CO 330 Combinatorial Enumeration

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Contents

Notation	Definition
$\mathbb{N}$	natural numbers $\{0, 1, 2, \dots\}$
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathcal{C}$	combinatorial class
c_n	counting sequence
C	generating function
$[x^n]f(x)$	coefficient of x^n in $f(x)$

1 Combinatorial Classes

Def Combinatorial Class: A set of objects $\mathcal{C}$ together with a weight function $w : \mathcal{C} \to \mathbb{N}$ (also known as a size function) such that for any $n \in \mathbb{N}$ the set

$$\mathcal{C}_n = \{\sigma \in \mathcal{C} : w(\sigma) = n\}$$

containing the elements of weight n is finite. The set $\mathcal{C}$ itself can be infinite.

Def Subclass: A subclass of a class $(\mathcal{C}, w)$ is any class whose objects form a subset of $\mathcal{C}$ and whose weight function is still w.

Def Counting Sequence: If $\mathcal{C}$ is a combinatorial class then the counting sequence (c_n) of $\mathcal{C}$ is the sequence where $c_n = |\mathcal{C}_n|$ equals the number of elements in $\mathcal{C}$ with weight n. The generating function of $\mathcal{C}$ is the series

$$C(x) = \sum_{n \geq 0} c_n x^n$$

Strings: Let Ω be a non-empty finite set. For any positive integer n the words or strings of length n over the alphabet Ω form the set

$$\Omega^n = \{(w_1, \ldots, w_n) : w_j \in \Omega\}$$

of tuples of length n whose elements come from Ω, and we let $\Omega^0 = \{\varepsilon\}$ be the set containing the empty string ε.

The words or strings over the alphabet Ω form the set

$$\Omega^* = \bigcup_{n \geq 0} \Omega^n$$

define this set a combinatorial class by defining the weight of a word to be its length.

Enumerating the combinatorial class $\mathcal{C} = \Omega^*$ is straightforward: an element of weight n is defined by picking n elements in Ω, so the counting sequence satisfies $c_n = |\Omega|^n$ and the generating function is always the geometric series

$$C(x) = \sum_{n \geq 0} |\Omega|^n x^n = \frac{1}{1 - |\Omega| x}$$

Subsets: Let Ω be a finite set with n elements. The subsets of Ω are the sets (including the empty set) whose elements come from Ω.

Fix a positive integer n and $[n] = \{1, 2, \ldots, n\}$. The class $\mathcal{C}^{[n]}$ of subsets of $[n]$ consists of all subsets of $[n]$ with weight defined by the number of elements in the subset.
Enumerating $\mathcal{C}^{[n]}$ is straightforward: there are $\binom{n}{k}$ ways to select k elements of $[n]$ for a subset of size k so $c_k^{[n]} = \binom{n}{k}$ and the generating function for this class is the polynomial

$$C^{[n]}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

Permutations: The permutations of Ω are the length n tuples containing each element of Ω exactly once.

The set $\mathcal{S}_n$ of permutations of length n consists of the permutations of $[n]$ and the class of permutations $\mathcal{S}$ consists of all permutations of all lengths with the weight function mapping a permutation to its length.

It is common to use Greek letters like π and σ for elements in $\mathcal{S}$. There are four ways to view a permutation $\pi \in \mathcal{S}_n$, all of which are equivalent

- One-line notation: The element π is expressed as the tuple $(\pi_1 \cdots \pi_n)$.
- Two-line notation: The element π is expressed as a table with two rows. The first row contains the numbers $[n]$ in order and the second contains the tuple π.
  E.g. $\pi = (35412)$

i	1	2	3	4	5
π_i	3	5	4	1	2

- As a Function: The element π is viewed as a function from $[n]$ to $[n]$ that takes $i \in [n]$ to the value $\pi_i \in [n]$.

  Under this interpretation $\mathcal{S}_n$ contains bijections from $[n]$ to itself and forms a permutation group of size n.
- Cycle Notation: If we consider π as a function then we can start with 1 and keep applying π to get a sequence of numbers $1, \pi(1), \pi(\pi(1)), \ldots$. Because π is a permutation, this sequence will eventually repeat, giving the cycle of π containing 1. Repeating this with the other elements of $[n]$ allows us to break π into disjoint cycles, which we write as consecutive tuples.

Enumerating unrestricted permutations is straightforward: the counting sequences $c_n = n!$ and the generating function of permutations is the series

$$S(x) = \sum_{n \geq 0} n! x^n$$

Def Lattice Paths: Given a finite set of steps $\mathcal{S} \subset \mathbb{Z}^d$, a restricting region $\mathcal{R} \subset \mathbb{R}^d$, a starting point $\mathbf{p} \in \mathbb{Z}^d \cap \mathcal{R}$, and a terminal set $\mathcal{T} \subset \mathcal{R}$, the lattice path model taking steps in $\mathcal{S}$, starting at $\mathbf{p}$, restricted to $\mathcal{R}$ and ending in $\mathcal{T}$ is the set of all finite tuples $(s_1, \ldots, s_r) \in \mathcal{S}_r$ of any length such that

$$\mathbf{p} + s_1 + \cdots + s_k \in \mathcal{R}, (1 \leq k \leq r)$$

and $\mathbf{p} + s_1 + \cdots + s_r \in \mathcal{T}$. The valid sequences in $\mathcal{S}_r$ form the walks or paths of the model, and can be visualized by starting at $\mathbf{p}$ and concatenating the vectors $s_1, \ldots, s_r$ in order. The steps of a walk $(s_1, \ldots, s_r)$ are the elements s_i.

By default we make a lattice path model a combinatorial class by taking the weight of a walk to be its number of steps.

In the unrestricted case, when $\mathcal{R} = \mathbb{R}^d$, enumeration is straightforward: there are $|S|$ possibilities for each step, so there are $|S|^n$ walks of length n. As a consequence, the generating function of an unrestricted walk model is always rational. The generating function of a model restricted to a halfspace $\mathbb{R}_{>0} \times \mathbb{R}^{d-1}$ can be irrational, however a powerful enumerative technique called the kernel method allows one to show that halfspace models always have algebraic generating functions. In contrast, there are two-dimensional walks restricted to the quadrant $\mathbb{R}^2_{>0}$ with generating functions that are transcendental but D-finite, non-D-finite but D-algebraic, and even series that are not D-algebraic (such a series is sometimes called hypertranscendental).

Integer Compositions: The combinatorial class of (integer) compositions consists of all tuples of positive integers

$$\mathcal{C} = \{(k_1, \dots, k_r) : r, k_j \in \mathbb{N}_{>0}\}$$

where the weight of a tuple is its sum:

$$|(k_1, \dots, k_r)| = k_1 + \cdots + k_r$$

Thus, the number of objects c_n in $\mathcal{C}$ of weight n is the number of ways to write n as a sum of positive integers where the order of the summands matters. The elements k_j in a composition $(k_1, \dots, k_r) \in \mathcal{C}$ are the parts or summands of the composition, and the number r of elements is the length or number of summands of the composition.

Lemma 1 There are 2^{n-1} compositions of size n.

Integer Partitions: The class $\mathcal{P}$ of (integer) partitions consists of all non-increasing tuples of positive integers $(k_1, \dots, k_r)$, meaning $k_1 \geq \cdots \geq k_r$. Equivalently, partitions are compositions where tuples with the same elements (in a different order) are identified. As for compositions, we usually write partitions as an unsimplified sum.

There is no short explicit formula for partitions, but a simple infinite product representation for the generating function of integer partitions is

$$P(x) = \prod_{k \geq 1} \frac{1}{1 - x^k}$$

Graphs and Trees: A (finite simple) graph G is a pair of finite sets $G = (V, E)$ where V defines the vertices or nodes of G and E contains the edges of G, which are sets defined by two distinct vertices of G.

The degree of a vertex is the number of edges it is contained in. A cycle in a graph is a sequence of at least three vertices connected by distinct edges that start and end at the same vertex. A walk in a graph is a sequence of vertices with each consecutive pair in the sequence connected by an edge, and a graph G is connected if for any pair of vertices a and b there is a walk in G starting at a and ending at b.

A connected graph with no cycles is called a tree. A rooted graph (usually a rooted tree) is a graph where one vertex, called the root, is specially marked. The height of a vertex v in a rooted tree is one less than the minimal number of vertices in a walk from the root to v (the root itself has height zero). It is common to draw rooted trees with the root on the top, the vertices of height one under the root, the vertices of height two under the vertices of height one, etc. The vertices of height $h + 1$ that are connected by an edge to a vertex v of height h are called the children of v, and v is called the parent of these vertices. A (rooted) k-ary tree is a tree where each vertex has at most k children and a complete (rooted) k-ary tree is a tree where each vertex has either no children or exactly k children. We often call 2-ary trees binary trees.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that any pair of distinct vertices a and b form an edge in E_1 if and only if the images $f(a)$

and $f(b)$ form an edge in E_2. Rooted graphs are isomorphic when there is an isomorphism that sends the root in one graph to the root of another.

A labelled graph class is a combinatorial class defined by a set $\mathcal{G}$ of graphs with vertex sets of the form $V = [n]$ for $n \in \mathbb{N}$ together with a weight function, typically taken as the number of vertices, although other common weightings include number of edges or number of vertices of degree one (when this does define a combinatorial class). An unlabelled graph class is a combinatorial class defined by a set $\mathcal{G}$ of graphs with vertex sets of the form $V = [n]$ for $n \in \mathbb{N}$ where isomorphic graphs are identified, together with an appropriate weight function.

Lemma 2 There are $2^{\binom{n}{2}}$ labelled graphs with n vertices.

Finally, we note that we can view graphs both as abstract structures and as drawings of those structures in the plane. This comes up for us in the enumeration of different kinds of trees. A class of trees is called planar if the order of the children of each node matters (i.e., must be respected when considering two graphs isomorphic) and non-planar if the order of the children does not matter.
Note: Planar here does not mean no edges crossing. All trees are already this kind of planar (can be drawn with no edges crossing).

2 Bijections and Combinatorial Proofs

Def Bijection: Let A and B be sets. A function $f : A \to B$ is injective if $f(a) = f(a')$ implies $a = a'$. A function $f : A \to B$ is surjective if $\forall b \in B$, there is some $a \in A$ such that $f(a) = b$. A function is bijective if it is both injective and surjective.

Lemma 1 The function $f : A \to B$ is a bijection if and only if it has an inverse $g : B \to A$ such that

$$g(f(a)) = a, \forall a \in A$$

$$f(g(b)) = b, \forall b \in B$$

We write $g = f^{-1}$.

Common Ways to Prove Bijection:

1. Define a map $f : A \to B$ and then prove
   - that f is well-defined (i.e., that it actually takes the elements of A to elements of B)
   - that f is injective
   - that f is surjective
2. Define two maps $f : A \to B$ and $g : B \to A$, and prove
   - that both maps are well-defined
   - that $g(f(a)) = a$ for all $a \in A$
   - that $f(g(b)) = b$ for all $b \in B$

Bijection of Combinatorial Classes: A bijection of combinatorial classes $(A, |\cdot|_A)$ and $(B, |\cdot|_B)$ is a bijection $f : A \to B$ between the set of objects in each class that preserves size, meaning $\forall a \in A$, $|a|_A = |f(a)|_B$. This implies $a_n = b_n$.

Def Permutation Matrix: A permutation matrix of size n is an $n \times n$ matrix with entries in $\{0, 1\}$ where every row and every column has exactly one 1.

Def Multiset: A multiset of size n with t types is a sequence $(m_1, \ldots, m_t) \in \mathbb{N}^t$ with $m_1 + \cdots + m_t = n$.

Lemma 2 If $n, t \in \mathbb{N}$ with $t \geq 1$ then there are

$$\binom{n+t-1}{t-1}$$

multisets of size n with t types.

Combinatorial Proofs: Bijections and counting methods allow us to give combinatorial proofs of identities. To prove an identity of the form

$$\text{LHS} = \text{RHS}$$

there are two common approaches.

1. Find two sets A and B where it is easy to prove $|A| = \text{LHS}$ and $|B| = \text{RHS}$ and then prove that A and B are in bijection.
2. Count a single set A in two different ways, one which shows that $|A| = \text{LHS}$ and the other of which shows $|A| = \text{RHS}$.

Sequence	**Combinatorial Interpretation**
$n!$	permutations of size n
2^n	binary strings of length n, subsets of $[n]$
A^n	strings of length n on alphabet of size A
$\binom{n}{k}$	subsets of $[n]$ with k elements
$\binom{a+b}{b}$	lattice paths from $(0, 0)$ to (a, b) using $(1, 0)$ and $(0, 1)$
$\binom{n+t-1}{t-1}$	multisets of size n with t types
$A \times B$	Cartesian product of sets with sizes A and B
$A + B$	union of disjoint sets with sizes A and B

3 Formal Series and Generating Functions

Def Ring: A ring is a set R together with two operations $+$ and $\times$ that take any $a, b \in R$ and return elements $a + b$ and $a \times b$ in R, together with two special distinct elements $0, 1 \in R$ such that for all $a, b, c \in R$,

- $(a + b) + c = a + (b + c)$
- $a + b = b + a$
- $a + 0 = a$
- there exists $(-a) \in R$ such that $a + (-a) = 0$
- $(a \times b) \times c = a \times (b \times c)$
- $a \times 1 = 1 \times a = a$
- $a \times (b + c) = (a \times b) + (a \times c)$
- $(b + c) \times a = (b \times a) + (c \times a)$

Algebraic Properties:

- R is commutative if $a \times b = b \times a$ for all $a, b \in R$
- R has no zero divisors if $a \times b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$

Def Integral Domain: A commutative ring with no zero divisors is called an integral domain.

Def Field: A field is an integral domain where for any $a \in R$ with $a \neq 0$ there is another element $a^{-1} \in R$ (the multiplicative inverse of a) with $a \times a^{-1} = 1$.

Def Formal Power Series: Let R be an integral domain. The ring of formal power series with variable x and coefficients in R is the set $R[[x]]$ of formal expressions

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

with each $c_j \in R$. Given

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \ B(x) = \sum_{n=0}^{\infty} b_n x^n$$

in $R[[x]]$ we define addition and multiplication for these formal expressions by

$$A(x) + B(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

and

$$A(x)B(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n$$

We write $[x^n]A(x)$ for the coefficient a_n of A, and also use the notation $A(0)$ for the constant term.

Lemma 1 Let R be an integral domain. A formal power series $A(x) = \sum_{n \geq 0} a_n x^n$ has a multiplicative inverse in $R[[x]]$ if and only if its constant a_0 is invertible in R.

Def Formal Laurent Series: Let R be an integral domain. The ring of formal Laurent series with variable x and coefficients in R is the set

$$R((x)) = \left\{ \sum_{n=l}^{\infty} c_n x^n : l \in \mathbb{Z}, c_j \in R \right\}$$

Given

$$A(x) = \sum_{n=l}^{\infty} a_n x^n, \ B(x) = \sum_{n=m}^{\infty} b_n x^n$$

in $R((x))$ we define addition and multiplication for these formal expressions by

$$A(x) + B(x) = \sum_{n=\min(l,m)}^{\infty} (a_n + b_n) x^n$$

and

$$A(x)B(x) = \sum_{n=l+m}^{\infty} \left(\sum_{k=l}^{n-m} a_k b_{n-k} \right) x^n$$

where we set $a_n = 0$ if $n < l$ and $b_n = 0$ if $n < m$. Formal Laurent series are like formal power series, except they are allowed to have a finite number of terms with negative powers.

Corollary If K is a field then $K((x))$ is a field.

Generalized Binomial Theorem Let K be a field contained in the complex numbers. If $\alpha = \frac{r}{s}$ for $r, s \in \mathbb{N}$ with $s > 0$ then the equation $y^s = (1+x)^r$ has a solution

$$y(x) = (1+x)^{\alpha} = \sum_{k \geq 0} \binom{\alpha}{k} x^k$$

in $K[[x]]$, where we extend the definition of the binomial coefficients to $\alpha \in \mathbb{Q}$ by setting

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

The equation $y^s = (1+x)^r$ has at most s distinct solutions, obtained by mulitplying the series $y(x)$ by any of the numbers in the set $\{t \in K : t^s = 1\}$. We write $(1+x)^{\alpha}$ for the series solution $y(x)$ that beings with the constant 1.

Def Generating Function: The generating function or generating series $C(x)$ of a sequence (c_n) to be the formal power series with coefficients c_n, and the generating function of a combinatorial class $\mathcal{C}$ to be the formal power series $C(x)$ in $\mathbb{Q}[[x]]$ defined by its counting sequence.

$$C(x) = \sum_{\sigma \in \mathcal{C}} x^{|\sigma|}$$

where $|\cdot|$ is the size function for the combinatorial class $\mathcal{C}$.

Def Negative Binomial Theorem:

$$\sum_{n \geq 0} \binom{n+t-1}{t-1} x^n = \frac{1}{(1-x)^t}$$

By the generalized binomial theorem

$$\binom{-t}{n} = (-1)^n \binom{n+t-1}{t-1}$$

Formal Power Series as a Metric Space: Consider a sequence $F_0(x), F_1(x), \ldots$ of formal power series in $R[[x]]$. We say this sequence formally converges if for every $n \in \mathbb{N}$ there exists some $L \in \mathbb{N}$ and $c_n \in R$ such that all coefficients $[x^n]F_k(x) = c_n$ whenever $k \geq L$. In other words, a sequence of formal power series formally converges if for each n the coefficient of x^n eventually stabilizes to a fixed value. When this holds we write

$$\lim_{k\to\infty} F_k(x) = \sum_{n\geq 0} c_n x^n$$

Def Valuation: The valuation $\mathrm{val}(F)$ of a formal power series $F(x) = \sum_{n\geq 0} f_n x^n$ is the smallest natural number i such that $f_i \neq 0$.

Infinite Summation of Formal Power Series: We say that a sequence (F_k) of formal power series is formally summable if the sequence (S_N) defined by the partial sums

$$S_N = \sum_{k=0}^{N} F_k(x)$$

formally converges. When (F_k) is formally summable then we define

$$\sum_{k\geq 0} F_k(x) = \lim_{N\to\infty} \left(\sum_{k=0}^{N} F_k(x) \right)$$

If $A, B \in R[[x]]$ then we define the composition $A(B(x))$ by

$$A(B(x)) = \sum_{n\geq 0} a_n B(x)^n$$

Theorem Let $A, B \in R[[x]]$. If A has only a finite number of non-zero coefficients then $A(B(x))$ always exists. If $A(x)$ has an infinite number of non-zero coefficients then $A(B(x))$ exists if and only if $B(0) = 0$ (B has no constant term).

Def Formal Derivative: Let $F(x) = \sum_{n\geq 0} f_n x^n$ be an element of $R[[x]]$. The formal derivative of F is defined as the formal power series

$$F'(x) = \sum_{n\geq 0} n f_n x^{n-1} = \sum_{n\geq 0} (n+1) f_{n+1} x^n$$

Def Formal Integral: If the positive integers have multiplicative inverses in R (for instance, if R is a field) then the formal integral of F is defined as the formal power series

$$\int F(x) = \sum_{n\geq 0} \frac{f_n}{n+1} x^{n+1}$$

Commutation Rule: If a sequence (F_k) of formal power series is formally summable then

$$\left(\sum_{k\geq 0} F_k(x)\right)' = \sum_{k\geq 0} F_k'(x)$$

Product Rule: If $A(x)$ and $B(x)$ are formal power series then the formal derivative satisfies

$$(A(x)B(x))' = A'(x)B(x) + A(x)B'(x)$$

Chain Rule: If $A(x)$ and $B(x)$ are formal power series and $B(0) = 0$ then

$$A(B(x))' = A'(B(x))B'(x)$$

Def Formal Exponential: If the postive integers have multiplicative inverses in R then the formal exponential in $R[[x]]$ is defined to be the series

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$$

Def Formal Logarithm: If the postive integers have multiplicative inverses in R then the formal logarithm in $R[[x]]$ is defined to be the series

$$\log(1+x) = -\int \frac{1}{1+x} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n$$

We can extend the generalized binomial theorem to

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n = \exp(\alpha \log(1+x))$$

in $\mathbb{C}[[x]]$ for any $\alpha \in \mathbb{C}$.

Rules for Coefficient Extraction:

- **Addition**
$$[x^n](A(x) + B(x)) = [x^n]A(x) + [x^n]B(x)$$
- **Multiplication**
$$[x^n](A(x)B(x)) = \sum_{k=0}^{n} [x^k]A(x) \cdot [x^{n-k}]B(x)$$
- **Coefficient Scaling**
$$[x^n]F(\rho x) = \rho^n [x^n]F(x)$$
- **Monomial Multiplication**
$$[x^n]x^k F(x) = \begin{cases} [x^{n-k}]F(x) & (k \leq n) \\ 0 & k > n \end{cases}$$

- **Re-Indexing** If $a, b \in \mathbb{N}$ with $b > a$ then

$$\sum_{n=a}^{b} f_n x^n = \sum_{n=0}^{b-a} f_{n+a} x^{n+a}$$

  If $b = \infty$, then both series have upper bound ∞.

- **Generalized Binomial Theorem** If $\alpha \in \mathbb{C}$ then

$$(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n$$

- **Negative Binomial Theorem** If t is a positive integer and $n \in \mathbb{N}$ then

$$\binom{-t}{n} = (-1)^n \binom{n+t-1}{t-1} = (-1)^n \binom{n+t-1}{n}$$

  combined with the generalized binomial theorem

$$\frac{1}{(1-x)^t} = \sum_{n \geq 0} \binom{n+t-1}{t-1} x^n = \sum_{n \geq 0} \binom{n+t-1}{n} x^n$$

- **Use of Convergent Series** If K is a field contained in the complex numbers and a convergent series with coefficients in K is found to satisfy a polynomial equation (by quadratic formula, Taylor series, etc.) then the formal series defined by the coefficients satisfies the same polynomial equation in $K[[x]]$.

4 Combinatorial Constructions

Def Atomic Class: The atomic class $\mathcal{Z} = \{\bullet\}$ is the combinatorial class containing one object of size 1 and no objects of any other size. The generating function of the atomic class is x.

Def Neutral Class: The neutral class ϵ is the combinatorial class containing one object of size 0 and no objects of any other size. The generating function of the neutral class is 1.

Def Sum: Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes with weight functions $|\cdot|_{\mathcal{A}}$ and $|\cdot|_{\mathcal{B}}$. The sum of $\mathcal{A}$ and $\mathcal{B}$ is the new combinatorial class $\mathcal{C} = \mathcal{A} + \mathcal{B}$,

- whose objects are the disjoint union of the objects in $\mathcal{A}$ and $\mathcal{B}$.
- whose weight function $|\cdot|_{\mathcal{C}}$ is defined by

$$|\sigma|_{\mathcal{C}} = \begin{cases} |\sigma|_{\mathcal{A}} & \text{if } \sigma \in \mathcal{A} \\ |\sigma|_{\mathcal{B}} & \text{if } \sigma \in \mathcal{B} \end{cases}$$

Lemma 1 If $\mathcal{C} = \mathcal{A} + \mathcal{B}$, then

$$C(x) = A(x) + B(x)$$

Def Product: Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes with weight functions $|\cdot|_{\mathcal{A}}$ and $|\cdot|_{\mathcal{B}}$. The product of $\mathcal{A}$ and $\mathcal{B}$ is the new combinatorial class $\mathcal{C} = \mathcal{A} \times \mathcal{B}$,

- whose objects consist of all pairs (α, β) with $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.
- whose weight function $|\cdot|_{\mathcal{C}}$ is defined by $|(\alpha, \beta)|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}$.

Lemma 2 If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then

$$C(x) = A(x)B(x)$$

Def k**th Power Class**: Iterating the product, for any positive integer K we define the kth power class

$$\mathcal{A}^k = \underbrace{A \times \cdots \times A}_{k}$$

By lemma 2, the generating function of $\mathcal{A}^k$ is $A(x)^k$.

Def Sequence: Let $\mathcal{A}$ be a combinatorial class with no elements of size 0. The sequence class

$$\mathrm{SEQ}(\mathcal{A}) = \epsilon + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \cdots$$

defined by $\mathcal{A}$ is the class

- whose objects consist of all finite length tuples $(\alpha_1, \ldots, \alpha_r) \in \mathcal{A}^r$ for any $r \in \mathbb{N}$.
- whose weight function is defined by $|(\alpha_1, \ldots, \alpha_r)|_{\mathcal{C}} = |\alpha_1|_{\mathcal{A}} + \cdots + |\alpha_r|_{\mathcal{A}}$.

Lemma 3 If $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$, then

$$C(x) = \frac{1}{1 - A(x)}$$

Notation of Sequences: It is convenient to use the notation $\mathrm{SEQ}_{\text{property}}(\mathcal{A})$ to denote the sum of powers $\mathcal{A}^k$ where k has a specific property.

- $\mathrm{SEQ}_{\leq r}(\mathcal{A}) = \epsilon + \mathcal{A} + \cdots + \mathcal{A}^r$
$$A(x) = \frac{1 - A(x)^{r+1}}{1 - A(x)}$$
- $\mathrm{SEQ}_{\geq r}(\mathcal{A}) = \mathcal{A}^r + \mathcal{A}^{r+1} + \cdots$
$$A(x) = \frac{A(x)^r}{1 - A(x)}$$
- $\mathrm{SEQ}_{\text{even}}(\mathcal{A}) = \epsilon + \mathcal{A}^2 + \mathcal{A}^4 + \cdots$
$$A(x) = \frac{1}{1 - A(x)^2}$$
- $\mathrm{SEQ}_{\text{odd}}(\mathcal{A}) = \mathcal{A} + \mathcal{A}^3 + \cdots$
$$A(x) = \frac{A(x)}{1 - A(x)^2}$$

Def Combinatorial Construction: A combinatorial construction is a function Φ that takes a collection of classes $\mathcal{A}_1, \ldots, \mathcal{A}_r$ of some type and returns a new combinatorial class $\Phi(\mathcal{A}_1, \ldots, \mathcal{A}_r)$. We call the construction Φ admissible if the counting sequence of $\Phi(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ depends only on the counting sequence of $\mathcal{A}_1, \ldots, \mathcal{A}_r$.

Def Specification of a Class: A specification of a class $\mathcal{A}$ is a system of admissible constructions

$$\begin{aligned}\mathcal{C}_1 &= \Phi_1(\mathcal{C}_1, \ldots, \mathcal{C}_r)\\ \mathcal{C}_2 &= \Phi_2(\mathcal{C}_1, \ldots, \mathcal{C}_r)\\ &\vdots\\ \mathcal{C}_r &= \Phi_r(\mathcal{C}_1, \ldots, \mathcal{C}_r)\end{aligned}$$

where $\mathcal{C}_1 = \mathcal{A}$.

Def Dependency Graph: The dependency graph of a specification is a directed graph whose vertices are the classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ in the specification, with an edge from the vertex corresponding to the class $\mathcal{C}_i$ to the vertex corresponding to the class $\mathcal{C}_j$ if $\mathcal{C}_j$ appears in $\Phi_i(\mathcal{C}_1, \ldots, \mathcal{C}_r)$. A specification is called iterative if its dependency graph has no directed cycles, and recursive if it has a directed cycle (including a self loop).

Remark: A combinatorial class may be specified in multiple ways, some are iterative and some are recursive. The class $\mathcal{B}$ of binary strings has iterative specification

$$\mathcal{B} = \mathrm{SEQ}(\mathcal{Z}_0 + \mathcal{Z}_1)$$

and recursive specification

$$\mathcal{B} = \epsilon + (\mathcal{Z}_0 + \mathcal{Z}_1) \times \mathcal{B}$$

Def Rational Specification: A rational specification is an iterative specification that uses only the sum, product, and sequence constructions together with the atmoic and neutral classes. A class described by a rational specification will always admit a rational generating function.

Restricted String Enumeration: A (binary) regular expression is any specification defined by the neutral class ϵ, the atomic classes $\mathcal{Z}_0$ and $\mathcal{Z}_1$, and the sum, product, and sequence constructions. By convention, when studying regular expressions we use the digits 0 and 1 for the atomic classes $\mathcal{Z}_0$ and $\mathcal{Z}_1$, drop the $\times$ symbol, and use the notation $\mathcal{A}^*$ for $\mathrm{SEQ}(\mathcal{A})$.

Any element in a combinatorial class defined by a regular expression is composed of a nested series of tuples of 0s and 1s. We say that a regular expression produces the class of binary strings constructed in this way, and any such class is called a regular language. A regular expression is ambiguous if it can produce the same binary string in at least two different ways and unambiguous if this does not happen. In particular, an anambiguous regular expression can never contain a sum of two sets of strings that share a common element.

Def Block: A block in a binary string is a maximal subsequence of adjacent elements that are all 0s or all 1s.
The unambiguous regular expressions $0^*(11^*00^*)^*1^*$ and $1^*(00^*11^*)^*0^*$ generate all binary strings by uniquely decomposing a string into its blocks of 0s and 1s.

Def Set Partitions: A set partition of size n is a decomposition of $[n] = \{1, \ldots, n\}$ into a disjoint union of non-empty sets called blocks. Fix a positive integer r. Let $\mathcal{S}_r$ be the class of set partitions with r blocks, and let $\mathcal{T}_r$ be the class of strings σ on the alphabet $\Omega = \{1, 2, \ldots, r\}$ such that

- each element of Ω appears at least once in σ, and

- for all $1 \leq i \leq r$ the first occurence of i in σ comes before the first occurrence of $i+1$.

If P is a set partition with r parts then we can uniquely sort its blocks $B_1, \ldots, B_r$ according to their minimal elements, and given $i \in [n]$ we write $b_P(i) = j$ for the index j such that $i \in B_j$. Define the map $f : \mathcal{S}_r \to \mathcal{T}_r$ that takes a set partition $P \in \mathcal{S}_r$ of size n to the string $f(S) = b_P(1) \cdots b_P(n)$.

$$\begin{aligned} \mathcal{T}_r = &\mathcal{Z}_1 + \mathrm{SEQ}(\mathcal{Z}_1) \\ &\times \mathcal{Z}_2 \times \mathrm{SEQ}(\mathcal{Z}_1 + \mathcal{Z}_2) \\ &\times \mathcal{Z}_3 \times \mathrm{SEQ}(\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3) \\ &\times \cdots \times \mathcal{Z}_r \times \mathrm{SEQ}(\mathcal{Z}_1 + \cdots + \mathcal{Z}_r) \end{aligned}$$

$$S_r(x) = \sum_{n \geq 0} \left\{ {n \atop r} \right\} x^n = \frac{x^r}{(1-x)(1-2x)\cdots(1-rx)}$$

$$\left\{ {n \atop r} \right\} = \frac{1}{r!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^n$$

Def Stirling Number of the Second Kind: The number of set partitions of size n with r blocks is known as the (n, r)-Stirling number of the second kind and denoted $\left\{ {n \atop r} \right\}$.

Recursive Specifications: Rational specification can only encode combinatorial classes with rational generating functions. Moving to recursive specifications thus allows a larger variety of combinatorial behaviour to be captured.

Def Rooted Planar Binary Tree: A rooted planar binary tree of size n is either empty or is a vertex followed by a left rooted planar binary tree and a right rooted planar binary tree (where the order of these subtrees matter). If $\mathcal{B}$ is the class of rooted planar binary trees then this definition implies the specification

$$\mathcal{B} = \epsilon + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$$

In particular, the generating function $B(x)$ of $\mathcal{B}$ satisfies the algebraic equation

$$B(x) = 1 + xB(x)^2$$

The quadratic formula has two solutions

$$y_{\pm}(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

The number of rooted planar binary trees are counted by this sequence of Catalan numbers

$$[x^n] \frac{1 - \sqrt{1-4x}}{2x} = \frac{1}{n+1} \binom{2n}{n}$$

Def Dyck Path: A Dyck path of length n is a lattice path using the steps $(1, 1)$ and $(1, -1)$ that starts at the origin $(0, 0)$, ends at $(2n, 0)$, and never moves to a point with negative y-coordinate. There are $\frac{1}{n+1}\binom{2n}{n}$ Dyck paths of length n.

Def Powerset: If $\mathcal{A}$ is a combinatorial class with no objects of size 0, then the powerset class $\mathrm{PSET}(\mathcal{A})$ defined by $\mathcal{A}$ is the combinatorial class whose objects consist of all finite sets of objects in $\mathcal{A}$ (without repetition). If $\mathcal{P} = \text{PSET}(\mathcal{A})$ then, since every element of $\mathcal{A}$ can either be included or not included in a subset, we have

$$P(x) = \prod_{\alpha \in \mathcal{A}} \left(1 + x^{|\alpha|}\right) = \prod_{n \geq 0} (1 + x^n)^{a_n}$$

By formal exponential and logarithm

$$P(x) = \exp\left(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} A(x^k)\right)$$

Def Multiset: If $\mathcal{A}$ is a combinatorial class with no objects of size 0, then the multiset class $\text{MSET}(\mathcal{A})$ defined by $\mathcal{A}$ is the combinatorial class whose objects consist of all finite sets of objects in $\mathcal{A}$ (with repetition allowed). If $\mathcal{M} = \text{MSET}(\mathcal{A})$ then, since every element of $\mathcal{A}$ can be included any number of times in a multiset, we have

$$M(x) = \prod \alpha \in \mathcal{A} \left(\frac{1}{1 - x^{|\alpha|}}\right) = \prod_{n \geq 0} (1 - x^n)^{-a_n}$$

By formal exponential and logarithm

$$M(x) = \exp\left(\sum_{k \geq 1} \frac{A(x^k)}{k}\right)$$

Def Pointed Class: If $\mathcal{A}$ is a combinatorial class with no objects of size 0, then the pointed class defined by $\mathcal{A}$ is the combinatorial class consisting of copies of the objects in $\mathcal{A}$ where one atom is marked. Because there are n ways to pick an atom to mark in an object of size n, and objects of size 0 cannot be marked, if $\mathcal{C} = \mathcal{A}'$ then

$$C(x) = \sum_{n \geq 1} n a_n x^n = x A'(x)$$

where $A'(x)$ denotes the formal derivative of $A(x)$.

5 Lagrange Implicit Function Theorem

Lagrange Implicit Function Theorem (LIFT) Let D be an integral domain containing $\mathbb{Q}$ (for instance, $\mathbb{R}, \mathbb{Q}[x]$, etc.). If $G(u)$ is a formal power series in $D[[u]]$ such that the constant term $G(0)$ is invertible in D then there is a unique element $R \in D[[x]]$ with $R(0) = 0$ such that

$$x = \frac{R(x)}{G(R(x))}$$

Furthermore, if $F(u)$ is any formal power series in $D[[u]]$ then

$$[x^n]F(R(x)) = \frac{1}{n}[u^{n-1}]F'(u)G(u)^n$$

for all $n \geq 1$.

Corollary If $R \in D[[x]]$ has no constant term and satisfies

$$x = \frac{R(x)}{G(R(x))}$$

for some power series $G \in D[[u]]$ whose constant term is invertible in D then

$$[x^n]R(x) = \frac{1}{n}[u^{n-1}]G(u)^n$$

Ex. If $\mathcal{T}$ is the class of nonempty planar rooted trees the

$$\mathcal{T} = \mathcal{Z} \times \text{SEQ}(\mathcal{T})$$

so

$$T(x) = \frac{x}{1 - T(x)} \implies T(x)(1 - T(x)) = x$$

and thus

$$x = \frac{T(x)}{G(T(x))}$$

where $G(u) = (1-u)^{-1}$. LIFT and the Negative Binomial Theorem imply

$$[x^n]T(x) = \frac{1}{n}[u^{n-1}](1-u)^{-n} = \frac{1}{n}(-1)^{n-1}\binom{-n}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

Ex. Fix a positive integer r. Find the number of rooted planar trees has either 0 or exactly r children.

The combinatorial description of the class $\mathcal{T}$ is the construction

$$\mathcal{T} = \mathcal{Z} \times (\epsilon + \mathcal{T}^r)$$

so

$$T(x) = x(1 + T(x)^r)$$

and thus,

$$x = \frac{T(x)}{G(T(x))}$$

where $G(u) = 1 + u^r$. LIFT and binomial theorem imply

$$[x^n]T(x) = \frac{1}{n}[u^{n-1}](1+u^r)^n = \begin{cases} 0 & \text{if } r \text{ doesn't divide } n-1 \\ \frac{1}{n}\binom{n}{\frac{n-1}{r}} & \text{if } r \text{ divides } n-1 \end{cases}$$

Ex. The class $\mathcal{F}$ of rooted planar 5-ary trees satisfies the specification

$$\mathcal{F} = \epsilon + \mathcal{Z} \times \mathcal{F}^5$$

This specification gives the algebraic equation

$$F(x) = 1 + xF(x)^5$$

however, this is not a form we can apply LIFT. We constructed $\mathcal{F}$ with an element of size 0, so $F(0) \neq 0$. To correct this, let $S(x) = F(x) - 1$ be the generating function for the number of elements in $\mathcal{F}$ of size at least 1. Then

$$x = \frac{S(x)}{(1+S(x))^5} = \frac{S(x)}{G(S(x))}$$

where $G(u) = (1+u)^5$, so if $n \geq 1$ then LIFT implies

$$[x^n]F(x) = [x^n]S(x) = \frac{1}{n}[u^{n-1}](1+u)^{5n} = \frac{1}{n}\binom{5n}{n-1}$$

Proof of LIFT

Def Residue: Let D be an integral domain containing the rational numbers and let $F(x) = \sum\limits_{n \geq k} f_n x^n$ be a formal Laurent series in $D((x))$. The residue of $F(x)$ is the coefficient

$$\operatorname{res}(F) = [x^{-1}]F(x) = f_{-1}$$

Def Formal Derivative of Laurent Series: Analogous to formal derivative of power series

$$\frac{d}{dx}F(x) = F'(x) = \sum_{n \geq k} n f_n x^{n-1}$$

Lemma 1 The derivative F' satisfies $\operatorname{res}(F') = 0$ for any Laurent series F.

Lemma 2 For any Laurent series $F, G \in D((x))$,

$$[x^{-1}]F'(x)G(x) = -[x^{-1}]F(x)G'(x)$$

Lemma Change of Variables Let D be an integral domain containing $\mathbb{Q}$, let $F(u)$ be a formal Laurent series in $D((x))$, and let $B(x)$ be a formal power series in $D[[x]]$ such that

$$B(x) = b_k x^k + b_{k+1}x^{k+1} + \cdots$$

for some $k > 0$ with b_k invertible in D. Then

$$[x^{-1}]F(B(x))B'(x) = k[u^{-1}]F(u)$$

Proof:
Step 1: Existence and Uniqueness
We first prove that there is a unique $R \in D[[x]]$ such that $R(x) = xG(R(x))$. Let $R(x) = \sum_{l \geq 0} r_l x^l$

and $G(u) = \sum_{k\geq 0} g_k u^k$ so that $r_0 = [x^0]xG(R(x)) = 0$. For any $n \geq 1$ we want to find the coefficient

$$\begin{aligned} r_n &= [x^n]xG(R(x)) \\ &= [x^{n-1}] \sum_{k\geq 0} g_k \left(\sum_{l\geq 1} r_l x^l \right)^k \\ &= [x^{n-1}] \sum_{k=0}^{n-1} g_k \left(\sum_{l\geq 1} r_l x^l \right)^k \\ &= \sum_{k=0}^{n-1} g_k \left(\sum_{|(l_1,\ldots,l_k)|=n-1} r_{l_1} \cdots r_{l_k} \right) \end{aligned}$$

where the final sum is over k-tuples of positive integers summing to $n-1$. This is a complicated expression, but it implies that there is a series $R(x)$ solving the stated equation whose coefficients r_n are uniquely determined inductively by a polynomial expression in $g_0, \ldots, g_{n-1}, r_0, \ldots, r_{n-1}$. We also note that $r_1 = g_0$ is invertible in D.

Step 2: Coefficients
Let $P(u) = uG(u)^{-1}$. If we set $u = R(x)$ then

$$x = R(x)G(R(x))^{-1} = uG(u)^{-1} = P(u)$$

so the substituion $x = P(u)$ is the inverse substitution to $u = R(x)$. In particular, if we define $H(u) = P(u)^{-n}F'(u)$ then $H(R(x)) = x^{-n}F'(R(x))$.
Now, if $n \geq 1$, then

$$\begin{aligned} [x^n]F(R(x)) &= [x^{-1}]x^{-1-n}F(R(x)) \\ &= \frac{-1}{n}[x^{-1}](x^{-n})'F(R(x)) && \text{(Def. of } (x^{-n})'\text{)} \\ &= \frac{1}{n}[x^{-1}]x^{-n}(F(R(x)))' && \text{(Lemma 2)} \\ &= \frac{1}{n}[x^{-1}]x^{-n}F'(R(x))R'(x) && \text{(Chain Rule)} \\ &= \frac{1}{n}[x^{-1}]H(R(x))R'(x) \end{aligned}$$

Since $R(x) = r_1 x + \cdots$ and $r_1 \neq 0$, the Change of Variables lemma applies with $k = 1$, giving

$$\begin{aligned} [x^n]F(R(x)) &= \frac{1}{n}[u^{-1}]H(u) \\ &= \frac{1}{n}[u^{-1}]P(u)^{-n}F'(u) \\ &= \frac{1}{n}[u^{-1}]u^{-n}G(u)^n F'(u) \\ &= \frac{1}{n}[u^{n-1}]G(u)^n F'(u) \end{aligned}$$

6 Parameters and Multivariate Generating Functions

Def Multivariate Formal Series: If D is an integral domain then the ring of formal power series with coefficients in D is also an integral domain. Thus, we can define the ring of d-variate formal power series inductively by

$$D[[x_1,\ldots,x_d]] = D[[x_1]][[x_2]]\cdots[[x_d]]$$

Ex. The elements of $D[[x,y]] = D[[x]][[y]]$ can be viewed as power series in y whose coefficients are power series in x. For instance substituting $y = (1+x)y$ into the geometric series $\frac{1}{1-y} = \sum\limits_{n\geq 0} y^n$ is valid as $(1+x)y$ has no constant term as a series in y, gives the expansion

$$\frac{1}{1-(1+x)y} = \sum_{n\geq 0}(1+x)^n y^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)y^n$$

in $D[[x,y]]$.

When the dimension d is understood write bold symbols for vectors $\mathbf{x} = (x_1,\ldots,x_d)$ and use the multi-index notation $\mathbf{x}^{\mathbf{i}} = x_1^{i_1}\cdots x_d^{i_d}$.

Def Coefficient Preserving Ring Isomorphism: Let $\Phi : D[[x,y]] \to D[[y,x]]$ be the map that takes a series in y with coefficients in $D[[x]]$ and returns the series in x with coefficients in $D[[y]]$ obtained by regrouping terms by common powers of x. Φ is a coefficient preserving ring isomorphism, or a bijection, and for all $F, G \in D[[x,y]]$,

- $\Phi(F+G) = \Phi(F) + \Phi(G)$
- $\Phi(FG) = \Phi(F)\Phi(G)$
- $[x^a][y^b]F = [y^b][x^a]\Phi(F)$ for all $a, b \in \mathbb{N}$.

We can write

$$F(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$$

as a series in $D[[x]]$ without worrying about the order of the variables, and we define $[\mathbf{x}^{\mathbf{i}}]F(\mathbf{x}) = f_i$ for all $\mathbf{i} \in \mathbb{N}^d$.

Def Iterated Laurent Series: We define the ring of iterated Laurent series

$$D((\mathbf{x})) = D((x_1))\cdots((x_d))$$

The order of the variables <u>does</u> matter.

Def Laurent Polynomials: An important subring of $D((\mathbf{x}))$ is the ring of Laurent polynomials $D[\mathbf{x},\overline{\mathbf{x}}]$ which consists of all elements of $D((\mathbf{x}))$ that contain only a finite number of non-zero coefficients. The order of the variables defining $D[\mathbf{x},\overline{\mathbf{x}}]$ does not affect coefficient extraction.

Def Classes with Parameters: For any $d \in \mathbb{N}$, a combinatorial class with d parameters is a set of objects $\mathcal{C}$ with a weight function $w : \mathcal{C} \to \mathcal{N}$ and a vector parameter function $\mathbf{p} : \mathcal{C} \to \mathbb{Z}^d$ such that for all $n \in \mathbb{N}$ only a finite number of objects in $\mathcal{C}$ have weight n.

We think of the weight function giving the size of an object while the parameter function tracks d parameters. The multivariate generating function of a combinatorial class with d parameters $(\mathcal{C}, w, \mathbf{p})$ is the formal power series

$$C(\mathbf{u},x) = \sum_{\sigma\in\mathcal{C}} \mathbf{u}^{\mathbf{p}(\sigma)}x^{w(\sigma)} = \sum_{n\geq 0}\left(\sum_{\mathbf{i}\in\mathbb{Z}^d} f_{\mathbf{i},n}\mathbf{u}^{\mathbf{i}}\right)x^n$$

in $\mathbb{Q}[\mathbf{u}, \overline{\mathbf{u}}][[x]]$, where

$$f_{\mathbf{i},n} = |\{\sigma \in \mathcal{C} : \mathbf{p}(\sigma) = \mathbf{i} \text{ and } w(\sigma) = n\}|$$

In particular, for any fixed n only a finite number of coefficients $f_{\mathbf{i},n}$ in the inner sum are non-zero (giving a Laurent polynomial).

Remark: Another potential generalization of the notion of a combinatorial class to the multivariate setting is a collection of objects $\mathcal{C}$ and a vector weight function $w : \mathcal{C} \to \mathbb{N}^d$ such that the set $\{\sigma \in \mathcal{C} : w(\sigma) = \mathbf{i}\}$ is finite for all $\mathbf{i} \in \mathbb{N}^d$. This approach is less useful.

Ex. If B is the class of binary strings where the number of zeroes is tracked as a parameter then, since there are $\binom{n}{k}$ binary strings with k zeroes, we have

$$B(u, x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} u^k \right) x^n = \frac{1}{1 - (1 + u)x}$$

Ex. Let P be the class of lattice paths in $\mathbb{Z}^2$ that start at the origin and take steps in

$$\{(-1, 0), (0, -1), (1, 0), (0, 1)\}$$

where the endpoint of a path in $\mathbb{Z}^2$ is tracked. Then

$$P(u_1, u_2, x) = \frac{1}{1 - (u_1 + 1/u_1 + u_2 + 1/u_2)x}$$

Def Combinatorial Class with Parameters: For any combinatorial class with parameters we can write

$$C(\mathbf{u}, x) = \sum_{n \geq 0} c_n(\mathbf{u}) x^n$$

where each c_n is a Laurent polynomial.

Because a Laurent polynomial only has a finite number of non-zero terms we can set $\mathbf{u} = \mathbf{1}$ to sum over all possible values of the parameters giving

$$C(\mathbf{1}, x) = C(x)$$

where $C(x)$ is the univariate generating function for the class $\mathcal{C}$ with no parameters tracked.

Similarly, if $C(\mathbf{u}, x)$ contains no terms of u_k with negative exponent then setting $u_k = 0$ in $C(\mathbf{u}, x)$ gives the generating function for the objects in $\mathcal{C}$ where the kth parameter is 0. This is useful for enumerating classes of objects where certain behaviours or patterns are forbidden.

Def Constructions and Tracking Classes: If $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are combinatorial classes with parameter functions $\mathbf{p}_1, \ldots, \mathbf{p}_r$ and $\mathcal{C} = \Phi(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ is an admissible consturction defining the combinatorial class $\mathcal{C}$ then we say that a parameter function on $\mathcal{C}$ is inherited through the construction if it is defined in terms of $\mathbf{p}_1, \ldots, \mathbf{p}_r$ in a natural way.

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with parameter functions $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$.

- If $\mathcal{C} = \mathcal{A} + \mathcal{B}$ then the inherited parameter function $\mathbf{p}_{\mathcal{C}}$ on $\mathcal{C}$ is defined by

  $$\mathbf{p}_{\mathcal{C}}(\sigma) = \begin{cases} \mathbf{p}_{\mathcal{A}}(\sigma) & \text{if } \sigma \in \mathcal{A} \\ \mathbf{p}_{\mathcal{B}}(\sigma) & \text{if } \sigma \in \mathcal{B} \end{cases}$$

  for all $\sigma \in \mathcal{C}$.

- If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ then the inherited parameter function $\mathbf{p}_\mathcal{C}$ on $\mathcal{C}$ is defined by

  $$\mathbf{p}_\mathcal{C}(\alpha, \beta) = \mathbf{p}_\mathcal{A}(\alpha) + \mathbf{p}_\mathcal{B}(\beta)$$

  for all $(\alpha, \beta) \in \mathcal{C}$.

- If $\mathcal{A}$ has no objects of size 0 and $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$ then the inherited parameter fucntion $\mathbf{p}_\mathcal{C}$ on $\mathcal{C}$ is defined by

  $$\mathbf{p}_\mathcal{C}(\alpha_1, \ldots, \alpha_r) = \mathbf{p}_\mathcal{A}(\alpha_1) + \cdots + \mathbf{p}_\mathcal{A}(\alpha_r)$$

  for all $(\alpha_1, \ldots, \alpha_r) \in \mathcal{C}$.

Lemma 1 Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes.

- If the parameter function of $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is inherited from the parameter functions of $\mathcal{A}$ and $\mathcal{B}$ then the multivariate generating functions with respect to these parameters satisfies

  $$C(\mathbf{u}, x) = A(\mathbf{u}, x) + B(\mathbf{u}, x)$$

- If the parameter function of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is inherited from the parameter functions of $\mathcal{A}$ and $\mathcal{B}$ then the multivariate generating functions with respect to these parameters satisfies

  $$C(\mathbf{u}, x) = A(\mathbf{u}, x)B(\mathbf{u}, x)$$

- If $\mathcal{A}$ has no objects of size zero and the parameter function of $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$ is inherited from the parameter function of $\mathcal{A}$ then the multivariate generating functions with respect to these parameters satisfies

  $$C(\mathbf{u}, x) = \frac{1}{1 - A(\mathbf{u}, x)}$$

Def Parameterized Neutral Class: To compute generating functions using constructions, it most useful to introduce new parameterized neutral classes $\mu_1, \mu_2, \ldots$ that have size 0 but contribute to the parameters being tracked. The parameterized specifications with these special types of neutral elements immediately give equations satisfied by the multivariate generating functions of classes with inherited parameters.

Ex. Let $\mathcal{B}$ be the class of binary strings enumerated by size and number of zeroes. Then

$$\mathcal{B} = \mathrm{SEQ}(\mathcal{Z}_0 \times \mu + \mathcal{Z}_1)$$

where $\mathcal{Z}_0, \mathcal{Z}_1$ are atomic classes corresponding to digits 0 and 1, and μ is a neutral class marking when a single 0 occurs. The atomic classes have generating functions equal to x while the neutral class μ has generating function u.

$$B(u, x) = \frac{1}{1 - (ux + x)} = \frac{1}{1 - (1 + u)x}$$

Ex. Find an algebraic equation satisfied by the bivariate generating function of planar rooted binary trees enumerated by size and number of leaves.

Our usual specification

$$\mathcal{T} = \epsilon + \mathcal{Z} \times \mathcal{T}^2$$

does not easily track leaves. Instead

$$\mathcal{N} = \underbrace{\mathcal{Z}}_{\text{no children}} + \underbrace{\mathcal{Z} \times \epsilon \times \mathcal{N}}_{\text{only right child}} + \underbrace{\mathcal{Z} \times \mathcal{N} \times \epsilon}_{\text{only left child}} + \underbrace{\mathcal{Z} \times \mathcal{N}^2}_{\text{two children}}$$

for the class $\mathcal{N}$ of nonempty rooted planar binary trees that decomposes a tree in terms of its nonempty subtrees. A leaf is a vertex with no children, so we have the parameterized specification

$$\mathcal{N} = \mathcal{Z} \times \mu + \mathcal{Z} \times \epsilon \times \mathcal{N} + \mathcal{Z} \times \mathcal{N} \times \epsilon + \mathcal{Z} \times \mathcal{N}^2$$

where μ is a neutral class marking when a leaf occurs. This gives

$$N(u,x) = ux + 2xN(u,x) + xN(u,x)^2$$

If we want to include the empty binary tree in the class, then the identity $T(u,x) = N(u,x) + 1$ implies

$$T(u,x) - 1 = ux + 2x(T(u,x) - 1) + x(T(u,x) - 1)^2$$

Quadratic formula gives single power series solution

$$N(u,x) = \frac{1 - 2x - \sqrt{1 - 4x + (1-u)4x^2}}{2x} = ux + (2u)x^2 + (u^2 + 4u)x^3 + \cdots$$

to the algebraic equation satisfied by $N(u,x)$. We can recover the generating funciton

$$N(1,x) = \frac{1 - \sqrt{1-4x}}{2x} - 1$$

for nonempty rooted planar binary trees, and the generating funciton

$$N(0,x) = \frac{1 - 2x - \sqrt{(1-2x)^2}}{2x} = 0$$

for the number of nonempty binary trees with no leaves (none).

Def Formal Partial Derivative: The formal partial derivative (with respect to u) of a bivariate generating series

$$C(u,x) = \sum_{n \geq 0} \left(\sum_{k \in \mathbb{Z}} c_{k,n} u^k \right) x^n$$

in $\mathbb{Z}[u, \overline{u}][[x]]$ is the series

$$\frac{d}{du} C(u,x) = C_u(u,x) = \sum_{n \geq 0} \left(\sum_{k \in \mathbb{Z}} k c_{k,n} u^{k-1} \right) x^n$$

Def Expected Value: The expected value or average value of a one-dimensional parameter $p : \mathcal{C} \to \mathbb{Z}$ on the objects of size n is defined as

$$\begin{aligned} \mathbb{E}_n[p] &= \sum_{k \in \mathbb{Z}} k \cdot [\text{probability that } p(\sigma) = k \text{ when } |\sigma| = n] \\ &= \sum_{k \in \mathbb{Z}} k \cdot \left[\frac{\#\text{ objects with } |\sigma| = n \text{ and } p(\sigma) = k}{\#\text{ objects with } |\sigma| = n} \right] \\ &= \sum_{k \in \mathbb{Z}} k \frac{c_{k,n}}{c_n} \end{aligned}$$

when this series exists, where c_n denotes the total number of objects of size n in $\mathcal{C}$.

Proposition 1 For all $n \in \mathbb{N}$,

$$\mathbb{E}_n[p] = \frac{[x^n]C_u(1,x)}{[x^n]C(1,x)}$$

Ex. Find average number of 0s among the binary strings of length n.

We have the bivariate generating function

$$B(u,x) = \frac{1}{1-(1+u)x}$$

enumerating binary strings by size and number of 0s. Thus,

$$B_u(1,x) = \left.\frac{x}{(1-(1+u)x)^2}\right|_{u=1} = \frac{x}{(1-2x)^2}$$

The Negative Binomial Theorem implies

$$[x^n]\frac{x}{(1-2x)^2} = [x^{n-1}](1-2x)^{-2} = 2^{n-1}\binom{n}{n-1} = n2^{n-1}$$

so

$$\frac{[x^n]\frac{x}{(1-2x)^2}}{[x^n]\frac{1}{1-2x}} = \frac{n2^{n-1}}{2^n} = \frac{n}{2}$$

Ex. Find average number of 1s among compositions of size n.

Let $\mathcal{C}$ be the class of compositions. We find the bivariate generating function enumerating $\mathcal{C}$ by size and number of 1s. We have previously used $\mathcal{C} = \mathrm{SEQ}(\mathrm{SEQ}_{\geq 1}(\mathcal{Z}))$ to enumerate but this makes it hard to track our parameter. Instead, we separate out the 1s in a composition using the specification

$$\mathcal{C} = \mathrm{SEQ}(\mathcal{Z} + \mathrm{SEQ}_{\geq 2}(\mathcal{Z}))$$

which decomposes a composition as a sequence of eleemnts that are either 1 or a positive integer greater than 1. If μ is a neutral class marking when a single 1 occcurs then we have the parameterized specification

$$\mathcal{C} = \mathrm{SEQ}(\mathcal{Z} \times \mu + \mathrm{SEQ}_{\geq 2}(\mathcal{Z}))$$

giving the bivariate generating function

$$C(u,x) = \frac{1}{1-\left(ux + \frac{x^2}{1-x}\right)} = \frac{1-x}{(1-x)(1-ux)-x^2}$$

For $n \geq 1$, we have

$$[x^n]C(1,x) = [x^n]\frac{1-x}{1-2x} = 2^{n-1}$$

Furthermore,

$$C_u(1,x) = \left.\frac{(1-x)^2 x}{((1-x)(1-ux)-x^2)^2}\right|_{u=1} = \frac{(1-x)^2 x}{(1-2x)^2}$$

so

$$[x^n]C_u(1,x) = [x^{n-1}](1-2x)^{-2} - 2[x^{n-2}](1-2x)^{-2} + [x^{n-3}](1-2x)^{-2}$$

The Negative Binomial Theorem implies

$$[x^k](1-2x)^{-2} = 2^k \binom{k+1}{k} = 2^k(k+1)$$

for all $k \geq 0$, so if $n \geq 3$ then

$$\begin{aligned}[x^n]C_u(1,x) &= 2^{n-1}n - 2^{n-1}(n-1) + 2^{n-3}(n-2) \\ &= 2^{n-1} + 2^{n-3}n - 2^{n-2} \\ &= 2^{n-2} + 2^{n-3}n\end{aligned}$$

The average number of 1s among the compositions of size n is thus

$$\frac{2^{n-2} + 2^{n-3}n}{2^{n-1}} = \frac{2+n}{4}$$

if $n \geq 3$, while the average is 0 when $n = 0$ and 1 if $n = 1, 2$.

Theorem

$$[x^n]\frac{d}{du}C(u,x) = \frac{d}{du}[x^n]C(u,x)$$

for any integral domain D and series $C(u,x) \in D[u,\overline{u}][[x]]$.

Ex. Find the average number of leaves among the rooted planar binary trees of size n.

We derived the algebraic equation

$$N(u,x) = ux + 2xN(u,x) + xN(u,x)^2$$

for the bivariate generating function $N(u,x)$ enumerating rooted planar binary trees by size and number of leaves. Thus,

$$\frac{N(u,x)}{u + 2N(u,x) + N(u,x)^2} = \frac{N(u,x)}{G(N(u,x))}$$

where $G(t) = u+2t+t^2$ for LIFT. The constant $G(0) = u$ is invertible in $\mathbb{Q}[u,\overline{u}]$ so the last exercise, LIFT, and the binomial theorem imply

$$\begin{aligned}[x^n]\left.\frac{d}{du}N(u,x)\right|_{u=1} &= \left.\frac{d}{du}[x^n]N(u,x)\right|_{u=1} \\ &= \left.\frac{d}{du}\left(\frac{1}{n}[t^{n-1}](u+2t+t^2)^n\right)\right|_{u=1} \\ &= [t^{n-1}]\left.\left(\frac{1}{n}\frac{d}{du}(u+2t+t^2)^n\right)\right|_{u=1} \\ &= [t^{n-1}]\left.(u+2t+t^2)^{n-1}\right|_{u=1} \\ &= [t^{n-1}](1+2t+t^2)^{n-1} \\ &= [t^{n-1}](1+t)^{2(n-1)} \\ &= \binom{2n-2}{n-1}\end{aligned}$$

for all $n \geq 1$. Thus, since there are $\frac{1}{n+1}\binom{2n}{n}$ rooted planar binary trees of size n, the average number of leaves among the trees of size $n \geq 1$ is

$$\begin{aligned}\frac{\binom{2n-2}{n-1}}{\frac{1}{n+1}\binom{2n}{n}} &= (n+1)\frac{(2n-2)!}{(2n)!}\cdot\frac{n!}{(n-1)!}\cdot\frac{n!}{(n-1)!}\\ &= \frac{n^2(n+1)}{2n(2n-1)}\end{aligned}$$

Note that as $n \to \infty$ this average grows like $n/4$, so approximately one quarter of the nodes in random binary trees of large sizes are leaves.

Def Variance: The variance of a univariate combinatorial parameter $p : \mathcal{C} \to \mathbb{Z}$ on the objects of size n in a combinatorial class $\mathcal{C}$ is

$$\mathrm{Var}_n[p] = \mathbb{E}_n[p^2] - \mathbb{E}_n[p]^2$$

where

$$\mathbb{E}_n[p^2] = \frac{[x^n](C_{uu}(1,x) + C_u(1,x))}{[x^n]C(1,x)}$$

Chebyshev's Inequality If the expected value μ and variance $\nu > 0$ of a parameter exist (and are finite) on the objects of size n, then for any $k > 0$ the probability that a uniformly randomly selected object σ of size n satisfies $|p(\sigma) - \mu| \geq k$ is at most $\frac{\nu}{k^2}$.

Corollary Let p be a univariate combinatorial parameter on a combinatorial class $\mathcal{C}$. If the expected values $\mathbb{E}_n[p]$ and variances $\mathrm{Var}_n[p]$ exist for all n and satisfy

$$\lim_{n\to\infty}\left(\frac{\mathrm{Var}_n[p]}{\mathbb{E}_n[p]^2}\right) = 0$$

then p is concentrated around its expected value, meaning that for any fixed $\epsilon > 0$ the probability that a uniformly randomly selected object $\sigma \in \mathcal{C}$ of size n satisfies

$$(1-\epsilon)\mathbb{E}_n[p] \leq p(\sigma) \leq (1+\epsilon)\mathbb{E}_n[p]$$

goes to 1 as $n \to \infty$.

7 q-Analogues

Def Inversion: Let $\pi = \pi_1 \ldots \pi_n$ be a permutation of $[n] = \{1, \ldots, n\}$. An inversion in π is a pair (i, j) with $1 \leq i < j \leq n$ such that $\pi_i > \pi_j$. We denote the number of inversions in π by $\mathrm{inv}(\pi)$.
$\pi = 123$ has no inversions, $\pi = 132$ has a single inversion 32 and $\pi = 321$ has three inversions from the pairs 32, 31, and 21.

Def q-Analogue of k: If $k \in \mathbb{N}$, then we define

$$[k]_q = 1 + q + \cdots + q^{k-1}$$

with $[0]_q = 1$.

Def q-Factorial: The q-factorial of $n \in \mathbb{N}$ is

$$[n]!_q = [n]_q \cdot [n-1]_q \cdots [0]_q$$

Remark: If k is a positive integer then setting $q = 1$ in $[k]_q$ recovers k and setting $q = 1$ in $[n]!_q$ recovers $n!$.

q-Factorial Theorem If S_n is the set of permutations of size n then

$$\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = [n]!_q$$

Def q-Binomial Coefficient: For any $n, k \in \mathbb{N}$ with $0 \leq k \leq n$ we define the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

Ex. The q-binomial coefficient can be computed by hand for small values

$$\begin{aligned} \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q &= \frac{(1+q)(1+q+q^2)(1+q+q^2+q^3)}{(1+q)(1+q)} \\ &= (1+q+q^2)(1+q^2) \\ &= q^4 + q^3 + 2q^2 + q + 1 \end{aligned}$$

Def Sum of Set: If $S = \{s_1, \ldots, s_k\}$ is a subset of $[n]$ then the sum of S is

$$\text{S} = s_1 + \cdots + s_k$$

q-Binomial Theorem For any $n \in \mathbb{N}$,

$$\prod_{k=1}^{n} (1 + q^k x) = \sum_{S \subset [n]} q^{\text{sum}(S)} x^{|S|} = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

Proof of q-Binomial Theorem

Proof of First Equality

Fix a natural number n and let $\mathcal{B}_n$ be the class of subsets of $[n]$ enumerated by size and subset sum. A subset S of $[n]$ is uniquely determined by the tuple $(\omega_1, \ldots, \omega_n)$ where $\omega_i = 1$ if $i \in S$ and $\omega_i = 0$ otherwise, so we have the specification

$$\mathcal{B}_n = (\epsilon + \mathcal{Z}_1) \times \cdots \times (\epsilon + \mathcal{Z}_n)$$

where $\mathcal{Z}_i$ is an atom marking that $i \in S$. Since any value $i \in S$ adds i to the parameter sum(S) we have the parameterized specification

$$\mathcal{B}_n = (\epsilon + \mu \times \mathcal{Z}_1) \times (\epsilon + \mu^2 \times \mathcal{Z}_2) \times \cdots \times (\epsilon + \mu^n \times \mathcal{Z}_n)$$

giving the claimed bivariate generating function

$$B_n(u, x) = \sum_{S \subset [n]} u^{\text{sum}(S)} x^{|S|} = (1 + ux)(1 + u^2 x) \cdots (1 + u^n x)$$

after setting $u = q$.

Proof of Second Equality
For any $n \in \mathbb{N}$ and $0 \leq k \leq n$ let $\mathcal{B}_n(k)$ denote the subsets of $[n]$ with k elements. Because we already know a relationship between the q-factorial and permutations, we define a bijection $\psi_{n,k} : S_n \to \mathcal{B}_n(k) \times S_k \times S_{n-k}$.

Remark: We know that
$$|\mathcal{B}_n(k)| = \frac{n!}{k!(n-k)!} = \frac{|S_n|}{|S_k| \cdot |S_{n-k}|}$$
but dealing with division is hard in a combinatorial point of view. So we construct a bijection between the sets S_n and $\mathcal{B}_n(k) \times S_k \times S_{n-k}$ of the same sizes.

For any sequence $a_1, \ldots, a_m$ of positive integers, let $P(a_1, \ldots, a_m)$ denote the permutation obtained by replacing each a_i with its relative order in the sequence. Ex. $P(3, 19, 5, 2, 32) = 2\ 4\ 3\ 1\ 5$.

Let $\psi_{n,k}$ be the function that takes $\sigma_1 \cdots \sigma_n \in S_n$ and returns the triple $(\alpha, \beta, \gamma) \in \mathcal{B}_n(k) \times S_k \times S_{n-k}$ where
$$\alpha = \{\sigma_1, \ldots, \sigma_k\},\ \beta = P(\sigma_1, \ldots, \sigma_k),\ \gamma = P(\sigma_{k+1}, \ldots, \sigma_n)$$
Ex. $\psi_{7,3}(2, 5, 4, 7, 1, 6, 3) = (\{2, 4, 5\}, 132, 4132)$.

Lemma 1 The map $\psi_{n,k}$ is a bijection.

Lemma 2 If $\sigma \in S_n$ with $\psi_{n,k}(\sigma) = (\alpha, \beta, \gamma)$ then
$$\operatorname{inv}(\sigma) = \left(\operatorname{sum}(\alpha) - \frac{k(k+1)}{2}\right) + \operatorname{inv}(\beta) + \operatorname{inv}(\gamma)$$

Combining Lemmas 1 and 2 with the q-factorial theorem gives
$$\begin{aligned}
[n]!_q &= \sum_{\sigma \in S_n} q^{\operatorname{inv}(\sigma)} \\
&= \sum_{(\alpha,\beta,\gamma) \in \mathcal{B}_n(k) \times S_k \times S_{n-k}} q^{\operatorname{sum}(\alpha) - \frac{k(k+1)}{2} + \operatorname{inv}(\beta) + \operatorname{inv}(\gamma)} \\
&= q^{-\frac{k(k+1)}{2}} \left(\sum_{\alpha \in \mathcal{B}_n(k)} q^{\operatorname{sum}(\alpha)}\right) \left(\sum_{\beta \in S_k} q^{\operatorname{inv}(\beta)}\right) \left(\sum_{\gamma \in \mathcal{S}_{n-k}} q^{\operatorname{inv}(\gamma)}\right) \\
&= q^{-\frac{k(k+1)}{2}} \left(\sum_{\alpha \in \mathcal{B}_n(k)} q^{\operatorname{sum}(\alpha)}\right) [k]!_q [n-k]!_q
\end{aligned}$$
$$\sum_{\alpha \in \mathcal{B}_n(k)} q^{\operatorname{sum}(\alpha)} = q^{-\frac{k(k+1)}{2}} \frac{[n]!_q}{[k]!_q [n-k]!_q} = q^{-\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$
The second equality in the q-Binomial Theorem follows immediately since
$$\sum_{\alpha \in \mathcal{B}_n(k)} q^{\operatorname{sum}(\alpha)} = [x^n] \sum_{S \subset [n]} q^{\operatorname{sum}(S)} x^{|S|}$$

Def Lattice Paths: Let $L(a, b)$ be the set of lattices paths from $(0, 0)$ to (a, b) consisting of East steps $E = (1, 0)$ and North steps $N = (0, 1)$.
$$|L(a, b)| = \binom{a+b}{a}$$

Def Area of Lattice Path: If $P \in L(a,b)$, the area of P is the number $\text{area}(P)$ of 1×1 boxes underneath the steps of P and above the x-axis.

q-Lattice Path Theorem For any $a, b \in \mathbb{N}$,

$$\sum_{P \in L(a,b)} q^{\text{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

q-Analogues

Object	**Parameter**	**Identity**
Permutation	Inversions	$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = [n]!_q$
Subsets	Sum of Elements	$\sum_{S \in \mathcal{B}_n(k)} q^{\text{sum}(S)} = q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$
Lattice Paths	Area	$\sum_{P \in L(a,b)} q^{\text{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$

Remark: Extension of the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Writing

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q \cdots [n-k+1]_q}{[k]!_q}$$

it is enough to generalize

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

when $n \notin \mathbb{N}$. This can be accomplished using a series expansion

$$\frac{1-q^n}{1-q} = n - \frac{n(n-1)}{2}(1-q) + \cdots$$

which connects to hypergeometric series.

Negative q-Binomial Theorem

$$\frac{1}{(1-x)(1-qx)\cdots(1-q^{t-1}x)} = \sum_{n \geq 0} \begin{bmatrix} t+n-1 \\ t \end{bmatrix}_q x^n$$