MA2101S Homework 4

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Question 1.

(a) *Solution.* For any $i, j, k, l \in \{1, \dots, n\}$, do note that in general,

$$\begin{aligned} E_{ij}E_{kl} &= \begin{cases} E_{il} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \\ &= \delta_{kj}E_{il} \end{aligned}$$

Then an expression for the commutator of E_{ij} and E_{kl} will be given by

$$\begin{aligned} [E_{ij}, E_{kl}] &= E_{ij}E_{kl} - E_{kl}E_{ij} \\ &= \delta_{kj}E_{il} - \delta_{il}E_{kj} \end{aligned}$$

□

(b) *Proof.* Take any matrix $A = (a_{ij}) \in \mathbb{M}_n(K)^0$, for off-diagonal entries, that is for $i, j \in \{1, \dots, n\}$ where $i \neq j$,

$$a_{ij}[E_{i1}, E_{1j}] = a_{ij}(\delta_{11}E_{ij} - \delta_{ij}E_{11}) = a_{ij}E_{ij}$$

For diagonal entries in A, when $i \in \{1, \dots, n-1\}$,

$$\begin{aligned} a_{ii}[E_{in}, E_{ni}] &= a_{ii}(\delta_{nn}E_{ii} - \delta_{ii}E_{nn}) \\ &= a_{ii}E_{ii} - a_{ii}E_{nn} \end{aligned}$$

Since $\mathrm{Tr}(A) = 0$,

$$\sum_{i=1}^{n} a_{ii} = 0 \implies a_{nn} = -\sum_{i=1}^{n-1} a_{ii}$$

Therefore

$$A = \sum_{i=1}^{n-1} a_{ii}[E_{in}, E_{ni}] + \sum_{i=1}^{n}\sum_{j\neq i} a_{ij}[E_{i1}, E_{1j}]$$

which means A belongs to subspace spanned by commutators.

Conversely given a finite indexing set I, take any matrix $C = \sum_{i\in I} c_i [A_i, B_i]$ where $(A_i, B_i)_{i\in I} \in \mathbb{M}_n(K), (c_i)_{i\in I} \in K$, then

$$\begin{aligned} \mathrm{Tr}(C) &= \mathrm{Tr}\left(\sum_{i\in I} c_i [A_i, B_i]\right) \\ &= \sum_{i\in I} c_i \mathrm{Tr}([A_i, B_i]) \\ &= \sum_{i\in I} c_i \mathrm{Tr}(A_iB_i - B_iA_i) \\ &= \sum_{i\in I} c_i (\mathrm{Tr}(A_iB_i) - \mathrm{Tr}(B_iA_i)) = 0 \end{aligned}$$

So $C \in \mathbb{M}_n(K)^0$. □

Question 2.

Proof. Take any arbitrary $X \in \mathbb{M}_n(K)^0 = \ker(\mathrm{Tr})$, from 1(b), we know X is a linear combination of commutators, so there exists a finite indexing set I, $(A_i, B_i)_{i \in I} \in \mathbb{M}_n(K), (c_i)_{i \in I} \in K$ such that

$$X = \sum_{i \in I} c_i \, [\, A_i, B_i \,] \,.$$

Then $X \in \ker(f)$, because

$$\begin{aligned} f(X) &= f\left(\sum_{i \in I} c_i \, [\, A_i, B_i \,]\right) \\ &= \sum_{i \in I} c_i f([\, A_i, B_i \,]) \\ &= \sum_{i \in I} c_i f(A_i B_i - B_i A_i) \\ &= \sum_{i \in I} c_i (f(A_i B_i) - f(B_i A_i)) = 0 \end{aligned}$$

So $\ker(\mathrm{Tr}) \subseteq \ker(f) \subseteq \mathbb{M}_n(K)$. Since $n > 0$ by assumption, application of rank-nullity theorem on Tr gives us $\mathrm{nullity}(\mathrm{Tr}) + 1 = n^2$, then counting dimensions, $n^2 - 1 \le \mathrm{nullity}(f) \le n^2$.

Case $\mathrm{nullity}(f) = n^2$, then f is the zero functional which uniquely determines that $f = 0 \, \mathrm{Tr}$.

Case $\mathrm{nullity}(f) = n^2 - 1 = \mathrm{nullity}(\mathrm{Tr})$, then by dimension of subspace, $\ker(\mathrm{Tr}) = \ker(f)$. Let $\{ \, u_1, \dots, u_{n^2-1} \, \}$ be a basis for $\ker(f)$, fix $v \in \mathbb{M}_n(K) \setminus \ker(f)$, then $\mathcal{B} := (u_1, \dots, u_{n^2-1}, v)$ is an ordered basis for $\mathbb{M}_n(K)$. Let $\mathcal{B}^\vee = (u_1^\vee, \dots, u_{n^2-1}^\vee, v^\vee)$ denote its dual basis. Now for any $x \in \mathbb{M}_n(K)$,

$$\begin{aligned} x &= \sum_{i=1}^{n^2-1} u_i^\vee(x) u_i + v^\vee(x) v \\ \mathrm{Tr}(x) &= v^\vee(x) \, \mathrm{Tr}(v) \\ f(x) &= v^\vee(x) f(v) \\ &= \frac{f(v)}{\mathrm{Tr}(v)} \, \mathrm{Tr}(x) \\ f &= \frac{f(v)}{\mathrm{Tr}(v)} \, \mathrm{Tr} \end{aligned}$$

as $v \notin \ker(\mathrm{Tr}) = \ker(f)$, $\mathrm{Tr}(v) \neq 0$ and $f(v) \neq 0$ so $c := \frac{f(v)}{\mathrm{Tr}(v)} \in K \setminus \{ \, 0 \, \}$ indeed exists.

Suppose there exists $c' \in K$ such that $f = c' \, \mathrm{Tr}$, evaluate at v, because $\mathrm{Tr}(v) \neq 0$,

$$\begin{aligned} f(v) = c \, \mathrm{Tr}(v) &= c' \, \mathrm{Tr}(v) \\ c &= c' \end{aligned}$$

completing the uniqueness proof. $\square$

Question 3.

Proof. $f^\vee : W^\vee \to V^\vee$ is defined as $f^\vee : x \mapsto x \circ f$. Take any $x \in \ker(f^\vee) \subseteq W^\vee$

$$\begin{aligned} & x \in \ker(f^\vee) \\ \iff & x \circ f = 0_{V^\vee} \\ \iff & \forall v \in V.\ (x \circ f)(v) = x(f(v)) = 0_K \\ \iff & \operatorname{Im}(f) \subseteq \ker(x) \end{aligned}$$

this means $\ker(f^\vee) = \{\, x \in W^\vee : x(\operatorname{Im}(f)) = \{0\} \,\} = \operatorname{Im}(f)^0$ the annihilator of $\operatorname{Im}(f)$. $\operatorname{Im}(f)$ is a subspace of W, so

$$\dim_K \operatorname{Im}(f) + \dim_K \operatorname{Im}(f)^0 = \dim_K W.$$ [1]

Rearranging terms gives us

$$\begin{aligned} \operatorname{nullity}(f^\vee) &= \dim_K \operatorname{Im}(f)^0 \\ &= \dim_K W - \dim_K \operatorname{Im}(f) \end{aligned}$$

by rank-nullity theorem on $f^\vee$,

$$\dim_K W^\vee - \dim_K \operatorname{Im}(f^\vee) = \dim_K W - \dim_K \operatorname{Im}(f)$$

because W is finite-dimensional, $\dim_K W^\vee = \dim_K W$ and cancelling terms gives us that

$$\dim_K \operatorname{Im}(f^\vee) = \dim_K \operatorname{Im}(f)$$ □

[1] An elementary proof of this statement can be found in Chapter 3, Hoffman & Kunze. (Theorem 16)

Question 4.

(a) *Proof.* Let $\mathcal{B}$ be a basis for $U \subsetneq V$. For any arbitrary $v \in V \setminus U$, since $v \notin U = \operatorname{span}(\mathcal{B})$, $\mathcal{B} \cup \{v\}$ is linearly independent, from maximal principle, there exists a maximal linearly independent subset of V that contains $\mathcal{B} \cup \{v\}$, denoted $\mathcal{C}$, which is a basis for V.

From the universal property of basis, there exists a (irrelevantly unique) linear functional φ defined on $\mathcal{C}$ as

$$\begin{aligned} \varphi : V &\to K \\ v &\mapsto 1 \\ ☺ &\mapsto 0 \end{aligned}$$

where ☺ is refers to any element in $\mathcal{C} \setminus \{v\}$. It then arises from the definition that for any $u \in U = \operatorname{span}(\mathcal{B})$, $\varphi(u) = 0_K$, while $\varphi(v) = 1_K$. □

(b) *Proof.* Fix the field $\mathbb{Q}$. Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Let $\alpha \in \mathbb{R} \setminus \{-1\}$ be arbitrary.

Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We know that $\mathbb{Q}$ is a subspace of $\mathbb{R}$ with a standard $\mathbb{Q}$-basis $\{1\}$. Since $\alpha \notin \mathbb{Q}$, $\{1, \alpha\}$ is linearly independent, similarly to part (a), there exists a maximally linearly independent subset of $\mathbb{R}$ containing $\{1, \alpha\}$, denoted $\mathcal{H}$, which is a $\mathbb{Q}$-basis for $\mathbb{R}$. Then from the universal property of basis, there exists a linear functional f_0 defined on the basis $\mathcal{H}$ as

$$\begin{aligned} f_0 : \mathbb{R} &\to \mathbb{Q} \\ 1 &\mapsto 1 \\ \alpha &\mapsto -1 \\ ☺ &\mapsto 0 \end{aligned}$$

where ☺ matches any element in $\mathcal{H} \setminus \{1, \alpha\}$. Now define $f : \mathbb{R} \to \mathbb{R}$ as the inclusion map of f_0. Then f will still be $\mathbb{Q}$-linear. It is also clear by construction that $f(1) = 1$ and $f(\alpha) = -1$.

Conversely suppose $\alpha \notin \mathbb{R} \setminus \mathbb{Q}$, so $\alpha \in \mathbb{Q}$. Suppose for a contradiction there exists a $\mathbb{Q}$-linear map $f : \mathbb{R} \to \mathbb{R}$ with $f(1) = 1$ and $f(\alpha) = -1$, then by $\mathbb{Q}$-linearity, since $\alpha \in \mathbb{Q}$,

$$\begin{aligned} \alpha \cdot f(1) &= \alpha \cdot 1 \\ f(\alpha) &= \alpha = -1 \end{aligned}$$

This contradicts fact that $\alpha \in \mathbb{R} \setminus \{-1\}$. □

Question 5.

Synopsis. I will express, $\sum_{i=1}^{k} v_f(R_i)$ in terms of the corners of each rectangle, and by carefully accounting for terms, show that the terms corresponding to internal corners all cancel out, leaving the terms corresponding to the 4 outermost corners, which gives $v_f(R)$.

Notation. Suppose $G = (V, E)$ is a simple graph with vertices V and edges E, for any vertex $v \in V$, $\deg(v)$ will denote the degree of the vertex.

Proof. Suppose $R = [a, b) \times [c, d)$ is a disjoint union,

$$R = \bigsqcup_{i \in I} R_i$$

where the indexing set $I = \{ 1, \dots, k \}$, and for any $i \in I$,

$$R_i = [a_i, b_i) \times [c_i, d_i).$$

The using additive property of f ($\mathbb{Q}$-linearity), the f-area of R can be expressed as

$$\begin{aligned} v_f(R) &= f(b - a) \cdot f(d - c) \\ &= (f(b) - f(a))(f(d) - f(c)) \\ &= f(a)f(c) + f(b)f(d) - f(b)f(c) - f(a)f(d) \end{aligned}$$

On the other hand, for any $i \in I$, similarly

$$v_f(R_i) = f(a_i)f(c_i) + f(b_i)f(d_i) - f(b_i)f(c_i) - f(a_i)f(d_i)$$

Consider this tabulation of the terms in $\sum_{i \in I} v_f(R_i)$,

$$\begin{gathered} f(a_1)f(c_1) + f(b_1)f(d_1) - f(b_1)f(c_1) - f(a_1)f(d_1) \\ + \cdots + \\ f(a_k)f(c_k) + f(b_k)f(d_k) - f(b_k)f(c_k) - f(a_k)f(d_k) \end{gathered}$$

Notice that for each rectangle R_i, there is a correspondence between the terms in the sum and the corners of the rectangle, like this

- $+\ f(a_i)f(c_i)$ corresponds to the lower-left corner of R_i, (a_i, c_i),
- $+\ f(b_i)f(d_i)$ corresponds to the upper-right corner of R_i, (b_i, d_i),
- $-\ f(b_i)f(c_i)$ corresponds to the lower-right corner of R_i, (b_i, c_i),
- $-\ f(a_i)f(d_i)$ corresponds to the upper-left corner of R_i, (a_i, d_i).

Consider the graph $G = (V, E)$ formed by tracing the edges of all the rectangles $(R_i)_{i \in I}$. V, the vertices of the graph, are to be formed by the corners of these rectangles.

For any $v \in V$, due to how the graph is formed, there is only a few cases.

- $\deg(v) = 2$, this means the point in correspondence to v has to be one of $\{ (a, c), (a, d), (b, c), (b, d) \}$, and whichever point it is, it belongs to exactly one rectangle in $(R_i)_{i \in I}$. Hence in the expression for $\sum_{i \in I} v_f(R_i)$, terms that correspond to any of the 4 outermost corners will appear exactly **once** (for each corner).

- $\deg(v) = 3$, this means that v is a corner shared by 2 rectangles in $(R_i)_{i \in I}$. So there exists distinct rectangles $R_p, R_q \in (R_i)_{i \in I}$ such that one of the below holds,
  - v is the upper-right corner of R_p and upper-left corner of R_q, or
  - v is the lower-right corner of R_p and lower-left corner of R_q, or
  - v is the upper-right corner of R_p and lower-right corner of R_q, or
  - v is the upper-left corner of R_p and lower-left corner of R_q.

  In any case, in the expression for $\sum_{i \in I} v_f(R_i)$, the two terms that correspond to v will have opposite signs (and obviously the same value), therefore cancelling each other.

- $\deg(v) = 4$, this means that v is a corner that is shared by 4 rectangles in $(R_i)_{i \in I}$. So there exists distinct rectangles $R_p, R_q, R_r, R_s \in (R_i)_{i \in I}$ such that
  1. v is the lower-left corner of R_p, and
  2. v is the lower-right corner of R_q, and
  3. v is the upper-left corner of R_r, and
  4. v is the upper-right corner of R_s.

  Similarly to previous case, in the expression for $\sum_{i \in I} v_f(R_i)$, the four terms that correspond to v will cancel each other out.

Since rectangles have right-angled corners, G will only has vertices with degrees 2,3 or 4, so the enumeration was exhaustive. Because every term in the expression for $\sum_{i \in I} v_f(R_i)$ corresponds to a corner of a rectangle in $(R_i)_{i \in I}$, which in turn is related to a vertex in V, we can conclude that by accounting for every $v \in V$, we have accounted for every individual term in $\sum_{i \in I} v_f(R_i)$.

Therefore, the only terms that remain in $\sum_{i \in I} v_f(R_i)$ are those in correspondence with $\{ (a, c), (a, d), (b, c), (b, d) \}$. It is very clear which corner of its respective rectangle each point belongs to, hence we can derive the sign of each remaining term, therefore

$$\sum_{i \in I} v_f(R_i) = f(a)f(c) + f(b)f(d) - f(a)f(d) - f(b)f(c) = v_f(R). \qquad \square$$

Question 6.

Proof. Suppose a rectangle $R = [a, b) \times [c, d)$ has the ratio $x(R)/y(R) \in \mathbb{Q}$. Let $p, q \in \mathbb{N}, q \neq 0$ such that p/q is the canonical fractional representation of $x(R)/y(R)$.

$$\frac{x(R)}{y(R)} = \frac{p}{q} \implies \frac{b-a}{p} = \frac{d-c}{q}$$

and $p > 0$ because $a < b$ by assumption. Proceed to partition R into pq squares of side length $\lambda := \frac{b-a}{p} = \frac{d-c}{q}$, by defining, for all $i \in \{0, \dots, p-1\}$, for all $j \in \{0, \dots, q-1\}$,

$$S_{ij} := [a + i\lambda, a + (i+1)\lambda) \times [c + j\lambda, c + (j+1)\lambda).$$

It is very clear that each of S_{ij} is actually a square (of side length λ). Now for any $(i, j), (k, l) \in \mathbb{N}_{<p} \times \mathbb{N}_{<q}$, where $(i, j) \neq (k, l)$,

$$\begin{aligned} S_{ij} &= [a + i\lambda, a + (i+1)\lambda) \times [c + j\lambda, c + (j+1)\lambda) \\ S_{kl} &= [a + k\lambda, a + (k+1)\lambda) \times [c + l\lambda, c + (l+1)\lambda) \end{aligned}$$

and because $i \neq k$ or $j \neq l$, $S_{ij} \cap S_{kl} = \emptyset$. Hence $(S_{ij})_{(i,j) \in \mathbb{N}_{<p} \times \mathbb{N}_{<q}}$ is a pairwise disjoint family of squares. Additionally, from associativity of set union,

$$\begin{aligned} \bigsqcup_{(i,j) \in \mathbb{N}_{<p} \times \mathbb{N}_{<q}} S_{ij} &= \bigsqcup_{i \in \mathbb{N}_{<p}} \bigsqcup_{j \in N_{<q}} S_{ij} \\ &= \bigsqcup_{i \in \mathbb{N}_{<p}} \bigsqcup_{j \in N_{<q}} [a + i\lambda, a + (i+1)\lambda) \times [c + j\lambda, c + (j+1)\lambda) \end{aligned}$$

in the inner $\bigsqcup$, i is fixed, then because $\times$ distributes over $\cup$ ($\sqcup$ in this case),

$$\begin{aligned} \bigsqcup_{(i,j) \in \mathbb{N}_{<p} \times \mathbb{N}_{<q}} S_{ij} &= \bigsqcup_{i \in \mathbb{N}_{<p}} [a + i\lambda, a + (i+1)\lambda) \times [c, c + q\lambda) \\ &= [a, a + p\lambda) \times [c, c + q\lambda) \end{aligned}$$

recall from our definition that $\lambda = \dfrac{b-a}{p} = \dfrac{d-c}{q}$, therefore

$$\bigsqcup_{(i,j) \in \mathbb{N}_{<p} \times \mathbb{N}_{<q}} S_{ij} = [a, b) \times [c, d) = R.$$

This concludes that R can be partitioned into a finite disjoint union of pq smaller squares.

Conversely, suppose a rectangle R is partitioned into a finite disjoint union, where $k \in \mathbb{N}$, $S_1, \dots, S_k \subseteq R$ are squares such that

$$R = \bigsqcup_{i=1}^{k} S_i.$$

Suppose for a contradiction the side length ratio $x(R)/y(R)$ does not lie in $\mathbb{Q}$. With possible scaling, we can assume without loss of generality that $y(R) = 1$, so $\alpha := x(R) \in \mathbb{R} \setminus \mathbb{Q}$. Since the side length is strictly positive, $\alpha > 0$, so in particular $\alpha \neq -1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be the $\mathbb{Q}$-linear map shown to exist in question 4(b), with property that $f(1) = 1$ and $f(\alpha) = -1$. From question 5, the f-area is additive, so

$$\begin{aligned} R &= \bigsqcup_{i=1}^{k} S_i \\ v_f(R) &= v_f\left(\bigsqcup_{i=1}^{k} S_i\right) \\ v_f(R) &= \sum_{i=1}^{k} v_f(S_i) \\ f(\alpha) \cdot f(1) &= \sum_{i=1}^{k} f(x(S_i)) \cdot f(y(S_i)) \end{aligned}$$

because S_i is a square with $x(S_i) = y(S_i)$,

$$\begin{aligned} -1 &= \sum_{i=1}^{k} f(x(S_i))^2 > 0 \\ -1 &> 0 \end{aligned}$$

which is a contradiction. Hence the side length ratio $x(R)/y(R)$ must be rational. □