Math 239 Introduction to Combinatorics

Keven Qiu

Contents

Standard Notation:

Set	Notation
natural numbers	$\mathbb{N} = \{0, 1, 2, \ldots\}$
integers	$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
rational numbers	$\mathbb{Q}$
real numbers	$\mathbb{R}$
complex numbers	$\mathbb{C}$
integers (modulo n)	$\mathbb{Z}_n = \{[0], [1], [2], \ldots [n-1]\}$
finite field of prime size	$\mathbb{F}_p = \mathbb{Z}_p$

Def Cardinality: The size of a set S is denoted by $|S|$

1 Basic Principles of Enumeration

Def Cartesian Product: Choosing an element of A and an element of B leads to $A \times B$ possibilities

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

with the cardinalities

$$|A \times B| = |A| \cdot |B|$$

Def Intersection: An element of A and B

$$A \cap B = \{c : c \in A \wedge c \in B\}$$

Def Union: Choosing an element of A or an element of B leads to

$$A \cup B = \{c : c \in A \vee c \in B\}$$

with the cardinalities

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Note: If the set is a disjoint union, that is $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

Def List: Contains all elements of S exactly once, in any order

Def Permutation: A list of $\{1, 2, \ldots n\}$ for some $n \in \mathbb{N}$, $\sigma : a_1, a_2, \ldots a_n$ can be interpreted as the function

$$\sigma : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}, \sigma(i) = a_i : 1 \leq i \leq n$$

Theorem 1.2: For $n \geq 1$, an $n-$ element set contains $n(n-1)\ldots(2)(1) = n!$ lists

Def Subset: A set of S containing some of the elements of S

Theorem 1.3: For $n \geq 0$, an $n-$ element set contains 2^n subsets

Def Partial List: A list of a subset of S

Theorem 1.4: For $n, k \geq 0$, an $n-$ element set contains $n(n-1)\ldots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!} = k!\binom{n}{k}$ partial sets of length k

Theorem 1.5: For $0 \leq k \leq n$, an $n-$ element set contains $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ subsets of length k

Example 1.6:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Example 1.7:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Note: $\binom{n}{k} = \binom{n}{n-k}$ and $\binom{n}{0} = \binom{n}{n} = 1$

Def Multiset: For $n \geq 0, t \geq 1 \in \mathbb{Z}$, a sequence of t elements $(n_1, n_2, \dots n_t)$ such that $n_1 + n_2 + \cdots + n_t = n$

Theorem 1.9: Let $n \geq 0, t \geq 1$, the number of $n-$element multisets is

$$|M(n,t)| = \binom{n+t-1}{t-1} = \frac{t(n+t)!}{(n+t)t!n!}$$

Def Surjective: For $f : A \to B$, for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$

Def Injective: For $f : A \to B$, for every $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$

Def Bijective: For $f : A \to B$, it is both surjective and injective

$$A \rightleftharpoons B$$

Proposition 1.11 Mutually Inverse Bijections: Let $f : A \to B, g : B \to A$ where $\forall a \in A, g(f(a)) = a$ and $\forall b \in B, f(g(b)) = b$. Then both f, g are bijections, and $f(a) = b$ if and only if $g(a) = b$

$$g = f^{-1}, f = g^{-1}$$

Theorem 1.15 Inclusion/Exclusion: Let $A_q, A_2, \dots A_m$ be finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{\emptyset \neq S \subseteq \{1,2,\dots m\}} (-1)^{|S|-1} |A_S|$$

Example 1.19 Vandermonde Convolution Formula: For $m, n, k \in \mathbb{N}$

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

2 The Idea of Generating Series

Example 2.1 Geometric Series: The simplest infinite power series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

Theorem 2.2 Binomial Theorem: Let $n \in \mathbb{N}$, then

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Theorem 2.4 Binomial Series: Let $t \geq 1 \in \mathbb{Z}$, then

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Def Weight Function: For a set A, the weight function is the function $w : A \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, there are only finitely many elements of weight n, that is

$$A_n = \omega^{-1}(n) = \{\alpha \in A : \omega(\alpha) = n\}$$

Def Generating Series: For a set A with ω, the generating series of A with respect to ω is

$$A(x) = \Phi_A^{\omega} = \sum_{\alpha \in A}^{\infty} x^{\omega(\alpha)}$$

Def Coefficient of Power Series: Let A be a set with $\omega : A \to \mathbb{N}$ and let

$$\Phi_A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

then for every $n \in \mathbb{N}$, the number of elements of A with weight n is $a_n = |A_n|$

Proposition 2.8 Let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be a power series, then for $k \in \mathbb{N}$

$$[x^k]G(x) = g_k$$

Lemma 2.10 Sum Lemma: Let A, B be disjoint sets such that $\omega : (A \cup B) \to \mathbb{N}$, then ω is a function of A and B where

$$\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$$

Lemma 2.11 Infinite Sum Lemma: Let $A_1, A_2, \ldots$ be pairwise disjoint sets for all combinations, and let $B = \cup_{j=0}^{\infty} A_j$ with $\omega : B \to \mathbb{N}$, then ω is a function of A_j where

$$\Phi_B(x) = \sum_{j=0}^{\infty} \Phi_{A_j}(x)$$

Lemma 2.12 Product Lemma: Let A, B be sets such that $\omega : A \to \mathbb{N}, \upsilon : B \to \mathbb{N}$. Define $\eta : A \times B \to \mathbb{N}$ such that $\eta(\alpha, \beta) = \omega(\alpha) + \upsilon(\beta)$, then η is a function on $A \times B$ such that

$$\Phi^{\eta}_{A\times B}(x) = \Phi^{\omega}_{A}(x) \cdot \Phi^{\upsilon}_{B}(x)$$

Lemma 2.13 Let A be a set where $\omega : A \to \mathbb{N}$ and let $A* = \cup_{k=0}^{\infty} A^k$ with $\omega* : A* \to \mathbb{N}$ where A^k is the Cartesian product of k copies of A, then $\omega*$ is a weight function of $A*$ if and only if there are no elements of A with weight 0

Lemma 2.14 String Lemma: Let A be a set where $\omega : A \to \mathbb{N}$ such that no elements of A have weight 0, then

$$\Phi_{A*}(x) = \frac{1}{1 - \Phi_A(x)}$$

Def Composition: A finite sequence of length $k \in \mathbb{N}$ positive integers $\gamma = (c_1, c_2, \ldots c_k)$ where parts $c_i \geq 1$ with size

$$|\gamma| = c_1 + c_2 + \cdots + c_k$$

Note: There is exactly one composition of length 0, $\epsilon = ()$

Lemma 2.17 Let $P = \{1, 2, 3, \ldots\}$ be the set of positive integers, then

a) The set of all compositions is $\mathcal{C} = P*$

b) The generating series $\mathcal{C}$ with respect to size is

$$\Phi_{\mathcal{C}}(x) = 1 + \frac{x}{1-2x} = \frac{1-x}{1-2x}$$

c) For each $n \in \mathbb{N}$, the number of compositions of size n is

$$|\mathcal{C}_n| = \begin{cases} 1 & \text{if } n = 0 \\ 2^{n-1} & \text{if } n \geq 1 \end{cases}$$

3 Binary Strings

Def Binary String: A finite sequence $\sigma = b_1 b_2 \dots b_n$ of length n, in which each bit is $b_i 1 \vee b_i = 0$. A Cartesian power $\{0,1\}^n$ where a binary string of arbitrary length is an element of $\cup_{n=0}^{\infty}\{0,1\}^n$

Note: There is exactly one binary string of length 0, $\epsilon = ()$

Def Regular Expression: A regular expression R can produce a subset $\mathcal{R} \subseteq \{0,1\}*$, or lead to a rational function $R(x)$

- $\epsilon, 0, 1$ are regular expressions
- If R, S are regular expressions, then so is $\mathsf{R} \cup \mathsf{S}$
- If R, S are regular expressions, then so is RS, where $\mathsf{R^k}$ is $\mathsf{R_1 R_2 \dots R_k}$
- If R is a regular expressions, then so is $\mathsf{R}*$

Def Concatenation: For binary strings $\alpha = a_1 a_2 \dots a_m, \beta = b_1 b_2 \dots b_n \in \{0,1\}*$, their concatenation is

$$\alpha\beta = a_1 a_2 \dots a_m b_1 b_2 \dots b_n$$

Def Concatenation Product: For sets of binary strings $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}*$, their concatenation product is

$$\mathcal{AB} = \{\alpha\beta : \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$$

Def Rational Language: A subset $\mathcal{R} \subseteq \{0,1\}^*$ of a regular expression R such that

- ϵ produces $\{\epsilon\}$, 0 produces $\{0\}$, 1 produces $\{1\}$
- If R produces $\mathcal{R}$ and S produces $\mathcal{S}$, then $\mathsf{R} \cup \mathsf{S}$ produces $\mathcal{R} \cup \mathcal{S}$
- If R produces $\mathcal{R}$ and S produces $\mathcal{S}$, then RS produces $\mathcal{RS}$
- If R produces $\mathcal{R}$, then $\mathsf{R}*$ produces $\mathcal{R}* = \cup_{k=0}^{\infty}\mathcal{R}^k$ where $\mathcal{R}^k$ is the concatenation product of $\mathcal{R}$

Def Unambiguous Expression: For a regular expression R, it is unambiguous if each string in $\mathcal{R}$ is produced exactly once

Lemma 3.9 Unambiguous Expression: Let R, S be unambiguous expressions producing $\mathcal{R}, \mathcal{S}$,

- ϵ, 0, 1 are unambiguous
- $\mathsf{R} \cup \mathsf{S}$ is unambiguous if and only if $\mathcal{R} \cap \mathcal{S} = \emptyset$
- RS is unambiguous if and only if there is a bijection $\mathcal{RS} \rightleftharpoons \mathcal{R} \times \mathcal{S}$, that is for every string $\alpha \in \mathcal{RS}$, there is exactly one way to write $\alpha = p\sigma$ with $p \in \mathcal{R}$ and $\sigma \in \mathcal{S}$
- $\mathsf{R}*$ is unambiguous if and only if each $\mathsf{R^k}$ is unambiguous and the union $\cup_{k=0}^{\infty}\mathcal{R}^k$ is disjoint

Def Rational Function: A function $R(x)$ of a regular expression R such that

- ϵ leads to 1, 0 leads to x, 1 leads to x

- $\mathsf{R} \cup \mathsf{S}$ leads to $R(x) + S(x)$
- RS leads to $R(x) \cdot S(x)$
- $\mathsf{R}*$ leads to $\frac{1}{1-R(x)}$

Theorem 3.13 Let R be a regular expression producing $\mathcal{R}$ and leading to $R(x)$, if R is unambiguous then $R(x) = \Phi_{\mathcal{R}}(x)$, the generating series for $\mathcal{R}$ with respect to length

Def Blocks of a String: For a binary string $\sigma = b_1 b_2 \dots b_n$ of length n, a block is a nonempty maximal subsequence of consecutive equal bits, thus cannot be made longer.

Proposition 3.17 Blocks Decompositions: The regular expressions

$$0^*(1^*10^*0)^*1^* \text{ and } 1^*(0^*01^*1)^*0^*$$

are unambiguous expressions for $\{0,1\}*$ that produce each binary string block by block

Def Prefix Decomposition: A regular expression in the form $\mathsf{A} * \mathsf{B}$

Def Postfix Decomposition: A regular expression in the form $\mathsf{AB}*$

Def Contains: For binary strings $k, \sigma \in \{0,1\}*$, σ contains k if there exists strings α, β such that $\sigma = \alpha k \beta$. Otherwise it avoids or excludes k

Theorem 3.26 Let $k \in \{0,1\}*$ be a non-empty string of length n with A being the set of strings that avoid k. Let C be the set of nonempty suffixes of γ of k such that $k\gamma = nk$ for some nonempty prefix n of k. Let $C(x) = \sum_{\gamma \in C} x^{l(\gamma)}$, then

$$A(x) = \frac{1 + C(x)}{(1-2x)(1+C(x)) + x^n}$$

4 Recurrence Relations

Proved The Fibonacci sequence is

$$f_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Def Homogeneous Linear Recurrence Relation: For an infinite sequence of complex numbers $\mathbf{g} = (g_0, g_1, \dots)$ with $a_1, a_2, \dots a_d \in \mathbb{C}$ and $N \geq d$, then $\mathbf{g}$ satisfies a homogeneous linear recurrence relation if for all $n \geq N$,

$$g_n + a_1 g_{n-1} + a_2 g_{n-2} + \cdots + a_d g_{n-d} = 0$$

Note: $g_0, g_1, \dots g_{N-1}$ are the initial conditions of the recurrence

Theorem 4.8 Let the sequence of complex numbers $\mathbf{g} = (g_0, g_1, \dots)$ have corresponding generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n$, then the following are equivalent

a) $\mathbf{g}$ satisfies a homogeneous linear recurrent relation with initial conditions $g_0, g_1, \dots g_{N-1}$

$$g_n + a_1 g_{n-1} + \cdots + a_d g_{n-d} = 0 \text{ for all } n \geq N$$

b) The series $G(x) = \frac{P(x)}{Q(x)}$ has

$$Q(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

and for all $0 \leq k \leq N-1$ with $g_n < 0$ for $n < 0$,

$$P(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$$

in which

$$b_k = g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$$

for all $0 \leq k \leq N-1$, with the convention that $g_n = 0$ for all $n < 0$.

Theorem 4.12 Partial Fractions: Let $G(x) = \frac{P(x)}{Q(x)}$ be a rational function with $\deg P < \deg Q$ and the constant term of $Q(x)$ is 1, for $\lambda_1, \dots \lambda_S$ distinct nonzero complex roots with $d_1 + \cdots + d_s = \deg Q$

$$Q(x) = (1-\lambda_1 x)^{d_1}(1-\lambda_2 x)^{d_2} \dots (1-\lambda_s x)^{d_s}$$

there are d complex numbers $(C_1^{(1)}, C_1^{(2)}, \dots C_1^{(d_1)}; C_2^{(1)}, C_2^{(2)}, \dots C_2^{(d_1)}; \dots C_s^{(1)}, C_s^{(2)}, \dots C_s^{(d_1)})$ uniquely determined such that

$$G(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{d_s} \frac{C_i^{(j)}}{(1-\lambda_i x)^j}$$

Theorem 4.14 Main Theorem: Let the sequence of complex numbers $\mathbf{g} = (g_0, g_1, \dots)$ have corresponding generating series $G(x) = \sum_{n=0}^{\infty} g_n x^n$, assuming that Theorem 4.8 holds and $G(x) = R(x) + \frac{P(x)}{Q(x)}$ with $\deg P(x) < \deg Q(x)$ and $Q(0) = 1$, then for $\lambda_1, \dots \lambda_S$ distinct nonzero complex roots with $d_1 + \cdots + d_s = \deg Q$ such that

$$Q(x) = (1 - \lambda_1 x)^{d_1}(1 - \lambda_2 x)^{d_2} \dots (1 - \lambda_s x)^{d_s}$$

there are polynomials $p_i(n)$ for $1 \le i \le s$ with $\deg p_i(n) < d_i$ such that for all $n > \deg R(x)$

$$g_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \cdots + p_s(n)\lambda_s^n$$

Theorem 4.18 Main Theorem: Let $\mathbf{g} = (g_0, g_1, \dots)$ be a sequence of complex numbers, then the following are equivalent

a) The sequence $\mathbf{g}$ satisfies a homogeneous linear recurrence relation (with initial conditions)

b) The sequence $\mathbf{g}$ satisfies a possibly inhomogeneous linear recurrence relation (with initial conditions) in which the RHS is an eventually polynomial function

c) The generating series $G(x) = \sum_{n=0} g_n x^n$ is a rational function

d) The sequence $\mathbf{g}$ is an eventually polyexp function

Catalan Classes: The n-th Catalan number is:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Binomial Theorem:

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Corollary:

$$(1+x)^{\frac{1}{2}} = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}\binom{2n-2}{n-1} x^n$$

Corollary:

$$(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n$$

5 Graph Theory

Def Graph: A graph G is a finite non-empty set $V(G)$ of vertex objects (p) with a set $E(G)$ of unordered pairs of distinct vertices called edges (q)

$$V(G) = \{1,2,3,4,5\} \text{ and } E(G) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

Digraph: A directed graph has ordered pairs of edges
Multigraph: A multigraph allows non-distinct vertices in edges

Def Adjacent: For the incident $e = \{u, v\}$, the vertices u and v (e joins u and v). $u \sim v$ is adjacent.

Def Neighbours: The vertices adjacent to a vertex u

$$N(u)$$

Def Planar: A graph which can be represented with no edges crossing

Def Isomorphism: Graphs G_1, G_2 are isomorphic if there exists a bijection $f : V(G_1) \to V(G_2)$ such that vertices u, v are adjacent in G_1 if and only if $f(u), f(v)$ are adjacent in G_2. $G_1 \cong G_2$
Isomorphic Class: All graphs isomorphic to G form the isomorphism class of G

Def Automorphism: The identity map on $V(G)$ is an isomorphism from G to itself

Def Degree: The number of edges incident with a vertex v

$$\deg(v)$$

Theorem 4.3.1 Handshaking Lemma/Degree-Sum Formula: For any graph G,

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Corollary 4.3.2: The number of vertices of odd degree in a graph is even

Corollary 4.3.3: The average degree of a vertex in the graph G is

$$\frac{2|E(G)|}{|V(G)|}$$

Def k-Regular Graph: A graph in which every vertex has degree k. There are $\frac{kn}{2}$ edges in a k-Regular graph.

Def Complete Graph: A graph in which all pairs of distinct vertices are adjacent, a complete graph with p vertices is K_p and $k-1-$Regular with $\binom{k}{2}$ edges.

Def Bipartite: A graph with a bipartition (A, B) into two sets A, B such that all edges join a vertex in A to a vertex in B
Complete Bipartite: A complete bipartite $K_{m,n}$ has all m vertices in A adjacent to all n vertices in B. There are mn edges in $K_{m,n}$.

Def $n-$Cube: For $n \geq 0$, the $n-$cube is the graph with the $\{0,1\}^n$ string vertices, such that two strings are adjacent if and only if they differ by exactly one position.

$$|V(G)| = 2^n$$

$$|E(G)| = n2^{n-1}$$

Def Adjacency Matrix: For a graph G with vertices $v_1, v_2, \dots v_p$, the $p \times p$ matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Note: A is a symmetric matrix with all diagonals equal to 0

Def Incidence Matrix: For a graph G with vertices $v_1, v_2, \dots v_p$ and edges $e_1, e_2, \dots e_q$, the $p \times q$ matrix $B = [b_{ij}]$ where

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with } e_j \\ 0, & \text{otherwise} \end{cases}$$

Note: Each column of B contains exactly two 1s

For the product BB^t, its $(i,j)^{th}$ element is

$$\sum_{k=1}^{q} b_{ik} b_{jk}$$

For $i \neq j$ this is the number of edges incident with v_i and v_j,
For $i = j$ this is the number of edges incident with v_i, thus $\deg(v_i)$
thus

$$BB^t = A + diag(\deg(v_1), \dots, \deg(v_p))$$

Def Subgraph: A graph such that its vertex set is a subset $U \subseteq V(G)$ and its edge set is the subset of edges of G such that both vertices belong to U

Spanning: If $V(H) = V(G)$ then H is the spanning subgraph of G
Proper: If $V(H) \subset V(G)$ then H is a proper subgraph of G

Def Empty Graph: A graph with no edges.

Def Walk: A v_0, v_n-walk of length n between v_0 and v_n is an alternating sequence of vertices and edges in G such that edge $e_i = \{v_{i-1}, v_i\}$

$$v_0 e_1 v_1 e_2 \dots v_{n-1} e_n v_n$$

Closed: A walk is closed if $v_0 = v_n$

Def Path: A v_0, v_n-path is a v_0, v_n-walk such that all vertices are distinct (edges are often omitted as they are given from distinct vertices)

$$v_0 v_1 v_2 \dots v_{n-1} v_n$$

Theorem 4.6.2 If there exists a walk in G from v_x to v_y, then there exists a path from v_x to v_y

Corollary 4.6.3 For x, y, z vertices of G, if there exists a path from x to y and from y to z, then there exists a path from x to z

Def Cycle: A $n-$cycle with length n is a graph G with n distinct vertices $v_0, v_1, \dots v_{n-1}$ and n distinct edges $\{v_0, v_1\}, \{v_1, v_2\}.\dots\{v_{n-1}, v_0\}$, thus a connected graph of degree two

Note: the smallest possible cycle is a $3-$cycle

Def Path: A subgraph from deleting one edge of a cycle

Theorem 4.6.4 If every vertex in G has degree at least 2, then G contains a cycle

Def Girth: For a graph G, the length of the shortest cycle $g(G)$
  Note: If G does not contain a cycle, then $g(G) = \infty$

Def Hamilton Cycle: A spanning cycle in a graph

Def Reflexive: A relation on S is reflexive if for $s \in S$, $s \approx s$

Def Symmetric: A relation on S is symmetric if for $s_1, s_2 \in S$, $s_1 \approx s_2 \to s_2 \approx s_1$

Def Transitive: A relation on S is transitive if for $s_1, s_2, s_3 \in S$, $s_1 \approx s_2 \wedge s_2 \approx s_3 \to s_1 \approx s_3$

Def Equivalence Relation: A relation which is reflexive, symmetric, and transitive
  Note: For $v \in V(G)$, v_i, v_j-walk is an equivalence relation

Def Connected: A graph G is connected if $\forall x, y \in V(G)$, there is a path from x to y

Theorem 4.8.2 For graph G with $v \in V(G)$, if $\forall w \in V(G)$ there is a $v, w-$path, then G is connected

Def Component: A subgraph C of G such that

- C is connected
- No subgraph of G that properly contains C is connected

Def Cut: Given a subset X of $V(G)$, the cut induced by X is the set of edges that have exactly one end in X

Theorem 4.8.5 A graph G is not connected if and only if there exists a proper non-empty subset X of $V(G)$ such that the cut induced by X is empty

Def Eulerian Circuit: A closed walk of the graph G that contains every edge of G exactly once

Theorem 4.9.2 For a connected graph G, G has a Eulerian circuit if and only if every vertex has even degree

Def Bridges: An edge e of a graph G is a cut-edge if $G - e$ has more components than G

Lemma 4.10.2 If $e = \{x, y\}$ is a bridge of a connected graph G, then $G - e$ has exactly two components where x and y are in different components

Theorem 4.10.3 An edge e is a bridge of a graph G if and only if it is not contained in any cycle of G

Corollary 4.10.4 If there are two distinct paths from vertex u to vertex v in a graph G, then G contains a cycle

6 Trees

Def Tree: A connected graph with no cycles

Def Forest: A graph with no cycles

Lemma 5.1.3 If u and v are vertices in a tree T, then there is a unique $u, v-$path in T

Lemma 5.1.4 Every edge of a tree T is a bridge

Theorem 5.1.5 If T is a tree, then $|E(T)| = |V(T)| - 1$

Corollary 5.1.6 If G is a forest with k components, then $|E(G)| = |V(G)| - k$

Def Leaf: A vertex in a tree of degree 1

Theorem 5.1.8 A tree with at least two vertices has at least two vertices of degree one a.k.a. two leaves

Alternative Proof of Theorem 5.1.8 For a tree T,

$$n_1 = 2 + \sum_{i=3}^{\infty}(i-2)n_i$$

Lemma A tree is bipartite

Def Spanning Tree: A spanning subgraph which is also a tree

Theorem 5.2.1 A graph G is connected if and only if it has a spanning tree

Corollary 5.2.2 If a graph G is connected with p vertices and $p-1$ edges, then G is a tree

Theorem 5.2.3 If T is a spanning tree of G and e is an edge not in T, then $T+e$ contains one cycle C. And if e' is an edge on C, then $T+e-e'$ is a spanning tree of G

Theorem 5.2.4 If T is a spanning tree of G and e is an edge in T, then $T-e$ has 2 components. If e' is in the cut induced by a component, then $T-e+e'$ is also a spanning tree of G

Def Odd Cycle: A cycle on an odd number of vertices

Lemma 5.3.1 An odd cycle is not bipartite

Theorem 5.3.2 Bipartite Characterization Theorem: A graph is bipartite if and only if it has no odd cycles

Theorem 5.6.1 Prim's algorithm produces a minimum spanning tree for G

Prim's Algorithm for Minimum Spanning Trees (MST) For a connected graph G and a weight function $w : E(G) \to \mathbb{R}$

1. For an arbitrary vertex v in G, let T be the tree consisting of v
2. While T is not spanning G
   (a) Let $e = uv$ be an edge with the smallest weight in the cut induced by $V(T)$ (where $u \in V(t), v \notin V(T)$)
   (b) Add u to $V(T)$ and add e to $E(T)$

7 Planar Graphs

Def Planar: A graph G has a planar embedding (map) if it can be drawn so that its edges intersect only at their ends and no two vertices coincide

Def Face: A connected region partitioned by the planar embedding such that it is surrounded by a boundary subgraph

Adjacent: Adjacent faces share an edge

Def Boundary Walk: A closed walk of the graph G around the perimeter of a face f

Degree: The number of edges in the boundary walk

Theorem 7.1.2 Faceshaking Lemma: For a planar embedding of a connected graph G with faces $f_1, \dots f_s$

$$\sum_{i=1}^{s} \deg(f_i) = 2|E(G)|$$

Corollary 7.1.3 For a planar embedding of a connected graph G with f faces, the average degree of a face is $\frac{2|E(G)|}{f}$

Theorem Lecture 7-1 Jordan Curve Theorem: Every planar embedding of a cycle separates the plane into two parts

Lemma Lecture 7-1 In a planar embedding, an edge e is a bridge if and only if the two sides of e are in the same face

Theorem 7.2.1 Euler's Formula: For a planar embedding with p vertices, q edges, f faces and c compoenents, we have

$$p - q + f = 1 + c$$

For a connected graph G with p vertices and q edges, if G has a planar embedding with f faces, then

$$p - q + f = 2$$

Theorem 7.3.1 A graph is planar if and only if it can be drawn on the surface of a sphere

Def Stereographic Projection: A drawing on a plane converted to be on a sphere. For a sphere that has point A tangent to the plane with antipodal point B, let the vertex x' on the sphere be the unique image of the point x that lines between x and B

Def Platonic Solids: A geometric solid such that all faces have the same degree and all vertices have the same degree

Note: These are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron

Def Platonic Graph: A graph which admits a planar embedding in which all vertices have the same degree $d \geq 3$ and all faces have the same degree $d^* \geq 3$

Theorem 7.4.1 There are exactly five platonic graphs

Lemma 7.4.2 Let G be a planar embedding with p vertices, q edges, s faces in which each vertex has degree $d \geq 3$ and each face has degree $d^* \geq 3$. Then (d, d^*) is one of 5 pairs

$$\{(3,3),(3,4),(4,3),(3,5),(5,3)\}$$

Lemma 7.4.3 If G is a platonic graph with p vertices, q edges and f faces where each vertex has degree d and each face degree d^*, then $p = 2\frac{q}{d}, f = 2\frac{q}{d^*}$ and

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

Lemma 7.5.1 If G contains a cycle, then in a planar embedding of G the boundary of each face contains a cycle

Lemma 7.5.2 Let G be a planar embedding with p vertices and q edges if each face of G has degree at least d^* then $(d^* - 2)q \leq d^*(p - 2)$

Theorem 7.5.3 In a planar graph G with $p \geq 3$ vertices and q edges,

$$q \leq 3p - 6$$

Corollary 7.5.4 K_5 is not planar

Corollary 7.5.5 A planar graph has a vertex of degree at most five

Theorem 7.5.6 In a bipartite planar graph G with $p \geq 3$ vertices and q edges,

$$q \leq 2p - 4$$

Lemma 7.5.7 $K_{3,3}$ is not planar

Def Edge Subdivision: An operation that builds paths of size $m > 1$ to replace each edge, such that $m - 1$ new vertices and edges are created for each path (does not change planarity)

Theorem 7.6.1 Kuratowski's Theorem: A graph is not planar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$

Def k-Colouring: A function from $V(G)$ to a set of k colours so that adjacent vertices have different colours

Note: If such a function exists it is k-colourable

Concise wording:
G is k-colourable if $\exists f : V(G) \to [k]$ such that $f(u) \neq f(v)$ for all $u \sim v$.
The smallest k such that G is k-colourable is called the chromatic number, $\chi(G)$.

Theorem 7.7.2 A graph is 2-colourable if and only if it is bipartite

Theorem 7.7.3 K_n is n-colourable and not k-colourable for any $k < n$. For K_n, $\chi(K_n) = n$.

Theorem 7.7.4 Every planar graph is 6-colourable

Def Edge-Contraction: For an edge $e = \{x, y\}$ in G, the graph G/e from contracting e for G has the vertex set $V(G) \setminus \{x, y\} \cup \{z\}$ and the edge set

$$\{\{u, v\} \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{\{u, z\} : u \notin \{x, y\}, \{u, w\} \in E(G) \text{ for some } w \in \{x, y\}\}$$

Remark: If G is planar and $e \in E(G)$, then G/e is planar.

Theorem 7.7.6 Every planar graph is 5-colourable

Theorem 7.7.7 Every planar graph is 4-colourable

Def Dual: For a planar embedding G, G^* is the planar embedding such that G^* has one vertex for each face of G and two vertices of G^* are adjacent when the faces of G have an edge in common. Vertices become faces, faces become vertices, and edges become edges.

Note: $(G^*)^* = G$, since face of degree k in G becomes a vertex of degree k in G^* and vice versa
Note: A bridge in G gives a loop in G^*
Note: Multiple edges between faces in G give a multigraph G^*

8 Matchings

Def Matching: A set of edges M for a graph G such that every vertex has degree at most 1, M saturates every vertex which is incident with an edge in M

Maximum: The largest possible matching for G

Maximal: A matching such that any added edges would invalidate the matching

Perfect: A matching such that every vertex is saturated, that is $|M| = \frac{p}{2}$

Def Alternating Path: With respect to a matching M, a path $v_0v_1 \dots v_n$ of G such that $\{v_i, v_{i+1}\} \in M$ if i is even and $\{v_i, v_{i+1}\} \notin M$ if i is odd, or
$\{v_i, v_{i+1}\} \notin M$ if i is even and $\{v_i, v_{i+1}\} \in M$ if i is odd

Def Augmenting Path: With respect to a matching M, an alternating path joining two distinct vertices which are not saturated by M

Lemma 8.1.1 M is a maximum matching in G if and only if there is no augmenting path

Def Cover: A set of C vertices for a graph G such that every edge of G has at least one end in C

Lemma 8.2.1 If M is a matching of G and C is a cover of G, then $|M| \leq |C|$

Lemma 8.2.2 If M is a matching of G and C is a cover of G, if $|M| = |C|$, then M is a maximum matching and C is a minimum cover

Theorem 8.3.1 Konig's Theorem: In a bipartite graph, the maximum size of a matching is the minimum size of a cover

XY Construction For an A, B bipartition of G with matching M, let X_0 be the set of vertices in A not saturated by M and let $v \in Z$ be the set of vertices in G joined to a vertex in X_0 by alternating path $P(v)$. With $X = A \cap Z$ and $Y = B \cap Z$, it follows that

- If $v \in X$, then $P(v)$ is even length and its last edge is in M
- If $v \in Y$, then $P(v)$ is odd length and its last edge is not in M
- If w is a vertex of $P(v)$ from X_0 to $v \in Z$, then $w \in Z$

X_0 is the set of unsaturated vertices in A, P is an augmenting path from X_0. $X = P \cap A$, $Y = P \cap B$, U is the set of vertices unsaturated in Y.

Lemma 8.3.2 For an A, B bipartition of G with matching M, and X, Y as defined above,

(a) There is no edge of G from X to $B \setminus Y$

(b) $C = Y \cup (A \setminus X)$ is a cover of G

(c) There is no edge of M from Y to $A \setminus X$

(d) $|M| = |C| - |U|$ where U is the set of unsaturated vertices in Y

(e) There is an augmenting path to each vertex in U

Matching Algorithm For an A, B bipartition of G with matching M, and X, Y as defined above,

(Step 1) If there is an unsaturated vertex $v \in Y$, construct a larger matching M' with augmenting path $P(v)$ until every vertex in Y is saturated

(Step 2) M is a maximal matching, and $C = Y \cup (A \setminus X)$ is a minimum cover

Bipartite Matching Algorithm For an A, B bipartition of G with matching M,

(Step 1) Let $\hat{X} = \{v \in A : v \text{ is unsaturated}\}$, $\hat{Y} = \emptyset$, and let $\text{pr}(v)$ be undefined for $v \in V(G)$

(Step 2) For $v \in B \setminus \hat{Y}$ such that there exists and edge $\{u, v\}$ where $u \in \hat{X}$, add v to $\hat{Y}$ and let $\text{pr}(v) = u$

(Step 3) If no vertices were added, return the maximum matching and minimum cover $C = \hat{Y} \cup (A \setminus \hat{X})$

(Step 4) If an unsaturated vertex v was added to $\hat{Y}$, use pr to trace an augmenting path from v to an unsaturated element of $\hat{X}$, producing a larger matching M' [Step 1]

(Step 5) Otherwise, for each vertex $v \in A \setminus \hat{X}$ such that $\{u, v\} \in M$ and $u \in \hat{Y}$, add v to $\hat{X}$ and set $\text{pr}(v) = u$. [Step 2]

Def Neighbour Set: For some graph G with $D \subseteq G$, the neighbour set $N(D)$ is the set of vertices adjacent to some vertex in D, that is

$$N(D) = \{v \in V(G) : \text{ there exists } u \in D \text{ with } \{u, v\} \in E(G)\}$$

$$E(D, D') = \{uv \in E(G) : u \in D, v \in D'\}$$

Theorem 8.4.1 Hall's Theorem: A bipartite graph G with bipartition A, B has a matching saturating every vertex in A if and only if every subset D of A satisfies

$$|N(D)| \geq |D|$$

Corollary 8.6.1 A bipartite graph G with bipartition A, B has a perfect matching if and only if $|A| = |B|$ and every subset D of A satisfies

$$|N(D)| \geq |D|$$

Theorem 8.6.2 If G is a k-regular bipartite graph with $k \geq 1$, then G has a perfect matching
Note: This holds if G contains multiple edges

Corollary The edges of a k-regular graph can be partitioned into k perfect matching