Geometric Transformation

COLLEGE OF COMPUTING

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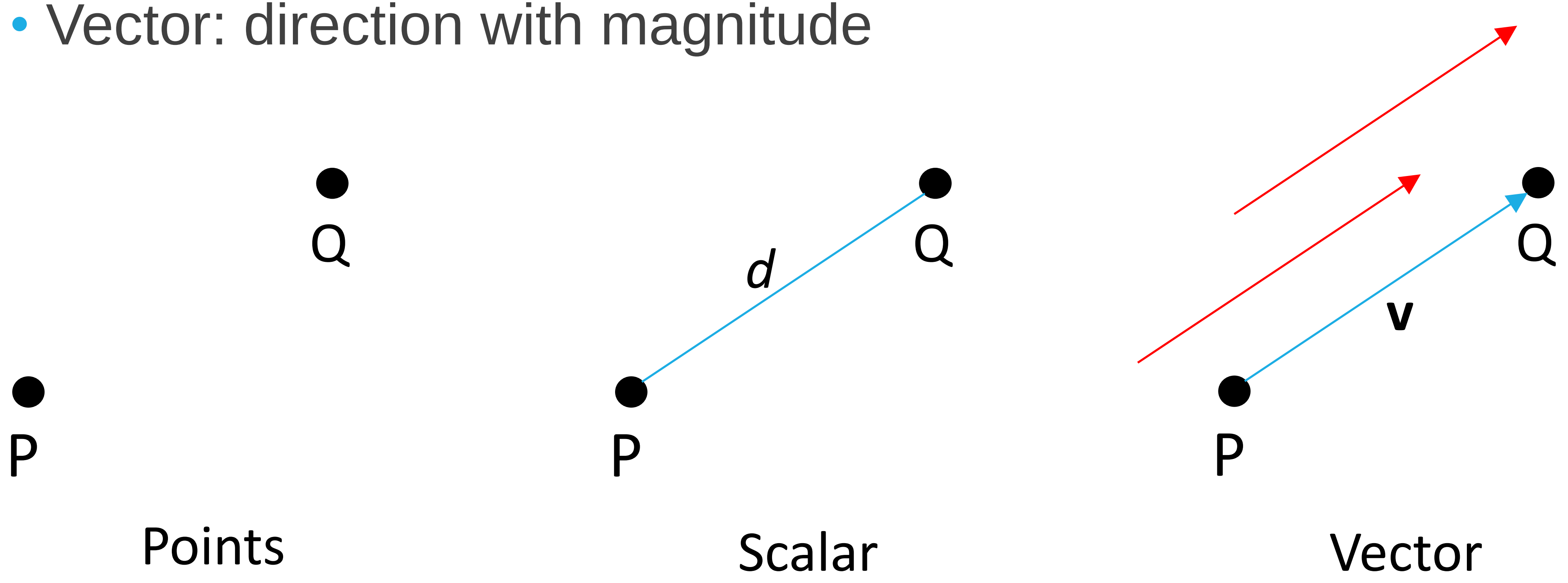
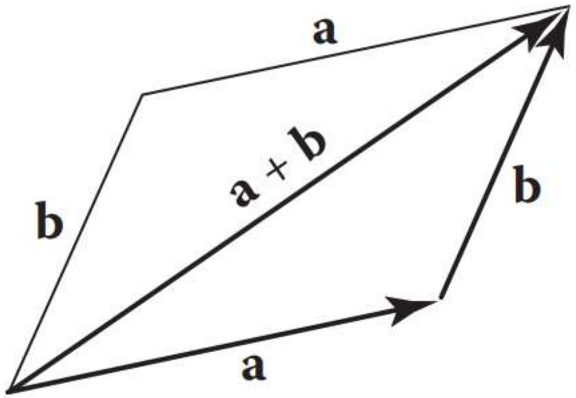
Recap on Math

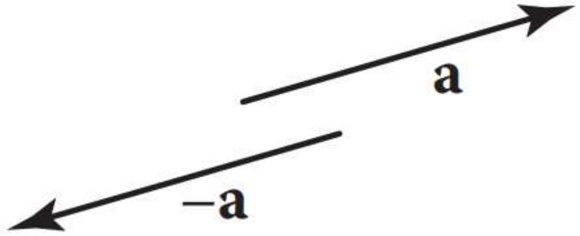
Points, Scalars and Vectors

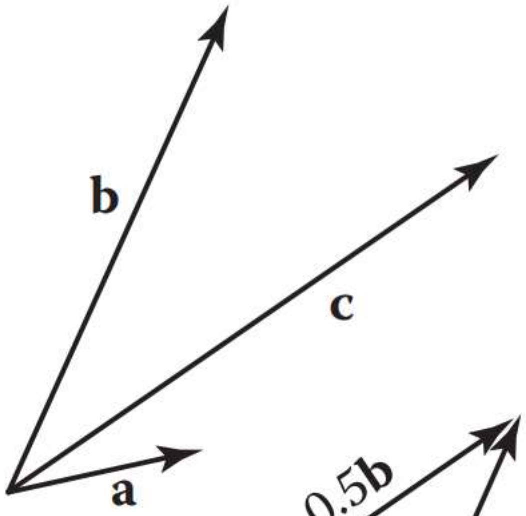
- Point: a location in space
- Scalar: real number, e.g. distance
- Vector: direction with magnitude

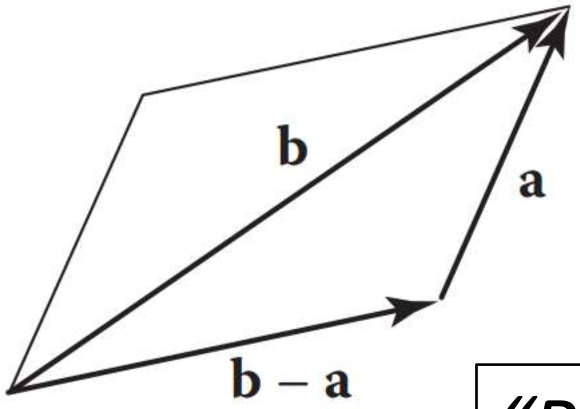


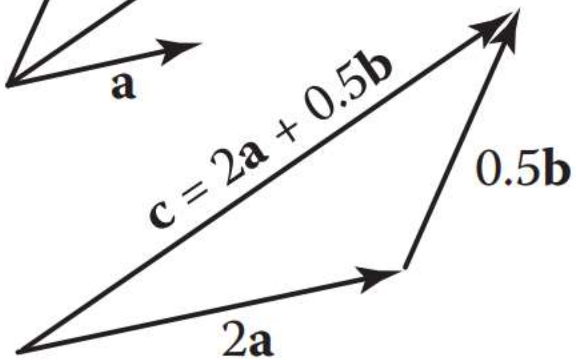
Vector Operations







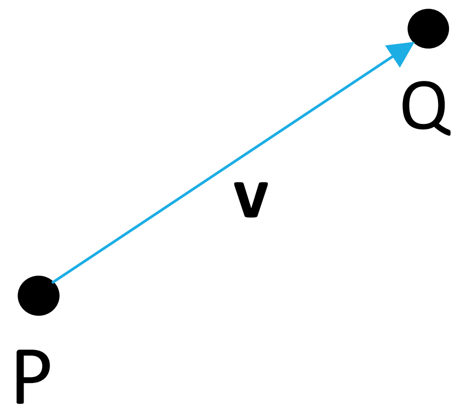




“Parallelogram rule”

Point-Vector Operations

- Point + Vector = Point (Q = P + **v**)
  (point-vector addition)
- Point – Point = Vector (**v** = Q – P)
  (point-point subtraction)



ex) P + 3**v** = ?

ex) 2P – Q + 3**v** = ?

ex) P + 3Q – **v** = ?

Line

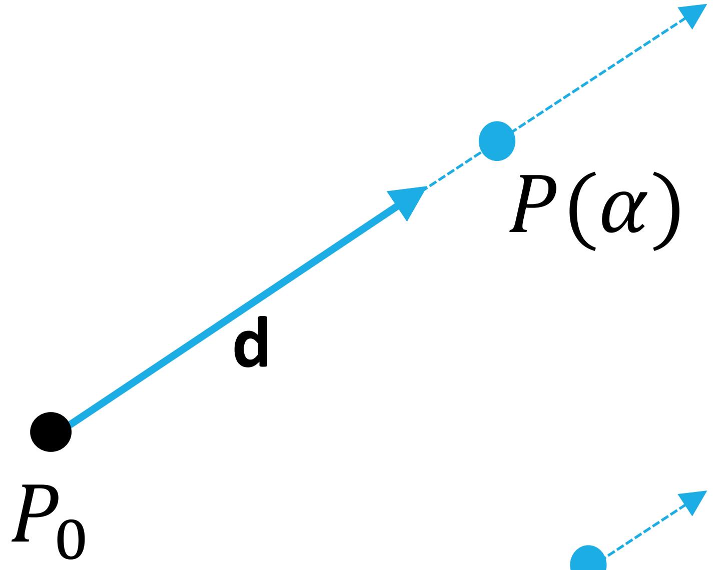
- Parametric form of a line:

$$P(\alpha) = P_0 + \alpha \boldsymbol{d},$$

P_0: an arbitrary point (origin)

α: an arbitrary scalar

$\boldsymbol{d}$: an arbitrary vector (direction)

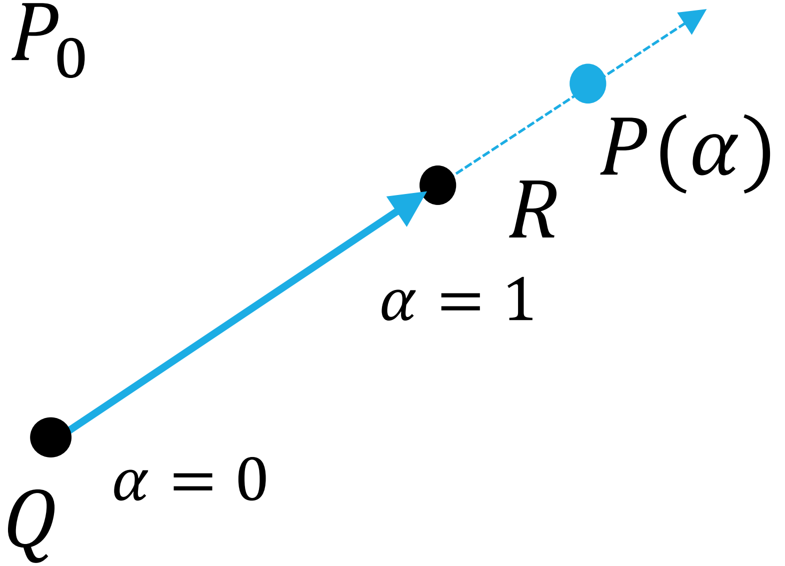


- Affine sum:

$$P = Q + \alpha(R - Q) = (1 - \alpha)Q + \alpha R$$

$P = \alpha_1 Q + \alpha_2 R,\ \alpha_1 + \alpha_2 = 1$ (affine sum)

Convex sum if $\alpha_1 \geq 0, \alpha_2 \geq 0$.

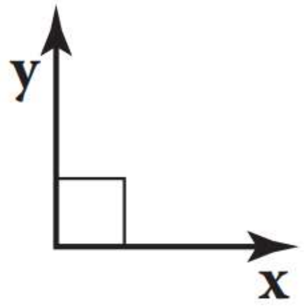


Cartesian Coordinates of Vectors

- A 2D vector can be written as a combination of any two nonzero vectors that are not parallel (linearly independent):

$$\boldsymbol{c} = a_c \boldsymbol{a} + b_c \boldsymbol{b}$$

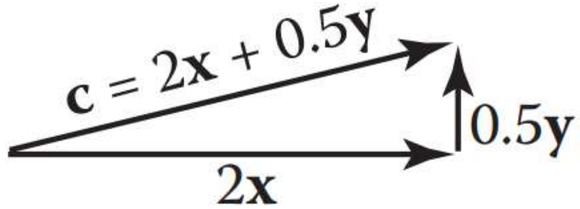
- If we use two orthonormal vectors $\boldsymbol{x}, \boldsymbol{y}$,



$$\boldsymbol{a} = x_a \boldsymbol{x} + y_a \boldsymbol{y}$$

The coordinates of $\boldsymbol{a} = (x_a, y_a)$, or written as

$$\boldsymbol{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

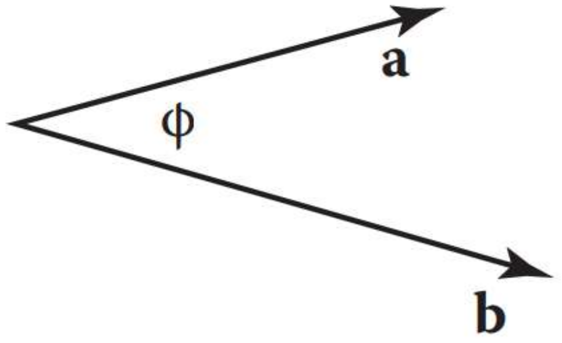


- Also applies to 3D, 4D,…

Dot Product (내적)

- Dot product (= scalar product = inner product):

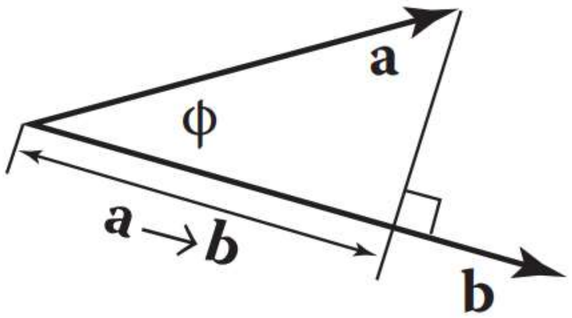
$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos \Phi$$



$\mathbf{a} \bullet \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos \phi$

- The projection of vector **a** onto vector **b**:

$$\boldsymbol{a} \longrightarrow \boldsymbol{b} = \|\boldsymbol{a}\| \cos \Phi = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|}$$



Dot Product (cont'd)

- Some dot product rules:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

$$(k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b}) = k\boldsymbol{a} \cdot \boldsymbol{b}$$

- $\boldsymbol{x} \cdot \boldsymbol{y} = 0$?
- If $\boldsymbol{a} = (x_a, y_a), \boldsymbol{b} = (x_b, y_b)$, then

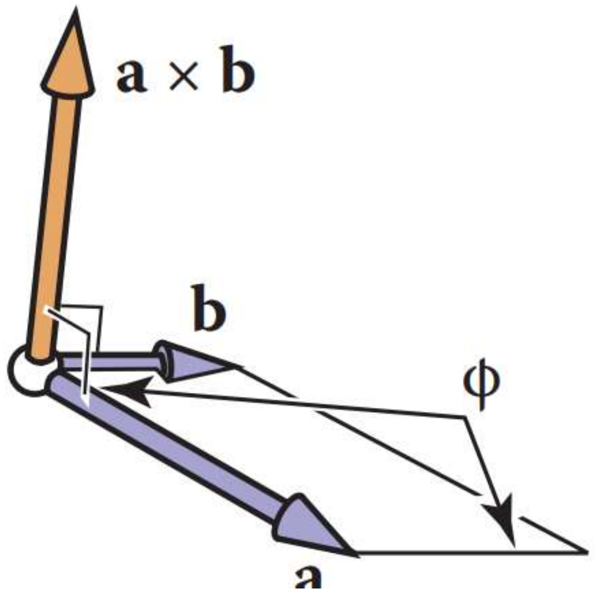
$$\boldsymbol{a} \cdot \boldsymbol{b} = x_a x_b + y_a y_b$$

Cross Product (외적)

- Cross Product (= vector product = exterior product):

$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin \Phi$$

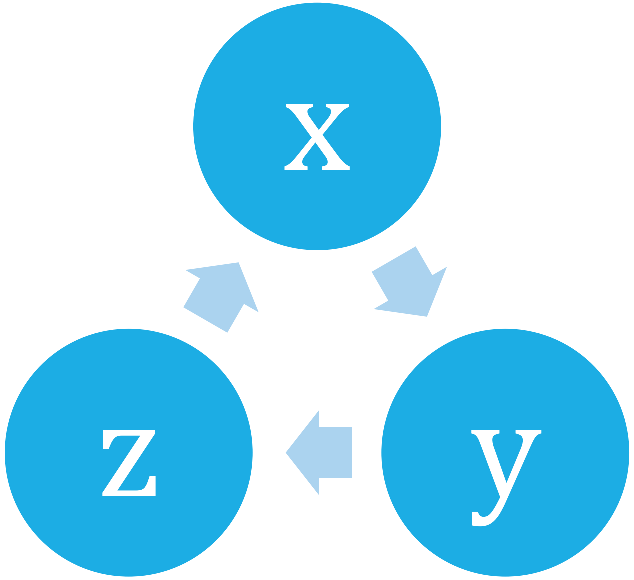
- $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$
- $\|\boldsymbol{a} \times \boldsymbol{b}\|$ = area of parallelogram made by $\boldsymbol{a}$ and $\boldsymbol{b}$

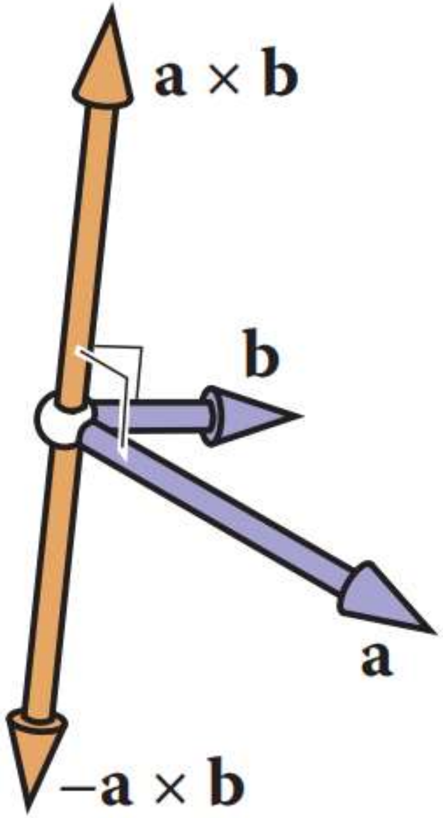


Cross Product (cont'd)

- Right-hand rule applies
- If $x = (1,0,0), y = (0,1,0), z = (0,0,1)$,

$$x \times y = +z$$
$$y \times x = -z$$
$$y \times z = +x$$
$$z \times y = -x$$
$$z \times x = +y$$
$$x \times z = -y$$





Cross Product (cont'd)

- Some cross product rules:

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$
$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$
$$(k\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (k\boldsymbol{b}) = k(\boldsymbol{a} \times \boldsymbol{b})$$

- If $\boldsymbol{a} = (x_a, y_a, z_a), \boldsymbol{b} = (x_b, y_b, z_b)$, then

$$\boldsymbol{a} \times \boldsymbol{b} = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b)$$

Coordinate Frames

- Orthonormal bases
  - In 2D, use **u**, **v** as bases such that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$ and $\boldsymbol{u} \cdot \boldsymbol{v} = 0$
  - In 3D, use **u**, **v**, **w** as bases such that

$$\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = \|\boldsymbol{w}\| = 1,$$

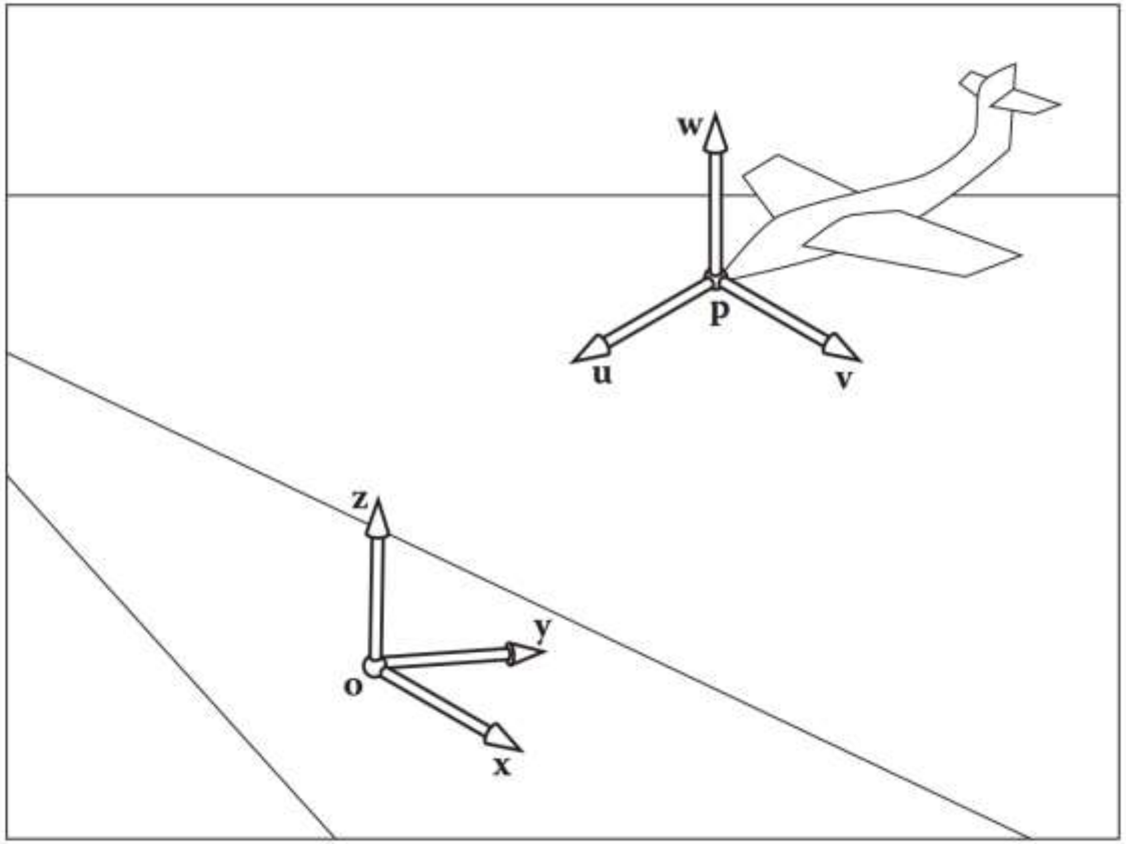
$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{u} = 0.$$

- The Cartesian canonical orthonormal basis is just one of infinitely many orthonormal bases (bases are not explicitly stored)
- The Cartesian canonical coordinate system: x,y,z-axes + o

Coordinate Systems

- The global model is typically stored in the canonical coordinate system (global/world coordinate system)

- We can define the model in another coordinate system (a frame of reference/coordinate frame)



Coordinate Systems

- $\boldsymbol{u} = x_u\boldsymbol{x} + y_u\boldsymbol{y} + z_u\boldsymbol{z}$

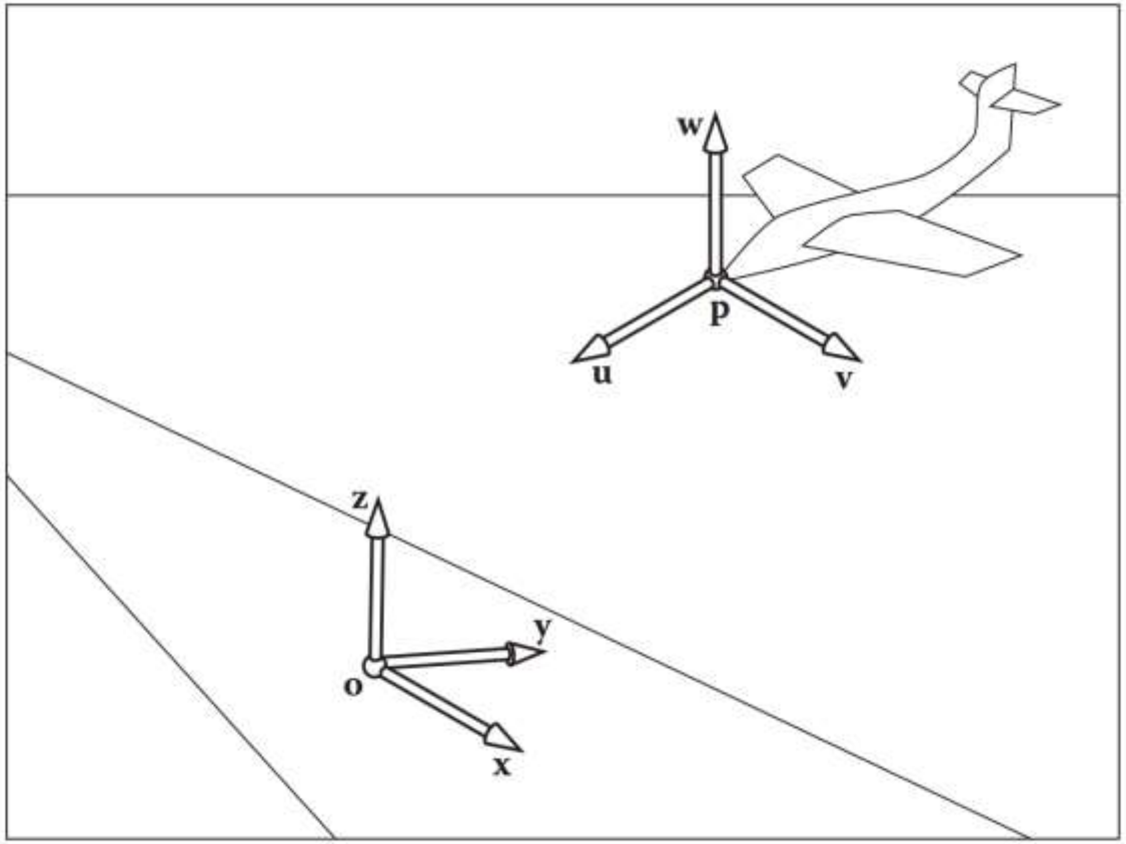
  $p = o + x_p\boldsymbol{x} + y_p\boldsymbol{y} + z_p\boldsymbol{z}$

- Express a vector a in the airplane's coordinate frame?

  $$\boldsymbol{a} = u_a\boldsymbol{u} + v_a\boldsymbol{v} + w_a\boldsymbol{w}$$

$\Rightarrow$ Get (u_a, v_a, w_a) by

$\Rightarrow$ $u_a = \boldsymbol{a} \cdot \boldsymbol{u}$, $v_a = \boldsymbol{a} \cdot \boldsymbol{v}$, $w_a = \boldsymbol{a} \cdot \boldsymbol{w}$



2D Geometric Transformation

2D Linear Transformation

- Linear transformation

⇒ Use 2 x 2 matrix to transform a 2D vector

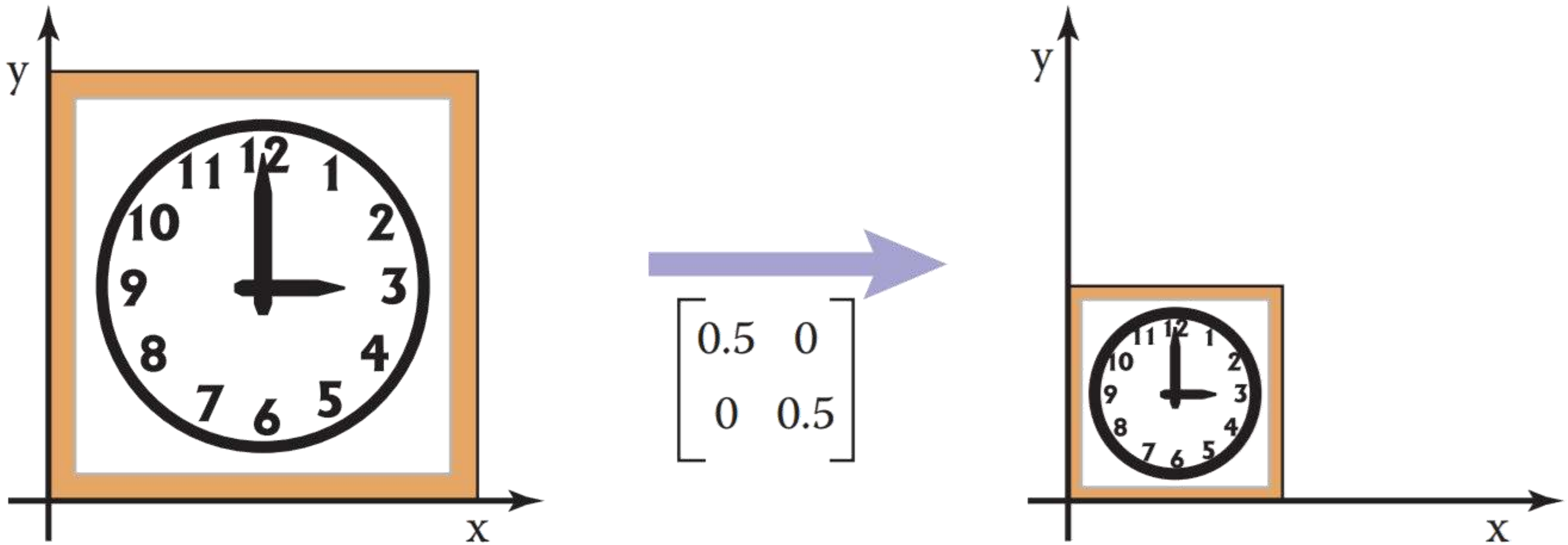
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

2D Scaling

- Scaling: changes length (and direction)

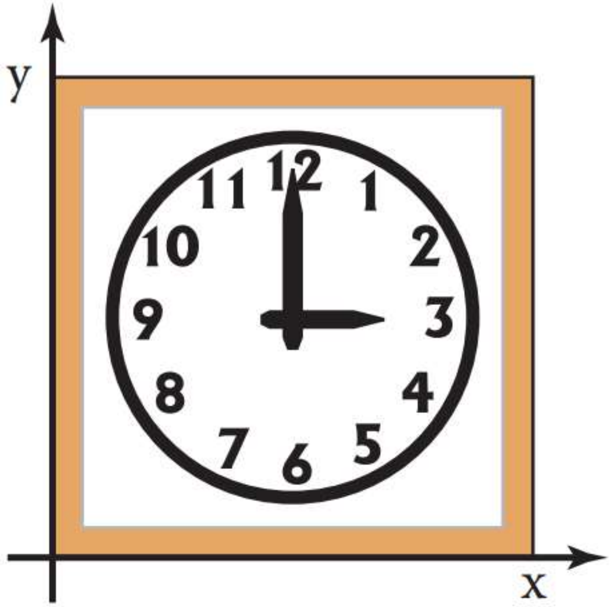
$$scale(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

- If $s_x = s_y$, uniform scaling

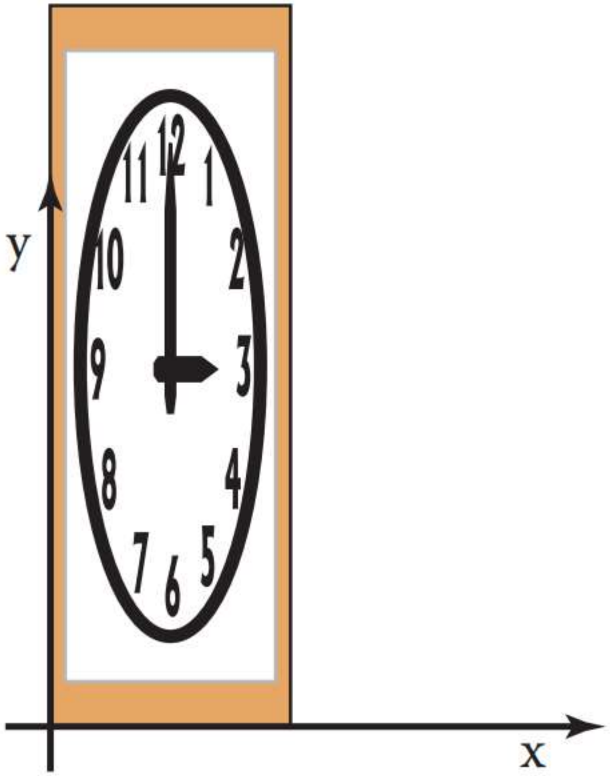


2D Scaling

- If $s_x \neq s_y$, nonuniform scaling



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$



2D Rotation

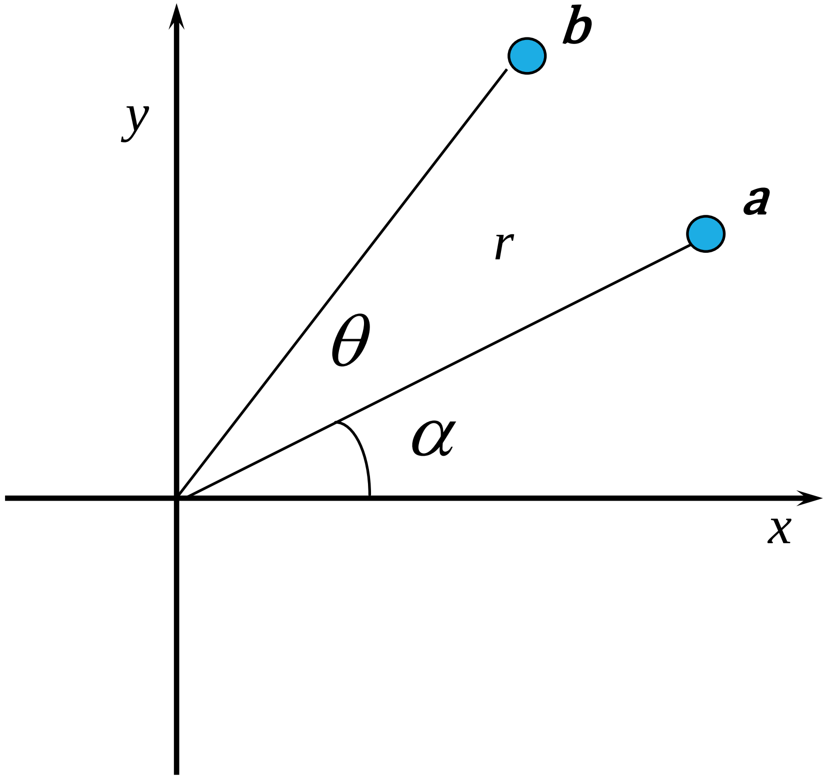
- A vector $\boldsymbol{a} = (x_a, y_a)$ can be written as a polar form

$$a = (x_a, y_a) = (r\cos\alpha\,, r\sin\alpha)$$

where $r = \sqrt{{x_a}^2 + {y_a}^2}$

- Rotating **a** counter-clockwise by θ to **b**:

$$x_b = r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$y_b = r\sin(\alpha+\theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$$



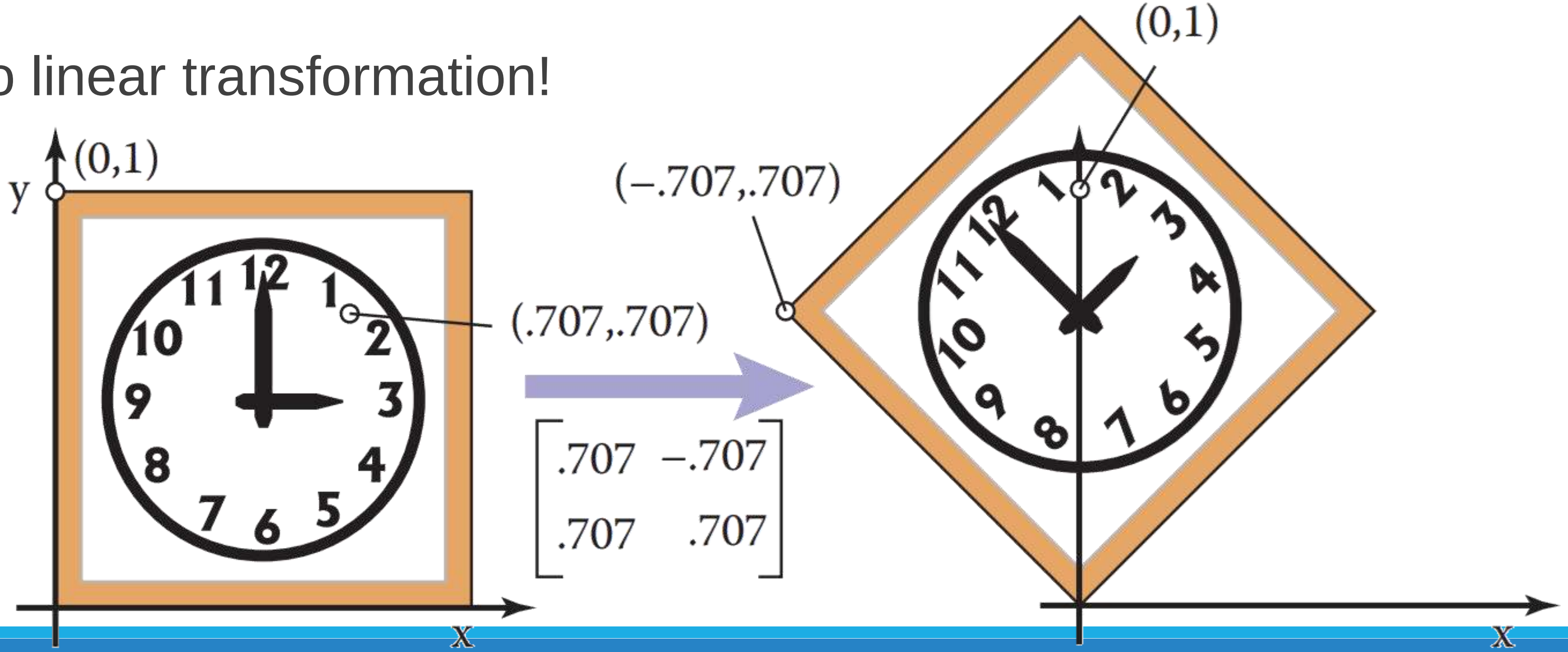
2D Rotation (cont'd)

$$x_b = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$
$$y_b = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta$$

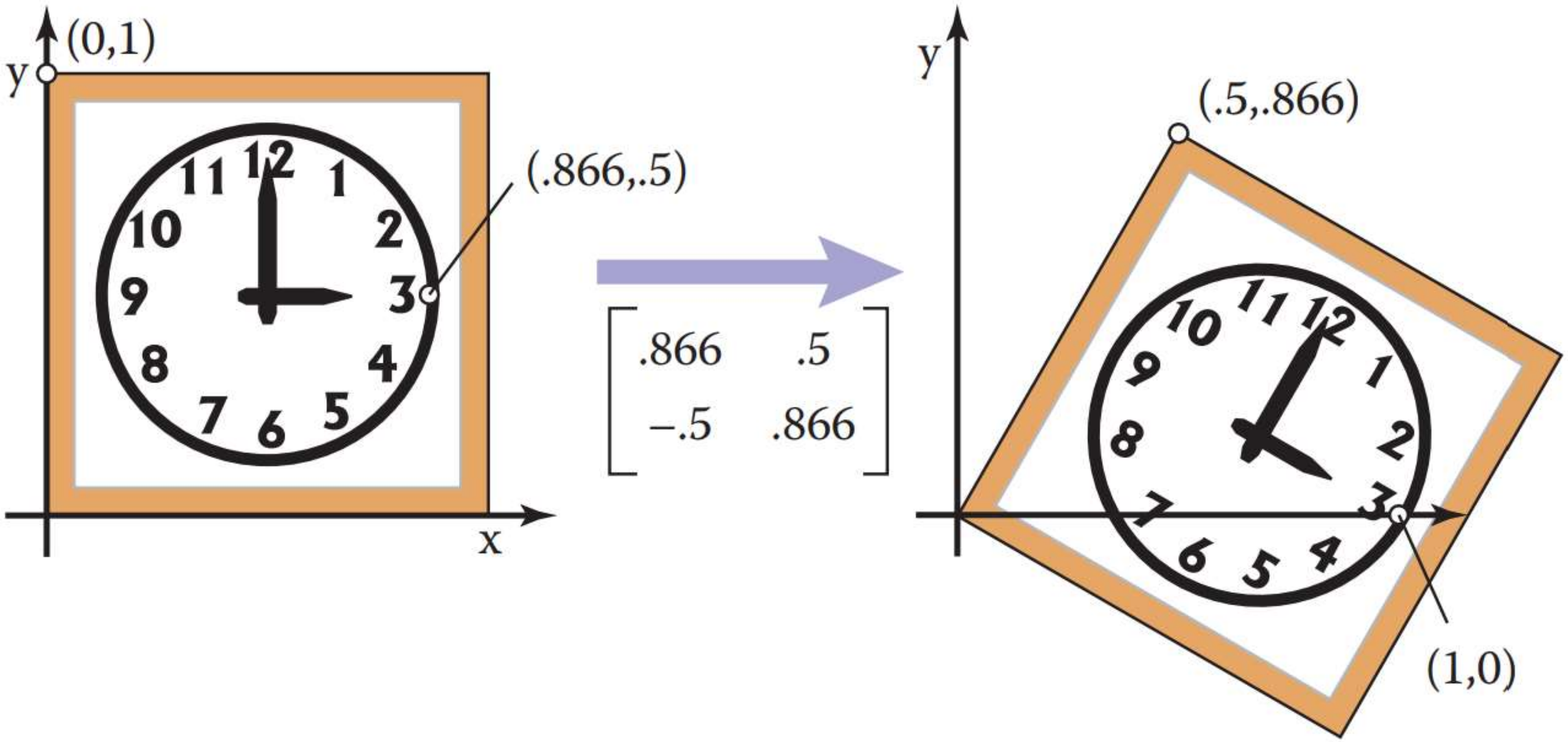
$$\begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

∴ Rotation is also linear transformation!



2D Rotation (cont'd)



Rotation Properties

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Orthogonal matrix: Two columns (rows) are orthogonal

$$(\cos\theta, \sin\theta) \cdot (-\sin\theta, \cos\theta) = 0$$

- What is the inverse matrix R^{-1}, of R ? $(R^{-1}R = RR^{-1} = I)$

Rotation Properties

- Matrix for rotating by –θ angle is as follows:

$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R^T$$

- Geometrically, $RR_{-\theta} = R_{-\theta}R = I$
- Therefore, $RR^T = R^TR = I$

$$\therefore R^{-1} = R^T$$

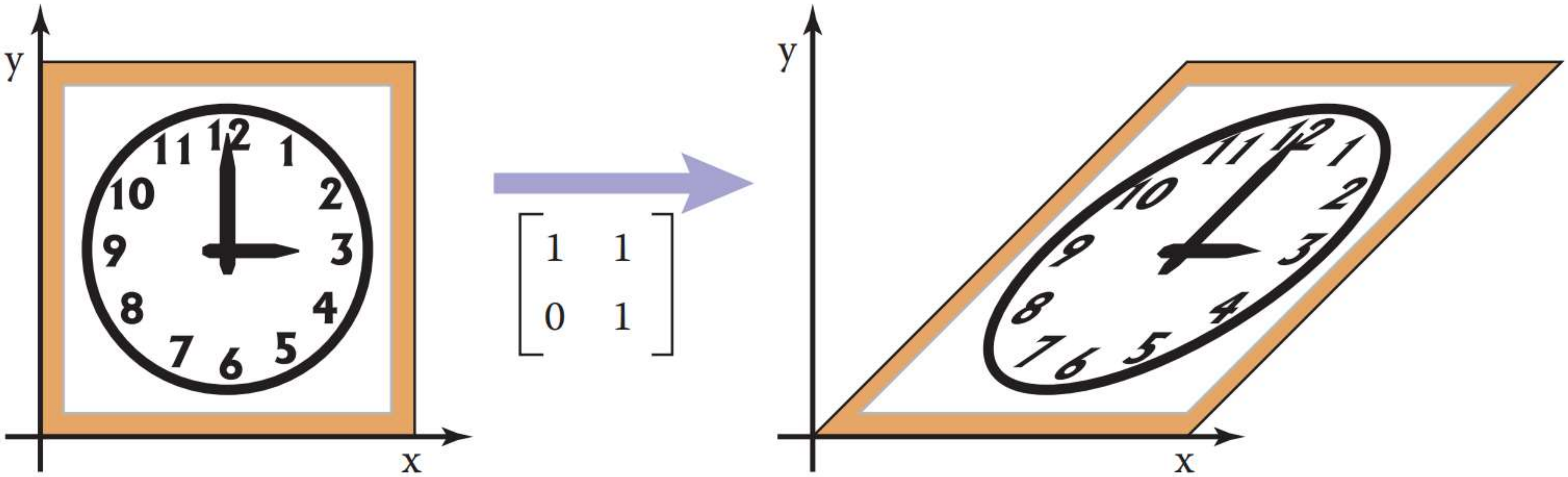
Other 2D Linear Transformations

Q) What do these transformations do?

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

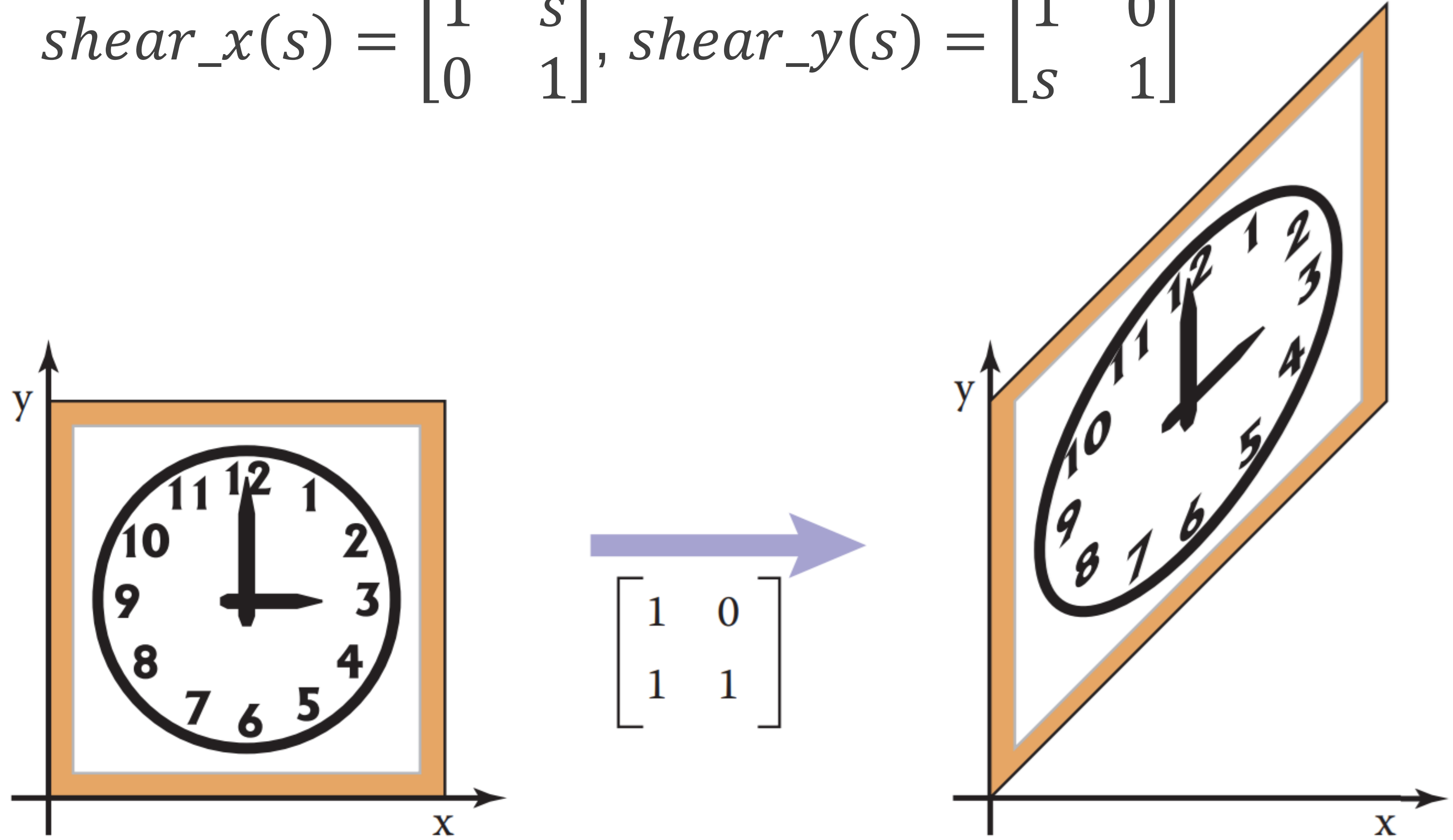
2D Shear

$$shear\_x(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \; shear\_y(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$



2D Shear

$$shear\_x(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \; shear\_y(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

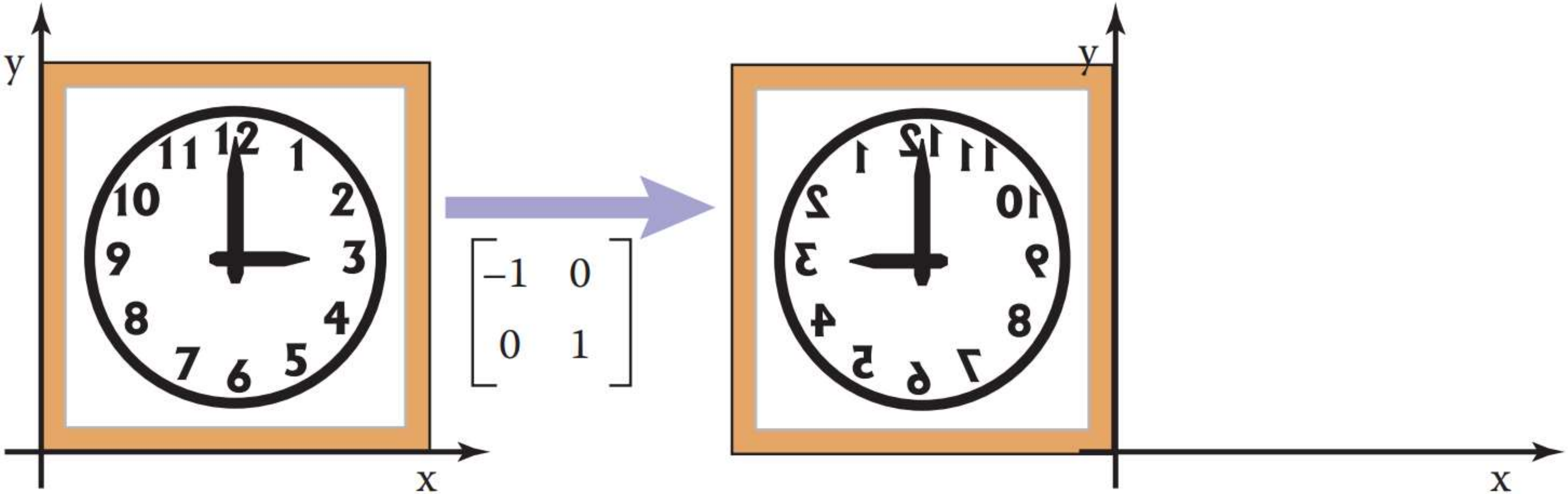


$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

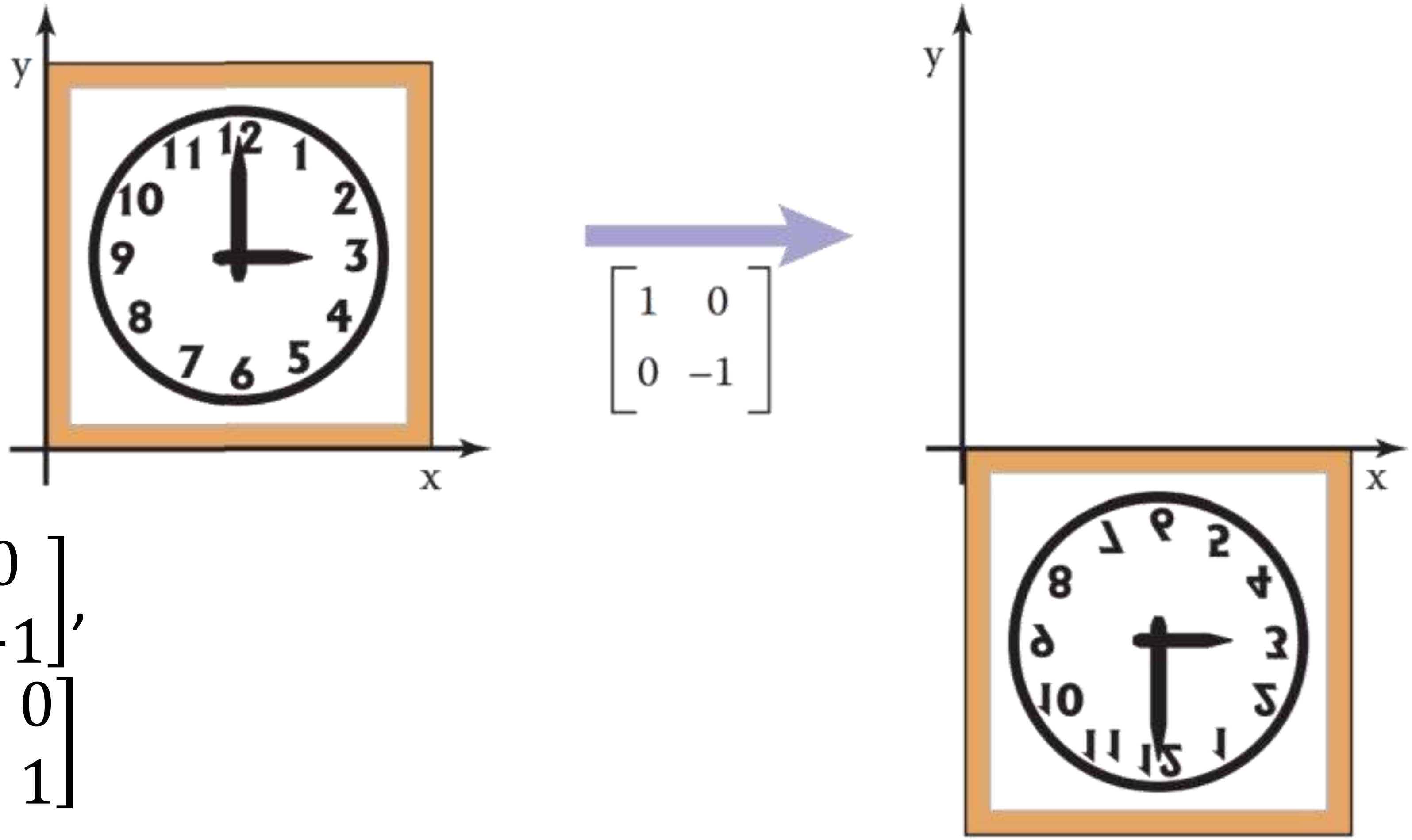
2D Reflection

- Reflection: mirror a vector across either of the coordinate axes

$$reflect\_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, reflect\_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



2D Reflection



$$reflect\_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$reflect\_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Composition of Linear Transformations

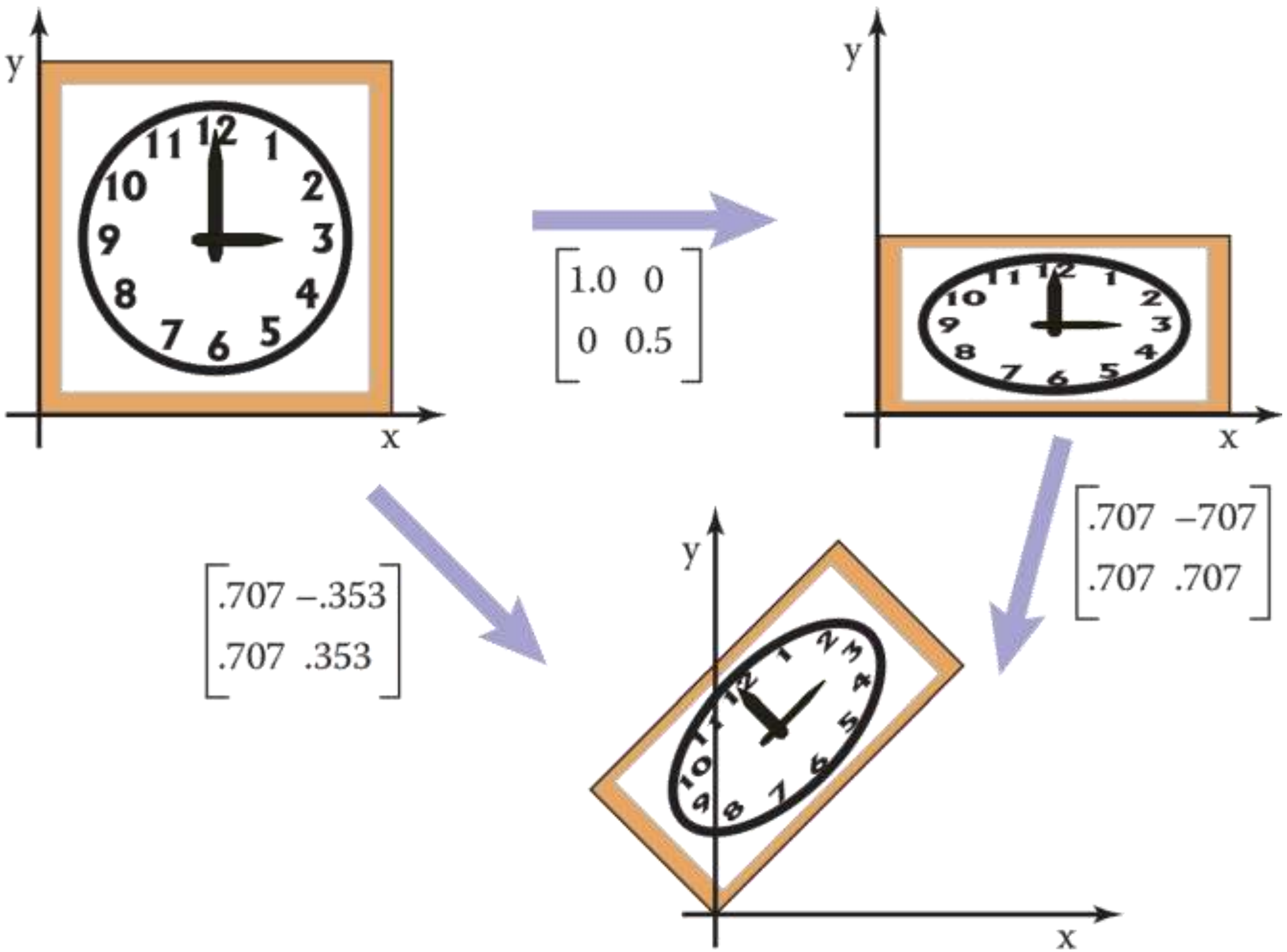
- Apply more than one transformation

$\Rightarrow$ Multiply transformation matrices

ex. $\boldsymbol{v}_2 = S\boldsymbol{v}_1$, then $\boldsymbol{v}_3 = R\boldsymbol{v}_2$

$\Rightarrow$ $\boldsymbol{v}_3 = R\boldsymbol{v}_2 = R(S\boldsymbol{v}_1) = (RS)\boldsymbol{v}_1$

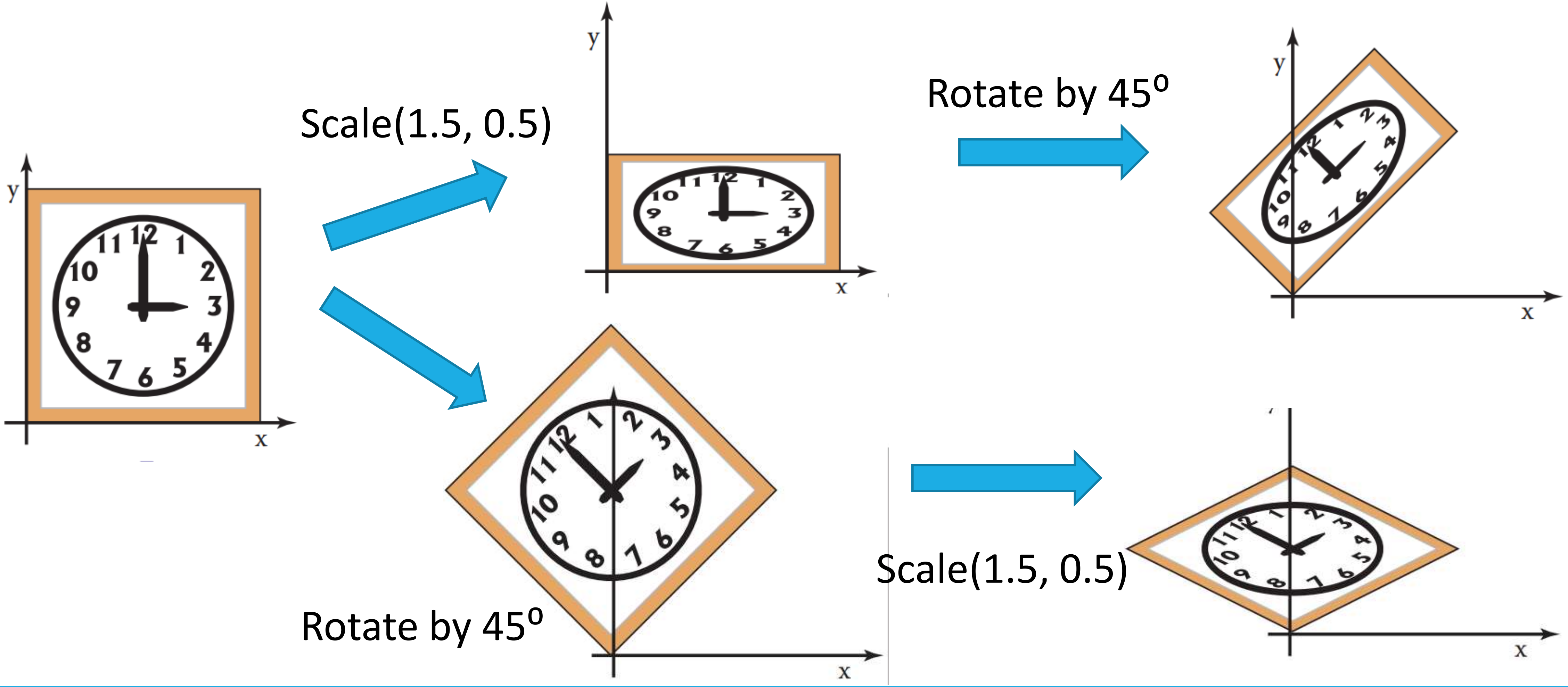
$\Rightarrow$ $M = RS$



First apply non-uniform scaling, and then rotate by 45 degrees

Composition of Transformations

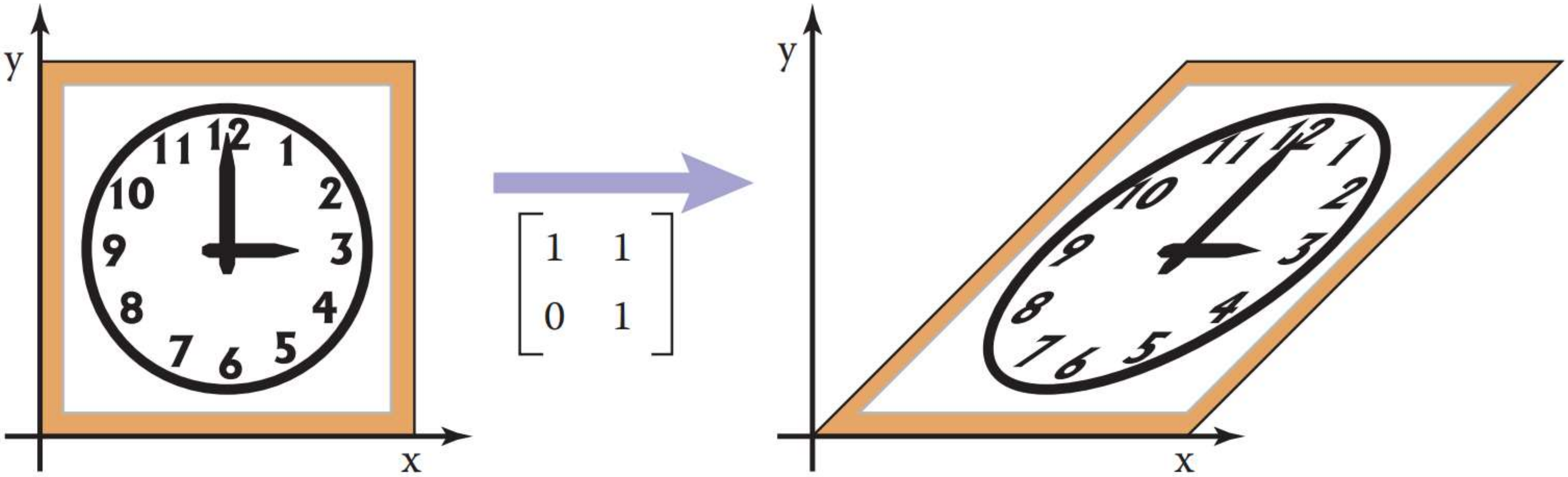
- What about changing the order of transformation?



Decomposition of Transformations

Q) Can we express shear transformation by the product of scaling and rotation transformation matrices?

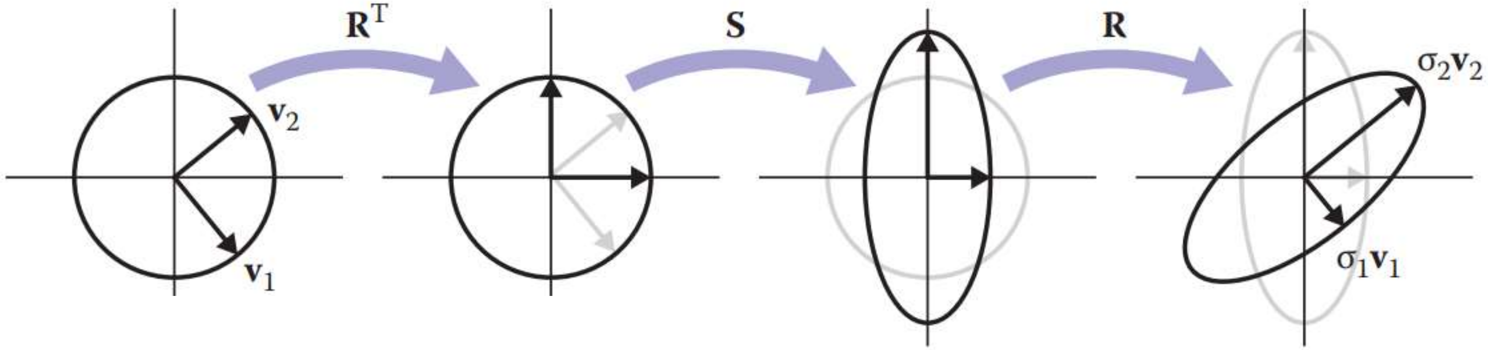
Q) What about arbitrary linear transformation?



Decomposition of Transformation

- For symmetric transformation, use eigen value decomposition

$$A = RSR^T = R\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}R^T$$



ex)

$$\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix} = \mathbf{R}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\mathbf{R}^T$$

$$= \begin{bmatrix}0.8507 & -0.5257\\ 0.5257 & 0.8507\end{bmatrix}\begin{bmatrix}2.618 & 0\\ 0 & 0.382\end{bmatrix}\begin{bmatrix}0.8507 & 0.5257\\ -0.5257 & 0.8507\end{bmatrix}$$

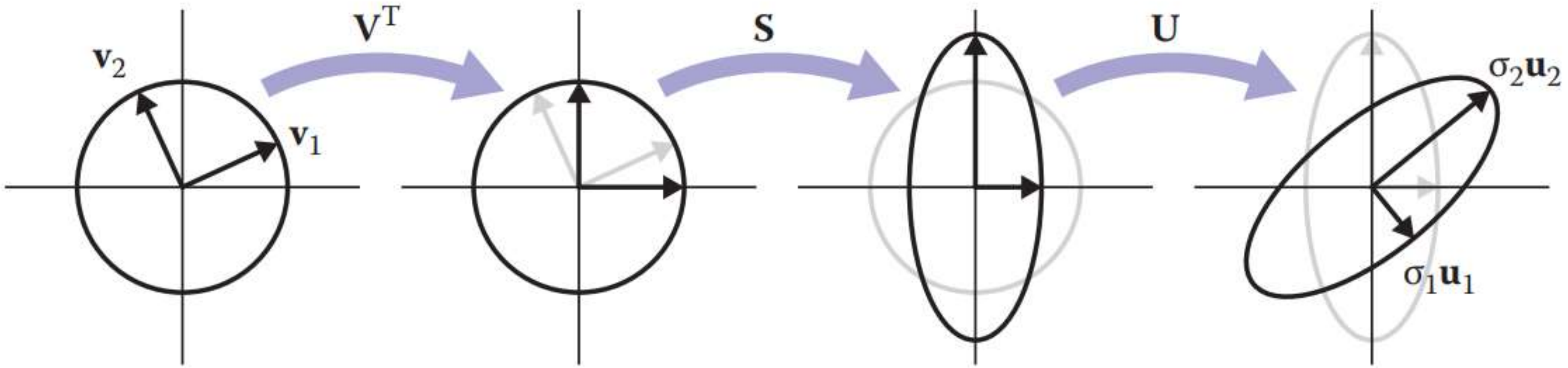
$$= \text{rotate}\,(31.7°)\ \text{scale}\,(2.618, 0.382)\ \text{rotate}\,(-31.7°).$$

Decomposition of Transformation

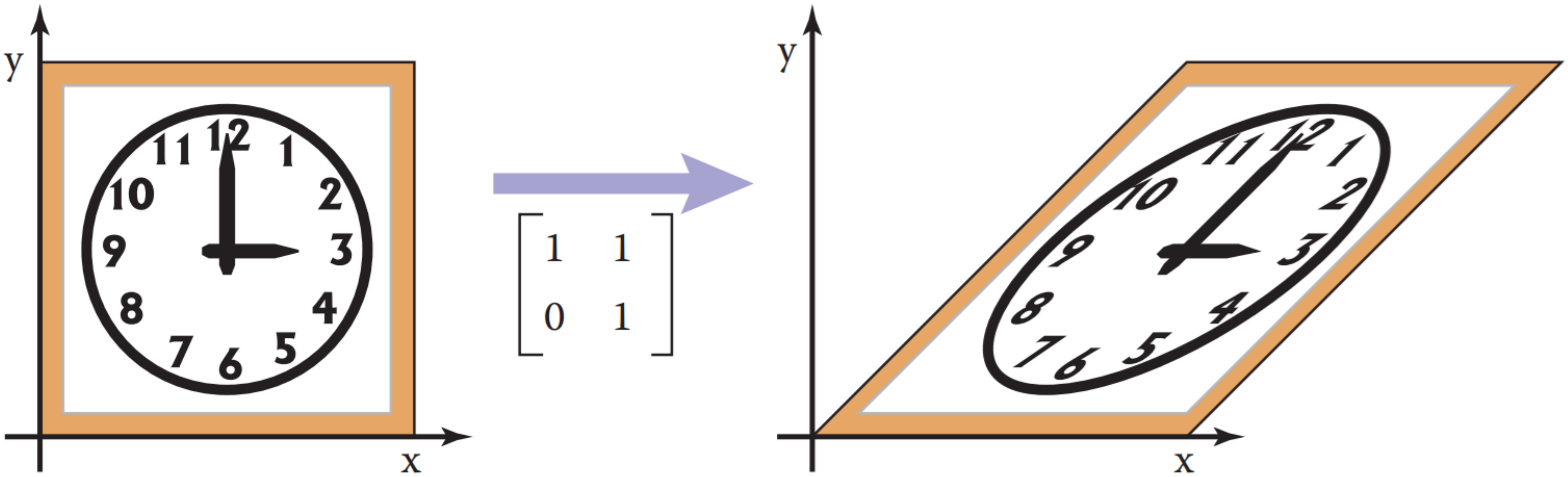
- For arbitrary transformation, use singular value decomposition (SVD)

$$A = USV^T$$

where U, V are orthogonal (rotation) matices, and S (eigen values) is a (non-uniform) scale matrix



Decomposition of Transformation



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \mathbf{R}_2 \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \mathbf{R}_1$$

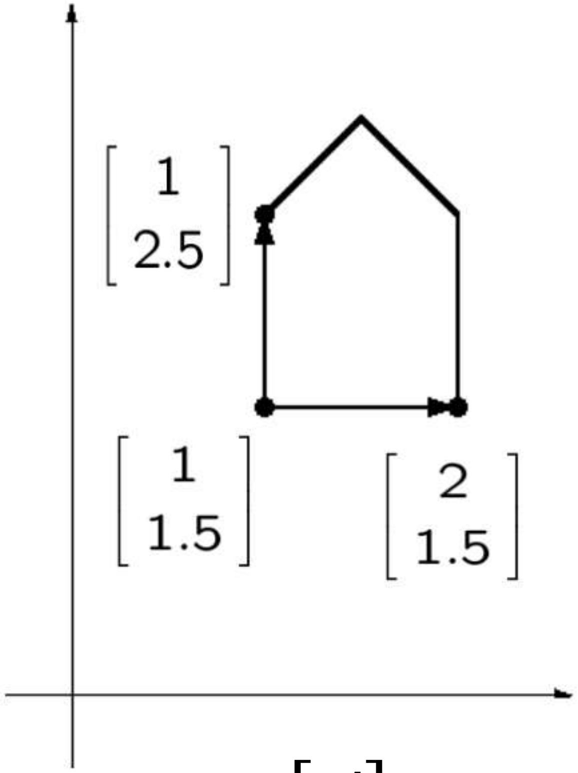
$$= \begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix} \begin{bmatrix} 1.618 & 0 \\ 0 & 0.618 \end{bmatrix} \begin{bmatrix} 0.5257 & 0.8507 \\ -0.8507 & 0.5257 \end{bmatrix}$$

$$= \text{rotate}\,(31.7^\circ)\ \text{scale}\,(1.618, 0.618)\ \text{rotate}\,(-58.3^\circ).$$

2D Translation

- Translation: move all points of an object by the same amounts

$$x' = x + x_t$$
$$y' = y + y_t$$



- Translation cannot be expressed by

2 x 2 linear transformation matrix

- How to solve this problem?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$$

Homogeneous Coordinate

- A point (x,y) in 2D is represented by

$$\begin{bmatrix} x \\ y \\ {\color{red}1} \end{bmatrix}$$

- Homogeneous coordinates can express both linear transformation and translation at the same time

$\Rightarrow$ Affine transformation

- Rigid-body transformation: translation + rotation

Homogeneous Coordinate

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

$\Rightarrow$

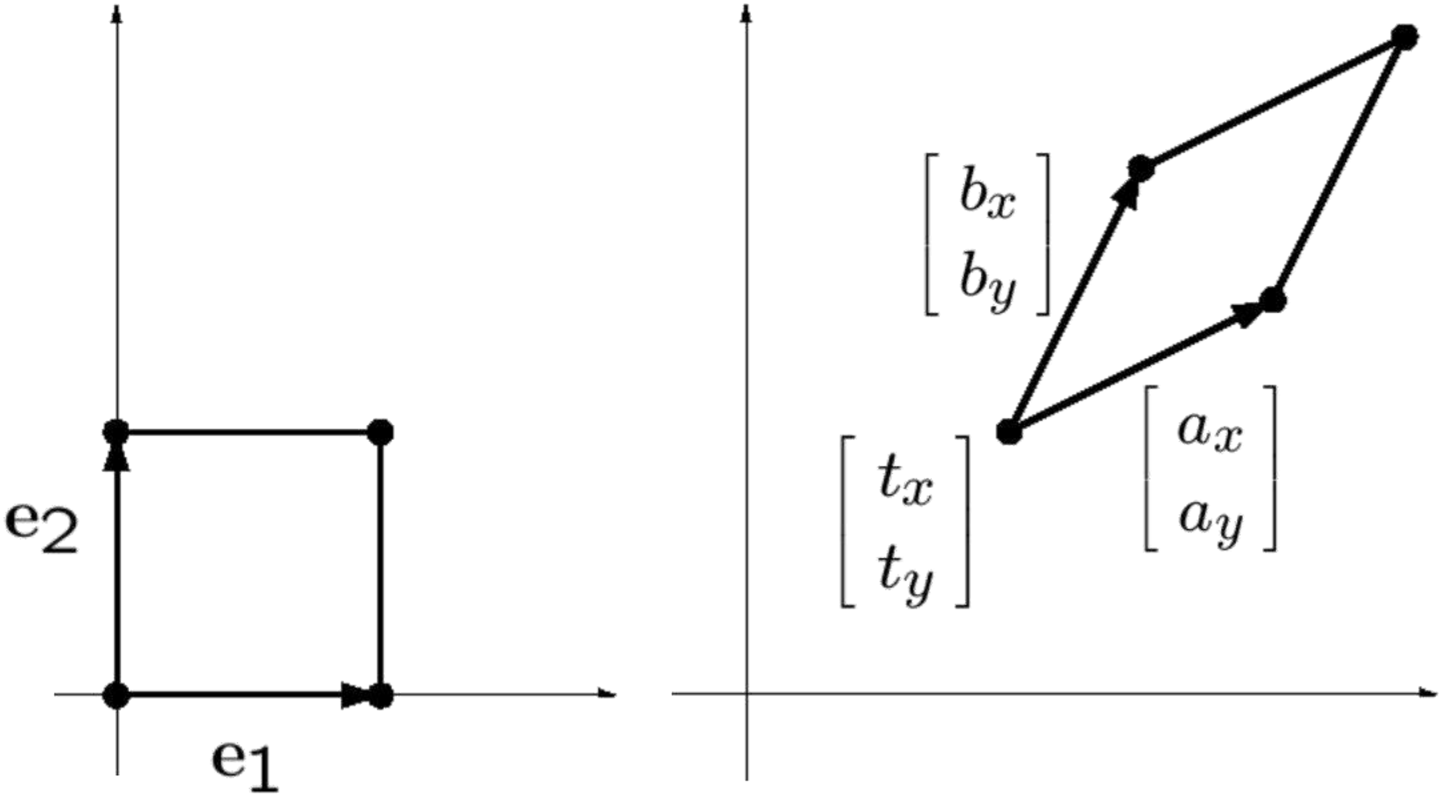
$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

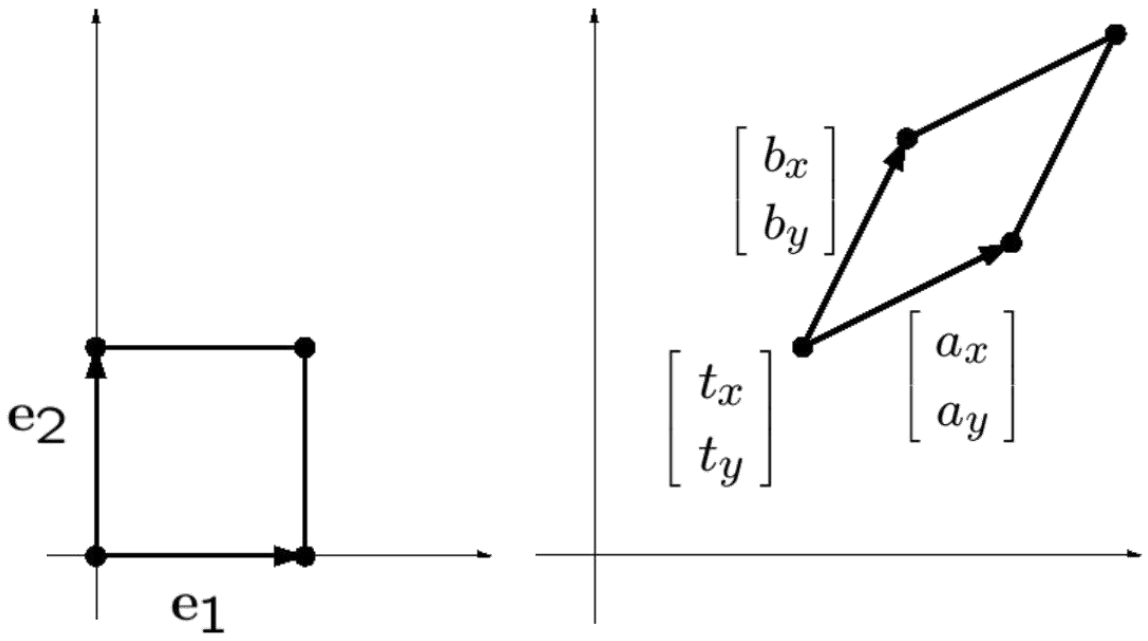
Q) Write a transformation matrix for the following transformation



Homogeneous Coordinates

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ 1 \end{bmatrix} = \begin{bmatrix} a_x & b_x & t_x \\ a_y & b_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

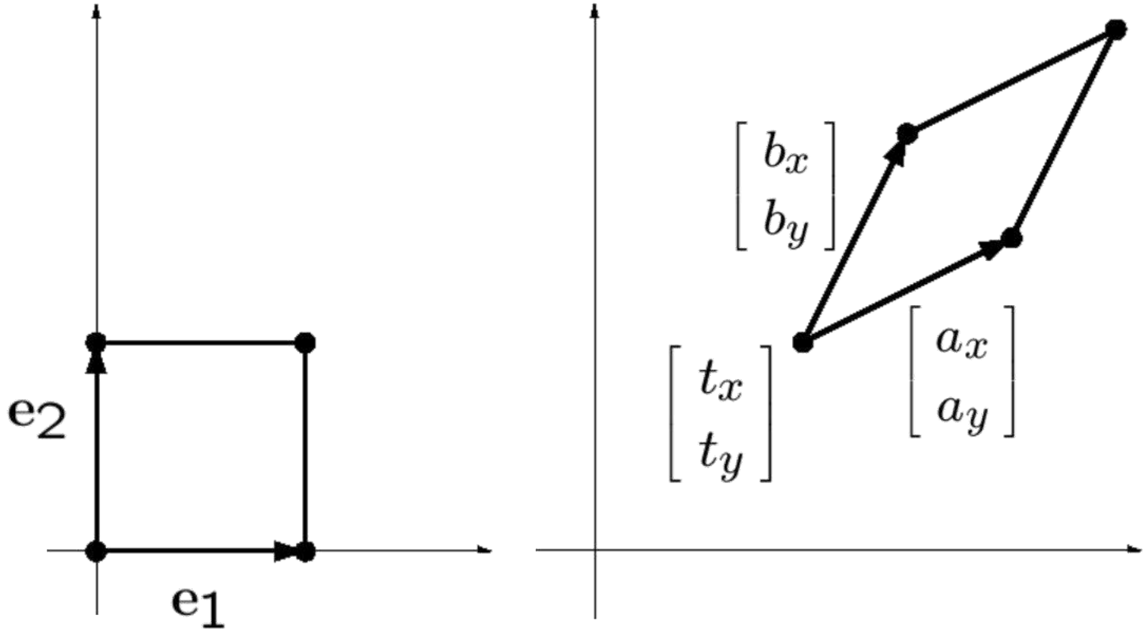


Homogeneous Coordinates

$$\begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix} = \begin{bmatrix} a_x & b_x & t_x \\ a_y & b_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ 0 \end{bmatrix} = \begin{bmatrix} a_x & b_x & t_x \\ a_y & b_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

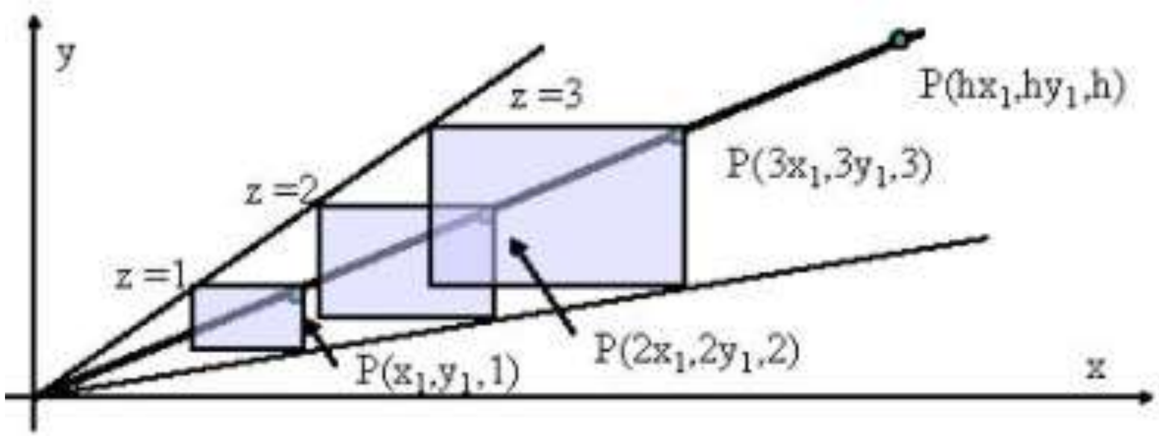
$$\begin{bmatrix} b_x \\ b_y \\ 0 \end{bmatrix} = \begin{bmatrix} a_x & b_x & t_x \\ a_y & b_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



Homogeneous Coordinates

- A point p is expressed as $\mathrm{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- A vector **v** is expressed as $\mathbf{v} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
- In homogeneous coordinates,

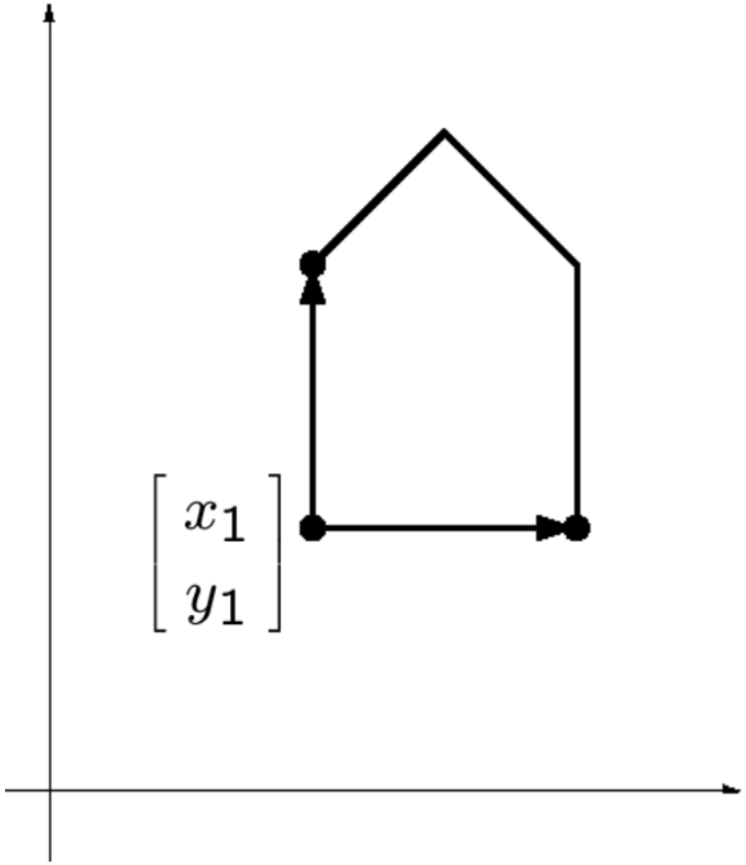
$$\mathrm{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} hx \\ hy \\ h \end{bmatrix}$$

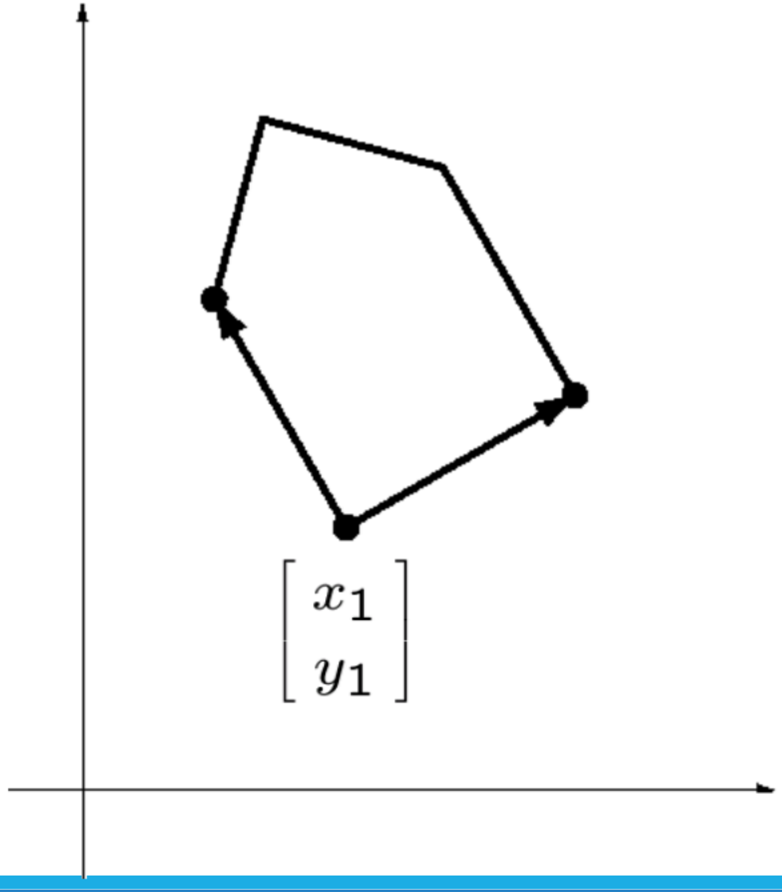


3D Representarion of homogeneous space

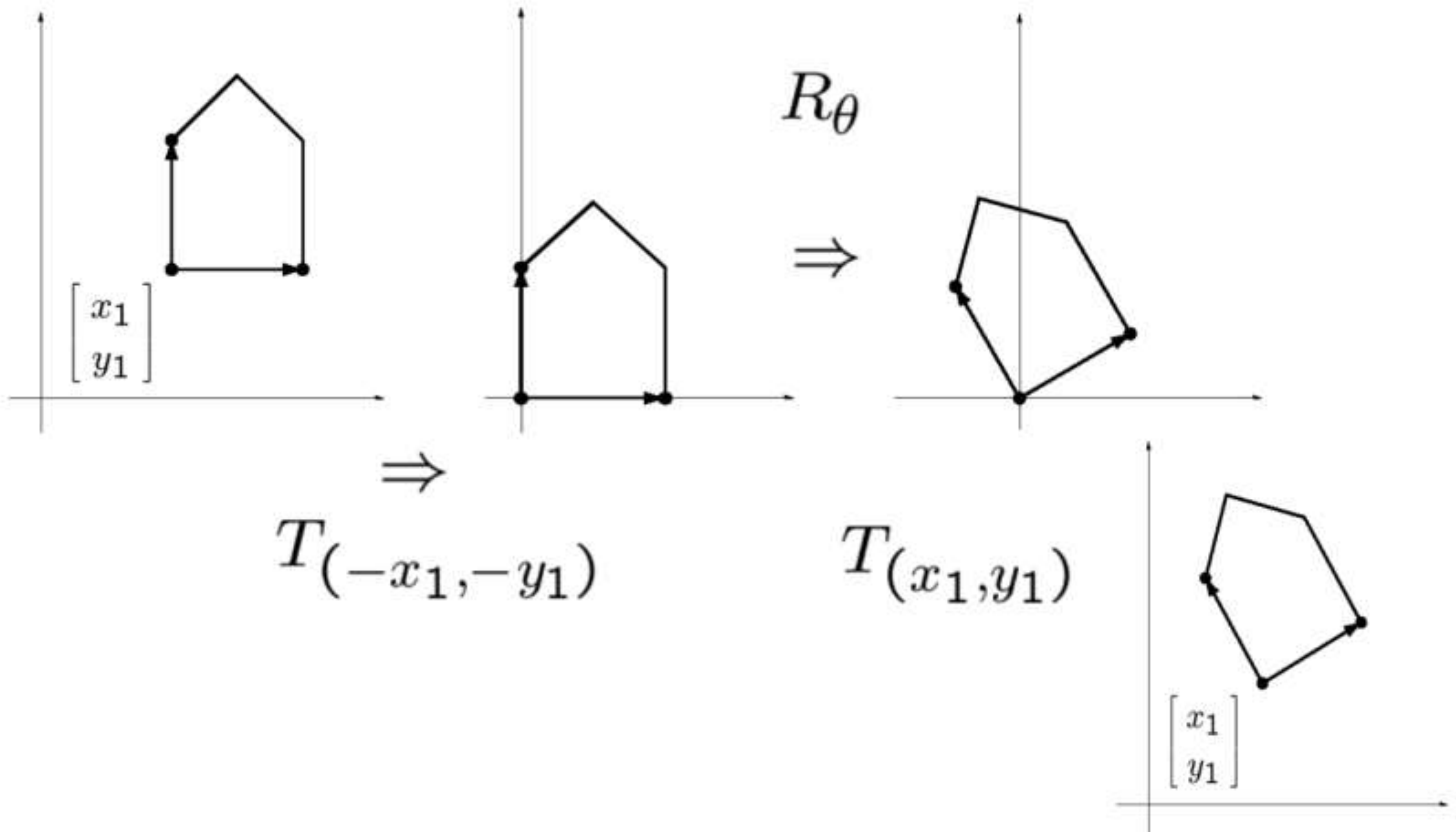
일반적인 2D Transformation

Q) Write a transformation matrix for the following transformation





일반적인 2D Transformation



일반적인 2D Transformation

$$\begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & -x_1\cos\theta + y_1\sin\theta \\ \sin\theta & \cos\theta & -x_1\sin\theta - y_1\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_1(1-\cos\theta) + y_1\sin\theta \\ \sin\theta & \cos\theta & y_1(1-\cos\theta) - x_1\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

Summary

- Points, scalars, vectors
- Coordinates, coordinate frames
- 2D transformation
- Homogeneous coordinates