

## MA2104 Assignment 2

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18th February 2018

### Question 1

(a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^3}{x^2 + 8y^6}.$

*Solution.* First take limit along the line  $y = 0$ ,

$$\begin{aligned} \lim_{(x,0) \rightarrow (0,0)} \frac{2x(0)^3}{x^2 + 8(0)^6} &= \lim_{x \rightarrow 0} \frac{0}{x^2} \\ &= 0 \end{aligned}$$

Next, take the limit along the curve  $x = y^3$ ,

$$\begin{aligned} \lim_{(y^3,y) \rightarrow (0,0)} \frac{2(y^3)y^3}{(y^3)^2 + 8y^6} &= \lim_{y \rightarrow 0} \frac{2y^6}{9y^6} \\ &= \frac{2}{9} \end{aligned}$$

Hence the limit does not exist. ■

(b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + 4x^2 + 2y^2}{2x^2 + y^2}.$

*Solution.*

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + 4x^2 + 2y^2}{2x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{2x^2 + y^2} + 2$$

Now it remains to compute  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{2x^2 + y^2}$ , note that

$$\begin{aligned} y^2 &\geq 0 \\ 2x^2 + y^2 &\geq 2x^2 \geq 0 \\ \frac{x^3}{2x^2 + y^2} &\leq \left| \frac{x^3}{2x^2 + y^2} \right| \leq \frac{|x^3|}{2x^2} = \frac{|x|}{2} \end{aligned}$$

and similarly

$$-\frac{|x|}{2} \leq -\left| \frac{x^3}{2x^2 + y^2} \right| \leq \frac{x^3}{2x^2 + y^2}$$

Since  $\lim_{(x,y) \rightarrow (0,0)} -\frac{|x|}{2} = \lim_{(x,y) \rightarrow (0,0)} \frac{|x|}{2} = 0$ , by Squeeze theorem,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{2x^2 + y^2} = 0,$$

then we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + 4x^2 + 2y^2}{2x^2 + y^2} = 2. \quad \blacksquare$$

## Question 2

(i)

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \\ &= f_u(u, v)(2xy) + f_v(u, v)(0) \\ &= 2xyf_u(u, v) \end{aligned} \quad \blacksquare$$

(ii)

$$\begin{aligned} \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) &= \frac{\partial}{\partial y} (2xyf_u(u, v)) \\ &= 2x \left( y \frac{\partial}{\partial y} (f_u(u, v)) + f_u(u, v) \right) \end{aligned}$$

Note that

$$\begin{aligned} \frac{\partial}{\partial y} (f_u(u, v)) &= f_{uu}(u, v) \frac{\partial u}{\partial y} + f_{uv}(u, v) \frac{\partial v}{\partial y} \\ &= x^2 f_{uu}(u, v) + 2y f_{uv}(u, v) \end{aligned}$$

So

$$\begin{aligned}\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) &= 2x (x^2 y f_{uu}(u, v) + 2y^2 f_{uv}(u, v) + f_u(u, v)) \\ &= 2x^3 y f_{uu}(u, v) + 4xy^2 f_{uv}(u, v) + 2x f_u(u, v)\end{aligned}$$

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### Question 3

*Solution.* Let  $f(x, y) = 2x^2 + y^2$  be the ellipse the insect moves on, then

$$\nabla f(x, y) = \langle 4x, 2y \rangle$$

$$\nabla f(1, 1) = \langle 4, 2 \rangle$$

By inspection, a unit tangent vector to  $f$  at  $(1, 1)$  will be

$$\mathbf{u} := \frac{1}{2\sqrt{5}} \langle 2, -4 \rangle = \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle.$$

Now compute  $\nabla T(1, 1)$ ,

$$\nabla T(x, y) = \langle 6x - 2y, -2x \rangle$$

$$\nabla T(1, 1) = \langle 4, -2 \rangle$$

Taking the directional derivative

$$\begin{aligned}D_{\mathbf{u}}T(1, 1) &= \nabla T(1, 1) \cdot \mathbf{u} \\ &= \langle 4, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle \\ &= \frac{8}{\sqrt{5}} \text{ } ^\circ\text{C cm}^{-1}\end{aligned}$$

Then by chain rule, rate of change of temperature will be

$$\frac{24}{\sqrt{5}} \text{ } ^\circ\text{C s}^{-1}.$$

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### Question 4

If  $\mathbf{r}(t) \neq \mathbf{0}$ , show that  $\frac{d}{dt} |\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t)$ .

*Proof.* Suppose  $\mathbf{r}(t) \neq \mathbf{0}$ ,

$$\frac{d}{dt} |\mathbf{r}(t)| = \frac{d}{dt} \left( \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} \right)$$

by chain rule outside and product rule inside,

$$\begin{aligned} &= \frac{1}{2\sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}} \cdot (\mathbf{r}(t)\mathbf{r}'(t) + \mathbf{r}'(t)\mathbf{r}(t)) \\ &= \frac{2\mathbf{r}(t)\mathbf{r}'(t)}{2\sqrt{|\mathbf{r}(t)|^2}} \\ &= \frac{1}{|\mathbf{r}(t)|} \mathbf{r}(t) \cdot \mathbf{r}'(t) \end{aligned}$$

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