CO 330 Combinatorial Enumeration

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Notation	**Definition**
$\mathbb{N}$	natural numbers $\{0, 1, 2, \dots\}$
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathcal{C}$	combinatorial class
c_n	counting sequence
C	generating function
$[z^n]f(z)$	coefficient of z^n in $f(z)$

Part I

Toolbox

Chapter 1

Combinatorial Classes

Def Combinatorial Class: A set of objects $\mathcal{C}$ together with a weight function $w : \mathcal{C} \to \mathbb{N}$ (also known as a size function) such that for any $n \in \mathbb{N}$ the set

$$\mathcal{C}_n = \{\sigma \in \mathcal{C} : w(\sigma) = n\}$$

containing the elements of weight n is finite. The set $\mathcal{C}$ itself can be infinite.

Def Subclass: A subclass of a class $(\mathcal{C}, w)$ is any class whose objects form a subset of $\mathcal{C}$ and whose weight function is still w.

Def Counting Sequence: If $\mathcal{C}$ is a combinatorial class then the counting sequence (c_n) of $\mathcal{C}$ is the sequence where $c_n = |\mathcal{C}_n|$ equals the number of elements in $\mathcal{C}$ with weight n. The generating function of $\mathcal{C}$ is the series

$$C(z) = \sum_{n \geq 0} c_n z^n$$

Strings: Let Ω be a non-empty finite set. For any positive integer n the words or strings of length n over the alphabet Ω form the set

$$\Omega^n = \{(w_1, \ldots, w_n) : w_j \in \Omega\}$$

of tuples of length n whose elements come from Ω, and we let $\Omega^0 = \{\varepsilon\}$ be the set containing the empty string ε.

The words or strings over the alphabet Ω form the set

$$\Omega^* = \bigcup_{n \geq 0} \Omega^n$$

define this set a combinatorial class by defining the weight of a word to be its length.

Let $\mathcal{C} = \Omega^*$, then $c_n = |\Omega|^n$ and the generating function is

$$C(z) = \sum_{n \geq 0} |\Omega|^n z^n = \frac{1}{1 - |\Omega| z}$$

Subsets: Let Ω be a finite set with n elements. The subsets of Ω are the sets (including the empty set) whose elements come from Ω.

The class $\mathcal{C}^{[n]}$ of subsets of $[n] = \{1, \ldots, n\}$ has counting sequence $c_k^{[n]} = \binom{n}{k}$ and generating function

$$C^{[n]}(z) = \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n$$

Permutations: The permutations of Ω are the length n tuples containing each element of Ω exactly once.

The set $\mathcal{S}_n$ of permutations of length n consists of the permutations of $[n]$ and the class of permutations $\mathcal{S}$ consists of all permutations of all lengths with the weight function mapping a permutation to its length.

It is common to use Greek letters like π and σ for elements in $\mathcal{S}$. There are four ways to view a permutation $\pi \in \mathcal{S}_n$, all of which are equivalent

- One-line notation: The element π is expressed as the tuple $(\pi_1 \cdots \pi_n)$.
- Two-line notation: The element π is expressed as a table with two rows. The first row contains the numbers $[n]$ in order and the second contains the tuple π.
  E.g. $\pi = (35412)$

  $$\begin{matrix} i & 1 & 2 & 3 & 4 & 5 \\ \pi_i & 3 & 5 & 4 & 1 & 2 \end{matrix}$$

- As a function: The element π is viewed as a function from $[n]$ to $[n]$ that takes $i \in [n]$ to the value $\pi_i \in [n]$.

  Under this interpretation $\mathcal{S}_n$ contains bijections from $[n]$ to itself and forms a permutation group of size n.
- Cycle Notation: If we consider π as a function then we can start with 1 and keep applying π to get a sequence of numbers $1, \pi(1), \pi(\pi(1)), \ldots$. Because π is a permutation, this sequence will eventually repeat, giving the cycle of π containing 1. Repeating this with the other elements of $[n]$ allows us to break π into disjoint cycles, which we write as consecutive tuples.

The counting sequence of $\mathcal{S}_n$ is $c_n = n!$ and the generating function is

$$S(z) = \sum_{n \geq 0} n! z^n$$

Def Lattice Paths: Given a finite set of steps $\mathcal{S} \subset \mathbb{Z}^d$, a restricting region $\mathcal{R} \subset \mathbb{R}^d$, a starting point $\mathbf{p} \in \mathbb{Z}^d \cap \mathcal{R}$, and a terminal set $\mathcal{T} \subset \mathcal{R}$, the lattice path model taking steps in $\mathcal{S}$, starting at $\mathbf{p}$, restricted to $\mathcal{R}$ and ending in $\mathcal{T}$ is the set of all finite tuples $(s_1, \ldots, s_r) \in \mathcal{S}_r$ of any length such that

$$\mathbf{p} + s_1 + \cdots + s_k \in \mathcal{R}, (1 \leq k \leq r)$$

and $\mathbf{p} + s_1 + \cdots + s_r \in \mathcal{T}$. The valid sequences in $\mathcal{S}_r$ form the walks or paths of the model, and can be visualized by starting at $\mathbf{p}$ and concatenating the vectors $s_1, \ldots, s_r$ in order. The steps of a walk $(s_1, \ldots, s_r)$ are the elements s_i.

By default we make a lattice path model a combinatorial class by taking the weight of a walk to be its number of steps.

In the unrestricted case, when $\mathcal{R} = \mathbb{R}^d$, enumeration is straightforward: there are $|S|$ possibilities for each step, so there are $|S|^n$ walks of length n. As a consequence, the generating function of an unrestricted walk model is always rational. The generating function of a model restricted to a halfspace $\mathbb{R}_{>0} \times \mathbb{R}^{d-1}$ can be irrational, however a powerful enumerative technique called the kernel method allows one to show that halfspace models always have algebraic generating functions. In contrast, there are two-dimensional walks restricted to the quadrant $\mathbb{R}^2_{>0}$ with generating functions that are transcendental but D-finite, non-D-finite but D-algebraic, and even series that are not D-algebraic (such a series is sometimes called hypertranscendental).

Integer Compositions: The combinatorial class of (integer) compositions consists of all tuples of positive integers

$$\mathcal{C} = \{(k_1, \dots, k_r) : r, k_j \in \mathbb{N}_{>0}\}$$

where the weight of a tuple is its sum:

$$|(k_1, \dots, k_r)| = k_1 + \dots + k_r$$

Thus, the number of objects c_n in $\mathcal{C}$ of weight n is the number of ways to write n as a sum of positive integers where the order of the summands matters. The elements k_j in a composition $(k_1, \dots, k_r) \in \mathcal{C}$ are the parts or summands of the composition, and the number r of elements is the length or number of summands of the composition.

Lemma 1 There are 2^{n-1} compositions of size n.

Integer Partitions: The class $\mathcal{P}$ of (integer) partitions consists of all non-increasing tuples of positive integers $(k_1, \dots, k_r)$, meaning $k_1 \geq \dots \geq k_r$. Equivalently, partitions are compositions where tuples with the same elements (in a different order) are identified. As for compositions, we usually write partitions as an unsimplified sum.

The generating function of integer partitions is

$$P(z) = \prod_{k \geq 1} \frac{1}{1 - z^k}$$

Def Graph: A (finite simple) graph G is a pair of finite sets $G = (V, E)$ where V defines the vertices or nodes of G and E contains the edges of G, which are sets defined by two distinct vertices of G.

Def Degree: The degree of a vertex is the number of edges incident to that vertex.

Def Cycle: A cycle in a graph is a sequence of at least three vertices connected by distinct edges that start and end at the same vertex.

Def Walk: A walk in a graph is a sequence of vertices with each consecutive pair in the sequence connected by an edge.

Def Connected: A graph G is connected if for any pair of vertices a and b there is a walk in G starting at a and ending at b.

Def Tree: A connected graph with no cycles is called a tree.

Def Rooted Graph: A rooted graph (usually a rooted tree) is a graph where one vertex, called the root, is specially marked.

Def Height: The height of a vertex v in a rooted tree is one less than the minimal number of vertices in a walk from the root to v (the root itself has height zero).

Def Rooted k-ary Tree: A (rooted) k-ary tree is a tree where each vertex has at most k children.

Def Complete k-ary tree: A complete (rooted) k-ary tree is a tree where each vertex has either no children or exactly k children. We often call 2-ary trees binary trees.

Def Isomorphic: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that any pair of distinct vertices a and b form an edge in E_1 if and only if the images $f(a)$ and $f(b)$ form an edge in E_2.

Labelled Graphs A labelled graph class is a combinatorial class defined by a set $\mathcal{G}$ of graphs with vertex sets of the form $V = [n]$ for $n \in \mathbb{N}$ together with a weight function, typically taken as the number of vertices.

Unlabelled Graphs An unlabelled graph class is a combinatorial class defined by a set $\mathcal{G}$ of graphs with vertex sets of the form $V = [n]$ for $n \in \mathbb{N}$ where isomorphic graphs are identified, together with an appropriate weight function.

Lemma 2 There are $2^{\binom{n}{2}}$ labelled graphs with n vertices.

Def Planar Tree: A class of trees is called planar if the order of the children of each node matters (i.e., must be respected when considering two graphs isomorphic) and non-planar if the order of the children does not matter.

Chapter 2

Bijections and Combinatorial Proofs

Def Bijection: Let A and B be sets. A function $f : A \to B$ is injective if $f(a) = f(a')$ implies $a = a'$. A function $f : A \to B$ is surjective if $\forall b \in B$, there is some $a \in A$ such that $f(a) = b$. A function is bijective if it is both injective and surjective.

Lemma 1 The function $f : A \to B$ is a bijection if and only if it has an inverse $g : B \to A$ such that

$$g(f(a)) = a, \forall a \in A$$

$$f(g(b)) = b, \forall b \in B$$

We write $g = f^{-1}$.

Common Ways to Prove Bijection:

1. Define a map $f : A \to B$ and then prove
   - that f is well-defined (i.e., that it actually takes the elements of A to elements of B)
   - that f is injective
   - that f is surjective
2. Define two maps $f : A \to B$ and $g : B \to A$, and prove
   - that both maps are well-defined
   - that $g(f(a)) = a$ for all $a \in A$
   - that $f(g(b)) = b$ for all $b \in B$

Bijection of Combinatorial Classes: A bijection of combinatorial classes $(A, |\cdot|_A)$ and $(B, |\cdot|_B)$ is a bijection $f : A \to B$ between the set of objects in each class that preserves size, meaning $\forall a \in A$, $|a|_A = |f(a)|_B$. This implies $a_n = b_n$.

Def Permutation Matrix: A permutation matrix of size n is an $n \times n$ matrix with entries in $\{0, 1\}$ where every row and every column has exactly one 1.

Def Multiset: A multiset of size n with t types is a sequence $(m_1, \ldots, m_t) \in \mathbb{N}^t$ with $m_1 + \cdots + m_t = n$.

Lemma 2 If $n, t \in \mathbb{N}$ with $t \geq 1$ then there are

$$\binom{n+t-1}{t-1} = \binom{n+t-1}{n}$$

multisets of size n with t types.

Combinatorial Proofs: Bijections and counting methods allow us to give combinatorial proofs of identities. To prove an identity of the form

$$\text{LHS} = \text{RHS}$$

there are two common approaches.

1. Find two sets A and B where it is easy to prove $|A| = \text{LHS}$ and $|B| = \text{RHS}$ and then prove that A and B are in bijection.

2. Count a single set A in two different ways, one which shows that $|A| = \text{LHS}$ and the other of which shows $|A| = \text{RHS}$.

Sequence	**Combinatorial Interpretation**
$n!$	permutations of size n
2^n	binary strings of length n, subsets of $[n]$
A^n	strings of length n on alphabet of size A
$\binom{n}{k}$	subsets of $[n]$ with k elements
$\binom{a+b}{b}$	lattice paths from $(0,0)$ to (a,b) using $(1,0)$ and $(0,1)$
$\binom{n+t-1}{t-1}$	multisets of size n with t types
$A \times B$	Cartesian product of sets with sizes A and B
$A + B$	union of disjoint sets with sizes A and B

Chapter 3

Formal Series and Generating Functions

Def Ring: A ring is a set R together with two operations $+$ and $\times$ that take any $a, b \in R$ and return elements $a + b$ and $a \times b$ in R, together with two special distinct elements $0, 1 \in R$ such that for all $a, b, c \in R$,

- $(a + b) + c = a + (b + c)$
- $a + b = b + a$
- $a + 0 = a$
- there exists $(-a) \in R$ such that $a + (-a) = 0$
- $(a \times b) \times c = a \times (b \times c)$
- $a \times 1 = 1 \times a = a$
- $a \times (b + c) = (a \times b) + (a \times c)$
- $(b + c) \times a = (b \times a) + (c \times a)$

Algebraic Properties:

- R is commutative if $a \times b = b \times a$ for all $a, b \in R$
- R has no zero divisors if $a \times b = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$

Def Integral Domain: A commutative ring with no zero divisors is called an integral domain.

Def Field: A field is an integral domain where for any $a \in R$ with $a \neq 0$ there is another element $a^{-1} \in R$ (the multiplicative inverse of a) with $a \times a^{-1} = 1$.

Def Formal Power Series: Let R be an integral domain. The ring of formal power series with variable z and coefficients in R is the set $R[[z]]$ of formal expressions

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

with each $c_j \in R$. Given

$$A(z) = \sum_{n=0}^{\infty} a_n z^n,\ \ B(z) = \sum_{n=0}^{\infty} b_n z^n$$

in $R[[z]]$ we define addition and multiplication for these formal expressions by

$$A(z) + B(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$

and

$$A(z)B(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n$$

We write $[z^n]A(z)$ for the coefficient a_n of A, and also use the notation $A(0)$ for the constant term.

Lemma 1 Let R be an integral domain. A formal power series $A(z) = \sum_{n\geq 0} a_n z^n$ has a multiplicative inverse in $R[[z]]$ if and only if its constant a_0 is invertible in R.

Def Formal Laurent Series: Let R be an integral domain. The ring of formal Laurent series with variable z and coefficients in R is the set

$$R((z)) = \left\{ \sum_{n=l}^{\infty} c_n z^n : l \in \mathbb{Z}, c_j \in R \right\}$$

Given

$$A(z) = \sum_{n=l}^{\infty} a_n z^n,\ \ B(z) = \sum_{n=m}^{\infty} b_n z^n$$

in $R((z))$ we define addition and multiplication for these formal expressions by

$$A(z) + B(z) = \sum_{n=\min(l,m)}^{\infty} (a_n + b_n) z^n$$

and

$$A(z)B(z) = \sum_{n=l+m}^{\infty} \left(\sum_{k=l}^{n-m} a_k b_{n-k} \right) z^n$$

where we set $a_n = 0$ if $n < l$ and $b_n = 0$ if $n < m$. Formal Laurent series are like formal power series, except they are allowed to have a finite number of terms with negative powers.

Corollary If K is a field then $K((z))$ is a field.

Generalized Binomial Theorem Let K be a field contained in the complex numbers. If $\alpha = \frac{r}{s}$ for $r, s \in \mathbb{N}$ with $s > 0$ then the equation $y^s = (1+z)^r$ has a solution

$$y(z) = (1+z)^{\alpha} = \sum_{k\geq 0} \binom{\alpha}{k} z^k$$

in $K[[z]]$, where we extend the definition of the binomial coefficients to $\alpha \in \mathbb{Q}$ by setting

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

The equation $y^s = (1+z)^r$ has at most s distinct solutions, obtained by multiplying the series $y(z)$ by any of the numbers in the set $\{t \in K : t^s = 1\}$. We write $(1+z)^\alpha$ for the series solution $y(z)$ that beings with the constant 1.

Def Generating Function: The generating function or generating series $C(z)$ of a sequence (c_n) to be the formal power series with coefficients c_n, and the generating function of a combinatorial class $\mathcal{C}$ to be the formal power series $C(z)$ in $\mathbb{Q}[[z]]$ defined by its counting sequence.

$$C(z) = \sum_{\sigma \in \mathcal{C}} z^{|\sigma|}$$

where $|\cdot|$ is the size function for the combinatorial class $\mathcal{C}$.

Def Negative Binomial Theorem:

$$\sum_{n \geq 0} \binom{n+t-1}{t-1} z^n = \frac{1}{(1-z)^t}$$

By the generalized binomial theorem

$$\binom{-t}{n} = (-1)^n \binom{n+t-1}{t-1}$$

Formal Power Series as a Metric Space: Consider a sequence $F_0(z), F_1(z), \ldots$ of formal power series in $R[[z]]$. We say this sequence formally converges if for every $n \in \mathbb{N}$ there exists some $L \in \mathbb{N}$ and $c_n \in R$ such that all coefficients $[z^n]F_k(z) = c_n$ whenever $k \geq L$. In other words, a sequence of formal power series formally converges if for each n the coefficient of z^n eventually stabilizes to a fixed value. When this holds we write

$$\lim_{k \to \infty} F_k(z) = \sum_{n \geq 0} c_n z^n$$

Def Valuation: The valuation $\mathrm{val}(F)$ of a formal power series $F(z) = \sum\limits_{n \geq 0} f_n z^n$ is the smallest natural number i such that $f_i \neq 0$.

Infinite Summation of Formal Power Series: We say that a sequence (F_k) of formal power series is formally summable if the sequence (S_N) defined by the partial sums

$$S_N = \sum_{k=0}^{N} F_k(z)$$

formally converges. When (F_k) is formally summable then we define

$$\sum_{k \geq 0} F_k(z) = \lim_{N \to \infty} \left(\sum_{k=0}^{N} F_k(z) \right)$$

If $A, B \in R[[z]]$ then we define the composition $A(B(z))$ by

$$A(B(z)) = \sum_{n \geq 0} a_n B(z)^n$$

Theorem Let $A, B \in R[[z]]$. If A has only a finite number of non-zero coefficients then $A(B(z))$ always exists. If $A(z)$ has an infinite number of non-zero coefficients then $A(B(z))$ exists if and only if $B(0) = 0$ (B has no constant term).

Def Formal Derivative: Let $F(z) = \sum_{n\geq 0} f_n z^n$ be an element of $R[[z]]$. The formal derivative of F is defined as the formal power series

$$F'(z) = \sum_{n\geq 0} n f_n z^{n-1} = \sum_{n\geq 0} (n+1) f_{n+1} z^n$$

Def Formal Integral: If the positive integers have multiplicative inverses in R (for instance, if R is a field) then the formal integral of F is defined as the formal power series

$$\int F(z) = \sum_{n\geq 0} \frac{f_n}{n+1} z^{n+1}$$

Commutation Rule: If a sequence (F_k) of formal power series is formally summable then

$$\left(\sum_{k\geq 0} F_k(z) \right)' = \sum_{k\geq 0} F_k'(z)$$

Product Rule: If $A(z)$ and $B(z)$ are formal power series then the formal derivative satisfies

$$(A(z)B(z))' = A'(z)B(z) + A(z)B'(z)$$

Chain Rule: If $A(z)$ and $B(z)$ are formal power series and $B(0) = 0$ then

$$A(B(z))' = A'(B(z))B'(z)$$

Def Formal Exponential: If the positive integers have multiplicative inverses in R then the formal exponential in $R[[z]]$ is defined to be the series

$$\exp(z) = \sum_{n\geq 0} \frac{z^n}{n!}$$

Def Formal Logarithm: If the positive integers have multiplicative inverses in R then the formal logarithm in $R[[z]]$ is defined to be the series

$$\log(1+z) = -\int \frac{1}{1+z} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} z^n$$

We can extend the generalized binomial theorem to

$$(1+z)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} z^n = \exp(\alpha \log(1+z))$$

in $\mathbb{C}[[z]]$ for any $\alpha \in \mathbb{C}$.

Rules for Coefficient Extraction:

- **Addition**
$$[z^n](A(z)+B(z)) = [z^n]A(z) + [z^n]B(z)$$

- **Multiplication**
$$[z^n](A(z)B(z)) = \sum_{k=0}^{n}[z^k]A(z)\cdot[z^{n-k}]B(z)$$

- **Coefficient Scaling**
$$[z^n]F(\rho z) = \rho^n[z^n]F(z)$$

- **Monomial Multiplication**
$$[z^n]z^kF(z) = \begin{cases}[z^{n-k}]F(z) & (k \le n)\\ 0 & k > n\end{cases}$$

- **Re-Indexing** If $a, b \in \mathbb{N}$ with $b > a$ then
$$\sum_{n=a}^{b} f_n z^n = \sum_{n=0}^{b-a} f_{n+a} z^{n+a}$$
If $b = \infty$, then both series have upper bound ∞.

- **Generalized Binomial Theorem** If $\alpha \in \mathbb{C}$ then
$$(1+z)^\alpha = \sum_{n\ge 0}\binom{\alpha}{n}z^n$$

- **Negative Binomial Theorem** If t is a positive integer and $n \in \mathbb{N}$ then
$$\binom{-t}{n} = (-1)^n\binom{n+t-1}{t-1} = (-1)^n\binom{n+t-1}{n}$$
combined with the generalized binomial theorem
$$\frac{1}{(1-z)^t} = \sum_{n\ge 0}\binom{n+t-1}{t-1}z^n = \sum_{n\ge 0}\binom{n+t-1}{n}z^n$$

- **Use of Convergent Series** If K is a field contained in the complex numbers and a convergent series with coefficients in K is found to satisfy a polynomial equation (by quadratic formula, Taylor series, etc.) then the formal series defined by the coefficients satisfies the same polynomial equation in $K[[z]]$.

Chapter 4

Combinatorial Constructions

Def Atomic Class: The atomic class $\mathcal{Z} = \{\bullet\}$ is the combinatorial class containing one object of size 1 and no objects of any other size. The generating function of the atomic class is z.

Def Neutral Class: The neutral class ϵ is the combinatorial class containing one object of size 0 and no objects of any other size. The generating function of the neutral class is 1.

Def Sum: Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes with weight functions $|\cdot|_{\mathcal{A}}$ and $|\cdot|_{\mathcal{B}}$. The sum of $\mathcal{A}$ and $\mathcal{B}$ is the new combinatorial class $\mathcal{C} = \mathcal{A} + \mathcal{B}$,

- whose objects are the disjoint union of the objects in $\mathcal{A}$ and $\mathcal{B}$.
- whose weight function $|\cdot|_{\mathcal{C}}$ is defined by

$$|\sigma|_{\mathcal{C}} = \begin{cases} |\sigma|_{\mathcal{A}} & \text{if } \sigma \in \mathcal{A} \\ |\sigma|_{\mathcal{B}} & \text{if } \sigma \in \mathcal{B} \end{cases}$$

Lemma 1 If $\mathcal{C} = \mathcal{A} + \mathcal{B}$, then

$$C(z) = A(z) + B(z)$$

Def Product: Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes with weight functions $|\cdot|_{\mathcal{A}}$ and $|\cdot|_{\mathcal{B}}$. The product of $\mathcal{A}$ and $\mathcal{B}$ is the new combinatorial class $\mathcal{C} = \mathcal{A} \times \mathcal{B}$,

- whose objects consist of all pairs (α, β) with $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.
- whose weight function $|\cdot|_{\mathcal{C}}$ is defined by $|(\alpha, \beta)|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}$.

Lemma 2 If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, then

$$C(z) = A(z)B(z)$$

Def kth Power Class: Iterating the product, for any positive integer K we define the kth power class

$$\mathcal{A}^k = \underbrace{A \times \cdots \times A}_{k}$$

By lemma 2, the generating function of $\mathcal{A}^k$ is $A(z)^k$.

Def Sequence: Let $\mathcal{A}$ be a combinatorial class with no elements of size 0. The sequence class

$$\mathrm{SEQ}(\mathcal{A}) = \epsilon + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \cdots$$

defined by $\mathcal{A}$ is the class

- whose objects consist of all finite length tuples $(\alpha_1, \ldots, \alpha_r) \in \mathcal{A}^r$ for any $r \in \mathbb{N}$.
- whose weight function is defined by $|(\alpha_1, \ldots, \alpha_r)|_{\mathcal{C}} = |\alpha_1|_{\mathcal{A}} + \cdots + |\alpha_r|_{\mathcal{A}}$.

Lemma 3 If $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$, then

$$C(z) = \frac{1}{1 - A(z)}$$

Notation of Sequences: It is convenient to use the notation $\mathrm{SEQ}_{\text{property}}(\mathcal{A})$ to denote the sum of powers $\mathcal{A}^k$ where k has a specific property.

- $\mathrm{SEQ}_{\leq r}(\mathcal{A}) = \epsilon + \mathcal{A} + \cdots + \mathcal{A}^r$
$$A(z) = \frac{1 - A(z)^{r+1}}{1 - A(z)}$$
- $\mathrm{SEQ}_{\geq r}(\mathcal{A}) = \mathcal{A}^r + \mathcal{A}^{r+1} + \cdots$
$$A(z) = \frac{A(z)^r}{1 - A(z)}$$
- $\mathrm{SEQ}_{\text{even}}(\mathcal{A}) = \epsilon + \mathcal{A}^2 + \mathcal{A}^4 + \cdots$
$$A(z) = \frac{1}{1 - A(z)^2}$$
- $\mathrm{SEQ}_{\text{odd}}(\mathcal{A}) = \mathcal{A} + \mathcal{A}^3 + \cdots$
$$A(z) = \frac{A(z)}{1 - A(z)^2}$$

Def Combinatorial Construction: A combinatorial construction is a function Φ that takes a collection of classes $\mathcal{A}_1, \ldots, \mathcal{A}_r$ of some type and returns a new combinatorial class $\Phi(\mathcal{A}_1, \ldots, \mathcal{A}_r)$. We call the construction Φ admissible if the counting sequence of $\Phi(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ depends only on the counting sequence of $\mathcal{A}_1, \ldots, \mathcal{A}_r$.

Def Specification of a Class: A specification of a class $\mathcal{A}$ is a system of admissible constructions

$$\begin{aligned}
\mathcal{C}_1 &= \Phi_1(\mathcal{C}_1, \ldots, \mathcal{C}_r) \\
\mathcal{C}_2 &= \Phi_2(\mathcal{C}_1, \ldots, \mathcal{C}_r) \\
&\vdots \\
\mathcal{C}_r &= \Phi_r(\mathcal{C}_1, \ldots, \mathcal{C}_r)
\end{aligned}$$

where $\mathcal{C}_1 = \mathcal{A}$.

Def Dependency Graph: The dependency graph of a specification is a directed graph whose vertices are the classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ in the specification, with an edge from the vertex corresponding to the class $\mathcal{C}_i$ to the vertex corresponding to the class $\mathcal{C}_j$ if $\mathcal{C}_j$ appears in $\Phi_i(\mathcal{C}_1, \ldots, \mathcal{C}_r)$. A specification is called iterative if its dependency graph has no directed cycles, and recursive if it has a directed cycle (including a self loop).

Remark: A combinatorial class may be specified in multiple ways, some are iterative and some are recursive. The class $\mathcal{B}$ of binary strings has iterative specification

$$\mathcal{B} = \text{SEQ}(\mathcal{Z}_0 + \mathcal{Z}_1)$$

and recursive specification

$$\mathcal{B} = \epsilon + (\mathcal{Z}_0 + \mathcal{Z}_1) \times \mathcal{B}$$

Def Rational Specification: A rational specification is an iterative specification that uses only the sum, product, and sequence constructions together with the atomic and neutral classes. A class described by a rational specification will always admit a rational generating function.

Restricted String Enumeration: A (binary) regular expression is any specification defined by the neutral class ϵ, the atomic classes $\mathcal{Z}_0$ and $\mathcal{Z}_1$, and the sum, product, and sequence constructions. By convention, when studying regular expressions we use the digits 0 and 1 for the atomic classes $\mathcal{Z}_0$ and $\mathcal{Z}_1$, drop the $\times$ symbol, and use the notation $\mathcal{A}^*$ for SEQ($\mathcal{A}$).

Any element in a combinatorial class defined by a regular expression is composed of a nested series of tuples of 0s and 1s. We say that a regular expression produces the class of binary strings constructed in this way, and any such class is called a regular language. A regular expression is ambiguous if it can produce the same binary string in at least two different ways and unambiguous if this does not happen. In particular, an unambiguous regular expression can never contain a sum of two sets of strings that share a common element.

Def Block: A block in a binary string is a maximal subsequence of adjacent elements that are all 0s or all 1s.
The unambiguous regular expressions $0^*(11^*00^*)^*1^*$ and $1^*(00^*11^*)^*0^*$ generate all binary strings by uniquely decomposing a string into its blocks of 0s and 1s.

Def Set Partitions: A set partition of size n is a decomposition of $[n] = \{1, \ldots, n\}$ into a disjoint union of non-empty sets called blocks. Fix a positive integer r. Let $\mathcal{S}_r$ be the class of set partitions with r blocks, and let $\mathcal{T}_r$ be the class of strings σ on the alphabet $\Omega = \{1, 2, \ldots, r\}$ such that

- each element of Ω appears at least once in σ, and
- for all $1 \leq i \leq r$ the first occurrence of i in σ comes before the first occurrence of $i + 1$.

If P is a set partition with r parts then we can uniquely sort its blocks $B_1, \ldots, B_r$ according to their minimal elements, and given $i \in [n]$ we write $b_P(i) = j$ for the index j such that $i \in B_j$. Define the map $f : \mathcal{S}_r \to \mathcal{T}_r$ that takes a set partition $P \in \mathcal{S}_r$ of size n to the string $f(S) = b_P(1) \cdots b_P(n)$.

$$\begin{aligned}\mathcal{T}_r = \mathcal{Z}_1 &+ \text{SEQ}(\mathcal{Z}_1)\\ &\times \mathcal{Z}_2 \times \text{SEQ}(\mathcal{Z}_1 + \mathcal{Z}_2)\\ &\times \mathcal{Z}_3 \times \text{SEQ}(\mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3)\\ &\times \cdots \times \mathcal{Z}_r \times \text{SEQ}(\mathcal{Z}_1 + \cdots + \mathcal{Z}_r)\end{aligned}$$

$$S_r(z) = \sum_{n \geq 0} \left\{ {n \atop r} \right\} z^n = \frac{z^r}{(1-z)(1-2z)\cdots(1-rz)}$$

$$\left\{ {n \atop r} \right\} = \frac{1}{r!} \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} j^n$$

Def Stirling Number of the Second Kind: The number of set partitions of size n with r blocks is known as the (n,r)-Stirling number of the second kind and denoted $\left\{ {n \atop r} \right\}$.

Recursive Specifications: Rational specification can only encode combinatorial classes with rational generating functions. Moving to recursive specifications thus allows a larger variety of combinatorial behaviour to be captured.

Def Rooted Planar Binary Tree: A rooted planar binary tree of size n is either empty or is a vertex followed by a left rooted planar binary tree and a right rooted planar binary tree (where the order of these subtrees matter). If $\mathcal{B}$ is the class of rooted planar binary trees then this definition implies the specification

$$\mathcal{B} = \epsilon + \mathcal{Z} \times \mathcal{B} \times \mathcal{B}$$

In particular, the generating function $B(z)$ of $\mathcal{B}$ satisfies the algebraic equation

$$B(z) = 1 + zB(z)^2$$

The quadratic formula has two solutions

$$y_{\pm}(z) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

The number of rooted planar binary trees are counted by this sequence of Catalan numbers

$$[z^n]\frac{1-\sqrt{1-4z}}{2z} = \frac{1}{n+1}\binom{2n}{n}$$

Def Dyck Path: A Dyck path of length n is a lattice path using the steps $(1,1)$ and $(1,-1)$ that starts at the origin $(0,0)$, ends at $(2n,0)$, and never moves to a point with negative y-coordinate. There are $\frac{1}{n+1}\binom{2n}{n}$ Dyck paths of length n.

Def Powerset: If $\mathcal{A}$ is a combinatorial class with no objects of size 0, then the powerset class $\text{PSET}(\mathcal{A})$ defined by $\mathcal{A}$ is the combinatorial class whose objects consist of all finite sets of objects in $\mathcal{A}$ (without repetition). If $\mathcal{P} = \text{PSET}(\mathcal{A})$ then, since every element of $\mathcal{A}$ can either be included or not included in a subset, we have

$$P(z) = \prod_{\alpha \in \mathcal{A}} \left(1 + z^{|\alpha|}\right) = \prod_{n \geq 0} (1+z^n)^{a_n}$$

By formal exponential and logarithm

$$P(z) = \exp\left(\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} A(z^k)\right)$$

Def Multiset: If $\mathcal{A}$ is a combinatorial class with no objects of size 0, then the multiset class $\text{MSET}(\mathcal{A})$ defined by $\mathcal{A}$ is the combinatorial class whose objects consist of all finite sets of objects in $\mathcal{A}$ (with repetition allowed). If $\mathcal{M} = \mathrm{MSET}(\mathcal{A})$ then, since every element of $\mathcal{A}$ can be included any number of times in a multiset, we have

$$M(z) = \prod \alpha \in \mathcal{A} \left(\frac{1}{1 - z^{|\alpha|}} \right) = \prod_{n \geq 0} (1 - z^n)^{-a_n}$$

By formal exponential and logarithm

$$M(z) = \exp \left(\sum_{k \geq 1} \frac{A(z^k)}{k} \right)$$

Def Pointed Class: If $\mathcal{A}$ is a combinatorial class with no objects of size 0, then the pointed class defined by $\mathcal{A}$ is the combinatorial class consisting of copies of the objects in $\mathcal{A}$ where one atom is marked. Because there are n ways to pick an atom to mark in an object of size n, and objects of size 0 cannot be marked, if $\mathcal{C} = \mathcal{A}'$ then

$$C(z) = \sum_{n \geq 1} n a_n z^n = zA'(z)$$

where $A'(z)$ denotes the formal derivative of $A(z)$.

Chapter 5

Lagrange Implicit Function Theorem

Lagrange Implicit Function Theorem (LIFT) Let D be an integral domain containing $\mathbb{Q}$ (for instance, $\mathbb{R}, \mathbb{Q}[z]$, etc.). If $G(u)$ is a formal power series in $D[[u]]$ such that the constant term $G(0)$ is invertible in D then there is a unique element $R \in D[[z]]$ with $R(0) = 0$ such that

$$z = \frac{R(z)}{G(R(z))}$$

Furthermore, if $F(u)$ is any formal power series in $D[[u]]$ then

$$[z^n]F(R(z)) = \frac{1}{n}[u^{n-1}]F'(u)G(u)^n$$

for all $n \geq 1$.

Corollary If $R \in D[[z]]$ has no constant term and satisfies

$$z = \frac{R(z)}{G(R(z))}$$

for some power series $G \in D[[u]]$ whose constant term is invertible in D then

$$[z^n]R(z) = \frac{1}{n}[u^{n-1}]G(u)^n$$

Proof of LIFT

Def Residue: Let D be an integral domain containing the rational numbers and let $F(z) = \sum\limits_{n \geq k} f_n z^n$ be a formal Laurent series in $D((z))$. The residue of $F(z)$ is the coefficient

$$\text{res}(F) = [z^{-1}]F(z) = f_{-1}$$

Def Formal Derivative of Laurent Series: Analogous to formal derivative of power series

$$\frac{d}{dz}F(z) = F'(z) = \sum_{n \geq k} n f_n z^{n-1}$$

Lemma 1 The derivative F' satisfies $\text{res}(F') = 0$ for any Laurent series F.

Lemma 2 For any Laurent series $F, G \in D((z))$,

$$[z^{-1}]F'(z)G(z) = -[z^{-1}]F(z)G'(z)$$

Lemma Change of Variables Let D be an integral domain containing $\mathbb{Q}$, let $F(u)$ be a formal Laurent series in $D((z))$, and let $B(z)$ be a formal power series in $D[[z]]$ such that

$$B(z) = b_k z^k + b_{k+1} z^{k+1} + \cdots$$

for some $k > 0$ with b_k invertible in D. Then

$$[z^{-1}]F(B(z))B'(z) = k[u^{-1}]F(u)$$

Proof:
Step 1: Existence and Uniqueness
We first prove that there is a unique $R \in D[[z]]$ such that $R(z) = zG(R(z))$. Let $R(z) = \sum_{l\geq 0} r_l z^l$ and $G(u) = \sum_{k\geq 0} g_k u^k$ so that $r_0 = [z^0]zG(R(z)) = 0$. For any $n \geq 1$ we want to find the coefficient

$$\begin{aligned}
r_n &= [z^n]zG(R(z)) \\
&= [z^{n-1}] \sum_{k\geq 0} g_k \left(\sum_{l\geq 1} r_l z^l \right)^k \\
&= [z^{n-1}] \sum_{k=0}^{n-1} g_k \left(\sum_{l\geq 1} r_l z^l \right)^k \\
&= \sum_{k=0}^{n-1} g_k \left(\sum_{|(l_1,\ldots,l_k)|=n-1} r_{l_1} \cdots r_{l_k} \right)
\end{aligned}$$

where the final sum is over k-tuples of positive integers summing to $n-1$. This is a complicated expression, but it implies that there is a series $R(z)$ solving the stated equation whose coefficients r_n are uniquely determined inductively by a polynomial expression in $g_0, \ldots, g_{n-1}, r_0, \ldots, r_{n-1}$. We also note that $r_1 = g_0$ is invertible in D.

Step 2: Coefficients
Let $P(u) = uG(u)^{-1}$. If we set $u = R(z)$ then

$$z = R(z)G(R(z))^{-1} = uG(u)^{-1} = P(u)$$

so the substitution $z = P(u)$ is the inverse substitution to $u = R(z)$. In particular, if we define $H(u) = P(u)^{-n}F'(u)$ then $H(R(z)) = z^{-n}F'(R(z))$.
Now, if $n \geq 1$, then

$$\begin{aligned}
[z^n]F(R(z)) &= [z^{-1}]z^{-1-n}F(R(z)) \\
&= \frac{-1}{n}[z^{-1}](z^{-n})'F(R(z)) && \text{(Def. of } (z^{-n})'\text{)} \\
&= \frac{1}{n}[z^{-1}]z^{-n}(F(R(z)))' && \text{(Lemma 2)} \\
&= \frac{1}{n}[z^{-1}]z^{-n}F'(R(z))R'(z) && \text{(Chain Rule)} \\
&= \frac{1}{n}[z^{-1}]H(R(z))R'(z)
\end{aligned}$$

Since $R(z) = r_1 z + \cdots$ and $r_1 \neq 0$, the Change of Variables lemma applies with $k = 1$, giving

$$\begin{aligned}
[z^n]F(R(z)) &= \frac{1}{n}[u^{-1}]H(u) \\
&= \frac{1}{n}[u^{-1}]P(u)^{-n}F'(u) \\
&= \frac{1}{n}[u^{-1}]u^{-n}G(u)^n F'(u) \\
&= \frac{1}{n}[u^{n-1}]G(u)^n F'(u)
\end{aligned}$$

Examples

Ex. If $\mathcal{T}$ is the class of nonempty planar rooted trees the

$$\mathcal{T} = \mathcal{Z} \times \mathrm{SEQ}(\mathcal{T})$$

so

$$T(z) = \frac{z}{1 - T(z)} \implies T(z)(1 - T(z)) = z$$

and thus

$$z = \frac{T(z)}{G(T(z))}$$

where $G(u) = (1-u)^{-1}$. LIFT and the Negative Binomial Theorem imply

$$[z^n]T(z) = \frac{1}{n}[u^{n-1}](1-u)^{-n} = \frac{1}{n}(-1)^{n-1}\binom{-n}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

Ex. Fix a positive integer r. Find the number of rooted planar trees has either 0 or exactly r children.

The combinatorial description of the class $\mathcal{T}$ is the construction

$$\mathcal{T} = \mathcal{Z} \times (\epsilon + \mathcal{T}^r)$$

so

$$T(z) = z(1 + T(z)^r)$$

and thus,

$$z = \frac{T(z)}{G(T(z))}$$

where $G(u) = 1 + u^r$. LIFT and binomial theorem imply

$$[z^n]T(z) = \frac{1}{n}[u^{n-1}](1+u^r)^n = \begin{cases} 0 & \text{if } r \text{ doesn't divide } n-1 \\ \frac{1}{n}\binom{n}{\frac{n-1}{r}} & \text{if } r \text{ divides } n-1 \end{cases}$$

Ex. The class $\mathcal{F}$ of rooted planar 5-ary trees satisfies the specification

$$\mathcal{F} = \epsilon + \mathcal{Z} \times \mathcal{F}^5$$

This specification gives the algebraic equation

$$F(z) = 1 + zF(z)^5$$

however, this is not a form we can apply LIFT. We constructed $\mathcal{F}$ with an element of size 0, so $F(0) \neq 0$. To correct this, let $S(z) = F(z) - 1$ be the generating function for the number of elements in $\mathcal{F}$ of size at least 1. Then

$$z = \frac{S(z)}{(1 + S(z))^5} = \frac{S(z)}{G(S(z))}$$

where $G(u) = (1 + u)^5$, so if $n \geq 1$ then LIFT implies

$$[z^n]F(z) = [z^n]S(z) = \frac{1}{n}[u^{n-1}](1 + u)^{5n} = \frac{1}{n}\binom{5n}{n-1}$$

Part II

Extensions

Chapter 6

Parameters and Multivariate Generating Functions

Def Multivariate Formal Series: If D is an integral domain then the ring of formal power series with coefficients in D is also an integral domain. Thus, we can define the ring of d-variate formal power series inductively by

$$D[[x_1, \ldots, x_d]] = D[[x_1]][[x_2]] \cdots [[x_d]]$$

Ex. The elements of $D[[x, y]] = D[[x]][[y]]$ can be viewed as power series in y whose coefficients are power series in x. For instance substituting $y = (1 + x)y$ into the geometric series $\frac{1}{1-y} = \sum\limits_{n\geq 0} y^n$ is valid as $(1 + x)y$ has no constant term as a series in y, gives the expansion

$$\frac{1}{1-(1+x)y} = \sum_{n\geq 0}(1+x)^n y^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)y^n$$

in $D[[x, y]]$.

When the dimension d is understood write bold symbols for vectors $\mathbf{x} = (x_1, \ldots, x_d)$ and use the multi-index notation $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_d^{i_d}$.

Def Coefficient Preserving Ring Isomorphism: Let $\Phi : D[[x, y]] \to D[[y, x]]$ be the map that takes a series in y with coefficients in $D[[x]]$ and returns the series in x with coefficients in $D[[y]]$ obtained by regrouping terms by common powers of x. Φ is a coefficient preserving ring isomorphism, or a bijection, and for all $F, G \in D[[x, y]]$,

- $\Phi(F + G) = \Phi(F) + \Phi(G)$
- $\Phi(FG) = \Phi(F)\Phi(G)$
- $[x^a][y^b]F = [y^b][x^a]\Phi(F)$ for all $a, b \in \mathbb{N}$.

We can write

$$F(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$$

as a series in $D[[x]]$ without worrying about the order of the variables, and we define $[\mathbf{x}^{\mathbf{i}}]F(\mathbf{x}) = f_i$ for all $\mathbf{i} \in \mathbb{N}^d$.

Def Iterated Laurent Series: We define the ring of iterated Laurent series

$$D((\mathbf{x})) = D((x_1)) \cdots ((x_d))$$

The order of the variables <u>does</u> matter.

Def Laurent Polynomials: An important subring of $D((\mathbf{x}))$ is the ring of Laurent polynomials $D[\mathbf{x}, \overline{\mathbf{x}}]$ which consists of all elements of $D((\mathbf{x}))$ that contain only a finite number of non-zero coefficients. The order of the variables defining $D[\mathbf{x}, \overline{\mathbf{x}}]$ does not affect coefficient extraction.

Def Classes with Parameters: For any $d \in \mathbb{N}$, a combinatorial class with d parameters is a set of objects $\mathcal{C}$ with a weight function $w : \mathcal{C} \to \mathcal{N}$ and a vector parameter function $\mathbf{p} : \mathcal{C} \to \mathbb{Z}^d$ such that for all $n \in \mathbb{N}$ only a finite number of objects in $\mathcal{C}$ have weight n.

We think of the weight function giving the size of an object while the parameter function tracks d parameters. The multivariate generating function of a combinatorial class with d parameters $(\mathcal{C}, w, \mathbf{p})$ is the formal power series

$$C(\mathbf{u}, z) = \sum_{\sigma \in \mathcal{C}} \mathbf{u}^{\mathbf{p}(\sigma)} z^{w(\sigma)} = \sum_{n \geq 0} \left(\sum_{\mathbf{i} \in \mathbb{Z}^d} f_{\mathbf{i},n} \mathbf{u}^{\mathbf{i}} \right) z^n$$

in $\mathbb{Q}[\mathbf{u}, \overline{\mathbf{u}}][[z]]$, where

$$f_{\mathbf{i},n} = |\{\sigma \in \mathcal{C} : \mathbf{p}(\sigma) = \mathbf{i} \text{ and } w(\sigma) = n\}|$$

In particular, for any fixed n only a finite number of coefficients $f_{\mathbf{i},n}$ in the inner sum are non-zero (giving a Laurent polynomial).

Remark: Another potential generalization of the notion of a combinatorial class to the multivariate setting is a collection of objects $\mathcal{C}$ and a vector weight function $w : \mathcal{C} \to \mathbb{N}^d$ such that the set $\{\sigma \in \mathcal{C} : w(\sigma) = \mathbf{i}\}$ is finite for all $\mathbf{i} \in \mathbb{N}^d$. This approach is less useful.

Def Combinatorial Class with Parameters: For any combinatorial class with parameters we can write

$$C(\mathbf{u}, z) = \sum_{n \geq 0} c_n(\mathbf{u}) z^n$$

where each c_n is a Laurent polynomial.

Because a Laurent polynomial only has a finite number of non-zero terms we can set $\mathbf{u} = \mathbf{1}$ to sum over all possible values of the parameters giving

$$C(\mathbf{1}, z) = C(z)$$

where $C(z)$ is the univariate generating function for the class $\mathcal{C}$ with no parameters tracked.

Similarly, if $C(\mathbf{u}, z)$ contains no terms of u_k with negative exponent then setting $u_k = 0$ in $C(\mathbf{u}, z)$ gives the generating function for the objects in $\mathcal{C}$ where the kth parameter is 0. This is useful for enumerating classes of objects where certain behaviours or patterns are forbidden.

Def Constructions and Tracking Classes: If $\mathcal{A}_1, \ldots, \mathcal{A}_r$ are combinatorial classes with parameter functions $\mathbf{p}_1, \ldots, \mathbf{p}_r$ and $\mathcal{C} = \Phi(\mathcal{A}_1, \ldots, \mathcal{A}_r)$ is an admissible construction defining the combinatorial class $\mathcal{C}$ then we say that a parameter function on $\mathcal{C}$ is inherited through the construction if it is defined in terms of $\mathbf{p}_1, \ldots, \mathbf{p}_r$ in a natural way.

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with parameter functions $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$.

- If $\mathcal{C} = \mathcal{A} + \mathcal{B}$ then the inherited parameter function $\mathbf{p}_{\mathcal{C}}$ on $\mathcal{C}$ is defined by

$$\mathbf{p}_{\mathcal{C}}(\sigma) = \begin{cases} \mathbf{p}_{\mathcal{A}}(\sigma) & \text{if } \sigma \in \mathcal{A} \\ \mathbf{p}_{\mathcal{B}}(\sigma) & \text{if } \sigma \in \mathcal{B} \end{cases}$$

  for all $\sigma \in \mathcal{C}$.

- If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ then the inherited parameter function $\mathbf{p}_{\mathcal{C}}$ on $\mathcal{C}$ is defined by

$$\mathbf{p}_{\mathcal{C}}(\alpha, \beta) = \mathbf{p}_{\mathcal{A}}(\alpha) + \mathbf{p}_{\mathcal{B}}(\beta)$$

  for all $(\alpha, \beta) \in \mathcal{C}$.

- If $\mathcal{A}$ has no objects of size 0 and $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$ then the inherited parameter function $\mathbf{p}_{\mathcal{C}}$ on $\mathcal{C}$ is defined by

$$\mathbf{p}_{\mathcal{C}}(\alpha_1, \ldots, \alpha_r) = \mathbf{p}_{\mathcal{A}}(\alpha_1) + \cdots + \mathbf{p}_{\mathcal{A}}(\alpha_r)$$

  for all $(\alpha_1, \ldots, \alpha_r) \in \mathcal{C}$.

Lemma 1 Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes.

- If the parameter function of $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is inherited from the parameter functions of $\mathcal{A}$ and $\mathcal{B}$ then the multivariate generating functions with respect to these parameters satisfies

$$C(\mathbf{u}, z) = A(\mathbf{u}, z) + B(\mathbf{u}, z)$$

- If the parameter function of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is inherited from the parameter functions of $\mathcal{A}$ and $\mathcal{B}$ then the multivariate generating functions with respect to these parameters satisfies

$$C(\mathbf{u}, z) = A(\mathbf{u}, z) B(\mathbf{u}, z)$$

- If $\mathcal{A}$ has no objects of size zero and the parameter function of $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$ is inherited from the parameter function of $\mathcal{A}$ then the multivariate generating functions with respect to these parameters satisfies

$$C(\mathbf{u}, z) = \frac{1}{1 - A(\mathbf{u}, z)}$$

Def Parameterized Neutral Class: To compute generating functions using constructions, it most useful to introduce new parameterized neutral classes $\mu_1, \mu_2, \ldots$ that have size 0 but contribute to the parameters being tracked. The parameterized specifications with these special types of neutral elements immediately give equations satisfied by the multivariate generating functions of classes with inherited parameters.

Def Formal Partial Derivative: The formal partial derivative (with respect to u) of a bivariate generating series

$$C(u, z) = \sum_{n \geq 0} \left(\sum_{k \in \mathbb{Z}} c_{k,n} u^k \right) z^n$$

in $\mathbb{Z}[u, \overline{u}][[z]]$ is the series

$$\frac{d}{du}C(u,z) = C_u(u,z) = \sum_{n\geq 0}\left(\sum_{k\in\mathbb{Z}} kc_{k,n}u^{k-1}\right)z^n$$

Def Expected Value: The expected value or average value of a one-dimensional parameter $p : \mathcal{C} \to \mathbb{Z}$ on the objects of size n is defined as

$$\begin{aligned}
\mathbb{E}_n[p] &= \sum_{k\in\mathbb{Z}} k \cdot [\text{probability that } p(\sigma) = k \text{ when } |\sigma| = n] \\
&= \sum_{k\in\mathbb{Z}} k \cdot \left[\frac{\#\text{ objects with } |\sigma| = n \text{ and } p(\sigma) = k}{\#\text{ objects with } |\sigma| = n}\right] \\
&= \sum_{k\in\mathbb{Z}} k\frac{c_{k,n}}{c_n}
\end{aligned}$$

when this series exists, where c_n denotes the total number of objects of size n in $\mathcal{C}$.

Proposition 1 For all $n \in \mathbb{N}$,

$$\mathbb{E}_n[p] = \frac{[z^n]C_u(1,z)}{[z^n]C(1,z)}$$

Theorem

$$[z^n]\frac{d}{du}C(u,z) = \frac{d}{du}[z^n]C(u,z)$$

for any integral domain D and series $C(u,z) \in D[u, \overline{u}][[z]]$.

Def Variance: The variance of a univariate combinatorial parameter $p : \mathcal{C} \to \mathbb{Z}$ on the objects of size n in a combinatorial class $\mathcal{C}$ is

$$\mathrm{Var}_n[p] = \mathbb{E}_n[p^2] - \mathbb{E}_n[p]^2$$

where

$$\mathbb{E}_n[p^2] = \frac{[z^n](C_{uu}(1,z) + C_u(1,z))}{[z^n]C(1,z)}$$

Chebyshev's Inequality If the expected value μ and variance $\nu > 0$ of a parameter exist (and are finite) on the objects of size n, then for any $k > 0$ the probability that a uniformly randomly selected object σ of size n satisfies $|p(\sigma) - \mu| \geq k$ is at most $\frac{\nu}{k^2}$.

Corollary Let p be a univariate combinatorial parameter on a combinatorial class $\mathcal{C}$. If the expected values $\mathbb{E}_n[p]$ and variances $\mathrm{Var}_n[p]$ exist for all n and satisfy

$$\lim_{n\to\infty}\left(\frac{\mathrm{Var}_n[p]}{\mathbb{E}_n[p]^2}\right) = 0$$

then p is concentrated around its expected value, meaning that for any fixed $\epsilon > 0$ the probability that a uniformly randomly selected object $\sigma \in \mathcal{C}$ of size n satisfies

$$(1-\epsilon)\mathbb{E}_n[p] \leq p(\sigma) \leq (1+\epsilon)\mathbb{E}_n[p]$$

goes to 1 as $n \to \infty$.

Examples

Ex. If B is the class of binary strings where the number of zeroes is tracked as a parameter then, since there are $\binom{n}{k}$ binary strings with k zeroes, we have

$$B(u,z) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}u^k\right)z^n = \frac{1}{1-(1+u)z}$$

Ex. Let P be the class of lattice paths in $\mathbb{Z}^2$ that start at the origin and take steps in

$$\{(-1,0),(0,-1),(1,0),(0,1)\}$$

where the endpoint of a path in $\mathbb{Z}^2$ is tracked. Then

$$P(u_1,u_2,z) = \frac{1}{1-(u_1+1/u_1+u_2+1/u_2)z}$$

Ex. Let $\mathcal{B}$ be the class of binary strings enumerated by size and number of zeroes. Then

$$\mathcal{B} = \mathrm{SEQ}(\mathcal{Z}_0 \times \mu + \mathcal{Z}_1)$$

where $\mathcal{Z}_0, \mathcal{Z}_1$ are atomic classes corresponding to digits 0 and 1, and μ is a neutral class marking when a single 0 occurs. The atomic classes have generating functions equal to z while the neutral class μ has generating function u.

$$B(u,z) = \frac{1}{1-(uz+z)} = \frac{1}{1-(1+u)z}$$

Ex. Find an algebraic equation satisfied by the bivariate generating function of planar rooted binary trees enumerated by size and number of leaves.

Our usual specification

$$\mathcal{T} = \epsilon + \mathcal{Z} \times \mathcal{T}^2$$

does not easily track leaves. Instead

$$\mathcal{N} = \underbrace{\mathcal{Z}}_{\text{no children}} + \underbrace{\mathcal{Z}\times\epsilon\times\mathcal{N}}_{\text{only right child}} + \underbrace{\mathcal{Z}\times\mathcal{N}\times\epsilon}_{\text{only left child}} + \underbrace{\mathcal{Z}\times\mathcal{N}^2}_{\text{two children}}$$

for the class $\mathcal{N}$ of nonempty rooted planar binary trees that decomposes a tree in terms of its nonempty subtrees. A leaf is a vertex with no children, so we have the parameterized specification

$$\mathcal{N} = \mathcal{Z}\times\mu + \mathcal{Z}\times\epsilon\times\mathcal{N} + \mathcal{Z}\times\mathcal{N}\times\epsilon + \mathcal{Z}\times\mathcal{N}^2$$

where μ is a neutral class marking when a leaf occurs. This gives

$$N(u,z) = uz + 2zN(u,z) + zN(u,z)^2$$

If we want to include the empty binary tree in the class, then the identity $T(u,z) = N(u,z)+1$ implies

$$T(u,z) - 1 = uz + 2z(T(u,z)-1) + z(T(u,z)-1)^2$$

Quadratic formula gives single power series solution

$$N(u,z) = \frac{1-2z-\sqrt{1-4z+(1-u)4z^2}}{2z} = uz + (2u)z^2 + (u^2+4u)z^3 + \cdots$$

to the algebraic equation satisfied by $N(u,z)$. We can recover the generating function

$$N(1,z) = \frac{1-\sqrt{1-4z}}{2z} - 1$$

for nonempty rooted planar binary trees, and the generating function

$$N(0,z) = \frac{1-2z-\sqrt{(1-2z)^2}}{2z} = 0$$

for the number of nonempty binary trees with no leaves (none).

Ex. Find average number of 0s among the binary strings of length n.

We have the bivariate generating function

$$B(u,z) = \frac{1}{1-(1+u)z}$$

enumerating binary strings by size and number of 0s. Thus,

$$B_u(1,z) = \left.\frac{z}{(1-(1+u)z)^2}\right|_{u=1} = \frac{z}{(1-2z)^2}$$

The Negative Binomial Theorem implies

$$[z^n]\frac{z}{(1-2z)^2} = [z^{n-1}](1-2z)^{-2} = 2^{n-1}\binom{n}{n-1} = n2^{n-1}$$

so

$$\frac{[z^n]\frac{z}{(1-2z)^2}}{[z^n]\frac{1}{1-2z}} = \frac{n2^{n-1}}{2^n} = \frac{n}{2}$$

Ex. Find average number of 1s among compositions of size n.

Let $\mathcal{C}$ be the class of compositions. We find the bivariate generating function enumerating $\mathcal{C}$ by size and number of 1s. We have previously used $\mathcal{C} = \mathrm{SEQ}(\mathrm{SEQ}_{\geq 1}(\mathcal{Z}))$ to enumerate but this makes it hard to track our parameter. Instead, we separate out the 1s in a composition using the specification

$$\mathcal{C} = \mathrm{SEQ}(\mathcal{Z} + \mathrm{SEQ}_{\geq 2}(\mathcal{Z}))$$

which decomposes a composition as a sequence of elements that are either 1 or a positive integer greater than 1. If μ is a neutral class marking when a single 1 occurs then we have the parameterized specification

$$\mathcal{C} = \mathrm{SEQ}(\mathcal{Z} \times \mu + \mathrm{SEQ}_{\geq 2}(\mathcal{Z}))$$

giving the bivariate generating function

$$C(u,z) = \frac{1}{1-\left(uz+\frac{z^2}{1-z}\right)} = \frac{1-z}{(1-z)(1-uz)-z^2}$$

For $n \geq 1$, we have

$$[z^n]C(1,z) = [z^n]\frac{1-z}{1-2z} = 2^{n-1}$$

Furthermore,

$$C_u(1,z) = \left.\frac{(1-z)^2 z}{((1-z)(1-uz)-z^2)^2}\right|_{u=1} = \frac{(1-z)^2 z}{(1-2z)^2}$$

so

$$[z^n]C_u(1,z) = [z^{n-1}](1-2z)^{-2} - 2[z^{n-2}](1-2z)^{-2} + [z^{n-3}](1-2z)^{-2}$$

The Negative Binomial Theorem implies

$$[z^k](1-2z)^{-2} = 2^k\binom{k+1}{k} = 2^k(k+1)$$

for all $k \geq 0$, so if $n \geq 3$ then

$$\begin{aligned}
[z^n]C_u(1,z) &= 2^{n-1}n - 2^{n-1}(n-1) + 2^{n-3}(n-2) \\
&= 2^{n-1} + 2^{n-3}n - 2^{n-2} \\
&= 2^{n-2} + 2^{n-3}n
\end{aligned}$$

The average number of 1s among the compositions of size n is thus

$$\frac{2^{n-2}+2^{n-3}n}{2^{n-1}} = \frac{2+n}{4}$$

if $n \geq 3$, while the average is 0 when $n = 0$ and 1 if $n = 1, 2$.

Ex. Find the average number of leaves among the rooted planar binary trees of size n.

We derived the algebraic equation

$$N(u,z) = uz + 2zN(u,z) + zN(u,z)^2$$

for the bivariate generating function $N(u,z)$ enumerating rooted planar binary trees by size and number of leaves. Thus,

$$\frac{N(u,z)}{u + 2N(u,z) + N(u,z)^2} = \frac{N(u,z)}{G(N(u,z))}$$

where $G(t) = u+2t+t^2$ for LIFT. The constant $G(0) = u$ is invertible in $\mathbb{Q}[u,\overline{u}]$ so the last exercise, LIFT, and the binomial theorem imply

$$\begin{aligned}
[z^n]\left.\frac{d}{du}N(u,z)\right|_{u=1} &= \left.\frac{d}{du}[z^n]N(u,z)\right|_{u=1} \\
&= \left.\frac{d}{du}\left(\frac{1}{n}[t^{n-1}](u+2t+t^2)^n\right)\right|_{u=1} \\
&= [t^{n-1}]\left.\left(\frac{1}{n}\frac{d}{du}(u+2t+t^2)^n\right)\right|_{u=1} \\
&= [t^{n-1}]\left.(u+2t+t^2)^{n-1}\right|_{u=1} \\
&= [t^{n-1}](1+2t+t^2)^{n-1} \\
&= [t^{n-1}](1+t)^{2(n-1)} \\
&= \binom{2n-2}{n-1}
\end{aligned}$$

for all $n \geq 1$. Thus, since there are $\frac{1}{n+1}\binom{2n}{n}$ rooted planar binary trees of size n, the average number of leaves among the trees of size $n \geq 1$ is

$$\begin{aligned}\frac{\binom{2n-2}{n-1}}{\frac{1}{n+1}\binom{2n}{n}} &= (n+1)\frac{(2n-2)!}{(2n)!} \cdot \frac{n!}{(n-1)!} \cdot \frac{n!}{(n-1)!} \\ &= \frac{n^2(n+1)}{2n(2n-1)}\end{aligned}$$

Note that as $n \to \infty$ this average grows like $n/4$, so approximately one quarter of the nodes in random binary trees of large sizes are leaves.

Chapter 7

q-Analogues

Def Inversion: Let $\pi = \pi_1 \dots \pi_n$ be a permutation of $[n] = \{1, \dots, n\}$. An inversion in π is a pair (i, j) with $1 \leq i < j \leq n$ such that $\pi_i > \pi_j$. We denote the number of inversions in π by $\text{inv}(\pi)$.
$\pi = 123$ has no inversions, $\pi = 132$ has a single inversion 32 and $\pi = 321$ has three inversions from the pairs 32, 31, and 21.

Def q-Analogue of k: If $k \in \mathbb{N}$, then we define

$$[k]_q = 1 + q + \cdots + q^{k-1}$$

with $[0]_q = 1$.

Def q-Factorial: The q-factorial of $n \in \mathbb{N}$ is

$$[n]!_q = [n]_q \cdot [n-1]_q \cdots [0]_q$$

Remark: If k is a positive integer then setting $q = 1$ in $[k]_q$ recovers k and setting $q = 1$ in $[n]!_q$ recovers $n!$.

q-Factorial Theorem If S_n is the set of permutations of size n then

$$\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = [n]!_q$$

Def q-Binomial Coefficient: For any $n, k \in \mathbb{N}$ with $0 \leq k \leq n$ we define the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

Def Sum of Set: If $S = \{s_1, \dots, s_k\}$ is a subset of $[n]$ then the sum of S is

$$\text{S} = s_1 + \cdots + s_k$$

q-Binomial Theorem For any $n \in \mathbb{N}$,

$$\prod_{k=1}^{n} (1 + q^k z) = \sum_{S \subset [n]} q^{\text{sum}(S)} z^{|S|} = \sum_{k=0}^{n} q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k$$

Proof of q-Binomial Theorem

Proof of First Equality

Fix a natural number n and let $\mathcal{B}_n$ be the class of subsets of $[n]$ enumerated by size and subset sum. A subset S of $[n]$ is uniquely determined by the tuple $(\omega_1, \ldots, \omega_n)$ where $\omega_i = 1$ if $i \in S$ and $\omega_i = 0$ otherwise, so we have the specification

$$\mathcal{B}_n = (\epsilon + \mathcal{Z}_1) \times \cdots \times (\epsilon + \mathcal{Z}_n)$$

where $\mathcal{Z}_i$ is an atom marking that $i \in S$. Since any value $i \in S$ adds i to the parameter $\text{sum}(S)$ we have the parameterized specification

$$\mathcal{B}_n = (\epsilon + \mu \times \mathcal{Z}_1) \times (\epsilon + \mu^2 \times \mathcal{Z}_2) \times \cdots \times (\epsilon + \mu^n \times \mathcal{Z}_n)$$

giving the claimed bivariate generating function

$$B_n(u, z) = \sum_{S \subset [n]} u^{\text{sum}(S)} z^{|S|} = (1 + uz)(1 + u^2 z) \cdots (1 + u^n z)$$

after setting $u = q$.

Proof of Second Equality

For any $n \in \mathbb{N}$ and $0 \leq k \leq n$ let $\mathcal{B}_n(k)$ denote the subsets of $[n]$ with k elements. Because we already know a relationship between the q-factorial and permutations, we define a bijection $\psi_{n,k} : S_n \to \mathcal{B}_n(k) \times S_k \times S_{n-k}$.

Remark: We know that

$$|\mathcal{B}_n(k)| = \frac{n!}{k!(n-k)!} = \frac{|S_n|}{|S_k| \cdot |S_{n-k}|}$$

but dealing with division is hard in a combinatorial point of view. So we construct a bijection between the sets S_n and $\mathcal{B}_n(k) \times S_k \times S_{n-k}$ of the same sizes.

For any sequence $a_1, \ldots, a_m$ of positive integers, let $P(a_1, \ldots, a_m)$ denote the permutation obtained by replacing each a_i with its relative order in the sequence. Ex. $P(3, 19, 5, 2, 32) = 2\ 4\ 3\ 1\ 5$.

Let $\psi_{n,k}$ be the function that takes $\sigma_1 \cdots \sigma_n \in S_n$ and returns the triple $(\alpha, \beta, \gamma) \in \mathcal{B}_n(k) \times S_k \times S_{n-k}$ where

$$\alpha = \{\sigma_1, \ldots, \sigma_k\},\ \beta = P(\sigma_1, \ldots, \sigma_k),\ \gamma = P(\sigma_{k+1}, \ldots, \sigma_n)$$

Ex. $\psi_{7,3}(2, 5, 4, 7, 1, 6, 3) = (\{2, 4, 5\}, 132, 4132)$.

Lemma 1 The map $\psi_{n,k}$ is a bijection.

Lemma 2 If $\sigma \in S_n$ with $\psi_{n,k}(\sigma) = (\alpha, \beta, \gamma)$ then

$$\text{inv}(\sigma) = \left(\text{sum}(\alpha) - \frac{k(k+1)}{2}\right) + \text{inv}(\beta) + \text{inv}(\gamma)$$

Combining Lemmas 1 and 2 with the q-factorial theorem gives

$$\begin{aligned}
[n]!_q &= \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \\
&= \sum_{(\alpha,\beta,\gamma)\in \mathcal{B}_n(k)\times S_k \times S_{n-k}} q^{\mathrm{sum}(\alpha)-\frac{k(k+1)}{2}+\mathrm{inv}(\beta)+\mathrm{inv}(\gamma)} \\
&= q^{-\frac{k(k+1)}{2}} \left(\sum_{\alpha\in\mathcal{B}_n(k)} q^{\mathrm{sum}(\alpha)}\right)\left(\sum_{\beta\in S_k} q^{\mathrm{inv}(\beta)}\right)\left(\sum_{\gamma\in\mathcal{S}_{n-k}} q^{\mathrm{inv}(\gamma)}\right) \\
&= q^{-\frac{k(k+1)}{2}} \left(\sum_{\alpha\in\mathcal{B}_n(k)} q^{\mathrm{sum}(\alpha)}\right)[k]!_q[n-k]!_q \\
\sum_{\alpha\in\mathcal{B}_n(k)} q^{\mathrm{sum}(\alpha)} &= q^{-\frac{k(k+1)}{2}} \frac{[n]!_q}{[k]!_q[n-k]!_q} = q^{-\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q
\end{aligned}$$

The second equality in the q-Binomial Theorem follows immediately since

$$\sum_{\alpha\in\mathcal{B}_n(k)} q^{\mathrm{sum}(\alpha)} = [z^n] \sum_{S\subset[n]} q^{\mathrm{sum}(S)} z^{|S|}$$

Def Lattice Paths: Let $L(a,b)$ be the set of lattices paths from $(0,0)$ to (a,b) consisting of East steps $E=(1,0)$ and North steps $N=(0,1)$.

$$|L(a,b)| = \binom{a+b}{a}$$

Def Area of Lattice Path: If $P \in L(a,b)$, the area of P is the number area(P) of 1×1 boxes underneath the steps of P and above the x-axis.

q-Lattice Path Theorem For any $a,b\in\mathbb{N}$,

$$\sum_{P\in L(a,b)} q^{\mathrm{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

q-Analogues

Object	**Parameter**	**Identity**
Permutation	Inversions	$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = [n]!_q$
Subsets	Sum of Elements	$\sum_{S\in\mathcal{B}_n(k)} q^{\mathrm{sum}(S)} = q^{\frac{k(k+1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$
Lattice Paths	Area	$\sum_{P\in L(a,b)} q^{\mathrm{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$

Remark: Extension of the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. Writing

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q \cdots [n-k+1]_q}{[k]!_q}$$

it is enough to generalize

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

when $n \notin \mathbb{N}$. This can be accomplished using a series expansion

$$\frac{1-q^n}{1-q} = n - \frac{n(n-1)}{2}(1-q) + \cdots$$

which connects to hypergeometric series.

Negative q-Binomial Theorem

$$\frac{1}{(1-z)(1-qz)\cdots(1-q^{t-1}z)} = \sum_{n\geq 0} \begin{bmatrix} t+n-1 \\ t \end{bmatrix}_q z^n$$

Examples

Ex. The q-binomial coefficient can be computed by hand for small values

$$\begin{aligned}\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q &= \frac{(1+q)(1+q+q^2)(1+q+q^2+q^3)}{(1+q)(1+q)} \\ &= (1+q+q^2)(1+q^2) \\ &= q^4 + q^3 + 2q^2 + q + 1\end{aligned}$$

Chapter 8

Integer Partitions

Def Partition: A partition of size n is a sequence of positive integers listed in weakly-decreasing order that sum to n.

If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a partition then following standard convention we write

$$n(\lambda) = |\lambda| = \lambda_1 + \cdots + \lambda_r$$

for the size of the partition and

$$k(\lambda) = r$$

for the number of parts or summands.

Def Partition Generating Function:

$$\Phi(z) = \sum_{\lambda \in \mathcal{Y}} z^{n(\lambda)} = \prod_{k \geq 0} \frac{1}{1 - z^k} = 1 + z + 2z^2 + 3z^3 + 5z^4 + 7z^5$$

Theorem The bivariate generating function enumerating partitions by size and number of parts is

$$\Phi(u, z) = \sum_{\lambda \in \mathcal{Y}} u^{k(\lambda)} z^{n(\lambda)} = \prod_{k \geq 0} \frac{1}{1 - uz^k}$$

Partition Generating Function Theorem For each $k \geq 1$, let M_k be a subset of $\mathbb{N}$. The bivariate generating function, tracking size and number of parts, for the subclass of partitions where the number of parts equal to k lie in M_k for all $k \geq 1$ is

$$\Phi(u, z) = \prod_{k \geq 1} \left(\sum_{j \in M_k} (uz^k)^j \right)$$

If we do not want to track the number of parts for a class of partitions, set $u = 1$.

Proof: Let $\mathcal{Y}_M$ be the subclass of partitions with the stated restrictions, and $\mathcal{S}_M$ be the subclass of $\mathcal{S}$ containing the sequences r where $r_k \in M_k$ for all k. The bijection $f : \mathcal{Y} \to \mathcal{S}$ described above restricts to a bijection $f : \mathcal{Y}_M \to \mathcal{S}_M$ so that

$$\begin{aligned}\Phi(u,z) = \sum_{\lambda \in \mathcal{Y}_M} u^{k(\lambda)} z^{n(\lambda)} &= \sum_{\mathbf{r} \in \mathcal{S}_M} u^{r_1+r_2+r_3+\cdots} z^{r_1+2r_2+3r_3+\cdots} \\ &= \left(\sum_{r_1 \in M_1} (uz)^{r_1} \right) \left(\sum_{r_2 \in M_2} (uz)^{2r_2} \right) \cdots \\ &= \prod_{k \geq 1} \left(\sum_{j \in M_k} (uz^k)^j \right)\end{aligned}$$

Corollary For any $n \geq 0$, the number of partitions of n with odd parts equals the number of partitions of n with distinct parts.

Proof: The Partition Generating Function implies that the generating function for the class of partitions with distinct parts is

$$\begin{aligned}\prod_{k \geq 1} (1+z^k) &= \prod_{k \geq 1} \left((1+z^k) \cdot \frac{1-z^k}{1-z^k} \right) \\ &= \frac{\prod_{k \geq 1}(1-z^{2k})}{\prod_{k \geq 1}(1-z^k)} \\ &= \prod_{k \geq 1} \frac{1}{1-z^{2k-1}}\end{aligned}$$

where the final line follows from the fact that all even powers of z in the denominator cancel with terms in the numerator. Also, the final line is the generating function for class of partitions with odd parts.

Def Ferrers Diagram: If $(\lambda_1, \ldots, \lambda_k)$ is a partition, then its Ferrers diagram consists of k rows of dots where the ith row has λ_i dots.

Def Young Diagram: A Young diagram or Young tableau uses boxes instead of dots.

Properties of Partitions and Diagrams

Partition	**Diagram**
size	number of dots/boxes
number of parts	number of rows
maximum part	number of columns

Def Conjugate Partition: If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n with largest part $l = \lambda_1$ then its conjugate partition λ' is the partition $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ of n where λ'_j equals the number of indices $i \in \{1, \ldots, r\}$ such that $\lambda_i \geq j$.
Ex. The conjugate of $\lambda = (4,2,2,1,1)$ is $\lambda' = (5,3,1,1)$.

Def Self-Conjugate: The map $\lambda \mapsto \lambda'$ defined by conjugation is a bijection and is its own inverse. A partition is called self-conjugate if $\lambda' = \lambda$.

Lemma For any $n \geq 0$ and positive integer k, the number of partitions of n with k parts equals the number of partitions of n with maximum part k.

Def Durfee Length: If λ is a partition, then its Durfee length $d(\lambda)$ is the number of indices i such that $\lambda_i \geq i$.
Visually, $d(\lambda)$ is the number of dots on the main diagonal of the Ferrers diagram of λ.

Def Durfee Square: The Durfee length is also the side-length of the largest square fitting in the Ferrers diagram.
Ex. If $\lambda = (5, 3, 3, 2, 1)$, then $d(\lambda) = 3$.

Proposition Let $\mathcal{S}$ be the class of self-conjugate partitions. Then the bivariate generating function enumerating $\mathcal{S}$ by size and Durfee length is

$$F(u, z) = \sum_{\lambda \in \mathcal{S}} u^{d(\lambda)} z^{n(\lambda)} = \prod_{k \geq 1} (1 + uz^{2k-1})$$

Euler's Identity

$$\prod_{k \geq 0} \frac{1}{1 - x^k y} = \sum_{d \geq 0} \frac{x^{d^2} y^d}{\prod_{i=1}^{d} (1 - x^i y)(1 - x^i)}$$

Proof: The LHS is the bivariate generating function for the class of all partitions enumerated by size and number of parts.

We can decompose the Ferrers diagram of λ into:

- D_λ
- Ferrers diagram of a partition α_λ below D_λ
- Ferrers diagram of a partition β_λ to the right of D_λ

Note that α_λ is a partition with maximum part at most $d(\lambda)$ and β_λ is a partition with at most $d(\lambda)$ parts. Furthermore, for any partitions α and β satisfying these two conditions, we can construct a partition λ such that $\alpha_\lambda = \alpha$ and $\beta_\lambda = \beta$.

Thus, we have a bijection g from the class $\mathcal{Y}$ of partitions to the disjoint union

$$\bigcup_{d \geq 0} (\{d\} \times A_d \times B_d)$$

where d is Durfee length of a partition, A_d is class of partitions with parts of size at most d, and B_d is class of partitions with at most d parts. If $g(\lambda) = (d, \alpha, \beta)$, then $n(\lambda) = d^2 + n(\alpha) + n(\beta)$ and

$k(\lambda) = d + k(\alpha)$, so

$$\begin{aligned}
\prod_{j\geq 0} \frac{1}{1 - x^j y} = \sum_{\lambda \in \mathcal{Y}} x^{n(\lambda)} y^{k(\lambda)} \\
&= \sum_{d\geq 0} \sum_{\alpha \in A_d} \sum_{\beta \in B_d} x^{d^2 + n(\alpha) + n(\beta)} y^{d + k(\alpha)} \\
&= \sum_{d\geq 0} x^{d^2} y^d \left(\sum_{\alpha \in A_d} x^{n(\alpha)} y^{k(\alpha)} \right) \left(\sum_{\beta \in B_d} x^{n(\beta)} \right) \\
&= \sum_{d\geq 0} x^{d^2} y^d \prod_{i=1}^{d} \left(\frac{1}{1 - x^i y} \right) \prod_{i=1}^{d} \left(\frac{1}{1 - x^i} \right) \\
&= \sum_{d\geq 0} \frac{x^{d^2} y^d}{\prod_{i=1}^{d} (1 - x^i y)(1 - x^i)}
\end{aligned}$$

Examples

Ex. The generating function for the class of partitions where every part is even is derived by taking

$$M_k = \begin{cases} \{0\} & \text{if } k \text{ is odd} \\ \mathbb{N} & \text{if } k \text{ is even} \end{cases}$$

so

$$\Phi(z) = \prod_{k\geq 1} \left(\sum_{j\geq 0} (z^{2k})^j \right) = \prod_{k\geq 1} \frac{1}{1 - z^{2k}}$$

Ex. Find the generating function for the classes of partitions where

(1) Each part appears at most twice.
$M_k = \{0, 1, 2\}$ so

$$\Phi(z) = \prod_{k\geq 1} (1 + z^k + z^{2k}) = \prod_{k\geq 1} \left(\frac{1 - z^{3k}}{1 - z^k} \right)$$

(2) Each part is divisible by 3.

$$M_k = \begin{cases} \{0\} & \text{if 3 does not divide } k \\ \mathbb{N} & \text{if } 3|k \end{cases} = 3\mathbb{Z} \implies \Phi(z) = \prod_{k\geq 1} \left(\sum_{j\geq 0} z^{3j\cdot k} \right) = \prod_{k\geq 1} \frac{1}{1 - z^{3k}}$$

(3) Every even part is divisible by 4 and odd part occurs an even number of times.

$$M_k = \begin{cases} \{0\} & \text{if } k \text{ is even but not divisible by 4} \\ \mathbb{N} & \text{if } 4|k \\ 2\mathbb{N} & \text{if } k \text{ is odd} \end{cases}$$

so

$$\Phi(z) = \prod_{k\geq 1}\left(\sum_{j\geq 0}(x^{4k})^j \cdot \sum_{j\geq 0}(x^{2k-1})^{2j}\right) = \prod_{k\geq 1}\frac{1}{(1-x^{4k})(1-x^{2(2k-1)})}$$

Chapter 9

Labelled Constructions

Def Labelled Objects: A combinatorial object of size n is labelled by giving each of its atomic parts a distinct label from $[n] = \{1, \ldots, n\}$.

Def Labelled Class: A combinatorial class is labelled by labelling each of its objects, where every object is repeated with all non-equivalent labellings.

Def Exponential Generating Function (EGF): The exponential generating function of a sequence (c_n) is the formal power series

$$C(z) = \sum_{n\geq 0} \frac{c_n}{n!} z^n$$

whose coefficients are scaled by $n!$.

Def Labelled Atomic Class: The labelled atomic class $\mathcal{Z} = \{1\}$ is the combinatorial class containing one object of size 1 and no objects of any other size.

Def Labelled Neutral Class: The neutral class ϵ is the combinatorial class containing one labelled object of size 0 and no objects of any other size.

Def Labelled Sum: As in the unlabelled case, the labelled sum $\mathcal{A}+\mathcal{B}$ of two labelled combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ is the new labelled class whose objects are the disjoint union of the objects in $\mathcal{A}$ and $\mathcal{B}$, and whose size function is inherited from $\mathcal{A}$ and $\mathcal{B}$.

Lemma 1 If $\mathcal{C} = \mathcal{A} + \mathcal{B}$ as labelled classes then their exponential generating functions satisfy $C(z) = A(z) + B(z)$.

Def Consistent Relabelling: A consistent relabelling of a labelled pair (a, b) of size $|a| + |b|$ is an assignment of $\{1, 2, \ldots, |a| + |b|\}$ to the atomic elements in a and b such that each number is used exactly once, and the new labels on the elements of a (and b) are in the same relative order as they were in the original labelling of a (and b). We denote $a \star b$ to be the set of all consistent relabellings of (a, b).

Def Labelled Product: The labelled product $\mathcal{A} \times \mathcal{B}$ of the labelled classes $\mathcal{A}$ and $\mathcal{B}$ is the class whose objects form the union

$$\mathcal{A} \star \mathcal{B} = \bigcup_{a\in\mathcal{A}, b\in\mathcal{B}} (a \star b)$$

of all consistent relabellings of the pairs of objects in $\mathcal{A}$ and $\mathcal{B}$, with size function

$$|(\alpha, \beta)| = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}$$

Lemma 2 If $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ as labelled classes, then their exponential generating functions satisfy $C(z) = A(z)B(z)$.

Def Labelled Power Class: For any positive integer k and labelled class $\mathcal{A}$ we define the labelled power class as

$$\mathcal{A}^k = \underbrace{\mathcal{A} \star \mathcal{A} \star \cdots \star \mathcal{A}}_{k}$$

and has exponential generating function

$$A(z)^k$$

Def Labelled Sequence Class: When $\mathcal{A}$ has no objects of size 0, the labelled sequence class is

$$\mathrm{SEQ}(\mathcal{A}) = \epsilon + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \cdots$$

and has exponential generating function

$$\frac{1}{1 - A(z)}$$

Def Labelled Set: Let $\mathcal{A}$ be a labelled combinatorial class. For any positive integer k, the class $\mathrm{SET}_k(\mathcal{A})$ consists of all objects in $\mathcal{A}^k$ where two tuples are considered equal if they are equal up to a permutation.

When $\mathcal{A}$ has no objects of size 0, then we define

$$\mathrm{SET}(\mathcal{A}) = \epsilon + \mathrm{SET}_1(\mathcal{A}) + \mathrm{SET}_2(\mathcal{A}) + \mathrm{SET}_3(\mathcal{A}) + \cdots$$

Lemma 3 If $\mathcal{C} = \mathrm{SET}(\mathcal{A})$, then the exponential generating functions of these classes satisfy

$$C(z) = \sum_{k \geq 0} \frac{A(z)^k}{k!} = \exp(A(z))$$

Def Labelled Cycle: Let $\mathcal{A}$ be a labelled combinatorial class. For any positive integer k, the class $\mathrm{CYC}_k(\mathcal{A})$ consists of all objects in $\mathcal{A}^k$ where two tuples are considered equal if they are equal up to a cyclic shift.

When $\mathcal{A}$ has no objects of size 0, then we define

$$\mathrm{CYC}(\mathcal{A}) = \mathrm{CYC}_1(\mathcal{A}) + \mathrm{CYC}_2(\mathcal{A}) + \mathrm{CYC}_3(\mathcal{A}) + \cdots$$

Lemma 4 If $\mathcal{C} = \mathrm{CYC}(\mathcal{A})$, then the exponential generating functions of these classes satisfy

$$C(z) = \sum_{k \geq 1} \frac{A(z)^k}{k} = \log\left(\frac{1}{1 - A(z)}\right)$$

Def Disjoint Cycle Permutation Notation: A permutation can be viewed in disjoint cycle notation as a set of disjoint labelled cycles. Thus, if $\mathcal{S}$ denotes the class of permutations then

$$\mathcal{S} = \mathrm{SET}(\mathrm{CYC}(\mathcal{Z}))$$

Note

$$S(z) = \exp\left(\log\left(\frac{1}{1-z}\right)\right) = \frac{1}{1-z}$$

Def Restricted Constructions: We can use the notation

$$\mathrm{SEQ}_x(\mathcal{A}), \mathrm{SET}_x(\mathcal{A}), \mathrm{CYC}_x(\mathcal{A})$$

where x is a property.

- $\mathrm{SET}_{\leq r}(\mathcal{A})$ contains the sets of elements in $\mathcal{A}$ with at most r elements, with generating function
$$\sum_{k=0}^{r} \frac{A(z)^k}{k!}$$
- $\mathrm{CYC}_{\leq r}(\mathcal{A})$ contains the cycles of elements in $\mathcal{A}$ with at most r elements, with generating function
$$\sum_{k=0}^{r} \frac{A(z)^k}{k}$$
- $\mathrm{SET}_{\mathrm{even}}(\mathcal{A})$ is the union of $\mathrm{SET}_k(\mathcal{A})$ for even k and contains all sets of elements in $\mathcal{A}$ with even size.
- $\mathrm{CYC}_{\mathrm{even}}(\mathcal{A})$ is the union of $\mathrm{CYC}_k(\mathcal{A})$ for odd k and contains all sets of elements in $\mathcal{A}$ with odd size.

Parity Restrictions Define the series

$$\cosh(z) = \frac{e^z + e^{-z}}{2},\ \sinh(z)\frac{e^z - e^{-z}}{2}$$

and let $\mathcal{A}$ be a combinatorial class with no objects of size 0. Define $\mathcal{B} = \mathrm{SET}_{\mathrm{even}}(\mathcal{A})$ and $\mathcal{C} = \mathrm{SET}_{\mathrm{odd}}(\mathcal{A})$, then

$$B(z) = \cosh(A(z))$$
$$C(z) = \sinh(A(z))$$

If $\mathcal{D} = \mathrm{CYC}_{\mathrm{even}}(\mathcal{A})$ and $\mathcal{E} = \mathrm{CYC}_{\mathrm{odd}}(\mathcal{A})$, then

$$D(z) = -\frac{1}{2}\log(1 - A(z)^2)$$
$$E(z) = \frac{1}{2}\log\left(\frac{1 + A(z)}{1 - A(z)}\right)$$

Def Restricted Permutations: Permutations can be viewed both as a labelled sequence (1-line notation) and a set of cycles (disjoint cycle notation). We can restrict permutations in our labelled constructions.

Def Fixed Point: A fixed point in a permutation is a cycle of length 1, or equivalently a map from $[n]$ to $[n]$, a fixed point is a point that is sent to itself.

Def Derangement: A derangement is a permutation with no fixed points.
The exponential generating function of the class $\mathcal{D}$ of derangements is

$$S_{\text{no fix}}(z) = \frac{e^{-z}}{1-z}$$

Set Partitions A set partition is a set of non-empty labelled sets. Thus, if $\mathcal{S}$ is the labelled class of set partitions then

$$\mathcal{S} = \text{SET}(\text{SET}_{\geq 1}(\mathcal{Z}))$$

so

$$S(z) = e^{e^z - 1}$$

If $\mathcal{S}^{(r)}$ is the labelled class of set partitions with r blocks, then

$$\mathcal{S}^{(r)} = \text{SET}_r(\text{SET}_{\geq 1}(\mathcal{Z}))$$

so

$$S^{(r)}(z) = \frac{(e^z - 1)^r}{r!}$$

Thus,

$$\begin{aligned}
\left\{ \begin{matrix} n \\ r \end{matrix} \right\} &= n![z^n]S^{(r)}(z) \\
&= n![z^n]\frac{(e^z-1)^r}{r!} \\
&= n![z^n]\sum_{j\geq 0}\binom{r}{j}(-1)^{r-j}e^{zj} \\
&= \sum_{j\geq 0}\binom{r}{j}(-1)^{r-j}j^n
\end{aligned}$$

Def Functional Graphs: A mapping of size n is a function $f : [n] \to [n]$. A direct counting argument proves there are n^n mappings of size n, but interpreting them using labelled constructions allow for more refined study.

The key to enumerating restricted mappings is that a mapping can be viewed as a directed graph on the nodes $[n]$ with an edge $x \to y$ if and only if $f(x) = y$.

LIFT Remark: To get $[z^n]T(z)^k$, LIFT implies

$$[z^n]T(z)^k = \frac{k}{n}[u^{n-k}]e^{nu} = \frac{k}{n} \cdot \frac{n^{n-k}}{(n-k)!}$$

Map Enumeration Theorem Let $\mathcal{M}$ be the class of mappings, $\mathcal{K}$ be the class of connected functional graphs, and $\mathcal{T}$ be the class of rooted labelled trees. Then $\mathcal{M} = \text{SET}(\mathcal{K})$ and $\mathcal{K} = \text{CYC}(\mathcal{T})$ so that

$$M(z) = \frac{1}{1 - T(z)}$$

where $T(z) = ze^{T(z)}$.

Def Idempotent Mapping: An idempotent mapping f is a mapping that satisfies $f(f(z)) = f(z)$ for all z. The class of functional graphs corresponding to idempotent maps consist of connected components that are stars: a potentially empty set of single vertices pointed at a loop.

Since the class $\mathcal{X}$ of stars has the specification $\mathcal{X} = \mathcal{Z}\times\mathrm{SET}(\mathcal{Z})$, we see that $\mathcal{I} = \mathrm{SET}(\mathcal{Z}\times\mathrm{SET}(\mathcal{Z}))$, so

$$I(z) = e^{ze^z}$$

Corollary For any positive integer n,

$$\sum_{k\geq 0}\binom{n-1}{k-1}k!n^{-k} = 1$$

Def Labelled Bivariate EGF: The labelled bivariate EGF of a labelled class $\mathcal{C}$ with respect to a parameter $p : \mathcal{C} \to \mathbb{N}$ is

$$C(u,z) = \sum_{\sigma\in\mathcal{C}} u^{p(\sigma)}\frac{z^{|\sigma|}}{n!} = \sum_{n,k\geq 0}\frac{c_{k,n}}{n!}u^k z^n$$

where $c_{k,n}$ is the number of objects in $\mathcal{C}$ with size n and parameter value k.

Def Expected Value: Similar to unlabelled objects, we have

$$C_u(1,z) = \sum_{n,k\geq 0}\frac{kc_{k,n}}{n!}z^n$$

so

$$\mathbb{E}_n[p] = \frac{[z^n]C_u(1,z)}{[z^n]C(1,z)} = \frac{\frac{1}{n!}\sum_{k\geq 0}kc_{k,n}}{\frac{1}{n!}c_n} = \sum_{k\geq 0}k\frac{c_{k,n}}{c_n}$$

Remark: Calculus rules of derivatives and compositions of exponential and logarithmic functions are valid.

Def Bell Number: The nth Bell number B_n is defined by the exponential generating function

$$e^{e^z-1} = \sum_{n\geq 0}\frac{B_n}{n!}z^n$$

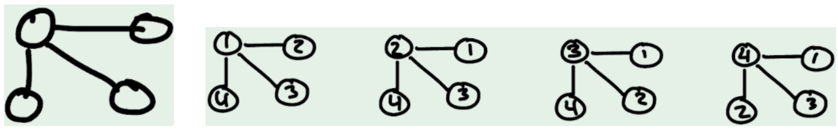
The Bell numbers also count the number of set partitions of size n, meaning that

$$B_n = \sum_{r=0}^{n}\left\{ {n \atop r} \right\}$$

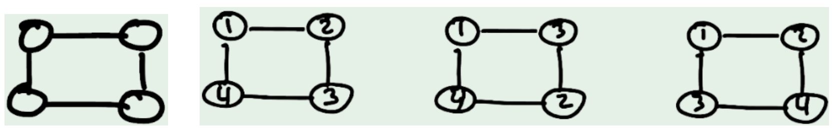
Def Double Factorial: The double factorial of n, denoted $n!!$ is the product of all the integers from 1 up to n that have the same parity.

Examples

Ex. Let $\mathcal{G}$ be the combinatorial class of abstract graphs with size given by number of vertices and consider the unlabelled graph with size 4. To assign the numbers $\{1, 2, 3, 4\}$ to the vertices of this graph it is sufficient to give a number to the unique vertex of degree 3, as all other vertices have the same edge relations. Therefore, this unlabelled graph corresponds to four labelled graphs.



Similarly, the unlabelled graph corresponds to the three labelled graphs.



Ex. The unlabelled class SEQ($\mathcal{Z}$) consists of all tuples of atoms, the labelled version contains the permutations of the tuples, so for each size of tuple in SEQ($\mathcal{Z}$), there are $n!$ objects.

Ex. The EGF for the class $\mathcal{S}$ of permutations is

$$S(z) = \sum_{n\geq 0} \frac{n!}{n!} z^n = \sum_{n\geq 0} z^n = \frac{1}{1-z}$$

Ex. The set $1 \star (1, 2, 3)$ has consistent relabelling

$$\{(1, (2, 3, 4)), (2, (1, 3, 4)), (3, (1, 2, 4)), (4, (1, 2, 3))\}$$

It does <u>not</u> have the element $(1, (3, 2, 4))$ since the labels in second object are not in increasing order.

Ex. The labelled product of the classes $\mathcal{A} = \{1, (1, 2)\}$ and $\mathcal{B} = \{1, (1, 2)\}$ is the class $\mathcal{A} \star \mathcal{B}$ containing the objects

$$(1, 2), (2, 1), (1, (2, 3)), (2, (1, 3)), (3, (1, 2)), ((1, 2), 3), ((1, 3), 2), ((2, 3), 1),$$
$$((1, 2), (3, 4)), ((1, 3), (2, 4)), ((1, 4), (2, 3)), ((2, 3), (1, 4)), ((2, 4), (1, 3)), ((3, 4), (1, 2))$$

Ex. All of elements of $\mathrm{SET}_3(\mathcal{A})$

$$(a_1, a_2, a_3), (a_1, a_3, a_2), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2), (a_3, a_2, a_1)$$

are all considered equal.

Ex. As elements of $\mathrm{CYC}_3(\mathcal{A})$ the objects

$$(a_1, a_2, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2)$$

are all considered equal, but are not equal to (a_1, a_3, a_2) since it is not a cyclic shift of these elements.

Ex. A non-planar rooted labelled tree of size n is a rooted tree on n vertices whose nodes are labelled with $[n]$ where the order of a vertices children do not matter. If $\mathcal{T}$ is the labelled class of non-planar rooted trees, then

$$\mathcal{T} = \mathcal{Z} \times \mathrm{SET}(\mathcal{T})$$

and the EGF $T(z)$ for the class of non-planar rooted labelled trees satisfies

$$z = \frac{T(z)}{e^{T(z)}} = \frac{T(z)}{G(T(z))}$$

where $G(u) = e^u$. LIFT implies that the number of non-planar rooted labelled trees on n nodes is

$$n![z^n]T(z) = \frac{n!}{n}[u^{n-1}]G(u)^n = (n-1)![u^{n-1}]e^{un} = n^{n-1}$$

Because there are n ways to root a tree on n nodes, the number of non-planar non-rooted labelled trees on N nodes is n^{n-2} when $n \geq 1$.

Ex. Find the EGF for the class $\mathcal{F}$ of permutations with no fixed points (derangements).

By definition $\mathcal{F}$ consists of permutations with no cycles of length 1, with specification

$$\mathcal{F} = \mathrm{SET}(\mathrm{CYC}_{\geq 2}(\mathcal{Z}))$$

Since the generating function of $\mathrm{CYC}_{\geq 2}(\mathcal{Z})$ is

$$\sum_{k\geq 2} \frac{z^k}{k} = \log\left(\frac{1}{1-z}\right) - z$$

we have

$$F(z) = \exp\left(\log\left(\frac{1}{1-z}\right) - z\right) = \frac{e^{-z}}{1-z}$$

Therefore, the number of derangements with size n is

$$\begin{aligned} n![z^n]F(z) &= n![z^n]\frac{\sum_{k\geq 0}\frac{(-z)^k}{k!}}{1-z} \\ &= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} \end{aligned}$$

The probability that a random derangement of size $n \to \infty$ is

$$\lim_{n\to\infty}\frac{n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}}{n!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1}$$

Ex. Fix a positive integer r.

1. Find the EGF for permutations with no cycles of length r.

   If $\mathcal{A}$ is the class of permutations with no r-cycles then $\mathcal{A} = \mathrm{SET}(\mathrm{CYC}_{\neq r}(\mathcal{Z}))$ then since $\mathrm{CYC}_{\neq r}(\mathcal{Z})$ has EGF

   $$\sum_{j\geq 1, j\neq r}\frac{z^j}{j} = \sum_{j\geq 1}\frac{z^j}{j} - \frac{z^r}{r} = \log\left(\frac{1}{1-z}\right) - \frac{z^r}{r}$$

   we have

   $$A(z) = \exp\left(\log\left(\frac{1}{1-z}\right) - \frac{z^r}{r}\right) = \frac{e^{-z^r/r}}{1-z}$$

2. Find the EGF for permutations where all cycles have length larger than r.
   If $\mathcal{B} = \mathrm{SET}(\mathrm{CYC}_{>r}(\mathcal{Z}))$ then since $\mathrm{CYC}_{>r}(\mathcal{Z})$ has EGF

$$\sum_{j>r} \frac{z^j}{j} = \log\left(\frac{1}{1-z}\right) - \sum_{j=1}^{r} \frac{z^j}{j}$$

   we have

$$B(z) = \frac{e^{-z-z^2/2-\cdots-z^r/r}}{1-z}$$

Ex. Find the EGF of mappings with no fixed points.

The functional graphs of mappings with with no fixed points are the function graphs with no cycles of length 1. So the class $\mathcal{F}$ of mappings with no fixed points satisfies

$$\mathcal{F} = \mathrm{SET}(\mathrm{CYC}_{\geq 2}(\mathcal{T}))$$

So

$$F(z) = \exp\left(\log\left(\frac{1}{1-T(z)} - T(z)\right)\right) = \exp\left(\log\left(\frac{1}{1-T(z)}\right)\right) \cdot \exp(-T(z)) = \frac{e^{-T(z)}}{1-T(z)}$$

Ex. A surjection of size n is any map f from $[n]$ onto a set of the form $[r]$. If $\mathcal{R}^{(r)}$ is the class of surjections onto $[r]$, then an element of $\mathcal{R}^{(r)}$ is defined by the pre-image (numbers sent) to each element of $[r]$.

$$f^{-1}(x) = \text{ set of numbers sent to } x$$

Thus, $\mathcal{R}^{(r)} = \mathrm{SEQ}_r(\mathrm{SET}_{\geq 1}(\mathcal{Z}))$. So

$$R^{(r)}(z) = (e^z - 1)^r$$

and the full class $\mathcal{R}$ of surjections satisfies

$$\mathcal{R} = \mathrm{SEQ}(\mathrm{SET}_{\geq 1}(\mathcal{Z})) \implies R(z) = \frac{1}{2-e^z}$$

Ex. A triangle tree is a labelled connected graph in which every edge is in exactly 1 cycle and that cycle has length 3. Find the number of triangle trees on n vertices.

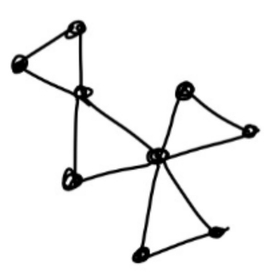


Figure: A triangle tree of size 9.

Let $\mathcal{R}$ be the class of rooted triangle trees. Removing the root of an element in $\mathcal{R}$ gives a set of connected components. Each can be viewed as a set of two rooted triangle trees.

$$\mathcal{R} = \mathcal{Z} \times \mathrm{SET}(\mathrm{SET}_2(\mathcal{R})) \implies R(z) = ze^{R(z)^2/2}$$

Using LIFT with $G(u) = e^{u^2/2}$, we get

$$[z^n]R(z) = \frac{1}{n}[u^{n-1}]e^{nu^2/2}$$

If n is even, then $[z^n]R(z) = 0$, otherwise if $n = 2k+1$, then

$$[z^n]R(z) = \frac{1}{n}[u^{n-1}]\sum_{k\geq 0}\frac{n^k u^{2k}}{2^k \cdot k!} = \frac{1}{n}\cdot\frac{n^k}{2^k \cdot k!} = \frac{n^{k-1}}{2^k \cdot k!} = \frac{(2k+1)^{k-1}}{2^k \cdot k!}$$

There are

$$n![z^n]R(z) = (2k+1)![z^n]R(z) = \frac{(2k)!(2k+1)^k}{2^k \cdot k!}$$

rooted triangle trees and there are

$$\frac{n!}{n}R(z) = \frac{(2k+1)!}{(2k+1)}[z^n]R(z) = \frac{(2k)!(2k+1)^{k-1}}{2^k \cdot k!}$$

triangle trees on n vertices.

Ex. Find the average number of cycles among the permutations of size n.

The chain rule implies that the generating function $S(u,z) = e^{u\log\frac{1}{1-z}}$ satisfies

$$\frac{d}{du}S(u,z)\bigg|_{u=1} = e^{u\log\frac{1}{1-z}}\log\frac{1}{1-z}\bigg|_{u=1} = \frac{1}{1-z}\log\frac{1}{1-z}$$

Since $\log\frac{1}{1-z} = \sum_{k\geq 1}\frac{z^k}{k}$, this means that the average number of cycles among the permutations of size n is

$$\frac{[z^n]S_u(1,z)}{[z^n]S(1,z)} = [z^n]\frac{1}{1-z}\log\frac{1}{1-z} = \sum_{k=1}^{n}\frac{1}{k}$$

We note that $H_n = \sum\limits_{k=1}^{n}\frac{1}{k}$ is the nth Harmonic number.

Ex. Find the average root degree among the rooted labelled trees with n vertices.

Recall the specification $\mathcal{T} = \mathcal{Z}\times \mathrm{SET}(\mathcal{T})$. So the parameterized specification is

$$\mathcal{R} = \mathcal{Z}\times\mathrm{SET}(\mu\times\mathcal{T})$$

We see that $T(z) = ze^{T(z)}$ and $R(u,z) = ze^{uT(z)}$ so that

$$\frac{d}{du}R(u,z)\bigg|_{u=1} = zT(z)e^{uT(z)}\Big|_{u=1} = zT(z)e^{T(z)} = T(z)^2$$

Thus, LIFT implies

$$[z^n]R_u(1,z) = [z^n]T(z)^2 = \frac{2}{n}[t^{n-2}]e^{nt} = \frac{2}{n}\cdot\frac{n^{n-2}}{(n-2)!}$$

so the average root degree is

$$\frac{\frac{2}{n}\cdot\frac{n^{n-2}}{(n-2)!}}{\frac{n^{n-1}}{n!}} = \frac{2(n-1)}{n}$$

Ex. Find the average number of blocks among the set partitions of size n in terms of B_n.

The specification for the class of set partitions is

$$\mathcal{S} = \mathrm{SET}(\mathrm{SET}_{\geq 1}(\mathcal{Z}))$$

Marking blocks gives the parameterized labelled specification

$$\mathcal{S} = \mathrm{SET}(\mu \times \mathrm{SET}_{\geq 1}(\mathcal{Z}))$$

corresponding to the EGF

$$S(u,z) = e^{u(e^z-1)}$$

which implies

$$\frac{d}{du}S(u,z)\Big|_{u=1} = (e^z-1)e^{u(e^z-1)}\Big|_{u=1} = e^z e^{e^z-1} - e^{e^z-1}$$

Since $[z^n]S(1,z) = \frac{B_n}{n!}$, we see that

$$[z^n]e^z e^{e^z} - 1 = [z^n]\frac{d}{dz}S(1,z) = \frac{B_{n+1}}{n!}$$

so the average number of blocks is

$$\frac{[z^n]e^z e^{e^z-1} - [z^n]e^{e^z-1}}{[z^n]e^{e^z-1}} = \frac{B_{n+1}}{B_n} - 1$$

Part III

Asymptotics

Chapter 10

Basics of Asymptotics

Def Eventually Positive: A sequence p_n is called eventually positive if there exists some $N \in \mathbb{N}$ such that $p_n > 0$ for all $n \geq N$.

Def Asymptotic: We write $f_n \sim g_n$ and say that f_n is asymptotic to g_n if

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 1$$

Def Big-O: We write $f_n = O(g_n)$ or $f_n \in O(g_n)$ and say that f_n is big-O of g_n if there exist $c > 0$ and $N \in \mathbb{N}$ such that

$$f_n \leq c \cdot g_n \text{ for all } n \geq N$$

Def Little-O: We write $f_n = o(g_n)$ or $f_n \in o(g_n)$ and say that f_n is little-O of g_n if

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 0$$

Proposition A sequence $f_n = o(1)$ if and only if $f_n \to 0$ as $n \to \infty$.

Def Stirling's Approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Asymptotic of Unlabelled Graphs The asymptotic behvaiour of g_n, which is the number of unlabelled graphs on n vertices is

$$g_n \sim \frac{2^{\binom{n}{2}}}{n!}$$

Asymptotic of Integer Partitions If p_n denotes the number of integer partitions of size n, then the asymptotic formula is

$$p_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

Lemma 1 Let f_n, g_n, a_n, b_n be eventually positive sequences.

- If $f_n = O(a_n)$ and $g_n = O(b_n)$, then

$$f_n + g_n = O(a_n + b_n) \text{ and } f_n g_n = O(a_n b_n)$$

- If $f_n = o(a_n)$ and $g_n = o(b_n)$, then

$$f_n + g_n = o(a_n + b_n) \text{ and } f_n g_n = o(a_n b_n)$$

Lemma 2 Let f_n, g_n, a_n, b_n be eventually positive sequences.

(a) $f_n = g_n(1 + o(1))$ if and only if $f_n \sim g_n$

(b) If $f_n = a_n + b_n$ with $a_n \sim g_n$ and $b_n = o(g_n)$, then $f_n \sim g_n$

(c) If $f_n = a_n + b_n$ where $a_n, b_n = O(g_n)$, then $f_n = O(g_n)$

Def Asymptotic Hierarchy:

Family Name	**General Member**
Constants	$f_1 = A$ for some $A > 0$
Log-Powers	$f_2 = (\log n)^B$ for some $B > 0$
Powers of n	$f_3 = n^C$ for some $C > 0$
Exponentials	$f_4 = D^n$ for some $D > 1$
n^n Powers	$f_5 = n^{En}$ for some $E > 0$

Lemma 3 In the asymptotic hierarchy, $f_k = o(f_{k+1})$ for $i = 1, 2, 3, 4$.

Theorem Let $a < b$ be integers and let $f(x)$ be a continuous function on the interval $[a-1, b+1]$.

(a) If f is increasing on $[a-1, b+1]$, then

$$\int_{a-1}^{b} f(x)\,dx \leq \sum_{k=a}^{b} f(k) \leq \int_{a}^{b+1} f(x)\,dx$$

(b) If f is decreasing on $[a-1, b+1]$, then

$$\int_{a-1}^{b} f(x)\,dx \geq \sum_{k=a}^{b} f(k) \geq \int_{a}^{b+1} f(x)\,dx$$

Corollary Let $a < b$ be integers and $f(x)$ be a continuous function on $I = [a-1, b+1]$. If f is either increasing or decreasing on I, and $|f(x)| \leq M$ on I, then

$$\left| \sum_{k=a}^{b} f(k) - \int_{a}^{b} f(x)\,dx \right| \leq M$$

Examples

Ex. If $f_n = n^2 + 1000n - 17$ and $g_n = n^2$, then $f_n \sim g_n$ since

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \lim_{n\to\infty} \frac{n^2 + 1000n - 17}{n^2} = 1$$

Ex. If $f_n = n^2 + 1000n - 17$ and $g_n = 2n^2$, then $f_n = O(g_n)$ since $f_n \le g_n$ for all $n \ge 1000$ because $1000n - 17 \le n^2$ when $n \ge 1000$.

Ex. If $f_n = 2n^2 + n$ and $g_n = n^2$, then $f_n = O(g_n)$ since $f_n \le 3g_n$ for all $n \ge 3$.

Ex. If $f_n = n^2 + 1000n - 17$ and $g_n = n^3$, then $f_n = o(g_n)$ since

$$\lim_{n\to\infty} \frac{f_n}{g_n} = \lim_{n\to\infty} \frac{n^2 + 1000n - 17}{n^3} = 0$$

Ex. Let $\alpha > 0$. Since the function $f(x) = x^\alpha$ has $f'(x) = \alpha x^{\alpha-1} > 0$ for any $x > 0$, we see that f is increasing and the corollary implies

$$\sum_{k=1}^{n} k^\alpha = \frac{n^{\alpha+1}}{\alpha+1} + O(n^\alpha)$$

Ex. Find the asymptotic behaviour of the Harmonic number $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$.

Because $f(x) = 1/x$ is decreasing for $x > 0$, we see that

$$H_n \ge \int_1^{n+1} \frac{1}{x}\,dx = \log(x)|_1^{n+1} = \log(n+1)$$

Since $f(x)$ is not defined at $x = 0$, we take out the first term of the sum and bound

$$H_n - 1 = \sum_{k=2}^{n} \frac{1}{k} \le \int_1^{n} \frac{1}{x}\,dx = \log n$$

Thus,

$$\log n < \log(n+1) \le H_n \le \log n + 1$$

so $H_n \sim \log n$.

Ex. Quicksort Let c_n = average number of comparisons quicksort does on a permutation on n. $c_0 = c_1 = 0$. If $n \ge 1$, we have $c_n = \frac{1}{n}\sum_{k=1}^{n}[(n-1) + c_{k-1} + c_{n-k}]$. So

$$c_n = \frac{1}{n}\sum_{k=1}^{n}[(n-1) + c_{k-1} + c_{n-k}]$$

$$nc_n = n(n-1) + \sum_{k=1}^{n} c_{k-1} + \sum_{k=1}^{n} c_{n-k}$$

$$nc_n = n(n-1) + 2\sum_{k=0}^{n-1} c_k$$

Substitute $n = n - 1$, we have

$$(n-1)c_{n-1} = (n-1)(n-2) + 2\sum_{k=0}^{n-2} c_k$$

Subtracting,

$$\begin{aligned}
nc_n - (n-1)c_{n-1} &= n(n-1) - (n-1)(n-2) + 2c_{n-1} \\
nc_n &= (n+1)c_{n-1} + 2(n-1) \\
\frac{c_n}{n+1} &= \frac{c_{n-1}}{n} + \frac{2}{n+1} - \frac{2}{n(n+1)} \\
&\le \frac{c_{n-1}}{n} + \frac{2}{n+1} \\
&\le \frac{c_{n-2}}{n-2} + \frac{2}{n} + \frac{2}{n+1} \\
&\vdots \\
&\le \frac{c_1}{2} + \sum_{k=2}^{n} \frac{2}{k+1} \\
&= 2\sum_{k=2}^{n} \frac{1}{k+1}
\end{aligned}$$

By theorem,

$$2\sum_{k=2}^{n} \frac{1}{k+1} \le 2\int_{1}^{n} \frac{1}{x+1}\,dx = 2(\log(n+1) - \log 2) \le 2\log(n+1)$$

so $c_n \le 2(n+1)\log(n+1) = O(n\log n)$ and $c_n \sim n\log n$.

Chapter 11

Rational Asymptotics

Def C-Finite Sequence: A C-finite sequence $(f_n) = f_1, f_2, \ldots$ is a sequence that satisfies a linear recurrence

$$f_n + c_1 f_{n-1} + \cdots + c_r f_{n-r} = 0, \ \forall n \geq N$$

where each $c_1, \ldots, c_r \in \mathbb{Q}$ with $c_r \neq 0$ and $N \geq r$. We say the integer r is the order of the sequence. The sequence is determined by the first N terms $f_0, \ldots, f_{N-1}$ called the initial condition.

Def Virahanka-Fibonacci Sequence: The C-finite sequence $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$ with initial conditions $f_0 = f_1 = 1$. We usually write $f_n - f_{n-1} - f_{n-2} = 0$ for $n \geq 2$.

Theorem The sequence (f_n) satisfies $f_n + c_1 f_{n-1} + \cdots + c_r f_{n-r} = 0$ if and only if its generating function $F(z) = \sum\limits_{n \geq 0} f_n z^n$ is the rational function $F(z) = \frac{G(z)}{H(z)}$ where

$$\begin{aligned} G(z) &= g_0 + g_1 z + \cdots + g_{N-1} z^{N-1} \\ H(z) &= 1 + c_1 z + \cdots + c_r z^r \end{aligned}$$

where

$$g_k = f_k + c_1 f_{k-1} + \cdots + c_r f_{k-r}$$

for all $0 \leq n \leq N-1$ and $f_k = 0$ if $k < 0$.

Corollary If (f_n) and (g_n) are C-finite, then so are $(f_n + g_n)$ and $\left(\sum\limits_{k=0}^{n} f_k g_{n-k}\right)$.

Corollary (f_n) is C-finite if and only if its generating function

$$F(z) = \frac{P(z)}{Q(z)}$$

is rational for polynomials P, Q where $Q(0) \neq 0$.

Fundamental Theorem of Algebra Any polynomial $P(x) \in \mathbb{Q}[x]$ factors into

$$P(z) = C(x - \lambda_1)^{d_1} \cdots (x - \lambda_s)^{d_s}$$

where $C \in \mathbb{C}$ and the $\lambda_i \in \mathbb{C}$ are distinct.
We call d_i the multiplicity of λ_i. A root of multiplicity 1 is a called single root.

Lemma Test For Multiplicity The multiplicity of a root λ_i of $P(x) \in \mathbb{Q}[x]$ is the smallest positive integer k where

$$P(\lambda_i) = P'(\lambda_i) = \cdots = P^{(k-1)}(\lambda_i) = 0$$

and

$$P^{(k)}(\lambda_i) \neq 0$$

Partial Fraction Decomposition If $F(z) = \frac{G(z)}{H(z)}$ where $\deg G < \deg H$ and

$$H(z) = C(z - \lambda_1)^{d_1} \cdots (z - \lambda_s)^{d_s}$$

where the $\lambda_i \in \mathbb{C}$. Then there exist $C_i^{(j)}$ such that

$$F(z) = \sum_{i=1}^{s} \sum_{j=1}^{d_i} \frac{C_i^{(j)}}{(1 - z/\lambda_i)^j}$$

C-Finite Coefficient Theorem Suppose that (f_n) is a C-finite sequence with rational generating function

$$F(z) = R(z) + \frac{G(z)}{H(z)}$$

where $\deg G < \deg H$ and $H(0) = 1$. If $\lambda_1, \ldots, \lambda_s$ form the distinct roots of H with multiplicities $d_1, \ldots, d_s$, then

$$f_n = p_1(n)\lambda_1^{-n} + \cdots + p_s(n)\lambda_s^{-n}$$

for all $n > \deg R$, where $\deg p_i(n) \leq d_i - 1$.
Note: $\deg P_i = d_i - 1$ if G and H are coprime.

Corollary If (f_n) satisfies

$$f_n + c_1 f_{n-1} + \cdots + c_r f_{n-r} = 0$$

for all $n \geq r$, then $f_n = p_1(n)\lambda_1^{-n} + \cdots + p_s(n)\lambda_s^{-n}$ for all $n \geq 0$, where the λ_i are the roots of the characteristic polynomial

$$H(z) = 1 + c_1 z + \cdots + c_r z^r$$

of the recurrent and $p_i(n)$ is a polynomial in n of degree one less than the multiplicity of λ_i as a root of H.

Lemma Assume the hypotheses of the C-Finite Coefficient Theorem. If λ_k is a simple root of $H(z)$ and $G(\lambda_k) \neq 0$, then the polynomial p_k in the explicit formula for f_n satisfies

$$p_k(n) = \frac{-G(\lambda_k)}{\lambda_k H'(\lambda_k)}$$

Lemma Assume the hypotheses of the C-Finite Coefficient Theorem. If λ_k is a root of multiplicity m of $H(z)$ and $G(\lambda_k) \neq 0$, then the polynomial p_k in the explicit formula for f_n satisfies

$$p_k(n) = \frac{m(-1)^m G(\lambda_k)}{\lambda_k^m H^{(m)}(\lambda_k)} n^{m-1} + O(n^{m-2})$$

Def Modulus: The modulus of a complex number $z = a + bi$ with $a, b \in \mathbb{R}$ is the non-negative real number $|z| = \sqrt{a^2 + b^2}$. If f_n is any sequence of complex numbers and g_n is any sequence of real numbers that is eventually positive, then we write

- $f_n = O(g_n)$ if $|f_n| = O(g_n)$.
- $f_n = o(g_n)$ if $|f_n| = o(g_n)$.
- Since $|z^n| = |z|^n$, then if $|z| < w$, then $|z^n| = |z|^n = o(w^n)$.

C-Finite Asymptotic Theorem Assume the hypotheses of the C-Finite Coefficient Theorem and the polynomials G, H are coprime.

1. If $|\lambda_1| < |\lambda_k|$ for all $k = 2, \ldots, s$, then
$$f_n = p_1(n)\lambda_1^{-n} + O(w^{-n})$$
where $w = \min_{2 \le k \le s} |\lambda_k| > |\lambda_1|$.
2. If $|\lambda_1| \le |\lambda_k|$ for all $k = 2, \ldots, s$ and λ_1 has higher multiplicity m than all other roots λ_k with $|\lambda_1| = |\lambda_k|$, then
$$f_n = \frac{m(-1)^m G(\lambda_1)}{\lambda_1^m H^{(m)}(\lambda_1)} n^{m-1} \lambda_1^{-n} + O(n^{m-2})$$

Skolem's Problem Is there an algorithm to decide if a C-finite sequence contains a zero term. Only decidable for order ≤ 4.

Ultimate Positivity Problem Is there an algorithm to decide if a C-finite sequence is eventually positive.

Def $\mathbb{N}$-Rational Functions: The set of $\mathbb{N}$-rational function is the set R of rational functions such that

- $0, 1, z \in R$.
- If $f, g \in R$, then $f + g \in R$.
- If $f, g \in R$, then $fg \in R$.
- If $f \in R$ and $f(0) = 0$, then $\frac{1}{1-f} \in R$.

We define the set of $\mathbb{N}$-rational functions to be the smallest set R of rational functions containing 1 and the variable z such that whenever $f, g \in R$, then the sum $f + g$, the product fg, and the pseudo-inverse $\frac{1}{1-zf}$ all in R.

Theorem Berstel If $F(z) = \frac{G(z)}{H(z)}$ is a $\mathbb{N}$-rational function and λ is a root of H with $|\lambda|$ minimal, then there exists a positive integer r such that $\lambda^r = |\lambda|^r$.

Corollary We can always decide asymptotic behaviour for $\mathbb{N}$-rational functions.

Meta-Principle Every rational generating function coming from a combinatorial problem is $\mathbb{N}$-rational.

Corollary There is no finite automata recognizing binary trees.

Computing Fibonacci Sequence

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix}$$

Examples

Ex. The generating function for binary strings with no blocks of size larger than k is

$$F(z) = \frac{1 - z^{k+1}}{1 - 2z + z^{k+1}}$$

Thus, the sequence satisfies $f_n = 2f_{n-1} - f_{n-k-1}$ for all $n \geq k + 2$.

Ex. Conway's Look and Say sequence:

$$(l_n) = 1, 11, 21, 1211, 111221, 312211, 13112221, \ldots$$

If we let d_n = number of digits in l_n, then

$$D(z) = \frac{G(z)}{H(z)} = \frac{1 + z + \cdots + 18z^{77} - 12z^{78}}{1 - z + \cdots - 9z^{71} + 6z^{72}}$$

Ex. If $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 0$, then a partial fraction decomposition shows

$$F(z) = \frac{1}{1 - z - z^2} = \frac{1}{\sqrt{5}} \left(\frac{1/\sigma}{1 - z/\sigma} - \frac{1/\tau}{1 - z/\tau} \right)$$

where $\sigma = \frac{-1+\sqrt{5}}{2}$ and $\tau = \frac{-1-\sqrt{5}}{2}$ are the roots of $1 - z - z^2$. Thus,

$$f_n = \frac{1}{\sigma\sqrt{5}} \sigma^{-n} - \frac{1}{\tau\sqrt{5}} \tau^{-n}$$

Ex. Find a closed form for (f_n) that satisfies $f_{n+3} = 3f_{n+1} - 2f_n$ for all $n \geq 0$ where $f_0 = f_1 = 4, f_2 = 13$.

The characteristic polynomial $H(z) = 1 - 3z^2 + 2z^3 = (1 - z)^2(1 + 2z)$, the C-Finite Coefficient Theorem implies there exist $A, B, C \in \mathbb{C}$ such that

$$f_n = (An + b) \cdot 1^n + C \cdot (-2)^n$$

Substituting $n = 0, 1, 2$ into the equation gets

$$\begin{aligned} &(n = 0) & 4 &= f_0 = B + C \\ &(n = 1) & 4 &= f_1 = A + B - 2C \\ &(n = 2) & 13 &= f_2 = 2A + B + 4C \end{aligned}$$

Solving this gets $(A, B, C) = (3, 3, 1)$ so that

$$f_n = 3(n + 1) + (-2)^n$$

Ex. The class of integer partitions with parts of size at most 3 has rational generating function

$$F(z) = \frac{1}{(1 - z)(1 - z^2)(1 - z^3)}$$

Find the asymptotic behaviour of p_n as $n \to \infty$.

Since the denominator of F is $H(z) = (1-z)^3(1+z)(1+z+z^2)$. The root $z = 1$ has multiplicity 3 and all other roots have multiplicity 1 (all have modulus 1). Also, $H^{(3)}(1) = -36$, so

$$p_n \sim \frac{3(-1)^3}{1^3 H^{(3)}(1)} \cdot 1^n \cdot n^{3-1} + O(n^{3-2}) = \frac{-3}{-36}n^2 + O(n) = \frac{n^2}{12}$$

Ex. Consider the look and say sequence d_n with generating function $D(z)$. $H(z)$ has a positive real root $\lambda = 0.7671\ldots$, and all other roots have larger modulus that λ, and $H'(\lambda), G(\lambda) \neq 0$. Therefore,

$$d_n \sim \frac{-G(\lambda)}{\lambda H'(\lambda)} \approx (2.04\ldots)(1.303\ldots)^n$$

Ex. The coefficients

$$[z^n]\left(\frac{1}{1-4z^2} + \frac{1}{1-z/2}\right) = \begin{cases} 2^n + 2^{-n} & \text{if } n \text{ even} \\ 2^{-n} & \text{if } n \text{ odd} \end{cases}$$

grow to infinity like 2^n when n is even, and decay to zero like 2^{-n} when n is odd.

Ex. If $F(z) = \frac{2(1+3z)^2}{(1-9z)(1+14z+81z^2)}$, then $F(z)$ has a series expansion with coefficients in $\mathbb{N}$, but the roots do not have a positive integer power being positive.

Ex. Prove the Catalan number sequence $c_n = \frac{1}{n+1}\binom{2n}{n}$ is not C-finite.

The Catalan generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ is not rational, so (c_n) is not C-finite.

Ex. Prove the series

$$F(z) = \sum_{n\geq 1} \frac{z^n}{n^5}$$

is irrational?

The sequence $f_n = n^{-5}$ is not C-finite, because it does not satisfy the conditions of the C-Finite Coefficient Theorem, so its generating function $F(z)$ is not rational.
It has been an open problem if

$$\zeta(5) = F(1) = \sum_{n\geq 1} \frac{1}{n^5}$$

is rational.

Ex. Let p_n be the nth prime number. Prove that (p_n) is not C-finite.

The prime number theorem implies

$$p_n = n\log n + n\log\log n + O(n)$$

and this asymptotic behaviour is not possible for C-finite sequences.

Chapter 12

Analytic Combinatorics

Def Subsequence: A subsequence of (f_n) is any sequence (f_{k_n}) with $0 \leq k_1 < k_2 < k_3 < \cdots$. It is obtained by selecting some terms of (f_n).

Def Limit Superior: The limit superior $\limsup\limits_{n\to\infty} x_n$ of any real sequence (x_n) is the supremum of all limits of all subsequences of (x_n).

Theorem (Root Test) Let $\rho = \limsup\limits_{n\to\infty} |a_n|^{1/n}$ and define

$$R = \begin{cases} 1/\rho & \text{if } 0 < \rho < \infty \\ \infty & \text{if } \rho = 0 \\ 0 & \text{if } \rho = \infty \end{cases}$$

Then, $F(z)$ converges whenever $|z| < R$ and diverges if $|z| > R$.

Def Radius of Convergence: We call the value R defined in the Root Test to be the radius of convergence of $F(z)$.

Theorem (Ratio Test) If $\lim\limits_{n\to\infty} \left|\frac{f_{n+1}}{f_n}\right|$ exists, then it equals the value ρ computed in the root test.

Def Exponential Growth: When we restrict tot he case of a series $F(z)$ with finite positive radius of convergence $R = 1/\rho$, we can write

$$f_n = \rho^n \cdot \theta(n) + O(\alpha^n)$$

for some function θ that grows slower than any exponential function and $0 < \alpha < \rho$.
We call ρ^n the exponential growth of (f_n) and $\theta(n)$ the sub-exponential growth of (f_n).

Def Singularity: A singularity of F is roughly a point where F is not analytic.
Division by zero, putting zero into a root, or putting zero into a logarithm.

Def Dominant Singularity: The singularities of F closest to the origin are called its dominant singularities. They dictate the dominant asymptotic behaviour of its coefficient sequence.

Determining Exponential Growth from Singularities The exponential growth of (f_n) can be deduced immediately from the first positive real singularity of $F(z)$. If the singularity is at $z = \rho$, then the exponential growth is ρ^{-n}.

Theorem If $F(z) = \sum_{n\geq 0} f_n z^n$ has a finite positive radius of convergence R, then

$$R = \text{ minimum modulus of the singularities of } F \text{ over the complex numbers}$$

Vivanti-Pringsheim Theorem If $F(z) = \sum_{n\geq 0} f_n z^n$ and $f_n \geq 0$ for all n, then the radius of convergence R of (f_n) is a minimal modulus singularity of $F(z)$.

Def Analytic Function: The complex function $F(z)$ is analytic at $z = a$ if its Taylor series

$$F(z) = \sum_{n\geq 0} \frac{F^{(n)}(n)}{n!}(z-a)^n$$

exists and has a positive radius of convergence.

- $e^z, \sin(z), \cos(z)$ are analytic in the entire complex plane.
- If $p(z)$ is analytic at $z = a$ and $p(a) \neq 0$, then $\sqrt{p(z)}$, $\frac{z}{p(x)}$, and $\log p(z)$ are analytic at $x = a$.
- If $f(z)$ and $g(z)$ are analytic at $z = a$, then $f(z) + g(z)$ and $f(z)g(z)$ are analytic at $z = a$.
- If $g(z)$ is analytic at $z = a$ and $f(z)$ is analytic at $z = g(a)$, then $f(g(z))$ is analytic at $z = a$.

Meromorphic Asymptotic Theorem Suppose $F(z) = \frac{G(z)}{H(z)}$ is the ratio of complex functions G and H that are analytic in $|z| \leq B$ for some $B > 0$. If

- $H(z)$ has a single zero $z = w$ with $|z| < B$ and no zeroes with $|z| = B$, and
- $H'(w) \neq 0$ and $G(w) \neq 0$

then

$$[z^n]F(z) = \frac{-G(w)}{w \cdot H'(w)} \cdot w^{-n} + O(B^{-n})$$

The Principles of Analytic Combinatorics

- **First Principle of Analytic Combinatorics**: The locations of the singularities of a generating function determine the exponential growth of its coefficient sequence.
- **Second Principle of Analytic Combinatorics**: The type of the singularities of a generating function determine the sub-exponential growth of its coefficient sequence.

The type of a singularity includes dividing by zero with a multiplicity, putting zero in a root, zero in a logarithm, etc.

Def Alternating Permutation: An alternating permutation is a permutation π such that $\pi_1 > \pi_2 < \pi_3 > \cdots$. The exponential generating function of alternating permutations is

$$T(z) = \tan(z)$$

Examples

Ex. If $x_n = \begin{cases} 2 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$, then $\limsup_{n\to\infty} x_n = 2$.

Ex.

- Since $\lim_{n\to\infty}(2^n)^{1/n} = 2$, the radius of convergence of

$$\frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

  is $\frac{1}{2}$.

- Since $\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$, the radius of convergence of

$$\sum_{n\geq 0} n! z^n$$

  is 0.

- Since $\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = 1$, the radius of convergence of

$$-\log(1-z) = \sum_{n\geq 0} \frac{z^n}{n}$$

  is 1.

- Since $\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = 0$, the radius of convergence of

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}$$

  is ∞.

Ex.

- The singularity of $F(z) = \frac{1}{1-2z}$ is $z = \frac{1}{2}$.
- The singularities of $F(z) = \frac{1}{1+z^2}$ are $z = i, -i$.
- The EGF of derangements is $F(z) = \frac{e^{-z}}{1-z}$ which has singularity at $z = 1$.
- The Catalan generating function $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ has a singularity at $z = \frac{1}{4}$. Even though $z = 0$ might be a singularity, the fact that the numerator vanishes at $z = 0$ implies this singularity can be removed.

Ex.

- If $F(z) = \frac{1}{1-2z}$, then f_n has exponential growth 2^n.

- If $F(z) = \frac{e^{-z}}{1-z}$, then f_n has exponential growth $1^n = 1$.
- If $F(z) = \frac{1-\sqrt{1-4z}}{2z}$, then f_n has exponential growth 4^n. We can also show using Stirling's approximation that $f_n \sim \frac{4^n}{\sqrt{n^3\pi}}$.
- The EGF of surjections is $F(z) = \frac{1}{2-e^z}$. Since $z = \log 2$ is the first and only positive real singularity of F, the exponential growth of f_n is $(\log 2)^{-n}$.
  F has an infinite number of singularities $\{\log 2 + k(2\pi i) : k \in \mathbb{Z}\}$ in the complex plane.

Ex. Find asymptotics for the number s_n of surjections of size n.

The EGF for surjections is $F(z) = \frac{1}{2-e^z}$. Both $G(z) = 1$ and $H(z) = 2 - e^z$ are analytic everywhere in the complex plane, and when $|z| \leq 1$, the only root of $H(z)$ is $z = \log 2$. $G(\log 2) = 1 \neq 0$ and $H'(\log 2) = -e^{\log 2} = -2 \neq 0$, so

$$[z^n]F(z) = \frac{-1}{(\log 2)(-2)} \cdot (\log 2)^{-n} + O(1) = \frac{1}{2(\log 2)^{n+1}} + O(1) \implies s_n \sim \frac{n!}{2(\log 2)^{n+1}}$$

Ex. Find asymptotics of the number of permutations with no cycle of length $\leq r$.

The EGF is $P(z) = \frac{e^{-z-z^2/2-\cdots-z^r/r}}{1-z}$. Since $H'(1) = -1 \neq 0$ and $G(1) = e^{-1-1/2-\cdots-1/r} \neq 0$,

$$[z^n]P(z) \sim 1^n \cdot \frac{-G(1)}{1 \cdot H'(1)} = e^{-1-1/2-\cdots-1/r} \implies p_n \sim n! e^{-1-1/2-\cdots-1/r}$$

Ex. The labelled class $\mathcal{A}$ of alignments has the labelled specification $\mathcal{A} = \mathrm{SEQ}(\mathrm{CYC}(\mathcal{Z}))$. Find asymptotics for the number of alignments of size n.

The specification implies

$$A(z) = \frac{1}{1 + \log(1-z)}$$

which has singularities when $z = 1$ (0 in logarithm) and $\log(1-z) = -1$ (divide by 0). The equation $\log(1-z) = -1$ has solution $z = 1 - e^{-1}$ which is the only dominant singularity of $A(z)$. Applying the Meromorphic Asymptotic Theorem gives

$$a_n = n![z^n]A(z) \sim \frac{n!}{e(1-e^{-1})^{n+1}}$$

Part IV

Random Generation

Chapter 13

Random Generation of Combinatorial Objects

Def Uniform Generation Algorithm: A uniform generation algorithm for a combinatorial class $\mathcal{C}$ is a randomized algorithm that takes $n \in \mathbb{N}$ and returns an element $c \in \mathcal{C}_n$ in $\mathcal{C}$ of size n, where every element of $\mathcal{C}_n$ is returned with probability $\frac{1}{c_n}$.

Random Function Use `rand` to generate a random integer in $\{0, \ldots, n\}$ by defining

$$\text{rand}(0, \ldots, n) = \lfloor (n+1) \cdot \text{rand}() \rfloor$$

Approaches to Random Generation

- Direct algorithm
- Bijections (or k-to-1 surjection)
- Recursive sampling (use specification)
- Rejection sampling (generate a larger set and repeat until desired property found)
- Ranking/unranking (order elements of c_n efficiently and compute kth element where k random)
- Boltzmann sampling (give up exact size to be more efficient)

Recursive Sampling Start with base cases, the neutral and atomic class.

- Combinatorial Sum: Suppose $\mathcal{A} = \mathcal{B} + \mathcal{C}$ and we have uniform generation algorithms genB(n) and genC(n). If $\alpha \in \mathcal{A}_n$, then $P(\alpha \in B_n) = \frac{b_n}{a_n} = \frac{b_n}{b_n + c_n}$.

  If $\alpha \in \mathcal{B}$, then it is returns with probability $\frac{b_n}{b_n+c_n} \cdot \frac{1}{b_n} = \frac{1}{b_n+c_n} = \frac{1}{a_n}$.

- Combinatorial Product: Suppose $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ and we have uniform generation algorithms genB(n) and genC(n). The probability that $\alpha \in \mathcal{A}_n$ is $\alpha = (\beta, \gamma)$, and $|\beta| = k$ is $\frac{b_k c_{n-k}}{a_n}$.

  The probability that $(\beta, \gamma) \in \mathcal{B}_k \times \mathcal{C}_{n-k}$ is returned is

  $$\frac{b_k c_{n-k}}{a_n} \cdot \frac{1}{b_k} \cdot \frac{1}{c_{n-k}} = \frac{1}{a_n}$$

- Sequence: If $\mathcal{B}$ has no objects of size 0, then $\mathcal{A} = \text{SEQ}(\mathcal{B})$ is equivalent to $\mathcal{A} = \epsilon + \mathcal{B} \times \mathcal{A}$, and so we can get a recursive algorithm.

Boltzmann Sampling Instead of saying we need an object of size n, we can look for an object of size $[k_1 n, k_2 n]$. This elegant framework is Boltzmann sampling.

Def Admissible Values: We call the points $v \in (0, R)$ where R is the radius of convergence of $F(z) = \sum\limits_{n \geq 0} f_n z^n$ the admissible values.

Def Boltzmann Model: Let $\mathcal{A}$ be a combinatorial class and v an admissible value for $A(x)$. The Boltzmann model at v assigns each $\alpha \in \mathcal{A}$ the probability

$$P_v()\alpha) = \frac{v^{|\alpha|}}{A(v)}$$

Def Boltzmann Generation Algorithm: A Boltzmann generation algorithm with parameter v is a randomized algorithm that returns $\alpha \in \mathcal{A}$ with probability $P_v(\alpha)$.
The expected size of the Boltzmann algorithm with parameter v is

$$\sum_{\alpha \in \mathcal{A}} |\alpha| \, P_v(\alpha) = \frac{\sum_{\alpha \in \mathcal{A}} |\alpha| \, v^{|\alpha|}}{A(v)} = \frac{vA'(v)}{A(v)}$$

Building Boltzmann Samplers The neutral class ϵ has generating function 1, so its Boltzmann generator returns a single ϵ with probability $\frac{v_0}{1} = 1$. The atomic class has generating function z, so its Boltzmann generator returns a single atom with probability $\frac{v^1}{v} = 1$.

- Combinatorial Sum: $\mathcal{A} = \mathcal{B} + \mathcal{C}$. The probability that $\alpha \in \mathcal{A}$ comes from $\mathcal{B}$ should be

  $$\sum_{\beta \in \mathcal{B}} \frac{v^{|\beta|}}{A(v)} = \frac{B(v)}{A(v)} \in (0, 1)$$

  Any admissible value for $A(z)$ is also admissible for $B(z)$.

- Combinatorial Product: $\mathcal{A} = \mathcal{B} \times \mathcal{C}$. If $\alpha = (\beta, \gamma)$, then

  $$\frac{v^{|\alpha|}}{A(v)} = \frac{v^{|\beta|+|\gamma|}}{A(v)} = \frac{v^{\beta}}{A(v)} \cdot \frac{v^{\gamma}}{A(v)}$$

  This means that each component is independent. Do not need recursive sampling.

- Sequence: $\mathcal{A} = \text{SEQ}(\mathcal{B})$. The probability that $(\beta_1, \ldots, \beta_k)$ appears is

  $$\frac{v^{|\beta_1|+\cdots+|\beta_k|}}{1/(1 - B(v))} = B(v)^k (1 - B(v))$$

  This is a geometric distribution.

Theorem Fix $\epsilon > 0$ and let $\mathcal{C}$ be a combinatorial class such that

- $C(z)$ has a finite radius of convergence $R > 0$

- $\lim\limits_{z \to R^-} \frac{zC'(z)}{C(z)} = \infty$
- a "technical" variance condition holds

Let z_n be the smallest positive root of $n = \frac{zC'(z)}{C(z)}$. Repeatedly call BOLT-C(z_n) until it returns something with size in $[(1-\epsilon)n, (1+\epsilon)n]$. With probability going to 1 as $n \to \infty$ and on average takes $O(n)$ total operations.

Def Expected Boltzmann Size: The function $E(z) = \frac{zC'(z)}{C(z)}$ tracks the expected Boltzmann size for $z \in (0, R)$.

Def Marked Rooted Binary Tree: A binary tree where either the root or children are marked.

Examples

Ex. Find a uniform generation algorithm to compute a random binary string of length n.

Directly generate binary digits one by one.

Ex. Find a uniform generation algorithm to compute a random subset of $\{1, \ldots, n\}$.

We have a bijection f from binary strings of length n to subsets of $[n]$ where $f(b_1, \ldots, b_n) = \{k : b_k = 1\}$.

Ex. For rooted binary trees with generating function $B(z) = \frac{1-\sqrt{1-4z}}{2z}$.

The expected size is 0.14 if $v = 0.1$, 0.62 if $v = 0.2$, 24.5 if $v = 0.249$.