Mathematical Model of Input for Peak-Detector

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May 20, 2016

1 Introduction

For the "freq_timing_est" block, frequency and timing information are supposed to be estimated. As discussed in the "freq_timing_tradeoff", we can use brutal search to get the correct value. But, before we make our decision by choosing the maximum peak, we still need to have a rigorous mathematic model for the peak.

This article will discuss this problem step to step.

2 Mathematical Model

Because the signal we received are discrete values, so we only consider about discrete signals, rather than continuous ones.

Notation	Description
s_k	received sample at time k
t_k	training sample at time k
d_k	data sample at time k
n_k	noise at time k
P_k^t	the k^{th} chip of spread code for one frame of training
P_k^d	the k^{th} chip of spread code for one frame of data
D_l^t	the l^{th} symbol for training
D_l^d	the l^{th} symbol for data
N_f	number of chips per frame
N_c	number of chips per symbol
N_s	number of symbols per frame
$N_{s'}$	number of symbols per shortened training sequence
σ^2	variance of Gaussian noise
k_0	number of samples of delay
f_0	frequency offset
T_c	duration between two chips

Table 1: Notation Table

First, the structure of training and data frame are like: The structure of training is like Fig. 1 and Fig. 2

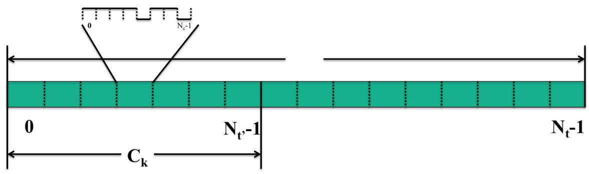

Figure 1: Training Frame Structure

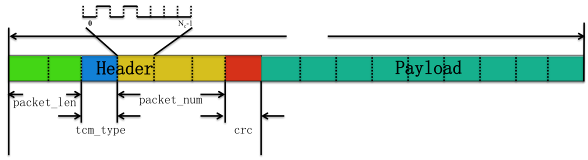

Figure 2: Data Frame Structure

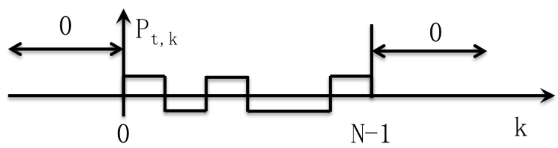

Figure 3: Sketch Map of P_k^t

2.1 Description of Variables

We first express each variables by mathematical languages.

According to the description in table 1, and also in order to simplify the following notation, the definition of P_y^x is defined as

$$P_k^x = \begin{cases} P_k^x & k = 0, \cdots, N_c - 1 \\ 0 & else \end{cases} \tag{1}$$
$$x = \text{``}t\text{''} or \text{``}d\text{''}$$

For instance, the figure for P_k^t is like Fig. 3

So, training chip at time k can be described as:

$$t_k = \sum_{k=0}^{N_c} D_0^t P_k^t \tag{2}$$
$$= D_0^t P_k^t \tag{3}$$

If we consider the frame-level periodicity, t_k would be

$$t_k = D^t_{\lfloor k/N_c \rfloor \% N_s} P^t_{k \% N_c} \tag{4}$$

Based on nearly the same idea, d_k can be written as:

$$d_k = D^d_{\lfloor k/N_c \rfloor} P^d_{k \% N_c} \tag{5}$$

Noise is simple. We simply choose IID Gaussian noise. So

$$n_k \sim \mathcal{N}(0, \sigma^2) \tag{6}$$

2.2 Description of Output of Matched Filter

Consider the time delay and frequency offset, the received signal would be

$$r_k = (t_{k-k_0} + d_{k-k_0})e^{j2\pi f_0 k T_c} + n_k \tag{7}$$

We first deal with the contribution of noise. Define the Pulse Response of Matched Filter to be h_{N_f-k}.

$$h_k = \begin{cases} P_k^t = D_{\lfloor k/N_c \rfloor}^t P_{k\%N_c}^t & k = 0, \cdots, N_f \\ 0 & else \end{cases} \tag{8}$$

Let $g_{k_0}^n = \sum_{k'=0}^{N_f-1} n_{k'} \cdot h_{k'}$

$$\begin{aligned}
\mathbb{E}\{g_{k_0}^n\} &= \mathbb{E}\{\sum_{k'=0}^{N_f-1} n_{k'} \cdot h_{k'}\} && (9)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{n_{k'} \cdot h_{k'}\} && (10)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{n_{k'}\}h_{k'} && (11)\\
&= \sum_{k'=0}^{N_f-1} 0 \cdot h_{k'} && (12)\\
&= 0 && (13)\\
\mathbb{E}\{(g_{k_0}^n)^2\} &= \mathbb{E}\{\sum_{k'=0}^{N_f-1} (n_{k'} \cdot h_{k'}\} && (14)\\
&= \mathbb{E}\{\sum_{k'=0}^{N_f-1} n_{k'}^2 \cdot \underbrace{(h_{k'})^2}_{\equiv 1}\} && (15)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{n_{k'}^2\} && (16)\\
&= N_f\sigma^2 && (17)
\end{aligned}$$

So $g_{k_0}^n \sim \mathcal{N}(0, N_f\sigma^2)$

The next step is to analyze the output of Matched Filter. Define the patten for Matched Filter for training frame to be P_{-k}^{t*}. The output of the Matched

Filter at sampled point $k = N_f$ is:

$$g_{k_0} = r_k * h_{N_f-k}\Big|_{k=N_f-1} \tag{18}$$

$$= (t_{k-k_0} + d_{k-k_0} + n_k) * h_{N_f-1-k}\Big|_{k=N_f-1} \tag{19}$$

$$= \underbrace{\sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'}}_{g^t_{k_0}} + \underbrace{\sum_{k'=0}^{N_f-1} d_{k'-k_0} \cdot h_{k'}}_{g^d_{k_0}} + \underbrace{\sum_{k'=0}^{N_f-1} n_{k'} \cdot h_{k'}}_{g^n_{k_0}} \tag{20}$$

$$= g^t_{k_0} + g^d_{k_0} + g^n_{k_0} \tag{21}$$

Now, we will discuss all the possible situations for timing. Let's formate the time offset k_0 in terms of N_s and N_c.

$$k_0 = \alpha \cdot N_c + \beta \quad \alpha = \mathbb{Z} \; and \; \beta = 0, \cdots, N_c - 1 \tag{22}$$

2.2.1 Perfectly Aligned

Perfectly Aligned means the sample at time k is also the first sample of patten for Matched Filter, as shown in fig. 4. For this situation, k_0 satisfies

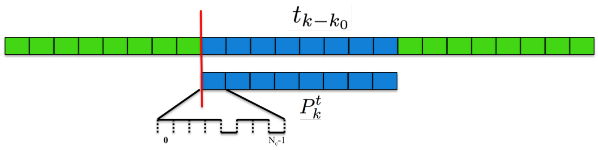

Figure 4: Perfectly Aligned

$$\begin{cases} \alpha = z \cdot N_s \; z \in \mathbb{Z} \\ \beta = 0 \end{cases} \Rightarrow k_0 = z \cdot N_s \cdot N_c \tag{23}$$

Then we can analyze $g^t_{k_0}$ and $g^d_{k_0}$ separately

$$g^t_{k_0} = \sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'} \tag{24}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-k_0)/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot h_{k'} \tag{25}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-z \cdot N_s \cdot N_c)/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot h_{k'} \tag{26}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot h_{k'} \tag{27}$$

$$\tag{28}$$

$$\because h_{k'} = D^t_{\lfloor k'/N_c \rfloor} P^t_{k'\% N_c} \tag{29}$$

$$\therefore g^t_{k_0} = \sum_{k'=0}^{N_f-1} D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k'\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k\% N_c} \tag{30}$$

$$= \sum_{k'=0}^{N_f-1} \underbrace{(D^t_{\lfloor k'/N_c \rfloor \% N_s})^2}_{\equiv 1} \underbrace{(P^t_{k'\% N_c})^2}_{\equiv 1} \tag{31}$$

$$= N_f \tag{32}$$

To analyzie g^s_k, we will first define a function to describe the convolution

$$C^n_s(f_1, f_2, k_0) = \sum_{k=s}^{s+n} f_1(k) \cdot f_2(k - k_0) \tag{33}$$

The diagram of the above formula is like Fig.5

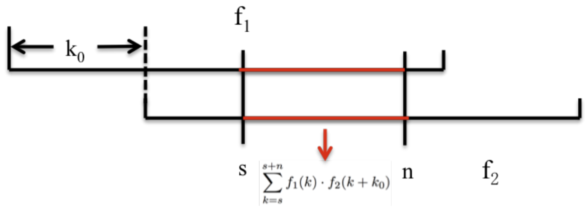

Figure 5: New Defined Convolution

$$g^d_{k_0} = \sum_{k'=0}^{N_f-1} d_{k'-k_0} \cdot h_{k'} \tag{34}$$

$$= \sum_{k'=0}^{N_f-1} D^d_{\lfloor (k'-k_0)/N_c \rfloor} P^d_{(k'-k_0)\% N_c} \cdot h_{k'} \tag{35}$$

$$= \sum_{k'=0}^{N_f-1} D^d_{\lfloor (k'-z \cdot N_s \cdot N_c)/N_c \rfloor} P^d_{(k'-z \cdot N_s \cdot N_c)\% N_c} \cdot h_{k'} \tag{36}$$

$$= \sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c \rfloor - z \cdot N_s} P^d_{k'\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k'\% N_c} \tag{37}$$

$$\tag{38}$$

For data symbols, we could only assume that

$$D^d_k = \begin{cases} +1 & p = 0.5 \\ -1 & p = 0.5 \end{cases} \tag{39}$$

Because the D^d_k here is a random variable, we could only exam the mean and variance for it, so do the $g^d_{k_0}$

$$\mathbb{E}\{D^d_k\} = 1 \times 0.5 + (-1) \times 0.5 = 0 \tag{40}$$

$$\mathbb{E}\{D^d_k)^2\} = 1 \times 0.5 + (-1)^2 \times 0.5 = 1 \tag{41}$$

and they are IID.

For $g^d_{k_0}$,

$$\mathbb{E}\{g^d_{k_0}\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor} P^t_{k'\%N_c}\} \tag{42}$$

$$= \sum_{k'=0}^{N_f-1} \mathbb{E}\{D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor} P^t_{k'\%N_c}\} \tag{43}$$

$$= \sum_{k'=0}^{N_f-1} P^d_{k'\%N_c} P^t_{k'\%N_c} \mathbb{E}\{D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} D^t_{\lfloor k'/N_c\rfloor}\} \tag{44}$$

$$= C_0^{N_c-1}(P^d_k, P^t_k, 0) \sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_{k''} \mathbb{E}\{D^d_{k''-z}\} \tag{45}$$

$$= 0 \tag{46}$$

$$\mathbb{E}\{(g^d_{k_0})^2\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor\%N_s} P^t_{k\%N_c}\}^2 \tag{47}$$

$$= \mathbb{E}\{\sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_{k''} D^d_{k''-z} \cdot \sum_{m=0}^{N_c} P^d_{m\%N_c} P^t_{m\%N_c}\}^2 \tag{48}$$

$$= \Big(\sum_{m=0}^{N_c} P^d_m P^t_m\Big)^2 \mathbb{E}\{\sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_{k''} D^d_{k''-z}\}^2 \tag{49}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \mathbb{E}\{\sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_{k''} D^d_{k''}\}^2 \tag{50}$$

$$\begin{aligned} &= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \Bigg(\mathbb{E}\{\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} (D^t_m D^d_m)^2\} \\ &+ \mathbb{E}\{\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} \sum_{n\neq m} D^t_m D^d_m D^t_n D^d_n\}\Bigg) \end{aligned} \tag{51}$$

$$\begin{aligned} &= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \Bigg(\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} \mathbb{E}\{D^d_m\}^2 \\ &+ \sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} \sum_{n\neq m} D^t_m D^t_n \mathbb{E}\{D^d_m\}\mathbb{E}\{D^d_n\}\Bigg) \end{aligned} \tag{52}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \Bigg(\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} \cdot 1\Bigg) \tag{53}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \cdot N_s \tag{54}$$

2.2.2 Chip-level Aligned

Chip-level Aligned means the start chip of received symbol is aligned with patten's symbol's first chip. But, the begining symbol of a frame for received data may not be aligned with patten's first symbol. Here is the diagram

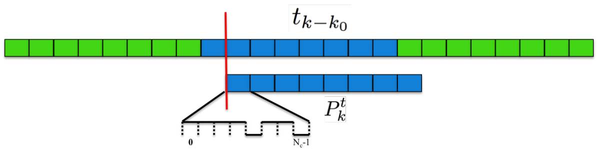

Figure 6: Chip-level Aligned

At this time

$$\begin{cases} \alpha = z & z \in \mathbb{Z} \\ \beta = 0 \end{cases} \Rightarrow k_0 = z \cdot N_c \tag{55}$$

For $g_{k_0}^t$:

$$\begin{aligned}
g_{k_0}^t &= \sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'} && (56)\\
&= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-k_0)/N_c \rfloor \% N_s} P^t_{k\% N_c} \cdot h_{k'} && (57)\\
&= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-z\cdot N_c)/N_c \rfloor \% N_s} P^t_{k\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k\% N_c} && (58)\\
&= \sum_{k'=0}^{N_f-1} D^t_{(\lfloor k'/N_c \rfloor - z)\% N_s} D^t_{\lfloor k'/N_c \rfloor} \underbrace{P^t_{k\% N_c} P^t_{k\% N_c}}_{\equiv 1} && (59)\\
&= \sum_{k'=0}^{N_f-1} D^t_{(\lfloor k'/N_c \rfloor - z)\% N_s} D^t_{\lfloor k'/N_c \rfloor} && (60)\\
&= C_z^{N_s-1}(D_k^t, D_k^t, z) + C_{N_s-z}^{N_s-1}(D_k^t, D_k^t, N_s - z) && (61)
\end{aligned}$$

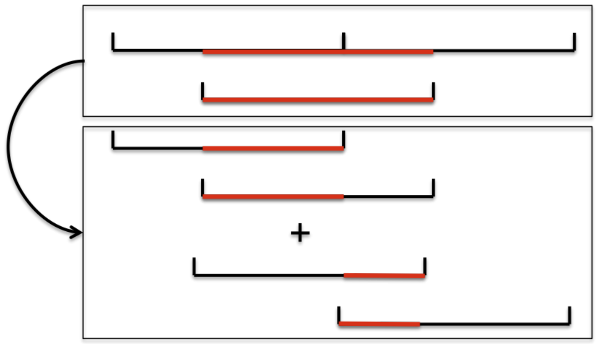

Figure 7: Diagram for Misaligned Convolution

For $g^d_{k_0}$:

$$\mathbb{E}\{g^d_{k_0}\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor \%N_s} P^t_{k\%N_c}\} \tag{62}$$

$$= \sum_{k'=0}^{N_f-1} \mathbb{E}\{D^d_{\lfloor k'/N_c\rfloor - z} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor \%N_s} P^t_{k\%N_c}\} \tag{63}$$

$$= \sum_{m=0}^{N_c} (P^d_m P^t_m) \sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_m \mathbb{E}\{D^d_{m-z}\} \tag{64}$$

$$= C_0^{N_c}(P^d_k, P^t_k, 0) \cdot \sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_m \cdot 0 \tag{65}$$

$$= 0 \tag{66}$$

$$\mathbb{E}\{(g^d_{k_0})^2\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor \%N_s} P^t_{k\%N_c}\}^2 \tag{67}$$

$$= (C_0^{N_c}(P^d_k, P^t_k, 0))^2 \mathbb{E}\{\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_{m-z} D^d_m\}^2 \tag{68}$$

$$= (C_0^{N_c}(P^d_k, P^t_k, 0))^2 \mathbb{E}\{\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_m D^d_m\}^2 \tag{69}$$

$$= (C_0^{N_c - 1}(P^d_k, P^t_k, 0))^2 \cdot N_s \tag{70}$$

The mean and variance of $g^d_{k_0}$ is the same as the situation of perfectly aligned.

2.2.3 Totally Misaligned

Totally Misaligned means the first chip of received data is not correspond to the first chip of patten. Like the Fig. 8

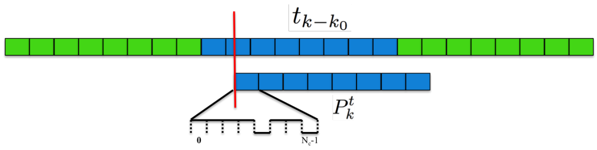

Figure 8: Totally Misaligned

$$k_0 = z,\ \ z \in \mathbb{Z} \tag{71}$$

For $g_{k_0}^t$:

$$g_{k_0}^t = \sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'} \tag{72}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-z)/N_c \rfloor \% N_s} P^t_{(k'-z)\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k\% N_c} \tag{73}$$

We will first deal with a smaller problem. If we only focus on patten of length N_c Define the result of the above situation to be $f^t(s_1, s_2, t_1, z)$

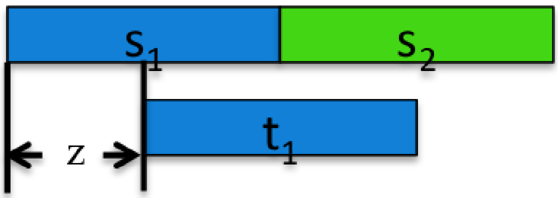

Figure 9: Chip-level Misaligned

$$f^t(s_1, s_2, t_1, z) = (s_1 \cdot t_1) C_z^{N_c-1}(P_k^t, P_k^t, z) + (s_2 \cdot t_1) C_{N_c-z}^{N_c-1}(P_k^t, P_k^t, N_c - z) \tag{74}$$

Now, back to $g_{k_0}^t$

$$g_{k_0}^t = \sum_{m=\lfloor(-k_0)/N_c\rfloor, n=0}^{\lfloor m=(-k_0)/N_c \rfloor + N_s, n=N_s} f^t_{m\% N_s, m+1\% N_s, n, z\% N_c} \tag{75}$$

For $g_{k_0}^d$

$$\mathbb{E}\{g_{k_0}^d\} = \sum_{k'=0}^{N_f-1} D^d_{\lfloor (k'-z)/N_c \rfloor \% N_s} P^d_{(k'-z)\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k\% N_c} \tag{76}$$

We can also define

$$f_1^d(s_1, s_2, t_1, z) = (s_1 \cdot t_1) C_z^{N_c-1}(P_k^d, P_k^t, z) + (s_2 \cdot t_1) C_{N_c-z}^{N_c-1}(P_k^d, P_k^t, N_c - z) \tag{77}$$

Then $g^d_{k_0}$ would be

$$g^d_{k_0} = \sum_{m=\lfloor(-k_0)/N_c\rfloor,n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s,n=N_s} f^d(m, m+1, n, z\%N_c) \tag{78}$$

But for the expectation, we may need to change a little bit. Because we want to split the patten symbol rather than the received symbol.

$$g^d_{k_0} = \sum_{m=\lfloor(-k_0)/N_c\rfloor,n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s,n=N_s} f^d(n, n+1, m, z\%N_c) \tag{79}$$

Then

$$\begin{aligned}
\mathbb{E}\{g^d_{k_0}\} &= \mathbb{E}\{\sum_{m=\lfloor(-k_0)/N_c\rfloor,n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s,n=N_s} f^d(n, n+1, m, z\%N_c)\} && (80)\\
&= \sum_{m=\lfloor(-k_0)/N_c\rfloor,n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s,n=N_s} \mathbb{E}\{f^d(n, n+1, m, z\%N_c)\} && (81)\\
&= 0 && (82)
\end{aligned}$$

$$\mathbb{E}\{(g^d_{k_0})^2\} = \left(C^{N_c-1}_{z\%N_c}(P^d_k, P^t_k, z\%N_c) + C^{z-1}_0(P^d_k, P^t_k, -z\%N_c)\right)^2 \cdot N_s \tag{83}$$

Because we can view it as a shifted training patten.