Trade-off of Frequency & Timing Estimation

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I. Introduction

Trade-off is common in the world. There is one famous theory called uncertainty principle[1]. Also, there is also a trade off for estimating frequency and timing.

In this scenario, we are trying to estimate the frequency offset of the received signal, and estimate the start point of the frame. We will prove that, the estimation cannot be perfect.

II. Mathematical Model

A. rectangle wave

The signal we use is $s(t)$:

$$s(t) = \begin{cases} \alpha & 0 \le t \le T_s \\ 0 & else \end{cases} \tag{1}$$

This is a very simple rectangle wave. For the receiver, we only consider about delay t_0 and frequency offset Δf, the received signal $r(t)$ would be

$$r(t) = s(t - t_0)e^{j2\pi\Delta f t} \tag{2}$$

But for the receiver, it doesn't know anything about the receiver, so the only tag receiver could choose for the matched filter is $h(t)$

$$h(t) = s(T_s - t) \tag{3}$$

Then, the output of the matched filter is $R(t)$, the result of convolution between $r(t)$ and $h(t)$.

$$R(t) = r(t) * h(t) = \int_{-\infty}^{+\infty} r(\tau)h(t-\tau)d\tau \tag{4}$$

$$= \int_{t_0}^{t} s(\tau - t_0)e^{j2\pi\Delta f\tau} s(\tau - (t - T_s))d\tau \tag{5}$$

$$= \int_{t_0}^{t} \alpha^2 e^{j2\pi\Delta f\tau} d\tau \tag{6}$$

$$= \alpha^2 \frac{e^{j2\pi\Delta f t}}{j2\pi\Delta f}\Bigg|_{t_0}^{t} \tag{7}$$

$$= \alpha^2 \frac{e^{j2\pi\Delta f t} - e^{j2\pi\Delta f t_0}}{j2\pi\Delta f} \tag{8}$$

Now, we will discuss three specific situations:

1) $\Delta f = 0$ and $t_0 = 0$: We will find that, the numerator and denominator are all 0, so we will turn to L'Hospital.

$$\lim_{\Delta f \to 0} \alpha^2 \frac{e^{j2\pi\Delta f t} - e^{j2\pi\Delta f t_0}}{j2\pi\Delta f} \tag{9}$$

$$= \lim_{\Delta f \to 0} \alpha^2 \frac{j2\pi t e^{j2\pi\Delta f t} - j2\pi t_0 e^{j2\pi\Delta f t_0}}{j2\pi} \tag{10}$$

$$= \alpha^2 (t - t_0) \tag{11}$$

When $t_0 = 0$, $R(t) = \alpha^2 t$. And we know that, because of the domain of t, we know that, when $t = T_s$, $R(t)$ will get its maximum value $R(Ts) = \alpha^2 T_s$.

2) $\Delta f = 0$, $t_0 = ?$: Similar to the procedure in part II-A1, also sample at $t = T_s$

$$R(t) = \alpha^2(T_s - t_0) \Rightarrow \hat{t_0} = T_s - \frac{R(t)}{\alpha^2} \tag{12}$$

3) $t_0 = 0$, but $\Delta f = ?$: Sample at $t = T_s$

$$R(t) = \alpha^2 \frac{e^{j2\pi\Delta f T_s} - 1}{j2\pi\Delta f} = \alpha^2 T_s e^{j\pi\Delta f T_s} sa(\Delta 2\pi f Ts) \tag{13}$$

So, we know that, when we get the correct Δf, we will get the largest value.

But we still haven't touch the situation where $\Delta f = ?$ and $t_0 = ?$. Our method is brute search. We enumerate all the possible pair of parameters $(t_0', \Delta f')$, and try them. With the help of above three theory, we know that, the correct/similar to true value will give out the largest $R(t)$.

References

[1] Uncertainty principle. Uncertainty principlen, 2016.