CO 250 Introduction to Optimization

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1 Introduction

1.1 Linear Programs

Def Affine Function: A function $f : \mathbb{R}^n \to \mathbb{R}$ is an affine function if $f(x) = a^T x + \beta$, where x and a are vectors with n entries and β is a real number. If $\beta = 0$, then f is a linear function. Thus, every linear function is affine, but the converse is not true.

Def Linear Program (LP): A linear program is the problem of maximizing or minimizng an affine function
$$f(x) = f(x_1, \ldots, x_n)$$
subject to a finite number of linear constraints of the form $g(x) \leq c$, $g(x) \geq c$, or $g(x) = c$.

- the x_i are real variables.

- the function f is the objective function

- the expressions $g(x) \leq c$, $g(x) \geq c$, and $g(x) = c$ are constraints

Ex.

$$
\begin{aligned}
\min \quad & 3x_1 + 2x_2 - x_3 + 5 \\
\text{s.t.} \quad & x_1 + x_3 \leq 7 \\
& 2x_1 \geq 10 \\
& x_1 + x_3 = 5 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Non Ex.

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 = 7 \\
& x_1 - x_2 < 0
\end{aligned}
$$

Ex. Employees at a compary have salary demands

- Tom wants $\geq \$20000$.

- Peter, Nina, Samir each want $\geq \$5000$ more than Tom.

- Gary wants at least that of Tom and Peter combined.

- Linda wants $\geq \$200$ more than Gary.

- Combined salary of Nina and Samir should be $\geq$ twice combined salary of Tom and Peter.

- Bob is at least as high as Peter and Samir.

- Bob and Peter should earn $\geq \$60000$ combined.

LP that will determine salaries minimizing total salary expenses.

Variables: $x_T, x_P, x_N, x_S, x_G, x_L, x_B$ be salaries of people.

$$
\begin{aligned}
\min \quad & x_T + x_P + x_N + x_S + x_G + x_L + x_B \\
\text{s.t.} \quad & x_T \geq 20000 \\
& x_P \geq x_T + 5000 \\
& x_N \geq x_T + 5000 \\
& x_S \geq x_T + 5000 \\
& x_G \geq x_T + x_P \\
& x_L \geq x_G + 200 \\
& x_N + x_S \geq 2(x_T + x_P) \\
& x_B \geq x_P \\
& x_B \geq x_S \\
& x_B + x_P \geq 60000 \\
& x_T, x_P, x_N, x_S, x_G, x_L, x_B \geq 0
\end{aligned}
$$

Say we want to minimize the highest paid employee/salary, we can use another variable y and set constraints of y to $y \geq x_1, \ldots, x_n$, then the objective function is $\min y$.

1.2 Integer Programs

Def Integer Program (IP): An integer program is a linear program together with constraints specifying that some particular non-empty subset of the variables are integers.

A mixed integer program (MIP) is an IP where not all variables are constrained to be integers.
A pure integer program is one where all variables have integer constraints.

Ex.

$$
\begin{aligned}
\min \quad & 2.1x_1 + x_2 - 3x_3 + 4 \\
\text{s.t.} \quad & x_1 + 2.5x_2 \leq 7.3 \\
& x_1 - x_3 \geq 4 \\
& x_1, x_2, x_3 \geq 0 \\
& x_1, x_2 \text{ integers}
\end{aligned}
$$

Ex. For a desert trek, $3.6m^3$ of cargo space is available. There are 7 items available each having a value and a volume: Given that only one of each item is available and items cannot be split up,

Item	1	2	3	4	5	6	7
Volume(m^3)	0.55	0.6	0.7	0.75	0.8	0.9	0.95
Value (\$)	250	300	500	700	750	900	950

formulate an IP for the maximize the total value of goods.

For item i, let x_i be a variable where $x_i = 1$ if we take item i, $x_i = 0$ otherwise.

$$\max \quad 250x_1 + 300x_2 + 500x_3 + 700x_4 + 750x_5 + 900x_6 + 950x_7$$
$$\text{s.t.} \quad 0.55x_1 + 0.6x_2 + 0.7x_3 + 0.75x_4 + 0.8x_5 + 0.9x_6 + 0.95x_7 \leq 3.6$$
$$x_1, \ldots, x_7 \geq 0$$
$$x_1, \ldots, x_7 \leq 1$$
$$x_1, \ldots, x_7 \text{ integers}$$

The last 3 constraints can be summarized to $x_1, \ldots, x_7 \in \{0, 1\}$. Alternatively, let v_i be the volume of item i, and n_i be the value.

$$\max \quad \sum_{i=1}^{7} n_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{7} v_i x_i \leq 3.6$$

$$x_1, \ldots, x_7 \in \{0, 1\}$$

Ex. The blocks represent resources. Say v_i is the value of resource i when mined. Suppose that

time available only allows 7 blocks to be mined, and each block must be either completely mined or not. Suppose also that, to mine a block, the blocks above it must be mined first. Ex. to mine block 12, blocks 3,4,5,8,9 must be mined. Formulate an IP to maximize the total value.
Let x_i be a vairable indicating whether i is mined or not.

$$\max \quad \sum_{i=1}^{12} v_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{12} \leq 7$$

$$x_1 + x_2 - 2x_6 \geq 0$$
$$x_2 + x_3 - 2x_7 \geq 0$$
$$x_3 + x_4 - 2x_8 \geq 0$$
$$x_4 + x_5 - 2x_9 \geq 0$$
$$x_6 + x_7 - 2x_{10} \geq 0$$
$$x_7 + x_8 - 2x_{11} \geq 0$$
$$x_8 + x_9 - 2x_{12} \geq 0$$
$$x_1, \ldots, x_{12} \in \{0, 1\}$$

This works because the constraint $x_i + x_j - 2x_k \geq 0$ where $x_i, x_j, x_k \in \{0, 1\}$ states that either $x_k = 0$ or $x_i = x_j = x_k = 1$. This translates to the statement that either block k is not mined, or

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blocks i and j are also mined. This captures the rule that each block can only be mined if those above it are mined too.

Remark: constraint $x \in \{0, 2\} \implies x - 2z = 0, \; z \in \mathbb{Z}, z \geq 0, z \leq 1$.

Techniques for integer program constraints

- If A then B is $x_b \geq x_a$

- If A then not B is $1 - x_b \geq x_a$

- If not A then B is $x_b \geq 1 - x_a$

- A if and only if B is $x_a = x_b$

- If A then B and C is $x_b \geq x_a, x_b \geq x_c$

- If A then B or C is $x_b + x_c \geq x_a$

- If B or C then A is $x_a \geq x_b, x_a \geq x_c$

- If B and C then A is $x_a \geq x_b + x_c - 1$

- If m or more of n items $x_2, \ldots, x_{n+1}$ then A is $\frac{x_2 + x_3 + \cdots + x_{n+1} - m + 1}{n - m + 1}$

1.3   Optimization Problems on Graphs

Def Graph: A graph is a pair $G = (V, E)$ where V is a finite set, and E is a suset of the unordered pairs of distinct elements of V.

A cost/weight/length function on a graph $G = (V, E)$ is a function $c : E \to \mathbb{R}$. We often write c_e instead of $c(e)$.

Def Bipartite Graph: A graph $G = (V, E)$ is bipartite if we can partition the vertices into two sets, say A and B, such that every edge has one endpoint in A and one endpoint in B.

Def Matching: A matching in a graph $G = (V, E)$ is a subset M of E such that no vertex is incident to more than one edge of M.

Def Perfect Matching: Matching M is a perfect matching of G if every vertex of G is incident to exactly one edge in the matching M. ($|M| = \frac{1}{2}|V|$)

Given a matching M in a graph G and a cost function $c : E \to \mathbb{R}$, the cost of M, written $c(M)$ is defined as $c(M) = \sum\limits_{e \in M} c_e$.

Minimum Cost Perfect Matching Problem: Given a graph $G = (V, E)$ and a cost function $c : E \to \mathbb{R}$, find the minimum cost of a perfect matching of G.

Define binary variables $x_e : e \in E$ where for each edge $e \in E$, $x_e = 0$ if $e \notin M$ and $x_e = 1$ if $e \in M$. We think of the variables $x_e : e \in E$ as forming a vector x indexed by edges, or equivalently a function $x : E \to \{0, 1\}$. x is the indicator function/vector for set M.

The objective function is

$$\sum_{e \in E} c_e x_e = \text{total cost of edges in matching with indicator function x}$$

we write $c(x)$ for this expression.

For each vertex v, write $\delta(v)$ for the set of edges that have v as an end (incident with v). The constraint that M is a perfect matching is equivalent to saying that for each $v \in V$, M contains exactly one edge in $\delta(v)$.
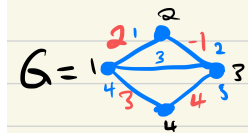
$$\sum_{e \in \delta(v)} x_e = 1$$

Translated to a statement about the vector x, this is saying that for each $v \in V$, we have

$$\sum_{e \in \delta(v)} x_e = 1 \iff x(\delta(v)) = 1$$

IP:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(v)} = 1 \qquad : v \in V \\
& x_e \in \{0,1\} : e \in E
\end{aligned}
$$

Ex.

$$
\begin{aligned}
\min \quad & 2x_1 - x_2 + 0x_3 + 3x_4 + 4x_5 \\
\text{s.t.} \quad & x_1 + x_3 + x_4 = 1 \\
& x_1 + x_2 = 1 \\
& x_2 + x_3 + x_5 = 1 \\
& x_4 + x_5 = 1 \\
& x_1, \ldots, x_5 \in \{0,1\}
\end{aligned}
$$

Def Path: A path in a graph $G = (V,E)$ is a sequence of edges of the form $v_1 v_2, v_2 v_3, \ldots, v_{k-1} v_k$ where $v_1, \ldots, v_k$ are distinct. We say this is a path from v_1 to v_k and we often think of it as a set rather than a sequence.

If $l : E \to \mathbb{R}$ is a length function, then the length of a path P is $\sum_{e \in P} l_e$.

Shortest Path Problem: Given a graph G and a length function $l : E \to \mathbb{R}$ and vertices s, t of G, find a minimum length st-path. We solve this problem when every length is positive.

Let $G = (V,E)$ be a graph and let $U \subseteq V$ be a suset of the vertices. We generalize the definition of δ. Let $\delta(U)$ denote the set of edges that have exactly one endpoint in U,

$$\delta(U) = \{uv \in E : u \in U, v \notin U\}$$

Def st-cut: An st-cut is a set of edges of the form $\delta(U)$, where $s \in U$ and $t \notin U$.

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Theorem 1.1 An st-path P intersects every st-cut.

Theorem 1.2 Let S be a set of edges that contains at least one edge from every st-cut. Then there exists an st-path P that is contained in the edges of S.

Variables: Introduce binary variable x_e for every edge e, where $x_e = 1$ is when edge e is selected to be part of our st-path, and $x_e = 0$ otherwise.

Constraints: By Theorem 1.2, in order to construct an st-path, it suffices to select one edge from every st-cut. Let $\delta(U)$ be an arbitrary st-cut of G. The number of edges we selected from the st-cut $\delta(U)$ is $\sum_{e \in \delta(U)} x_e$. Since we wish to select at least one edge from $\delta(U)$, the constraint should be

$$\sum_{e \in \delta(U)} x_e \geq 1$$

Objective function: We wish to minimize the total length of the edges in the st-path P we selected. If e is an edge of P then we will have $x_e = 1$ and we should contribute c_e to the objective function.

$$\sum_{e \in E} c_e x_e$$

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(U)} x_e \geq 1 : (U \subseteq V, s \in U, t \notin U) \\
& x_e \geq 0 : (e \in E) \\
& x_e \text{ integer}
\end{aligned}
$$

1.4   Nonlinear Programs

Def Nonlinear Program (NLP): Consider functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, for every $i \in \{1, 2, \ldots, m\}$. A nonlinear program is an optimization problem of the form

$$
\begin{aligned}
\min \quad & z = f(x) \\
\text{s.t.} \quad & g_1(x) \leq 0 \\
& g_2(x) \leq 0 \\
& \vdots \\
& g_m(x) \leq 0
\end{aligned}
$$

In order to maximize the objective function, we take $z = -f(x)$.

Techniques for nonlinear program constraints

- $x_1 \in \mathbb{Z}$ is $x_1 - \lceil x_1 \rceil = 0$

- $g_1(x) \leq 0$ or $g_2(x) \leq 0$ is $\min(g_1(x), g_2(x))$

- If $g_1(x) \leq 0$ then $g_2(x) \leq 0$. Define $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(y) = \begin{cases} 1 & (y \leq 0) \\ 0 & (y > 0) \end{cases}$$

Then the constraint is $f(g_1(x)) \cdot g_2(x) \leq 0$.

2 Solving Linear Programs

2.1 Possible Outcomes

There are three possible outcomes for an LP:

- Unbounded

$$\begin{aligned} \max \quad & x_1 \\ \text{s.t.} \quad & x_1 \geq 0 \end{aligned}$$

- Infeasible

$$\begin{aligned} \max \quad & x_1 \\ \text{s.t.} \quad & x_1 \geq 1 \\ & x_1 \leq -1 \end{aligned}$$

- Optimal (solution exists)

$$\begin{aligned} \max \quad & x_1 \\ \text{s.t.} \quad & x_1 \leq 4 \end{aligned}$$

Claim the following has no feasible solution ($\nexists x_1, x_2, x_3$ satisfying all the constraints).

$$\begin{aligned} \min \quad & x_1 + x_2 + 7x_3 \\ \text{s.t.} \quad & 2x_1 - x_2 + x_3 = 3 \quad A \\ & x_1 - 3x_2 - x_3 = 2 \quad B \\ & x_1 + x_2 + 3x_3 = 3 \quad C \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Consider the constraint $2A - 2B - C$. This says

$$2(2x_1 - x_2 + x_3) - 2(x_1 - 3x_2 - x_3) - (x_1 + x_2 + 3x_3) = 2(3) - 2(2) - 3 \Leftrightarrow x_1 + 3x_2 + x_3 = -1$$

but $x_1, x_2, x_3 \geq 0$ for every feasible solution so $0 \leq x_1 + 3x_2 + x_3 = -1 < 0$. This is a contradiction, so there is no feasible solution.

Proposition Consider the LP
$$\max\{c^T x : Ax = b, x \geq 0\}$$

If there is a vector y such that $y^T A \geq 0$ and $y^T b < 0$, then the LP is infeasible (no x that satisfies the constraints) and y provides a certificate (proof of this fact).

Proof: Suppose that such a y exists, and x is a feasible solution. Then $Ax = b$ so $y^T Ax = y^T b$. But $y^T A \geq 0$ by assumption, and $x > 0$ because x is feasible so $(y^T A)x \geq 0$ because it is a dot product of non-negative vectors. This is a contradiction, because $0 \leq (y^T A)x = y^T b < 0$.

From previous example, $y = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$, then $y^T A = \begin{pmatrix} 1 & 3 & 1 \end{pmatrix} \geq 0$, $y^T b = -1 < 0$. So infeasible by proposition.

Def Infeasible: An LP is infeasible if it has no feasible solution.

Def Unbounded: An LP is unbounded if, for all k, there exists a feasible solution x with object value better than k (less for minimization, more for maximization).

Ex. Consider the LP
$$\max\{z(x) = c^T x : Ax = b, x \geq 0\}$$

where

$$A = \begin{pmatrix} 1 & 1 & -3 & 1 & 2 \\ 0 & 1 & -2 & 2 & -2 \\ -2 & -1 & 4 & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix}, c = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 7 \\ 1 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$$

Let $\bar{x} = (2, 0, 0, 1, 2)^T$ and $d = (1, 2, 1, 0, 0)^T$. Note that $A\bar{x} = b$, so $\bar{x}$ is feasible. Note also that $Ad = 0$. Therefore, for all $t \in \mathbb{R}$, we have $A(\bar{x} + td) = A\bar{x} + tAd = b + 0 = b$.

Now $c^T \bar{x} = 7$ and $c^T d = 2$, so $c^T(\bar{x} + td) = 7 + 2t$. It follows that for all $t \geq 0$, the point $x(t) = \bar{x} + td$ is feasible with objective value $7 + 2t$. Since $7 + 2t \to \infty$ as $t \to \infty$, it follows that the LP is unbounded.

Proposition Consider the LP
$$\max\{c^T x : Ax = b, x \geq 0\}$$

If there exists a vector $d \geq 0$ with $Ad = 0$ and $c^T d > 0$, and there exists $\bar{x} \geq 0$ with $A\bar{x} = b$, then the LP is unbounded (for all M, there is a feasible solution x' with $c^T x' > M$) and $\bar{x}, d$ provide a certificate of this.

Proof: If such an $\bar{x}, d$ exists, let $x(t) = \bar{x} + td$, for each $t \geq 0$. Since $\bar{x} \geq 0, d \geq 0$, we have $\bar{x} + td \geq 0$, so $x(t) \geq 0$ and $A(x(t)) = A(\bar{x} + td) = A\bar{x} + tAd = A\bar{x} = b$. So $x(t)$ is feasible for all $t \geq 0$. Now
$$c^T(x(t)) = c^T(\bar{x} + td) = c^T \bar{x} + t(c^T d)$$

This function is the objective value at $x(t)$, but since $c^T d > 0$, it approaches ∞ as $t \to \infty$. So the LP is unbounded.

A proof of optimality. Consider (L1)

$$\max \quad (-3, 0, -1, 0, -2)x + 8$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ -2 & 3 & 4 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$x \geq 0$$

Claim: $x = (0, 1, 0, 2, 0)$ is optimal. Proof: Note that $(0, 1, 0, 2, 0)$ gives an objective value of 8, and that any $x \geq 0$ satisfies $(-3, 0, -1, 0, -2)x \leq 0 + 0 + 0 + 8 = 8$, so no feasible solution can have a larger objective value. So $(0, 1, 0, 2, 0)$ is optimal.

Consider (L2)

$$\max \quad (1, 0, -5, 8, 10)x - 8$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ -2 & 3 & 4 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$x \geq 0$$

Claim: $x = (0, 1, 0, 2, 0)$ is optimal. Proof: If x satisfy the constraints, then $x_1 - x_3 + 2x_4 + 3x_5 = 4$ (first row of matrix). So

$$(-3, 0, -1, 0, -2)x + 8 = (-3, 0, -1, 0, -2)x + 8 + 4 \overbrace{((1, 0, -1, 2, 3)x - 4)}^{=0}$$
$$= (1, 0, -5, 8, 10)x - 8 \qquad \text{(objective function)}$$

So (L1) and (L2) have the same feasible solutions, and also every feasible solution has the same objective values for (L1) and (L2).

2.2 Standard Equality Form

Def Standard Equality Form (SEF): A linear program is said to be in standard equality form if it is of the form

$$\max\{c^T x + \bar{z} : Ax = b, x \geq 0\}$$

where $\bar{z}$ denotes some constant. In other words, an LP in SEF satisfies these conditions

- it is a maximization problem

- other than the nonnegativity constraints, all constraints are equality constraints

- every variable has a nonnegativity constraint

Ex.

$$\max \quad (2, -3, 4)x$$
$$\text{s.t.} \quad x_1 + x_2 = 7$$
$$x_2 + x_3 = 9$$
$$x \geq 0$$

Def Equivalent Linear Programs: Two LPs (P1) and (P2) are equivalent if

- (P1) is infeasible if and only if (P2) is infeasible

- (P1) is unbounded if and only if (P2) is unbounded

- Given any optimal solution for (P1), we can construct an optimal solution for (P2), and vice versa.

We argue that every LP is equivalent to some LP in SEF.

Let (P) be an arbitary LP. If (P) is a minimization problem, then replacing the objective function with its negative and changing minimization to maximization gives an equivalent maximization problem.

To change inequality constraints to equality constraints, we introduce a slack variable y representing how far the inequality $a^T x \leq b$ is from being tight. The two LPs are equivalent because whenever x is feasible for (P1), $(x, b - a^T x)$ is feasible for (P2) with the same objective value. and whenever (x, y) is feasible for (P2), x is feasible for (P1) with the same objective value.

$$\begin{aligned} \max \quad & (2, 3, 1)x + 1 \\ \text{s.t.} \quad & x_1 + x_2 = 7 \\ & x_1 - x_3 \leq 6 \\ & x \geq 0 \end{aligned}$$

is the same as

$$\begin{aligned} \max \quad & (2, 3, 1)x + 1 \\ \text{s.t.} \quad & x_1 + x_2 = 7 \\ & x_1 - x_3 + y = 6 \\ & x, y \geq 0 \end{aligned}$$

Another one is if $x_k \leq 0$ then we can change it to $x_k + y = 0, y \geq 0$.

Equivalent LPs for nonnegativity constraint:

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + x_3 = 17 \\ & x_1 - x_2 \leq 5 \\ & x_1, x_3 \geq 0 \end{aligned}$$

is the same as

$$\begin{aligned} \max \quad & x_1 + (x_2' - x_2'') + x_3 \\ \text{s.t.} \quad & 2x_1 + x_3 = 17 \\ & x_1 - (x_2' - x_2'') \leq 5 \\ & x_1, x_2', x_2'', x_3 \geq 0 \end{aligned}$$

For any feasible solution x to (P1), if $x_2 \geq 0$, then $(x_1, x_2, 0, x_3)$ is feasible for $(P2)$ with same objective value and if $x_2 < 0$, then $x(x_1, 0, -x_2, x_3)$ is feasible for (P2) with the same objective value.

Conversely, if (x_1, x_2', x_2'', x_3) is feasible for (P2), then $(x_1, x_2' - x_2'', x_3)$ is feaisble for (P1) with same objective value. So (P1) and (P2) are equivalent.

Changing an LP to SEF: Given an arbitrary LP (P), we can find an equivalent LP (P') in SEF by

(1) Changing minimization to maximization and negativizing objective function if (P) is a minimization problem

(2) a. For each $\leq$ constraint, $a_i^T x \leq b_i$, introducing a slack variable y_i, and replacing $a_i^T x \leq b_i$ with $a_i^T x + y_i = b_i, y_i \geq 0$.

  b. For each $\geq$ constraint, $a_i^T x \geq b_i$, introducing a slack variable y_i, and replacing $a_i^T x \geq b_i$ with $a_i^T x - y_i = b_i, y_i \geq 0$.

(3) For each variable x_i that does not have an $x_i \geq 0$ constraint, replacing it everywhere with $x_i' - x_i''$ for new variables x_i', x_i'', and adding constraints $x_i', x_i'' \geq 0$.

2.3 Bases and Canonical Forms

Consider the LP

$$
\begin{aligned}
\max \quad & 2x_1 + 7x_3 + 6 \\
\text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 0 & 0 & -2 \\ -2 & 0 & 1 & 0 & 3 \\ 3 & 0 & 0 & 1 & 7 \end{pmatrix} x = \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} \\
& x \geq 0
\end{aligned}
$$

Note that $x = (0, 2, 6, 5, 0)^T$ is a feasible solution with objective value 6 because $2e_1 + 6e_2 + 5e_e = (2, 6, 5)^T$ from the identity matrix inside A.

Idea: Starting with $(0, 2, 6, 5, 0)^T$, try to increase the first entry, keep the fifth entry the same, and alter entries $2, 3, 4$ to remain feasible.

Let $t \geq 0$, and let $x = (t, x_2, x_3, x_4, 0)^T$; try to find x_2, x_3, x_4 to make this x feasible. So we need $Ax = b$, i.e.

$$
\begin{aligned}
t + x_2 = 2 &\implies x_2 = 2 - t \\
-2t + x_3 = 6 &\implies x_3 = 6 + 2t \\
3t + x_4 = 5 &\implies x_4 = 5 - 3t
\end{aligned}
$$

So $x = (t, 2 - t, 6 + 2t, 5 - 3t, 0)^T$ satisfies $Ax = b$ for each $t \geq 0$.

$$
x = \begin{pmatrix} 0 \\ 2 \\ 6 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \\ -3 \\ 0 \end{pmatrix}
$$

For x to be feasible, we need $x \geq 0$ also. This is equivalent to

$$2 - t \geq 0 \Leftrightarrow t \leq 2$$
$$6 + 2t \geq 0 \Leftrightarrow \text{always true for } t \geq 0$$
$$5 - 3t \geq 0 \Leftrightarrow t \leq \frac{5}{3}$$

So for all $t \leq \frac{5}{3}, x \geq 0$, x is feasible. We want $t = x_1$ to be as large as possible, so we set $t = \frac{5}{3}$, and this gives

$$x = \begin{pmatrix} 0 \\ 2 \\ 6 \\ 5 \\ 0 \end{pmatrix} + \frac{5}{3} \begin{pmatrix} 1 \\ -1 \\ 2 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ \frac{28}{3} \\ 0 \\ 0 \end{pmatrix}$$

This x has objective value $2x_1 + 7x_5 + 6 = \frac{10}{3} + 6 > 6$.

Def Basis: Given a real matrix A with a set of column indices $\mathcal{I} = \{1, \ldots, k\}$, a basis of A is a set $B \subseteq \mathcal{I}$ for which the submatrix A_B is a square nonsingular matrix. A_B denotes the submatrix obtained from A by only including columns in B.

Ex. $A = \begin{pmatrix} 2 & 1 & 2 & 3 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 & 0 \\ -3 & 0 & 4 & -3 & 0 & 1 \end{pmatrix}$. $\{2, 5, 6\}$ is a basis in the matrix A because $A_{\{2,5,6\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is nonsingular and square.

Ex. $\{1, 2, 4\}$ is not a basis in A because $A_{\{1,2,4\}}$ is singular. $((1, 1, -1)^T$ is in nullspace).

Note: If the number of rows of $A > \text{rank}(A)$, then A has no bases; otherwise A has ≥ 1 basis, and it can be found by row reductions.

Proposition If A and A' are row-equivalent and have the same number of rows, then they have the same bases.

Proof: Elementary row operations preserve whether linear combinations of columns are zero, and hence nonsingularity.

If $B \subseteq \mathcal{I}$ is a basis for A, then a vector x is a basic solution to the equation $Ax = b$ if $x_i = 0$ for all $i \in \mathcal{I} \setminus B$.

Ex. $x = (0, 2, 0, 0, 3, 4)^T$ is a basic solution to $Ax = (2, 3, 4)^T$, for the basis $\{2, 5, 6\}$ of A.

Ex. $x = (1, 0, 1, 0, 0, 0)^T$ is a basic solution to $Ax = (4, 3, 1)^T$ for the basis $\{1, 3, 6\}$ and also for the basis $\{1, 3, 5\}$.

Given a basis $B \subseteq \mathcal{I}$, the indices in B are basic and the indices outside B are nonbasic. Similarly, the entries x_i are basic/nonbasic. If x is a basic solution to $Ax = b$ for some basis B, and also $x \geq 0$, then x is a basic feasible solution to $\max\{c^T x : Ax = b, x \geq 0\}$.

Proposition For each basis b of A and vector b, there is exactly one basic solution to $Ax = b$. (It may or may not be feasible)

Proof: Solving $Ax = b$ with the restriction that $x_i = 0$ for all $i \notin B$ is equivalent to solving $A_B x = b$; this has a unique solution because A_B is a nonsingular square matrix.

Recall: If $A, b, c, B \subseteq \{\text{cols of } A\}$ are such that

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- A_B is an identity matrix

- $c_i = 0$ for all $i \in B$ (equivalently $c_B = 0$)

- x is a feasible solution that is basic for B

then there is a way to find a feasible solution to $\max\{c^T x + d : Ax = b, x \geq 0\}$ with a better objective value than x.

Def Canonical Form: Consider the LP in SEF:

$$\max\{c^T x + z : Ax = b, x \geq 0\}$$

Let B be a basis of A. We say that the LP is in canonical form for B if the following conditions are satisfied:

- A_B is an identity matrix

- $c_B = 0$

Ex. Converting to canonical form

$$
\begin{aligned}
\max \quad & x_1 + 2x_3 - x_3 + 2x_4 + 3x_5 + 6 \\
\text{s.t.} \quad & \begin{pmatrix} 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 1 & 0 & 0 \\ -2 & -4 & -1 & 1 & -7 \end{pmatrix} x = \begin{pmatrix} 13 \\ 2 \\ -25 \end{pmatrix} \\
& x \geq 0
\end{aligned}
$$

We want to convert(without changing the set of feasible solutions or the value of objective function) this to canonical form for the basis $B = \{1, 3, 5\}$.

Goal: Row-reduce to change $Ax = b$ into $A'x = b'$, where $A'_B = I_3$

$$
\begin{pmatrix} 1 & 3 & 0 & 2 & 4 & | & 13 \\ 0 & 2 & 1 & 0 & 0 & | & 2 \\ -2 & -4 & -1 & 1 & -7 & | & -25 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 2 & 4 & | & 13 \\ 0 & 2 & 1 & 0 & 0 & | & 2 \\ 0 & 2 & -1 & 5 & 1 & | & 1 \end{pmatrix}
$$

$$
\sim \begin{pmatrix} 1 & 3 & 0 & 2 & 4 & | & 13 \\ 0 & 2 & 1 & 0 & 0 & | & 2 \\ 0 & 4 & 0 & 5 & 1 & | & 3 \end{pmatrix}
$$

$$
\sim \begin{pmatrix} 1 & -9 & 0 & -18 & 0 & | & 1 \\ 0 & 2 & 1 & 0 & 0 & | & 2 \\ 0 & 4 & 0 & 5 & 1 & | & 3 \end{pmatrix}
$$

So $Ax = b$ has the same solution set as $A'x = b'$ where $A' = \begin{pmatrix} 1 & -9 & 0 & -18 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & 0 & 5 & 1 \end{pmatrix}$ and $b' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

So what we esentially did was $A_B^{-1} Ax = A_B^{-1} b$

So an equivalent LP is

$$\max \quad x_1 + 2x_3 - x_3 + 2x_4 + 3x_5 + 6$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & -9 & 0 & -18 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & 0 & 5 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$x \geq 0$$

We wish to change the form of the objective function $x_1 + 2x_3 - x_3 + 2x_4 + 3x_5 + 6$ to have the coefficients of x_1, x_3, x_5 to be equal to 0 without changing its value. We do this by using the equation $A'x = b'$ to eliminate x_1, x_3, x_5 for all feasible solutions.

Note that every feasible solution x satisfies

$$x_1 - 9x_2 - 18x_4 = 1$$
$$2x_2 + x_3 = 2$$
$$4x_2 + 5x_4 + x_5 = 3$$

So for feasible x, objective function is

$$
\begin{aligned}
x_1 + 2x_3 - x_3 + 2x_4 + 3x_5 + 6 = {} & x_1 + 2x_3 - x_3 + 2x_4 + 3x_5 + 6 \\
& - 1(x_1 - 9x_2 - 18x_4 - 1) \\
& + 1(2x_2 + x_3 - 2) \\
& - 3(4x_2 + 5x_4 + x_5 - 3) \\
= {} & x_2 + 5x_4 + 14
\end{aligned}
$$

So the canonical form LP is

$$\max \quad x_2 + 5x_4 + 14$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & -9 & 0 & -18 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & 0 & 5 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$x \geq 0$$

The basic feasible solution is $x = (1, 0, 2, 0, 3)^T$.

Proposition Supose an LP

$$\max\{c^T x + \bar{z} : Ax = b, x \geq 0\}$$

and a basis B of A are given. Then the following LP is an equivalent LP in canonical form for the basis B, and $y = c_B^T A_B^{-1}$ (row vector), then:

$$\max \quad (c^T - yA)x + (yb + \bar{z})$$
$$\text{s.t.} \quad A_B^{-1} A x = A_B^{-1} b$$
$$x \geq \mathbf{0}$$

Proof: First note that, since A_B is nonsingular, we have $Ax = b$ if and only if $A_B^{-1} A x = A_B^{-1} b$, so feasible solutions are the same.

Second, note that $(A_B^{-1}A)_B = A_B^{-1}A_B = I$.
Third, for each feasible x, we have

$$(c^T - yA)x + yb + \overline{z}$$
$$= c^T x - c_B^T A_B^{-1} x + c_B^T A_B^{-1} b + \overline{z}$$
$$= c^T x + \overline{z} - \underbrace{c_B^T A_B^{-1}(Ax - b)}_{=0}$$
$$= c^T x + \overline{z}$$

Finally, $(c^T - yA)_B = (c^T - c_B^T A_B^{-1}A)_B = c_B^T - c_B^T A_B^{-1}A_B = 0$ so $c^T - yA$ has all its B-entries zero.

2.4   The Simplex Algorithm

Ex. Exploiting Canonical form

$$\max \quad (0, 2, 0, -1, 0)x + 10$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & 0 & \frac{1}{2} & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix}$$

$$x \geq 0$$

This is in canonical form for $B = \{1, 3, 5\}$ and $x = (5, 0, 1, 0, 9)$ is a basic feasible solution for B.

We want to find $t \geq 0$ and a vector $d = (-\frac{1}{2}t, t, -\frac{1}{2}t, 0, -\frac{3}{2}t)$ such that $x + d$ is feasible with better objective value than x. This amounts to finding a vector $x_B = (x_1, x_3, x_5)^T$ s.t.

$$x_B = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} - t \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} \geq 0$$

and $(5, 0, 1, 0, 9) + t(-\frac{1}{2}, 1, -\frac{1}{2}, 0, -\frac{3}{2}) \geq 0$. We want to find the largest t s.t. this is true. This value is $t = 2$ where we have $1 - \frac{t}{2} = 0$ (x_3 becomes 0).

This gives a new value for x of $(5, 0, 1, 0, 9) + 2(-\frac{1}{2}, 1, -\frac{1}{2}, 0, -\frac{3}{2}) = (4, 2, 0, 0, 6)$. This has objective value $14 > 10$.

$x_1 = (4, 2, 0, 0, 6)$ is not a basic solution for $B = \{1, 3, 5\}$ because it has a nonzero entry in position 2. But x_1 is a basic feasible solution for $B_1 = \{1, 2, 5\}$. So to try to increase objective value again, we convert the LP to canonical form for $B_1 = \{1, 2, 5\}$.

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & | & 5 \\ 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 0 & | & 1 \\ 0 & \frac{3}{2} & 0 & \frac{1}{2} & 1 & | & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & | & 5 \\ 0 & 1 & 2 & -1 & 0 & | & 2 \\ 0 & \frac{3}{2} & 0 & \frac{1}{2} & 1 & | & 9 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & | & 4 \\ 0 & 1 & 2 & -1 & 0 & | & 2 \\ 0 & 0 & -3 & 2 & 1 & | & 6 \end{pmatrix}$$

and

$$(0, 2, 0, -1, 0)x + 10 = (0, 2, 0, -1, 0)x + 10$$
$$- 2\underbrace{((0, 1, 2, -1, 0)x - 2)}_{=0}$$
$$= (0, 0, -4, 1, 0)x + 14$$

New LP:

$$\max \quad (0, 0, -4, 1, 0)x + 14$$
$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -3 & 2 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$$
$$x \geq 0$$

new $x_1 = (4, 2, 0, 0, 6)^T$ is a basic feasible solution for $B_1 = \{1, 2, 5\}$.

As before, we want $x_1 + t(-1, 1, 0, 1, -2) \geq 0$. $d = (1, -1, 0, 1, 2)$ and this vector has to satisfy $Ad = 0$ so that $A(x + td) = Ax$.

$t = \min\{4/1, -, 6/2\} = 3$. This gives a better feasible solution

$$x_2 = (4, 2, 0, 0, 6)^T + 3(-1, 1, 0, 1, -2)^T = (1, 5, 0, 3, 0)^T$$

This is a basic feasible solution for $B_2 = \{1, 2, 4\}$. Converting the problem into canonical form for B_2 gives

$$\max \quad (0, 0, -\frac{5}{2}, 0, -\frac{1}{2}) + 17$$
$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & 1 & \frac{1}{2} \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$$
$$x \geq 0$$

We cannot continue by choosing a positive entry in c because $c \leq 0$ but x_2 is optimal, because $c^T x_2 + 17 = 17$ and for all feasible x, $c^T x + 17 \leq 17$, because $c \leq 0, x \geq 0$ so $c^T x \leq 0$. So we can stop, as we have found an optimal solution.

Suppose that we have an LP in canonical form for a basis B

$$\max \quad c_N^T x_N + \overline{z}$$
$$\text{s.t.} \quad x_B + A_N x_N = b$$
$$x \geq 0$$

where c_N is the non basic entries in c and let $\overline{x}$ be the basic feasible solution for B, i.e. $\overline{x}_N = 0$.

Proposition If $c_N \leq 0$ (equivalently $c \leq 0$), then $\overline{x}$ is optimal.

Proof: $\overline{x}$ is feasible, and has objective function $c_N^T x_N + \overline{z} = 0 + \overline{z} = \overline{z}$ because $\overline{x}_N = 0$. For an arbitrary feaisble solution x, we have $x \geq 0$ and $c_N \leq 0$, so $c_N^T x_N \leq 0$. So x has objective value $\leq 0 + \overline{z} = \overline{z}$. So $\overline{x}$ is optimal.

Proposition If $k \in N$ (N is non basis) is such that $c_k > 0$ and $A_k \leq 0$, then the problem is unbounded.

Proof: Let $d \in \mathbb{R}^n$ with $d_k = 1$, $d_j = 0$ for all $j \in N \setminus \{k\}$, and $d_{b_i} = -A_{k,i}$ for $1 \leq i \leq m$, where b_i is the ith entry of B. ex. $d = (0, 0, 0, \ldots, 1, 0, 0, \ldots, -a_1, -a_2, \ldots, -a_m)$.

Since all entries of A_k are ≤ 0, we have $d \geq 0$. Since $d_k = 1, c_k > 0$, and every entry other than k is either 0 in c or 0 in d, we have $c^T d = c_k \cdot 1 > 0$.

Finally,

$$Ad = \underbrace{\sum_{i \in N} d_i A_i}_{d_k A_k} + \sum_{j \in B} d_j A_j$$

$$= d_k A_k + \sum_{i=1}^{m} (-A_{k,i}) \cdot e_j$$

$$= 1 \cdot A_k + (-A_k)$$

$$= 0$$

Since $\bar{x}$ is feasible, $d \geq 0, c^T d > 0, Ad = 0$, then the problem is unbounded.

If the hypotheses for neither proposition holds, then there is some $k \in N$ for which $c_k > 0$, and $A_{j,k} > 0$ for some $j \in \{1, \ldots, m\}$. If this is the case, then we can choose $j \in \{1, \ldots, m\}$ such that $A_{j,k} > 0$ while $\frac{b_j}{A_{j,k}}$ is as small as possible, set $B' = (B \setminus \{j\}) \cup \{k\}$, convert into canonical form for B', and get a feasible solution with objective value at least as large.

One-Phase Simplex Algorithm: Starting with a basic feasible solution for a basis B_0, convert to canonical form for B_0, If the hypotheses for proposition 1 or 2 hold, then STOP because either $\bar{x}$ is optimal (Prop 1) or the problem is unbounded (Prop 2). Otherwise, choose j and k as above, and set $B_1 = (B_0 \setminus \{j\}) \cup \{k\}$ and repeat.

Theorem Bland's Rule Choose the smallest allowed j and k for which $c_k > 0$ and j for which the minimum $\frac{b_j}{A_{j,k}}$ occurs, then the algorithm terminates.

The one-phase simplex method requires a basis B for which that basic solution $\bar{x}$ is feasible to get started.

Ex. (P):

$$\max \quad (1, 6, 4)x + 2$$
$$\text{s.t.} \quad \begin{pmatrix} 7 & 1 & 3 \\ -1 & 2 & -4 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$x \geq 0$$

(P')

$$\min \quad x_4 + x_5$$
$$\text{s.t.} \quad \begin{pmatrix} 7 & 1 & 3 & 1 & 0 \\ -1 & 2 & -4 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$
$$x \geq 0$$

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Algorithm 1 Simplex Algorithm

$$\max\{c^T x : Ax = b, x \geq \mathbf{0}\}$$

Input: Linear program (P) and feasible basis B

Output: An optimal solution of (P) or a certificate proving that (P) is unbounded.

1: Rewrite (P) so that it is in canonical form for the basis B. Let $\bar{x}$ be the basic feasible solution for B
2: **if** $c_N \leq \mathbf{0}$ **then**
3:     Stop ($\bar{x}$ is optimal)
4: Select $k \in N$ such that $c_k > 0$
5: **if** $A_k \leq \mathbf{0}$ **then**
6:     Stop ((P) is unbounded)
7: Let r be any index i where the following minimum is attained:

$$t = \min\left\{\frac{b_i}{A_{i,k}} : A_{i,k} > 0\right\}$$

8: Let l be the rth basis element
9: Set $B = B \cup \{k\} \setminus \{l\}$
10: Go to step 1

2.5   Finding Feasible Solutions

Note that

- $B = \{4, 5\}$ is a basis in (P') and that $(0, 0, 0, 2, 3)^T$ is a corresponding basic feasible solution.

- No solution to (P') has objective value < 0 (so optimum ≥ 0)

- If (x_1, x_2, x_3) is feasible in (P), then $(x_1, x_2, x_3, 0, 0)$ is feasible in (P') and has objective value 0, so is optimal. Hence, if (P') has optimum > 0, then (P) is infeasible.

- If 0 is the optimum for (P'), and $(x_1, x_2, x_3, x_4, x_5)$ is an optimal solution that is basic for some basis B, then $x_4 = x_5 = 0$ and hence (x_1, x_2, x_3) is a feasible solution for (P) that is basic for some basis.

Two-Phase Simplex Algorithm: Given an LP (P) in SEF,

$$\begin{aligned}
\max \quad & c^T x + \bar{z} \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}$$

construct (P') by scaling rows of A and entries of b to make $b \geq 0$, then writing

$$\min \quad x_{n+1} + \cdots + x_{n+m}$$

$$\text{s.t.} \quad \begin{pmatrix} A & | & I \end{pmatrix} \begin{pmatrix} x \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = b$$

$$x \geq 0$$

Now $(0|b^T)$ is a basic feasible solution for (P'): run one-phase simplex on (P').

- If (P') has optimum > 0, then STOP: (P) is infeasible.

- If (P') has optimum 0 and $\bar{x}$ is an optimal soltuion given by simplex, then $\bar{x}_{n+1}, \ldots, \bar{x}_{n+m} = 0$; now $(\bar{x}_1, \ldots, \bar{x}_n)$ is a basic feasible solution for (P). Run one-phase simplex on (P) to either find an optimal solution, or output unbounded.

Fundamental Theorem of Linear Programming Let (P) be an LP problem. Then exactly one of the following holds:

(1) (P) is infeasible

(2) (P) is unbounded

(3) (P) has an optimal solution

2.6   Geometry

The feasible region of an LP is the set of feasible solutions if there are n variables $x_1, \ldots, x_n$, this region is a subset of $\mathbb{R}^n$.

Ex.

$$a_1^T x = b_1$$
$$a_2^T x \leq b_2$$
$$a_3^T x \geq b_3$$

Then the subsets $S_1 = \{x : a_1^T x = b_1\}, S_2 = \{x : a_2^T x \leq b_2\}, S_3 = \{x : a_3^T x \geq b_3\}$. Then the feasible region is $S_1 \cap S_2 \cap S_3$. S_1 is a plane through $\mathbb{R}^3$.

Def Euclidean Norm: The Euclidean norm of x is defined as

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Proposition Let $a, b \in \mathbb{R}^n$. Then $a^T b = \|a\| \, \|b\| \cos(\theta)$, where θ is the angle between a and b. Therefore, for every pair of nonzero vectors a, b, we have:

- $a^T b = 0$ if and only if a, b are orthogonal.

- $a^T b > 0$ if and only if the angle between a, b is less than $90°$.

- $a^T b < 0$ if and only if the angle between a, b is larger than $90°$.

Def Hyperplane and Halfspace: Let $a \in \mathbb{R}^n$ be a nonzero vector and let $b \in \mathbb{R}$, we define:

(1) $H = \{x \in \mathbb{R}^n : a^T x = b\}$ is a hyperplane.

(2) $F = \{x \in \mathbb{R}^n : a^T x \leq b\}$ is a halfspace.

Def Feasible Region: The feasible region of an optimization problem is the set of feasible solutions.

Proposition The feasible region of an LP is a polyhedron or equivalently the intersection of a finite number of halfspaces.

Proof: For each constraint $a^T x \leq b$, let $F = \{x : a^T x \leq b\}$. For each constraint $a^T x \geq b$, let $F = \{x : -a^T x \leq -b\}$. For each constraint $a^T x = b$, let $F_1 = \{x : a^T x \leq b\}$ and $F_2 = \{x : -a^T x \leq -b\}$. Taking the intersection of F (or F_1, F_2) over all constraints expresses the feasible region as a finite intersection of halfspaces.

Def Polyhedron: A polyhedron is a finite intersection of halfspaces. (Note: this differs from various more informal notions of polyhedron).

Remark: A hyperplane in $\mathbb{R}^n$ is a translation of an $n - 1$-dimensional subspace.

Proof: Given $H = \{x \in \mathbb{R}^n : a^T x = b\}$, let H_0 be the subspace of $\{x \in \mathbb{R}^n : a^T x = 0\}$ of $\dim = n-1$. Fix $x_0 \in H$. THen $x \in H$ iff $a^T x = b$ iff $a^T x = a^T x_0$ iff $a^T(x - x_0) = 0$ iff $x - x_0 \in H_0$. So $x \in H$ iff $x - x_0 \in H_0$; thus H is a translation of H_0 by x_0.

Given a halfspace $F = \{x \in \mathbb{R}^n : a^x \leq b\}$, fix $x_0 \in \mathbb{R}$ such that $a^T x_0 = b$, then $x \in F$

- iff $a^T x \leq b$

- iff $a^T x \leq a^T x_0$

- iff $a^T(x - x_0) \leq 0$

- iff $\|a\| \|x - x_0\| \cos(\theta) \leq 0$ where θ is the angle between a and $x - x_0$, $0° \leq \theta < 180°$

- iff $\cos(\theta) \leq 0$

- iff $\theta \geq 90°$

Thus, F is the set of all points at angle $\geq 90°$ from a, translated by x_0 (stuff on one side of the hyperplane H).

Proposition Let $\bar{x} \in H$.

- H is the set of points x for which a and $x - \bar{x}$ are orthogonal.

- F is the set of points x for which a and $x - \bar{x}$ form an angle of at least $90°$.

Def Convex: A set $S \subseteq \mathbb{R}^n$ is said to be convex if for every pair of points $x, y \in S$, the line segment with ends x, y is included in S.

$$\{\lambda x + (1 - \lambda)y : 0 \le \lambda \le 1\}$$

Proposition If $\mathcal{S}$ is a (possibly infinite) collection of convex sets, then the intersection of the sets in $\mathcal{S}$, $\bigcap_{s \in \mathcal{S}} s$ is convex. Note that $\bigcap \mathcal{S} = \bigcap_{s \in \mathcal{S}} s$.

Proof: Let $x, y \in \bigcap \mathcal{S}$, and let L be the line segment from x to y. For each $s \in \mathcal{S}$, we have $x, y \in s$ and s is convex, so $L \subseteq s$. So $L \subseteq s$ for all $s \in \mathcal{S}$, so $L \subseteq \bigcap \mathcal{S}$, therefore $\bigcap \mathcal{S}$ is convex.

Proposition Halfspaces are convex.

Proof: Let $F = \{x \in \mathbb{R}^n : a^T x \le b\}$ be a halfspace. Let $x, y \in F$, and $0 \le \lambda \le 1$; we want to show that $\lambda x + (1 - \lambda)y \in F$, or equivalently that $a^T(\lambda x + (1 - \lambda)y) \le b$.

$$a^T(\lambda x + (1 - \lambda)y) = \lambda a^T x + (1 - \lambda)a^T y \qquad (a^T x \le b, a^T y \le b)$$
$$\le \lambda b + (1 - \lambda)b = b$$

Corollary Polyhedra (or equivalently, the feasible region of LPs) are convex.

Def Properly Contained: We say that a point x is properly contained in a line segment if it is in the line segment but is distinct from its ends.

A square has 4 extreme points, a semi circle has an infinite amount of extreme points.

Def Extreme Point: Given a set S, a point $x \in S$ is <u>not</u> an extreme point of S if and only if there exists distinct $x_1, x_2 \in S$, and $0 < \lambda < 1$, for which

$$x = \lambda x_1 + (1 - \lambda)x_2$$

Equivalently, a point x is an extreme point in S if it does not lie in the interior of a nontrivial line segment that is contained in S.

Proposition If $S = \{v \in \mathbb{R}^2 : \|v\| \le 1\}$, then the extreme points are the points x for which $\|x\| = 1$.

Proof: If $\|x\| < 1$, let $\epsilon = \frac{1}{2}(1 - \|x\|)$, and $x_1 = x + \epsilon x$ and $x_2 = x - \epsilon x$. We can show that $x_1, x_2 \in S$, and we have $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$, so x is not extreme.

Theorem Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a polyhedron and let $\overline{x} \in P$. Let $A^= x = b^=$ be the set of tight constraints for $\overline{x}$. Then $\overline{x}$ is an extreme point of P if and only if $\text{rank}(A^=) = n$.

Theorem Let A be a matrix where the rows are linearly indepedent and let b be a vector. Let $P = \{x : Ax = b, x \ge 0\}$ and let $\overline{x} \in P$. Then $\overline{x}$ is an extreme point of P if and only if $\overline{x}$ is a basic feasible solution of $Ax = b$.

3 Duality Theory

3.1 Weak Duality

Def Dual: We define the dual of a linear program

$$\min\{c^T x : Ax \ge b : x \ge 0\} \implies \max\{b^T y : A^T y \le c, y \ge 0\}$$

and
$$\min\{c^T x : Ax = b, x \geq 0\} \implies \max\{b^T y : A^T y \leq c, y \text{ free}\}$$

Theorem Weak Duality Let (P) and (D) be a primal-dual pair where (P) is a maximization problem and (D) a minimization problem. Let $\overline{x}$ and $\overline{y}$ be feasible solutions for (P) and (D) respectively:

1. Then $c^T \overline{x} \leq b^T \overline{y}$.

2. If $c^T \overline{x} = b^T \overline{y}$, then $\overline{x}$ is an optimal solution to (P) and $\overline{y}$ is an optimal solution to (D).

Corollary Let (P) and (D) be a primal-dual pair. Then:

(1) If (P) is unbounded, then (D) is infeasible.

(2) If (D) is unbounded, then (P) is infeasible.

(3) If (P) and (D) are both feasible, then (P) and (D) both have optimal solutions.

3.2 Strong Duality

Theorem Strong Duality Let (P) and (D) be a primal-dual pair. If there exists an optimal solution $\overline{x}$ of (P), then there exists an optimal solution $\overline{y}$ of (D). Moreover, the value of $\overline{x}$ in (P) equals the value of $\overline{y}$ in (D).

Theorem Strong Duality-Feasibility Let (P) and (D) be a primal-dual pair. If (P) and (D) are both feaisble, then there exists an optimal solution $\overline{x}$ of (P) and an optimal solution $\overline{y}$ of (D) and the objective value of $\overline{x}$ in (P) equals the objective value of $\overline{y}$ in (D).

3.3 Complementary Slackness

Theorem Complementary Slackness Let (P) and (D) be an arbitrary primal-dual pair. Let $\overline{x}$ be a feasible solution to (P) and let $\overline{y}$ be a feasible solution to (D). Then $\overline{x}$ is an optimal solution to (P) and $\overline{y}$ is an optimal solution to (D) if and only if the complementary slackness conditions hold.

- For every variable x_j of $(P_{\max})$, $\overline{x}_j = 0$ or the corresponding constraint j of $(P_{\min})$ is satisfied with equality.

- For every variable y_i of $(P_{\min})$, $\overline{y}_i = 0$ or the corresponding constraint i of $(P_{\max})$ is satisfied with equality.

3.4 Geometry of Optimality

Def Cone: Let $a_1, \ldots, a_k$ be a set of vectors in $\mathbb{R}^n$. We define the cone generated by $a_1, \ldots, a_k$ to be the set
$$C = \text{cone}\{a_1, \ldots, a_k\} = \left\{ \sum_{i=1}^{k} \lambda_i a_i : \lambda_i \geq 0, \forall i = 1, \ldots, k \right\}$$

Theorem Let $\bar{x}$ be a feasible solution to $\max\{c^T x : Ax \leq b\}$. Then $\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.

3.5 Farkas' Lemma

Theorem Farkas' Lemma Let A be an $m \times n$ matrix and let b be a vector with m entries. Then exactly one of the following statements is true:

(1) The system $Ax = b, x \geq 0$ has a solution.

(2) There exists a vector y such that $A^T y \geq 0$ and $b^T y < 0$.