Week 8 Exercises

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1 ACA_0 is arithmetically conservative over PA^U

Let $M = (|M|, U_M, 0_M, 1_M, +_M, \cdot_M, \exp_M, <_M) \vDash \mathbf{PA}^U$. For each arithmetical formula $\varphi(x)$ in $\mathcal{L}_2(|M| \cup \{U\})$ we associate a witness $X_\varphi \subseteq |M|$ satisfying $\forall x[\varphi(x) \leftrightarrow x \in X_\varphi]$. Now collect all such witnesses

$$\mathcal{S} = \left\{X_\varphi : \varphi(x) \text{ arithmetical}\right\}$$

and we show that

$$M' = (|M|, \mathcal{S}, 0_M, 1_M, +_M, \cdot_M, \exp_M, <_M) \vDash \mathbf{ACA}_0.$$

Firstly by our choices of X_φ it's easy to see that $M' \vDash$ arithmetical-$\mathbf{CA}$. The second order induction axiom

$$\forall X\, [0 \in X \wedge \forall x\, (x \in X \to x + 1 \in X) \to \forall x\, (x \in X)]$$

is also satisfied by M' as we have only added arithmetical sets to $\mathcal{S}$ and for any $X_\varphi \in \mathcal{S}$,

$$\varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1)) \to \forall x(\varphi(x))$$

is an instance of $\mathbf{IND}$ in $\mathbf{PA}^U$.

So suppose F is an arithmetic formula and $\mathbf{ACA}_0 \vdash F$, let $M = (|M|, U_M, \dots) \vDash \mathbf{PA}^U$ and use construction above to get $M' = (|M|, \mathcal{S}_M, \dots) \vDash \mathbf{ACA}_0$. As $M' \vDash F$ and F, being arithmetic, only quantifies over $|M|$, $M \vDash F$ too.

2

Find strength of $\mathbf{PA} + I\Sigma_n$.

Let $\mathbf{PA}_n q \vdash^m_k \Gamma \Rightarrow \Delta$ also be like $\mathbf{PA}_n \vdash^m_k \Gamma \Rightarrow \Delta$ where we replace cut rank with $q\text{-rnk}$.

2.1 (a)

> If $\mathbf{PA}_n q \vdash^{m}_{n+2+r+1} \Gamma \Rightarrow \Delta$ then $\mathbf{PA}_n q \vdash^{4^m}_{n+2+r} \Gamma \Rightarrow \Delta$.

Given φ is a formula with $q\text{-rnk}(\varphi) \geq n+1$ and we want to eliminate cuts involving φ.

Given Σ_n induction, any formula produced by an instance our induction axiom $I\Sigma_n$ is Π_{n+1}, so if we have φ a formula with $q\text{-rnk}(\varphi) > n+1$,

- φ is too complicated (say Π_{n+2}) to ever be involved with an instance of induction axiom, then we directly apply reduction lemma.
- We can also directly apply reduction lemma until φ itself originated in an instance of induction axiom.
- For cases where φ is simple enough (Π_{n+1}) but has an inflated $q\text{-rnk}$, let's just restrict our induction axiom to not work on φ but only its prenex normal form, this should not affect the strength.

2.2 (b)

> If $\mathbf{PA}_n q \vdash^{m}_{n+2} \Gamma \Rightarrow \Delta$ then $\mathbf{PA}_\omega q \vdash^{\omega+k}_{n+2} \Gamma' \Rightarrow \Delta'$.

I think the stronger bound given by the First Interpretation Theorem still works here and the weaker bound in Pohler 7.3.19 ($\omega \cdot (m+1)$) is due to induction being weakened into a structural *rule* from an *axiom*.

2.3 (c)

> If $\mathbf{PA}_\omega q \vdash^{\alpha}_{1+m+1} \Gamma \Rightarrow \Delta$ then $\mathbf{PA}_\omega q \vdash^{4^\alpha}_{1+m} \Gamma \Rightarrow \Delta$.

Proof of cut elimination theorem almost exactly applies here.

2.4 (d)

> If $\mathbf{PA}_\omega q \vdash^{\alpha}_{1} \Gamma \Rightarrow \Delta$, then $\mathbf{PA}_\omega \vdash^{\omega \cdot \alpha}_{1} \Gamma \Rightarrow \Delta$.

I still cannot figure out how to reduce quantifier-free cuts to atomic cuts only introducing a $\omega\cdot$ factor so I will just skip and black-box.

2.5 (e)

> Compute $\|\mathbf{PA}_n\|$.

In general, this is useful,

$$4^{\omega\cdot\alpha} = (4^\omega)^\alpha = \omega^\alpha.$$

Whenever $\mathbf{PA}_n \vdash \Gamma \Rightarrow \Delta$, we have

$$\begin{aligned}
&\mathbf{PA}_n q \vdash^{m}_{n+2} \Gamma \Rightarrow \Delta \\
&\mathbf{PA}_\omega q \vdash^{\omega+k}_{n+2} \Gamma \Rightarrow \Delta \\
&\mathbf{PA}_\omega q \vdash^{4^{\omega+k}}_{n+1} \Gamma \Rightarrow \Delta \\
&\mathbf{PA}_\omega q \vdash^{\omega\cdot l}_{n+1} \Gamma \Rightarrow \Delta \text{ where } l = 4^k \\
&\mathbf{PA}_\omega q \vdash^{\omega^l}_{n} \Gamma \Rightarrow \Delta \text{ as } 4^{\omega\cdot l} = \omega^l \\
&\mathbf{PA}_\omega q \vdash^{\omega^{\omega^{l-1}}}_{n-1} \Gamma \Rightarrow \Delta \text{ as } 4^{\omega^l} = \omega^{\omega^{l-1}}
\end{aligned}$$

and so on, so in general we have

$$\mathbf{PA}_\omega q \vdash^{\alpha}_{1} \Gamma \Rightarrow \Delta$$

where

$$\alpha = \omega^{\omega^{\cdot^{\cdot^{\cdot^{m}}}}}$$

where ω^{-} is iterated n times starting from some finite m. Now by part (d), $\omega \cdot \alpha = \alpha$ when $n > 1$ and $\omega \cdot \alpha = \omega^{m+1}$ when $n = 1$, and we have

$$\mathbf{PA}_\omega \vdash^{\alpha}_{1} \Gamma \Rightarrow \Delta.$$

We should be able to measure $\|\mathbf{PA}_n\|$ with

$$\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\} n \text{ iterated exponentiations from } \omega$$

but I still cannot convince myself how allowing atomic cuts are fine (which is Pohler Exercise 7.3.18?).