Topological response in ferromagnets

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We present a theory of the intrinsic anomalous Hall effect in a model of a doped Weyl semimetal, which serves here as the simplest toy model of a generic three-dimensional metallic ferromagnet with Weyl nodes in the electronic structure. We analytically evaluate the anomalous Hall conductivity as a function of doping, which allows us to explicitly separate the Fermi-surface and non-Fermi-surface contributions to the Hall conductivity by carefully evaluating the zero-frequency and zero wave-vector limits of the corresponding response function. We show that this separation agrees with the one suggested a long time ago in the context of the quantum Hall effect by Streda [J. Phys. C **15**, L717 (1982)].

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I. INTRODUCTION

There is significant current interest in the topologically nontrivial properties of the electronic structure of solid materials. While the recent explosive expansion of this field is tied to the discovery of topological insulators (TIs) [1], there is a growing realization that topologically nontrivial properties occur in a significantly broader class of systems, not exhausted by TIs. In particular, as has been anticipated some time ago [2] and fully realized recently, certain gapless systems exhibiting point touching nodes between nondegenerate bands, or Weyl nodes, are topologically nontrivial [3–6]. Such *Weyl semimetals* realize massless chiral fermions in a condensed-matter setting and possess unique physical properties. Most of these may be thought of as being consequences of chiral anomaly, a phenomenon that is characteristic of chiral fermions in odd spatial dimensions (and thus not observed in graphene, for example) [7–11]. The very recent experimental realization of *Dirac semimetals* [12–14] paves the way for the realization of Weyl semimetals in the near future.

Some of the interesting physics of Weyl semimetals, however, in fact occurs in an even broader class of materials. In particular, the electronic structure of (almost) any three-dimensional (3D) metallic ferromagnet possesses Weyl nodes, even if they are generally not aligned with the Fermi level, as would be the case in a Weyl semimetal [15]. These nodes play an important role in the phenomenon of the anomalous Hall effect (AHE) [16–24], the interest in which has grown sharply in the last decade with the realization [17,18] that topological properties of the electronic structure play an important, if not defining, role in it, making it a close relative of the quantum Hall effect.

In this paper, we study a simple model of a Weyl semimetal, introduced by us before [5], focusing on the dependence of the intrinsic AHE on doping. This model of a doped Weyl semimetal will serve here as the simplest possible toy model of the electronic structure of a 3D metallic ferromagnet with Weyl nodes, in which nearly everything can be calculated analytically, thus making our arguments especially clear. In particular, we will demonstrate explicitly that it is possible to separate contributions to the intrinsic anomalous Hall conductivity into Fermi-surface and non-Fermi-surface contributions by taking the low-frequency and long-wavelength limits of the corresponding response function in different order, which corresponds to evaluating either a transport or a thermodynamic equilibrium property. We demonstrate that Weyl nodes significantly affect the non-Fermi-surface contribution to the AHE.

II. θ TERM IN DOPED WEYL SEMIMETAL

We will consider a specific realization of a doped Weyl semimetal, based on a topological-insulator–normal-insulator (TI-NI) heterostructure, doped with magnetic impurities [5]. The main advantage of this model of Weyl semimetal is its simplicity: all calculations can be done analytically in this case. Our final results, however, will be of general relevance and to some extent independent of the specifics of a particular model of a metallic ferromagnet. We start from the imaginary-time action of electrons in the Weyl semimetal, coupled to the electromagnetic field,

$$S = \int d\tau d^3r \{\Psi^\dagger(\mathbf{r},\tau)[\partial_\tau - \mu + ieA_0(\mathbf{r},\tau) + \hat{H}]\Psi(\mathbf{r},\tau)\}, \tag{1}$$

where $A_0(\mathbf{r},\tau)$ is the scalar potential and

$$\hat{H} = v_F \tau^z (\hat{z}\times\boldsymbol{\sigma})\cdot(-i\boldsymbol{\nabla} + e\mathbf{A}) + \hat{\Delta} + b\sigma^z \tag{2}$$

is the Hamiltonian of noninteracting electrons in the Weyl semimetal, minimally coupled to the vector potential $\mathbf{A}$. We will ignore the z component of the vector potential, as it will not play any role in what follows. Throughout this paper, we will use the units in which $\hbar = c = 1$. $\hat{\Delta} = \Delta_S \tau^x \delta_{i,j} + \Delta_D(\tau^+\delta_{j,i+1} + \text{H.c.})/2$ is the operator, describing tunneling of electrons between the top (pseudospin $\uparrow$) and bottom (pseudospin $\downarrow$) surfaces of same or neighboring TI layers in the heterostructure [5]. We will assume that Δ_S is positive for concreteness, while Δ_D can have any sign. The $b\sigma^z$ term describes the spin splitting due to uniform magnetization in the z direction.

Turning the electromagnetic field off for a moment, the Hamiltonian $\hat{H}$ is easily diagonalized in several simple steps. After Fourier transforming to momentum space and a canonical transformation of the spin and pseudospin variables $\sigma^\pm \to \tau^z\sigma^\pm$, $\tau^\pm \to \sigma^z\tau^\pm$, we obtain

$$H(\mathbf{k}) = v_F(\hat{z}\times\boldsymbol{\sigma})\cdot\mathbf{k} + [b + \hat{\Delta}(k_z)]\sigma^z. \tag{3}$$

The tunneling operator $\hat{\Delta}(k_z)$ can now be diagonalized separately. Its eigenvalues are given by $t\Delta(k_z)$, where $t = \pm$ and $\Delta(k_z) = \sqrt{\Delta_S^2 + \Delta_D^2 + 2\Delta_S\Delta_D\cos(k_z d)}$. The corresponding eigenvectors are

$$\left|u_{k_z}^t\right\rangle = \frac{1}{\sqrt{2}}\left[1, t\frac{\Delta_S + \Delta_D e^{-ik_z d}}{\Delta(k_z)}\right]. \tag{4}$$

Note that the eigenvectors of the tunneling operator depend only on k_z. Introducing $m_t(k_z) = b + t\Delta(k_z)$, we can now rewrite the momentum-space Hamiltonian of Eq. (3) as

$$H_t(\mathbf{k}) = v_F(\hat{z}\times\boldsymbol{\sigma})\cdot\mathbf{k} + m_t(k_z)\sigma^z. \tag{5}$$

$H_t(\mathbf{k})$ may now be viewed in a more general way as the simplest description of the electronic structure of a 3D ferromagnet. The "mass term" $m_+(k_z)$ is always positive, while $m_-(k_z)$ may change sign from positive to negative at the Weyl nodes, whose locations along the $k_x = k_y = 0$ line are determined by the solutions of the equation

$$b - \Delta(k_z) = 0. \tag{6}$$

When $b = 0$, $m_\pm(k_z) = \pm\Delta(k_z)$ and Eq. (5) describes two pairs of Kramers-degenerate bands. Kramers degeneracy when $b = 0$ is the reason that a minimal model of a ferromagnet with spin-orbit interactions must include four bands.

Diagonalizing $H_t(\mathbf{k})$ in the space of the spin operators, we obtain its eigenvalues,

$$\epsilon_{st}(\mathbf{k}) = s\sqrt{v_F^2\left(k_x^2 + k_y^2\right) + m_t^2(k_z)} = s\epsilon_t(\mathbf{k}), \tag{7}$$

and eigenvectors,

$$\left|v_{\mathbf{k}}^{st}\right\rangle = \frac{1}{\sqrt{2}}\left[\sqrt{1 + s\frac{m_t(k_z)}{\epsilon_t(\mathbf{k})}}, -ise^{i\varphi_{\mathbf{k}}}\sqrt{1 - s\frac{m_t(k_z)}{\epsilon_t(\mathbf{k})}}\right], \tag{8}$$

where $s = \pm$ and $e^{i\varphi_{\mathbf{k}}} = k_+/\sqrt{k_x^2 + k_y^2}$. The vortexlike structure of the spinor $|v_{\mathbf{k}}^{st}\rangle$ due to the presence of the $e^{i\varphi_{\mathbf{k}}}$ phase factor is behind the "topological" physical properties of this system that we will discuss below. The full eigenfunctions of H thus have the form of a tensor product of two spinors,

$$\left|z_{\mathbf{k}}^{st}\right\rangle = \left|v_{\mathbf{k}}^{st}\right\rangle \otimes \left|u_{k_z}^t\right\rangle. \tag{9}$$

We can now integrate out electron variables in Eq. (1) and obtain an effective action for the electromagnetic field, induced by coupling to the electrons. This action will contain two distinct kinds of contributions. The first kind will contain terms, proportional to $\mathbf{E}^2$ and $\mathbf{B}^2$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. These terms describe the electric and magnetic polarizability of the material. The second kind contains the "topological" contribution, which has the form of a "3D Chern-Simons term" (which may, alternatively, be thought of as the $\mathbf{E}\cdot\mathbf{B}$ term, but with a spatially dependent coefficient [8]). Adopting the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$, the topological contribution to the electromagnetic field action takes the following form:

$$S = \sum_{\mathbf{q}, i\Omega} \epsilon^{z0\alpha\beta}\Pi(\mathbf{q}, i\Omega)A_0(-\mathbf{q}, -i\Omega)\hat{q}_\alpha A_\beta(\mathbf{q}, i\Omega), \tag{10}$$

where $\hat{q}_\alpha = q_\alpha/q$ and summation over repeated indices is implicit. The z direction in Eq. (10) is picked out by the magnetization b. The response function $\Pi(\mathbf{q}, i\Omega)$ is given by

$$\Pi(\mathbf{q}, i\Omega) = \frac{ie^2 v_F}{V}\sum_{\mathbf{k}}\frac{n_F[\xi_{s't'}(\mathbf{k})] - n_F[\xi_{st}(\mathbf{k}+\mathbf{q})]}{i\Omega + \xi_{s't'}(\mathbf{k}) - \xi_{st}(\mathbf{k}+\mathbf{q})} \times \left\langle z_{\mathbf{k}+\mathbf{q}}^{st}\middle|z_{\mathbf{k}}^{s't'}\right\rangle\left\langle z_{\mathbf{k}}^{s't'}\middle|\boldsymbol{\sigma}\cdot\hat{q}\middle|z_{\mathbf{k}+\mathbf{q}}^{st}\right\rangle, \tag{11}$$

where $\xi_{st}(\mathbf{k}) = \epsilon_{st}(\mathbf{k}) - \epsilon_F$ and summation over repeated band indices s,t is implicit.

To evaluate $\Pi(\mathbf{q}, i\Omega)$ explicitly, it is convenient to rotate coordinate axes so that $\mathbf{q} = q\hat{x}$ and assume that $\epsilon_F > 0$. The $\epsilon_F < 0$ result is evaluated analogously. There exist two kinds of contributions to $\Pi(\mathbf{q}, i\Omega)$: *interband*, with $s \neq s'$, and *intraband*, with $s = s' = +$. Let us first evaluate the interband contributions. In this case, we can set $i\Omega = 0$ in the denominator of Eq. (11) from the start. Since the pseudospin part of the eigenvectors $|z_{\mathbf{k}}^{st}\rangle$ depends only on k_z, i.e., is independent of $\mathbf{q} = q\hat{x}$, the scalar product of the pseudospin wave functions simply gives us $\delta_{tt'}$. We then note that $\langle z_{\mathbf{k}+\mathbf{q}}^{\pm t}|z_{\mathbf{k}}^{\mp t}\rangle \to 0$ when $\mathbf{q} \to 0$, while $\langle z_{\mathbf{k}+\mathbf{q}}^{\pm t}|\sigma^x|z_{\mathbf{k}}^{\mp t}\rangle$ remains finite in this limit. To leading order in $\mathbf{q}$, we can then expand $\langle z_{\mathbf{k}+\mathbf{q}}^{\pm t}|z_{\mathbf{k}}^{\mp t}\rangle$ to first order in $\mathbf{q}$, while setting $\mathbf{q} = 0$ everywhere else. We obtain

$$\left\langle z_{\mathbf{k}+\mathbf{q}}^{\pm t}\middle|z_{\mathbf{k}}^{\mp t}\right\rangle = \mp\frac{v_F q}{2\epsilon_t(\mathbf{k})\sqrt{k_x^2 + k_y^2}}\left[\pm ik_y + \frac{m_t(k_z)}{\epsilon_t(\mathbf{k})}k_x\right], \tag{12}$$

and

$$\left\langle z_{\mathbf{k}}^{\pm t}\middle|\sigma^x\middle|z_{\mathbf{k}}^{\mp t}\right\rangle = \frac{1}{\sqrt{k_x^2 + k_y^2}}\left[\pm ik_x - \frac{m_t(k_z)}{\epsilon_t(\mathbf{k})}k_y\right]. \tag{13}$$

Substituting this into Eq. (11) and leaving only the terms that will survive angular integration, we obtain

$$\Pi^{\text{inter}}(\mathbf{q}, i\Omega) = \frac{e^2 v_F^2 q}{2V}\sum_t\sum_{\mathbf{k}}\frac{1 - n_F[\epsilon_t(\mathbf{k}) - \epsilon_F]}{\epsilon_t^3(\mathbf{k})}m_t(k_z). \tag{14}$$

Introducing $x = v_F^2(k_x^2 + k_y^2)$, we can rewrite this as

$$\Pi^{\text{inter}}(\mathbf{q}, i\Omega) = -\frac{e^2 q}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\int_0^\infty dx\,\Omega_z^t(x, k_z) \times \left\{1 - n_F\left[\sqrt{x + m_t^2(k_z)} - \epsilon_F\right]\right\}, \tag{15}$$

where

$$\Omega_z^t(x, k_z) = -\frac{m_t(k_z)}{2\left[x + m_t^2(k_z)\right]^{3/2}} \tag{16}$$

is the z component of the Berry curvature of the $s = -, t$ bands. The first term in Eq. (15) is the contribution to $\Pi^{\text{inter}}(\mathbf{q}, i\Omega)$ of the two $s = -$ bands, which are completely filled when $\epsilon_F > 0$. The second term is the contribution of the incompletely filled $s = +$ bands. Integrating over x, we obtain

$$\Pi^{\text{inter}}(\mathbf{q}, i\Omega) = \frac{e^2 q}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\ \text{sign}[m_t(k_z)] - \frac{e^2 q}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\text{sign}[m_t(k_z)] \times \left[1 - \frac{|m_t(k_z)|}{\epsilon_F}\right]\Theta[\epsilon_F - |m_t(k_z)|], \tag{17}$$

where $\Theta(x)$ is the Heaviside step function. The first term in Eq. (17) is the contribution of the completely filled $s=-$ bands. The second term is the contribution of incompletely filled $s=+$ bands and, as first pointed out by Haldane [22], can be expressed in terms of an integral of the Berry connection field over the Fermi surface. Indeed, components of the Berry connection for the $s=+$ bands are given by

$$A_x^t(\mathbf{k}) = -i\langle v_{\mathbf{k}}^{+t}|\partial_{k_x}|v_{\mathbf{k}}^{+t}\rangle = -\frac{v_F^2 k_y}{2\epsilon_t(\mathbf{k})[\epsilon_t(\mathbf{k})+m_t(k_z)]},$$
$$A_y^t(\mathbf{k}) = -i\langle v_{\mathbf{k}}^{+t}|\partial_{k_y}|v_{\mathbf{k}}^{+t}\rangle = \frac{v_F^2 k_x}{2\epsilon_t(\mathbf{k})[\epsilon_t(\mathbf{k})+m_t(k_z)]}. \tag{18}$$

Integrating the Berry connection over a 2D section of the Fermi surface, corresponding to a fixed k_z, i.e., $v_F^2(k_x^2+k_y^2)=\epsilon_F^2-m_t^2(k_z)$, we obtain

$$\begin{aligned}\phi_t(k_z) &= \oint d\mathbf{k}\cdot\mathbf{A}^{+t}(\mathbf{k}) = \pi[1-m_t(k_z)/\epsilon_F] \\ &= \pi\{\text{sign}[m_t(k_z)] - m_t(k_z)/\epsilon_F\} \\ &\quad + \pi\{1-\text{sign}[m_t(k_z)]\},\end{aligned} \tag{19}$$

where $\phi_t(k_z)$ is the total Berry phase, accumulated along the corresponding section of the Fermi surface. Using this, we can rewrite Eq. (17) in the following way:

$$\begin{aligned}\Pi^{\text{inter}}(\mathbf{q},i\Omega) &= \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z\ \text{sign}[m_t(k_z)] \\ &\quad - \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \frac{\phi_t(k_z)}{\pi}\Theta[\epsilon_F-|m_t(k_z)|] \\ &\quad + \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z\{1-\text{sign}[m_t(k_z)]\} \\ &\quad \times \Theta[\epsilon_F-|m_t(k_z)|].\end{aligned} \tag{20}$$

The interpretation of Eq. (20) is, unfortunately, not unambiguous due to the fact that the Berry phase $\phi_t(k_z)$ is only defined modulo 2π. Due to this ambiguity, one has some freedom in how to group various terms in Eq. (20) and assign meaning to them. In particular, note that both the first and the last term in Eq. (20) contribute either 0 or 2π to ϕ_t. Thus we could absorb both terms into ϕ_t and obtain

$$\Pi^{\text{inter}}(\mathbf{q},i\Omega) = -\frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \frac{\phi_t(k_z)}{\pi}\Theta[\epsilon_F-|m_t(k_z)|], \tag{21}$$

which means that $\Pi^{\text{inter}}(\mathbf{q},i\Omega)$ is regarded as entirely a Fermi-surface property. This is closely related to Haldane's well-known theorem that the anomalous Hall conductivity is a Fermi-surface property [22]. Another, equally reasonable, possibility, however, is to absorb only the last term into ϕ_t. In this case, we have

$$\begin{aligned}\Pi^{\text{inter}}(\mathbf{q},i\Omega) &= \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m_t(k_z)] \\ &\quad - \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \frac{\phi_t(k_z)}{\pi}\Theta[\epsilon_F-|m_t(k_z)|].\end{aligned} \tag{22}$$

This means that we identify the entire contribution of *unfilled* bands with the Fermi surface, while the contribution of filled bands remains separate. This is the separation employed by us in Ref. [15]. Finally, perhaps the most natural and most physically motivated interpretation, as will become clear below, results from the following regrouping of terms:

$$\begin{aligned}&\Pi^{\text{inter}}(\mathbf{q},i\Omega) \\ &= \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z\ \text{sign}[m_t(k_z)] \\ &\quad \times [1-\Theta(\epsilon_F-|m_t(k_z)|)] \\ &\quad - \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \frac{\phi_t(k_z)-\pi}{\pi}\Theta[\epsilon_F-|m_t(k_z)|].\end{aligned} \tag{23}$$

To understand the meaning of Eq. (23), let us now evaluate the *intraband* contribution to $\Pi(\mathbf{q},i\Omega)$. In this case we have $s=s'=+$ in Eq. (11). It is then clear that, unlike in the case of the interband contribution, evaluated above, the value of the intraband contribution depends crucially on the order in which the limits of $\mathbf{q}\to 0$ and $\Omega\to 0$ are taken. If the limit is taken so that $\Omega/v_F|\mathbf{q}|\to\infty$, then one is evaluating the dc limit of a transport quantity (the optical Hall conductivity). In this case, the intraband contribution vanishes identically. On the other hand, if the limit is taken so that $\Omega/v_F|\mathbf{q}|\to 0$, then one is evaluating an equilibrium thermodynamic property, whose physical meaning will become clear below. In this case, the intraband contribution is not zero and is given by

$$\begin{aligned}\Pi^{\text{intra}}(\mathbf{q},i\Omega) &= \frac{ie^2 v_F}{V}\sum_t\sum_{\mathbf{k}} \left.\frac{dn_F(x)}{dx}\right|_{x=\epsilon_t(\mathbf{k})-\epsilon_F} \\ &\quad \times \langle z_{\mathbf{k}+\mathbf{q}}^{+t}|z_{\mathbf{k}}^{+t}\rangle\langle z_{\mathbf{k}}^{+t}|\boldsymbol{\sigma}\cdot\hat{q}|z_{\mathbf{k}+\mathbf{q}}^{+t}\rangle.\end{aligned} \tag{24}$$

The derivative of the Fermi distribution function in Eq. (24) expresses the important fact that $\Pi^{\text{intra}}(\mathbf{q},i\Omega)$ is associated entirely with the Fermi surface. Rotating coordinates so that $\mathbf{q}=q\hat{x}$ and expanding to leading order in q, as above, we obtain

$$\begin{aligned}\Pi^{\text{intra}}(\mathbf{q},i\Omega) &= -\frac{e^2 v_F^2 q}{2}\sum_t \int \frac{d^3k}{(2\pi)^3} \\ &\quad \times \left.\frac{dn_F(x)}{dx}\right|_{x=\epsilon_t(\mathbf{k})-\epsilon_F} \frac{m_t(k_z)}{\epsilon_t^3(\mathbf{k})}.\end{aligned} \tag{25}$$

Evaluating the integral over $k_{x,y}$ as above, we finally obtain

$$\begin{aligned}&\Pi^{\text{intra}}(\mathbf{q},i\Omega) \\ &= -\frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \frac{m_t(k_z)}{\epsilon_F}\Theta[\epsilon_F-|m_t(k_z)|] \\ &= \frac{e^2 q}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \frac{\phi_t(k_z)-\pi}{\pi}\Theta[\epsilon_F-|m_t(k_z)|],\end{aligned} \tag{26}$$

i.e., the intraband contribution to $\Pi(\mathbf{q},i\Omega)$ is equal to the second term in the interband contribution given by Eq. (23) in magnitude, but opposite in sign. Combining the inter- and intraband contributions to $\Pi(\mathbf{q},i\Omega)$, we thus obtain

$$\Pi(\mathbf{q},i\Omega)=\frac{e^2q}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\ \mathrm{sign}[m_t(k_z)]\times\{1-\Theta[\epsilon_F-|m_t(k_z)|]\},\qquad(27)$$

i.e., the second term in Eq. (23) cancels out when the low-frequency, long-wavelength limit is taken in such a way that $\Omega/v_F|\mathbf{q}|\to 0$. On the other hand, when $\Omega/v_F|\mathbf{q}|\to\infty$, the intraband contribution vanishes and

$$\Pi(\mathbf{q},i\Omega)=\Pi^{\mathrm{inter}}(\mathbf{q},i\Omega).\qquad(28)$$

This physical difference in the kind of response the system exhibits is precisely the basis of Streda's separation of contributions to the Hall conductivity into σ_{xy}^I and σ_{xy}^{II} [25]. Our analysis makes it clear that this is, in fact, the most physically meaningful separation of contributions to the intrinsic anomalous Hall conductivity.

Generalizing the above results to arbitrary sign of ϵ_F, we finally obtain

$$S=-i\epsilon^{z0\alpha\beta}\sigma_{xy}\int d^3r\,d\tau\,A_0(\mathbf{r},\tau)\partial_\alpha A_\beta(\mathbf{r},\tau),\qquad(29)$$

where, if the low-frequency, long-wavelength limit is taken so that $\Omega/v_F|\mathbf{q}|\to\infty$, then

$$\sigma_{xy}=\frac{e^2}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\ \mathrm{sign}[m_t(k_z)]\times\{\Theta[\epsilon_F+|m_t(k_z)|]-\Theta[\epsilon_F-|m_t(k_z)|]\}+\frac{e^2}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\frac{m_t(k_z)}{|\epsilon_F|}\Theta[|\epsilon_F|-|m_t(k_z)|].\qquad(30)$$

This expression corresponds to the full dc anomalous Hall conductivity. On the other hand, when the low-frequency, long-wavelength limit is taken so that $\Omega/v_F|\mathbf{q}|\to 0$, we obtain

$$\sigma_{xy}=\sigma_{xy}^{II}=\frac{e^2}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\ \mathrm{sign}[m_t(k_z)]\times\{\Theta[\epsilon_F+|m_t(k_z)|]-\Theta[\epsilon_F-|m_t(k_z)|]\}.\qquad(31)$$

This is precisely the Streda's σ_{xy}^{II} contribution to the Hall conductivity, which is a thermodynamic equilibrium quantity, equal to

$$\sigma_{xy}^{II}=e\left(\frac{\partial N}{\partial B}\right)_\mu,\qquad(32)$$

where N is the total electron number. This relation follows immediately from Eq. (29) and the order of limits $\Omega/v_F|\mathbf{q}|\to 0$, which corresponds to thermodynamic equilibrium. Correspondingly, the σ_{xy}^I contribution is given by

$$\sigma_{xy}^I=\sigma_{xy}-\sigma_{xy}^{II}=\frac{e^2}{8\pi^2}\sum_t\int_{-\pi/d}^{\pi/d}dk_z\frac{m_t(k_z)}{|\epsilon_F|}\times\Theta[|\epsilon_F|-|m_t(k_z)|].\qquad(33)$$

As is clear from the above analysis, σ_{xy}^I is the contribution to σ_{xy} that can be associated with states on the Fermi surface. This contribution is nonuniversal, i.e., it depends on details of the electronic structure and is a continuous function of the Fermi energy. σ_{xy}^{II}, on the other hand, is the contribution of all states below the Fermi energy and is a thermodynamic equilibrium property of the ferromagnet. It attains a universal value, which depends only on the distance between the Weyl nodes, when the Fermi energy coincides with the nodes, i.e., when $\epsilon_F=0$,

$$\sigma_{xy}^{II}=\frac{e^2\mathcal{K}}{4\pi^2},\qquad(34)$$

where

$$\mathcal{K}=\frac{2}{d}\arccos\left(\frac{\Delta_S^2+\Delta_D^2-b^2}{2\Delta_S|\Delta_D|}\right)\qquad(35)$$

is the distance between the Weyl nodes. When $b>|\Delta_S+|\Delta_D||$, the Weyl nodes annihilate at the edges of the Brillouin zone and a gap opens up. In this case, $\mathcal{K}=2\pi/d$, i.e., a reciprocal lattice vector and σ_{xy}^{II} is quantized as long as the Fermi level is in the gap [26]. Both contributions, along with the total anomalous Hall conductivity σ_{xy}, are plotted as a function of the Fermi energy in Figs. 1 and 2 in two different cases: when Weyl nodes are present and when they are not. The former occurs when $|\Delta_S-|\Delta_D||<b<|\Delta_S+|\Delta_D||$. As can be seen from Fig. 1, Weyl nodes provide the dominant contribution to σ_{xy}^{II} and to the total Hall conductivity σ_{xy}, if the Fermi level is not too far from the nodes.

It is also worthwhile to note the following interesting property, which is evident from Fig. 1. Both the total

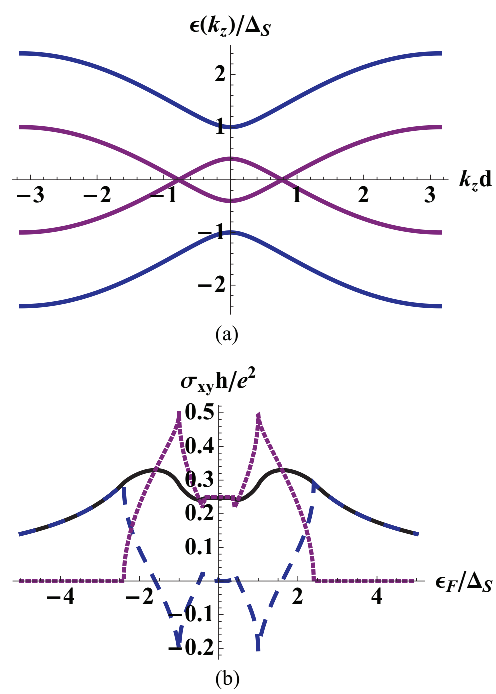

FIG. 1. (Color online) (a) The band edges along the z direction in momentum space. The parameters are such that two Weyl nodes are present. (b) Total intrinsic anomalous Hall conductivity (solid line), σ_{xy}^I (dashed line), and σ_{xy}^{II} (dotted line). Note that the van Hove-like singularities in σ_{xy}^I and σ_{xy}^{II}, associated with band edges, mutually cancel and the total Hall conductivity σ_{xy} is a smooth function of the Fermi energy.

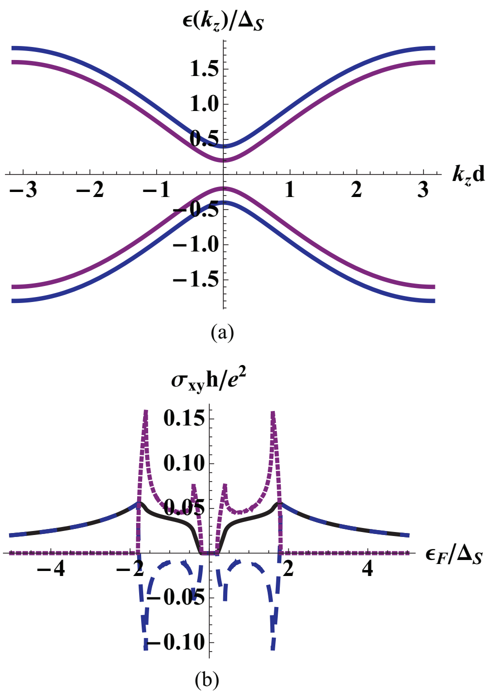

FIG. 2. (Color online) (a) The band edges along the z direction in momentum space. The spin splitting is not large enough for the Weyl nodes to appear (i.e., $b < |\Delta_S - |\Delta_D||$) and the spectrum has a full gap. (b) Total intrinsic anomalous Hall conductivity (solid line), σ_{xy}^I (dashed line), and σ_{xy}^{II} (dotted line).

anomalous Hall conductivity σ_{xy} and the two distinct contributions to it $\sigma_{xy}^{I,II}$ appear to exhibit a quasiplateau behavior when ϵ_F is not too far from the Weyl nodes. To understand the origin of this behavior, consider the derivative of σ_{xy}^{II} with respect to the Fermi energy,

$$\frac{\partial \sigma_{xy}^{II}}{\partial \epsilon_F} = -\frac{e^2}{8\pi^2}\sum_t \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m_t(k_z)]\delta[\epsilon_F - |m_t(k_z)|]. \tag{36}$$

Unlike Eq. (31), this is straightforward to evaluate analytically. We obtain

$$\begin{aligned}\frac{\partial \sigma_{xy}^{II}}{\partial \epsilon_F} &= -\frac{e^2}{8\pi^2}\int_{-\pi/d}^{\pi/d} dk_z \\ &\quad \times\{\delta[\Delta(k_z) - b + \epsilon_F] - \delta[\Delta(k_z) - b - \epsilon_F]\} \\ &= \frac{e^2}{4\pi^2}(1/\tilde{v}_{F+} - 1/\tilde{v}_{F-}),\end{aligned} \tag{37}$$

where we have assumed that ϵ_F is sufficiently close to zero, so that only the $t = -$ bands contribute to the integral, and

$$\tilde{v}_{F\pm} = \frac{d}{2(b \pm \epsilon_F)}\sqrt{[(b \pm \epsilon_F)^2 - (\Delta_S - |\Delta_D|)^2][(\Delta_S + |\Delta_D|)^2 - (b \pm \epsilon_F)^2]} \tag{38}$$

are the two Fermi velocities, corresponding to two pairs of solutions of the equation $|b - \Delta(k_z)| = \epsilon_F$. It is then clear that as long as $\Delta_S - |\Delta_D| \ll b \pm \epsilon_F \ll \Delta_S + \Delta_D$, both Fermi velocities are independent of the Fermi energy and thus $\partial\sigma_{xy}^{II}/\partial\epsilon_F$ vanishes. This simply means that as long as the band dispersion near the Weyl nodes may be regarded as linear, the Fermi velocity is independent of the Fermi energy. By a nearly identical calculation, it is easy to show that $\partial\sigma_{xy}^{I}/\partial\epsilon_F$ also vanishes when ϵ_F is sufficiently close to zero. This is the origin of the quasiplateau behavior in Fig. 1. This result implies that the intrinsic anomalous Hall conductivity is equal to its thermodynamic equilibrium part, σ_{xy}^{II}, not just when the Fermi energy coincides with the Weyl nodes, but even away from them as long as the band dispersion may be assumed to be linear.

III. DISCUSSION AND CONCLUSIONS

The main message of this paper is that it is possible to separate out two distinct contributions to the intrinsic anomalous Hall conductivity: a contribution that can be associated with states on the Fermi surface, σ_{xy}^I, and a contribution that cannot be assigned to states on the Fermi surface and is instead associated with all filled states, σ_{xy}^{II}. This separation coincides with the one proposed a long time ago by Streda [25] in the context of the quantum Hall effect. The physical basis for this is that while σ_{xy}^{II} is a thermodynamic equilibrium property of the material, the Fermi-surface contribution σ_{xy}^I is a purely transport property, which disappears in equilibrium. This physical distinction between the two contributions may, in principle, allow one to separate them experimentally.

σ_{xy}^I and σ_{xy}^{II} will also be affected very differently by disorder. The role of impurity scattering in the AHE has long been a highly controversial issue [24]. What complicates matters especially is the existence of the so-called side-jump contribution to the anomalous Hall conductivity, which, while arising from impurity scattering, is, paradoxically, independent of the impurity concentration. This contribution is thus always of the same order as the intrinsic contribution and may even cancel it exactly in some models [27,28]. We can expect, however, that only the Fermi-surface part of the intrinsic anomalous Hall conductivity σ_{xy}^I will be significantly affected by the impurity scattering. The thermodynamic equilibrium contribution σ_{xy}^{II} should be, at least approximately, independent of disorder, making its role especially important in real materials. We leave detailed investigation of these issues to future work.

Finally, it would also be interesting to extend the above considerations to more realistic models of metallic ferromagnets, such as the model discussed by us before in Ref. [15].

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