FE5112 CA2

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1 Introduction

This assignment discusses the pricing of an exotic Asian American put option using the Least Squares Monte Carlo (LSMC) method. We will also introduce variance reduction methods and highlight the improvement in variance.

2 LSMC Without Variance Reduction

Given that the market is complete and arbitrage free. The price of a non-dividend paying stock S_t at time t with risk free rate 8% is governed by the following SDE under the risk neutral probability measure Q:

$$dS_t = 0.08 S_t dt + \sigma(S_t) S_t dW_t, S_0 = 100; \tag{1}$$

where W_t is the brownian motion under Q and $\sigma(\cdot)$ is a function as follows:

$$\sigma(x) = \begin{cases} 0.25 + 0.02(1 - x/S_0) & \text{for } x \leq 100 \\ max(0.001, 0.25 - 0.01(x/S_0 - 1) & \text{for } x > 100 \end{cases} \tag{2}$$

The one-year-non-exercise American Asian put option has a strike price $K = 108$ and the maturity $T = 2$ years. The option can only be exercised on the first day of the second year onwards. If the option is exercised on day n, $366 \leq n \leq 730$, the option buyer will receive a payout of $(K - A_n)^+$ such that

$$A_n = \frac{1}{60} \sum_{i=-60}^{-1} S_{\frac{n+i}{365}}. \tag{3}$$

2.1 Algorithm

Least Squares Monte Carlo (LSMC) method without variance reduction was used for pricing the option. For the simplicity of the simulation assignment, we set $T = 2$ with $N = 365 * 2$, $dt = T/N = 1/365$, such that we have daily time intervals. However, the code can take care of different dt values if needed.

1. Generate M sample price paths.

2. Populate matrix A which contains the value of A_n, such that $A(i, j)$ refers to the ith sample path on the jth day.

3. Before we proceed with the backward propagation, we calculate the payoff on the final day of exercise, $n = 730$.

4. During backward propagation, we check if the Asian American put option is In-the-Money (ITM).

   (a) If it is, we store A_n in an array together with its corresponding discount future value at on the $(n + 1)$th day.

   (b) Else we simply discount the payoff backwards and save it.

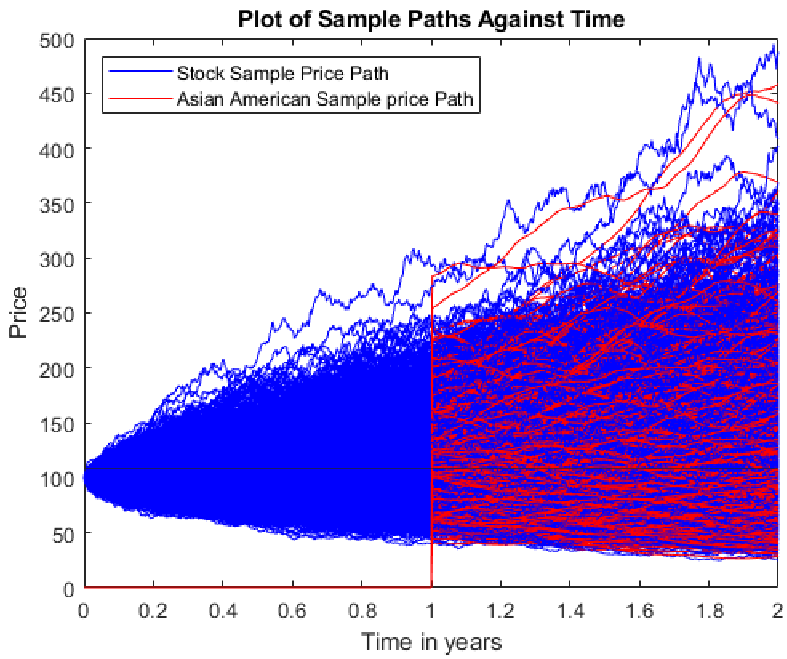
5. We do so for all M sample paths and build a polynomial degree of 2 using the points we have collected. The polynomial is constructed using the in built MATLAB function `polyfit`[1]. `Polyfit` is a function that constructs a polynomial of degree n using the least squares regression method.

6. Lastly, we discount all sample prices to get the fair value of the option.

[1] https://www.mathworks.com/help/matlab/ref/polyfit.html

3 Computational Results

Setting $M = 10^4$, we arrived at a mean value of 10.6785 and a variance of 148.5283 for the price of the Asian American put option. From Figure 1, we can see that the sample price paths have a very large variance, resulting in a large variance for A_t as well. The horizontal black line is set at $K = 108$, and all red paths below the horizontal line is considered to be ITM. For the interest of checking our algorithm, we implemented a European put option version of this exotic option and found the mean price to be 9.8694 with a variance of 192.4845, the code can be found in the zipped file as well.



3.1 Analysis

Out of curiosity, we decided to investigate the mean and variance of the terminal stock price S_T and terminal sixty day mean A_T as well. As we can see in Table 1, the variance is significantly larger than its mean. Both blue and red paths have a very large spread, ranging from about 20 to 450. For the option price, its mean is close to its standard deviation, hence we cannot be very certain about the option price. However, to be fair, the volatility from the SDE is relatively high as well. Thus, we can try to minimise the variance of the sample price paths which will ultimately minimise the variance of the option price from LSMC. Elapsed time for the algorithm excluding plotting of graph and setting number of sample paths $M = 10^4$ is about 20 seconds.

	Mean	Variance
Terminal Stock Price S_T	99.9430	1336.0
Terminal 60 day mean A_T	99.2423	1229.1
Asian American put Option Value	10.6785	148.5283

Table 1: Mean and Variance of Prices

3.2 Conclusion

To improve the precision of this simulation, we have to employ variance reduction methods. When it comes to pricing an option, we want to minimise our variance as much as possible. In the next section, we will use various variance reduction methods to reduce the variance of our sample price paths, which will result in a lower variance for our final option price. Our simulation gives us with a final present value of 10.6785 for the Asian American put option.

4 LSMC with Variance Reduction

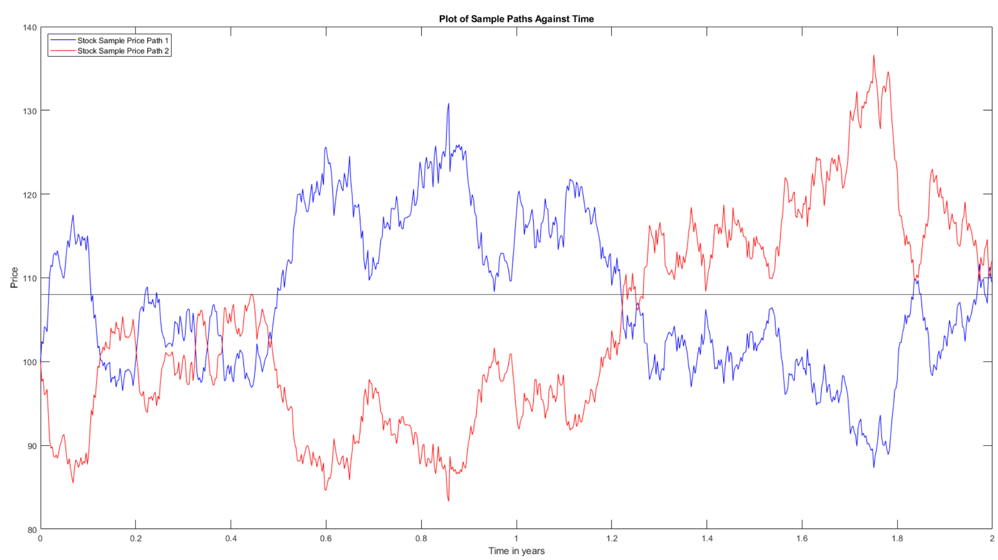
In this section, we employ three different methods of variance reduction, namely the Antithetic Variate Method, Control Variate Method and Importance Sampling Method. We will compare the difference in performance when variance reduction is introduced and discuss our preferred method. We will also discuss the efficacy of all three methods.

4.1 Antithetic Variate Method

The antithetic variate method is very easy to implement but takes up more memory and computational power as we have observed in our simulation. In addition to the regular LSMC done in section 2, we simply introduce mirroring matrices for sample price paths, sixty day mean price paths and payoffs. The brief algorithm is as follows,

1. Generate two M sample price paths.
   (a) One price path uses the standard white noise generated .
   (b) The other price path uses the negative of the same standard white noise generated.
2. Follow algorithm in subsection 2.1 but applying steps to both paths.
3. Finally, take the average of the two path payoffs.

An example of the sample price paths obtained by the antithetic variate method can be seen in Figure 2 below.



4.1.1 Analysis of Results

Using LSMC with the antithetic variate method, we arrived at a mean option price of 10.4925 with a variance of 24.9109. LSMC without variance reduction gave us a mean option price of 10.4896 with a variance of 141.5304. The ratio of their variances is 5.7462. We observe that the variance has been reduced significantly, while the mean is approximately the same as before.

Elapsed time for the algorithm excluding plotting of graph and setting number of sample paths $M = 10^4$ is about 40 seconds, which is two times longer than if we did LSMC without any variance reduction methods. This was expected as two times more memory was needed to store the extra sample paths for the antithetic variate method.

	Mean	Variance
Option Price without Variance Reduction	10.4896	141.5304
Option Price with the Antithetic Variate Method	10.4925	24.9109

Table 2: Mean and Variance after using the Antithetic variate method

4.2 Control Variate Method

The control variate is easy to implement as well and is most commonly used when we have a closed pricing formula for our control variate. The idea behind the control variate method is that if the pricing from the LSMC misprices the option, then it will also misprice an option with a closed form formula by a similar amount. Then we can reduce the variance caused by mispricing by using the following formula,

$$\tilde{X} = X + c(Y - E(Y)) \tag{4}$$

for some constant c. By simple calculus, we can show that the optimal value of c that minimises the variance of $\tilde{X}$ is such that

$$\tilde{c} = -\frac{Cov(X, Y)}{Var(Y)}.$$

However, for our stock in question, the volatility is not constant and thus there does not exist a closed form solution for the pricing of the stock with respect to any options.
We choose to use a European put option as the control variate. We choose the European put option because it can be seen as a similar option to the Asian American put option. At the same time, we choose a simplistic vanilla option because we want to minimise the error between the theoretical value and the mean calculated by our simulation. As there exists no closed form solution for the pricing the option of this stock with non constant volatility, we try our best to get as close as possible to the theoretical value by simply using LSMC for a regular European put option. In an ideal world, we have the closed form solution for this stock with varying volatility.

4.2.1 Analysis of Results

Using the control variate method, we arrived at a mean option price of 10.6186 with a variance of 75.4395. LSMC without the control variate method gave us a mean option price of 10.6186 with a variance of 148.5502. The ratio of their variance is 1.9691. We can see that the variance has been reduced by almost half. Solving for $\tilde{c}$, we have $\tilde{c} = -0.7347$. The mean of both option prices are the same, which is what we expect to see when we take the expectation of both sides of the equation (4), we get $E(\tilde{X}) = E(X)$.
Elapsed time for the algorithm without graph plots and setting sample paths $M = 10^4$ is about 20 seconds, which is half the time of the antithetic variate method.

	Mean	Variance
Option Price without Variance Reduction	10.6186	148.5502
Option Price with the Control Variate Method	10.6186	75.4395

Table 3: Mean and Variance after using the Antithetic variate method

4.3 Importance Sampling Method

The importance sampling method works by changing the probability distribution such that there is a higher probability of a desired event occurring. We can do so by using the Radon-Nikodym derivative and Girsanov's Theorem. By doing so, we are able to modify the drift term in our current probability measure Q. Recall from our problem that our stock follows the SDE under the risk neutral probability measure Q with W_t Brownian motion.

Since we are dealing with an Asian American put option, to use the importance sampling to our favour, we should lower the drift r so that while simulating sample path prices, we have smaller stock prices, thus resulting in ITM Asian American put options.

Recall that,

$$dS_t = rS_t dt + \sigma(S_t) S_t dW_t, \tag{5}$$

where W_t is a Brownian motion under Q. Then we choose θ such that

$$\theta = \frac{r - r_1}{\sigma(S_t)}, \tag{6}$$

where r_1 is the new desired constant drift term and $r_1 < r$. By Girsanov's Theorem, there exists an equivalent probability measure $\tilde{Q}$ such that $d\tilde{W}_t$ is a $\tilde{Q}$-Brownian motion and Radon-Nikodym derivative $\frac{d\tilde{Q}}{dQ}$ such that,

$$E\left(\frac{d\tilde{Q}}{dQ}|F_t\right) = e^{-\frac{\theta^2 t}{2} - \theta W_t}$$

and $\tilde{W} = W_t + \theta t$. So we have,

$$\begin{aligned}\frac{dQ}{d\tilde{Q}} &= e^{\frac{\theta^2 T}{2} + \theta W_T} \\ &= e^{\frac{\theta^2 T}{2} + \theta(\tilde{W}_T - \theta T)} \\ &= e^{\frac{-\theta^2 T}{2} + \theta \tilde{W}_T}\end{aligned}$$

and for the new SDE we have,

$$\begin{aligned}dS_t &= rS_t dt + \sigma(S_t) S_t dW_t \\ &= rS_t dt + \sigma(S_t) S_t d\tilde{W}_t - \theta\sigma(S_t) S_t dt \\ &= \left(r - \frac{r - r_1}{\sigma(S_t)}\sigma(S_t)\right) S_t dt + \sigma(S_t) S_t d\tilde{W}_t \\ &= r_1 S_t dt + \sigma(S_t) S_t d\tilde{W}_t.\end{aligned}$$

Therefore, the new option price can be calculated as such,

$$\begin{aligned}e^{-rT} E^Q(p) &= e^{-rT} E^{\tilde{Q}}\left(p\frac{dQ}{d\tilde{Q}}\right) \\ &= e^{-rT} E^{\tilde{Q}}\left(pe^{\frac{-\theta^2 T}{2} + \theta \tilde{W}_T}\right),\end{aligned}$$

where p is the payoff of the exotic Asian American put option.

4.3.1 Analysis of Results

An important point to note is that the importance sampling method is only very useful when dealing with highly unlikely events during the life of an option. However, for our problem, we notice that the strike price is $K = 108$ with the initial strike price $S_0 = 100$. Therefore, with a drift term of 0.08, we should not really consider this Asian American put option to be deep out of the money. Nevertheless, we use the importance sampling method for the purpose of this assignment.

Setting sample paths $M = 10^4$, we vary the decrement in the drift term r_1 as seen in Table 4. μ and σ^2 is the mean and variance of the price of the option without using any variance reduction methods, which gives us the same results as what we observed in LSMC section 3.1. We use % of ITM options at terminal time T because we want to observe the percentage of guaranteed non-negative payoff for the sample path price. We can see that as we r_1 gets smaller and more negative, the percentage of ITM options increases, this is because of the more negative drift term causing a larger portion of stock prices to be smaller, thus we have a larger portion of A_T being ITM.
μ_1 and σ_1^2 is the mean and variance of the price of the option simulated with the importance sampling method. Our results contradict what we expected to observe. As r_1 decreases, the mean price increases while the variance increases as well. Hence, ratios decrease when r_1 decrease. This is likely due to the fact that we are using importance sampling method on an event that is not very unlikely to occur and ultimately backfired[2]. We can see that the percentage of ITM options at the beginning is very close to 50%. Increasing the percentage of ITM options with importance sampling resulted in about 70% non-negative payoff paths. A simplistic interpretation of an increase in price of this exotic option is the result of a more negative risk-free rate, causes investors to expect prices to fall, increasing the value of put options. Overall, the importance sampling method did not improve the variance of our LSMC.
Elapsed time for each simulation was about 40 seconds for $M = 10^4$.

	$r_1 = r - 0.01$	$r_1 = r - 0.05$	$r_1 = 0$	$r_1 = r - 0.1$	$r_1 = r - 0.5$
% of ITM options at time T without IS	47.53	47.56	48.13	47.65	47.98
% of ITM options at time T with IS	49.81	58.68	65.65	69.30	99.66
μ	10.5137	10.551	10.5748	10.6503	10.6809
σ^2	145.3841	145.9188	142.9381	145.7823	145.3806
μ_1	11.029	13.143	15.4016	16.6765	54.0742
σ_1^2	150.4027	174.9105	185.8208	198.2641	186.3143
Ratio	0.9666	0.8342	0.7692	0.7353	0.7803

Table 4: Mean and Variance after using the Importance Sampling method

We understand that importance sampling method is used to reduce variance, but our results contradict the very purpose of this method. To explain this, we must realise that choosing a good density for importance sampling is non-trivial, but for this case, we have placed too much weight on the non-negative payoff paths and too little for the tails of the payoff distribution such that our option is almost always in the money. However, it is not impossible to see an increase in variance[3].

[2] https://statweb.stanford.edu/ owen/mc/Ch-var-is.pdf
[3] http://www.columbia.edu/ mh2078/MonteCarlo/MCS$_{V}ar_{R}ed_{A}dvanced.pdf Page5$

5 Conclusion

In conclusion, we can see that the antithetic variate method is undeniably the most effective in reducing variance for the option price and was very easy to implement. Antithetic variate method had a ratio of 5.7462 which was impressive. On the other hand, as the stock price has a complicated and non constant volatility, using the control variate method was challenging. Due to the lack of closed form solution for pricing the stock with non constant volatility, it was not as straightforward as expected to implement the control variate method. We used a European put option as the control variate as it was the simplest to price thus minimising the error between simulated value and theoretical value and arrived at a ratio of 1.9691. Lastly, we tried the importance sampling method to reduce variance but our results contradicted our expectations. We believe that for such a likely event for our exotic option to be in the money, using the importance sampling method only increased our variance. Thus, performing worse than the LSMC without variance reduction and giving us an average ratio of 0.8171 when we vary the new drift term r_1.

	Ratio
Antithetic Variate Method	5.7462
Control Variate Method	1.9691
Importance Sampling Method	0.8171

Table 5: Conclusion of Variance Reduction Methods

Ultimately, we believe that the antithetic variate method is most appropriate for this exotic option because of two reasons.

1. Lack of closed form solution for Control Variate Method.
2. The fact that this exotic option with is strike price $K = 108$ is not a rare event, causes the importance sampling method to place too much weight on non-negative payoff paths.

References

[1] Choe, G. H. (2016). Stochastic Analysis for Finance with Simulations (Universitext) (1st ed. 2016 ed.). Springer.

[2] Importance Sampling and possibility of increasing variance from misuse.
`http://www.columbia.edu/ mh2078/MonteCarlo/MCS_Var_Red_Advanced.pdf`

[3] Importance Sampling
`https://statweb.stanford.edu/ owen/mc/Ch-var-is.pdf`

[4] Monte Carlo Variance Reduction
`http://www.goddardconsulting.ca/option-pricing-monte-carlo-vr.html`