Models of Set Theory without Choice

Final Presentation

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Slides

https://m5th.b0ss.net/MA4199/slides-final.pdf

Recap

Axiom of Choice

$$\forall A\ \exists R\,(R \text{ well-orders } A)\,.$$

"Useful" consequences of choice

- comparability of cardinalities
- every vector space has a basis
- product of compact space is compact
- ...

Less "nice" consequences of choice

- Banach-Tarski paradox
- existence of non-Lebesgue measurable set

Constructible Universe

Gödel (1938) produced a model of ZFC from ZF, showing that

$$\mathrm{Con}(\mathsf{ZF}) \to \mathrm{Con}(\mathsf{ZFC}).$$

Fraenkel-Mostowski Model

Working in ZF set theory with atoms, Fraenkel (1922) produced a model where choice fails, the methods were refined by Mostowski (1938). By permuting the atomic members in the universe, they showed that

$$\mathrm{Con}(\mathsf{ZFA}) \to \mathrm{Con}(\mathsf{ZFA}) + \neg\mathsf{Choice}.$$

Cohen Model

Cohen (1963) invented forcing and showed that

$$\mathrm{Con}(\mathsf{ZF}) \Rightarrow \mathrm{Con}(\mathsf{ZF}) + \neg\mathsf{Choice}$$

which completes the independence argument.

In addition, in Cohen second model, there is a countable family of pairs without choice function, showing that

$$\mathrm{Con}(\mathsf{ZF}) \Rightarrow \mathrm{Con}(\mathsf{ZF}) + \neg\mathsf{Countable\ Choice}.$$

Solovay Model

Result

Theorem (Solovay)**.** Suppose there is a model of ZFC with an inaccessible cardinal, then there is a model of ZF in which Choice fails, and all subsets of reals are Lebesgue-measurable.

Overview

- Start with ground model M where $M \vDash \kappa$ is inaccessible.
- Force with Lévy collapse.
- Look at inner-model of all objects definable from countable sequence of ordinals.
- Random real forcing is related to membership in a null set.
- Any countable-ordinal-definable subset of reals $A \subseteq \mathbb{R}$ is witnessed by a countable sequence of ordinals s.
- Perform random-real forcing over $M[s]$ and use the information to produce a Borel set X such that $X \triangle A$ is measure zero.

Lévy Collapse

Definition

Definition. For any $S \subseteq$ **ORD**, the *Lévy collapsing order* for S is

$$\mathrm{Lv}(S) = \left\{ \begin{aligned} p : p &\text{ is a function} \\ &\wedge |p| < \omega \\ &\wedge \operatorname{dom}(p) \subseteq S \times \omega \\ &\wedge \forall (\alpha, n) \in \operatorname{dom}(p)\, [p(\alpha, n) = 0 \vee p(\alpha, n) \in \alpha] \end{aligned} \right\}$$

ordered by reverse inclusion.

Properties of Lévy Collapse

1. $\mathrm{Lv}(S)$ is **almost homogeneous**, for any p, q, there is an automorphism π on $\mathrm{Lv}(S)$ such that $\pi(p)$ is compatible with q.
2. If $p \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$, then $1_{\mathbb{P}} \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$; thus, either $1_{\mathbb{P}} \Vdash \varphi(\check{x}_1, \dots, \check{x}_n)$ or $1_{\mathbb{P}} \Vdash \neg\varphi(\check{x}_1, \dots, \check{x}_n)$.
3. Suppose that $S = X \cup Y$ is a disjoint union, and set $\mathbb{P}_0 = \mathrm{Lv}(X)$ and $\mathbb{P}_1 = \mathrm{Lv}(Y)$. Then G is $\mathrm{Lv}(S)$-generic iff G is of the form

$$G = \{p \cup q : p \in G_0 \wedge q \in G_1\}$$

where G_0 is $\mathbb{P}_0$-generic, $(\mathbb{P}_1)^{M[G_0]} = \mathbb{P}_1$, and G_1 is $\mathbb{P}_1$-generic over $M[G_0]$.

Properties of Lévy Collapse

4. When κ is regular and countable, $\mathrm{Lv}(\kappa)$ has κ-c.c., forcing with it preserves all cardinals $\geq \kappa$.
5. Suppose κ is regular, $\mathrm{Lv}(\kappa)$ has the κ-c.c., and G is $\mathrm{Lv}(\kappa)$-generic.
   For any $s \in M[G]$ with $s : \gamma \to$ **ORD** where $\gamma < \kappa$, there is a $\delta < \kappa$ such that $s \in M[G \cap \mathrm{Lv}(\delta)]$.

Intuition. Forcing with $\mathrm{Lv}(\kappa)$ "gently" collapses all cardinals strictly between ω and κ, which causes $M[G] \vDash \kappa = \omega_1$.

"Gentle" in the sense that any $< \kappa$ sequence of ordinals is introduced at some earlier stage of the collapse.

Properties of Lévy Collapse

6. Suppose $M \vDash |\mathbb{P}| \leq |\alpha|$ and
$$1_{\mathbb{P}} \Vdash \exists f\,(f : \omega \to \alpha \text{ is onto})\,.$$
Then there is a dense subset $D_\alpha \subseteq \mathrm{Lv}(\{\alpha\})$ and an injective embedding $D_\alpha \to \mathbb{P}$ whose image is dense.

Intuition. $\mathrm{Lv}(\{\kappa\})$ is a way to collapse κ into ω, this is a partial converse. If $\mathbb{P}$ collapses α and renders it countable, then forcing with $\mathbb{P}$ is equivalent to forcing with $\mathrm{Lv}(\{\alpha\})$.

An important lemma

Characterisation of $M[G]$, it allows us to "enlarge" the ground model to "absorb" any countable sequence of ordinals. More precisely,

Lemma.

Suppose $\kappa > \omega$ is regular and G is $\mathrm{Lv}(\kappa)$-generic. For any $s : \omega \to \mathbf{ORD}$, $s \in M[G]$, there is a $\mathrm{Lv}(\kappa)$-generic filter H over $M[s]$ such that $M[G] = M[s][H]$.

Proof of important lemma

- Split G as follows, where δ is given by property 5,

$$G_0 = G \cap \mathrm{Lv}(\delta),$$
$$G_1 = G \cap \mathrm{Lv}(\{\delta\}),$$
$$G_2 = G \cap \mathrm{Lv}(\kappa \setminus (\delta + 1)).$$

  and $s \in M[G_0]$.

- By intermediate model property, $M[G_0] = M[s][H_0]$ where $\mathbb{P} \in M[s]$ and H_0 is generic over $\mathbb{P}$.

- Let $\mathbb{Q} = \mathbb{P} \times \mathrm{Lv}(\{\delta\}) \in M[s]$, by product forcing, $M[s][H_0][G_1]$ is $\mathbb{Q}$-generic over $M[s]$. Now

$$|\mathbb{Q}| \leq |\mathrm{Lv}(\delta) \times \mathrm{Lv}(\{\delta\})| = |\mathrm{Lv}(\delta + 1)| = |\delta|$$

  and since $\mathbb{Q}$ contains a copy of $\mathrm{Lv}(\{\delta\})$,

$$1_{\mathbb{Q}} \Vdash \exists f \, (f : \omega \to \delta \text{ is onto}) \, .$$

- Applying property 6, in $M[s]$ forcing with $\mathbb{Q}$ is equivalent to forcing with $\mathrm{Lv}(\{\delta\})$, so there is a $\mathrm{Lv}(\{\delta\})$-generic H_1 such that

$$M[s][H_1] = M[s][H_0][G_1].$$

- As $1_{\mathrm{Lv}(\delta+1)} \Vdash \exists f\,(f : \omega \to \delta$ is onto$)$, the same argument gives a $\mathrm{Lv}(\delta+1)$-generic (over $M[s]$) filter H_2 such that

$$M[s][H_2] = M[s][H_1].$$

- Now we can repeatedly apply property 3 to get

$$\begin{aligned} M[G] = M[G_0][G_1][G_2] &= M[s][H_0][G_1][G_2] \\ &= M[s][H_1][G_2] = M[s][H_2][G_2] \end{aligned}$$

and it happens that $H_2 \cup G_2$ is of the form in property 3, and thus generic over $M[s]$.

Random Real Forcing

Definitions

- "Reals" denote 2^ω the Cantor space, definitions and results apply to any other standard space (like ω^ω and real line), mutatis mutandis.
- For a binary string $s \in 2^{<\omega}$, denote $O(s) = \{f \in 2^\omega : f \supseteq s\}$. $\{O(s) : s \in 2^{<\omega}\}$ is a base for the topology on 2^ω.
- For each $s \in 2^{<\omega}$, the Lebesgue measure of its corresponding *basic open set* is the coin-flip measure,

$$m_L(O(s)) = \frac{1}{2^{|s|}}.$$

- We fix a "nice" enumeration $\langle \mathbf{s}_i : i \in \omega \rangle$.
- Force with $\mathcal{B}^* = \{X : X \text{ is a non-null Borel set}\}$, ordered by inclusion.

Coding Borel Sets

Definition.

Let each $c : \omega \to \omega$ can encodes a Borel set

$$A_c = \begin{cases} \bigcup \{O(\mathbf{s}_i) : c(i+1) = 1\} & \text{if } c(0) = 0, \\ 2^\omega \setminus \bigcup \{O(\mathbf{s}_i) : c(i+1) = 1\} & \text{if } c(0) = 1, \\ \bigcap_{n \in \omega} \bigcup \{O(\mathbf{s}_i) : c(2^n 3^{i+1}) = 1\} & \text{otherwise.} \end{cases}$$

Properties of Coding scheme

i. If $c(0) = 0$, A_c is open;
ii. if $c(0) = 1$, A_c is closed;
iii. if $c(0) > 1$, A_c is G_δ.
iv. Each open, closed and G_δ set is indexed by some code c.
v. The following notions are absolute for M:

$$A_c \quad ; \quad A_c = 0 \quad ; \quad A_c \subseteq A_d \quad ; \quad A_c \subseteq (2^\omega \setminus A_d) \quad ; \quad A_c \cap A_d.$$

vi. The Lebesgue measure is absolute in the sense that for any code $c \in M$,

$$m_L^M(A_c^M) = m_L(A_c).$$

Characterizations of $\mathcal{B}^*$-forcing

First characterization. $\mathcal{B}^*$-forcing is akin to adding a single real.

Theorem.

Suppose G is $\mathcal{B}^*$-generic, then there is a unique real $x \in 2^\omega$ such that for any closed code $c \in M$,

$$x \in A_c^{M[G]} \text{ iff } A_c^M \in G,$$

and $M[x] = M[G]$.

Proof

- Work in $M[G]$, there is a unique real $x \in 2^\omega$ specified by

$$\{x\} = \bigcap \left\{ A_c^{M[G]} : c \in M \text{ is a closed code and } A_c^M \in G \right\}.$$

The intersection is nonempty due to G being a filter.
The result is a singleton because for each $n \in \omega$, this set is dense

$$\{C \in \mathcal{B}^* : C \text{ is closed} \wedge (\exists k \in 2)\,(\forall f \in C)\,(f(n) = k)\}.$$

- For closed code $c \in M$, if $A_c^M \in G$ then $x \in A_c^{M[G]}$ follows from definition of x.

Proof

- Conversely suppose $x \in A_c^{M[G]}$, need to show $A_c^M \in G$. We just check that A_c^M meets each closed set in G as the following set is dense,

$$\{C \in \mathcal{B}^* : C \text{ is closed} \wedge (C \subseteq A \vee C \cap A = 0)\}.$$

- Let d be a closed code with $A_d^M \in G$,
by definition of x, $x \in A_c^{M[G]} \cap A_d^{M[G]}$
by absoluteness, $x \in A_c^M \cap A_d^M$
so A_c^M meets every closed set in G.

Proof

We already defined x from G, to show $M[x] = M[G]$ we recover G from x,

$$G = \left\{p \in \mathcal{B}^* : \exists c \left(c \in M \text{ is closed code} \wedge x \in A_c^{M[x]} \wedge A_c^M \subseteq p\right)\right\}$$

Characterizations of $\mathcal{B}^*$-forcing

The x that was added is known as *random real*.

Second characterization. A random real avoids all null sets.

Intuition. Think of Cantor space 2^ω as a universal probability space, then a random element would be one that avoids all probability zero (null) sets.

Theorem

A real x is random over M iff $x \notin A_c$ for any $c \in M$ which encodes a G_δ null set.

Proof ($\Rightarrow$)

- Let x be random and G be $\mathcal{B}^*$-generic such that $M[x] = M[G]$.
- Let c be a G_δ code for a null set, need $x \notin A_c$.
- Write down the dense set
  $D = \{C \in \mathcal{B}^* : C \text{ closed} \wedge C \subseteq (2^\omega \setminus A_c)\}$.
- $G \cap D \neq 0$ so let $d \in M$ be a closed code such that
  $A_d^M \in G \cap D$.
- By definition of x, $x \in A_d^{M[G]}$ so by absoluteness $x \in A_d$ and
  $x \notin A_c$

Proof ($\Leftarrow$)

- Suppose for all G_δ codes for null set $c \in M$, $x \notin A_c$.
- Let G be recovered from x in manner of previous theorem. We show G is $\mathcal{B}^*$-generic over M
- Let $D \in M$ be a dense set, by definition of G it suffices to show there exists $p \in D$ and a closed code $c \in M$ satisfying $x \in A_c$ and $A_c^M \subseteq p$.
- Let $\mathcal{A} \subseteq \{C : C \text{ closed} \wedge \exists p \in D\,(C \subseteq p)\}$ be maximal antichain in $\mathcal{B}^*$.
- As $\mathcal{B}^*$ algebra satisfies c.c.c., $|\mathcal{A}| \leq \omega$, let $\langle c_n : n \in \omega \rangle$ such that

$$\left\langle A_{c_n} : n \in \omega \right\rangle \text{ enumerates } \mathcal{A}.$$

Proof ($\Leftarrow$)

- Now define c,

$$c(2^n 3^{i+1}) = 1 \text{ iff } O(\mathbf{s}_i) \cap A_{c_n} = 0.$$

- For each $n \in \omega$, A_c includes all basic open sets that avoids A_{c_n}, so

$$A_c = \bigcap_{n \in \omega} \left(2^\omega \setminus A_{c_n}\right) = 2^\omega \setminus \left(\bigcup_{n \in \omega} A_{c_n}\right).$$

- A_c is null by maximality of $\mathcal{A}$.
- By hypothesis, $x \notin A_c$, then $x \in A_{c_n}$ for some closed code $c_n \in M$.

Solovay Model

Definitions

- Assume $\kappa \in M$ and $M \vDash \kappa$ is an inaccessible cardinal.
- A set X is *definable from a countable sequence of ordinals* iff for some $s : \omega \to \mathbf{ORD}$ and some formula $\varphi(x_1, x_2)$,

$$y \in X \text{ iff } \varphi(s, y).$$

Denote as $X \in \mathbf{COD}$.

Proposition.

The collection of all hereditarily ω-ordinal definable sets $\mathbf{HCOD} = \{X : \mathrm{trcl}(X) \subseteq \mathbf{COD}\}$ is an inner model of ZF.

Key Lemma

This result is built on the previously-established properties of the Lévy collapse.

Lemma.

For each formula $\varphi(v)$, there is a $\tilde{\varphi}(v)$ such that for any $s : \omega \to \mathbf{ORD}, s \in M[G]$,

$$M[G] \vDash \varphi(s) \text{ iff } M[s] \vDash \tilde{\varphi}(s).$$

Proof of Key Lemma

- For any $s \in \mathbf{ORD}^\omega \cap M[G]$, by "important lemma", there is a $\mathbb{P} = \mathrm{Lv}(\kappa)$-generic filter H over $M[s]$ such that $M[G] = M[s][H]$.
- Apply property 2 of Lévy collapse (taking ground model to be $M[s]$), we have

$$M[s][H] \vDash \varphi(s) \text{ iff } M[s] \vDash \ulcorner 1_{\mathbb{P}} \Vdash \varphi(\check{s}) \urcorner .$$

- As forcing is definable in the ground model, take $\tilde{\varphi}$ to be the statement that encapsulates the forcing assertion.

Solovay Theorem

Theorem.

Let G be $\mathrm{Lv}(\kappa)$-generic, then in $M[G]$, each subset of reals definable from a countable sequence of ordinals is Lebesgue measurable.

In particular, there is a **Solovay model** N, with $M \subseteq N \subseteq M[G]$, containing precisely $\mathbf{HCOD}^{M[G]}$ where every subset of reals is Lebesgue measurable.

Proof

Work in $M[G]$.

- Let $A \subseteq 2^\omega$ be a ω-ordinal definable subset of reals, then for some $s : \omega \to \mathbf{ORD}$ and formula $\varphi(v_1, v_2)$,

$$x \in A \text{ iff } \varphi(s, x).$$

- By Key Lemma, we have $\tilde{\varphi}(v_1, v_2)$ such that

$$x \in A \text{ iff } M[s][x] \models \tilde{\varphi}(s, x).$$

Proof

- By 2nd characterization of $\mathcal{B}^*$-forcing over $M[s]$,

$$\{x \in 2^\omega : x \text{ is not random over } M[s]\} \\ = \bigcup \{A_c : c \in M[s] \text{ is a } G_\delta \text{ code for a null set}\}.$$

- RHS is countable union of null sets as κ remains inaccessible in $M[s]$ (due to property 5), and $\omega^\omega \cap M[s]$ is countable.
- **Objective.** Find Borel set X such that $A \triangle X$ only has non-random reals.

Proof.

- Now for $\mathcal{B}^*$-forcing argument over $M[s]$, let $\mathring{r}$ denote the canonical name for the random real.
- Let $\mathcal{Y} \subseteq \{C : C \text{ closed} \wedge (C \Vdash \tilde{\varphi}(\check{s}, \mathring{r}) \vee C \Vdash \neg\tilde{\varphi}(\check{s}, \mathring{r}))\}$ be maximal antichain in $(\mathcal{B}^*)^{M[s]}$.
- If x is random over $M[s]$, $x \in A$ iff $M[s][x] \vDash \tilde{\varphi}(s, x)$ iff $p \Vdash \tilde{\varphi}(\check{s}, \mathring{r})$ for some $p \in H$ and $(\mathcal{B}^*)^{M[s]}$-generic H.
- H meets some $q \in \mathcal{Y}$ which by first characterization gives

$$x \in A \text{ iff } x \in \bigcup \left\{ A_c : A_c^{M[s]} \in \mathcal{Y} \wedge A_c^{M[s]} \Vdash \tilde{\varphi}(\check{s}, \mathring{r}) \right\}.$$

- By c.c.c. of $\mathcal{B}^*$, $\mathcal{Y}$ is countable, so the expression on the right is in fact a Borel set, which witnesses X.

Thank you