MA2101S Homework 2

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Problem 1. Let K be any field, let V be a K-vector space, and let $T : V \to V$ be a K-linear endomorphism. Suppose $v \in V$ and $n \in \mathbb{N}_{>0}$ such that

$$T^n v = 0 \quad \text{but} \quad T^{n-1} V \neq 0 \quad \text{in } V.$$

Show that the n vectors $v, Tv, \dots, T^{n-1}v$ in V are linearly independent over K.

Proof. Consider the equation

$$c_1 v + c_2 Tv + \cdots + c_{n-1} T^{n-1} v = 0 \tag{1.1}$$

where $c_1, c_2, \dots, c_{n-1} \in K$.
Then applying T^{n-1} to both sides, we get, by linearity of T,

$$\begin{aligned} T^{n-1}\left(c_1 v + c_2 Tv + \cdots + c_{n-1} T^{n-1} v\right) &= T^{n-1} 0 \\ T^{n-1}(c_1 v) + T^{n-1}(c_2 Tv) + \cdots + T^{n-1}(c_{n-1} T^{n-1} v) &= 0 \\ c_1 T^{n-1} v + \underbrace{c_2 T^n v + \cdots + c_{n-1} T^{2n-2} v}_{0} &= 0 \\ c_1 T^{n-1} v &= 0 \end{aligned}$$

and because $T^{n-1} v \neq 0$, we have $c_1 = 0$. Now rewrite (1.1) and apply T^{n-2} to both sides, again by linearity of T,

$$\begin{aligned} T^{n-2}\left(c_2 Tv + \cdots + c_{n-1} T^{n-1} v\right) &= T^{n-2} 0 \\ T^{n-2}(c_2 Tv) + \cdots + T^{n-2}(c_{n-1} T^{n-1} v) &= 0 \\ c_2 T^{n-1} v + \underbrace{c_3 T^n v + \cdots + c_{n-1} T^{2n-3} v}_{0} &= 0 \\ c_2 T^{n-1} v &= 0 \end{aligned}$$

we have $c_1 = c_2 = 0$.
The other $n-3$ cases are analogous. So $c_1 = c_2 = \cdots = c_{n-1} = 0$, linear independence shown. ∎

Problem 2. Let $V := \mathrm{Maps}(\mathbb{R}, \mathbb{R})$ denote the $\mathbb{R}$-vector space of $\mathbb{R}$-valued functions on $\mathbb{R}$. Show that for any $n \in \mathbb{N}$ and for any pairwise distinct real numbers $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, the n exponential functions in the variable $t \in \mathbb{R}$ given by

$$e^{\alpha_1 t}, \dots, e^{\alpha_n t} \in V$$

are linearly independent over $\mathbb{R}$.

Proof. Consider the equation

$$f : t \mapsto c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} + \cdots + c_n e^{\alpha_n t} = 0_V \tag{2.1}$$

where $c_1, c_2, \dots, c_n \in \mathbb{R}$. $\alpha_1, \dots, \alpha_n$ are pairwise distinct. By reordering terms, we can assume $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Then rewrite as follows

$$\begin{gathered} \alpha_2 = \alpha_1 + d_2 \\ \cdots \\ \alpha_n = \alpha_1 + d_n \end{gathered}$$

and because $\alpha_1 < \cdots < \alpha_n$ by assumption, $d_2 < \cdots < d_n$ and they are all strictly positive in $\mathbb{R}$. Then for any $t \in \mathbb{R}$, from (2.1)

$$\begin{aligned} c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} + \cdots + c_n e^{\alpha_n t} &= 0 \\ c_1 e^{\alpha_1 t} + c_2 e^{(\alpha_1 + d_2) t} + \cdots + c_n e^{(\alpha_1 + d_n) t} &= 0 \\ c_1 e^{\alpha_1 t} + c_2 e^{\alpha_1 t} e^{d_2 t} + \cdots + c_n e^{\alpha_1 t} e^{d_n t} &= 0 \\ e^{\alpha_1 t} \left(c_1 + c_2 e^{d_2 t} + \cdots + c_n e^{d_n t} \right) &= 0 \end{aligned}$$

because $e^t \neq 0$ for all $t \in \mathbb{R}$,

$$c_1 + c_2 e^{d_2 t} + \cdots + c_n e^{d_n t} = 0 \tag{2.2}$$

Now take limit as $t \to -\infty$, it is known that $\lim_{t \to -\infty} e^t = 0$,

$$\begin{aligned} \lim_{t \to -\infty} \left(c_1 + c_2 e^{d_2 t} + \cdots + c_n e^{d_n t} \right) &= 0 \\ \lim_{t \to -\infty} c_1 + \lim_{t \to -\infty} \left(c_2 e^{d_2 t} + \cdots + c_n e^{d_n t} \right) &= 0 \\ c_1 + 0 &= 0 \end{aligned}$$

so $c_1 = 0$.
As $d_2 < \cdots < d_n$, from (2.2) we can repeat the same process and factor out $e^{d_2 t}$, then take the limit as $t \to -\infty$ again to get $c_2 = 0$.
The other $n - 2$ cases are analogous. So $c_1 = c_2 = \cdots = c_n = 0$, linear independence shown. $\square$

Problem 3. Let K be a field, and let V and W be K-vector spaces. Let $T, U \in \operatorname{Hom}_K(V, W)$ be K-linear maps $V \to W$. Suppose $\operatorname{Im}(T) \cap \operatorname{Im}(U) = \{ 0_W \}$ and T, U are non-zero. Show that T and U are linearly independent in $\operatorname{Hom}_K(V, W)$.

Proof. Consider the equation

$$cT + dU = 0_{\operatorname{Hom}_K(V,W)} \tag{3.1}$$

where $c, d \in K$. Suppose for a contradiction T, U are linearly dependent, so c, d nonzero, then take any $v \in V$ where $U(v) \neq 0$,

$$\begin{aligned} (cT + dU)(v) &= 0_{\operatorname{Hom}_K(V,W)}(v) \\ cT(v) + dU(v) &= 0_W \\ T(v) &= -c^{-1} dU(v) \end{aligned}$$

So we have $-c^{-1}dU(v) \in \operatorname{Im}(T)$, by subspace property of the image of a linear map,

$$(-d^{-1}c)(-c^{-1}dU(v)) \in \operatorname{Im}(T) \implies U(v) \in \operatorname{Im}(T).$$

Clearly $U(v) \in \operatorname{Im}(U)$, this means $U(v) \in \operatorname{Im}(T) \cap \operatorname{Im}(U) \implies U(v) = 0_W$, which is a contradiction. $\square$

Problem 4. Let K be a field, and let X be a K-vector space.

(a) Let V and W be finite dimensional K-subspaces of X. Show that

$$\dim_K(V) + \dim_K(W) = \dim_K(V + W) + \dim_K(V \cap W)$$

Proof. Let $\alpha = \{ u_1, \dots, u_r \}$ be a basis for $V \cap W$. First expand α to be a basis for V, similar to the proof of existence of basis (for finite-dimensional vector spaces).
Set $\beta := \emptyset$, while $\operatorname{span}(\alpha \cup \beta) \neq V$, choose vector $v \in V, v \notin \operatorname{span}(\alpha \cup \beta)$, and set $\beta := \beta \cup \{ v \}$. $\alpha \cup \beta$ is now a basis for V.
Set $\gamma := \emptyset$, while $\operatorname{span}(\alpha \cup \gamma) \neq W$, choose vector $v \in W, v \notin \operatorname{span}(\alpha \cup \gamma)$, and set $\gamma := \gamma \cup \{ v \}$. $\alpha \cup \gamma$ is now a basis for W. The algorithms halt due as V, W are finite-dimensional.

Claim. $\alpha \cup \beta \cup \gamma = \{ u_1, \dots, u_r, v_1, \dots, v_m, w_1, \dots, w_n \}$ is a basis for $V + W$.
Take any arbitary vector in $x \in V + W$, by definition, $\exists v \in V, w \in W$. $x = v + w$.
$\alpha \cup \beta$ is a basis for V so $\exists c_1, \dots, c_{r+m} \in K$,

$$v = \sum_{i=1}^{r} c_i u_i + \sum_{i=1}^{m} c_{r+i} v_i.$$

Also, $\alpha \cup \gamma$ is a basis for W so $\exists d_1, \dots, d_{r+n} \in K$,

$$v = \sum_{i=1}^{r} d_i u_i + \sum_{i=1}^{n} d_{r+i} w_i.$$

Then because $x = u + w$,

$$\begin{aligned} x &= \sum_{i=1}^{r} c_i u_i + \sum_{i=1}^{m} c_{r+i} v_i + \sum_{i=1}^{r} d_i u_i + \sum_{i=1}^{n} d_{r+i} w_i \\ &= \sum_{i=1}^{r} (c_i + d_i) u_i + \sum_{i=1}^{m} c_{r+i} v_i + \sum_{i=1}^{n} d_{r+i} w_i \end{aligned}$$

Therefore $\alpha \cup \beta \cup \gamma$ generates $V + W$.

To show linear independence, consider the equation

$$\sum_{i=1}^{r} c_i u_i + \sum_{i=1}^{m} d_i v_i + \sum_{i=1}^{n} e_i w_i = 0 \tag{4.1}$$

where $c_1, \dots, c_r, d_1, \dots, d_m, e_1, \dots, e_n \in K$. Then

$$\underbrace{\sum_{i=1}^{r} c_i u_i + \sum_{i=1}^{m} d_i v_i}_{\text{in } V} = \underbrace{-\sum_{i=1}^{n} e_i w_i}_{\text{in } W} \tag{4.2}$$

so $-\sum_{i=1}^{n} e_i w_i \in V \cap W$, since $V \cap W$ has basis α, exist scalars $b_1, \dots, b_r$ such that

$$\begin{aligned} -\sum_{i=1}^{n} e_i w_i &= \sum_{i=1}^{r} b_i u_i \\ 0 &= \sum_{i=1}^{r} b_i u_i + \sum_{i=1}^{n} e_i w_i \end{aligned}$$

from linear independence of $\alpha \cup \gamma$, $b_1 = \cdots = b_r = e_1 = \cdots = e_n = 0$. Then RHS of (4.2) is zero, and by linear independence of $\alpha \cup \beta$, we have $c_1 = \cdots = c_r = d_1 = \cdots = d_m = 0$. This completes the proof of the claim.

Then by counting the sizes of α, β, γ, we get

$$\begin{aligned} \dim_K(V) + \dim_K(W) &= |\alpha \cup \beta| + |\alpha \cup \gamma| \\ &= r + m + r + n = r + m + n + r \\ &= |\alpha \cup \beta \cup \gamma| + |\alpha| \\ &= \dim_K(V + W) + \dim_K(V \cap W) \end{aligned}$$

which completes the proof. $\square$

(b) Let U, V and W be finite dimensional K-subspaces of X. Show that

$$\begin{aligned}&\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U+V+W)\\&\quad\geqslant \max\left(\dim_K(U\cap V),\dim_K(V\cap W),\dim_K(W\cap U)\right)\end{aligned}$$

Proof. Firstly, subspace addition is commutative and associative, a property inherited from vector addition. Then by applying result of part (a), compute $\dim_K(U+V+W)$ in 3 different ways. Firstly,

$$\begin{aligned}&\dim_K(U+V+W)\\&=\dim_K((U+V)+W)\\&=\dim_K(U+V)+\dim_K(W)-\dim_K((U+V)\cap W)\\&=\dim_K(U)+\dim_K(V)-\dim_K(U\cap V)+\dim_K(W)-\dim_K((U+V)\cap W)\end{aligned}$$

Rearranging terms,

$$\begin{aligned}&\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U+V+W)\\&\qquad=\dim_K(U\cap V)+\dim_K((U+V)\cap W).\end{aligned}$$

In particular, $\dim_K((U+V)\cap W)\geqslant 0$, so

$$\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U+V+W)\geqslant\dim_K(U\cap V). \tag{4.3}$$

Similarly,

$$\begin{aligned}\dim_K(U+V+W)&=\dim_K(U+(V+W))\\&=\dim_K(U)+\dim_K(V+W)-\dim_K(U\cap(V+W))\\\dim_K(U+V+W)&\leqslant\dim_K(U)+\dim_K(V+W)\\\dim_K(U+V+W)&\leqslant\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(V\cap W)\\\dim_K(V\cap W)&\leqslant\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U+V+W)\qquad(4.4)\end{aligned}$$

Finally,

$$\begin{aligned}\dim_K(U+V+W)&=\dim_K(V+(U+W))\\&=\dim_K(V)+\dim_K(U+W)-\dim_K(V\cap(U+W))\\\dim_K(U+V+W)&\leqslant\dim_K(V)+\dim_K(U+W)\\\dim_K(U+V+W)&\leqslant\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U\cap W)\\\dim_K(U\cap W)&\leqslant\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U+V+W)\qquad(4.5)\end{aligned}$$

(4.3), (4.4) and (4.5) all hold true, therefore combining inequalities,

$$\begin{aligned}&\dim_K(U)+\dim_K(V)+\dim_K(W)-\dim_K(U+V+W)\\&\quad\geqslant\max\left(\dim_K(U\cap V),\dim_K(V\cap W),\dim_K(W\cap U)\right)\end{aligned}$$

□

Problem 5. Let $V := \text{Maps}(\mathbb{N}, \mathbb{R})$ denote the $\mathbb{R}$-vector space of all sequences in $\mathbb{R}$ indexed by $\mathbb{N}$, and let $W \subseteq V$ denote the subset of sequences $(x_0, x_1, \ldots, x_n, \ldots) \in V$ satisfying

$$x_n = x_{n-1} + x_{n-2} \quad \text{for all } n \in \mathbb{N}_{\geqslant 2}.$$

Notation. Let $K_0 : \mathbb{N} \to \mathbb{R}$ denote the zero sequence, where $\forall n \in \mathbb{N}.\ K_0(n) = 0_{\mathbb{R}}$. Also throughout Questions 5 and 6, functional notation instead of subscripts will be used to access members of a sequence.

(a) Show that W is an $\mathbb{R}$-subspace of V.

Proof. $0_V \in V$ is the zero sequence, K_0. For any $n \in \mathbb{N}_{\geqslant 2}$, $K_0(n) = 0$ and $K_0(n-1) + K_0(n-2) = 0 + 0 = 0$. Therefore $0_V \in W$.
To show closure under vector addition, take any $f, g \in W$, then for any $n \in \mathbb{N}_{\geqslant 2}$,

$$\begin{aligned}
(f+g)(n) &= f(n) + g(n) \\
&= f(n-1) + f(n-2) + g(n-1) + g(n-2) \\
&= f(n-1) + g(n-1) + f(n-2) + g(n-2) \\
&= (f+g)(n-1) + (f+g)(n-2)
\end{aligned}$$

so $f + g \in W$. To show closure under scalar multiplication, take any $f \in W, x \in \mathbb{R}$, and for any $n \in \mathbb{N}_{\geqslant 2}$,

$$\begin{aligned}
(xf)(n) &= x \cdot f(n) \\
&= x \cdot (f(n-1) + f(n-2)) \\
&= x \cdot f(n-1) + x \cdot f(n-2) \\
&= (xf)(n-1) + (xf)(n-2)
\end{aligned}$$

so $xf \in W$. Therefore W is a subspace of V. $\square$

(b) Show that an $\mathbb{R}$-basis of W is given by the two sequences

$$(a_0, a_1, \dots) \quad \text{and} \quad (a_1, a_2, \dots)$$

where $a_0, a_1, a_2, \dots$ are the *Fibonacci numbers* defined inductively by:

$$a_0 := 0, \quad a_1 := 1, \quad a_n := a_{n-1} + a_{n-2} \quad \text{for all } n \in \mathbb{N}_{\geqslant 2}.$$

Exercise 5.1. *The map $T : W \to \mathbb{R}^2$ as defined by $f \mapsto (f(0), f(1))$ is a $\mathbb{R}$-linear isomorphism.*

Proof. To show linearity, for any $f, g \in W$, $a, b \in \mathbb{R}$. Consider $T(af + bg)$,

$$\begin{aligned} T(af+bg) &= ((af+bg)(0), (af+bg)(1)) \\ &= (af(0)+bg(0), af(1)+bg(1)) \\ &= (af(0), af(1)) + (bg(0), bg(1)) \\ &= a\,(f(0), f(1)) + b\,(g(0), g(1)) \\ &= aT(f) + bT(g) \end{aligned}$$

Next, consider the kernel of T, so suppose $f \in W$, $T(f) = (0,0) \in \mathbb{R}^2$, then from definition of T, $f(0) = 0$ and $f(1) = 0$, using characterising property of W, it means f has to be the zero sequence K_0, therefore T has a trivial kernel (T injects).
Now consider the range of T, for any $(x_0, x_1) \in \mathbb{R}^2$, define a sequence $f : \mathbb{N} \to \mathbb{R}$ inductively as follows,

$$f(0) := x_0, \quad f(1) := x_1, \quad f(n) = f(n-1) + f(n-2) \quad \text{for all } n \in \mathbb{N}_{\geqslant 2}.$$

By construction, $f \in W$, and it is clear that $T(f) = (x_0, x_1)$, therefore T maps onto $\mathbb{R}^2$. Hence T is an $\mathbb{R}$-linear isomorphism. □

Proposition. *An $\mathbb{R}$-basis of W is given by the two sequences*

$$f := (a_0, a_1, \dots) \quad \textit{and} \quad g := (a_1, a_2, \dots)$$

where a_i denotes the i-th Fibonacci number.

Proof. $T(f) = (0,1)$ and $T(g) = (1,1)$. From MA1101R, an easy computation gives us that $\{ (0,1), (1,1) \}$ is a basis for $\mathbb{R}^2$. Therefore as isomorphisms preserve structure, $\{ T^{-1}(0,1), T^{-1}(1,1) \} = \{ f, g \}$ is a basis for W. □

Problem 6. Preserving the notation as in the previous question.

(a) Determine (distinct) real numbers $\alpha, \beta \in \mathbb{R}$ such that the two sequences

$$(\alpha^0, \alpha^1, \alpha^2, \dots) \quad \text{and} \quad (\beta^0, \beta^1, \beta^2, \dots)$$

also form an $\mathbb{R}$-basis of W.

Solution. Firstly, the two sequences must be in W. So we have to solve for a geometric sequence $f = (x^0, x^1, x^2, \dots)$ satisfying the property that for all $n \in \mathbb{N}_{\geqslant 2}$,

$$x^n = x^{n-1} + x^{n-2}. \tag{6.1}$$

Since we want f to be part of an $\mathbb{R}$-basis of W, f should not be the zero sequence, so take $x \neq 0$. Then (6.1) reduces to the following

$$\begin{aligned} x^2 &= x^0 + x^1 \\ x^2 - x - 1 &= 0 \end{aligned} \tag{6.2}$$

Solving for roots in (6.2), we can see that setting

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$

we obtain the only two nonzero values for $\alpha, \beta \in \mathbb{R}$ such that the sequences $(\alpha^0, \alpha^1, \alpha^2, \dots)$ and $(\beta^0, \beta^1, \beta^2, \dots)$ lie in W. ■

Claim. The sequences form a $\mathbb{R}$-basis for W.

Proof. By Exercise 5.1, it suffices to check if $\{ (\alpha^0, \alpha^1), (\beta^0, \beta^1) \}$ form a basis for $\mathbb{R}^2$,

$$\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and we're done. □

(b) Show that the Fibonacci numbers are given by the closed formula

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Proof. Define $a, f, g \in W$ as

$$\begin{aligned} a &= (a_0, a_1, \dots) \\ f &= (\alpha^0, \alpha^1, \alpha^2, \dots) \\ g &= (\beta^0, \beta^1, \beta^2, \dots) \end{aligned}$$

where again a_i denotes the i-th Fibonacci number, keeping α, β from part (a). Let T be the isomorphism $W \to \mathbb{R}^2$ defined in 5.1.
Since $a \in W$ and $\{ f, g \}$ is a basis for W (part (a)), then there exists unique $c, d \in \mathbb{R}$ where $a = cf + dg$, so solving for c, d.

$$\begin{aligned} a &= cf + dg \\ T(a) &= T(cf + dg) \\ T(a) &= cT(f) + dT(g) \\ (0, 1) &= c(1, \alpha) + d(1, \beta) \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & 1 \end{array}\right] \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & -\frac{1}{\sqrt{5}} \end{array}\right]$$

$$c = \frac{1}{\sqrt{5}}, \quad d = -\frac{1}{\sqrt{5}}.$$

Since $a = cf + dg$, applying this equation pointwise, for any $n \in \mathbb{N}$,

$$\begin{aligned} a(n) &= cf(n) + dg(n) \\ a_n &= \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{\sqrt{5}}\beta^n \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \end{aligned}$$

obtaining the closed formula for the Fibonacci numbers. □