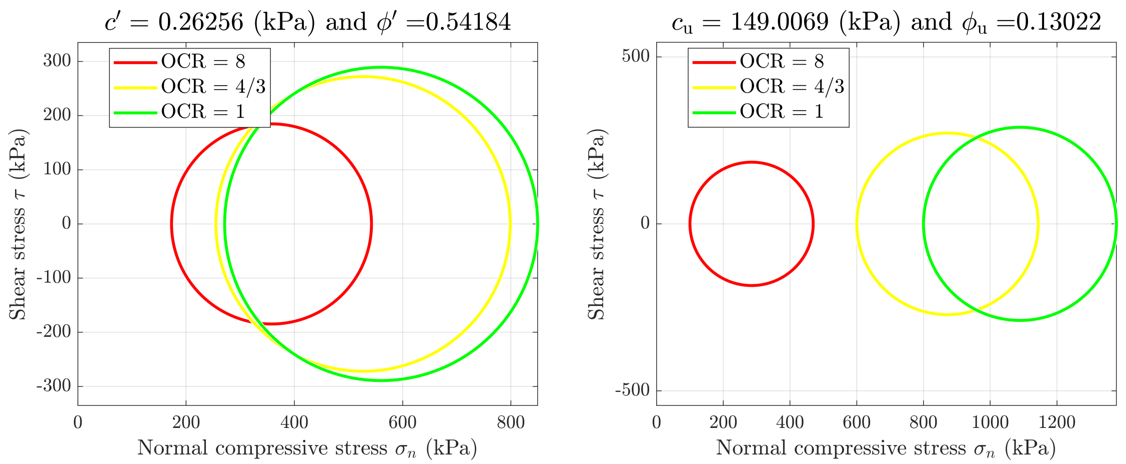


Figure 4: Numerical simulation results of a triaxial **drained** compression test for soil with **OCR = 4/3**. The initial confining pressure is 600 kPa ($p'|_{t=0} = -600$ kPa). By default, the remaining parameters will be the same as those in Figure 1.

Phi is in radian, not degree



(a) Effective stress Mohr's circle (b) Total stress Mohr's circle (a translation)

Figure 5: Failure envelopes for the triaxial undrained compression test. Different values of OCR correspond to different values of initial confining pressure, but besides this, other parameters just follow from Figure 1.

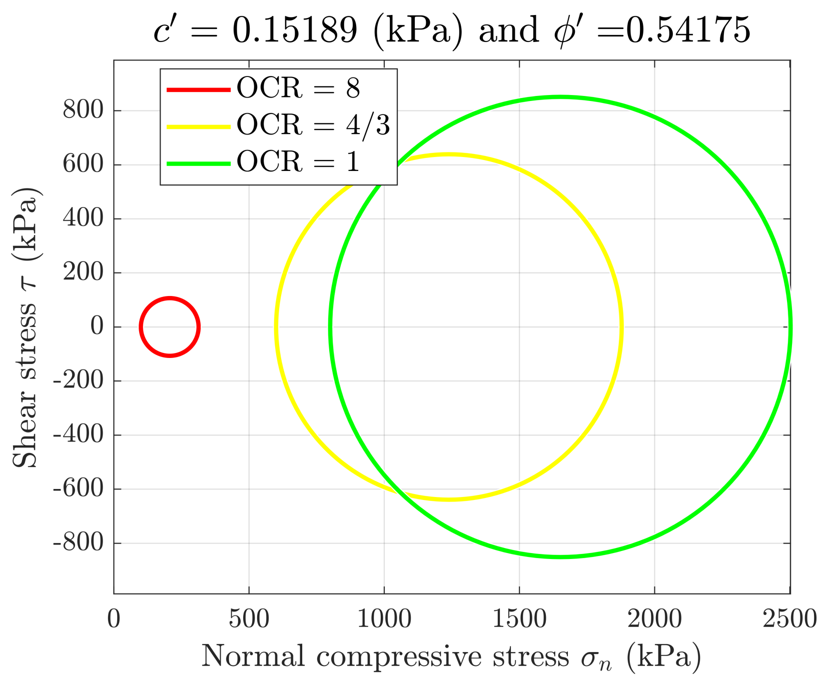


Figure 6: Failure envelope for the triaxial drained compression test. Different values of OCR correspond to different values of initial confining pressure, but besides this, other parameters just follow from Figure 1.