Randomization and Ambiguity Aversion

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Abstract

We propose a model of preferences in which the effect of randomization on ambiguity depends on how the unknown probability law is determined. We adopt the framework of **?** and relax the axioms. In the resulting representation of the individual's preference, the individual has a collection of sets of priors $\mathcal{M}$. She believes that before she moves, nature has chosen an unknown scenario (a set of priors) from $\mathcal{M}$, and from that scenario, nature will choose a prior after she moves. The representation illustrates how randomization may partially eliminate the effect of ambiguity.

1 Introduction

It is often assumed in economics that an individual has a unique subjective prior (probability measure) over states of the world, even though the true probability law—if one is already determined—is rarely provided. **?** introduces intuitive thought experiments to show that, on the contrary, an individual usually faces *ambiguity* and is *ambiguity-averse*; that is, she does not have a unique prior and dislikes betting on an event without a unique subjective probability assessment. In one experiment, assuming that a correct bet yields $100 and a wrong bet $0, it is predicted that an individual facing a risky urn (that contains 50 red and 50 black balls) and an ambiguous urn (with 100 red and black balls, but in an unknown proportion) will prefer a bet on the color of a randomly drawn ball from the risky urn over that from the ambiguous urn. Nonetheless, fixing any urn, the individual will be indifferent between betting on either color of a randomly drawn ball.[1] This prediction is confirmed by many subsequent studies.

In a seminal paper, **?** introduce the *maxmin expected utility* (MEU) model to capture ambiguity aversion. In the model, the individual has multiple priors. To evaluate a choice object, such as a bet, she adopts the prior that minimizes its utility; to choose from multiple choice objects, she maximizes the minimum utility.[2] The interpretation is that with multiple priors in mind, the individual is conservative and focuses on the worst-case scenario—or, equivalently, it is as if the individual believes that nature plays against her by choosing the worst probability law from a fixed set of laws.

The MEU model, or more generally the maxmin principle, has been useful in various areas of economics: contract theory, mechanism design, macroeconomics, finance, etc.[3] Surpris-

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[1]This implies that the individual must not have a unique prior, even if we consider (subjective) non-expected utility models (see **?**).

[2]In the previous experiment, suppose the individual has three priors about the composition of the ambiguous urn: The number of red balls is either 30, 50, or 70. Then, to evaluate the bet on red (black) drawn from the ambiguous urn, she believes that there are only 30 red (black) balls.

[3]See, among many others, **?**; **?**; **?**; **?**; **?**; and **?**. In the MEU model, minimization is over the individual's multiple priors. The maxmin principle has also been adopted when minimization is based on regret, over multiple mechanisms, over multiple production technologies, over opponents' preferences, etc.

ingly, however, in these applications, it is usually unclear how a basic concept in economics, *randomization/mixed strategies*, should be modeled when an individual behaves according to the MEU model or any other maxmin criterion.[4] Randomization is modeled differently in different applications or simply ruled out; sometimes a discussion of the difficulty of introducing randomization and various ways to model randomization is provided, but there is not yet a general principle that helps us better understand how randomization should be modeled in a given application.[5]

In this paper, we introduce a theory of randomization under ambiguity that may help address this issue. We focus on (randomization under) the MEU model, but the insights we derive may be applied to other ambiguity models and models with other maxmin criteria.

1.1 How Should Randomization Eliminate the Effect of Ambiguity?

Suppose there are two services, A and B. An individual needs to purchase one of them. There are two states, s_A and s_B. In state s_A (s_B), service A (B) yields payoff 1 and the other service yields 0. Although both services seem equally good to the individual, the individual does not have a unique probability assessment of s_A or s_B. However, some details of services A and B determine the probability of s_A and s_B. Consider two situations. In the first, those details are precisely stated in contracts, although the individual never plans to read them. In the second, service providers do not explain those details in contracts. If the individual chooses the services randomly, will randomization interact with ambiguity in any way in either situation?

To understand the answer to this question, we first return to the urn experiment, in which the individual is indifferent between betting on red and black being drawn from the

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[4]In the literature on ambiguity, two types of mixture operations have been studied. Here, by randomization, we mean the mixture operation implemented before the state is revealed to the individual. In Section 2.1, we will introduce the other type of mixture operation, state-wise randomization.

[5]See Section 5 for more details.

ambiguous urn. **?** points out that randomization renders ambiguity irrelevant. Suppose the individual tosses a fair coin, and bets on red being drawn from the ambiguous urn if heads and black if tails. Then, no matter what the number of red/black balls is in the ambiguous urn and which color is drawn, the individual will bet on red and black with equal probability; that is, she always receives a 50-50 lottery between \$100 and \$0 (see the right-hand side of Figure 1), which is what she will receive if she bets on red/black being drawn from the risky urn.[6] Therefore, if randomization is perceived in this way, ambiguity becomes irrelevant.

However, randomization can be viewed in other ways. A popular alternative view (the left-hand side of Figure 1) is as follows: Regardless of the outcome of the coin toss, the individual must bet on either red or black being drawn from the ambiguous urn; that is, after the coin toss, she always ends up with a bet from the ambiguous urn that she dislikes in the first place. As **?** argues, if ambiguity is perceived in this way, its effect should be unaffected by randomization. To date, in applications that apply the maxmin principle, if mixed strategies are not ruled out, either the first view or the second is adopted.[7]

Figure 1: There are two states, in which s_R (s_B) means that red (black) is drawn from the ambiguous urn. Randomization is represented by solid lines, with "R" ("B") representing betting on red (black). The number along a solid line indicates the objective probability of betting on the corresponding color.

A few papers have noted that the main difference between the two views lies in *subjective*

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[6]The right-hand side of Figure 1 is in fact a simplified version of this idea. See Figure 3.

[7]See Section 5 for a more detailed discussion.

timing.[8] This is easier to explain under the MEU model. Suppose the individual believes that the number of red balls in the ambiguous urn is either 30 or 70. Recall that an as-if interpretation of the MEU model is that the individual believes that nature plays against her. Does she believe that she randomizes before nature moves or after? If she believes that she moves first, nature can choose the probability law *based on the outcome of the coin toss*: If heads (betting on red), the number of red balls will be 30; otherwise, it is 70. This is consistent with the second view: Regardless of the outcome of the coin toss, the individual is affected by ambiguity as in the case without the coin.

This is not true, however, if the individual believes that nature moves first. If nature moves first, although the number of red balls is unknown, it *cannot* depend on the outcome of the coin toss. This is therefore consistent with the first view: Fixing an arbitrary number of red balls, the coin toss gives the individual a 50-50 lottery between $100 and $0. Hence, the interaction between randomization and ambiguity should depend on the individual's belief about how the unknown probability law is determined.

Now, return to the service example. In the first situation, before the individual randomizes, the contracts have pinned down the probability of s_A and s_B, although the probability is unknown to the individual. The view that randomization renders ambiguity irrelevant is more suitable. In the second situation, it seems more plausible that the effect of ambiguity is unaffected by randomization: The individual may worry that after randomization and choosing a service, the details of the service can still be manipulated.

In the service example, first, there is a natural objective timing of when the probability law is determined. This is often not the case, and even if there is a natural objective timing, the individual is free to hold a different view. Second, the timing is extreme, in the sense that the probability law is completely pinned down either before randomization or after. Again, this rarely happens in practice. For example, in the first situation of the service example, if contracts use words that can be interpreted in many ways, and "The service provider

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[8]Among others, see **?**; **?**; **?**; **?**; and **?**. Note that subjective timing does not have to be identical to objective timing (if there is any).

reserves all rights for final explanation" is added to both contracts, how does it change the individual's view of randomization?

Similarly, consider a stylized investment example in which the individual can either long or short a firm. Suppose she believes that the probability that the firm succeeds depends on some unknown long-term factor that has been determined and some unknown short-term factor that will be determined after she makes a decision. How should randomization interact with ambiguity?

1.2   Preview of Results

To study randomization under ambiguity, we adopt the choice domain of **?** (henceforth AA) and analyze an individual's preference.[9] The domain consists of lotteries over acts (*lotteries* for short henceforth). A lottery is a probability measure (mixed strategy) over acts. An act assigns to each state of the world a *prize*. A prize is a probability measure over consequences. Note that this domain does not require us to describe objectively how the probability law is determined; this will be subjective and revealed from the individual's preference.

Figure 2: A lottery P that yields act f with probability $P(f)$ and act g with $P(g)$, in which f_i (g_i) is the prize associated with state s_i by f (g).

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[9]We use the original AA choice domain that has two types of mixture operations. In many papers using the AA choice domain, only one mixture operation, which will be called state-wise randomization in our paper, is involved, because they assume implicitly that the individual is indifferent between the two types of mixture operations. See Section 5 for more details.

We incorporate ideas from **?** and first- and second-order stochastic dominance to relax AA's axioms. Our main theorem characterizes the following representation of the individual's preference over lotteries, the *double maxmin expected utility* (DMEU) representation: For each lottery P,

$$W(P) = \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP,$$

in which each $M \in \mathcal{M}$ is a set of priors, and $u(f(s))$ is the utility of the prize that an act f assigns to state s. Each $M \in \mathcal{M}$ is called a *scenario*. The individual believes that before she randomizes, nature, which plays against her, has chosen a scenario (although unknown to her) from $\mathcal{M}$. After she randomizes, nature will choose a prior μ from that scenario.

Figures 3–5 illustrate how the DMEU representation reveals the individual's belief about the determination of the unknown probability law and about the interaction between randomization and ambiguity. Figure 3 depicts the urn example assuming that the individual believes that randomization eliminates the effect of ambiguity, in which every scenario $M \in \mathcal{M}$ is a singleton. Figure 4 depicts the urn example assuming that the individual believes that the effect of ambiguity is unaffected by randomization, in which case $\mathcal{M}$ is a singleton and $M \in \mathcal{M}$ is not.

Figure 5 is the investment example, in which the probability that the firm succeeds depends on some unknown long-term and short-term factors. This example shows that randomization may partially eliminate the effect of ambiguity—it helps eliminate the effect of ambiguity across scenarios, but not across probability laws within a scenario. The individual believes that one unknown scenario in $\mathcal{M} = \{M_G, M_N, M_B\}$ has occured before she chooses. Both scenarios M_G (good long-term factor) and M_B (bad long-term factor) are singletons— the long-term factor alone determines the probability law. In scenario M_N, how likely the firm is to succeed also depends on the short-term factor, which is not yet determined. In this case, randomization eliminates the effect of ambiguity across scenarios M_G and M_B, but not within M_N.[10]

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[10]A more detailed discussion follows Theorem 2.

Figure 3: There are two scenarios. For example, $(0.3, 0.7)$ is the scenario with 30 red balls and 70 black balls in the ambiguous urn. There is only one prior in each scenario. In state s_R (s_B), the ball drawn is red (black). In each scenario, randomization and state revelation constitute a compound lottery that yields $100 and $0 with equal probability.

We discuss the related literature below, and more discussion can be found in Section 5. The closest paper to ours is **?**. Saito also studies an axiomatic model in which the ambiguity-averse individual may have a preference for randomization. His model uses a convex combination to combine the two extreme timing beliefs. We will show that his model is a special case of ours. In addition to models of ambiguity aversion, there are other models of preference for randomization under different choice domains with different motivations, such as **?**; **?**; **?**; and **?**, among others.

Several papers have also examined representations of preferences that involve collections

Figure 4: The only scenario contains two priors (either 30 or 70 red balls in the ambiguous urn). The worst prior within the scenario depends on the realization of randomization.

of sets of priors. **?** study a representation called the multiple multiple-priors representation, which allows for violations of completeness and transitivity. **?** propose the Boolean expected utility representation, in which the belief the individual uses to evaluate an act is the most pessimistic prior from the most optimistic set of priors.

Empirical studies have examined whether individuals strictly prefer to randomize. For example, **?**; **?**; **?**; **?**; **?**; and **?**. Many of them find a nonnegligible number of subjects who strictly prefer to randomize, but some do not. This is consistent with our theory: The individual may strictly prefer to randomize in some choice problems but not in others, and how desirable randomization appears to her depends on her timing beliefs.

Preference for randomization is also an important topic in preference elicitation. For example, our main idea that the individual's evaluation of a lottery depends on her belief about how the unknown probability law is determined has also appeared in the research on random incentive mechanisms. **?** and **?** argue that ambiguity-averse participants may use the randomization device to hedge, and thus the incentive compatibility of the random incentive mechanism may be affected. **?** also points out the difficulty of preference elicitation when

individuals face ambiguity: Individuals with non-SEU preferences may end up behaving as if they are SEU maximizers in preference elicitation choice problems.

Finally, many papers have studied ambiguity and robustness in different areas of economics. As mentioned earlier, how mixed strategies should be introduced and evaluated has been an issue. Our results offer some new insights to this issue. We leave the detailed discussion to the end of the paper.

The paper proceeds as follows. Section 2 introduces the choice domain, mixture operations, and a variant of AA's characterization of the subjective expected utility representation. Section 3 presents the main axioms, the DMEU representation, and the main results. Section 4 examines three special cases of the DMEU. Section 5 discusses the related literature.

2 Preliminaries

The set of consequences is a compact Polish space $\mathcal{X}$.[11] Let $\Delta(\mathcal{X})$ be the set of Borel probability measures on $\mathcal{X}$, endowed with the topology of weak convergence. Elements of $\Delta(\mathcal{X})$ are called *prizes*. Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a finite set of states. An *act* $f : \mathcal{S} \to \Delta(\mathcal{X})$ is a function that assigns a prize to each state. For each state s_i, we write f_i instead of $f(s_i)$ for simplicity. Let $\mathcal{F}$ denote the set of all acts, endowed with the product topology. Let $\Delta(\mathcal{F})$, the set of Borel probability measures on $\mathcal{F}$, denote the set of all *lotteries*, endowed with the topology of weak convergence. Henceforth, when we say a set of acts, we mean a Borel measurable subset of $\mathcal{F}$. The support of a lottery $P \in \Delta(\mathcal{F})$, denoted by $\mathrm{supp}(P)$, is the smallest closed set of acts $F \subset \mathcal{F}$ such that $P(F) = 1$. The individual has a binary relation/preference $\succsim$ on $\Delta(\mathcal{F})$. We denote the asymmetric part of $\succsim$ by $\succ$ and the symmetric part by $\sim$.

A lottery represents randomization over acts. Following randomization, the individual receives an act. Next, a state is *revealed to the individual*, after which the act assigns the prize associated with the realized state to the individual. Finally, as the prize's risk resolves,

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[11]A Polish space is a complete separable metric space.

the individual receives a consequence (see Figure 6).

Lotteries are denoted by P, Q, R, acts are denoted by f, g, h, and prizes are denoted by p, q, r. A degenerate lottery that assigns probability 1 to an act f is identified with f. If an act yields the same prize p in all states, the act is called a *constant act*, and is identified with p. If a prize assigns probability 1 to a consequence x, we denote the prize by δ_x.

We assume throughout the paper that $\succsim$ is nontrivial: There exist some lotteries P and Q such that $P \succ Q$.

2.1 Randomization and State-wise Randomization

We distinguish between two kinds of mixture operations. First, the *randomization* between lotteries P and Q with probability $\alpha \in [0, 1]$, denoted by $\alpha P + (1 - \alpha)Q$, is a lottery that assigns probability $\alpha P(F) + (1 - \alpha)Q(F)$ to each set of acts $F \subset \mathcal{F}$. The second kind of mixture operation is state by state and for acts. The *state-wise randomization* between acts f and g with probability $\alpha \in [0, 1]$, denoted by $\alpha f +_{sw} (1 - \alpha)g$, is an act that assigns the prize $\alpha f_i + (1 - \alpha)g_i$ to each state s_i (see Figure 7).[12]

The difference between the two mixture operations is obvious, but one useful way to understand it is to consider randomization as a mixture performed *before* the state is revealed to the individual, and state-wise randomization a mixture performed *after* the state is revealed. In the urn experiment, before a ball is drawn, the individual tosses a fair coin and bets on red being drawn from the ambiguous urn if heads and black if tails. This is randomization and is done before the color of the drawn ball is revealed to the individual. As discussed in Section 1, whether such randomization eliminates the effect of ambiguity depends on the individual's subjective timing belief.

In contrast, suppose the individual first observes the color of the ball drawn, and then tosses the coin and bets accordingly. This will generate an act that yields a 50-50 lottery

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[12]Here, $\alpha f_i + (1 - \alpha)g_i$ is the standard mixture between prizes (probability measures over consequences) used in expected utility theory. With an abuse of notation, this mixture is also denoted by "+." In most other places of the paper, however, "+" represents randomization between lotteries.

between \$100 and \$0 in all states, s_R (red is drawn) and s_B (black is drawn), which is exactly the 50-50 state-wise randomization between betting on red and black (see also the right-hand side of Figures 1 and 7). Since the prize is identical in all states, this 50-50 state-wise randomization eliminates the effect of ambiguity, *regardless of the individual's belief about how the unknown probability law is determined.* Intuitively, if the mixture happens after the state is revealed, nature must have determined the probability law. Hence, the individual's belief about how the probability law is determined no longer matters; more importantly, under the same mixture probability, state-wise randomization should be (weakly) more effective in eliminating the effect of ambiguity than randomization.

2.2 Subjective Expected Utility

AA provide a characterization of the subjective expected utility representation. We introduce a variant of it that will be useful in motivating our axioms in Section 3. The first four axioms are restatements of AA's axioms and assumptions using our terminology and notation.

Axiom 1 *(Weak Order) The preference $\succsim$ is complete and transitive.*

Axiom 2 *(Continuity) For any $P \in \Delta(\mathcal{F})$, $\{Q \in \Delta(\mathcal{F}) : Q \succsim P\}$ and $\{Q \in \Delta(\mathcal{F}) : P \succsim Q\}$ are closed.*

The first two axioms are basic. The next three axioms are the ones we want to relax later.

Axiom 3 *(Independence) For any $P, Q, R \in \Delta(\mathcal{F})$ and $\alpha \in (0,1)$, $P \succsim Q$ if and only if $\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$.*

Axiom 4 *(State-wise Independence) For any $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \succsim g$ if and only if $\alpha f +_{sw} (1-\alpha)h \succsim \alpha g +_{sw} (1-\alpha)h$.*

To state the last axiom, we need the following notation. For each lottery P, let f^P denote the act such that

$$f_i^P(X) = \int_{\mathcal{F}} f_i(X) \, dP, \tag{1}$$

for any $i \in \{1, \ldots, n\}$ and measurable set of consequences $X \subset \mathcal{X}$. By construction, the lottery P is converted into an act f^P state by state via reduction of compound lotteries. For example, suppose the support of P is finite. Fixing any state s_i, f_i^P is the mixture of prizes weighted by P, $\sum_{f \in \mathrm{supp}(P)} P(f) \cdot f_i$, which is a standard compound lottery in expected utility theory. To put it differently, if we convert all randomization in P into state-wise randomization, we obtain f^P.

Axiom 5 *(Strong Dominance) For any $P, Q \in \Delta(\mathcal{F})$, if $f_i^P \succsim f_i^Q$ for any i, $P \succsim Q$.*

This axiom combines AA's two axioms, *Monotonicity in Prizes* and *Reversal of Order in Compound Lotteries*. *Monotonicity in Prizes* requires that for any two acts f and g, if $f_i \succsim g_i$ for every i, $f \succsim g$. *Reversal of Order in Compound Lotteries* requires that $f^P \sim P$. Therefore, *Strong Dominance* requires that the individual be indifferent between randomization and state-wise randomization. The lemma below emphasizes this, and illustrates some basic relation between *Independence*, *State-wise Independence*, and *Strong Dominance*. All omitted proofs can be found in the Appendix.

Lemma 1 *Suppose $\succsim$ satisfies Weak Order. The following statements are true:*

1. *For any lotteries P, Q and $\alpha \in [0, 1]$, Strong Dominance implies that $\alpha P + (1 - \alpha)Q \sim f^{\alpha P + (1-\alpha)Q} \sim \alpha f^P + (1-\alpha)f^Q \sim \alpha f^P +_{sw} (1-\alpha)f^Q$;*

2. *Under Strong Dominance, Independence is equivalent to State-wise Independence.*

The axioms above characterize the subjective expected utility representation. A function $u : \Delta(\mathcal{X}) \to \mathbb{R}$ is an *expected utility function* if for any prize p,

$$u(p) = \int_{\mathcal{X}} u(\delta_x) \, dp.$$

Let $\Delta(\mathcal{S})$ denote the set of probability measures on $\mathcal{S}$.

Definition 1 *The individual's preference $\succsim$ has a subjective expected utility (SEU) representation if there exists a continuous expected utility function $u : \Delta(\mathcal{X}) \to \mathbb{R}$ and $\mu \in \Delta(\mathcal{S})$ such that the following function W represents $\succsim$:*

$$W(P) = \int_{\mathcal{F}} \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dP. \tag{2}$$

Since we assume that $\succsim$ is nontrivial, implicitly in the definition above, u is not constant. We will not mention this again for our other representations, but similar implicit assumptions apply to them as well.

When $\succsim$ has an SEU representation, the individual is indifferent between randomization and state-wise randomization, which is reflected in the following equations:

$$W(P) = \int_{\mathcal{F}} \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dP = \int_{\mathcal{S}} \left(\int_{\mathcal{F}} u(f) \, dP \right) \, d\mu = \int_{\mathcal{S}} u(f^P) \, d\mu = W(f^P).$$

We say that an SEU representation is unique if in (2), u is unique up to a positive affine transformation and μ is unique. The theorem below restates AA's main representation result.

Theorem 1 *(?) The individual's preference $\succsim$ has an SEU representation if and only if it satisfies Weak Order, Continuity, Independence, State-wise Independence, and Strong Dominance. Furthermore, the SEU representation is unique.*

Because *Strong Dominance* holds, Lemma 1 implies that one of the two axioms, *Independence* or *State-wise Independence*, is redundant. Stating the theorem in this way, however, helps us motivate the relaxations we introduce below.

3 Axioms and the Representation

Three axioms will be relaxed. Observing that state-wise randomization may render ambiguity irrelevant because of hedging (as discussed in Section 2.1), **?** propose a relaxation of their independence assumption, which is *State-wise Independence* in our setting, to capture this. They allow the individual's preference to be convex so that she may strictly prefer state-wise randomization; meanwhile, constant acts should continue to satisfy the independence assumption, because constant acts do not provide hedging opportunities. We relax *State-wise Independence* in the same way.

We also want to relax *Independence* so that the individual may strictly prefer randomization. In the urn experiment, if the individual agrees with Raiffa's (1961) argument, or in the service example, if the details of the services are precisely stated in contracts beforehand, randomization can render ambiguity irrelevant. In the investment example in Figure 5, randomization partially eliminates the effect of ambiguity, which also violates *Independence*. We relax *Independence* following the way **?** relax *State-wise Independence*.

We say that a lottery P preserves *Independence* if for any lotteries Q, R and $\alpha \in (0,1)$, $Q \succsim R$ if and only if $\alpha Q + (1-\alpha)P \succsim \alpha R + (1-\alpha)P$. We say that an act f preserves *State-wise Independence* if for any acts g, h and $\alpha \in (0,1)$, $g \succsim h$ if and only if $\alpha g +_{sw} (1-\alpha)f \succsim \alpha h +_{sw} (1-\alpha)f$.

Axiom 6 *(Preference for Randomization)*

  (i) For any $P, Q \in \Delta(\mathcal{F})$ and $\alpha \in [0,1]$, $P \succsim Q$ implies $\alpha P + (1-\alpha)Q \succsim Q$;

  (ii) Every constant act preserves Independence.

Axiom 7 *(Preference for State-wise Randomization)*

  (i) For any $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $f \succsim g$ implies $\alpha f +_{sw} (1-\alpha)g \succsim g$;

  (ii) Every constant act preserves State-wise Independence.

Next, we replace *Strong Dominance* with three weaker axioms. *Strong Dominance* requires that the individual be indifferent between randomization and state-wise randomization. Such indifference is problematic, because the individual may not agree that randomization is as effective in eliminating the effect of ambiguity as state-wise randomization. For instance, suppose an individual believes that the effect of ambiguity is unaffected by randomization in the urn experiment. Since the 50-50 state-wise randomization renders ambiguity irrelevant in the urn experiment (as discussed in Section 2.1), this individual must have viewed randomization and state-wise randomization differently. Hence, *Strong Dominance* is violated.[13]

Indifference between the two types of mixture operations arises because to compare two lotteries, P and Q, *Strong Dominance* first converts them to two acts, state by state, via reduction of compound lotteries (see (1)). The conversion turns randomization into state-wise randomization and implicitly identifies the two types of mixture operations. Therefore, a natural weakening is to compare P and Q without the conversion. We compare them *realization by realization* and state by state. The relation between realization-by-realization comparison of lotteries and first-order stochastic dominance is well known. Following **?**, we say that a set of acts $F \subset \mathcal{F}$ is *increasing* if whenever $f_i \succsim g_i$ for any i and $g \in F$, we have $f \in F$.

Definition 2 *We say that P first-order stochastically dominates Q if for any increasing set of acts $F \subset \mathcal{F}$, $P(F) \geq Q(F)$.*

For any two acts f and g, we think of f as an improvement of g if $f_i \succsim g_i$ for every $i \in \{1, \ldots, n\}$. The condition in the above definition implies that we can view P as a realization-by-realization improvement of Q, similar to the classic results on first-order

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[13]An individual can simultaneously violate *Independence*, *State-wise Independence*, and *Strong Dominance*. To see this, consider an individual who believes that randomization eliminates the effect of ambiguity in the service example with precise contracts (which suggests that *Independence* is violated), and at the same time believes that ambiguity is unaffected by randomization in the urn experiment. Since the 50-50 state-wise randomization renders ambiguity irrelevant in the urn experiment, both *State-wise Independence* and *Strong Dominance* are violated.

stochastic dominance from the theory of risk. The axiom below, *FOSD*, says that if P is a realization-by-realization improvement of Q, P is preferred to Q.[14]

Axiom 8 *(FOSD) For any $P, Q \in \Delta(\mathcal{F})$, if P first-order stochastically dominates Q, then $P \succsim Q$.*

The next two axioms compare randomization with state-wise randomization. The first is related to *second-order stochastic dominance*. The discussion in Sections 1 and 2.1 shows that the extent to which randomization eliminates the effect of ambiguity depends on the individual's subjective timing belief, while the extent to which state-wise randomization eliminates the effect of ambiguity does not. Moreover, under the same mixture probability, state-wise randomization should be weakly more effective in eliminating the effect of ambiguity than randomization, regardless of the individual's subjective timing belief. Therefore, given any lottery, the individual should prefer to replace randomization with state-wise randomization whenever possible.

What choice behavior reveals that the individual prefers to replace randomization with state-wise randomization? First, we should have

$$\alpha f +_{sw} (1 - \alpha)g \succsim \alpha f + (1 - \alpha)g \tag{3}$$

for any acts f, g and $\alpha \in [0, 1]$.

The comparison between $\alpha f +_{sw} (1 - \alpha)g$ and $\alpha f + (1 - \alpha)g$ closely resembles the comparison between $\delta_{\$(\alpha x + (1-\alpha)y)}$ and $\alpha\delta_{\$x} + (1 - \alpha)\delta_{\$y}$ $(x, y \in \mathbb{R})$ in expected utility theory. In particular, $\alpha f +_{sw} (1-\alpha)g$ and $\alpha f + (1-\alpha)g$ share the same expected utility (of prizes) fixing any state, similar to how $\delta_{\$(\alpha x + (1-\alpha)y)}$ and $\alpha\delta_{\$x} + (1 - \alpha)\delta_{\$y}$ share the same expected amount of money. In expected utility theory, $\alpha\delta_{\$x} + (1 - \alpha)\delta_{\$y}$ is a simple *mean-preserving spread* of

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[14]One can show that *FOSD* is weaker than the relaxation of *Strong Dominance* in **?**. In the context of social choice, **?** considers a similar relaxation. The idea behind the monotonicity conditions in **?** and **?**, which are also in the context of social choice, is similar to *FOSD*.

$\delta_{\$(\alpha x+(1-\alpha)y)}$ (see **?**). Below, we define mean-preserving spreads in our setting following how they are defined in Rothschild and Stiglitz.

Definition 3 *We say that Q is a simple mean-preserving spread of P if $P = \beta[\alpha f +_{sw} (1-\alpha)g] + (1-\beta)R$ and $Q = \beta[\alpha f + (1-\alpha)g] + (1-\beta)R$ for some acts f, g, lottery R, and $\alpha, \beta \in [0,1]$. We say that Q is a mean-preserving spread of P if we can find lotteries $P_1, \ldots, P_l$ such that $P_1 = P$, $P_l = Q$, and P_{j+1} is a simple mean-preserving spread of P_j, $j = 1, \ldots, l-1$.*

Following Theorem 1(b) of **?**, we define second-order stochastic dominance in our setting.

Definition 4 *We say that P second-order stochastically dominates Q if there exist two sequences of finite-support lotteries $\{P_j\}_{j=1}^{\infty}$ and $\{Q_j\}_{j=1}^{\infty}$ such that (i) P_j converges to P and Q_j converges to Q; and (ii) every Q_j is a mean-preserving spread of P_j.*

The idea of the next axiom is as follows: If in the limit, P can be obtained by replacing some part of randomization of Q with state-wise randomization, P should be more effective in eliminating the effect of ambiguity than Q and hence be preferred.

Axiom 9 *(SOSD) For any $P, Q \in \Delta(\mathcal{F})$, if P second-order stochastically dominates Q, then $P \succsim Q$.*

The last axiom states that for constant acts, the individual remains indifferent between randomization and state-wise randomization, as in subjective expected utility theory.[15] To understand this, note that under $SOSD$, $P \succsim Q$ if $Q = \beta[\alpha f + (1-\alpha)g] + (1-\beta)R$ is a simple mean-preserving spread of $P = \beta[\alpha f +_{sw} (1-\alpha)g] + (1-\beta)R$. Strict preference $P \succ Q$ occurs, however, only if $\alpha f +_{sw} (1-\alpha)g$ in P is *strictly* more effective in eliminating the effect of ambiguity than $\alpha f + (1-\alpha)g$ in Q. Following **?**, we have assumed that when g is a constant act, neither $\alpha f +_{sw} (1-\alpha)g$ nor $\alpha f + (1-\alpha)g$ eliminates the effect of ambiguity.[16] Thus, if

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[15] A similar idea appears in an axiom called *Indifference to Probability Mixture* in **?**.

[16] See part (ii) of *Preference for Randomization* and part (ii) of *Preference for State-wise Randomization*.

g is a constant act, $\alpha f +_{sw} (1 - \alpha)g$ will not be strictly more effective than $\alpha f + (1 - \alpha)g$ in eliminating the effect of ambiguity, and $P \succ Q$ should not occur.

Formally, we say that Q is a *constant-act simple mean-preserving spread* of P if

$$P = \beta[\alpha f +_{sw} (1 - \beta)p] + (1 - \beta)R \text{ and } Q = \beta[\alpha f + (1 - \alpha)p] + (1 - \beta)R$$

for some act f, constant act p, lottery R, and $\alpha, \beta \in [0, 1]$. We say that Q is a *constant-act mean-preserving spread* of P if we can find lotteries $P_1, \ldots, P_l$ such that $P_1 = P$, $P_l = Q$, and P_{j+1} is a constant-act simple mean-preserving spread of P_j, $j = 1, \ldots, l - 1$.

Definition 5 *We say that P can be obtained by modifying Q's mixture timing of constant acts if there exist two sequences of finite-support lotteries $\{P_j\}_{j=1}^{\infty}$ and $\{Q_j\}_{j=1}^{\infty}$ such that (i) P_j converges to P and Q_j converges to Q; and (ii) every Q_j is a constant-act mean-preserving spread of P_j, or every P_j is a constant-act mean-preserving spread of Q_j.*

The last axiom is as follows.

Axiom 10 *(Indifference to Mixture Timing of Constant Acts) For any $P, Q \in \Delta(\mathcal{F})$, if P can be obtained by modifying Q's mixture timing of constant acts, then $P \sim Q$.*

This axiom rules out the possibility that, for example, the individual prefers early risk resolution or has different risk attitudes before and after the state is revealed. We can incorporate these possibilities in our analysis, but we leave them out to emphasize that our findings are not driven by these possibilities.

The following lemma complements Lemma 1 and shows that under some of our new axioms, *Independence* and *State-wise Independence* imply *Strong Dominance*. Together with Lemma 1, this means that it is impossible to only violate one axiom among *Independence*, *State-wise Independence*, and *Strong Dominance*. Any two of those three axioms imply the third. This finding will be useful later.

Lemma 2 *Suppose $\succsim$ satisfies Weak Order, Continuity, FOSD, and Indifference to Mixture Timing of Constant Acts. Independence and State-wise Independence imply Strong Dominance.*

Our main theorem will show that the axioms above are equivalent to the following representation of the individual's preference. Let $\mathcal{K}(\Delta(\mathcal{S}))$ be the collection of all nonempty closed subsets of $\Delta(\mathcal{S})$ endowed with the Hausdorff metric topology. Since $\Delta(\mathcal{S})$ is compact and Polish, $\mathcal{K}(\Delta(\mathcal{S}))$ is also compact and Polish (see Theorem 3.85 of **?**).

Definition 6 *The individual's preference $\succsim$ has a double maxmin expected utility (DMEU) representation if there exist a continuous expected utility function $u : \Delta(\mathcal{X}) \to \mathbb{R}$ and a compact collection $\mathcal{M}$ of compact subsets of $\Delta(\mathcal{S})$ such that $\succsim$ can be represented by*

$$W(P) = \min_{M \in \mathcal{M}} \ \int_{\mathcal{F}} \left(\min_{\mu \in M} \ \int_{\mathcal{S}} u(f) \ d\mu \right) \ dP. \tag{4}$$

If the individual's preference $\succsim$ has a representation described in the above definition, we say that $\succsim$ has a DMEU representation $(\mathcal{M}, u)$. Below is the main theorem.

Theorem 2 *The individual's preference $\succsim$ has a DMEU representation if and only if it satisfies Weak Order, Continuity, Preference for Randomization, Preference for State-wise Randomization, FOSD, SOSD, and Indifference to Mixture Timing of Constant Acts.*

The uniqueness of the representation will be studied in Section 3.2, and we provide some comparative static analysis for the DMEU representation in the Appendix.

In the DMEU representation, each $M \in \mathcal{M}$ is a set of priors called a *scenario*. The interpretation of the representation is that the individual who is conservative and focuses on the worst case believes that before she randomizes (trivially or not), a scenario $M \in \mathcal{M}$ unknown to her has occurred, and after the realization of her randomization, an unknown probability law from M will be determined. Alternatively, the individual believes that before she randomizes, nature, which plays against her, has chosen a scenario from $\mathcal{M}$, and after she

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randomizes, nature will choose a probability law from the scenario based on the realization of her randomization.

Randomization may only partially eliminate the effect of ambiguity. It may help the individual hedge across scenarios, but not across probability laws within a scenario. Figure 5 is a simple example. If the individual chooses "Long" deterministically, the worst scenario will be M_B. If she chooses "Short" deterministically, the worst scenario will be M_G. Either way, her utility will be zero. However, if she randomizes between "Long" and "Short" with equal probability, the effect of ambiguity induced by M_G and M_B is eliminated: In scenarios M_G and M_B, the individual receives the same prize with utility 0.5. The worst scenario now is M_N, which contains two priors. Randomization cannot eliminate the effect of ambiguity induced by the priors in M_N, and the individual's utility will be 0.3 under M_N.

For any two acts f and g, randomization between f and g may be strictly preferred only if f and g have different minimizing scenarios. With trivial randomization—that is, if the individual chooses an act f with probability 1—(4) becomes

$$W(f) = \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu,$$

in which $\mathbb{M} = \bigcup_{M \in \mathcal{M}} M$ pools all priors together.

3.1   Sketch of the Proof

To prove the sufficiency of the axioms, we begin by finding an expected utility representation for the individual's preference over constant acts, and use it to construct a utility representation $W(P)$ for her preference over lotteries.

Next, we define the *expected utility core* $\succsim^*$ of $\succsim$: For any lotteries P, Q, $P \succsim^* Q$ if for any $R \in \Delta(\mathcal{F})$ and $\alpha \in (0, 1]$, $\alpha P + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)R$. Our notion of the expected utility core is similar to that of **?**, **?**, and **?**.[17] The choice domains, research questions, and

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[17]Similar concepts have also appeared in earlier papers, including **?** and **?**, which can be viewed as the dual version of the type of expected utility core that we use.

key axioms of those papers are different from ours.

We verify that $\succsim^*$ satisfies the axioms in **?** and then apply their main theorem to find a closed set of continuous functions $\mathbb{V}$ such that $P \succsim^* Q$ if and only if

$$\int_{\mathcal{F}} V(u(f_1), \ldots, u(f_n))\ dP \geq \int_{\mathcal{F}} V(u(f_1), \ldots, u(f_n))\ dQ$$

for every $V \in \mathbb{V}$.

We show that if P first-order or second-order stochastically dominates Q, $P \succsim^* Q$. In addition, if P can be obtained by modifying Q's mixture timing of constant acts, $P \sim^* Q$. According to the definition of $\succsim^*$, these observations imply that every $V \in \mathbb{V}$ satisfies properties similar to those in Gilboa and Schmeidler's (1989) Lemma 3.3, and therefore every $V \in \mathbb{V}$ has an MEU representation. We use the fact that every V has an MEU representation to establish that $\mathbb{V}$ is bounded and equicontinuous. According to the Arzelá–Ascoli theorem, $\mathbb{V}$ is compact.

Finally, we show that

$$W(P) = \min_{V \in \mathbb{V}}\ \int_{\mathcal{F}} V(u(f_1), \ldots, u(f_n))\ dP; \tag{5}$$

that is, the representation of the individual's preference $W(P)$ we initially constructed is equal to the minimum of $\int_{\mathcal{F}} V(u(f_1), \ldots, u(f_n))\ dP$.[18] Note the for any lotteries P and Q, $P \succsim^* Q \Rightarrow P \succsim Q$, but the converse is not true in general. However, our axioms ensure that the converse holds under certain conditions, which is a key property of $\succsim^*$ to proving (5): For any lottery P and constant act p,

$$P \succsim p \Rightarrow P \succsim^* p.$$

A similar property of the expected utility core is also used in **?**.

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[18]The right-hand side of (5) is similar to the representation in **?**, but our choice domain, proof strategy, and research questions are different from his.

3.2 Uniqueness of the DMEU Representation

Suppose the individual's preference is represented by the DMEU representation W as in (4). In general, we can add redundant scenarios to $\mathcal{M}$ without changing W, and hence $\mathcal{M}$ will not be unique. For example, suppose $\mathcal{M} = \{\{\mu_1, \mu_2\}\}$ for some distinct $\mu_1, \mu_2 \in \Delta(\mathcal{S})$. If we replace $\mathcal{M}$ with $\{\{\mu_1, \mu_2\}, \{\mu_1\}\}$, because

$$\int_{\mathcal{F}} \left(\min_{\mu \in \{\mu_1, \mu_2\}} \int_{\mathcal{S}} u(f) \, d\mu \right) dP \leq \int_{\mathcal{F}} \left(\min_{\mu \in \{\mu_1\}} \int_{\mathcal{S}} u(f) \, d\mu \right) dP,$$

the function W will be unaffected.

However, once we remove the redundant scenarios from $\mathcal{M}$, $\mathcal{M}$ will be unique up to the closed convex hull. In other words, there is a unique minimal DMEU representation. The uniqueness of the expected utility function u is standard: It is unique up to a positive affine transformation.

Theorem 3 *Suppose the individual's preference $\succsim$ satisfies Weak Order, Continuity, Preference for Randomization, Preference for State-wise Randomization, FOSD, SOSD, and Indifference to Mixture Timing of Constant Acts. Then, $\succsim$ has a DMEU representation $\left(\widehat{\mathcal{M}}, u \right)$ such that the following statements hold:*

1. *If $\succsim$ has a DMEU representation $(\mathcal{M}, v)$, then $\overline{co}\left(\widehat{\mathcal{M}} \right) \subset \overline{co}(\mathcal{M})$ and $v = \alpha u + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$.*

2. *The expected utility core $\succsim^*$ of $\succsim$ satisfies the following: For any $P, Q \in \Delta(\mathcal{F})$, $P \succsim^* Q$ if and only if*

$$\int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) dP \geq \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) dQ$$

   for every $M \in \widehat{\mathcal{M}}$.

Moreover, $\widehat{\mathcal{M}}$ is unique up to the closed convex hull.

Our notion of uniqueness is similar to that of **?**. In their continuous Cautious Expected Utility representation, the minimal set of certainty equivalent functions is also unique up to the closed convex hull.

4   Special Cases of the DMEU

We examine three special cases of the DMEU. In the first, it is as if the individual always believes that nature moves before she randomizes; in the second, it is as if she always believes that nature moves after she randomizes; and the third is **?**. The first two have also been discussed in **?** and **?**.[19]

To obtain the DMEU representation, we simultaneously relax *Independence*, *State-wise Independence*, and *Strong Dominance*. A natural question is what the representation of the individual's preference looks like if we only relax some of them. Lemmas 1 and 2 tell us that imposing any two of these three axioms is equivalent to imposing all of them, which leads to the SEU representation. Therefore, the remaining possibility is to impose exactly one of the three axioms on a preference with a DMEU representation.

First, we show that if *Strong Dominance* is imposed, it is as if the individual always believes that nature moves before she randomizes, which corresponds to **?**.

Theorem 4 *Suppose the individual's preference $\succsim$ has a DMEU representation. Then, there exists a continuous expected utility function $u : \Delta(\mathcal{X}) \to \mathbb{R}$ and a compact set $\mathbb{M} \subset \Delta(\mathcal{S})$ such that*

$$W^{ea}(P) = \min_{\mu \in \mathbb{M}} \int_{\mathcal{F}} \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dP \tag{6}$$

represents $\succsim$ if and only if $\succsim$ satisfies Strong Dominance.

We call (6) the *Ex Ante MEU* representation. This representation is equivalent to the DMEU representation whose scenarios are all singletons, and the union of whose scenarios

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[19]**?** characterize two representations that are similar to the first two special cases of the DMEU representation discussed in this section.

is $\mathbb{M}$. In the Ex Ante MEU representation, for each prior $\mu \in \mathbb{M}$, the individual evaluates a lottery P using the standard SEU formula. Therefore, it is as if the individual believes that before she randomizes, a fixed but unknown probability law has already been determined.

To understand why the Ex Ante MEU representation satisfies *Strong Dominance*, recall that an important consequence of *Strong Dominance* is $P \sim f^P$, which is in general not true in the DMEU representation. However, in the Ex Ante MEU representation, $P \sim f^P$ always holds:

$$
\begin{aligned}
W^{ea}(P) &= \min_{\mu \in \mathbb{M}} \int_{\mathcal{F}} \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dP \\
&= \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} \left(\int_{\mathcal{F}} u(f) \, dP \right) \, d\mu \\
&= \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f^P) \, d\mu = W^{ea}(f^P).
\end{aligned}
$$

Next, we turn to *Independence*. If *Independence* is imposed instead, we will obtain a representation in which it is as if the individual always believes that nature moves after she randomizes.

Theorem 5 *Suppose the individual's preference $\succsim$ has a DMEU representation. Then, there exists a continuous expected utility function $u : \Delta(\mathcal{X}) \to \mathbb{R}$ and a compact set $\mathbb{M} \subset \Delta(\mathcal{S})$ such that*

$$
W^{ep}(P) = \int_{\mathcal{F}} \left(\min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP \tag{7}
$$

represents $\succsim$ if and only if $\succsim$ satisfies Independence.

We call (7) the *Ex Post MEU* representation. This representation is equivalent to the DMEU representation that has only one scenario, $\mathbb{M}$. In the Ex Post MEU representation, to evaluate a lottery P, for each act that is a realization of the lottery, the individual takes the worst prior from $\mathbb{M}$ to evaluate the act. Therefore, it is as if the individual believes that nature moves only after the realization of her randomization.

The reason the Ex Post MEU representation satisfies *Independence* is simple: $W^{ep}(P)$ is linear in P.

Finally, if we impose *State-wise Independence* on a preference with a DMEU representation, we will return to the SEU representation. To see this, recall that in a DMEU representation, the utility of an act (degenerate lottery) f is $W(f) = \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f)\, d\mu$, in which $\mathbb{M} = \bigcup_{M \in \mathcal{M}} M$ pools together all priors from all scenarios. This is the MEU representation of preferences over acts from **?**. If *State-wise Independence* holds, $W(f)$ must be linear. Then, there can only be one prior in $\mathbb{M}$, which means that the DMEU representation must be an SEU representation.

4.1 ? and the DMEU Representation

? observes that in the urn experiment, there are two intuitive ways to evaluate a lottery. Using our language, the two ways correspond to the cases in which it is as if the individual always believes that nature moves before and after, respectively, she randomizes. Saito characterizes a representation in which a lottery P is evaluated by a convex combination of the Ex Ante MEU function and the Ex Post MEU function; that is, the utility of P is equal to

$$W(P) = \delta W^{ea}(P) + (1 - \delta)W^{ep}(P) \tag{8}$$

for some $\delta \in [0, 1]$ and some Ex Ante MEU function W^{ea} and Ex Post MEU function W^{ep} that share the same set of priors $\mathbb{M}$. The parameter δ measures how useful lotteries are in the elimination of ambiguity's effect. Clearly, the main idea of this representation is closely related to that of our DMEU representation, and we show below that in fact, Saito's (2015) representation is a special case of the DMEU representation.

Saito's (2015) representation generalizes the Ex Ante MEU and the Ex Post MEU representations, but is not flexible enough to capture the following example. Consider an individual who, facing the urn experiment, believes that nature moves completely before she

moves; facing the service example without any contract provided, believes that nature moves completely after she moves. In (8), this individual must use $\delta = 1$ in the first choice problem and $\delta = 0$ in the second, which is not allowed in Saito. The DMEU representation, however, allows the individual to hold the subjective timing belief described above.

In terms of Saito's (2015) axioms, because Saito studies preferences over sets of acts, it is not straightforward to see which of his axioms is violated by this example, but intuitively, the violated axiom should be *Dominance*. Denote the 50-50 randomization between betting on red and black being drawn from the ambiguous urn in the urn experiment by P, and the 50-50 randomization between choosing the two service providers in the service example without contracts provided by Q. Suppose the payoffs are matched: The utility of getting the good (bad) service is equal to the utility of \$100 (\$0). Now, imagine that nature moves before the individual randomizes in *both* choice problems. In that case, P and Q will be equally good. Similarly, imagine that nature moves after she randomizes in *both* choice problems. In that case, P and Q will again be equally good. Since Saito's *Dominance* compares lotteries only based on these two (hypothetical) extreme timing beliefs, *Dominance* should imply that the individual is indifferent between P and Q. In our discussion, however, we have argued that it is reasonable that the individual strictly prefers P to Q.

It is probably not obvious why the objective function (8) is a special case of the DMEU representation. To see this directly from the representation, given a representation as in (8), for each $\hat{\mu} \in \mathbb{M}$, let $M_{\hat{\mu}} := \{\delta\hat{\mu} + (1 - \delta)\mu' : \mu' \in \mathbb{M}\}$. Let $\mathcal{M} := \{M_{\hat{\mu}} : \hat{\mu} \in \mathbb{M}\}$. Since $\mathbb{M}$ is compact, verifying that $\mathcal{M}$ is compact is straightforward. Now, consider the following DMEU representation:

$$W(P) = \min_{M_{\hat{\mu}} \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M_{\hat{\mu}}} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP.$$

We have

$$
\begin{aligned}
W(P) &= \min_{M_{\hat{\mu}} \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu' \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d(\delta \hat{\mu} + (1-\delta)\mu') \right) \, dP \\
&= \delta \min_{\hat{\mu} \in \mathbb{M}} \int_{\mathcal{F}} \int_{\mathcal{S}} u(f) \, d\hat{\mu} \, dP + (1-\delta) \int_{\mathcal{F}} \left(\min_{\mu' \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu' \right) \, dP \\
&= \delta W^{ea}(P) + (1-\delta) W^{ep}(P).
\end{aligned}
$$

Thus, the individual's objective function in **?** is a special case of the DMEU representation.

The observation above proves the following result.

Theorem 6 *Suppose the individual's preference $\succsim$ can be represented by*

$$
W(P) = \delta \min_{\mu \in \mathbb{M}} \int_{\mathcal{F}} \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dP + (1-\delta) \int_{\mathcal{F}} \left(\min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP
$$

for some compact set $\mathbb{M} \subset \Delta(\mathcal{S})$ and $\delta \in [0,1]$. Then, $\succsim$ has a DMEU representation $(\mathcal{M}, u)$, in which $\mathcal{M} = \{M_{\hat{\mu}} : \hat{\mu} \in \mathbb{M}\}$ and $M_{\hat{\mu}} = \{\delta \hat{\mu} + (1-\delta)\mu' : \mu' \in \mathbb{M}\}$ for each $\hat{\mu} \in \mathbb{M}$.

This finding allows us to use the DMEU representation to offer a new interpretation of Saito's (2015) representation. Consider an individual who believes that the set of possible priors is a convex set $\mathbb{M}$. She believes that before she randomizes, nature has partially determined the probability law by choosing a subset of $\mathbb{M}$ or, equivalently, shrinking $\mathbb{M}$ to some subset of it. In particular, she believes that nature can shrink $\mathbb{M}$ by a fixed fraction, $1 - \delta$, centered at any $\hat{\mu} \in \mathbb{M}$. It is not difficult to see that shrinking $\mathbb{M}$ by $1 - \delta$ centered at $\hat{\mu} \in \mathbb{M}$ exactly generates the scenario $M_{\hat{\mu}}$ we construct above. Therefore, if an individual's preference can be represented by (8), it is as if she holds the above belief about how the probability law is determined.

5 Discussion

Some related papers have been discussed at the end of the Introduction. We provide additional discussion in this section. In addition to **?**, **?** and **?** are also closely related to our paper. Seo characterizes the *second-order subjective expected utility* (SOSEU) representation, in which the individual has a probability measure $m \in \Delta(\Delta(\mathcal{S}))$ over probability measures on the state space (μ's from $\Delta(\mathcal{S})$), and she evaluates a lottery P according to

$$W(P) = \int_{\mathcal{F}} \left[\int_{\Delta(\mathcal{S})} v \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dm(\mu) \right] \, dP.^{20}$$

In terms of axioms, there are two main differences between the SOSEU representation and the DMEU representation. First, the former maintains *Independence*, while the latter relaxes it. Thus, in Seo, the individual always disagrees with Raiffa's argument that randomization renders ambiguity irrelevant and has no preference for randomization. Second, the DMEU representation satisfies *Indifference to Mixture Timing of Constant Acts*, while the SOSEU representation reduces to the SEU representation if a similar additional axiom is imposed (see Seo's Corollary 5.2). Hence, the only intersection between Seo's representation and ours is the SEU representation. Finally, as the function v in Seo's representation becomes arbitrarily concave, the SOSEU representation will converge to the Ex Post MEU representation.

Noting that there are two popular ways to evaluate randomization under ambiguity, which correspond to what we call the Ex Ante MEU and the Ex Post MEU representations, **?** introduces a representation that combines them (see (8)). This representation later appears in **?**, which, by analyzing preferences over sets of acts, characterizes the behavior of an individual who anticipates choosing randomly from a set of acts. Section 4.1 discusses some main differences between Saito's model and ours. In particular, since the DMEU representation nests Saito's representation as a special case, the latter must also satisfy *SOSD*. In other words, fixing any mixture probability, state-wise randomization must be more

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[20]This representation is similar to the smooth ambiguity representation introduced by **?**.

effective in eliminating the effect of ambiguity than randomization in Saito's representation.

The choice domain we use is from AA. By analyzing the individual's preference in the choice domain $\Delta(\mathcal{F})$, AA characterize the SEU representation. If their axiom, *Reversal of Order in Compound Lotteries*—or, in our version, *Strong Dominance*—holds, the individual does not distinguish between randomization and state-wise randomization. Then, the choice domain $\Delta(\mathcal{F})$ can be reduced to $\mathcal{F}$, in which case only state-wise randomization needs to be considered. Many papers on ambiguity have been built on this reduced version of AA's domain, including **?**; **?**; **?**; **?**; **?**; **?**; **?**; and **?**. In contrast, **?**, **?**, and our paper adopt the original choice domain of AA without imposing *Strong Dominance*.

A large amount of research has studied ambiguity and robustness (some maxmin principles) in various areas of economics. However, how mixed strategies should be introduced and evaluated has been an issue. For example, most papers that study monopoly sales problems have assumed that randomization can help individuals hedge (similar to the case in the Ex Ante MEU representation).[21] On the other hand, most papers on contracts/auctions/mechanisms with ambiguity or robustness concerns have either ruled out randomization/mixed strategies or assumed that randomization does not help individuals hedge (similar to the case in the Ex Post MEU representation).[22] Some of these papers point out that randomization should help individuals hedge, but this case will be more difficult to deal with.[23] An exception is **?**, who examine both extreme cases. Mixed strategies are ruled out in most papers on robustness or ambiguity in macroeconomics and finance. At least in some of these papers, it is natural to consider mixed strategies.[24]

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[21]See, among others, **?**; **??**; **?**; and **?**.

[22]See, among others, **?**; **?**; **?**; **??**; **?**; **??**; **?**; **?**; and **?**.

[23]Indeed, according to our analysis, some of these papers (e.g., **?**) should have assumed that randomization helps individuals hedge.

[24]See, among others, **?**; **?**; **?**; **?**; and **?**.

6   Conclusion

How randomization/mixed strategies should be modeled under the presence of ambiguity or when some maxmin principle is used has been an unsettled issue in many research areas of economics. Two simple yet opposite ways to model randomization are well known. In the classic urn experiment of **?**, one corresponds to **?**—by which randomization should render ambiguity irrelevant—and the other argues that the effect of ambiguity is unaffected by randomization.

We point out that the individual's belief about how the unknown probability law is determined is important in answering this question. If the individual believes that before she randomizes, the probability law—although unknown to her—has been determined, then the first way to model randomization is reasonable. If, however, she believes that the probability law is determined only after her randomization finishes, then the second way is reasonable.

The subjective belief about how the unknown probability law is determined, however, is usually more complicated than these two extreme cases. For example, the probability that an investment is successful may depend on some long-term factor and some short-term factor. The individual may believe that before she makes any decision (possibly randomized), the unknown long-term factor has been determined, while the short-term factor will be determined much later. In this case, neither of the above two ways to model randomization seems appropriate.

To address these issues, we adopt the classic framework of **?** and relax their axioms. A new representation is derived. In the representation, the individual has a collection of sets of priors, $\mathcal{M}$. She behaves as if she believes that before she randomizes (trivially or not), nature has chosen an unknown scenario (a set of priors) $M \in \mathcal{M}$, and after the randomization, nature will choose an unknown prior from M. Thus, the individual's preference reveals her belief about how the unknown probability law is determined, and the representation allows randomization to partially eliminate the effect of ambiguity: Randomization may help the individual hedge across scenarios, but not across probability laws within a scenario.

A Appendix

PROOF OF LEMMA 1: It is straightforward to verify the first statement using equation (1) and *Weak Order*. Suppose *Independence* and *Strong Dominance* hold. Take any acts f, g, h and $\alpha \in (0, 1)$ such that $f \succsim g$. By *Independence*, we know that $\alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h$. Using the first statement of the lemma, $\alpha f + (1 - \alpha)h \sim \alpha f +_{sw} (1 - \alpha)h$ and $\alpha g + (1 - \alpha)h \sim \alpha g +_{sw} (1 - \alpha)h$. Therefore, $\alpha f +_{sw} (1 - \alpha)h \succsim \alpha g +_{sw} (1 - \alpha)h$. Showing that $\alpha f +_{sw} (1 - \alpha)h \succsim \alpha g +_{sw} (1 - \alpha)h$ implies $f \succsim g$ is similar. Next, suppose *State-wise Independence* and *Strong Dominance* hold. Take any lotteries P, Q, R and $\alpha \in (0, 1)$ such that $P \succsim Q$. Due to the first statement of the lemma, $f^P \succsim f^Q$. By *State-wise Independence*, $\alpha f^P +_{sw} (1 - \alpha)f^R \succsim \alpha f^Q +_{sw} (1 - \alpha)f^R$. Applying the first statement of the lemma again, we know that $\alpha f^P +_{sw} (1 - \alpha)f^R \sim \alpha P + (1 - \alpha)R$ and $\alpha f^Q +_{sw} (1 - \alpha)f^R \sim \alpha Q + (1 - \alpha)R$. Therefore, $\alpha P + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)R$. Showing that $\alpha P + (1 - \alpha)R \succsim \alpha Q + (1 - \alpha)R$ implies $P \succsim Q$ is similar.

∎

PROOF OF LEMMA 2: Consider two lotteries P and Q such that $f_i^P \succsim f_i^Q$ for each $i \in \{1, \ldots, n\}$. By *FOSD*, $f^P \succsim f^Q$. For each $f \in \mathcal{F}$, by *Weak Order* and *Continuity*, we can find a constant act p^f such that $f \sim p^f$ (see Lemma 4). Let $\{P_l\}_{l=1}^{\infty}$ be a sequence of finite-support lotteries that converges to P. Take an arbitrary P_l. By applying *Independence* finitely many times, we know that the lottery $\bar{P}_l := \sum_{f \in \text{supp}(P_l)} P_l(f) \cdot p^f \in \Delta(\Delta(\mathcal{X}))$ is indifferent to $P_l = \sum_{f \in \text{supp}(P_l)} P_l(f) \cdot f$. Similarly, due to *State-wise Independence*, state-wise randomization $\sum_{sw}_{f \in \text{supp}(P_l)} P_l(f) \cdot p^f$ is indifferent to $\sum_{sw}_{f \in \text{supp}(P_l)} P_l(f) \cdot f = f^{P_l}$. Moreover, we can obtain $\sum_{sw}_{f \in \text{supp}(P_l)} P_l(f) \cdot p^f$ by modifying $\bar{P}_l$'s mixture timing of constant acts. Thus, *Indifference to Mixture Timing of Constant Acts* implies that $P_l \sim \bar{P}_l \sim f^{P_l}$. Since f^{P_l} converges to f^P according to (1), by *Continuity*, $P \sim f^P$. The same arguments apply to Q. Therefore, $P \succsim Q$.

∎

PROOF OF THEOREM 2: We first prove the sufficiency of the axioms. *Preference for State-wise Randomization* implies that for any $\alpha \in (0,1)$ and $p, q, r \in \Delta(\mathcal{X})$, $p \succ q$ implies $\alpha p +_{sw} (1-\alpha)r \succ \alpha q +_{sw} (1-\alpha)r$. Since $\mathcal{X}$ is Polish, $\Delta(\mathcal{X})$ is Polish, and hence $\mathcal{F}$ is closed in $\Delta(\mathcal{F})$ and the set of constant acts is closed in $\Delta(\mathcal{F})$ (see Chapter 15 of **?**). Therefore, by *Continuity*, for any $p \in \Delta(\mathcal{X})$, $\{q \in \Delta(\mathcal{X}) : q \succsim p\}$ and $\{q \in \Delta(\mathcal{X}) : p \succsim q\}$ are closed. According to expected utility theory, there exists a continuous expected utility function $u : \Delta(\mathcal{X}) \to \mathbb{R}$ such that $p \succsim q$ if and only if $u(p) \geq u(q)$, in which the mixture operation is "$+_{sw}$."

Similarly, when restricting $\succsim$ to $\Delta(\Delta(\mathcal{X}))$, $\succsim$ also satisfies *Independence*.

Lemma 3 *Every lottery in $\Delta(\Delta(\mathcal{X}))$ preserves Independence.*

Proof. We show that for any $p, q \in \Delta(\mathcal{X})$ and $\alpha \in (0,1)$, $\alpha p + (1-\alpha)q$ also preserves *Independence*. Take any $P, Q \in \Delta(\mathcal{F})$. Because p preserves *Independence*, for any $\beta \in (0,1)$, $P \succsim Q$ if and only if $\beta P + (1-\beta)p \succsim \beta Q + (1-\beta)p$. Because q preserves *Independence*, for any $\gamma \in (0,1)$, $\beta P + (1-\beta)p \succsim \beta Q + (1-\beta)p$ if and only if

$$\gamma(\beta P + (1-\beta)p) + (1-\gamma)q \succsim \gamma(\beta Q + (1-\beta)p) + (1-\gamma)q. \tag{9}$$

In particular, let $\gamma = \frac{\alpha}{1-\beta+\alpha\beta}$, which implies that

$$\frac{\gamma(1-\beta)}{1-\gamma} = \frac{\alpha}{1-\alpha}.$$

Clearly, $\gamma \in (\alpha, 1)$ and

$$\beta\gamma = \frac{\alpha\beta}{1-\beta+\alpha\beta}$$

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can take any value in $(0, 1)$ since α and β are arbitrary numbers in $(0, 1)$. Then, (9) becomes

$$\beta\gamma P + (1 - \beta\gamma)(\alpha p + (1 - \alpha)q) \succsim \beta\gamma Q + (1 - \beta\gamma)(\alpha p + (1 - \alpha)q)$$

for any $\beta\gamma \in (0, 1)$. Therefore, $\alpha p + (1 - \alpha)q$ preserves *Independence*.

A standard induction argument implies that all lotteries in $\Delta(\Delta(\mathcal{X}))$ (randomization over constant acts $\Delta(\mathcal{X})$) with finite support perserve *Independence*. Since $\mathcal{X}$ is Polish, $\Delta(\mathcal{X})$ is Polish, and therefore lotteries with finite support are dense in $\Delta(\Delta(\mathcal{X}))$ (Theorem 15.10 of ?). By *Continuity*, all lotteries in $\Delta(\Delta(\mathcal{X}))$ preserve *Independence*. ∎

Since $\Delta(\Delta(\mathcal{X}))$ is a closed subset of $\Delta(\mathcal{F})$ (Corollary 15.6 of ?), $\succsim$ on $\Delta(\Delta(\mathcal{X}))$ satisfies the standard continuity axiom. Thus, there exists a continuous expected utility function $u^{ea} : \Delta(\Delta(\mathcal{X})) \to \mathbb{R}$ such that for any $P, Q \in \Delta(\Delta(\mathcal{X}))$, $P \succsim Q$ if and only if $u^{ea}(P) \geq u^{ea}(Q)$, in which the mixture operation is "+."

Because $\mathcal{X}$ is compact, u and u^{ea} are both continuous expected utility functions, and they represent the same preference on $\Delta(\mathcal{X})$, we must be able to find a best x_h and a worst x_l such that $u(\delta_{x_h}) = u^{ea}(\delta_{x_h}) = 1$ and $u(\delta_{x_l}) = u^{ea}(\delta_{x_l}) = 0$, and $u(\Delta(\mathcal{X})) = u^{ea}(\Delta(\mathcal{X})) = [0, 1]$.[25] For any $p \in \Delta(\mathcal{X})$, we can find a unique $\alpha^p \in [0, 1]$ such that $p \sim \alpha^p \delta_{x_h} + (1 - \alpha^p)\delta_{x_l}$ (see Lemma 4 for more details). By *Indifference to Mixture Timing of Constant Acts*, $\alpha^p \delta_{x_h} + (1 - \alpha^p)\delta_{x_l} \sim \alpha^p \delta_{x_h} +_{sw} (1 - \alpha^p)\delta_{x_l}$. Then, we have $u(p) = u^{ea}(p)$ any $p \in \Delta(\mathcal{X})$, which in turn implies that for any $P \in \Delta(\Delta(\mathcal{X}))$, $u(f^P) = u^{ea}(P)$. Therefore, from here on, with an abuse of notation, we use u to denote u^{ea}; that is, u is a continuous expected utility representation of $\succsim$ restricted to $\Delta(\Delta(\mathcal{X}))$ under the mixture operation "+," and also of $\succsim$ restricted to $\Delta(\mathcal{X})$ under the mixture operation "$+_{sw}$." For each act f, let $\mathbf{u}(f)$ denote the n-tuple $(u(f_1), \ldots, u(f_n)) \in [0, 1]^n$.

Each lottery $P \in \Delta(\mathcal{F})$ induces a Borel probability measure m^P on $[0, 1]^n$. Take any Borel set $\Lambda \subset [0, 1]^n$. Recall that $\mathcal{F}$ is a closed subset of $\Delta(\mathcal{F})$. Since u is continuous, $\{f \in \mathcal{F} : \mathbf{u}(f) \in \Lambda\}$ is also Borel in $\Delta(\mathcal{F})$. Then, we define $m^P(\Lambda) := P(\{f \in \mathcal{F} : \mathbf{u}(f) \in \Lambda\})$

---

[25]Note that u is linear on $\Delta(\mathcal{X})$, and u^{ea} is linear on $\Delta(\Delta(\mathcal{X}))$.

for each Borel set $\Lambda \subset [0,1]^n$. Verifying that m^P is a probability measure is standard. This definition implies that for any $P, Q \in \Delta(\mathcal{F})$, $\alpha \in (0,1)$, and Borel set $\Lambda \subset [0,1]^n$,

$$
\begin{aligned}
m^{\alpha P + (1-\alpha)Q}(\Lambda) &= [\alpha P + (1-\alpha)Q](\{f \in \mathcal{F} : \mathbf{u}(f) \in \Lambda\}) \qquad (10) \\
&= \alpha P(\{f \in \mathcal{F} : \mathbf{u}(f) \in \Lambda\}) + (1-\alpha)Q(\{f \in \mathcal{F} : \mathbf{u}(f) \in \Lambda\}) \\
&= \alpha m^P(\Lambda) + (1-\alpha)m^Q(\Lambda);
\end{aligned}
$$

that is, $m^{\alpha P + (1-\alpha)Q}$ and $\alpha m^P + (1-\alpha)m^Q$ are the same measure.

Clearly, $\{m^P : P \in \Delta(\mathcal{F})\}$ is a subset of $\Delta([0,1]^n)$, the set of all Borel probability measures on $[0,1]^n$. The converse also holds. For each $(\alpha_1, \ldots, \alpha_n) \in [0,1]^n$, we can find an act $f^{(\alpha_1, \ldots, \alpha_n)}$ such that $f_i^{(\alpha_1, \ldots, \alpha_n)} = \alpha_i \delta_{x_h} + (1-\alpha_i)\delta_{x_l}$.[26] By definition, $\mathbf{u}(f^{(\alpha_1, \ldots, \alpha_n)}) = (\alpha_1, \ldots, \alpha_n)$. Note that $F^{hl} := \{f^{(\alpha_1, \ldots, \alpha_n)} : (\alpha_1, \ldots, \alpha_n) \in [0,1]^n\}$ is closed in $\mathcal{F}$ and thus in $\Delta(\mathcal{F})$. Take an arbitrary $m \in \Delta([0,1]^n)$. For any Borel measurable set $F \subset \mathcal{F}$, define $\Lambda^F := \{(\alpha_1, \ldots, \alpha_n) \in [0,1]^n : (\alpha_1, \ldots, \alpha_n) = \mathbf{u}(f)$ for some $f \in F \cap F^{hl}\}$. Since $F \cap F^{hl}$ is Borel and u is continuous, Λ^F is Borel measurable. Let $P^m(F) = m(\Lambda^F)$. Therefore, $\{m^P : P \in \Delta(\mathcal{F})\} = \Delta([0,1]^n)$.

For simplicity, for any $x \in \mathcal{X}$, we write x instead of δ_x. Note that δ_x may be treated as a degenerate prize, a constant act, or a degenerate lottery, depending on the context. Next, we use u to construct a representation of $\succsim$.

Lemma 4 *There exists a continuous function $I : \Delta([0,1]^n) \to \mathbb{R}_+$ such that for any $P, Q \in \Delta(\mathcal{F})$, (a) $P \succsim Q$ if and only if $I(m^P) \geq I(m^Q)$; (b) $I(m^P) = u(P)$ if $P \in \Delta(\Delta(\mathcal{X}))$; and (c) $I(\Delta([0,1]^n)) = [0,1]$.*

Proof. Take any $P \in \Delta(\mathcal{F})$. By *Continuity*, $\{\alpha \in [0,1] : \alpha x_h + (1-\alpha)x_l \succsim P\}$ and $\{\alpha \in [0,1] : P \succsim \alpha x_h + (1-\alpha)x_l\}$ are closed in $[0,1]$. Neither is empty, because *FOSD* implies that $x_h \succsim P \succsim x_l$ and hence 1 belongs to the former set and 0 belongs to the latter.

---

[26] The mixture used here is the standard mixture operation from expected utility theory.

Because the union of those two sets is a connected set $[0,1]$, we know that the two sets cannot have empty intersection. Therefore, for any P, we can find some α^P such that

$$\alpha^P x_h + (1 - \alpha^P) x_l \sim P.$$

Since $\succsim$ is nontrivial, $x_h \succ x_l$ and for any $\alpha > \alpha'$, $\alpha x_h + (1 - \alpha) x_l \succ \alpha' x_h + (1 - \alpha') x_l$. Therefore, α^P is unique for each $P \in \Delta(\mathcal{F})$. Define a function $\tilde{I} : \Delta(\mathcal{F}) \to [0, 1]$ such that

$$\tilde{I}(P) = \alpha^P.$$

For constant acts x_h and x_l, $\tilde{I}(x_h) = 1 = u(x_h)$ and $\tilde{I}(x_l) = 0 = u(x_l)$. It is standard to verify that $\tilde{I}$ represents $\succsim$.

Note that if $P \in \Delta(\Delta(\mathcal{X}))$, $P \sim \alpha^P x_h + (1 - \alpha^P) x_l$ implies that $u(P) = u(\alpha^P x_h + (1 - \alpha^P) x_l)$. Therefore, $\tilde{I}(P) = u(P)$ for any $P \in \Delta(\Delta(\mathcal{X}))$.

For each $m \in \Delta([0,1]^n)$, we can find some $P \in \Delta(\mathcal{F})$ such that $m = m^P$. Define $I(m) = \tilde{I}(P)$ for each $m \in \Delta([0,1]^n)$. Then, we obtain a function $I : \Delta([0,1]^n) \to [0,1]$. Note that I is well defined: For any $P, Q \in \Delta(\mathcal{F})$ such that $m^P = m^Q$, we must have $P \sim Q$ and hence $\tilde{I}(P) = \tilde{I}(Q)$. This is because when $m^P = m^Q$, according to our definition, P first-order stochastically dominates Q and Q first-order stochastically dominates P. Therefore, by *FOSD*, $P \sim Q$.

By *Continuity*, I is continuous. To see this, take any sequence of Borel probability measures on $[0,1]^n$, $\{m_l\}_{l=1}^\infty$, that converges to $m \in \Delta([0,1]^n)$. The sequence of lotteries $\{P^{m_l}\}_{l=1}^\infty$ must converge to P^m. Based on how we define $\tilde{I}$, $\tilde{I}(P^{m_l})$ must converge to $\tilde{I}(P^m)$.

Last, since $I(\Delta([0,1]^n))$ is connected, $I(\Delta([0,1]^n)) = [0,1]$. ∎

The function I satisfies two additional useful properties.

Lemma 5 *For any $P, Q \in \Delta(\mathcal{F})$ and $\alpha \in [0,1]$, $I(\alpha m^P + (1 - \alpha) m^Q) \geq \alpha I(m^P) + (1 - \alpha) I(m^Q)$.*

Proof. We know from Lemma 4 that $\alpha^P x_h + (1-\alpha^P)x_l \sim P$ and $\alpha^Q x_h + (1-\alpha^Q)x_l \sim Q$. By Lemma 3, we know that both $\alpha^P x_h + (1-\alpha^P)x_l$ and $\alpha^Q x_h + (1-\alpha^Q)x_l$ preserve *Independence*. Therefore,

$$\alpha P + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]$$
$$\sim \quad \alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]$$
$$\sim \quad \alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)Q.$$

By part (i) of *Preference for Randomization*,

$$\frac{1}{2}(\alpha P + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]) + \frac{1}{2}(\alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)Q)$$
$$\succsim \quad \alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)Q$$
$$\sim \quad \alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l].$$

Thus,

$$\frac{1}{2}(\alpha P + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]) + \frac{1}{2}(\alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)Q)$$
$$= \quad \frac{1}{2}(\alpha P + (1-\alpha)Q) + \frac{1}{2}(\alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l])$$
$$\succsim \quad \alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l],$$

which implies that $\alpha P + (1-\alpha)Q \succsim \alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]$ since $\alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]$ preserves *Independence*. Therefore,

according to parts (a) and (b) of Lemma 4,

$$
\begin{aligned}
I(\alpha m^P + (1-\alpha)m^Q) &= I(m^{\alpha P + (1-\alpha)Q}) \\
&\geq I(m^{\alpha[\alpha^P x_h + (1-\alpha^P)x_l] + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]}) \\
&= \alpha I\left(m^{\alpha^P x_h + (1-\alpha^P)x_l}\right) + (1-\alpha)I\left(m^{\alpha^Q x_h + (1-\alpha^Q)x_l}\right) \\
&= \alpha I(m^P) + (1-\alpha)I(m^Q).
\end{aligned}
$$

∎

Lemma 6 *For any $P \in \Delta(\Delta(\mathcal{X}))$, $Q \in \Delta(\mathcal{F})$, and $\alpha \in [0,1]$, $I(\alpha m^P + (1-\alpha)m^Q) = \alpha u(P) + (1-\alpha)I(m^Q)$.*

Proof. According to Lemma 4, $Q \sim \alpha^Q x_h + (1-\alpha^Q)x_l$. Since P preserves *Independence* (Lemma 3), $\alpha P + (1-\alpha)Q \sim \alpha P + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]$. Then,

$$
\begin{aligned}
I(\alpha m^P + (1-\alpha)m^Q) &= I(m^{\alpha P + (1-\alpha)Q}) \\
&= I(m^{\alpha P + (1-\alpha)[\alpha^Q x_h + (1-\alpha^Q)x_l]}) \\
&= I\left(\alpha m^P + (1-\alpha)m^{\alpha^Q x_h + (1-\alpha^Q)x_l}\right) \\
&= \alpha u(P) + (1-\alpha)I(m^{\alpha^Q x_h + (1-\alpha^Q)x_l}) \\
&= \alpha u(P) + (1-\alpha)I(m^Q).
\end{aligned}
$$

The fourth equality follows from the fact that both $\alpha^Q x_h + (1-\alpha^Q)x_l$ and P are in $\Delta(\Delta(\mathcal{X}))$, and part (b) of Lemma 4. ∎

Next, define $\succsim^*$ on $\Delta([0,1]^n)$ as follows: For any $P, Q \in \Delta(\mathcal{F})$,

$$
m^P \succsim^* m^Q \text{ if } \alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R \tag{11}
$$

for any $R \in \Delta(\mathcal{F})$ and $\alpha \in (0,1]$. Note that if there is another lottery P' such that $m^{P'} = m^P$, by *FOSD*, $\alpha P + (1-\alpha)R \sim \alpha P' + (1-\alpha)R$. Hence, $\succsim^*$ is well defined. We have the following

observations.

Lemma 7 *The following statements are true: (a) $\succsim^*$ is reflexive and transitive, and for any $m \in \Delta([0,1]^n)$, $\{m' \in \Delta([0,1]^n) : m' \succsim^* m\}$ and $\{m' \in \Delta([0,1]^n) : m \succsim^* m'\}$ are closed, (b) for any $P, Q, R \in \Delta(\mathcal{F})$ and $\alpha \in (0,1)$, $m^P \succsim^* m^Q$ implies $\alpha m^P + (1-\alpha)m^R \succsim^* \alpha m^Q + (1-\alpha)m^R$, and (c) for any $P \in \Delta(\mathcal{F})$ and $Q \in \Delta(\Delta(\mathcal{X}))$, $P \succsim Q$ implies $m^P \succsim^* m^Q$.*

Proof. For part (a), reflexivity and transitivity of $\succsim^*$ are immediate. Since $\Delta([0,1]^n)$ is Polish, to verify the continuity property of $\succsim^*$, it suffices to show that for any two sequences of lotteries $\{P_l\}$ and $\{Q_l\}$ such that P_l converges to P, Q_l converges to Q, and $m^{P_l} \succsim^* m^{Q_l}$ for any $l \in \mathbb{N}$, we have $m^P \succsim^* m^Q$. Fixing any $R \in \Delta(\mathcal{F})$ and $\alpha \in (0,1]$, since $\alpha P_l + (1-\alpha)R \succsim \alpha Q_l + (1-\alpha)R$, by *Continuity* of $\succsim$, we have $\alpha P + (1-\alpha)R \succsim \alpha Q + (1-\alpha)R$.

For part (b), take any $R' \in \Delta(\mathcal{F})$ and $\beta \in (0,1]$. We want to show that if $m^P \succsim^* m^Q$, we have

$$\beta(\alpha P + (1-\alpha)R) + (1-\beta)R' \succsim \beta(\alpha Q + (1-\alpha)R) + (1-\beta)R'.$$

To see this, we only need to notice that $m^P \succsim^* m^Q$ and

$$\beta(\alpha P + (1-\alpha)R) + (1-\beta)R' = \alpha\beta P + (1-\alpha\beta)\left(\frac{\beta(1-\alpha)}{1-\alpha\beta}R + \frac{1-\beta}{1-\alpha\beta}R'\right)$$

and

$$\beta(\alpha Q + (1-\alpha)R) + (1-\beta)R' = \alpha\beta Q + (1-\alpha\beta)\left(\frac{\beta(1-\alpha)}{1-\alpha\beta}R + \frac{1-\beta}{1-\alpha\beta}R'\right).$$

To show (c), take any $R \in \Delta(\mathcal{F})$. For any $\alpha \in (0,1]$, Lemmas 5 and 6 imply that

$$
\begin{aligned}
I(m^{\alpha P + (1-\alpha)R}) &= I(\alpha m^P + (1-\alpha)m^R) \\
&\geq \alpha I(m^P) + (1-\alpha)I(m^R) \\
&\geq \alpha I(m^Q) + (1-\alpha)I(m^R) \\
&= I(m^{\alpha Q + (1-\alpha)R}).
\end{aligned}
$$

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∎

Consider the vector space of continuous functions from $[0,1]^n$ to $\mathbb{R}$, $C([0,1]^n)$, endowed with the sup norm. According to the above lemma and **?**, there exist a closed (and convex) subset $\mathbb{V}$ of $C([0,1]^n)$ such that $m \succsim^* m'$ if and only if

$$\int_{[0,1]^n} V \; dm \geq \int_{[0,1]^n} V \; dm' \text{ for every } V \in \mathbb{V}.$$

Without loss of generality, assume that for every $V \in \mathbb{V}$, $V(0,\ldots,0) = 0$ and $V(1,\ldots,1) = 1$.

Next, we show that every V in $\mathbb{V}$ satisfies a few properties. For any $(\alpha_1,\ldots,\alpha_n) \in [0,1]^n$, let $\delta_{(\alpha_1,\ldots,\alpha_n)}$ denote the Dirac measure on $(\alpha_1,\ldots,\alpha_n)$.

Lemma 8 *Every V in $\mathbb{V}$ is weakly increasing, concave, and satisfies $V(\lambda\alpha_1 + (1-\lambda)\beta,\ldots,\lambda\alpha_n + (1-\lambda)\beta) = \lambda V(\alpha_1,\ldots,\alpha_n) + (1-\lambda)\beta$ for any $(\alpha_1,\ldots,\alpha_n) \in [0,1]^n$ and $\beta,\lambda \in [0,1]$.*

Proof. For any $(\alpha_1,\ldots,\alpha_n),(\beta_1,\ldots,\beta_n) \in [0,1]^n$, we can find some f and g such that $\mathbf{u}(f) = (\alpha_1,\ldots,\alpha_n)$ and $\mathbf{u}(g) = (\beta_1,\ldots,\beta_n)$. *SOSD* implies that for any $R \in \Delta(\mathcal{F})$, $\lambda \in [0,1]$, and $\gamma \in (0,1]$,

$$\gamma(\lambda f +_{sw} (1-\lambda)g) + (1-\gamma)R \succsim \gamma(\lambda f + (1-\lambda)g) + (1-\gamma)R.$$

This means that $\lambda f +_{sw} (1-\lambda)g \succsim^* \lambda f + (1-\lambda)g$, and hence $\delta_{(\lambda\alpha_1 + (1-\lambda)\beta_1,\ldots,\lambda\alpha_n + (1-\lambda)\beta_n)} \succsim^* \lambda\delta_{(\alpha_1,\ldots,\alpha_n)} + (1-\lambda)\delta_{(\beta_1,\ldots,\beta_n)}$. Therefore, for any $V \in \mathbb{V}$,

$$V(\lambda\alpha_1 + (1-\lambda)\beta_1,\ldots,\lambda\alpha_n + (1-\lambda)\beta_n) \geq \lambda V(\alpha_1,\ldots,\alpha_n) + (1-\lambda)V(\beta_1,\ldots,\beta_n).$$

Following similar steps, *FOSD* implies that every $V \in \mathbb{V}$ is weakly increasing. Finally, suppose $p \in \Delta(\mathcal{X})$ satisfies $u(p) = \beta$. *Indifference to Mixture Timing of Constant Acts* implies that for any $R \in \Delta(\mathcal{F})$ and $\gamma \in (0,1]$,

$$\gamma(\lambda f +_{sw} (1-\lambda)p) + (1-\gamma)R \sim \gamma(\lambda f + (1-\lambda)p) + (1-\gamma)R;$$

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that is, $\delta_{(\lambda\alpha_1+(1-\lambda)\beta,\ldots,\lambda\alpha_n+(1-\lambda)\beta)} \sim^* \lambda\delta_{(\alpha_1,\ldots,\alpha_n)} + (1-\lambda)\delta_{(\beta,\ldots,\beta)}$. This means that for any $V \in \mathbb{V}$,

$$V(\lambda\alpha_1 + (1-\lambda)\beta, \ldots, \lambda\alpha_n + (1-\lambda)\beta) = \lambda V(\alpha_1, \ldots, \alpha_n) + (1-\lambda)V(\beta, \ldots, \beta).$$

We only need to show that $V(\beta, \ldots, \beta) = \beta$. According to our construction of u, $u(p) = \beta$ implies that $\beta x_h + (1-\beta)x_l \sim p$. By Lemma 3, we have $\beta x_h + (1-\beta)x_l \sim^* p$. Thus, for any $V \in \mathbb{V}$,

$$\beta V(1, \ldots, 1) + (1-\beta)V(0, \ldots, 0) = \beta = V(\beta, \ldots, \beta).$$

∎

The lemma above implies that every $V \in \mathbb{V}$ is positively homogeneous (let $\beta = 0$) and $V([0,1]^n) = [0,1]$. Therefore, $\mathbb{V}$ is bounded. Below, we show that $\mathbb{V}$ is compact.

Lemma 9 $\mathbb{V}$ *is compact.*

Proof. Since $[0,1]^n$ is Polish, $C([0,1]^n)$ is also Polish (see Chapter 3.19 of **?**). Therefore, to show that $\mathbb{V}$ is compact, we only need to show that it is sequentially compact. Since $[0,1]^n$ is compact and $\mathbb{V}$ is bounded, by the Arzelá–Ascoli theorem, if we can show that $\mathbb{V}$ is equicontinuous, we know that for any sequence in $\mathbb{V}$, there is a subsequence that converges in sup norm. Of course, the limit is in $\mathbb{V}$, because $C([0,1]^n)$ is complete, $\mathbb{V}$ is closed, and hence $\mathbb{V}$ is complete.

Suppose $\mathbb{V}$ is not equicontinuous. Then, there exists some $(\alpha_1, \ldots, \alpha_n) \in [0,1]^n$ such that for some $\varepsilon > 0$, for any $\omega \in (0,1)$, we can find some $(\beta_1^\omega, \ldots, \beta_n^\omega)$ and $V_\omega \in \mathbb{V}$ such that

$$|V_\omega(\beta_1^\omega, \ldots, \beta_n^\omega) - V_\omega(\alpha_1, \ldots, \alpha_n)| > \varepsilon, \tag{12}$$

in which $(\beta_1^\omega, \ldots, \beta_n^\omega)$ is within the ω-neighborhood of $(\alpha_1, \ldots, \alpha_n)$. Take a sequence of $\left\{ \left(\beta_1^{\omega_j}, \ldots, \beta_n^{\omega_j} \right) \right\}_j$ such that (12) holds and ω_j converges to zero as j goes to infinity.

Suppose $\alpha_i = 0$ or 1 for some $i \in \{1, \ldots, n\}$. Because we have $V(\lambda\alpha_1 + (1-\lambda)\gamma, \ldots, \lambda\alpha_n + (1-\lambda)\gamma) = \lambda V(\alpha_1, \ldots, \alpha_n) + (1-\lambda)\gamma$ for every $V \in \mathbb{V}$ and $\lambda, \gamma \in (0,1)$, we can multiply $(\alpha_1, \ldots, \alpha_n)$ and each $\left(\beta_1^{\omega_j}, \ldots, \beta_n^{\omega_j}\right)$ by λ and add some $(\gamma, \ldots, \gamma)$ to them so that (i) $\lambda\alpha_i + (1-\lambda)\gamma \in (0,1)$ for every $i \in \{1, \ldots, n\}$ and (ii) we have a new version of (12) with the right-hand side being $\lambda\varepsilon$. Therefore, without loss of generality, assume that $\alpha_i \in (0,1)$ for every $j \in \{1, \ldots, n\}$. Then, we can also assume without loss of generality that for any $j \in \mathbb{N}$, $\beta_i^{\omega_j} > 0$ for any $i \in \{1, \ldots, n\}$.

Since $V(\gamma, \ldots, \gamma) = \gamma$ and V is weakly increasing for any $V \in \mathbb{V}$, we know that $V_{\omega_j}(\alpha_1, \ldots, \alpha_n) > 0$. Since every $V \in \mathbb{V}$ is bounded by 1, (12) implies that for each $j \in \mathbb{N}$,

$$\frac{\left|V_{\omega_j}(\alpha_1, \ldots, \alpha_n) - V_{\omega_j}\left(\beta_1^{\omega_j}, \ldots, \beta_n^{\omega_j}\right)\right|}{V_{\omega_j}(\alpha_1, \ldots, \alpha_n)} = \left|\frac{V_{\omega_j}\left(\beta_1^{\omega_j}, \ldots, \beta_n^{\omega_j}\right)}{V_{\omega_j}(\alpha_1, \ldots, \alpha_n)} - 1\right| > \varepsilon. \tag{13}$$

However, because every $V \in \mathbb{V}$ is positively homogeneous and weakly increasing,

$$\min_{i \in \{1, \ldots, n\}} \frac{\beta_i^{\omega_j}}{\alpha_i} \leq \frac{V\left(\beta_1^{\omega_j}, \ldots, \beta_n^{\omega_j}\right)}{V(\alpha_1, \ldots, \alpha_n)} \leq \max_{i \in \{1, \ldots, n\}} \frac{\beta_i^{\omega_j}}{\alpha_i}.$$

Therefore, as j goes to infinity, $\frac{V\left(\beta_1^{\omega_j}, \ldots, \beta_n^{\omega_j}\right)}{V(\alpha_1, \ldots, \alpha_n)}$ must converge to 1. We have a contradiction. ∎

Note that for any $V \in \mathbb{V}$ and $P \in \Delta(\mathcal{F})$, $\int_{\mathcal{F}} V(\mathbf{u}(f)) \, dP = \int_{[0,1]^n} V \, dm^P$. Next, we show that

$$I(m^P) = \min_{V \in \mathbb{V}} \int_{[0,1]^n} V \, dm^P.$$

If we can show this, according to the proof of Lemma 4,

$$W(P) = \min_{V \in \mathbb{V}} \int_{\mathcal{F}} V(\mathbf{u}(f)) \, dP \tag{14}$$

represents $\succsim$.

First, we show that $I(m^P) \leq \min_{V \in \mathbb{V}} \int_{[0,1]^n} V \, dm^P$. If $\alpha := \min_{V \in \mathbb{V}} \int_{[0,1]^n} V \, dm^P = 1$, the inequality holds according to part (c) of Lemma 4. If $\alpha < 1$, let β be an arbitrary number

such that $\alpha < \beta < 1$, and suppose $u(p) = \beta$ for some $p \in \Delta(\mathcal{X})$. Then, for some $V \in \mathbb{V}$,

$$\int_{[0,1]^n} V \ dm^P < \beta = V(\beta, \dots, \beta);$$

that is, $P \not\succsim^* p$. By part (c) of Lemma 7, this implies that $p \succ P$. Therefore, $I(\delta_{(\beta,\dots,\beta)}) = \beta > I(m^P)$. Let β converge to α. We have $I(m^P) \leq \alpha$.

Second, we show that $I(m^P) \geq \alpha = \min_{V \in \mathbb{V}} \int_{[0,1]^n} V \ dm^P$. Suppose $u(q) = \alpha$ for some $q \in \Delta(\mathcal{X})$. Then, according to the representation of $\succsim^*$, $P \succsim^* q$. According to the definition of $\succsim^*$, $P \succsim q$; that is, $I(m^P) \geq u(q) = \alpha$.

Notice that each $V \in \mathbb{V}$ induces a preference over $\mathcal{F}$. Because V is continuous, weakly increasing, concave, and satisfies $V(\lambda\alpha_1 + (1-\lambda)\beta, \dots, \lambda\alpha_n + (1-\lambda)\beta) = \lambda V(\alpha_1, \dots, \alpha_n) + (1-\lambda)\beta$, we can follow the steps in **?** to show that $V(f) = \min_{\mu \in M} \int_{\mathcal{S}} u(f) \ d\mu$ for some compact subset of $\Delta(\mathcal{S})$.

More specifically, denote the preference induced by V on $\mathcal{F}$ by $\succsim^V$; that is, $f \succsim^V g$ if $V(\mathbf{u}(f)) \geq V(\mathbf{u}(g))$. The only mixture operation considered in this domain is state-wise randomization, $+_{sw}$. We know that $(\mathcal{F}, +_{sw})$ is a mixture space. Clearly, $\succsim^V$ is complete and transitive, and for any act f, the sets $\{g \in \mathcal{F} : g \succsim^V f\}$ and $\{g \in \mathcal{F} : f \succsim^V g\}$ are closed because V is continuous and, as shown previously, $\mathcal{F}$ is a closed subset of $\Delta(\mathcal{F})$. Concavity of V and $V(\lambda\alpha_1 + (1-\lambda)\beta, \dots, \lambda\alpha_n + (1-\lambda)\beta) = \lambda V(\alpha_1, \dots, \alpha_n) + (1-\lambda)\beta$ imply that constant acts preserve *State-wise Independence*, and that for any f, g and $\alpha \in (0,1)$, $f \succsim^V g$ implies $\alpha f +_{sw} (1-\alpha)g \succsim^V g$. Lastly, *FOSD* implies that if for any $i \in \{1, \dots, n\}$, $f_i \succsim^V g_i$, then $f \succsim^V g$.

Then, we can apply the main theorem of **?** to show that there exists a compact (and convex) set M of probability measures over $\mathcal{S}$ such that

$$V(f) = \min_{\mu \in M} \int_{\mathcal{S}} u(f) \ d\mu.$$

Compactness follows from the fact that $\mathcal{S}$ is finite, and $\left\{ \min_{\mu \in M} \int_{\mathcal{S}} u(f)\,d\mu : M \in \mathcal{M} \right\}$ is compact by Lemma 9.

XXX M is compact XXX

Proving the necessity of the axioms is standard. We only verify *FOSD* and *SOSD* below. Suppose

$$W(P) = \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f)\,d\mu \right)\,dP$$

represents $\succsim$ such that $\mathbb{V} = \left\{ \min_{\mu \in M} \int_{\mathcal{S}} u(f)\,d\mu : M \in \mathcal{M} \right\}$ is compact.

We verify *FOSD* for a more general representation described in (14), in which $\mathbb{V}$ is compact and every $V \in \mathbb{V}$ is continuous and weakly increasing. This representation nests the DMEU representation as a special case. Thus, if we can show that every $\succsim$ that has such a representation satisfies *FOSD*, then whenever $\succsim$ has a DMEU representation, *FOSD* must also hold.

Suppose P first-order stochastically dominates Q. For each $V \in \mathbb{V}$, $\int V(\mathbf{u}(f))\,dP \geq \int V(\mathbf{u}(f))\,dQ$ because V is weakly increasing (see **?**). Therefore,

$$\min_{V \in \mathbb{V}} \int V(\mathbf{u}(f))\,dP = \int V^P(\mathbf{u}(f))\,dP \geq \int V^P(\mathbf{u}(f))\,dQ \geq \min_{V \in \mathbb{V}} \int V(\mathbf{u}(f))\,dQ,$$

which means that *FOSD* holds.

Next, let Q be a simple mean-preserving spread of P; that is, $P = \beta[\alpha f +_{sw} (1-\alpha)g] + (1-\beta)R$ and $Q = \beta[\alpha f + (1-\alpha)g] + (1-\beta)R$ for some $\beta \in [0,1]$ and $R \in \Delta(\mathcal{F})$. We need to verify that $W(P) \geq W(Q)$. If this is true, the continuity of W and the fact that finite-support lotteries are dense in $\Delta(\mathcal{F})$ will imply that *SOSD* holds (see **?** for a similar proof). We have

$$
\begin{aligned}
W(P) &= \min_{M \in \mathcal{M}} \left\{ \beta \left(\min_{\mu \in M} \int_{\mathcal{S}} u((\alpha f +_{sw} (1-\alpha)g))\,d\mu \right) + (1-\beta) \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(h)\,d\mu \right)\,dR \right\} \\
&= \min_{M \in \mathcal{M}} \left\{ \beta \left(\min_{\mu \in M} \int_{\mathcal{S}} [\alpha u(f) + (1-\alpha)u(g)]\,d\mu \right) + (1-\beta) \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(h)\,d\mu \right)\,dR \right\},
\end{aligned}
$$

and

$$W(Q) = \min_{M \in \mathcal{M}} \left\{ \begin{array}{c} \alpha\beta \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \ d\mu \right) + (1-\alpha)\beta \left(\min_{\mu \in M} \int_{\mathcal{S}} u(g) \ d\mu \right) \\ +(1-\beta) \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(h) \ d\mu \right) \ dR \end{array} \right\}.$$

Notice that for each $M \in \mathcal{M}$,

$$\min_{\mu \in M} \int_{\mathcal{S}} [\alpha u(f) + (1-\alpha)u(g)] \ d\mu \geq \alpha \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \ d\mu \right) + (1-\alpha) \left(\min_{\mu \in M} \int_{\mathcal{S}} u(g) \ d\mu \right).$$

Therefore, $W(P) \geq W(Q)$.

∎

PROOF OF THEOREM 3: The proof of Theorem 2 shows that we can construct a compact set $\widehat{\mathbb{V}} \subset C([0,1]^n)$ such that (i) for any $m, m' \in \Delta([0,1]^n)$, $m \succsim^* m'$ if and only if

$$\int_{[0,1]^n} V \ dm \geq \int_{[0,1]^n} V \ dm' \ \text{for every } V \in \widehat{\mathbb{V}};$$

(ii) $W(P) = \min_{V \in \widehat{\mathbb{V}}} \int_{\mathcal{F}} V(\mathbf{u}(f)) \ dP$ represents $\succsim$; and (iii) for each $V \in \widehat{\mathbb{V}}$, there exists a compact set $M_V \subset \Delta(\mathcal{S})$ such that $V(f) = \min_{\mu \in M_V} \int_{\mathcal{S}} u(f) \ d\mu$.

Note that $\succsim^*$ defined in (11) is different from the expected utility core we define in Section 3.1, because $\succsim^*$ is a binary relation on $\Delta([0,1]^n)$ and the expected utility core is a binary relation on $\Delta(\mathcal{F})$. With an abuse of notation, let $\succsim^*$ also denote the expected utility core of $\succsim$. According to the definition of the expected utility core, we immediately have $P \succsim^* Q \iff m^P \succsim^* m^Q$ for any $P, Q \in \Delta(\mathcal{F})$. Since for any $V \in \widehat{\mathbb{V}}$ and $P \in \Delta(\mathcal{F})$, $\int_{\mathcal{F}} V(\mathbf{u}(f)) \ dP = \int_{[0,1]^n} V \ dm^P$, we know that for any $P, Q \in \Delta(\mathcal{F})$, $P \succsim^* Q$ if and only if for every $V \in \widehat{\mathbb{V}}$,

$$\int_{\mathcal{F}} V(\mathbf{u}(f)) \ dP \geq \int_{\mathcal{F}} V(\mathbf{u}(f)) \ dQ$$
$$\int_{\mathcal{F}} \left(\min_{\mu \in M_V} \int_{\mathcal{S}} u(f) \ d\mu \right) \ dP \geq \int_{\mathcal{F}} \left(\min_{\mu \in M_V} \int_{\mathcal{S}} u(f) \ d\mu \right) \ dQ.$$

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Let $\widehat{\mathcal{M}} = \left\{ M_V : V \in \widehat{\mathbb{V}} \right\}$. Then, the DMEU representation $\left(\widehat{\mathcal{M}}, u \right)$ of $\succsim$ satisfies the second part of the theorem.

Suppose there is another DMEU representation $(\mathcal{M}, v)$ of $\succsim$. Since u and v are both expected utility representations of $\succsim$ on $\Delta(\mathcal{X})$, there exist $\alpha > 0$ and $\beta \in R$ such that $v = \alpha u + \beta$. Because

$$
\begin{aligned}
W_{\mathcal{M},v}(P) &= \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} v(f)\, d\mu \right)\, dP \\
&= \alpha \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f)\, d\mu \right)\, dP + \beta = \alpha W_{\mathcal{M},u}(P) + \beta,
\end{aligned}
$$

$(\mathcal{M}, v)$ is a DMEU representation of $\succsim$ if and only if $(\mathcal{M}, u)$ is a DMEU representation of $\succsim$. Thus, without loss of generality, let $v = u$. Let

$$
\mathbb{V} = \left\{ V \in C([0,1]^n) : \text{for some } M \in \mathcal{M}, \ V(\mathbf{u}(f)) = \min_{\mu \in M} \int_{\mathcal{S}} u(f)\, d\mu \ \text{for any } f \in \mathcal{F} \right\}.
$$

We need to show that the closed convex hull of $\widehat{\mathcal{M}}$, $\overline{\mathrm{co}}\left(\widehat{\mathcal{M}} \right) \subset \overline{\mathrm{co}}\left(\mathcal{M} \right)$.

First, suppose $(\mathcal{M}, u)$ also satisfies the second part of the theorem. Then, for any $P, Q \in \Delta(\mathcal{F})$, the following three statements are equivalent: (i) $P \succsim^* Q$; (ii) for every $V \in \widehat{\mathbb{V}}$, $\int_{\mathcal{F}} V(\mathbf{u}(f))\, dP \geq \int_{\mathcal{F}} V(\mathbf{u}(f))\, dQ$; and (iii) for every $V \in \mathbb{V}$, $\int_{\mathcal{F}} V(\mathbf{u}(f))\, dP \geq \int_{\mathcal{F}} V(\mathbf{u}(f))\, dQ$. According to the uniqueness result from **?**, we know that

$$
\mathrm{cl}\left(\mathrm{cone}\left(\widehat{\mathbb{V}} \right) + \{ \gamma \mathbf{1}_{[0,1]^n} \}_{\gamma \in \mathbb{R}} \right) = \mathrm{cl}\left(\mathrm{cone}\left(\mathbb{V} \right) + \{ \gamma \mathbf{1}_{[0,1]^n} \}_{\gamma \in \mathbb{R}} \right). \tag{15}
$$

XXX

We want to show that $\overline{\mathrm{co}}\left(\widehat{\mathcal{M}} \right) = \overline{\mathrm{co}}\left(\mathcal{M} \right)$. If this is not true, without loss of generality, suppose there exists some $M \in \overline{\mathrm{co}}\left(\widehat{\mathcal{M}} \right) \setminus \overline{\mathrm{co}}\left(\mathcal{M} \right)$. Let V_M be the function that satisfies $V_M(\mathbf{u}(f)) = \min_{\mu \in M} \int_{\mathcal{S}} u(f)\, d\mu$ for any $f \in \mathcal{F}$. Due to (15), $V_M \in \mathrm{cl}\left(\mathrm{cone}\left(\mathbb{V} \right) + \{ \gamma \mathbf{1}_{[0,1]^n} \}_{\gamma \in \mathbb{R}} \right)$. This shows that $\widehat{\mathcal{M}}$ is unique up to the closed convex hull.

Next, suppose $(\mathcal{M}, u)$ may not satisfy the second part of the theorem.

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PROOF OF THEOREM 4: We only prove the sufficiency of the axioms. Part 1 of Lemma 1 shows that for any lottery P, $P \sim f^P$. Therefore, for any DMEU representation of $\succsim$,

$$W(P) = \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP,$$

it must be true that $W(P) = W(f^P)$. For an act f, we know that

$$W(f) = \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu,$$

in which $\mathbb{M} = \bigcup_{M \in \mathcal{M}} M$. Therefore,

$$
\begin{aligned}
W(P) &= W(f^P) \\
&= \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} \left(\int_{\mathcal{F}} u(f) \, dP \right) \, d\mu \\
&= \min_{\mu \in \mathbb{M}} \int_{\mathcal{F}} \left(\int_{\mathcal{S}} u(f) \, d\mu \right) \, dP.
\end{aligned}
$$

Since the set of probability measures is a closed and bounded subset of $\mathbb{R}^n$, $\mathbb{M}$ must be compact.

■

PROOF OF THEOREM 5: We only show that if $\succsim$ has a DMEU representation and satisfies *Independence*, then it has an Ex Post MEU representation. Since *Weak Order*, *Continuity*, and *Independence* hold, there exists a continuous expected utility function $V' : \mathcal{F} \to \mathbb{R}$ such that

$$W^{ep}(P) = \int_{\mathcal{F}} V'(f) \, dP$$

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represents $\succsim$.

Define $u(p) = W^{ep}(p)$ for any constant act p. Recall that x_l is one of the worst lotteries and x_h is one of the best lotteries. Since *FOSD* holds, whenever two acts f and g satisfy $W^{ep}(f_i) \geq W^{ep}(g_i)$ for any i, it must be true that $W^{ep}(f) \geq W^{ep}(g)$. Therefore, there exists a function $V^* : [u(x_l), u(x_h)]^n \to \mathbb{R}$ such that

$$V^*(u(f_1), \ldots, u(f_n)) = V'(f).$$

Notice that for any constant act p,

$$V^*(u(p), \ldots, u(p)) = V'(p) = W^{ep}(p) = u(p).$$

Futhermore, due to *Indifference to Mixture Timing of Constant Acts*, for any $p \in \Delta(\mathcal{X})$,

$$u(p) = W^{ep}(p) = W^{ep}\left(\int_{\mathcal{X}} \delta_x \, dp\right) = \int_{\mathcal{X}} V'(\delta_x) \, dp = \int_{\mathcal{X}} u(\delta_x) \, dp.$$

Since W is continuous, u is a continuous expected utility representation of $\succsim$ on $\Delta(\mathcal{X})$.

We know that $\succsim$ has a DMEU representation $(\mathcal{M}, v)$. Since both u and v are continuous expected utility representations of $\succsim$ on $\Delta(\mathcal{X})$, v must be equal to some positive affine transformation of u. Therefore, without loss of generality, we can assume that $\succsim$'s DMEU representation is $(\mathcal{M}, u)$; that is,

$$W(P) = \min_{M \in \mathcal{M}} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu\right) \, dP$$

represents $\succsim$. Note that for any P, $\alpha^P x_h + (1 - \alpha^P)x_l \sim P$, as in Lemma 4. We have $W(P) = \alpha^P u(x_h) + (1 - \alpha^P)u(x_l) = W^{ep}(P)$. In particular, for any act f,

$$W^{ep}(f) = V^*(f) = W(f) = \min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu,$$

in which $\mathbb{M} = \bigcup_{M \in \mathcal{M}} M$; that is,

$$W^{ep}(P) = \int \left(\min_{\mu \in \mathbb{M}} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP.$$

Since the set of probability measures is a closed and bounded subset of $\mathbb{R}^n$, $\mathbb{M}$ must be compact.

∎

COMPARATIVE STATICS: Finally, we provide some comparative static results for the DMEU representation. Suppose there are two individuals whose preferences over lotteries are $\succsim_1$ and $\succsim_2$, respectively. If $\succsim_i$ has a DMEU representation $(\mathcal{M}_i, u_i)$, we use $\mathbb{M}_i$ to denote $\bigcup_{M \in \mathcal{M}_i} M$, $i = 1, 2$. The proposition below provides a sufficient condition for when $\succsim_2$ always thinks that randomization is less effective in eliminating the effect of ambiguity than $\succsim_1$.

Proposition 1 *Suppose $\succsim_1$ and $\succsim_2$ have DMEU representations $(\mathcal{M}_1, u)$ and $(\mathcal{M}_2, u)$, respectively, and $\mathbb{M}_1 = \mathbb{M}_2$. If for any $M_1 \in \mathcal{M}_1$, there exists an $M_2 \in \mathcal{M}_2$ such that $M_1 \subset M_2$, then $P \succsim_2 f$ implies $P \succsim_1 f$ for any lottery P and act f.*

PROOF OF PROPOSITION 1: For each lottery P, let $M(P)$ denote an element of $\mathcal{M}_1$ that minimizes $\int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) \, dP$. For each $M_1 \in \mathcal{M}_1$, let $M_2(M_1)$ denote an element of $\mathcal{M}_2$ that contains M_1. Since the two individuals share the same u and $\mathbb{M}_1 = \mathbb{M}_2$, they must

also share the same evaluation of every act. Then, the proposition follows from

$$
\begin{aligned}
W_1(P) &= \min_{M \in \mathcal{M}_1} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) dP \\
&= \int_{\mathcal{F}} \left(\min_{\mu \in M(P)} \int_{\mathcal{S}} u(f) \, d\mu \right) dP \\
&\geq \int_{\mathcal{F}} \left(\min_{\mu \in M_2(M(P))} \int_{\mathcal{S}} u(f) \, d\mu \right) dP \\
&\geq \min_{M \in \mathcal{M}_2} \int_{\mathcal{F}} \left(\min_{\mu \in M} \int_{\mathcal{S}} u(f) \, d\mu \right) dP \\
&= W_2(P).
\end{aligned}
$$

■

The converse of the proposition above holds when we focus on DMEU representations with minimal sets of scenarios.

Proposition 2 *TBW*

PROOF OF PROPOSITION 2: TBW

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Figure 5: In the good scenario M_G, the firm will profit (state s_p) with probability 1. In the bad scenario M_B, the firm will profit with probability 0. In the neutral scenario M_N, the probability that the firm profits is either 0.3 or 0.7, which will be determined after randomization.

Figure 6: An example of a lottery P, in which f and g are two acts, $\{s_1, s_2\}$ is the state space, and f_i (g_i) is the prize associated with state s_i by f (g).

Figure 7: Suppose there are two states: s_1 and s_2. The left-hand lottery is the randomization with probability α between acts (degenerate lotteries) f and g. The right-hand act is the state-wise randomization with probability α between acts f and g.