Mathematical Model of Input for Peak-Detector

Author

August 31, 2016

Contents

1 Introduction **2**

2 Frequency and TIming Trade-off in Estimation **2**

2.1 Mathematical Model 2

2.1.1 $\Delta f = 0$ and $t_0 = 0$ 3

2.1.2 $\Delta f = 0$, $t_0 =?$ 3

2.1.3 $t_0 = 0$, but $\Delta f =?$ 3

3 Input Signal of Peak Detector **4**

3.1 Mathematical Model 4

3.1.1 Description of Variables 4

3.1.2 Description of Output of Matched Filter 5

3.1.2.1 Perfectly Aligned 6

3.1.2.2 Chip-level Aligned 9

3.1.2.3 Totally Misaligned 11

3.1.3 Ideal Situation 12

1 Introduction

CDMA system is famous for its large user capacity and superior ability of keeping security. In practical CDMA system, the diagram of the whole system is like

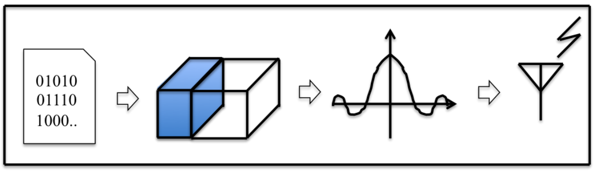

Figure 1: TX model

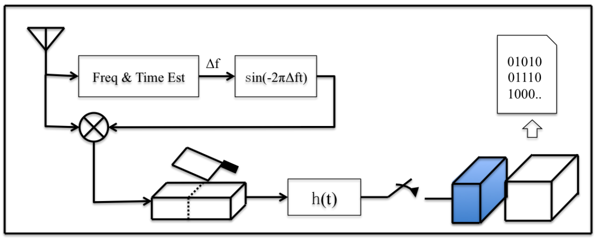

Figure 2: RX model

From the diagrams, we can see that the complexity of RX model is much more complicated then TX model. The reason is that, the channel will add several uncertainty to the signal, while the transmitter doesn't have to deal with that. So, the main block in the RX model is the "freq_&time_est", which will give out the frequency and timing information of the received signal. This article is mainly focus on the performance of this block.

2 Frequency and TIming Trade-off in Estimation

Trade-off is common in the world. There is one famous theory called uncertainty principle[1]. Also, there is also a trade off for estimating frequency and timing.

In this scenario, we are trying to estimate the frequency offset of the received signal, and estimate the start point of the frame. We will prove that, the estimation cannot be perfect.

2.1 Mathematical Model

The signal we use is $s(t)$:

$$s(t) = \begin{cases} \alpha & 0 \leq t \leq T_s \\ 0 & else \end{cases} \tag{1}$$

This is a very simple rectangle wave. For the receiver, we only consider about delay t_0 and frequency offset Δf, the received signal $r(t)$ would be

$$r(t) = s(t - t_0)e^{j2\pi\Delta f t} \tag{2}$$

But for the receiver, it doesn't know anything about the receiver, so the only tag receiver could choose for the matched filter is $h(t)$

$$h(t) = s(T_s - t) \tag{3}$$

Then, the output of the matched filter is $R(t)$, the result of convolution between $r(t)$ and $h(t)$.

$$\begin{align} R(t) = r(t) * h(t) &= \int_{-\infty}^{+\infty} r(\tau)h(t-\tau)d\tau \tag{4}\\ &= \int_{t_0}^{t} s(\tau - t_0)e^{j2\pi\Delta f\tau}s(\tau - (t - T_s))d\tau \tag{5}\\ &= \int_{t_0}^{t} \alpha^2 e^{j2\pi\Delta f\tau}d\tau \tag{6}\\ &= \alpha^2 \frac{e^{j2\pi\Delta f t}}{j2\pi\Delta f}\bigg|_{t_0}^{t} \tag{7}\\ &= \alpha^2 \frac{e^{j2\pi\Delta f t} - e^{j2\pi\Delta f t_0}}{j2\pi\Delta f} \tag{8} \end{align}$$

Now, we will discuss three specific situations:

2.1.1 $\Delta f = 0$ and $t_0 = 0$

We will find that, the numerator and denominator are all 0, so we will turn to L'Hospital.

$$\begin{align} &\lim_{\Delta f \to 0} \alpha^2 \frac{e^{j2\pi\Delta f t} - e^{j2\pi\Delta f t_0}}{j2\pi\Delta f} \tag{9}\\ = &\lim_{\Delta f \to 0} \alpha^2 \frac{j2\pi t e^{j2\pi\Delta f t} - j2\pi t_0 e^{j2\pi\Delta f t_0}}{j2\pi} \tag{10}\\ = &\alpha^2(t - t_0) \tag{11} \end{align}$$

When $t_0 = 0$, $R(t) = \alpha^2 t$. And we know that, because of the domain of t, we know that, when $t = T_s$, $R(t)$ will get its maximum value $R(Ts) = \alpha^2 T_s$.

2.1.2 $\Delta f = 0$, $t_0 =?$

Similar to the procedure in part 2.1.1, also sample at $t = T_s$

$$R(t) = \alpha^2(T_s - t_0) \Rightarrow \hat{t_0} = T_s - \frac{R(t)}{\alpha^2} \tag{12}$$

2.1.3 $t_0 = 0$, but $\Delta f =?$

Sample at $t = T_s$

$$R(t) = \alpha^2 \frac{e^{j2\pi\Delta f T_s} - 1}{j2\pi\Delta f} = \alpha^2 T_s e^{j\pi\Delta f T_s} sa(\Delta 2\pi f Ts) \tag{13}$$

So, we know that, when we get the correct Δf, we will get the largest value.

But we still haven't touch the situation where $\Delta f =?$ and $t_0 =?$. Our method is brute search. We enumerate all the possible pair of parameters $(t_0', \Delta f')$, and try them. With the help of above three theory, we know that, the correct/similar to true value will give out the largest $R(t)$.

3 Input Signal of Peak Detector

For the "freq_timing_est" block, frequency and timing information are supposed to be estimated. As discussed in the "freq_timing_tradeoff", we can use brutal search to get the correct value. But, before we make our decision by choosing the maximum peak, we still need to have a rigorous mathematic model for the peak.

This section will discuss the mathematic model for "True" peak and other points.

3.1 Mathematical Model

Because the signal we received are discrete values, so we only consider about discrete signals, rather than continuous ones.

Notation	Description
s_k	received sample at time k
t_k	training sample at time k
d_k	data sample at time k
n_k	noise at time k
P^t_k	the k^{th} chip of spread code for one frame of training
P^d_k	the k^{th} chip of spread code for one frame of data
D^t_l	the l^{th} symbol for training
D^d_l	the l^{th} symbol for data
N_f	number of chips per frame
N_c	number of chips per symbol
N_s	number of symbols per frame
$N_{s'}$	number of symbols per shortened training sequence
σ^2	variance of Gaussian noise
k_0	number of samples of delay
f_0	frequency offset
T_c	duration between two chips

Table 1: Notation Table

First, the structure of training and data frame are like: The structure of training is like Fig. 3 and Fig. 4

3.1.1 Description of Variables

We first express each variables by mathematical languages.

According to the description in table 1, and also in order to simplify the following notation, the definition of P^x_y is defined as

$$\begin{aligned} P^x_k &= \begin{cases} P^x_k & k = 0, \cdots, N_c - 1 \\ 0 & else \end{cases} \\ x &= “t'' or “d'' \end{aligned} \tag{14}$$

For instance, the figure for P^t_k is like Fig. 5

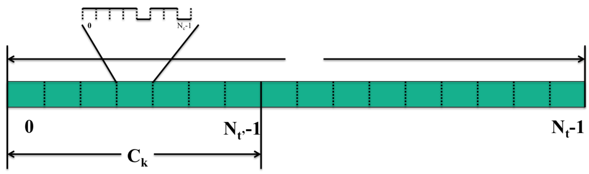

Figure 3: Training Frame Structure

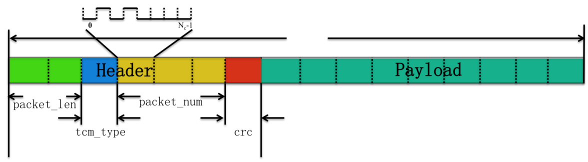

Figure 4: Data Frame Structure

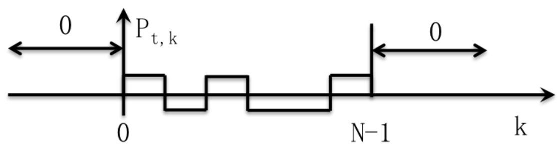

Figure 5: Sketch Map of P_k^t

So, training chip at time k can be described as:

$$t_k = \sum_{k=0}^{N_c} D_0^t P_k^t \tag{15}$$

$$= D_0^t P_k^t \tag{16}$$

If we consider the frame-level periodicity, t_k would be

$$t_k = D^t_{\lfloor k/N_c \rfloor \% N_s} P^t_{k\% N_c} \tag{17}$$

Based on nearly the same idea, d_k can be written as:

$$d_k = D^d_{\lfloor k/N_c \rfloor} P^d_{k\% N_c} \tag{18}$$

Noise is simple. We simply choose IID Gaussian noise. So

$$n_k \sim \mathcal{N}(0, \sigma^2) \tag{19}$$

3.1.2 Description of Output of Matched Filter

Consider the time delay and frequency offset, the received signal would be

$$r_k = (t_{k-k_0} + d_{k-k_0}) e^{j2\pi f_0 k T_c} + n_k \tag{20}$$

We first deal with the contribution of noise. Define the Pulse Response of Matched Filter to be $h_{N_f - k}$.

$$h_k = \begin{cases} P_k^t = D^t_{\lfloor k/N_c \rfloor} P^t_{k\% N_c} & k = 0, \cdots, N_f \\ 0 & else \end{cases} \tag{21}$$

Let $g^n_{k_0} = \sum_{k'=0}^{N_f-1} n_{k'} \cdot h_{k'}$

$$\begin{aligned}
\mathbb{E}\{g^n_{k_0}\} &= \mathbb{E}\{\sum_{k'=0}^{N_f-1} n_{k'} \cdot h_{k'}\} && (22)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{n_{k'} \cdot h_{k'}\} && (23)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{n_{k'}\} h_{k'} && (24)\\
&= \sum_{k'=0}^{N_f-1} 0 \cdot h_{k'} && (25)\\
&= 0 && (26)\\
\mathbb{E}\{(g^n_{k_0})^2\} &= \mathbb{E}\{\sum_{k'=0}^{N_f-1} (n_{k'} \cdot h_{k'}\} && (27)\\
&= \mathbb{E}\{\sum_{k'=0}^{N_f-1} n^2_{k'} \cdot \underbrace{(h_{k'})^2}_{\equiv 1}\} && (28)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{n^2_{k'}\} && (29)\\
&= N_f \sigma^2 && (30)
\end{aligned}$$

So $g^n_{k_0} \sim \mathcal{N}(0, N_f\sigma^2)$

The next step is to analyze the output of Matched Filter. Define the patten for Matched Filter for training frame to be P^{t*}_{-k}. The output of the Matched Filter at sampled point $k = N_f$ is:

$$\begin{aligned}
g_{k_0} &= r_k * h_{N_f-k}\Big|_{k=N_f-1} && (31)\\
&= (t_{k-k_0} + d_{k-k_0} + n_k) * h_{N_f-1-k}\Big|_{k=N_f-1} && (32)\\
&= \underbrace{\sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'}}_{g^t_{k_0}} + \underbrace{\sum_{k'=0}^{N_f-1} d_{k'-k_0} \cdot h_{k'}}_{g^d_{k_0}} + \underbrace{\sum_{k'=0}^{N_f-1} n_{k'} \cdot h_{k'}}_{g^n_{k_0}} && (33)\\
&= g^t_{k_0} + g^d_{k_0} + g^n_{k_0} && (34)
\end{aligned}$$

Now, we will discuss all the possible situations for timing. Let's formate the time offset k_0 in terms of N_s and N_c.

$$k_0 = \alpha \cdot N_c + \beta \quad \alpha = \mathbb{Z} \; and \; \beta = 0, \cdots, N_c - 1 \tag{35}$$

3.1.2.1 Perfectly Aligned Perfectly Aligned means the sample at time k is also the first sample of patten for Matched Filter, as shown in fig. 6. For this

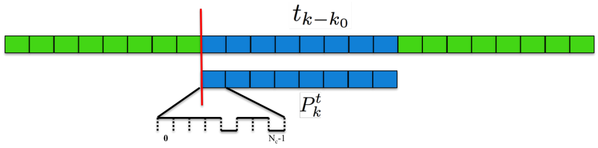

Figure 6: Perfectly Aligned

situation, k_0 satisfies

$$\begin{cases} \alpha = z \cdot N_s \ z \in \mathbb{Z} \\ \beta = 0 \end{cases} \Rightarrow k_0 = z \cdot N_s \cdot N_c \tag{36}$$

Then we can analyze $g_{k_0}^t$ and $g_{k_0}^d$ separately.
For $g_{k_0}^t$:

$$\begin{aligned}
g_{k_0}^t &= \sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'} && (37)\\
&= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-k_0)/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot h_{k'} && (38)\\
&= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-z \cdot N_s \cdot N_c)/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot h_{k'} && (39)\\
&= \sum_{k'=0}^{N_f-1} D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot h_{k'} && (40)\\
& && (41)
\end{aligned}$$

$$\begin{aligned}
\because h_{k'} &= D^t_{\lfloor k'/N_c \rfloor} P^t_{k' \% N_c} && (42)\\
\therefore g_{k_0}^t &= \sum_{k'=0}^{N_f-1} D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k' \% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k \% N_c} && (43)\\
&= \sum_{k'=0}^{N_f-1} \underbrace{(D^t_{\lfloor k'/N_c \rfloor \% N_s})^2}_{\equiv 1} \underbrace{(P^t_{k' \% N_c})^2}_{\equiv 1} && (44)\\
&= N_f && (45)
\end{aligned}$$

To analyzie g_k^s, we will first define a function to describe the convolution

$$C_s^n(f_1, f_2, k_0) = \sum_{k=s}^{s+n} f_1(k) \cdot f_2(k - k_0) \tag{46}$$

The diagram of the above formula is like Fig.7 Then $g_{k_0}^d$:

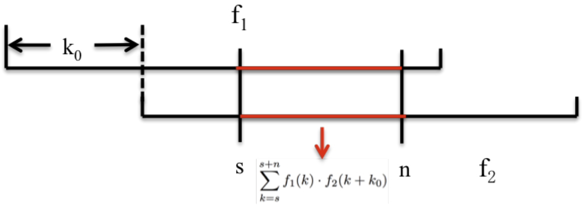

Figure 7: New Defined Convolution

$$
\begin{aligned}
g^d_{k_0} &= \sum_{k'=0}^{N_f-1} d_{k'-k_0}\cdot h_{k'} && (47)\\
&= \sum_{k'=0}^{N_f-1} D^d_{\lfloor (k'-k_0)/N_c\rfloor} P^d_{(k'-k_0)\%N_c}\cdot h_{k'} && (48)\\
&= \sum_{k'=0}^{N_f-1} D^d_{\lfloor (k'-z\cdot N_s\cdot N_c)/N_c\rfloor} P^d_{(k'-z\cdot N_s\cdot N_c)\%N_c}\cdot h_{k'} && (49)\\
&= \sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c}\cdot D^t_{\lfloor k'/N_c\rfloor} P^t_{k'\%N_c} && (50)
\end{aligned}
$$

For data symbols, we could only assume that

$$
D^d_k = \begin{cases} +1 & p = 0.5 \\ -1 & p = 0.5 \end{cases} \quad (51)
$$

Because the D^d_k here is a random variable, we could only exam the mean and variance for it, so do the $g^d_{k_0}$

$$
\begin{aligned}
\mathbb{E}\{D^d_k\} &= 1\times 0.5 + (-1)\times 0.5 = 0 && (52)\\
\mathbb{E}\{D^d_k)^2\} &= 1\times 0.5 + (-1)^2\times 0.5 = 1 && (53)
\end{aligned}
$$

and they are IID.

For $g^d_{k_0}$,

$$
\begin{aligned}
\mathbb{E}\{g^d_{k_0}\} &= \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c}\cdot D^t_{\lfloor k'/N_c\rfloor} P^t_{k'\%N_c}\} && (54)\\
&= \sum_{k'=0}^{N_f-1} \mathbb{E}\{D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c}\cdot D^t_{\lfloor k'/N_c\rfloor} P^t_{k'\%N_c}\} && (55)\\
&= \sum_{k'=0}^{N_f-1} P^d_{k'\%N_c} P^t_{k'\%N_c} \mathbb{E}\{D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} D^t_{\lfloor k'/N_c\rfloor}\} && (56)\\
&= C_0^{N_c-1}(P^d_k, P^t_k, 0) \sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor + N_s - 1} D^t_{k''}\mathbb{E}\{D^d_{k''-z}\} && (57)\\
&= 0 && (58)
\end{aligned}
$$

$$\mathbb{E}\{(g^d_{k_0})^2\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c\rfloor - z\cdot N_s} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c\rfloor\%N_s} P^t_{k\%N_c}\}^2 \tag{59}$$

$$= \mathbb{E}\{\sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} D^t_{k''} D^d_{k''-z} \cdot \sum_{m=0}^{N_c} P^d_{m\%N_c} P^t_{m\%N_c}\}^2 \tag{60}$$

$$= \Big(\sum_{m=0}^{N_c} P^d_m P^t_m\Big)^2 \mathbb{E}\{\sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} D^t_{k''} D^d_{k''-z}\}^2 \tag{61}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \mathbb{E}\{\sum_{k''=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} D^t_{k''} D^d_{k''}\}^2 \tag{62}$$

$$\begin{aligned} &= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \Bigg(\mathbb{E}\{\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} (D^t_m D^d_m)^2\} \\ &+ \mathbb{E}\{\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} \sum_{n\neq m} D^t_m D^d_m D^t_n D^d_n\}\Bigg) \end{aligned} \tag{63}$$

$$\begin{aligned} &= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \Bigg(\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} \mathbb{E}\{D^d_m\}^2 \\ &+ \sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} \sum_{n\neq m} D^t_m D^t_n \mathbb{E}\{D^d_m\}\mathbb{E}\{D^d_n\}\Bigg) \end{aligned} \tag{64}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \Bigg(\sum_{m=\lfloor k'/N_c\rfloor}^{\lfloor k'/N_c\rfloor+N_s-1} \cdot 1\Bigg) \tag{65}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \cdot N_s \tag{66}$$

3.1.2.2 Chip-level Aligned Chip-level Aligned means the start chip of received symbol is aligned with patten's symbol's first chip. But, the begining symbol of a frame for received data may not be aligned with patten's first symbol. Here is the diagram

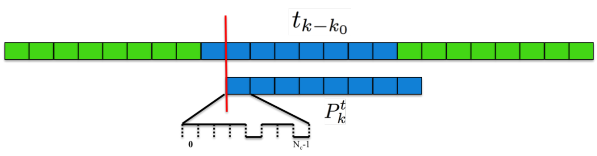

Figure 8: Chip-level Aligned

At this time

$$\begin{cases} \alpha = z & z \in \mathbb{Z} \\ \beta = 0 \end{cases} \Rightarrow k_0 = z \cdot N_c \tag{67}$$

For $g^t_{k_0}$:

$$g^t_{k_0} = \sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'} \tag{68}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-k_0)/N_c \rfloor \% N_s} P^t_{k\% N_c} \cdot h_{k'} \tag{69}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-z\cdot N_c)/N_c \rfloor \% N_s} P^t_{k\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k\% N_c} \tag{70}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{(\lfloor k'/N_c \rfloor - z)\% N_s} D^t_{\lfloor k'/N_c \rfloor} \underbrace{P^t_{k\% N_c} P^t_{k\% N_c}}_{\equiv 1} \tag{71}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{(\lfloor k'/N_c \rfloor - z)\% N_s} D^t_{\lfloor k'/N_c \rfloor} \tag{72}$$

$$= C_z^{N_s-1}(D^t_k, D^t_k, z) + C_{N_s-z}^{N_s-1}(D^t_k, D^t_k, N_s - z) \tag{73}$$

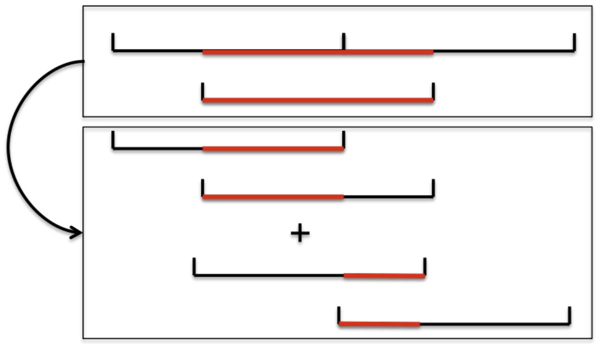

Figure 9: Diagram for Misaligned Convolution

For $g^d_{k_0}$:

$$\mathbb{E}\{g^d_{k_0}\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c \rfloor - z} P^d_{k'\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k\% N_c}\} \tag{74}$$

$$= \sum_{k'=0}^{N_f-1} \mathbb{E}\{D^d_{\lfloor k'/N_c \rfloor - z} P^d_{k'\% N_c} \cdot D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k\% N_c}\} \tag{75}$$

$$= \sum_{m=0}^{N_c} (P^d_m P^t_m) \sum_{m=\lfloor k'/N_c \rfloor}^{\lfloor k'/N_c \rfloor + N_s - 1} D^t_m \mathbb{E}\{D^d_{m-z}\} \tag{76}$$

$$= C_0^{N_c}(P^d_k, P^t_k, 0) \cdot \sum_{m=\lfloor k'/N_c \rfloor}^{\lfloor k'/N_c \rfloor + N_s - 1} D^t_m \cdot 0 \tag{77}$$

$$= 0 \tag{78}$$

$$\mathbb{E}\{(g^d_{k_0})^2\} = \mathbb{E}\{\sum_{k'=0}^{N_f-1} D^d_{\lfloor k'/N_c \rfloor - z} P^d_{k'\%N_c} \cdot D^t_{\lfloor k'/N_c \rfloor \% N_s} P^t_{k\%N_c}\}^2 \tag{79}$$

$$= (C_0^{N_c}(P^d_k, P^t_k, 0))^2 \mathbb{E}\{\sum_{m=\lfloor k'/N_c \rfloor}^{\lfloor k'/N_c \rfloor + N_s - 1} D^t_{m-z} D^d_m\}^2 \tag{80}$$

$$= (C_0^{N_c}(P^d_k, P^t_k, 0))^2 \mathbb{E}\{\sum_{m=\lfloor k'/N_c \rfloor}^{\lfloor k'/N_c \rfloor + N_s - 1} D^t_m D^d_m\}^2 \tag{81}$$

$$= (C_0^{N_c-1}(P^d_k, P^t_k, 0))^2 \cdot N_s \tag{82}$$

The mean and variance of $g^d_{k_0}$ is the same as the situation of perfectly aligned.

3.1.2.3 Totally Misaligned Totally Misaligned means the first chip of received data is not correspond to the first chip of patten. Like the Fig. 10

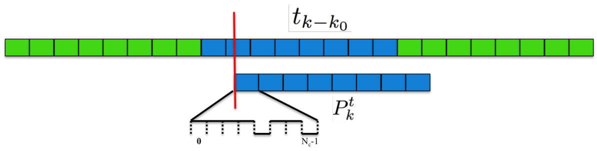

Figure 10: Totally Misaligned

$$k_0 = z, \ z \in \mathbb{Z} \tag{83}$$

For $g^t_{k_0}$:

$$g^t_{k_0} = \sum_{k'=0}^{N_f-1} t_{k'-k_0} \cdot h_{k'} \tag{84}$$

$$= \sum_{k'=0}^{N_f-1} D^t_{\lfloor (k'-z)/N_c \rfloor \% N_s} P^t_{(k'-z)\%N_c} \cdot D^t_{\lfloor k'/N_c \rfloor} P^t_{k\%N_c} \tag{85}$$

We will first deal with a smaller problem. If we only focus on patten of length N_c Define the result of the above situation to be $f^t(s_1, s_2, t_1, z)$

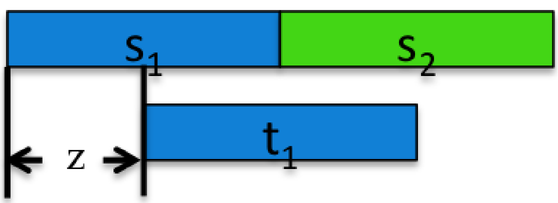

Figure 11: Chip-level Misaligned

$$f^t(s_1, s_2, t_1, z) = (s_1 \cdot t_1)C_z^{N_c-1}(P_k^t, P_k^t, z) + (s_2 \cdot t_1)C_{N_c-z}^{N_c-1}(P_k^t, P_k^t, N_c - z) \tag{86}$$

Now, back to $g_{k_0}^t$

$$g_{k_0}^t = \sum_{m=\lfloor(-k_0)/N_c\rfloor, n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s, n=N_s} f_{m\%N_s, m+1\%N_s, n, z\%N_c}^t \tag{87}$$

For $g_{k_0}^d$

$$\mathbb{E}\{g_{k_0}^d\} = \sum_{k'=0}^{N_f-1} D_{\lfloor(k'-z)/N_c\rfloor\%N_s}^d P_{(k'-z)\%N_c}^d \cdot D_{\lfloor k'/N_c\rfloor}^t P_{k\%N_c}^t \tag{88}$$

We can also define

$$f_1^d(s_1, s_2, t_1, z) = (s_1 \cdot t_1)C_z^{N_c-1}(P_k^d, P_k^t, z) + (s_2 \cdot t_1)C_{N_c-z}^{N_c-1}(P_k^d, P_k^t, N_c - z) \tag{89}$$

Then $g_{k_0}^d$ would be

$$g_{k_0}^d = \sum_{m=\lfloor(-k_0)/N_c\rfloor, n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s, n=N_s} f^d(m, m+1, n, z\%N_c) \tag{90}$$

But for the expectation, we may need to change a little bit. Because we want to split the patten symbol rather than the received symbol.

$$g_{k_0}^d = \sum_{m=\lfloor(-k_0)/N_c\rfloor, n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s, n=N_s} f^d(n, n+1, m, z\%N_c) \tag{91}$$

Then

$$\begin{aligned} \mathbb{E}\{g_{k_0}^d\} &= \mathbb{E}\{\sum_{m=\lfloor(-k_0)/N_c\rfloor, n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s, n=N_s} f^d(n, n+1, m, z\%N_c)\} && (92) \\ &= \sum_{m=\lfloor(-k_0)/N_c\rfloor, n=0}^{\lfloor m=(-k_0)/N_c\rfloor+N_s, n=N_s} \mathbb{E}\{f^d(n, n+1, m, z\%N_c)\} && (93) \\ &= 0 && (94) \end{aligned}$$

$$\mathbb{E}\{(g_{k_0}^d)^2\} = \left(C_{z\%N_c}^{N_c-1}(P_k^d, P_k^t, z\%N_c) + C_0^{z-1}(P_k^d, P_k^t, -z\%N_c)\right)^2 \cdot N_s \tag{95}$$

Because we can view it as a shifted training patten.

3.1.3 Ideal Situation

The formulas above are for practical situations. For idea situations, we would like to let spread codes have properities like below:

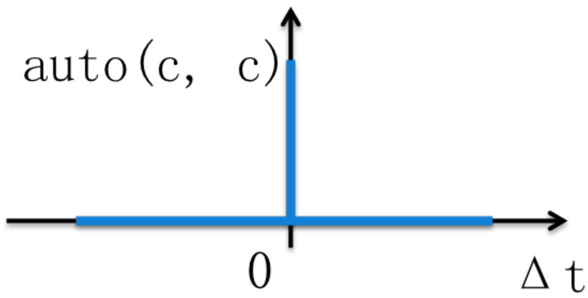

Figure 12: Auto-correlation for certain spread codes

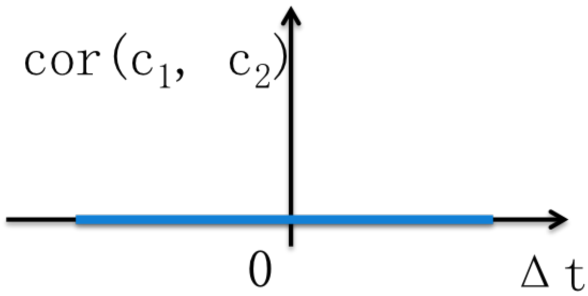

Figure 13: Cross-correlation for different spread codes

1. Different spread codes are orthogonal;
2. Autocorrelation is a δ function.

So the auto-correlation and cross-correlation arelike:
The mathematical description for this property would be:

for different spread codes $C_i(t)$and$C_j(t), i \neq j$means they are different.

$$c_i(t) * c_i(-t + \Delta t) = \begin{cases} N_c & \Delta t = k \cdot N_c, k = \mathbb{Z} \\ 0 & else \end{cases} \tag{96}$$

$$c_i(t) * c_j(-t + \Delta t) = 0, \forall k \in \mathbb{Z} \tag{97}$$

Compared the ideal situation with the analysis did in subsection 3.1.2, because the spread code for training and data are orthogonal, or say different, so

$$g_{k_0}^d \equiv 0 \tag{98}$$

Because the auto-correlation funciton has perfect property, eq 3.1.2.2 and eq 3.1.2.3 should be as close to 0 as possible.

References

[1] Uncertainty principle. Uncertainty principlen, 2016.